# Nahm's equations in hyperkähler geometry

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## Contents

# 1 Introduction

Nahm's equations are a system of non-linear ordinary differential equations naturally appearing in gauge theory. They are especially important for *hyperkähler geometry*, i.e. the study of Riemannian manifolds $(M, g)$ with three complex structures $I, J, K$ that are Kähler with respect to $g$ and satisfy the quaternionic identity $IJK = -1$. Nahm's equations are an essential tool to construct many non-trivial examples. They depend on a choice of a compact Lie group $G$, and have been used to construct hyperkähler structures on various manifolds associated to $G$, such as the cotangent bundle $T^*G_{\mathbb{C}}$ of its complexification $G_{\mathbb{C}}$ [38], and all coadjoint orbits in $\mathfrak{g}_{\mathbb{C}}^*$ [40, 39, 37, 7].

Concretely, Nahm's equations are the first-order system of ODEs for quadruples of maps $A_0, A_1, A_2, A_3$ from an interval $I \subseteq \mathbb{R}$ to the Lie algebra $\mathfrak{g}$ of a compact Lie group $G$ given by

$$\begin{aligned}
\dot{A}_1 + [A_0, A_1] + [A_2, A_3] &= 0\\
\dot{A}_2 + [A_0, A_2] + [A_3, A_1] &= 0\\
\dot{A}_3 + [A_0, A_3] + [A_1, A_2] &= 0.
\end{aligned}$$

There is also a natural action of the group of smooth maps $I \to G$ on the set of solutions to Nahm's equations. Hence, by choosing different intervals $I$ and imposing suitable boundary conditions, we can construct various *moduli spaces* of solutions to Nahm's equations. In many cases, those moduli spaces are finite-dimensional hyperkähler manifolds. The hyperkähler structures ultimately descend from writing $A = A_0 + A_1 i + A_2 j + A_3 k$ where $i, j, k \in \mathbb{H}$ are the quaternions and viewing the moduli space as an infinite-dimensional *hyperkähler quotient* [30], a concept which we will review in §3.

Nahm's equations can be seen as a special case of the anti-self-dual (ASD) equations from gauge theory. Those are non-linear partial differential equations that first appeared in physics, namely, Yang–Mills theory. They are the condition that a connection $A$ on a principal $G$-bundle over a four-manifold $M$ has anti-self-dual curvature $F_A$, i.e. $*F_A + F_A = 0$, where $*$ is the Hodge-star operator. We will review in §2 how Nahm's equations can be seen as the ASD equation for an $\mathbb{R}^3$ invariant connection on $\mathbb{R}^4$, as an example of dimensional reduction. The concept of dimensional reduction of the ASD equations has been extremely useful for getting interesting hyperkähler moduli spaces. Other examples include moduli spaces of monopoles [28, 1, 32, 16, 25] and Higgs bundles [29], as we will review below.

We will then describe various moduli spaces of solutions to Nahm's equations and explain a few interesting properties, focusing on the cases where the interval $I$ on which Nahm's equations are defined is bounded. More precisely, in §4, we describe the most basic case where $I$ is a compact interval $[a, b]$ and solutions are smooth throughout. This gives Kronheimer's hyperkähler metric [38] on the cotangent bundle $T^*G_{\mathbb{C}}$. Then, in Section 5, we discuss Nahm's equations on an open interval $(a, b)$ with prescribed poles at the boundary points, following Bielawski [4]. This gives other interesting hyperkähler manifolds associated to $G$ generalizing those relevant for monopole moduli spaces [28, 1, 32, 16], and can also be used to construct new hyperkähler manifolds from old by the so-called *hyperkähler slice construction.*

# 2 The ASD equations and its reductions

In this section, we review the anti-self-duality (ASD) equations and its reductions, including Nahm's equations. See, for example, [17, 2, 19] for introductions to gauge theory and the ASD equations.

## 2.1 The general ASD equations

Let $M$ be an oriented Riemannian 4-manifold, $G$ a compact Lie group, and $P \to M$ a principal $G$-bundle over $M$. Let $A \in \Omega^1_P(\mathfrak{g})$ be a connection on $P$ and $F_A \in \Omega^2_M(\mathrm{ad}(P))$ its curvature. Then, because $M$ is 4-dimensional, the Hodge star operator maps two-forms to two-forms. Hence, it makes sense to require that

$$*F_A + F_A = 0.$$

This equation is called the *anti-self-dual Yang-Mills equation*, or the *ASD equation* for short. The self-dual version $*F_A = F_A$ is equivalent to the ASD equation by switching the orientation of $M$. If we simply ask that $*F_A$ is proportional to $F_A$, say, $*F_A = \lambda F_A$, then because $*^2 = 1$, we get $\lambda^2 = 1$ so $\lambda = \pm 1$.

To see where this equation comes from, suppose first that $M$ is compact and consider the Yang–Mills functional

$$\mathrm{YM} : A \longmapsto \int_M \|F_A\|^2,$$

from the space of connections on $P$ to $\mathbb{R}$, where $\|F_A\|^2$ is computed using the Riemannian metric on $M$ and a choice of invariant inner-product on $\mathfrak{g} := \mathrm{Lie}(G)$. The Euler-Lagrange equation for the Yang–Mills functional is

$$d_A * F_A = 0 \in \Omega^3_M(\mathrm{ad}(P)),$$

which is a second order PDE called the Yang–Mills equations. If $*F_A = \pm F_A$ then we automatically get the Yang–Mills equations $d_A * F_A = 0$ by the Bianchi identity. In fact, solutions to $*F_A = \pm F_A$ are the absolute minimum of the Yang–Mills functional. The advantage of the ASD equations over the full Yang–Mills equations is that they are first-order PDEs and hence their analysis is much simpler.

Let $\mathrm{Aut}(P)$ be the group of automorphisms of $P$, i.e. $G$-equivariant smooth maps $\varphi : P \to P$ commuting with the projections to $M$. Then,

$\mathrm{Aut}(P)$ acts on the space of ASD connections by pullbacks. More precisely, for $\varphi \in \mathrm{Aut}(P)$ and a connection $A \in \Omega^1_P(\mathfrak{g})$, we define $\varphi \cdot A \coloneqq (\varphi^{-1})^* A \in \Omega^1_P(\mathfrak{g})$, which is another connection, and it satisfies the ASD equation if $A$ does. (We take $(\varphi^{-1})^* A$ instead of $\varphi^* A$ so that this defines a left action.)

To find a *reduction* of the ASD equation, we start by taking the model space $M = \mathbb{R}^4$ and look for solutions to the ASD equations that are invariant under some subgroup $\Gamma$ of translations of $\mathbb{R}^4$. For example, $\Gamma$ could be translations in the last three coordinates. Then, we try to interpret the connection $A$ as a connection on $\mathbb{R}^4/\Gamma$ plus some extra “fields”. The next step is to rewrite these equations in a coordinate independent way, so that it makes sense for principal bundles $P$ on other spaces $M$ modelled on $\mathbb{R}^4/\Gamma$. Then, the gauge group of $P$ acts on the space of connections and fields to give another moduli space, which is often hyperkähler. More precisely, if $\mathcal{A}$ is the space of connections and Higgs fields on $M$ and $\mathcal{G}$ the group of gauge transformations, there is often a subset $\mathcal{A}^0 \subseteq \mathcal{A}$ satisfying appropriate boundary conditions helping us defining a metric, and a subgroup $\mathcal{G}^0 \subseteq \mathcal{G}$ acting freely on $\mathcal{A}^0$ and whose hyperkähler moment map condition is given by the reduction of the ASD connection, giving us a hyperkähler moduli space. Let’s see examples.

## 2.2 ASD on $\mathbb{R}^4$ — instantons

Consider the special case of $M = \mathbb{R}^4$ with the flat metric and the trivial principal bundle $P = M \times G$. Then, connections can be written $A = \sum_{i=0}^3 A_i dx^i$ and the curvature as $F = \sum_{i,j} \frac{1}{2} F_{ij} dx^i \wedge dx^j$, where

$$F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j].$$

Then, the ASD equation is

$$\begin{aligned} F_{01} + F_{23} &= 0 \\ F_{02} + F_{31} &= 0 \\ F_{03} + F_{12} &= 0. \end{aligned}$$

Expanding everything, we get the non-linear PDE

$$\begin{aligned} \partial_0 A_1 - \partial_1 A_0 + \partial_2 A_3 - \partial_3 A_2 + [A_0, A_1] + [A_2, A_3] &= 0 \\ \partial_0 A_2 - \partial_2 A_0 + \partial_3 A_1 - \partial_1 A_3 + [A_0, A_2] + [A_3, A_1] &= 0 \\ \partial_0 A_3 - \partial_3 A_0 + \partial_1 A_2 - \partial_2 A_1 + [A_0, A_3] + [A_1, A_2] &= 0. \end{aligned}$$

Since $P$ is trivial, the group of gauge transformations can be identified with the group $\mathcal{G}$ of smooth maps from $\mathbb{R}^4$ to $G$. The action in this gauge reads

$$g \cdot A = \mathrm{Ad}_g\, A - (dg)g^{-1},$$

or in coordinates,

$$(g \cdot A)_i = \mathrm{Ad}_g\, A_i - (\partial_i g)g^{-1}.$$

Here the notation $(\partial_i g)g^{-1}$ makes sense for a matrix Lie group, where it is simply matrix multiplication. In general, we view $\partial_i g(x) \in T_{g(x)}G$ for $x \in \mathbb{R}^4$ and then $(\partial_i g(x))g(x)^{-1}$ is defined as $dR_{g(x)^{-1}}(\partial_i g(x)) \in T_1 G = \mathfrak{g}$.

## 2.3 Reduction to 3D — the Bogomolny equations

Choose a splitting $\mathbb{R}^4 = \mathbb{R} \times \mathbb{R}^3$ into time and space and consider connections which are time-invariant, i.e. invariant under translations in the first $\mathbb{R}$-factor of $\mathbb{R} \times \mathbb{R}^3$. Then, we can write the connection as $\Phi dx^0 + \sum_{i=1}^3 A_i dx^i$, where $\Phi, A_1, A_2, A_3$ are maps $\mathbb{R}^3 \to \mathfrak{g}$. We can view this more invariantly as a connection $A = \sum_{i=1}^3 A_i dx^i$ on the trivial principal $G$ bundle $P$ over $\mathbb{R}^3$ and a section $\Phi$ of $\mathrm{ad}(P)$ called the Higgs field. Then, the ASD equation is equivalent to

$$*d_A \Phi = F_A.$$

This equation makes sense on any principal $G$-bundle $P$ over a 3-manifold $M$, where $A$ is a connection on $P$ and $\Phi$ a section of $\mathrm{ad}(P)$. This is called the *Bogomolny equation* and its solutions are called *monopoles* [25, 1]. These equations have been studied in great details especially for the trivial principal SU(2)-bundle over $\mathbb{R}^3$ [28, 16, 1] and for other groups [31, 32, 48]. For compact 3-manifolds, all monopoles are trivial, but we can consider other non-compact Riemannian 3-manifolds such as, for example, the hyperbolic three-space [51].

The group of gauge transformations of $P \to \mathbb{R}^3$ preserves the Bogomolny equation. Thus, by imposing suitable boundary conditions at infinity, we can construct a moduli space of monopoles, which is often hyperkähler.

## 2.4 Reduction to 2D — the Hitchin equations

Now, consider $M = \mathbb{R}^2 \times \mathbb{R}^2$ and look for connections which are invariant under translations in the second $\mathbb{R}^2$-factor. Write such connections as $A_0 dx^0+$

$A_1dx^1 + \phi_2dx^2 + \phi_3dx^3$, where $A_i, \phi_i : \mathbb{R}^2 \to \mathfrak{g}$. Again, we view $A = A_0dx^0 + A_1dx^1$ as a connection on $\mathbb{R}^2$. To interpret $(\phi_2, \phi_3)$ in a coordinate-invariant way, identify the first $\mathbb{R}^2$-factor with $\mathbb{C}$ via $z = x^0 + ix^1$ and define

$$\Phi = \frac{1}{2}(\phi_1 - i\phi_2)dz \in \Omega^{1,0}_{\mathbb{C}}(\mathrm{ad}(P) \otimes \mathbb{C}),$$

where $P$ is the trivial principal $G$-bundle over $\mathbb{C}$. Then, the ASD equations become

$$\begin{aligned} F_A + [\Phi, \Phi^*] &= 0 \\ \bar{\partial}_A \Phi &= 0, \end{aligned}$$

which are the celebrated *Hitchin equations* [29, 26]. These equations are invariant under conformal transformations, and hence make sense on a compact Riemann surface $\Sigma$. Through an adaptation of the Kobayachi–Hitchin correspondence, the moduli space of solutions to the Hitchin equations over $\Sigma$ is isomorphic to the moduli space of (semistable) *Higgs bundles* over $\Sigma$, i.e. holomorphic principal $G_{\mathbb{C}}$-bundles $P$ over $\Sigma$ together with a section of $\mathrm{ad}(P) \otimes T^*\Sigma$.

## 2.5 Reduction to 1D — the Nahm equations

Consider $M = \mathbb{R} \times \mathbb{R}^3$ and connections which are invariant under translations by $\mathbb{R}^3$. Such a connection can be written $A_0dx^0 + A_1dx^1 + A_2dx^2 + A_3dx^3$, where $A_i : \mathbb{R} \to \mathfrak{g}$. Again, we can view this as a connection $A$ on the trivial $G$-bundle $P$ over $\mathbb{R}$ and a Higgs field $\Phi = A_1i + A_2j + A_3k : \mathbb{R} \to \mathfrak{g} \otimes \mathrm{Im}\,\mathbb{H}$, viewed as a section of the vector bundle $\mathrm{ad}(P) \otimes \mathbb{H}$ over $\mathbb{R}$. Then, the ASD equations are equivalent to

$$*d_A\Phi + \frac{1}{2}[\Phi, \Phi] = 0.$$

In components, this is

$$\begin{aligned} \dot{A}_1 + [A_0, A_1] + [A_2, A_3] &= 0 \\ \dot{A}_2 + [A_0, A_2] + [A_3, A_1] &= 0 \\ \dot{A}_3 + [A_0, A_3] + [A_1, A_2] &= 0 \end{aligned}$$

and those are called *Nahm's equations*, first introduced by Nahm in [49] (but reinterpreted as a reduction of the ASD equation as above by Donaldson

[16]). They play a central rôle in the study of monopoles [28, 16, 32, 50], but also in the construction of non-trivial hyperkähler structures on manifolds associated to $G$ [40, 39, 37, 7, 38, 4, 12, 8, 14, 11], which is the main focus of these lecture notes. See [6, 12, 4] for other reviews.

The group of gauge transformations of $P \to \mathbb{R}$ acts on $(A, \Phi)$ preserving Nahm's equations. In coordinates, a gauge transformation is a map $g : \mathbb{R} \to G$ and acts by

$$g \cdot (A_0, A_1, A_2, A_3) = (\operatorname{Ad}_g A_0 - \dot{g}g^{-1}, \operatorname{Ad}_g A_1, \operatorname{Ad}_g A_2, \operatorname{Ad}_g A_3).$$

The notation $\dot{g}g^{-1}$ makes sense for a matrix Lie group (and every compact Lie group is one), but it can also be made invariantly by viewing $\dot{g}(t) \in T_{g(t)}G$ and using right translation $R_{g(t)^{-1}} : G \to G$ to map this to $dR_{g(t)^{-1}}(\dot{g}(t)) \in T_1G = \mathfrak{g}$. Then, the map $\mathbb{R} \to \mathfrak{g} : t \mapsto dR_{g(t)^{-1}}(\dot{g}(t))$ is smooth. Equivalently, this map is $t \mapsto \theta(\dot{g}(t))$ where $\theta \in \Omega^1_{T^*G}(\mathfrak{g})$ is the right-invariant Maurer–Cartan form.

When $G = \mathrm{U}(n)$, there is another useful way of thinking about Nahm's equations. Namely, a principal $\mathrm{U}(n)$-bundle over $I$ is equivalent to a Hermitian vector bundle $E$ of rank $n$ on $I$. Then, the $A_0$-component can be viewed as a connection $\nabla : \Omega^0_I(E) \to \Omega^1_I(E)$ and the Higgs field components $A_1, A_2, A_3$ as three skew-adjoint sections of $\operatorname{End}(E)$. Then, Nahm's equations are equivalent to

$$\begin{aligned}
\nabla_{\partial_t} A_1 + [A_2, A_3] &= 0 \\
\nabla_{\partial_t} A_2 + [A_3, A_1] &= 0 \\
\nabla_{\partial_t} A_3 + [A_1, A_2] &= 0.
\end{aligned}$$

The gauge group $\mathcal{G}$ is the group of automorphisms of $E$ preserving the hermitian metric and acts on $(\nabla, A_1, A_2, A_3)$ by

$$g \cdot (\nabla, A_1, A_2, A_3) = (g\nabla g^{-1}, gA_1g^{-1}, gA_2g^{-1}, gA_3g^{-1}).$$

To see examples of solutions to Nahm's equations, consider the case $G = \mathrm{SU}(2)$ and let $\sigma_1, \sigma_2, \sigma_3 \in \mathfrak{su}(2)$ be a basis with

$$[\sigma_1, \sigma_2] = \sigma_3, \quad [\sigma_2, \sigma_3] = \sigma_1, \quad [\sigma_3, \sigma_1] = \sigma_2.$$

Then, $A = (0, \sigma_1/t, \sigma_2/t, \sigma_3)$ is a solution with a first-order pole at $t = 0$. More generally, for any $0 \leq k \leq 1$ and $0 \leq D < 2K(k)$, where $K$ is the complete elliptic integral of the first kind, we may take $A = (0, f_1\sigma_1, f_2\sigma_2, f_3\sigma_3)$, where

$$
\begin{aligned}
f_1(t) &= \frac{D\,\mathrm{cn}_k(Dt)}{\mathrm{sn}_k(Dt)} &&= \frac{1}{t} + \frac{D^2(k-2)}{6}t \;+ O(t^3)\\
f_2(t) &= \frac{D\,\mathrm{dn}_k(Dt)}{\mathrm{sn}_k(Dt)} &&= \frac{1}{t} + \frac{D^2(1-2k)}{6}t + O(t^3)\\
f_3(t) &= \frac{D}{\mathrm{sn}_k(Dt)} &&= \frac{1}{t} + \frac{D^2(k+1)}{6}t \;+ O(t^3),
\end{aligned}
$$

and $\mathrm{sn}_k, \mathrm{cn}_k, \mathrm{dn}_k$ are the Jacobi elliptic functions with elliptic modulus $k$. We recover the first case by letting $D \to 0$. When $D = 2K(k)$, we get another pole at $t = 1$ with residues $\sigma$. Also, by letting $k \to 0$, we get

$$
\begin{aligned}
f_1(t) &= \frac{D\cos(Dt)}{\sin(Dt)}\\
f_2(t) &= \frac{D}{\sin(Dt)}\\
f_3(t) &= \frac{D}{\sin(Dt)}.
\end{aligned}
$$

By [14], every solution to the SU(2)-Nahm equations with a first-order pole of residues $\sigma$ at $t = 0$ can be obtained from such a solution by acting by the gauge group $\mathcal{G} = \{g : I \to G : g(0) = 1\}$ and by an SO(3)-action which will be described in §4.3.8.

# 3 Hyperkähler manifolds and their quotients

We now give a brief introduction to hyperkähler manifolds and their quotients. See, for example, [15, 27, 30] for other reviews.

## 3.1 Hyperkähler manifolds

A *hyperkähler manifold* is a smooth manifold $M$ with a Riemannian metric $g$ and three complex structures $I, J, K$ that are Kähler with respect to $g$ and satisfy $IJK = -1$. In other words, $I, J, K$ behave like the quaternions $i, j, k \in \mathbb{H}$, and hence hyperkähler manifolds can be viewed as manifolds modelled on the quaternions. Each tangent space $T_pM$ is an $\mathbb{H}$-module through the action of $I, J, K$ and hence is isomorphic to $\mathbb{H}^n$ for some $n$. In particular, hyperkähler manifolds always have dimension a multiple of four.

Hyperkähler structures are much more rigid than Kähler structures, and also much rarer. The space $\mathbb{H}^n$ of $n$-tuples of quaternions is of course an example, but non-trivial examples are quite difficult to construct. Hyperkähler manifolds can equivalently be described as $4n$-dimensional Riemannian manifolds with holonomy contained in the compact symplectic group $\mathrm{Sp}(n) \subseteq \mathrm{GL}(4n, \mathbb{R})$. Hence, their existence was anticipated in the 1950s by Berger's list [3] of possible holonomy groups, which contains $\mathrm{Sp}(n)$ for all $n$. However, the first non-trivial examples only came 20 years later, namely, Calabi [9] showed that $T^*\mathbb{CP}^n$ has a Riemannian metric with holonomy $\mathrm{Sp}(n)$.

One way to see the rigidity of hyperkähler structures and the reason it took so long to construct examples, is to look at hyperkähler submanifolds. In Kähler geometry, we have the two main examples, $\mathbb{C}^n$ and $\mathbb{CP}^n$, and non-singular subvarieties of those are new, often highly non-trivial, Kähler manifolds. In hyperkähler geometry, however, this method does not work. It is is natural at first sight to expect that using "quaternionic polynomials" of some sort we can produce interesting hyperkähler submanifolds of $\mathbb{H}^n$. But this cannot be the case as the only hyperkähler submanifolds of $\mathbb{H}^n$ are copies of $\mathbb{H}^k$ ($k \leq n$) embedded as affine linear quaternionic subspace [21, Theorem 5]. Also, $\mathbb{HP}^n$ is not hyperkähler (but it is quaternionic-Kähler; see Swann) and does not have any non-trivial hyperkähler submanifold either [21].

So taking subspaces does not give anything new, but there is another way of getting new hyperkähler manifolds from old, which does give interesting results even for $\mathbb{H}^n$. This the notion of hyperkähler quotients [30] (or hyperkähler reduction), an extension of the notion of Kähler quotients (or symplectic reduction).

## 3.2 Riemannian quotients

We begin by reviewing the notion of Riemannian quotients, which is the first step in the hierarchy of quotients of which hyperkähler quotients is the third. At each step of the hierarchy, a new idea will be introduced.

Let $(M, g)$ be a Riemannian manifold and $G$ a compact Lie group acting freely on $M$ by preserving the metric $g$. Then, the orbit space $M/G$ has a unique smooth structure such that the quotient map $M \to M/G$ is a smooth submersion. Moreover, there is a canonical Riemannian metric induced on $M/G$. One way to see this structure is to take for each point $p \in M$ the orthogonal complement $H_p$ of $V_p := \ker d\pi_p$ so that $T_pM = V_p \oplus H_p$. Then, the restriction $d\pi_p : H_p \to T_{\pi(p)}(M/G)$ is a linear isomorphism, so we can use the inner-product on $H_p$ to get one on $T_{\pi(p)}(M/G)$. The $G$-invariance of the Riemannian metric on $M$ ensures that this is well-defined, and this gives a Riemannian metric $\bar{g}$ on $M/G$. Equivalently, $\bar{g}$ is uniquely characterized by the formula

$$\bar{g}(X,Y) \circ \pi = g(X^*, Y^*) \tag{3.1}$$

for all vector fields $X, Y$ on $M/G$, where $X^*, Y^*$ are the horizontal lifts of $X, Y$ on $M$.

More generally, a smooth map between Riemannian manifolds $\pi : (M, g) \to (N, \bar{g})$ satisfying (3.1) is called a *Riemannian submersion*. Any such map has the property that if $v \in T_pM$ is horizontal (i.e. orthogonal to $\ker d\pi_p$) then the unique geodesic $\gamma$ starting at $v$ is horizontal at each point and $\pi \circ \gamma$ is a geodesic on $N$ (see e.g. [45]). In particular, if $M$ is a *complete* Riemannian manifold (i.e. geodesics are defined for all times) then so is $N$.

We summarize the discussion in the following theorem.

**Theorem 3.1.** *Let $(M, g)$ be a Riemannian manifold and $G$ be a compact Lie group acting freely and by isometries on $M$. Then, the quotient space $M/G$ inherits a Riemannian metric $\bar{g}$ uniquely characterized by* (3.1)*. Moreover, if $(M, g)$ is complete, then so is $(M/G, \bar{g})$.*

## 3.3 Kähler quotients

We now recall the notion of Kähler quotients, following [30, §3(C)]. Let $(M, g, \omega, I)$ be a Kähler manifold and $G$ a compact Lie group acting freely on $M$ by preserving the Kähler structure. The theory works more generally for non-free actions, but we would have to discuss stratified symplectic spaces and complex-analytic spaces [54, 53, 23], which we omit for conciseness.

Let $\mathfrak{g}$ be the Lie algebra of $G$. For $x \in \mathfrak{g}$ we denote by $x^\#$ the vector field on $M$ generated by this action. Since $G$ preserves the Kähler structure, we have $\mathcal{L}_{x^\#}\omega = 0$ and by Cartan's formula, this gives $i_{x^\#}d\omega + d(i_{x^\#}\omega) = 0$. Since $\omega$ is closed, we get that all one-forms $i_{x^\#}\omega$ are closed. Suppose that they are in fact exact and write $i_{x^\#}\omega = d\mu_x$ for some smooth function $\mu_x : M \to \mathbb{R}$. In other words, we suppose that $x^\#$ is a Hamiltonian vector field and that $\mu_x$ is the Hamiltonian function. Such an action is called *Hamiltonian* and in that case we can arrange all smooth functions $\mu_x$ into a single smooth map $\mu : M \to \mathfrak{g}^*$. If we can arrange $\mu$ to be $G$-equivariant with respect to the coadjoint action on $\mathfrak{g}^*$, we call the result a *moment map*. We say that a *Hamiltonian Kähler manifold* is a triple $(M, G, \mu)$, where $M$ and $G$ are as above and $\mu$ is a fixed choice of moment map.

In that case, it is easy to show that $\mu$ is a submersion so $Z \coloneqq \mu^{-1}(0) \subseteq M$ is a smooth submanifold. If we restrict the two-form $\omega$ on $Z$, i.e. consider $i^*\omega$ where $i : Z \hookrightarrow M$, we get in general a degenerate two-form. However, the spaces $(T_pZ)^\omega$ where the symplectic form vanishes on all tangent vectors of $Z$ is precisely the tangent space $T_p(G \cdot p)$ of the orbit through $p$ (which is contained in $Z$ by equivariance). Hence, the quotient manifold $Z/G$ (which is smooth by the freeness assumption) inherits a non-degenerate closed two-form, i.e. it is symplectic, and is called the *symplectic reduction* of $M$ by $G$ with respect to $\mu$, a concept introduced by Marsden–Weinstein [41]. The symplectic form $\bar{\omega}$ on $Z/G$ is uniquely characterized by the fact that $\pi^*\bar{\omega} = i^*\omega$, where $\pi : Z \to Z/G$ is the quotient map.

All the above discussion only uses the symplectic form, but we can also incorporate the full Kähler structure. First, we get an induced Riemannian metric $\bar{g}$ on $Z/G$ using the subspace metric on $Z$, which is $G$-invariant. It turns out that $\bar{g}$ and $\bar{\omega}$ are always compatible with a complex structure $\bar{I}$, making $Z/G$ into a Kähler manifold. The complex structure $\bar{I}$ on $Z/G$ sends a vector $v \in T_pZ/G$ to $d\pi(Iv^*)$ where $v^* \in T_pZ$ is the horizontal lift ($I$ preserves the horizontal bundle $H$, so this formula makes sense). The integrability of this almost complex structure can be verified by computing directly that it is covariantly constant with respect to the Levi-Civita connection of $\bar{g}$. See [30, §3(C)] for details.

Moreover, if $M$ is geodesically complete, then so is the Kähler quotient $\mu^{-1}(0)/G$. To see this, we use the Hopf–Rinow theorem to observe that $\mu^{-1}(0)$ is geodesically complete since distances on $\mu^{-1}(0)$ are greater than or equal to distances on $M$. Then, $\mu^{-1}(0)/G$ is complete since $\mu^{-1}(0) \to \mu^{-1}(0)/G$ is a Riemannian submersion.

To sum up, we have:

**Theorem 3.2.** *Let $(M, g, I, \omega)$ be a Kähler manifold, $G$ a compact Lie group acting freely on $M$ by preserving the Kähler structure, and $\mu : M \to \mathfrak{g}^*$ a moment map with respect to $\omega$. Then, $\mu^{-1}(0)/G$ is a smooth manifold and has a unique Kähler structure $(\bar{g}, \bar{I}, \bar{\omega})$ such that $\pi^*\bar{\omega} = i^*\omega$ and $\bar{g}$ is the quotient metric of $i^*g$, where $\pi : \mu^{-1}(0) \to \mu^{-1}(0)/G$ and $i : \mu^{-1}(0) \hookrightarrow M$. Moreover, if $g$ is complete, then so is $\bar{g}$.*

Now, in most cases, the complex structure can also be viewed by a diffeomorphism with a quotient of $M$ by a complex Lie group. For that, we need to add a few extra assumptions. First, that the action of $G$ extends to a holomorphic action of its complexification $G_{\mathbb{C}}$. Recall that $G_{\mathbb{C}}$ is a complex Lie group (i.e. a group with a complex manifold structure such that the multiplication and inversion maps are holomorphic) uniquely determined by the property that $\mathrm{Lie}(G_{\mathbb{C}}) = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} =: \mathfrak{g}_{\mathbb{C}}$ and $G \subseteq G_{\mathbb{C}}$ is a maximal compact subgroup. This assumption is equivalent to the requirement that the vector fields $Ix^{\#}$ for $x \in \mathfrak{g}$ are all complete, so this is an integrability assumption and does not incorporate any additional choice.

There is a general theorem [53, 23] which says that $\mu^{-1}(0)/G \cong M^{\mathrm{ss}}/G_{\mathbb{C}}$ for some $G_{\mathbb{C}}$-invariant open set $M^{\mathrm{ss}} \subseteq M$ called the set of *semistable points*. Moreover, this works even for non-free actions if we replace the complex quotient by a categorical quotient in the category of complex-analytic spaces. But we will omit this theory and argue from first principle that $\mu^{-1}(0)/G \cong M/G_{\mathbb{C}}$ under some assumptions (so we will have $M^{\mathrm{ss}} = M$ here). See, for example, [57, 17, 56, 33, 52, 35, 47, 22, 53, 23, 34, 44] for more on this subject.

The fist assumption is that we have a global Kähler potential $f : M \to \mathbb{R}$ (i.e. $\omega = 2i\partial\bar{\partial}f$) which induces the moment map in the sense that $\mu(p)(x) = df(Ix_p^{\#})$ for all $p \in M$ and $x \in \mathfrak{g}$. Note that if we start with a $G$-manifold $M$ with any $G$-invariant Kähler potential $f$, then this formula always defines a moment map. The point is that if $f$ is proper and bounded below on each $G_{\mathbb{C}}$-orbit, then the inclusion $\mu^{-1}(0) \subseteq M$ descends to a biholomorphism $\mu^{-1}(0)/G \cong M/G_{\mathbb{C}}$. Here, $M/G_{\mathbb{C}}$ inherits the structure of a complex manifold since the action of $G_{\mathbb{C}}$ on $M$ is free and proper. To prove this diffeomorphism, we have to show that every $G_{\mathbb{C}}$-orbit through a point $p$ in $M$ intersects $\mu^{-1}(0)$, and that this intersection is precisely a $G$-orbit. To do so, we consider the map

$$\mathcal{F} : G_{\mathbb{C}} \longrightarrow \mathfrak{g}, \quad \mathcal{F}(g) = f(g \cdot p).$$

Then, the formula for $\mu$ in terms of $f$ implies that if $g \in G_{\mathbb{C}}$ is a critical point of $\mathcal{F}$ then $\mu(g \cdot p) = 0$. Thus, we wish to minimize $\mathcal{F}$. In fact, since $f$ is $G$-invariant, we may view this as a map $\mathcal{F} : \mathcal{H} \to \mathfrak{g}$ where $\mathcal{H} \coloneqq G_{\mathbb{C}}/G$ is the set left cosets. Now, recall that $\mathcal{H} \cong \mathfrak{g}$ is a Riemannian symmetric space of non-compact type, with non-positive curvature. In particular, it is geodesically convex (every two points are joined by a unique minimizing geodesic). Moreover, the function $\mathcal{F} : \mathcal{H} \to \mathbb{R}$ is geodesically convex, i.e. for every geodesic $\gamma$, the composition $\mathcal{F} \circ \gamma$ is convex in the usual sense. To show this, it suffices to observe that geodesics are of the form $G\exp(itx)g$ for $x \in \mathfrak{g}$ and $g \in G_{\mathbb{C}}$ and we have

$$\frac{d^2}{dt^2}\mathcal{F}(G\exp(itx)g) = \|x^{\#}_{e^{itx}g\cdot p}\|^2,$$

where the norm is computed using the Riemannian metric on $M$. Moreover, since $f$ is proper and bounded below on $G_{\mathbb{C}}$-orbits, $\mathcal{F}$ attains a minimum, and by geodesic convexity, this minimum is unique. Here we are using that $G_{\mathbb{C}}$-orbits are closed and hence embedded. The closedness of the orbits is true for any complex-algebraic action, but also more generally for any Hamiltonian action [20]. In other words, we have shown that for all $p \in M$ there is $g \in G_{\mathbb{C}}$ such that $\mu(g \cdot p) = 0$ and that any other such $g' \in G_{\mathbb{C}}$ is of the form $g' = kg$ for $k \in G$. Thus, the map

$$\mu^{-1}(0)/G \longrightarrow M/G_{\mathbb{C}}$$

is a bijection. It is also easy to check that it is holomorphic with respect to $\bar{I}$ and hence is a biholomorphism. Moreover, the Kähler potential $f : M \to \mathbb{R}$ descends to a Kähler potential $\bar{f} : \mu^{-1}(0)/G \to \mathbb{R}$.

The above discussion can be generalized by having, instead of a Kähler potential $f$, a holomorphic line bundle $L \to M$ with a unitary structure whose curvature induces the Kähler form $\omega$. However, for most applications on hyperkähler manifolds, the above version suffices.

We summarize the above discussion in the following theorem.

**Theorem 3.3.** *Let $(M, g, I, \omega)$ be a Kähler manifold with a global Kähler potential $f : M \to \mathbb{R}$ and let $G$ be a compact Lie group acting freely on $M$ by preserving $f$ and the Kähler structure. Then, there is a moment map $\mu : M \to \mathfrak{g}^*$ for this action given by $\mu(p)(x) = df(Ix^{\#}_p)$ for $p \in M$ and $x \in \mathfrak{g}$. Moreover, if $f$ is proper and bounded below on each $G_{\mathbb{C}}$-orbits, then the inclusion $\mu^{-1}(0) \subseteq M$ descends to a biholomorphism*

$$\mu^{-1}(0)/G \xrightarrow{\cong} M/G_{\mathbb{C}}$$

*with respect to the Kähler structure on $\mu^{-1}(0)/G$ obtained from reduction and the complex structure on $M/G_{\mathbb{C}}$ descending from $M$.*

## 3.4 Hyperkähler quotients

We now review the notion of hyperkähler quotients introduced by Hitchin–Karlhede–Lindström–Roček [30].

Suppose that $(M, g, I, J, K)$ is a hyperkähler manifold and that $G$ is a compact Lie group acting on $M$ by preserving the hyperkähler structure. We will still assume that the $G$-action is free, but the whole story extends to the non-free action with a certain notion of stratified hyperkähler spaces [13, 42].

A *hyperkähler moment map* is a map $\mu = (\mu_I, \mu_J, \mu_K) : M \to \mathfrak{g}^* \otimes \operatorname{Im}\mathbb{H}$ whose three components $\mu_I, \mu_J, \mu_K$ are moment maps with respect to the Kähler forms $\omega_I, \omega_J, \omega_K$ of $I, J, K$.

In that case the map

$$\mu_{\mathbb{C}} \coloneqq \mu_J + i\mu_K : M \longrightarrow \mathfrak{g}_{\mathbb{C}}^*$$

is holomorphic with respect to $I$ and is a submersion. Hence, the level set $\mu_{\mathbb{C}}^{-1}(0) \subseteq M$ is a complex submanifold with respect to $I$ and hence also a Kähler manifold. Moreover, the moment map $\mu_I$ restricts to a moment map $\mu_{\mathbb{C}}^{-1}(0) \to \mathfrak{g}^*$ for the action of $G$ on $\mu_{\mathbb{C}}^{-1}(0)$. Thus, we can perform the Kähler quotient to get a Kähler manifold $\mu_I^{-1}(0) \cap \mu_{\mathbb{C}}^{-1}(0)/G$. But $\mu_I^{-1}(0) \cap \mu_{\mathbb{C}}^{-1}(0)/G = \mu^{-1}(0)/G$ so the situation is entirely symmetric and we can switch the roles of $I, J, K$ to get three Kähler structures $\bar{I}, \bar{J}, \bar{K}$ on $\mu^{-1}(0)/G$ with the same Riemannian metric $\bar{g}$ and satisfying $\bar{I}\bar{J}\bar{K} = -1$. In other words, $\mu^{-1}(0)/G$ is a hyperkähler manifold, called the *hyperkähler quotient* of $M$ by $G$ with respect to $\mu$. We can check that the Kähler forms $\bar{\omega}_I$, $\bar{\omega}_J$, $\bar{\omega}_K$ on $\mu^{-1}(0)/G$ satisfy $\pi^*\bar{\omega}_I = i^*\omega_I$, $\pi^*\bar{\omega}_J = i^*\omega_J$, $\pi^*\bar{\omega}_K = i^*\omega_K$, where $\pi : \mu^{-1}(0) \to \mu^{-1}(0)/G$ and $i : \mu^{-1}(0) \hookrightarrow M$, and this characterizes them uniquely. Moreover, a hyperkähler structure is completely determined by its three Kähler forms (we have, e.g. $I = \omega_3^{-1}\omega_2$), so this condition determines the whole hyperkähler structure. Also, by the same argument as for Kähler quotients, if $M$ is complete, then so is the hyperkähler quotient.

**Theorem 3.4** ([30])**.** *Let $(M, g, I, J, K)$ be a hyperkähler manifold, $G$ a compact Lie group acting freely on $M$ by preserving the hyperkähler structure, and $\mu : M \to \mathfrak{g}^* \otimes \operatorname{Im}\mathbb{H}$ be a hyperkähler moment map. Then, the quotient $\mu^{-1}(0)/G$ is a smooth manifold and has a unique hyperkähler structure $(\bar{g}, \bar{I}, \bar{J}, \bar{K})$ such that $\pi^*\bar{\omega}_I = i^*\omega_I$, $\pi^*\bar{\omega}_J = i^*\omega_J$, $\pi^*\bar{\omega}_K = i^*\omega_K$, where $\pi : \mu^{-1}(0) \to \mu^{-1}(0)/G$ and $i : \mu^{-1}(0) \hookrightarrow M$. Moreover, if $M$ is complete, then so is $\mu^{-1}(0)/G$.*

**Definition 3.5.** In the setup of Theorem 3.4, the quotient $\mu^{-1}(0)/G$ is called the *hyperkähler quotient* of $M$ by $G$ with respect to $\mu$ and is denoted

$$M /\!\!/\!\!/_{\mu} G \coloneqq \mu^{-1}(0)/G,$$

or simply $M /\!\!/\!\!/ G$ if the moment map $\mu$ is understood.

Now, just like the Kähler case, we can express this quotient in the category of complex manifolds by supposing that the $G$-action extends to an $I$-holomorphic $G_{\mathbb{C}}$-action and that the first Kähler structure $(I, \omega_I)$ on $M$ is induced by a $G$-invariant global Kähler potential $f : M \to \mathbb{R}$ such that $\mu_I(p)(x) = df(Ix_p^{\#})$. Then, Theorem 3.3 implies that the inclusion $\mu^{-1}(0) \subseteq \mu_{\mathbb{C}}^{-1}(0)$ descends to a biholomorphism

$$\mu^{-1}(0)/G \xrightarrow{\cong} \mu_{\mathbb{C}}^{-1}(0)/G_{\mathbb{C}}. \tag{3.2}$$

But this biholomorphism respects more structures. First, note that $\omega_{\mathbb{C}} \coloneqq \omega_J + i\omega_K$ is a non-degenerate holomorphic two-form on $(M, I)$, i.e. $(M, I, \omega_{\mathbb{C}})$ is a complex-symplectic manifold. Moreover, $G_{\mathbb{C}}$ preserves $\omega_{\mathbb{C}}$ and $\mu_{\mathbb{C}}$ is a holomorphic moment map for the action of $G_{\mathbb{C}}$ on $(M, I, \omega_{\mathbb{C}})$. Hence, the right-hand side of (3.2) has a natural complex-symplectic structure, coming from a holomorphic version of symplectic reduction. But the left-hand side, being a hyperkähler manifold $(\mu^{-1}(0)/G, \bar{g}, \bar{I}, \bar{J}, \bar{K})$ also has a complex-symplectic form, namely, $\bar{\omega}_J + i\bar{\omega}_K$. Then, the biholomorphism (3.2) is in fact an isomorphism of complex-symplectic manifolds. We summarize this in the following theorem.

**Theorem 3.6.** *Let $(M, g, I, J, K)$ be a hyperkähler manifold and $G$ a compact Lie group acting freely on $M$ by preserving the hyperkähler structure. Suppose that there is a $G$-invariant global Kähler potential $f : M \to \mathbb{R}$ for $(M, g, I)$ and a hyperkähler moment map $\mu : M \to \mathfrak{g}^* \otimes \operatorname{Im} \mathbb{H}$ whose first component $\mu_I$ is induced by $f$. Suppose also that the $G$-action extends to an $I$-holomorphic $G_{\mathbb{C}}$-action and that $f$ is proper and bounded below on each $G_{\mathbb{C}}$-orbit. Then, the inclusion $\mu^{-1}(0) \hookrightarrow \mu_{\mathbb{C}}^{-1}(0)$ descends to an isomorphism*

$$\mu^{-1}(0)/G \xrightarrow{\cong} \mu_{\mathbb{C}}^{-1}(0)/G_{\mathbb{C}}.$$

*of complex-symplectic manifolds, where the left-hand side has the complex-symplectic structure $(\bar{I}, \bar{\omega}_J + i\bar{\omega}_K)$ induced from the hyperkähler structure obtained from the hyperkähler quotient, and the right-hand side has the complex-symplectic structure obtained from complex-symplectic reduction.*

Again, this theorem can be generalized to the case where $G$ does not necessarily acts freely, but we have to replace $\mu_{\mathbb{C}}^{-1}(0)/G_{\mathbb{C}}$ by a categorical quotient in the category of complex analytic spaces, or in many situations, by a GIT quotient.

## 3.5 Twistor spaces

Twistor theory for hyperkähler geometry gives a one-to-one correspondence between hyperkäher manifolds and certain complex manifolds with additional holomorphic data. This was introduced in [30]. The advantage of this method is to encode the Riemannian metric and other $C^\infty$ data with purely holomorphic objects in complex geometry.

We begin by recalling a few facts about the quaternions. Let $\mathrm{Sp}(1) \subseteq \mathbb{H}$ be the group of unit-norm quaternions. Recall that $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$ via

$$\mathrm{Sp}(1) \longrightarrow \mathrm{SU}(2), \quad u + vj \longmapsto \begin{pmatrix} u & -v \\ \bar{v} & \bar{u} \end{pmatrix},$$

where we use that every quaternion $q \in \mathbb{H}$ can be uniquely expressed as $q = u + vj$ for $u, v \in \mathbb{C}$. This description of $\mathrm{SU}(2)$ makes the double cover $\mathrm{Sp}(1) \to \mathrm{SO}(3)$ easy to see. Indeed, for every $q \in \mathrm{Sp}(1)$, the map

$$A_q : \mathbb{H} \longrightarrow \mathbb{H}, \quad p \longmapsto qpq^{-1}.$$

is an isometry. Moreover, we have $\mathrm{Re}(qpq^*) = \mathrm{Re}(p)\|q\|^2$, so $A_q$ restricts to a map $\mathrm{Im}\,\mathbb{H} \to \mathrm{Im}\,\mathbb{H}$ and hence $A_q \in \mathrm{SO}(\mathrm{Im}\,\mathbb{H})$. Thus, using the standard basis $i, j, k \in \mathrm{Im}\,\mathbb{H}$, we have a natural Lie group homomorphism map $\mathrm{Sp}(1) \to \mathrm{SO}(3)$, whose kernel is $\{\pm 1\} \subseteq \mathrm{Sp}(1)$.

To make this map more explicit, identify $\dot{\mathbb{C}}^2 := \mathbb{C}^2 - \{0\}$ with $\mathbb{H}^*$ via $(u, v) \mapsto u + vj$ and consider the composition

$$\dot{\mathbb{C}}^2 \longrightarrow \mathrm{Sp}(1) \longrightarrow \mathrm{SO}(3),$$

where the first map is the projection on the unit-quaternions using $\mathbb{H}^* \to \mathrm{Sp}(1) : q \mapsto q/\|q\|$. A straightforward computation shows that this map is

$$(u, v) \longmapsto \frac{1}{|u|^2 + |v|^2} \begin{pmatrix} |u|^2 - |v|^2 & 2\,\mathrm{Im}(u\bar{v}) & 2\,\mathrm{Re}(u\bar{v}) \\ 2\,\mathrm{Im}(uv) & \mathrm{Re}(u^2 + v^2) & -\mathrm{Im}(u^2 - v^2) \\ -2\,\mathrm{Re}(uv) & \mathrm{Im}(u^2 + v^2) & \mathrm{Re}(u^2 - v^2) \end{pmatrix}.$$

Note that the first row is $\mathbb{C}^*$-invariant and descends to a diffeomorphism $\mathbb{CP}^1 \to S^2$, which is explicitly given by

$$\varphi : \mathbb{CP}^1 \longrightarrow S^2, \quad [u : v] \longmapsto \left( \frac{|u|^2 - |v|^2}{|u|^2 + |v|^2}, \frac{2\,\mathrm{Im}(u\bar{v})}{|u|^2 + |v|^2}, \frac{2\,\mathrm{Re}(u\bar{v})}{|u|^2 + |v|^2} \right).$$

Now, the definition of the twistor space of a hyperkähler manifold $M$ is as follows. Let $M$ be a hyperkähler manifold with complex structures $I, J, K$ and Riemannian metric $g$. Then, for any $(a, b, c)$ in the unit two-sphere, $aI + bJ + cK$ is another integrable complex structure which is Kähler with respect to $g$. Using the diffeomorphism $\mathbb{CP}^1 \to S^2$ above, let $\mathbf{I}_\zeta$ for $\zeta \in \mathbb{CP}^1$ be the corresponding complex structure.

Then, $Z \coloneqq M \times \mathbb{CP}^1$ can be given a complex structure which at a point $(p, \zeta)$ is given by $\mathbf{I}_{(p,\zeta)} \coloneqq (\mathbf{I}_\zeta, i)$ where $i$ is the standard complex structure on $\mathbb{CP}^1$. Then, $\mathbf{I}$ is integrable, making $Z$ into a complex manifold encoding all three complex structures on $M$. We can recover all complex structures by noting that the projection $\pi : Z \to \mathbb{CP}^1$ is holomorphic and the fibre at $\zeta \in \mathbb{CP}^1$ is biholomorphic to $(M, \mathbf{I}_\zeta)$.

Moreover, each point $p \in M$ gives rise to a holomorphic section of $\pi$, namely, $\sigma_p : \mathbb{CP}^1 \to Z : \zeta \mapsto (p, \zeta)$. These section have more special properties. First, they are compatible with the real structure $\tau : Z \to Z : (p, \zeta) \mapsto (p, -1/\bar{\zeta})$ on $Z$. The holomorphic sections $\sigma_p$ have the property that $\sigma_p \circ \tau' = \tau \circ \sigma_p$ where $\tau'$ is the antipodal map on $\mathbb{CP}^1$. Another property satisfied by any such section $\sigma_p$ is that its normal bundle $N \to \mathbb{CP}^1$ is isomorphic to $\mathcal{O}(1)^{\oplus 2n}$, where $\dim M = 4n$. Then, $M$ is in bijection with the set of holomorphic sections of $\pi : Z \to \mathbb{CP}^1$ compatible with $\tau$ in this way and with normal bundle $\mathcal{O}(1)^{\oplus 2n}$. Such sections are called *twistor lines*. Moreover, by a theorem of Kodaira [36] in deformation theory, the set of twistor lines can be given the structure of a smooth manifold and hence the manifold $M$ is completely encoded by $Z$.

In fact, the metric $g$ can also be encoded into holomorphic data on $Z$. We first observe that $g$ is determined by the symplectic forms $\omega_I$, $\omega_J$, $\omega_K$. Then, recall also that $\omega_J + i\omega_K$ is a complex-symplectic form with respect to $I$. More generally, for any oriented orthonormal frame in $\mathbb{R}^3$, we have another triple $I', J', K'$ of complex structures and hence another $I'$-holomorphic symplectic form $\omega_{J'} + i\omega_{K'}$. In particular, for each fixed $\zeta \in \mathbb{CP}^1$, we have a circle of holomorphic-symplectic structures on $M$. In fact, we can multiply this by a real number and get another complex-symplectic form, so we can say that there is a canonical $\mathbb{C}^*$-family of complex-symplectic form on each fibre of

$Z \to \mathbb{CP}^1$. There is no canonical element in this family however, so we have to be more careful if we want to express this globally on $Z$. To do so, define $\mathcal{V}_Z = \ker d\pi \subseteq TZ$, the vertical bundle of the fibration $Z \to \mathbb{CP}^1$. Then, each member of this family is an element of $\Lambda^2 \mathcal{V}_Z^*$. We cannot find a holomorphic section of $\Lambda^2 \mathcal{V}_Z^*$ over $Z$ which will pick a single choice of complex-symplectic form, but the symplectic forms do determine a holomorphic section of $\Lambda^2 T_F^* \otimes \pi^* \mathcal{O}(2)$ where $\mathcal{O}(2)$ is line bundle of degree 2 on $\mathbb{CP}^1$. To see this note that choosing a frame $I', J', K'$ is equivalent to choosing an element of $\mathrm{SO}(3)$. Now, using the covering $\dot{\mathbb{C}}^2 = \mathbb{H}^* \to \mathrm{Sp}(1) \to \mathrm{SO}(3)$ explained above, the holomorphic-symplectic form corresponding to the image of $(u, v) \in \dot{\mathbb{C}}^2$ in $\mathrm{SO}(3)$ is

$$\frac{1}{|u|^2 + |v|^2} \left( u^2(\omega_2 + i\omega_3) - 2iuv\omega_1 + v^2(\omega_2 - i\omega_3) \right).$$

Now, multiplying by constant $|u|^2 + |v|^2$ we get

$$u^2(\omega_2 + i\omega_3) - 2iuv\omega_1 + v^2(\omega_2 - i\omega_3).$$

which depends holomorphically on $u, v$. Since this is a homogeneous polynomial of degree 2, it defines a section of $\Lambda^2 \mathcal{V}_Z^* \otimes \pi^* \mathcal{O}(2)$.

We call $Z$ together with the fibration $\pi : Z \to \mathbb{CP}^1$, the real structure $\tau : Z \to Z$, and the section of $\Lambda^2 \mathcal{V}_Z^*$ the *twistor space* of $M$. Then, all this holomorphic data completely encodes $M$ and its hyperkähler structure.

**Theorem 3.7** ([30]). *Let $Z$ be a complex manifold of dimension $2n + 1$ equipped with*

(i) *a holomorphic section $\pi : Z \to \mathbb{CP}^1$,*

(ii) *a holomorphic section $\omega$ of $\Lambda^2 \mathcal{V}_Z^* \otimes \pi^* \mathcal{O}(2)$ where $\mathcal{V}_Z = \ker d\pi \subseteq TZ$, and*

(iii) *an anti-holomorphic involution $\tau : Z \to Z$ covering the antipodal map $\tau' : \mathbb{CP}^1 \to \mathbb{CP}^1$ and satisfying $\tau^* \omega = \omega$.*

*Then, the set $M$ of twistor line, i.e. holomorphic sections $\sigma$ of $\pi$ with normal bundle $\mathcal{O}(1)^{\oplus 2n}$ and such that $\sigma \circ \tau' = \tau \circ \sigma$, is a $4n$-dimensional smooth manifold with three complex structure $I, J, K$ satisfying $IJK = -1$. Moreover, there is a natural pseudo-Riemannian metric $g$ on $M$ of type $(4k, 4n - 4k)$ for* *some $k$ which is pseudo-Kähler with respect to $I, J, K$, i.e. $I^*g = g$, $J^*g = g$, $K^*g = g$, and the two-forms $gI$, $gJ$, $gK$ are closed. In particular, if $k = n$, then $M$ is hyperkähler.*

*Moreover, if $M'$ is any hyperkähler manifold, the hyperkähler manifold $M$ associated to the twistor space $Z$ of $M$ as above has a connected component isomorphic to $M'$.*

## 3.6 Hyperkähler rotation

Note that for any $q \in \mathrm{Sp}(1) \subseteq \mathbb{H}$, $qiq^{-1}$ is another complex structure on $\mathbb{H}$, i.e. $(qiq^{-1})^2 = -1$, and it is of the form $qiq^{-1} = ai + bj + ck$ for $(a, b, c)$ in the two-sphere. More generally, if $\mathbf{I} = ai + bj + ck \in \mathbb{H}$ with $(a, b, c) \in S^2$, then $\mathbf{I}^2 = -1$ and $q\mathbf{I}q^{-1}$ is also a complex structure on $\mathbb{H}$ of the form $a'i + b'j + c'k$ where $(a', b', c') \in S^2$ is the rotation of $(a, b, c)$ by the image of $q$ in $\mathrm{SO}(3)$.

Similarly, on a hyperkähler manifold $(M, g, I, J, K)$, we can define for any $q \in \mathrm{Sp}(1)$ another frame of complex structures $I' = qIq^{-1}$, $J' = qJq^{-1}$, $K' = qKq^{-1}$ satisfying $I'J'K' = -1$, by letting $I' = aI + bJ + cK$ where $(a, b, c)$ are the coefficients appearing in the formula $qiq^{-1} = ai + bj + ck \in \mathbb{H}$, and similarly for $J'$ and $K'$. Thus, we can think of $\mathrm{Sp}(1)$ as acting on the two-sphere of complex structures of $M$ by rotation. A hyperkähler rotation is, roughly speaking, a geometric realization of this by an action on $M$.

**Definition 3.8.** Let $(M, g, I, J, K)$ be a hyperkähler manifold. A *hyperkähler rotation* is an isometric action of $\mathrm{Sp}(1)$ on $M$ such that for all $q \in \mathrm{Sp}(1)$, the map

$$M \longrightarrow M, \quad m \longmapsto q \cdot m$$

is holomorphic with respect to $I$ on the left and $qIq^{-1}$ on the right. If the action descends to $\mathrm{SO}(3)$ (i.e. $\{\pm 1\} \subseteq \mathrm{Sp}(1)$ acts trivially), we call it an $\mathrm{SO}(3)$*-hyperkähler rotation*.

The main point about hyperkähler rotation is that the Kähler manifolds $(M, g, I)$, $(M, g, J)$, and $(M, g, K)$ are all isomorphic. More generally, all Kähler manifolds $(M, g, aI + bJ + cK)$ for $(a, b, c) \in S^2$ are isomorphic, i.e. there is no special element in the two-sphere of complex-structures.

For example, $\mathbb{H}^n$ has a hyperkähler rotation given by

$$q \cdot (p_1, \ldots, p_n) \longmapsto (qp_1, \ldots, qp_n)$$

for $q \in \mathrm{Sp}(1)$. This is not an SO(3)-hyperkähler rotation, but we can also define

$$q \cdot (p_1, \dots, p_n) \longmapsto (qp_1q^{-1}, \dots, qp_nq^{-1})$$

which is an SO(3)-hyperkähler rotation.

But not every hyperkähler manifold has a hyperkähler rotation. For instance, moduli spaces of solutions to the Hitchin equations [29] do not have any hyperkähler rotation, since the complex structures $I$ and $J$ are not isomorphic.

# 4 Nahm's equations on a compact interval

We now focus on the moduli space of solutions to Nahm's equations on a compact interval $I = [a, b]$. We will explain how this gives us a $G \times G$-invariant hyperkähler metric on $T^*G_{\mathbb{C}}$ for any compact Lie group $G$, as shown in Kronheimer [38] and Dancer–Swann [12]. See also [4, §2], [6], and [55].

Let us first observe how natural it is that $T^*G_{\mathbb{C}}$ has a hyperkähler structure. Recall this if $M$ is any smooth manifold, the cotangent bundle $T^*M$ inherits a canonical symplectic form, namely, $\omega = -d\alpha$ where $\alpha$ is the tautological one-form on $T^*M$ given by $\alpha(v \in T_\xi T^*M) = \xi(d\pi(v))$ where $\pi : T^*M \to M$ is the bundle map. This also works in the holomorphic setting, so $T^*G_{\mathbb{C}}$ is a complex-symplectic manifold. As we have seen in §3.4, all hyperkähler manifolds have an underlying complex-symplectic structure. Moreover, by the holomorphic version of the Darboux theorem, on a complex-symplectic manifold, there is a hyperkähler structure on a neighbourhood of every point (but it may not exist globally). Thus, we generally expect complex-symplectic manifolds to have hyperkähler structures, although they are in general difficult to construct.

But there is another reason to expect a hyperkähler structure on $T^*G_{\mathbb{C}}$. If we start with the compact Lie group $G$ together with an invariant inner-product on $\mathfrak{g} = \mathrm{Lie}(G)$ (which always exist by averaging), we have a canonical $G \times G$-invariant Riemannian metric on $G$. Next, the complexification $G_{\mathbb{C}}$ is a complex manifold and we have the polar decomposition, saying that the map $G \times \mathfrak{g} \to G_{\mathbb{C}} : (k, x) \mapsto ke^{ix}$ is a diffeomorphism. Thus, using the invariant inner-product to identify $\mathfrak{g}^* \cong \mathfrak{g}$, we have a sequence of diffeomorphisms $G_{\mathbb{C}} = G \times \mathfrak{g} = G \times \mathfrak{g}^* = T^*G$, and hence $G_{\mathbb{C}}$ also has a canonical symplectic form. It turns out that this symplectic form is compatible with the complex structure, making $G_{\mathbb{C}}$ into a Kähler manifold, and this structure is also invariant under the action of $G \times G$. Thus, $T^*G$ can be viewed as a complexification of $G$. The next step is then to consider $T^*G_{\mathbb{C}}$, which should be a kind of quaternionification of $G$, and hence it is natural to expect that it has a $G \times G$-invariant hyperkähler structure. This is true, but this structure is much less obvious than the case of $G$ and $G_{\mathbb{C}}$, and we have to rely on infinite-dimensional quotients to construct it.

But it is interesting that this infinite-dimensional construction also has more basic analogues which can be used to construct the bi-invariant Riemannian metric on $G$ and the Kähler structure on $T^*G = G_{\mathbb{C}}$. We first discuss this, as most of the ideas are contained in these easier examples. This approach is due to Bielawski [6].

## 4.1 Bi-invariant metric on a compact Lie group from an infinite-dimensional quotient

Let $\mathcal{A}$ be the space of connections on the trivial principal $G$-bundle over $I = [0,1]$ and let $\mathcal{G}$ be the gauge group. More precisely, we identify $\mathcal{A} = C^1(I, \mathfrak{g})$ and $\mathcal{G} = C^2(I, G)$, and the action of $\mathcal{G}$ on $\mathcal{A}$ is $g \cdot A = gAg^{-1} - \dot{g}g^{-1}$. The set $\mathcal{A}$ is a Banach space (with norm $\|A\| = \|A\|_\infty + \|\dot{A}\|_\infty$) and $\mathcal{G}$ is a Banach Lie group. We can also view $\mathcal{A}$ as an infinite-dimensional Banach manifold, where each tangent space is identified with $\mathcal{A}$. Moreover, we can endow $\mathcal{A}$ with a Riemannian metric by the $L^2$ inner-product

$$\langle X, Y \rangle = \int_0^c \langle X(t), Y(t) \rangle dt,$$

for tangent vectors $X, Y \in \mathcal{A}$. Moreover, this metric is invariant under the action of $\mathcal{G}$. The normal subgroup $\mathcal{G}^0 = \{g \in \mathcal{G} : g(0) = g(1) = 1\}$ acts freely on $\mathcal{A}$ with finite-dimensional slices, so the quotient space $\mathcal{A}/\mathcal{G}^0$ is a finite-dimensional Riemannian manifold.

To see this, take $A \in \mathcal{A}$. Then, elements of $\mathcal{G}^0$ can be written $e^x$ for $x : I \to \mathfrak{g}$ with $x(0) = x(1) = 1$. Thus, we get that the tangent space through the orbit of $A$ is the set of $[x, A] - \dot{x}$. Then, using integration by part, we see that $X \in \mathcal{A}$ is orthogonal $[x, A] - \dot{x}$ if and only if $\int_0^c \langle X, [x, A] - \dot{x} \rangle = \int_0^c \langle \dot{X} + [A, X], x \rangle = 0$ and hence the orthogonal complement to $T_A(\mathcal{G}_0 \cdot A)$ is the space of solutions to the linear ODE $\dot{X} + [A, X] = 0$, which is finite-dimensional, of dimension $\mathfrak{g}$. Moreover, if $b \in C^1(I, \mathfrak{g})$ small enough the $\mathcal{G}_0$-orbit through $A + b$ intersects this slice exactly once (see Kronheimer).

The metric on $\mathcal{A}$ is invariant under the larger group $\mathcal{G}$ and there is a residual action of $\mathcal{G}/\mathcal{G}^0 = G \times G$ on $\mathcal{A}/\mathcal{G}^0$, so the Riemannian metric on $\mathcal{A}/\mathcal{G}^0$ is $G \times G$-invariant. Now, by taking the holonomy of a connection, we get a map $\mathcal{A} \to G$. More concretely, if $A \in \mathcal{A}$, viewed as a map $A : I \to \mathfrak{g}$, then there is a unique solution to the equation $gAg^{-1} - \dot{g}g^{-1} = 0$ with $g(0) = 0$, i.e. each connection can be gauged by $\mathcal{G}$ to the trivial connection. Indeed, we can put the equation in the form $\dot{g} = gA$, which is a linear ODE. Then, $g(1) \in G$ is uniquely determined by $A$, giving us a map $\mathcal{A} \to G$. In fact, this map is $\mathcal{G}^0$-invariant and descends to a diffeomorphism $\mathcal{A}/\mathcal{G}^0 \to G$ which intertwines the $\mathcal{G}/\mathcal{G}^0 = G \times G$-action on $\mathcal{A}/\mathcal{G}^0$ with the $G \times G$-action on $G$ by left and right multiplications. This constructs a bi-invariant metric on $G$, which is, in fact, the usual one.

**Remark 4.1.** If $G$ is simply-connected, the condition that two elements of $\mathcal{A}$ are in the same $\mathcal{G}^0$-orbit can be defined intrinsically in $\mathcal{A}$ without mentioning $G$. Moreover, we can define a Lie group structure directly on $\mathcal{A}/\sim$ without reference to $G$, and such that the above diffeomorphism $\mathcal{A}/\sim \,\cong G$ is an isomorphism of Lie groups. Since $\mathcal{A}/\sim$ and its Lie group structure then depends only on the Lie algebra $\mathfrak{g}$, this is one way of proving that every Lie algebra integrates to a simply connected Lie group. This is the approach taken by Duistermaat and Kolk [18].

## 4.2 Kähler structure on a complex reductive group from the "baby Nahm equations"

To get the Kähler metric on $T^*G \cong G_{\mathbb{C}}$, we complexify, that is, we consider the space $\mathcal{A}_{\mathbb{C}} = \mathcal{A} \otimes_{\mathbb{R}} \mathbb{C}$ of $C^1$ maps $A = A_0 + iA_1 : I \to \mathfrak{g}_{\mathbb{C}}$. We can view this as an infinite-dimensional Kähler manifold by considering the Riemannian metric on $\mathcal{A}_{\mathbb{C}}$ given by

$$\langle X, Y \rangle := \int_I \Big( \langle X_0(t), Y_0(t) \rangle + \langle X_1(t), Y_1(t) \rangle \Big) dt$$

for $X, Y \in \mathcal{A}_{\mathbb{C}}$ and the complex structure $I(X_0, X_1) = (-X_1, X_0)$. The group $\mathcal{G}$ acts on $\mathcal{A}_{\mathbb{C}}$ by

$$g \cdot (A_0, A_1) = (gA_0g^{-1} - \dot{g}g^{-1}, gA_1g^{-1})$$

and this action preserves the Kähler structure. In other words, we can view $A_0$ as a connection on the trivial principal $G$-bundle $P$ on interval $I$ and $A_1$

as a "Higgs field", i.e. a section of $\mathrm{ad}(P)$, and this is the natural action by automorphisms of $P$. As before, the subgroup $\mathcal{G}^0 \subseteq \mathcal{G}$ acts freely on $\mathcal{A}_{\mathbb{C}}$. Now, we want to take the *Kähler quotient* of $\mathcal{A}_{\mathbb{C}}$ by $\mathcal{G}^0$. First, we have a moment map for the $\mathcal{G}^0$-action given by

$$\mu : \mathcal{A}_{\mathbb{C}} \longrightarrow C^0(I, \mathfrak{g}), \quad A \longmapsto \dot{A}_1 + [A_0, A_1].$$

The interpretation of this as a moment map for the $\mathcal{G}_0$-action is through the non-degenerate pairing $C^0(I, \mathfrak{g}) \times \mathrm{Lie}(\mathcal{G}_0) \to \mathbb{R} : (x, y) \mapsto \int_I \langle x, y \rangle$, where $\mathrm{Lie}(\mathcal{G}_0) = \{x \in C^2(I, \mathfrak{g}) : x(0) = x(1) = 0\}$. To see this, let $x \in \mathrm{Lie}(\mathcal{G}_0)$ and let $X, Y \in T_A\mathcal{A}_{\mathbb{C}} = \mathcal{A}_{\mathbb{C}}$. Then, the Kähler form $\omega$ on $\mathcal{A}_{\mathbb{C}}$ is given by

$$\omega(X, Y) = \langle IX, Y \rangle = \int_I \langle -X_1, Y_0 \rangle + \langle X_0, Y_1 \rangle$$

and

$$x_A^{\#} = \frac{d}{ds}\bigg|_{s=0} e^{sx} \cdot A = ([x, A_0] - \dot{x}, [x, A_1]).$$

Thus,

$$\begin{aligned} i_{x_A^{\#}}\omega(Y) &= \int_I \langle -[x, A_1], Y_0 \rangle + \langle [x, A_0] - \dot{x}, Y_1 \rangle \\ &= \int_I \langle [Y_0, A_1] + [A_0, Y_1], x \rangle - \langle \dot{x}, Y_1 \rangle \end{aligned}$$

By integration by part and the fact that $x(0) = x(1) = 0$, we get $-\int_0^1 \langle \dot{x}, Y_1 \rangle = \int_0^1 \langle x, \dot{Y}_1 \rangle$ and hence

$$i_{x_A^{\#}}\omega(Y) = \int_I \langle \dot{Y}_1 + [Y_0, A_1] + [A_0, Y_1], x \rangle = \int_I \langle d\mu_A(Y), x \rangle,$$

where $\mu(A) = \dot{A}_1 + [A_0, A_1]$.

Moreover, $\mu$ is a submersion and hence the version of the inverse function theorem for Banach manifolds shows that $\mu^{-1}(0)$ is a Banach submanifold of $\mathcal{A}_{\mathbb{C}}$. To see this, we must show that for all $A \in \mathcal{A}_{\mathbb{C}}$, the derivative $d\mu_A : \mathcal{A}_{\mathbb{C}} \to C^0(I, \mathfrak{g})$ is surjective. In other words, we must show that for all $x \in C^0(I, \mathfrak{g})$ and $A \in \mathcal{A}_{\mathbb{C}}$, we can solve the equation

$$\dot{X}_1 + [X_0, A_1] + [A_0, X_1] = x$$

for $X \in \mathcal{A}_{\mathbb{C}}$. But this is a first order linear inhomogenous ODE, so we can always find a solution.

Moreover, $\mathcal{G}^0$ acts freely and a similar argument as in the preceding section shows that this action has finite-dimensional slices so that the quotient $\mu^{-1}(0)/\mathcal{G}^0$ is a finite-dimensional Kähler manifold canonically associated to $G$.

Now, by an infinite-dimensional version of the equivalence between Kähler and complex quotients stated in Theorem 3.3, we have a biholomorphism

$$\mu^{-1}(0)/\mathcal{G}^0 \longrightarrow \mathcal{A}_{\mathbb{C}}/\mathcal{G}^0_{\mathbb{C}}$$

where $\mathcal{G}_{\mathbb{C}} = C^2(I, G_{\mathbb{C}})$ is thought of as the complexification of $\mathcal{G}$, and $\mathcal{G}^0_{\mathbb{C}}$ is the subgroup of $g \in \mathcal{G}_{\mathbb{C}}$ which are trivial at the boundary points.

To see this, we imitate the proof of the finite-dimensional case (Theorem 3.3), so we need a $\mathcal{G}_0$-invariant Kähler potential. This can be taken to be

$$f : \mathcal{A}_{\mathbb{C}} \longrightarrow \mathbb{R}, \quad f(A) = \frac{1}{2}\int_I \|A_1\|^2.$$

Moreover, the moment map $\mu$ defined above is the moment map associated to $f$. We need to show that for every $A \in \mathcal{A}_{\mathbb{C}}$ there exists $g \in \mathcal{G}^0_{\mathbb{C}}$ such that $g \cdot A \in \mu^{-1}(0)$ and that those are unique up to the action of $\mathcal{G}^0$. As in the finite-dimensional case, the trick is to consider the "Kempf–Ness function"

$$F : \mathcal{G}^0_{\mathbb{C}} \longrightarrow \mathbb{R}, \quad g \longmapsto f(g \cdot A).$$

Then, if $g$ is a critical point of $F$ we have $\mu(g \cdot A) = 0$ as expected. Now, note that $F$ is invariant under left multiplication by $\mathcal{G}^0$, so it descends to a map $\mathcal{F} : C^2(I, \mathcal{H})^0 \to \mathbb{R}$, where $\mathcal{H} = G_{\mathbb{C}}/G$ and $C^2(I, \mathcal{H})^0$ is the set of $C^2$ maps $h : I \to \mathcal{H}$ such that $h(0) = h(1) = G$. Then, the uniqueness part amounts to show that critical points of $\mathcal{F}$ are unique. Again, we show this by showing that $\mathcal{F}$ has a unique global minimum by a convexity argument.

First, as in the Riemannian case, we can gauge $A$ to zero, i.e. there is $g \in \mathcal{G}_{\mathbb{C}}$ such that $g \cdot A = 0$ and $g(0) = 1$. Indeed, this reduces to the linear ODE $\dot{g} = gA$. Thus, we can assume without loss of generality that $A = 0$, but we have to enlarge the domain of $\mathcal{F}$ to $\mathcal{G}_{\mathbb{C}}/\mathcal{G} = C^2(I, \mathcal{H})$ and now we must show that for any $a, b \in G_{\mathbb{C}}$ there is a unique $h \in C^2(I, \mathcal{H})$ which minimizes $\mathcal{F}$ with the constraint that $h(0) = Ga$ and $h(c) = Gb$. But the map $\mathcal{F}$ can be expressed as

$$\mathcal{F} : C^2(I, \mathcal{H}) \longrightarrow \mathbb{R}, \quad \mathcal{F}(h) = \frac{1}{2}\int_I \|\dot{h}(t)\|^2_{\mathcal{H}} dt,$$

where the norm is computed in the $G$-invariant metric on the symmetric space $\mathcal{H}$. In other words, minimums of the functional $\mathcal{F}$ are precisely geodesics on $\mathcal{H}$. But $\mathcal{H}$ is a simply-connected complete Riemannian manifold of sectional curvature $\leq 0$ (for two orthogonal vectors $x, y \in \mathfrak{g} \cong T_1(G_{\mathbb{C}}/G)$ the sectional curvature is given by $K(x,y) = -\frac{1}{4}\|[x,y]\|^2$, see, e.g. [24]), so there is a unique geodesic through any two points (a theorem of Cartan). (In fact, in this case all geodesics are of the form $t \mapsto G\exp(itx)g$ for $x \in \mathfrak{g}$ and $g \in G_{\mathbb{C}}$). This implies that for all $A \in \mathcal{A}_{\mathbb{C}}$ there is $g \in \mathcal{G}^0_{\mathbb{C}}$ such that $\mu(g \cdot A) = 0$ and, moreover, that if $g' \in \mathcal{G}^0_{\mathbb{C}}$ has the same property, then $g' = gk$ for some $k \in \mathcal{G}^0$. Hence, we get a homeomorphism

$$\mu^{-1}(0)/\mathcal{G}^0 \longrightarrow \mathcal{A}_{\mathbb{C}}/\mathcal{G}^0_{\mathbb{C}}.$$

Now, using the holonomy map again, we have a biholomorphism $\mathcal{A}_{\mathbb{C}}/\mathcal{G}^0_{\mathbb{C}} \cong G_{\mathbb{C}}$ and the composition $\mu^{-1}(0)/\mathcal{G}^0 \to G_{\mathbb{C}}$ is a biholomorphism which intertwines the residual $\mathcal{G}/\mathcal{G}^0 = G \times G$ action on $\mu^{-1}(0)/\mathcal{G}^0$ with the action of $G \times G$ on $G_{\mathbb{C}}$ by left and right multiplications. This gives the bi-invariant Kähler structure on $G_{\mathbb{C}}$. It can also be checked directly from the infinite-dimensional construction that the Riemannian metric is complete.

We can also identify the symplectic quotient $\mu^{-1}(0)/\mathcal{G}^0$ as $T^*G$. Indeed, we can always find $g \in \mathcal{G}$ which transforms $A_0$ to 0, i.e. $gA_0g^{-1} - \dot{g}g^{-1} = 0$ with $g(0) = 1$. Then, since the moment map is $\mathcal{G}$-equivariant, the new element $B = g \cdot A$ satisfies $\dot{B}_1 = 0$ so $g \cdot A = (0, x)$ for some constant $x \in \mathfrak{g}$. Then, the map

$$\mu^{-1}(0)/\mathcal{G}_0 \longrightarrow G \times \mathfrak{g}, \quad A \longmapsto (g(1), A_1(0))$$

where $\dot{g} = gA_0$ and $g(0) = 1$ is a diffeomorphism. Moreover, by identifying $G \times \mathfrak{g} = G \times \mathfrak{g}^* = T^*G$ using right translations, this diffeomorphism is a symplectomorphism with respect to the canonical symplectic form on $T^*G$. This morphism is also compatible with the natural $G \times G$-action on $T^*G$. Recall that if $M$ is any smooth manifold with an action of a Lie group $G$, then this lifts to an action of $G$ on $T^*M$ which preserves the canonical symplectic form and has a canonical moment map (given by $\mu(\xi)(x) = \alpha(x^{\#}_{\xi})$ for $\xi \in T^*M$ and $x \in \mathfrak{g}$ where $\alpha \in \Omega^1_M(\mathfrak{g})$ is the tautological 1-form). We can use this general principle with the action of $G \times G$ on $G$ by left and right multiplication to get a canonical Hamiltonian $G \times G$-action on $T^*G$. Then, the symplectomorphism $\mu^{-1}(0)/\mathcal{G}_0 \to T^*G$ intertwines the residual $\mathcal{G}/\mathcal{G}^0 = G \times G$-action and this canonical $G \times G$-actions. Moreover, the moment map on $T^*G$ has a simple expression on $\mu^{-1}(0)/\mathcal{G}_0$, namely, it is given by

$$\mu^{-1}(0)/\mathcal{G}_0 \longrightarrow \mathfrak{g}^* \times \mathfrak{g}^*, \quad A \longmapsto (A_1(0), -A_1(1)),$$

where we identify $\mathfrak{g}^*$ with $\mathfrak{g}$ using the invariant inner-product.

By going through the definitions of the maps above, we can see that the diffeomorphisms $G \times \mathfrak{g} \to G_{\mathbb{C}}$ is given by $(k, x) \mapsto ke^{ix}$. Hence, a corollary of our discussion is a proof of the polar decomposition for complex reductive groups.

Moreover, by general theory, the Kähler potential $f : \mathcal{A}_{\mathbb{C}} \to \mathbb{R}$ descends to a Kähler potential on $\mu^{-1}(0)/\mathcal{G}^0$. In terms of $T^*G = G \times \mathfrak{g}$, we can see that this potential is given by

$$G \times \mathfrak{g} \longrightarrow \mathbb{R}, \quad (g, x) \longmapsto \frac{1}{2}\|x\|^2.$$

Hence, the Kähler structure on $T^*G = G_{\mathbb{C}}$ can be been obtained purely by finite-dimensional methods, simply by using the polar decomposition $G \times \mathfrak{g} = G_{\mathbb{C}}$ and defining the potential $f$ by this formula.

On the other hand, when we take the next step and quaternionise to get the hyperkähler structure on $T^*G_{\mathbb{C}}$, the Kähler potential will not admit such a simple finite-dimensional formula, and hence this approach will give genuinely new and non-trivial structures, which can only be seen by going to the infinite-dimensional setting.

## 4.3 Hyperkähler structure on the cotangent bundle of a complex reductive group from Nahm's equations

Finally, we quaternionize, i.e. let $\mathcal{A}_{\mathbb{H}}$ be the set of $C^1$ maps $A = A_0 + A_1 i + A_2 j + A_3 k : I \to \mathfrak{g}_{\mathbb{H}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{H}$. We will view this as an infinite-dimensional Banach hyperkähler manifold with an action of $\mathcal{G}^0$, and study the hyperkähler quotient. The moment map equations will then be precisely the Nahm equations and the hyperkähler quotient will be identified with $T^*G_{\mathbb{C}}$.

### 4.3.1 The moduli space

We will restrict to the interval $I = [0, 1]$ for simplicity of notation, but all results are valid for a general interval $[a, b]$. We will discuss the effect of varying the interval in the next section. As before, we see $\mathcal{A}_{\mathbb{H}}$ as a Banach manifold with Riemannian metric

$$\langle X, Y \rangle := \int_0^1 \sum_{i=0}^{3} \langle X_i(t), Y_i(t) \rangle dt.$$

But now we also have three complex structures induced by left multiplications by $i, j, k$ and hence $\mathcal{A}_{\mathbb{H}}$ is a Banach hyperkähler manifold. Again, this structure is preserved by the gauge group $\mathcal{G}$, which acts by

$$g \cdot (A_0, A_1, A_2, A_3) = (gA_0g^{-1} - \dot{g}g^{-1}, gA_1g^{-1}, gA_2g^{-1}, gA_3g^{-1}).$$

Then, $\mathcal{G}^0$ acts freely on $\mathcal{A}_{\mathbb{H}}$ and we have a hyperkähler moment map

$$\mu : \mathcal{A}_{\mathbb{H}} \longrightarrow C^0(I, \mathfrak{g})^3, \quad A \longmapsto \begin{pmatrix} \dot{A}_1 + [A_0, A_1] + [A_2, A_3] \\ \dot{A}_2 + [A_0, A_2] + [A_3, A_1] \\ \dot{A}_3 + [A_0, A_3] + [A_1, A_2] \end{pmatrix}. \tag{4.1}$$

The proof is similar to the one of the preceding section. Moreover, the same ideas using slices show Kronheimer's theorem:

**Theorem 4.2** (Kronheimer [38])**.** *The quotient*

$$\mathcal{M} := \mu^{-1}(0)/\mathcal{G}^0$$

*is a finite-dimensional smooth manifold with a complete hyperkähler structure invariant under the $G \times G$-action.*

*Proof.* (Sketch) To show that $\mu^{-1}(0)$ is a smooth Banach submanifold of $\mathcal{A}_{\mathbb{H}}$ we use the inverse function theorem, i.e. it suffices to show that for all $A \in \mathcal{A}_{\mathbb{H}}$, the derivative $d\mu_A : \mathcal{A}_{\mathbb{H}} \to C^0(I, \mathfrak{g})^3$ is surjective. In other words, we must show that for all $x \in C^0(I, \mathfrak{g})^3$ and $A \in \mathcal{A}_{\mathbb{H}}$, we can solve the equation

$$\begin{aligned}
\dot{X}_1 + [X_0, A_1] + [A_0, X_1] + [X_2, A_3] + [A_2, X_3] &= x_1 \\
\dot{X}_2 + [X_0, A_2] + [A_0, X_2] + [X_3, A_1] + [A_3, X_1] &= x_2 \\
\dot{X}_3 + [X_0, A_3] + [A_0, X_3] + [X_1, A_2] + [A_1, X_2] &= x_3
\end{aligned}$$

for $X \in \mathcal{A}_{\mathbb{H}}$. As before, this is a first order linear inhomogenous equation, so we can always find a solution.

Now, to show that the quotient $\mu^{-1}(0)/\mathcal{G}^0$ is smooth we must find slices for the $\mathcal{G}^0$-action. For that, we can use the $L^2$-inner-product. At each $A \in \mu^{-1}(0)$, the orthogonal complement to the $\mathcal{G}^0$-orbit through $A$ is the set of $X \in \mathcal{A}_{\mathbb{H}}$ such that

$$\dot{X}_0 + [A_0, X_0] + [A_1, X_1] + [A_2, X_2] + [X_3, X_3] = 0 \tag{4.2}$$

$$\dot{X}_1 + [A_0, X_1] + [X_0, A_1] + [A_2, X_3] + [X_2, A_3] = 0 \tag{4.3}$$

$$\dot{X}_2 + [A_0, X_2] + [X_0, A_2] + [A_3, X_1] + [X_3, A_1] = 0 \tag{4.4}$$

$$\dot{X}_3 + [A_0, X_3] + [X_0, A_3] + [A_1, X_2] + [X_1, A_2] = 0. \tag{4.5}$$

The first equation is the condition that $X$ is orthogonal to the $\mathcal{G}^0$-orbit and the last three equations are the condition that $X$ is tangent to $\mu^{-1}(0)$. This is a linear ODE whose space of solutions has dimension $4 \dim \mathfrak{g}$. Moreover, in a neighbourhood of $A$, each $\mathcal{G}^0$-orbit meets $A + H_A \subseteq \mathcal{A}_{\mathbb{H}}$ exactly once, so $\mathcal{M}$ is a smooth manifold and inherits a hyperkähler structure by the hyperkähler quotient construction.

To see completeness of the metric, let $x_n \in \mathcal{M}$ be a Cauchy sequence. Then, we can show that $\nu^{-1}(0) \to \mathcal{M}$ has the horizontal lift property and this implies that there is a sequence of representatives $A_n \in \nu^{-1}(0)$ that are bounded in $\mathcal{A}$. In particular, they have bounded $L^2$ norm. Then, there exists unique $g_n \in \mathcal{G}$ such that $g_n \cdot A_n = (0, *)$. But then $\|g_n \cdot A_n\|_2 = \|A_n\|_2$, so without loss of generality we have representatives of the form $A_n = (0, *)$ with bounded $L^2$-norm. But $A_n$ are then infinitely differentiable and the Nahm's equations imply that all derivatives have bounded $L^2$ norm so $A_n \to A$ where $A$ is smooth (by the Arzela–Ascoli theorem) and satisfies Nahm's equations. Moreover, since Riemannian submersions shorten distances, $x_n$ converges to $x = \pi(A)$ in $\mathcal{M}$. □

### 4.3.2 Varying the interval

Denote by $\mathcal{M}_{[a,b]}$ the moduli space of solutions to Nahm's equations on the compact interval $I = [a, b]$. Then all results of the preceding section remain true, showing that $\mathcal{M}_{[a,b]}$ is a finite-dimensional hyperkähler manifold isomorphic to $T^*G_{\mathbb{C}}$ as a complex-symplectic manifold.

For all $c \in \mathbb{R}$, the translation map $[a, b] \to [a + c, b + c]$ induces an isomorphism $\mathcal{M}_{[a,b]} \to \mathcal{M}_{[a+c,b+c]}$ of hyperkähler manifolds, so $\mathcal{M}_{[a,b]}$ depends only on the length of the interval. Hence, without loss of generality, we can restrict to the intervals of the form $[0,a]$ for $a > 0$ and denote the corresponding moduli space by $\mathcal{M}_a$.

Now, for $a, b > 0$ there is a scaling map $\mathcal{M}_a \to \mathcal{M}_b$ obtained by sending a solution $A : [0,a] \to \mathfrak{g}_{\mathbb{H}}$ in $\mathcal{M}_a$ to the solution $B : [0,b] \to \mathfrak{g}_{\mathbb{H}}$ in $\mathcal{M}_b$ defined by $B(t) = \frac{a}{b} A(\frac{a}{b} t)$. This map is holomorphic with respect to all three complex structures, but scales the metric by a constant factor:

**Proposition 4.3.** *For all $a, b > 0$, the scaling map $\mathcal{M}_a \to \mathcal{M}_b$ is conformal with conformal factor $a/b$, i.e. if $g_a$ and $g_b$ are the Riemannian metrics on $\mathcal{M}_a$ and $\mathcal{M}_b$ respectively, then the pullback of $g_b$ is $\frac{a}{b} g_a$.*

### 4.3.3 Complex-symplectic description

We now identify the complex structure:

**Theorem 4.4** (Kronheimer [38]). *There is a $G \times G$-equivariant diffeomorphism*

$$\mathcal{M} \longrightarrow T^*G_{\mathbb{C}},$$

*which is also an isomorphism of complex-symplectic manifolds with respect to $(I, \omega_J + i\omega_K)$ on $\mathcal{M}$ and the canonical complex-symplectic structure on $T^*G_{\mathbb{C}}$.*

This uses an infinite-dimensional version of the equivalence between hyperkähler quotients and complex-symplectic quotients as stated in Theorem 3.6. We now explain the proof of this thereom. Let

$$\alpha = A_0 + iA_1, \quad \beta = A_2 + iA_3,$$

viewed as maps $I \to \mathfrak{g}_{\mathbb{C}}$. In other words, we identify $\mathcal{A}_{\mathbb{H}}$ with $\mathcal{A}_{\mathbb{C}}^2$, where the latter is viewed as an infinite-dimensional complex-symplectic manifold. Using the decomposition $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$, we have a conjugation operation $\mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}} : x \mapsto \bar{x}$ such that $\bar{x} = x$ if and only if $x \in \mathfrak{g}$. But it is more customary to use the "conjugate transpose" operation $\mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}} : x \mapsto x^*$ defined by $x^* := -\bar{x}$. If we have an embedding $G \subseteq U(n)$ (which always exists) then $x^*$ is the usual conjugate transpose. Then, splitting the hyperkähler moment map (4.1) into a complex part $\mu_{\mathbb{C}} = \mu_J + i\mu_K : \mathcal{A}_{\mathbb{C}}^2 \to C^0(I, \mathfrak{g}_{\mathbb{C}})$ and a real part $\mu_I : \mathcal{A}_{\mathbb{H}} \to C^0(I, \mathfrak{g})$, we see that Nahm's equations are equivalent to

$$\begin{aligned} \dot{\beta} + [\alpha, \beta] &= 0 \\ \dot{\alpha} + \dot{\alpha}^* + [\alpha, \alpha^*] + [\beta, \beta^*] &= 0. \end{aligned}$$

The first of these is called the *complex Nahm equation* and the second is called the *real Nahm equation*. As expected from the finite-dimensional picture, the complex part $\mu_{\mathbb{C}}$ is a complex-moment with respect to the action of the complexified group $\mathcal{G}_{\mathbb{C}}$. Explicitly, $\mathcal{G}_{\mathbb{C}}$ acts on $\mathcal{A}_{\mathbb{C}}^2$ by

$$g \cdot (\alpha, \beta) = (\mathrm{Ad}_g \alpha - \dot{g}g^{-1}, \mathrm{Ad}_g \beta).$$

Hence, the complex Nahm equation is preserved by $\mathcal{G}_{\mathbb{C}}$ and, as for the finite-dimensional case, we wish to show that $\mu^{-1}(0)/\mathcal{G}^0$ is biholomorphic to $\mu_{\mathbb{C}}^{-1}(0)/\mathcal{G}_{\mathbb{C}}^0$. In other words, we want the following result of Donaldson:

**Theorem 4.5** (Donaldson [16])**.** *For every solution $(\alpha, \beta)$ to the complex Nahm equation, there exists $g \in \mathcal{G}_{\mathbb{C}}^0$ such that $g \cdot (\alpha, \beta)$ solves the real Nahm equation. Moreover, $g$ is unique up to left multiplication by $\mathcal{G}^0$.*

This theorem implies that the map $\mathcal{M} \to \mathcal{N}$, where

$$\begin{aligned} \mathcal{N} &:= \mu_{\mathbb{C}}^{-1}(0)/\mathcal{G}_{\mathbb{C}}^0 \\ &= \{(\alpha, \beta) \in \mathcal{A}_{\mathbb{C}}^2 : \dot{\beta} + [\alpha, \beta] = 0\}/\mathcal{G}_{\mathbb{C}}^0 \end{aligned}$$

is an isomorphism of complex-symplectic manifolds. The proof of Donaldson's theorem is very neat and will be given below. But assuming it for the time being, we can prove Theorem 4.4 quite easily.

As before, the $\mathcal{G}_{\mathbb{C}}$-freedom allows us to gauge $\alpha$ to zero, i.e. there exists a unique $g \in \mathcal{G}_{\mathbb{C}}$ such that $g \cdot (\alpha, \beta) = (0, \tilde{\beta})$ and $g(0) = 1$, as this is equivalent to the linear ODE $\dot{g} = g\alpha$. Now, since the complex Nahm equation is preserved by the $\mathcal{G}_{\mathbb{C}}$-action, the new solution $(0, \tilde{\beta})$ satisfies $\dot{\tilde{\beta}} = 0$ and hence $g \cdot (\alpha, \beta) = (0, x)$ for some constant $x \in \mathfrak{g}_{\mathbb{C}}$. Thus, up to the action of $\mathcal{G}_{\mathbb{C}}^0$, the solution $(\alpha, \beta)$ is determined by $x$ and $g(1)$. Hence, we have a biholomorphism

$$G_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}} \longrightarrow \mathcal{N}, \quad (a, x) \longmapsto g_a \cdot (0, x),$$

where $g_a$ is any smooth map $I \to G_{\mathbb{C}}$ with $g(1) = a^{-1}$. The inverse map is

$$(\alpha, \beta) \longmapsto (g(1), \beta(a))$$

where $\dot{g} = g\alpha$ and $g(0) = 1$. Now, using right translations and the invariant inner-product, we have that

$$G_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}} \cong G_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}}^* \cong T^* G_{\mathbb{C}}.$$

Moreover, at a point $(a,x) \in G_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}}$ and using $T_{(a,x)}(G_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}}) \cong \mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}}$ the canonical holomorphic-symplectic form on $T^*G_{\mathbb{C}}$ can be expressed as

$$\omega_{(a,x)}((u_1,v_1),(u_2,v_2)) = \langle u_1, v_2\rangle - \langle u_2, v_1\rangle - \langle x, [u_1,u_2]\rangle,$$

where $\langle \cdot, \cdot \rangle : \mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}} \to \mathbb{C}$ is the non-degenerate invariant bilinear form associated to our choice of invariant inner-product on $\mathfrak{g}$.

**Proposition 4.6.** *The map $G_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}} \to \mathcal{N}$ is an isomorphism of complex-symplectic manifolds.*

*Proof.* We can find local lifts $U \subseteq G_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}} \to \mu_{\mathbb{C}}^{-1}(0)$ whose composition with the projection to $\mathcal{N}$ is the isomorphism in question. To simplify the notation, we suppose there is a lift $\widehat{\varphi} : G_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}} \to \mu_{\mathbb{C}}^{-1}(0)$. Let $\pi : \mu_{\mathbb{C}}^{-1}(0) \to \mathcal{N}$ be the quotient map, $i : \mu_{\mathbb{C}}^{-1}(0) \hookrightarrow \mathcal{A}_{\mathbb{H}}$ the inclusion, and $\varphi : G_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}} \to \mathcal{N}$ the bijection. We have $\varphi = \pi\widehat{\varphi}$. Let $\omega$ be the symplectic form on $\mathcal{A}_{\mathbb{H}}$ and $\eta$ the one on $G_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}}$. Then it suffices to show that $\widehat{\varphi}^* i^* \omega = \eta$ since the symplectic form $\bar{\omega}$ on $\mathcal{N}$ satisfies $\pi^*\bar{\omega} = i^*\omega$ and hence $\widehat{\varphi}^* i^*\omega = \widehat{\varphi}^*\pi^*\bar{\omega} = \varphi^*\bar{\omega}$.

Now, let $(a,x) \in G_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}}$ and choose tangent vectors $(u_1,v_1),(u_2,v_2) \in T_{(a,x)}(G_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}})$. Since we use right translation, $u_1$ can be identified with a tangent vector of $G_{\mathbb{C}}$ at $a$ via $\frac{d}{dt}|_{t=0} e^{tx}a$. Then,

$$d\varphi_{(a,x)}(u,v) = \frac{d}{ds}\Big|_{s=0} \varphi(e^{su}a, x + sv)$$

and we can take $g_{e^{su}a}(t) = g_a(t)e^{-tsu}$. Hence,

$$d\varphi_{(a,x)}(u,v) = (\mathrm{Ad}_{g_a} u, \mathrm{Ad}_{g_a}(t[x,u] + v)).$$

Now, we have

$$\omega((X_1,Y_1),(X_2,Y_2)) = \int_0^1 \langle X_1, Y_2\rangle - \langle Y_1, X_2\rangle$$

so

$$\begin{aligned}(\varphi^*\omega)((u_1,v_1),(u_2,v_2)) &= \int_0^1 \langle u_1, t[x,u_2] + v_2\rangle - \langle u_2, t[x,u_1] + v_1\rangle \\ &= \langle u_1, v_2\rangle - \langle u_2, v_1\rangle - \langle x, [u_1,u_2]\rangle\end{aligned}$$

as claimed. ∎

We now explain the proof of Donaldson's theorem. As in the finite-dimensional case, we need a Kähler potential for the first complex structure on $\mathcal{A}_{\mathbb{H}}$. Here, we take

$$f : \mathcal{A}_{\mathbb{H}} = \mathcal{A}_{\mathbb{C}}^2 \longrightarrow \mathbb{R}, \quad f(\alpha, \beta) = \frac{1}{8} \int_I \|\alpha + \alpha^*\|^2 + 2\|\beta\|^2, \tag{4.6}$$

or equivalently

$$f(A) = \frac{1}{4} \int_I 2\|A_1\|^2 + \|A_2\|^2 + \|A_3\|^2.$$

It is a global Kähler potential with the property that $\mu_I(A)(x) = df(Ix_A^\#)$ for $x \in \mathrm{Lie}(\mathcal{G}^0)$. Hence, we argue as in the finite-dimensional case, i.e. we fix a solution $(\alpha, \beta)$ to the complex Nahm equation and consider the functional

$$F : \mathcal{G}_{\mathbb{C}}^0 \longrightarrow \mathbb{R}, \quad F(g) = f(g \cdot (\alpha, \beta)).$$

Hence $F$ is a "Kempf–Ness function", i.e. a critical point $g$ of $F$ has the property that $g \cdot (\alpha, \beta)$ satisfies the real Nahm equation. Moreover, if $k \in \mathcal{G}^0$ then $F(kg) = F(g)$ so we can view $F$ as a function on $C^2(I, \mathcal{H})^0$, i.e. the set of $C^2$ maps $h : I \to \mathcal{H} = G_{\mathbb{C}}/G$ with fixed points $h(0) = h(1) = 0$. As before, we can gauge $\alpha$ to zero using $\mathcal{G}_{\mathbb{C}}$, i.e. there exists $g \in \mathcal{G}_{\mathbb{C}}$ such that $g \cdot (\alpha, \beta) = (0, x)$ for some $x \in \mathfrak{g}$. Then, we can assume that $(\alpha, \beta) = (0, x)$, but must find a minimum $h$ with boundary points instead given by $h(0) = Gg(0)^{-1}$ and $h(c) = Gg(1)^{-1}$. Then, the problem reduces to showing that for all boundary points $h_0, h_1 \in \mathcal{H}$ the functional

$$\mathcal{F} : C^2(I, \mathcal{H}) \longrightarrow \mathbb{R}, \quad \mathcal{F}(h) = f(h \cdot (0, x))$$

has a unique minimum with $h(0) = h_0$ and $h(1) = h_1$. This functional takes the form

$$\mathcal{F}(h) = \int_I \|\dot{h}\|_{\mathcal{H}}^2 + V_x(h),$$

where $V_x(h) = \|hxh^{-1}\|^2 \geq 0$. This is a problem in the calculus of variations. A minimum of $\mathcal{F}$ can be thought of as a particle moving under the potential $-V_x$ in the Riemannian manifold $\mathcal{H}$. Moreover, the function $V_x : \mathcal{H} \to \mathbb{R}$ is geodesically convex and $\mathcal{H} \cong \mathfrak{g}$ is simply connected and of negative curvature. Then, it follows from the direct method in the calculus of variations that $\mathcal{F}$ for any points $h_0, h_1 \in \mathcal{H}$, there is a unique path from $h_0$ to $h_1$ minimizing $\mathcal{F}$. This is, in essence, the proof of Donaldson's theorem.

### 4.3.4 Tri-Hamiltonian action

Now, the larger group $\mathcal{G}$ still acts on $\mathcal{A}_{\mathbb{H}}$ by preserving the hyperkähler structure and solutions to Nahm's equations. Thus, there is a residual action of $\mathcal{G}/\mathcal{G}^0 = G\times G$ on $\mathcal{M}$ preserving the hyperkähler structure. Going through the definitions of the isomorphisms with $G_{\mathbb{C}}\times\mathfrak{g}_{\mathbb{C}}$ and $T^*G_{\mathbb{C}}$ it is easy to verify the following description.

**Proposition 4.7.** *Under the diffeomorphism $\mathcal{M}\to G_{\mathbb{C}}\times\mathfrak{g}_{\mathbb{C}}$, the action of $G\times G$ is given by*

$$(g,h)\cdot(a,x) = (gah^{-1}, \mathrm{Ad}_g\, x).$$

*Under the isomorphism $G_{\mathbb{C}}\times\mathfrak{g}_{\mathbb{C}}\cong T^*G_{\mathbb{C}}$, this corresponds to the lift of the action of $G\times G$ on $G_{\mathbb{C}}$ by $(g,h)\cdot a = gah^{-1}$.*

As explained earlier, if a Lie group $G$ acts on a smooth manifold $M$, the lift of this action to $T^*M$ has a canonical moment. Thus, by the previous result, we have a moment map for $G\times G$ action on $\mathcal{M}$ with respect to the first Kähler structure, and by symmetry, there must be a hyperkähler moment map on $\mathcal{M}$. As in the preceding section, this map has a simple expression by evaluation solutions to Nahm's equations at the boundary points. That is, by identifying $\mathfrak{g}^*$ with $\mathfrak{g}$ using the invariant inner-product and we can view $(\mathfrak{g}\times\mathfrak{g})^*\otimes\mathrm{Im}\,\mathbb{H} = (\mathfrak{g}\times\mathfrak{g})^3$ as the space of $2\times 3$ matrices with coordinates in $\mathfrak{g}$, and the hyperkähler moment map

$$\mu : \mathcal{M} \longrightarrow (\mathfrak{g}\times\mathfrak{g})^*\otimes\mathrm{Im}\,\mathbb{H}$$

is given by

$$\mu(A) = \begin{pmatrix} A_1(0) & A_2(0) & A_3(0) \\ -A_1(1) & -A_2(1) & -A_3(1)\end{pmatrix}.$$

We now prove this formula, following [12]. We will only prove that $A\mapsto -A_1(b)$ is a moment map with respect to $\omega_I$ for the right action of $G$ on $\mathcal{M}$. The rest follows by similar arguments. Take $x\in\mathfrak{g}$ and $A\in\mathcal{M}$. Then, a lift $X$ to $\mathcal{A}_{\mathbb{H}}$ of the vector $x^\#_A\in T_A\mathcal{M}$ can be computed by noting that the gauge transformation $t\mapsto e^{stx}$ acts via $e^{tx}$ on $\mathcal{M}$, so

$$X = \frac{d}{ds}\Big|_{s=0} e^{stx}\cdot A = ([tx,A_0]-x, [tx,A_1],[tx,A_2],[tx,A_3]).$$

Now, let $y \in T_A\mathcal{M}$ and take any horizontal lift $Y$ to $y$. Then, $\omega_I(x_A^\#, y) = \langle IX, Y\rangle$. Note that $IX$ is not necessarily horizontal, but any vertical component will be killed by $Y$ anyway. Hence,

$$\begin{aligned}\omega_I(x_A^\#, y) &= \int_0^1 -\langle [tx, A_1], Y_0\rangle + \langle [tx, A_0] - x, Y_1\rangle - \langle [tx, A_3], Y_2\rangle + \langle [tx, A_2], Y_3\rangle \\ &= \int_0^1 \langle tx, [Y_0, A_1] + [A_0, Y_1] + [Y_2, A_3] + [A_2, Y_3]\rangle - \langle x, Y_1\rangle\end{aligned}$$

Now, using (4.3), we get

$$\omega_I(x_A^\#, y) = \int_0^1 -\langle tx, \dot{Y_1}\rangle - \langle x, Y_1\rangle$$

and by integration by part this is $-\langle x, Y_1(1)\rangle$. But letting $\mu(A) = -A_1(1)$ we have $\langle \mu, x\rangle = -\langle x, A_1(1)\rangle$ so $(d\langle \mu, x\rangle)_A(y) = -\langle x, Y_1(1)\rangle$. This concludes the proof of the moment map formula for the action of $G \times G$ on $\mathcal{M}$.

Now, the complex part of the moment map $\mu_\mathbb{C} = \mu_J + i\mu_K : \mathcal{M} \to \mathfrak{g}_\mathbb{C} \times \mathfrak{g}_\mathbb{C}$ is holomorphic with respect to $I$. Hence, it can be viewed as a holomorphic map $\mu_\mathbb{C} : G_\mathbb{C} \times \mathfrak{g}_\mathbb{C} \to \mathfrak{g}_\mathbb{C} \times \mathfrak{g}_\mathbb{C}$ and it turns out it has a simple algebraic expression, namely,

$$\mu_\mathbb{C}(a, x) = (x, -\operatorname{Ad}_{a^{-1}} x). \tag{4.7}$$

This follows easily from the definition of the isomorphism $\mathcal{M} \to G_\mathbb{C} \times \mathfrak{g}_\mathbb{C}$. As expected, this is the canonical complex moment map for the action of $G_\mathbb{C} \times G_\mathbb{C}$ on $T^*G_\mathbb{C}$.

### 4.3.5 Hyperkähler quotients

By the preceding section for any closed subgroup $H$ of $G\times G$, we can consider the hyperkähler quotient of $\mathcal{M}$ by $H$. Moreover, we have a global Kähler potential $f$ descending from (4.6). It can be shown that this potential is proper [43]. Hence, if the $H$-action is free, Theorem 3.6 applies to show that it is isomorphic to the complex-symplectic quotient of $T^*G_\mathbb{C}$ by $H_\mathbb{C} \subseteq G_\mathbb{C} \times G_\mathbb{C}$. For example, for any closed subgroup $H \subseteq G$, we get a hyperkähler structures on $T^*(G_\mathbb{C}/H_\mathbb{C})$ by viewing $H$ as a subgroup of the left factor of $G \times G$.

This works more generally for non-free actions, where $\mathcal{M} /\!\!/\!\!/ H$ is homeomorphic to the affine GIT quotient $\nu_\mathbb{C}^{-1}(0) /\!\!/ H_\mathbb{C} = \operatorname{Spec} \mathbb{C}[\nu_\mathbb{C}^{-1}(0)]^{H_\mathbb{C}}$, where $\nu_\mathbb{C} : T^*G_\mathbb{C} \to \mathfrak{h}_\mathbb{C}^*$ is given by composing (4.7) with the restriction map

$\mathfrak{g}_{\mathbb{C}}^* \times \mathfrak{g}_{\mathbb{C}}^* \to \mathfrak{h}_{\mathbb{C}}^*$. We can also shift the first moment map $\mu_I$ by the differential of a character $\chi : H \to S^1$ to realize $\mathcal{M}/\!\!/\!\!/_{\mu - d\chi} H$ as a quasi-projective complex algebraic variety, namely, $\operatorname{Proj} \bigoplus_{n=0}^{\infty} \mathbb{C}[\nu_{\mathbb{C}}^{-1}(0)]^{H_{\mathbb{C}}, \chi^n}$, and this can be used to construct explicit resolution of singularities of $\mathcal{M}/\!\!/\!\!/ H$ [44].

### 4.3.6 Concatenation

As in §4.3.2, let $\mathcal{M}_a$ be the moduli space of solutions to Nahm's equations on $[0, a]$. Let $a, b > 0$. Then, we can consider the action of $G$ on $\mathcal{M}_a$ by right translations and on $\mathcal{M}_b$ by left translations. This gives a free action of $G$ on $\mathcal{M}_a \times \mathcal{M}_b$ and hence we can consider the hyperkähler quotient $(\mathcal{M}_a \times \mathcal{M}_b)/\!\!/\!\!/ G$. Note that since the hyperkähler moment maps are evaluation of a solution to Nahm's equations at the boundary points, the zero-level set of the hyperkähler moment map is the set of $(A, B) \in \mathcal{M}_a \times \mathcal{M}_b$ such that $A_i(a) = B_i(0)$ for $i = 1, 2, 3$. Thus, by thinking of elements of $\mathcal{M}_a$ as solutions on $[0, a]$ and elements of $\mathcal{M}_b$ as solutions of $[a, a+b]$, there is a concatenation map which gives a well-defined smooth map

$$(\mathcal{M}_a \times \mathcal{M}_b)/\!\!/\!\!/ G \longrightarrow \mathcal{M}_{a+b}.$$

This map is in fact an isomorphism of hyperkähler manifolds.

### 4.3.7 The reduced Nahm equations

In our discussion of the "baby Nahm equation", we had two descriptions of $\mu^{-1}(0)/\mathcal{G}^0$: the complex version $\mathcal{A}_{\mathbb{C}}/\mathcal{G}_{\mathbb{C}}^0 \cong G_{\mathbb{C}}$ and the symplectic version $\mu^{-1}(0)/\mathcal{G}^0 \cong T^*G$. So far, we have only described the complex version of Nahm's equations, but there is also an analogue of the diffeomorphism $\mu^{-1}(0)/\mathcal{G}^0 \cong T^*G$, due to Dancer–Swann [12]. This gives us another finite-dimensional representation of $\mathcal{M}$, but this one does not depend on a choice of complex structure.

As for the Kähler quotient, we can always gauge $A_0$ to 0, i.e. for each solution $A \in \mathcal{A}_{\mathbb{H}}$ to Nahm's equations there is a unique $g \in \mathcal{G}^0$ such that $g(0) = 1$ and $g \cdot A = (0, P_1, P_2, P_3)$ for some $P_i \in \mathcal{A}$. Indeed, $g$ is the unique solution to the linear ODE $\dot{g} = gA_0$ with $g(0) = 1$. Then, the $P_i$'s satisfy the so-called *reduced Nahm equations*

$$\begin{aligned}\dot{P}_1 + [P_2, P_3] &= 0\\ \dot{P}_2 + [P_3, P_1] &= 0\\ \dot{P}_3 + [P_1, P_1] &= 0.\end{aligned}$$

Note that now the number of unknowns is equal to the number of equations, so solutions are uniquely determined by their initial conditions. Moreover, $P_i(0) = A_i(0)$ since $g(0) = 1$, so, by the smooth dependence of solutions to linear ODEs on the initial conditions, we have:

**Proposition 4.8.** *Define*

$$\varphi : \mathcal{M} \longrightarrow G \times \mathfrak{g}^3, \quad A \longmapsto (g(1), A_1(0), A_2(0), A_3(0))$$

*where $g$ is the unique solution solution to $\dot{g} = gA$ with $g(0) = 1$. Then, $\varphi$ is a diffeomorphism onto $G \times W$ for some open set $W \subseteq \mathfrak{g}^3$ which is star-shaped about $(0, 0, 0)$. Moreover, this diffeomorphism intertwines the action of $G \times G$ on $\mathcal{M}$ with the action on $G \times W$ given by $(g, h) \cdot (a, x) = (gah^{-1}, \operatorname{Ad}_g x)$.*

We can also describe the Kähler potential on this finite-dimensional manifold, namely,

$$f : G \times W \longrightarrow \mathbb{R}, \quad f(a, x) = \frac{1}{4} \int_0^1 2\|P_1^x\|^2 + \|P_2^x\|^2 + \|P_3^x\|^2,$$

where $P^x$ is the unique solution to the reduced Nahm equation with $P^x(0) = x$.

### 4.3.8 Hyperkähler rotation

There is an additional phenomenon on $\mathcal{M}$ which has no analogue on $G$ and $T^*G = G_{\mathbb{C}}$, namely an additional SO(3)-symmetry, called a *hyperkähler rotation* (see §3.6).

The easiest way to describe this is to view $\mathcal{A}_{\mathbb{H}}$ as a bi-$\mathbb{H}$-module by left and right multiplications and consider for $q \in \mathrm{Sp}(1) \subseteq \mathbb{H}^*$ the map

$$\mathcal{A}_{\mathbb{H}} \longrightarrow \mathcal{A}_{\mathbb{H}}, \quad A \longmapsto q \cdot A := qAq^{-1}.$$

In other words, viewing $\mathcal{A}_{\mathbb{H}} = \mathcal{A} \otimes_{\mathbb{R}} \mathbb{H}$ this is the unique linear extension of the map $A \otimes p \mapsto A \otimes qpq^{-1}$ for $A \in \mathcal{A}$ and $p \in \mathbb{H}$. We have $(-q) \cdot A = q \cdot A$, so this descends to an action of SO(3). Explicitly, an element of SO(3) acts on $A = (A_0, A_1, A_2, A_3)$ by fixing $A_0$ and rotating $A_1, A_2, A_3$.

Then, for each $q \in \mathrm{Sp}(1)$, the map $A \to q \cdot A$ is an isometry, and rotates the complex structures, in the sense that if $\mathbf{I}$ is any element of the two-sphere of complex structures, then $q : \mathcal{A}_{\mathbb{H}} \to \mathcal{A}_{\mathbb{H}}$ holomorphic with respect to $\mathbf{I}$ on the left and $q\mathbf{I}q^{-1}$ on the right.

The remarkable fact is that Nahm's are invariant under this action, which follows from the fact that the hyperkähler moment map $\mu : \mathcal{A}_{\mathbb{H}} \to C^0(I, \mathfrak{g}) \otimes \operatorname{Im}\mathbb{H}$ is equivariant with respect to the action of $\mathrm{Sp}(1)$ on $\mathcal{A}_{\mathbb{H}}$ and on $\operatorname{Im}\mathbb{H}$. Moreover, the action of SO(3) on $\mathcal{A}_{\mathbb{H}}$ commutes with the action of $\mathcal{G}$, so this descends to an action of SO(3) on $\mathcal{M}$ commuting with the action of $G \times G$.

Hence, for any choice of frames of complex structures $I', J', K'$, the hyperkähler manifold $(\mathcal{M}, g, I', J', K')$ is isomorphic as a complex-symplectic manifold to $T^*G_{\mathbb{C}}$.

### 4.3.9 Functorial property

Let $M$ be any hyperkähler manifold with a tri-Hamiltonian action of $G$ with moment map $\mu : M \to \mathfrak{g}^* \otimes \operatorname{Im}\mathbb{H}$. Then, there is canonical operation we can do on $M$, which is to consider the diagonal action of $G$ on $M \times T^*G_{\mathbb{C}}$, where $G$ acts on $T^*G_{\mathbb{C}}$ by left translations, and take the hyperkähler quotient. This action is free, so $(M \times T^*G_{\mathbb{C}}) /\!/\!/ G$ is always another smooth hyperkähler manifold. What is it?

**Proposition 4.9.** *The hyperkähler manifold $(M \times T^*G_{\mathbb{C}}) /\!/\!/ G$ is isomorphic as a complex-symplectic manifold to an open subset $U$ of $M$ (but the hyperkähler structure might differ). If $M$ has a $G$-invariant Kähler potential $f$ for $\omega_1$ which is bounded below on $G_{\mathbb{C}}$-orbits, and $\mu_1$ is the moment map associated to $f$, then $U = M$.*

To prove this, we first prove the complex-symplectic version.

**Proposition 4.10.** *Let $(M, I, \omega, G_{\mathbb{C}}, \mu)$ be a complex-Hamiltonian manifold. Then, the complex-symplectic quotient of $M \times T^*G_{\mathbb{C}}$ by $G_{\mathbb{C}}$ is a complex-symplectic manifold and is isomorphic to $M$.*

*Proof.* The action of $G_{\mathbb{C}}$ on $T^*G_{\mathbb{C}}$ is free and proper, and hence so is the action on $M \times T^*G_{\mathbb{C}}$. Thus, the holomorphic version of the Marsden–Weinstein reduction theorem applies and the reduction of $M \times T^*G_{\mathbb{C}}$ by

$G_{\mathbb{C}}$ is a smooth complex-symplectic manifold. Now, the moment map is $\nu : (p,(a,x)) \mapsto \mu(p) + x$, so the quotient is

$$\nu^{-1}(0)/G_{\mathbb{C}} = \{(p,(a,x)) \in M \times G_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}} : x = -\mu(p)\}/G_{\mathbb{C}},$$

where $G_{\mathbb{C}}$ acts by $g \cdot (p,(a,x)) = (g \cdot p, (ga, \mathrm{Ad}_g\, x))$. Now, the inclusion $j : M \hookrightarrow \nu^{-1}(0) : p \mapsto (p,(1,-\mu(p)))$ descends to a biholomorphism $\varphi : M \to \nu^{-1}(0)/G_{\mathbb{C}}$. Moreover, if $\eta$ denotes the complex-symplectic form on $T^*G_{\mathbb{C}}$ and $\zeta$ the one on $\nu^{-1}(0)/G_{\mathbb{C}}$ we need to show that $\varphi^*\zeta = \omega$. The form $\zeta$ is uniquely characterized by $\pi^*\zeta = i^*(\omega+\eta)$, where $\pi : \nu^{-1}(0) \to \nu^{-1}(0)/G_{\mathbb{C}}$ and $i : \nu^{-1}(0) \hookrightarrow M \times T^*G_{\mathbb{C}}$. Now, $j^*i^*(\omega+\eta) = \omega$, so $j^*\pi^*\zeta = \omega$, but $j^*\pi^* = \varphi^*$. To show that $j^*i^*(\omega+\eta) = \omega$, consider the composition $k : M \to M \times T^*G_{\mathbb{C}} : p \mapsto (p,(1,-\mu(p)))$. Then, for $u, v \in T_pM$ we have $dk_p(u) = (u,(0,-d\mu_p(u)))$ so

$$(k^*(\omega+\eta))(u,v) = \omega(u,v) + \eta((0,-d\mu_p(u)),(0,-d\mu_p(v))) = \omega(u,v)$$

since the fibres of $T^*G_{\mathbb{C}}$ are Lagrangian. □

*Proof of Proposition 4.10.* By the the preceding result and Theorem 3.6, it suffices to show that $M \times T^*G_{\mathbb{C}}$ has a $G$-invariant Kähler potential which is proper on $G_{\mathbb{C}}$-orbits. This follows from the fact that the sum of a proper and bounded below function and one which is bounded below, is proper. □

There is a categorical interpretation of the result of this section [46]. Take a category whose objects are complex reductive groups and such that a morphism from $G_{\mathbb{C}}$ to $H_{\mathbb{C}}$ is an isomorphism class of complex-symplectic manifolds with a complex-Hamiltonian action of $G_{\mathbb{C}} \times H_{\mathbb{C}}$. The composition between $M : G_{\mathbb{C}} \to H_{\mathbb{C}}$ and $N : H_{\mathbb{C}} \to K_{\mathbb{C}}$ is the symplectic reduction $(M \times N)/\!\!/ H_{\mathbb{C}}$. Then, we have found the identity morphism in that category: $T^*G_{\mathbb{C}}$. Note, however, that we cannot enhance this to a category whose morphisms are hyperkähler manifolds since $\mathcal{M} = T^*G_{\mathbb{C}}$ would not be an identity since the hyperkähler structures are changed. For example, we saw that $(\mathcal{M}_a \times \mathcal{M}_b)/\!\!/\!\!/ G \cong \mathcal{M}_{a+b}$.

#### 4.3.10 Twistor space

We can give an explicit description of the twistor space $Z_{\mathcal{M}}$ of $\mathcal{M}$. Since there is a hyperkähler rotation, we know that the fibres of $Z$ will all be copies of $T^*G_{\mathbb{C}}$, so the twistor space will be obtained by two copies of $T^*G_{\mathbb{C}} \times \mathbb{C}$ glued in a certain way. This was computed by Kronheimer in [38], but the proof we will give is inspired from Biquard [8].

For a solution to Nahm's equations $A$ and $\zeta \in \mathbb{CP}^1 - \{\infty\}$, set

$$\begin{aligned}\alpha(A,\zeta) &= (A_0 + iA_1) - \zeta(A_2 - iA_3)\\ \beta(A,\zeta) &= (A_2 + iA_3) - 2i\zeta A_1 + \zeta^2(A_2 - iA_3).\end{aligned}$$

View $\mathcal{A}_{\mathbb{H}} \times \mathbb{C}$ is an open subset of the twistor space of $\mathcal{A}_{\mathbb{H}}$. Then, the map

$$\mathcal{A}_{\mathbb{H}} \times \mathbb{C} \longrightarrow \mathcal{A}_{\mathbb{C}}^2 \times \mathbb{C}, \quad (A,\zeta) \longmapsto (\alpha(A,\zeta), \beta(A,\zeta), \zeta)$$

is a trivialization, in the sense that for each fixed $\zeta$ it is complex-symplectomorphism from $\mathcal{A}_{\mathbb{H}}$ with the complex-symplectic structure $(\mathrm{I}_\zeta, (\omega_2 + i\omega_3) - 2i\zeta\omega_1 + \zeta^2(\omega_2 - i\omega_3))$ to $\mathcal{A}_{\mathbb{C}}^2$ with the standard complex-symplectic structure (viewing $\mathcal{A}_{\mathbb{C}}^2 = T^*\mathcal{A}_{\mathbb{C}}$)). Moreover, this isomorphism intertwines the action of $\mathcal{G}^0$ on $\mathcal{A}_{\mathbb{H}}$ with the standard action on $\mathcal{A}_{\mathbb{C}}^2$. Hence, it descends to a trivialization of $\mathcal{M} \times \mathbb{C} \subseteq Z_{\mathcal{M}}$ to the complex-symplectic quotient $\{(\alpha,\beta) : \dot{\beta} + [\alpha,\beta] = 0\}/\mathcal{G}_{\mathbb{C}}^0$. As above, we can identify the latter with $G_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}}$, so we have a biholomorphism

$$\pi^{-1}(U) \subseteq Z_{\mathcal{M}} \longrightarrow G_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}} \times \mathbb{C},$$

where $\pi : Z_{\mathcal{M}} \to \mathbb{CP}^1$ and $U = \mathbb{CP}^1 - \{\infty\}$. Similarly, by setting for $\tilde{\zeta} \in \mathbb{CP}^1 - \{0\}$

$$\begin{aligned}\tilde{\alpha}(A,\tilde{\zeta}) &= (A_0 - iA_1) + \tilde{\zeta}(A_2 + iA_3)\\ \tilde{\beta}(A,\tilde{\zeta}) &= \tilde{\zeta}^2(A_2 + iA_3) - 2i\tilde{\zeta}A_1 + (A_2 - iA_3)\end{aligned}$$

we get a biholomorphism

$$\pi^{-1}(V) \subseteq Z_{\mathcal{M}} \longrightarrow G_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}} \times \mathbb{C}.$$

Then, we want to compute the transition function

$$G_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}} \times \mathbb{C}^* \longrightarrow G_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}} \times \mathbb{C}^*.$$

To do this, observe the relation

$$\begin{aligned}\tilde{\alpha}(A,\zeta^{-1}) &= \alpha(A,\zeta) + \zeta^{-1}\beta(A,\zeta)\\ \tilde{\beta}(A,\zeta^{-1}) &= \zeta^{-2}\beta(A,\zeta).\end{aligned}$$

Let $(a, x, \zeta) \in G_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}} \times \mathbb{C}^*$ and $(\tilde{a}, \tilde{x}, \tilde{\zeta}) \in G_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}} \times \mathbb{C}^*$ be related by this transition function. We have $(\alpha, \beta) = g \cdot (0, x)$ where $g(0) = 1$ and $g(1) = a^{-1}$. Similarly, $(\tilde{\alpha}, \tilde{\beta}) = \tilde{g} \cdot (0, \tilde{x})$. Then, the relations above reduce to

$$\begin{aligned} -\dot{\tilde{g}}\tilde{g}^{-1} &= -\dot{g}g^{-1} + \zeta^{-1} \operatorname{Ad}_g x \\ \operatorname{Ad}_{\tilde{g}} \tilde{x} &= \zeta^{-2} \operatorname{Ad}_g x. \end{aligned}$$

Upon inspection, we find that if $\tilde{g}^{-1}g = e^{\lambda x}$ for some function $\lambda$ then the first equation holds with $\tilde{x} = x/\zeta^2$. This means $\tilde{g} = ge^{-\lambda x}$ and since $\tilde{g}(0) = 1$ it suggest trying $\lambda = tc$ for some constant $c$, i.e. $\tilde{g} = ge^{-tcx}$. Plugging this ansatz in the first equation we get

$$-\dot{g}g^{-1} + gcxg^{-1} = -\dot{g}g^{-1} + \zeta^{-1}gxg^{-1}$$

and hence $c = 1/\zeta$ solves it. Thus, $\tilde{a}^{-1} = (a^{-1}e^{-x/\zeta})^{-1} = e^{x/\zeta}a$ and we find that

$$(\tilde{a}, \tilde{x}, \tilde{\zeta}) = (e^{x/\zeta}a, x/\zeta^2, 1/\zeta).$$

**Theorem 4.11.** *The twistor space is obtained from two copies of $G_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}} \times \mathbb{C}$ with their standard complex-symplectic structures and glued by $(a, x, \zeta) \sim (\tilde{a}, \tilde{x}, \tilde{\zeta})$ if and only if $(\tilde{a}, \tilde{x}, \tilde{\zeta}) = (e^{x/\zeta}a, x/\zeta^2, 1/\zeta)$. Moreover, over the patch $\zeta \neq \infty$, the real structure for $\zeta \neq 0$ is given by*

$$G \times \mathfrak{g} \times \mathbb{C}^* \longrightarrow G \times \mathfrak{g} \times \mathbb{C}^*, \quad (a, x, \zeta) \longmapsto (e^{\bar{x}/\bar{\zeta}}\bar{a}, \bar{x}/\bar{\zeta}^2, -1/\bar{\zeta}),$$

*and the elements $(a, x, 0)$ are sent to $(\bar{a}, \bar{x}, 0)$ in the other patch.*

# 5 Nahm's equations with poles

We now discuss solutions to Nahm's equations on an open interval $(a, b)$ with first-order poles of fixed residues at $t = a, b$, following Bielawski [4].

## 5.1 Boundary conditions

As before, we first consider the case $I = (0, 1)$ to simplify the notation. In this section, we discuss the possible boundary conditions for a solution $A$ to Nahm's equations on $(0, 1)$. Note that since we can always gauge $A_0$ to 0, this component cannot have a pole. But $A_1, A_2, A_3$ might have poles, so let's write

$$A_i = \frac{\sigma_i}{t} + B_i$$

near $t = 0$ for some $\sigma_i$, where $\sigma_0 = 0$. Then, Nahm's equations are

$$\begin{aligned}
-\frac{\sigma_1}{t^2} + \dot{B}_1 + \left[B_0, \frac{\sigma_1}{t} + B_1\right] + \left[\frac{\sigma_2}{t} + B_2, \frac{\sigma_3}{t} + B_3\right] = 0 \\
-\frac{\sigma_2}{t^2} + \dot{B}_2 + \left[B_0, \frac{\sigma_2}{t} + B_2\right] + \left[\frac{\sigma_3}{t} + B_3, \frac{\sigma_1}{t} + B_1\right] = 0 \\
-\frac{\sigma_3}{t^2} + \dot{B}_3 + \left[B_0, \frac{\sigma_3}{t} + B_3\right] + \left[\frac{\sigma_1}{t} + B_1, \frac{\sigma_2}{t} + B_2\right] = 0
\end{aligned}$$

Collecting the terms in $1/t^2$, we get

$$\begin{aligned}
\sigma_1 &= [\sigma_2, \sigma_3] \\
\sigma_2 &= [\sigma_3, \sigma_1] \\
\sigma_3 &= [\sigma_1, \sigma_2].
\end{aligned}$$

This defines an $\mathfrak{su}(2)$-subalgebra in $\mathfrak{g}$. Indeed, the above relations are precisely the ones satisfied by the matrices

$$\sigma_1 = \begin{pmatrix} i/2 & 0 \\ 0 & -i/2 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -1/2 \\ 1/2 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & -i/2 \\ -i/2 & 0 \end{pmatrix}$$

in $\mathfrak{su}(2)$, so this gives a Lie algebra embedding

$$\rho : \mathfrak{su}(2) \to \mathfrak{g}.$$

Thus, boundary conditions for Nahm's equations are classified by $\mathfrak{su}(2)$-subalgebras. Equivalently, $\rho$ complexifies to an embedding $\mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{g}_{\mathbb{C}}$ and every such complex embedding comes from a real one $\mathfrak{su}(2) \to \mathfrak{g}$. So we also use the notation $\rho$ for the complex version

$$\rho : \mathfrak{sl}(2, \mathbb{C}) \longrightarrow \mathfrak{g}_{\mathbb{C}}.$$

Equivalently, this is an $\mathfrak{sl}(2, \mathbb{C})$-triple, i.e. a triple of elements $(e, h, f)$ in $\mathfrak{g}_{\mathbb{C}}$ such that

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

We can go from an $\mathfrak{sl}(2, \mathbb{C})$-triple to an $\mathfrak{su}(2)$-triple and conversely by the relation

$$e = -\sigma_2 + i\sigma_3, \quad h = -2i\sigma_1, \quad f = \sigma_2 + i\sigma_3.$$

In particular, when $G = U(n)$, pole conditions are classified by faithful representations of $\mathfrak{sl}(2, \mathbb{C})$ on $\mathbb{C}^n$ and hence there are finitely many possibilities.

More generally, the $\mathfrak{sl}(2,\mathbb{C})$-triple up to conjugations are in one-to-one correspondences with the set of nilpotent orbits in $\mathfrak{g}_\mathbb{C}$ and there are only finitely many of them.

Thus, when choosing boundary conditions for Nahm's equations we have finitely many choices. There is a canonical choice, which is the $\mathfrak{sl}_2$-triple corresponding to the largest nilpotent orbit, or in other words, any $\mathfrak{sl}_2$ triple $(e,h,f)$ such that $e$ is regular (and hence $h$ and $f$ are also regular). This is called a *principal* $\mathfrak{sl}_2$*-triple*.

Fix two $\mathfrak{sl}_2$-triple $\rho_0=(e_0,h_0,f_0)$ and $\rho_1=(e_1,h_1,f_1)$ in $\mathfrak{g}_\mathbb{C}$ or equivalently $\mathfrak{su}(2)$-triples $\sigma_0$, $\sigma_1$ in $\mathfrak{g}$. Let $\mathcal{A}_\mathbb{H}(\rho_0,\rho_1)$ be the set of smooth maps $(0,1)\to\mathfrak{g}\otimes\mathbb{H}$ which near $t=0$ are of the form $A(t)=\sigma_0/t+B(t)$ where $B$ has an expansion in non-negative powers of $t^{1/2}$, and similarly at $t=1$. The fractional powers are needed because if $\mathfrak{g}=\mathfrak{h}\oplus\bigoplus_{\alpha\in\Phi}\mathfrak{g}_\alpha$ is a Cartan decomposition with $h_0\in\mathfrak{h}$, then for all $x_\alpha\in\mathfrak{g}_\alpha$, the maps

$$(\alpha(t),\beta(t))=\left(-\frac{h_0}{2t},\frac{f_0}{t}+\sum_{\alpha\in\Phi}t^{\alpha(h)/2}x_\alpha\right)$$

are solutions to the complex Nahm equation with the appropriate pole condition at $t=0$, and we would like to include them even when some $\alpha(h)$ are not even (they are always integers). On the other hand, if $\rho$ is principal, no fractional power is needed, since $\alpha(h)=2\operatorname{level}(\alpha)$ for all $\alpha\in\Phi$.

Since the poles are fixed at both endpoints, the $L^2$-metric is well-defined, endowing $\mathcal{A}_\mathbb{H}(\rho_0,\rho_1)$ with the structure of an infinite-dimensional hyperkähler manifold. As before, we will view Nahm's equations as a hyperkähler moment map condition and take the hyperkähler quotient.

## 5.2 The moduli space

Define the centralizer $G_\rho=\{g\in G:\operatorname{Ad}_g\sigma_i=\sigma_i\}$. For example, $G_\rho=Z_G$ if $\rho$ is principal, and $G_\rho=G$ if $\rho$ is trivial. Moreover, $(G_\rho)_\mathbb{C}=(G_\mathbb{C})_\rho=\{g\in G:\operatorname{Ad}_g e=e,\operatorname{Ad}_g h=h,\operatorname{Ad}_g f=f\}$. Let $\mathcal{G}$ be the group of smooth maps $g:[0,1]\to G$ which are analytic near $t=0,1$, $g(0)\in G_{\rho_0}$, and $g(1)\in G_{\rho_1}$.

Let $\mathcal{G}^0$ be the subgroup of $g\in\mathcal{G}$ such that $g(0)=g(1)=1$. Then, as before, the map

$$\mu:\mathcal{A}_\mathbb{H}(\rho_0,\rho_1)\longrightarrow C^0(I,\mathfrak{g})^3,\quad A\longmapsto\begin{pmatrix}\dot{A}_1+[A_0,A_1]+[A_2,A_3]\\ \dot{A}_2+[A_0,A_2]+[A_3,A_1]\\ \dot{A}_3+[A_0,A_3]+[A_1,A_2]\end{pmatrix}$$

can be viewed as a hyperkähler moment map for the action of $\mathcal{G}^0$ on $\mathcal{A}_{\mathbb{H}}(\rho_0, \rho_1)$, and the quotient

$$\mathcal{M}(\rho_0, \rho_1) \coloneqq \mu^{-1}(0)/\mathcal{G}^0$$

is a finite-dimensional smooth manifold with a complete hyperkähler structure. Moreover, there is a residual action of $\mathcal{G}/\mathcal{G}^0 \cong G_{\rho_0} \times G_{\rho_1}$ on $\mathcal{M}(\rho_0, \rho_1)$ which preserves the hyperkähler structure. We recover the results of the preceding section by setting $\rho_0 = \rho_1 = 0$, i.e $\mathcal{M}(0, 0) = \mathcal{M} \cong T^*G_{\mathbb{C}}$.

More generally, we may consider the moduli spaces $\mathcal{M}_a(\rho_0, \rho_1)$ where $a > 0$ is the length of the interval $I$ on which Nahm's equations are defined. Then, by an argument similar to the one in §4.3.6, we have:

**Proposition 5.1.** *For all $a, b > 0$ there is an isomorphism of hyperkähler manifolds*

$$(\mathcal{M}_a(\rho_0, 0) \times \mathcal{M}_b(0, \rho_1)) /\!/\!/ G \longrightarrow \mathcal{M}_{a+b}(\rho_0, \rho_1),$$

*where $G$ acts on $\mathcal{M}_a(\rho_0, 0)$ as the right factor of $G_{\rho_0} \times G$ and on $\mathcal{M}_b(0, \rho_1)$ as the left factor of $G \times G_{\rho_1}$.*

*Moreover, there is an isomorphism of hyperkähler manifolds $\mathcal{M}_b(0, \rho_1) \cong \mathcal{M}_b(\rho_1, 0)$ obtained by sending a solution $A$ to Nahm's equation on $(0, b)$ to the solution $t \mapsto -A(1-t)$ in $\mathcal{M}_b(\rho_1, 0)$.*

Hence, it suffices to describe

$$\mathcal{M}_\rho \coloneqq \mathcal{M}(\rho, 0).$$

The first thing to observe is that $\mathcal{M}_\rho$ has a holomorphic description as before, namely, it is the quotient of solutions of $\dot{\beta} + [\alpha, \beta] = 0$ by $\mathcal{G}^0_{\mathbb{C}}$. This is proved by first using the Donaldson's Theorem 4.5 on $[\varepsilon, 1]$ for $\varepsilon > 0$ to find $g_\varepsilon$ such that $g_\varepsilon(\varepsilon) = g(1) = 1$ and $g_\varepsilon \cdot (\alpha, \beta)$ satisfies the real Nahm equation on $[\varepsilon, 1]$ and carefully letting $\varepsilon \to 0$.

## 5.3 Complex-symplectic description

We first collect a few Lie algebraic facts. For an $\mathfrak{sl}_2$-triple $\rho = (e, h, f)$, we define the *Slodowy slice*

$$\mathcal{S}_\rho \coloneqq f + Z_{\mathfrak{g}_{\mathbb{C}}}(e),$$

which is an affine subspace of $\mathfrak{g}_{\mathbb{C}}$ of dimension $\dim Z_{\mathfrak{g}_{\mathbb{C}}}(e) \geq \operatorname{rk} \mathfrak{g}_{\mathbb{C}}$. The case where $\dim \mathcal{S}_\rho = \operatorname{rk} \mathfrak{g}_{\mathbb{C}}$ corresponds to the case where $\rho$ is principal. The

Slodowy slice has the property of intersecting all adjoint orbit transversely, which is equivalent to the linear-algebraic fact that for all $x \in \mathcal{S}_\rho$ we have

$$\mathfrak{g}_{\mathbb{C}} = [\mathfrak{g}_{\mathbb{C}}, x] + Z_{\mathfrak{g}_{\mathbb{C}}}(e).$$

This can be proved by decomposing $\mathfrak{g}_{\mathbb{C}}$ into irreducible representations of $\mathfrak{sl}(2,\mathbb{C}) = \operatorname{span}\{e, h, f\}$.

**Theorem 5.2** (Bielawski)**.** *For all $\rho$, $G_{\mathbb{C}} \times \mathcal{S}_\rho$ is a complex-symplectic submanifold of $G_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}} \cong T^*G_{\mathbb{C}}$ and there is an isomorphism of complex-symplectic manifolds $\mathcal{M}_\rho \cong G_{\mathbb{C}} \times \mathcal{S}_\rho$.*

We now give an idea of the proof. First, there exists $g \in \mathcal{G}^0_{\mathbb{C}}$ such that $g\cdot(\alpha,\beta) = (\alpha',\beta')$ has the property that $\alpha' = \frac{i\sigma_1}{t} = -\frac{h}{2t}$ near $t=0$. Now, decompose $\mathfrak{g} = \bigoplus_{\lambda\in\mathbb{Z}} \mathfrak{g}_\lambda$, where $\mathfrak{g}_\lambda$ are the weight spaces for the representation $\mathfrak{sl}(2,\mathbb{C}) \to \operatorname{End}(\mathfrak{g}_{\mathbb{C}})$. Then, near $t=0$, the complex Nahm equation implies that $\beta' = \frac{f}{t} + \sum_{\lambda\in\mathbb{Z}} t^{\lambda/2} x_\lambda$ for some $x_\lambda \in \mathfrak{g}_\lambda$, and since $\beta'$ has a simple pole with residues $f/t$, we have $x_\lambda = 0$ for $\lambda < 0$. Moreover, we can further gauge by some $g$ to kill all $x_\lambda$ except the highest weight ones, i.e. those $x_\lambda$ such that $[e, x_\lambda] = 0$. Let $x_1, \ldots, x_k \in \mathfrak{g}_{\mathbb{C}}$ be a basis of highest weights vectors of weights $\lambda_1, \ldots, \lambda_k$. Note that $\operatorname{span}\{x_1, \ldots, x_k\} = Z_{\mathfrak{g}_{\mathbb{C}}}(e)$. Hence, there exists $\varepsilon > 0$ and $g : [0,\varepsilon) \to G_{\mathbb{C}}$ such that

$$g\cdot(\alpha,\beta) = \left(-\frac{h}{2t}, \frac{f}{t} + \sum_{i=1}^{k} t^{\lambda_i/2} x_i\right).$$

Define

$$D(t) = \exp\left(\frac{\log(t)}{2} h\right) : (0,1] \to G_{\mathbb{C}}.$$

Then, $-\dot{D}D^{-1} = -\frac{h}{2t}$ and $\operatorname{Ad}_{D(t)} x_\lambda = t^{\lambda/2} x_\lambda$ if $x_\lambda \in \mathfrak{g}_\lambda$, so

$$g\cdot(\alpha,\beta) = D\cdot(0, f + x)$$

near $t = 0$. Hence, every $(\alpha,\beta) \in \mathcal{M}_\rho$ has a representative such that $(\alpha,\beta) = D\cdot(0,x)$ near $t=0$ for a unique $x \in \mathcal{S}_\rho$. Hence, $D^{-1}\cdot(\alpha,\beta)$ is a smooth solution on $[0,1]$ and hence there exists a unique $g \in \mathcal{G}_{\mathbb{C}}$ such that $g(0) = 1$ and $gD^{-1}\cdot(\alpha,\beta) = (0,x)$. Then, we have an isomorphism of complex-symplecitc manifolds

$$G \times S_\rho \longrightarrow \mathcal{M}_\rho, \quad (a,x) \mapsto Dg_a\cdot(0,x),$$

where $g_a(0) = 1$ and $g_a(1) = a^{-1}$.

## 5.4 Tri-Hamiltonian action

From its construction, $\mathcal{M}(\rho_0, \rho_1)$ has a residual action of $\mathcal{G}/\mathcal{G}^0 = G_{\rho_0} \times G_{\rho_1}$ which preserves the hyperkähler structure. To describe the moment map, let

$$\pi_0 : \mathfrak{g} \longrightarrow \mathfrak{g}_{\rho_0}, \quad \pi_1 : \mathfrak{g} \longrightarrow \mathfrak{g}_{\rho_1}$$

be the orthogonal projections. Note that if $A \in \mathcal{A}_{\mathbb{H}}(\rho_0, \rho_1)$ then $\pi_0(A)$ is smooth near $t = 0$ and $\pi_1(A)$ is smooth near $t = 1$. This follows from the fact that if $\rho = (\sigma_1, \sigma_2, \sigma_3)$ is an $\mathfrak{su}(2)$-triple, then $\sigma_i \in \mathfrak{g}_\rho^\perp$ for all $i$ since $\mathfrak{g}_\rho = \{x \in \mathfrak{g} : [x, \sigma_i] = 0, i = 1, 2, 3\}$ and hence for all $x \in \mathfrak{g}_\rho$ we have

$$\langle \sigma_1, x \rangle = \langle [\sigma_2, \sigma_3], x \rangle = \langle \sigma_2, [\sigma_3, x] \rangle = 0$$

and similarly for $\sigma_2$ and $\sigma_3$. Hence, we can define

$$\mu : \mathcal{M}(\rho_0, \rho_1) \longrightarrow (\mathfrak{g}_{\rho_0} \times \mathfrak{g}_{\rho_1})^3$$

by

$$\mu(A) = \begin{pmatrix} \pi_0(A_1(0)) & \pi_0(A_2(0)) & \pi_0(A_3(0)) \\ -\pi_1(A_1(1)) & -\pi_1(A_2(1)) & -\pi_1(A_3(1)) \end{pmatrix}$$

The same proof as in §4.3.4 shows that this is a hyperkähler moment map for the action of $G_{\rho_0} \times G_{\rho_1}$ on $\mathcal{M}(\rho_0, \rho_1)$.

## 5.5 Twistor space

An analysis similar to the one carried in §4.3.10 for $T^*G_{\mathbb{C}}$ shows that the twistor space of $\mathcal{M}_\rho$ can be identified as follows.

**Theorem 5.3.** *The twistor space of $\mathcal{M}_\rho$ is biholomorphic to two copies of $G \times \mathcal{S}_\rho \times \mathbb{C}$ glued via $(a, x, \zeta) \sim (\tilde{a}, \tilde{x}, \tilde{\zeta})$ if and only if*

$$\begin{aligned} \tilde{\zeta} &= \zeta^{-1} \\ \tilde{x} &= \zeta^{-2} \operatorname{Ad}_{\zeta^{-h}} x \\ \tilde{a} &= \zeta^{-h} e^{x/\zeta} a, \end{aligned}$$

*where $\zeta^{-h} := \exp(-\log(\zeta) h)$ (which is well-defined for $\zeta \in \mathbb{C}^*$).*

## 5.6 Hyperkähler slices

Let $M$ be a hyperkähler manifold and $G$ a compact Lie group acting on $M$ by preserving the hyperkähler structure and with hyperkähler moment map $\mu : M \to \mathfrak{g}^* \otimes \operatorname{Im} \mathbb{H}$.

Then, for any $\mathfrak{su}(2)$-triple $\rho$, we have an action of $G$ on $\mathcal{M}_\rho$ and hence we can consider the hyperkähler quotient

$$(M \times \mathcal{M}_\rho) /\!\!/\!\!/ G,$$

which is a smooth hyperkähler manifold. This generalizes the discussion in §4.3.9, where we have shown that $(M \times \mathcal{M}_\rho) /\!\!/\!\!/ G \cong M$ when $\rho = 0$. More generally, we have:

**Theorem 5.4** (Bielawski [4])**.** *Let $(M, g, I, J, K)$ be a hyperkähler manifold and $G$ a compact Lie group acting on $M$ by preserving the hyperkähler structure and with hyperkähler moment map $\mu : M \to \mathfrak{g}^* \otimes \operatorname{Im} \mathbb{H}$ and suppose that the action extends to an $I$-holomorphic action of $G_{\mathbb{C}}$. Let $\mu_{\mathbb{C}} = \mu_J + i\mu_K : M \to \mathfrak{g}_{\mathbb{C}} \cong \mathfrak{g}_{\mathbb{C}}^*$. Then, $\mu_{\mathbb{C}}^{-1}(\mathcal{S}_\rho)$ is a complex-symplectic submanifold of $M$ and the hyperkähler manifold $(M \times \mathcal{M}_\rho) /\!\!/\!\!/ G$ is isomorphic as a complex-symplectic manifold to an open subset $U$ of $\mu_{\mathbb{C}}^{-1}(\mathcal{S}_\rho)$. Moreover, if there is a global Kähler potential for $(M, g, I)$ which is bounded below on $G_{\mathbb{C}}$-orbits, then $U = \mu_{\mathbb{C}}^{-1}(\mathcal{S}_\rho)$.*

## 5.7 Poles at both endpoints

Coming back to the original moduli space $\mathcal{M}(\rho_0, \rho_1)$ with poles at both endpoints, we can use Bielawski's theorem on

$$\mathcal{M}(\rho_0, \rho_1) = (\mathcal{M}_{\rho_0} \times \mathcal{M}_{\rho_1}) /\!\!/\!\!/ G$$

to get that $\mathcal{M}(\rho_0, \rho_1)$ can be described as a complex-symplectic submanifold of $G_{\mathbb{C}} \times \mathcal{S}_{\rho_0}$, namely,

$$\mathcal{M}(\rho_0, \rho_1) = \{(a, x) \in G_{\mathbb{C}} \times \mathcal{S}_{\rho_0} : -\operatorname{Ad}_{a^{-1}} x \in \mathcal{S}_{\rho_1}\}.$$

Since $G_{\mathbb{C}} \times \mathcal{S}_{\rho_0}$ is a complex-symplectic submanifold of $T^* G_{\mathbb{C}}$, so is $\mathcal{M}(\rho_0, \rho_1)$.

Indeed, we apply Theorem 5.4 with $\rho = \rho_1$ and $M = \mathcal{M}_{\rho_0}$ and note that, by (4.7), the complex moment map $\mu_{\mathbb{C}} : M = G_{\mathbb{C}} \times \mathcal{S}_{\rho_0} \to \mathfrak{g}_{\mathbb{C}}$ is given by $(a, x) \mapsto -\operatorname{Ad}_{a^{-1}} x$.

## 5.8 Hyperkähler rotation

There is also a hyperkähler rotation on the spaces $\mathcal{M}(\rho_0, \rho_1)$. To describe it, let $\varphi_0 : \mathrm{Sp}(1) \to G$ be the Lie group homomorphism whose derivative is $\rho_0 : \mathfrak{su}(2) \to \mathfrak{g}$ and similarly for $\varphi_1 : \mathrm{Sp}(1) \to G$. For $q \in \mathrm{Sp}(1)$ and $A \in \mathcal{A}_{\mathbb{H}}(\rho_0, \rho_1)$, take any element $g \in \mathcal{G}$ such that $g(0) = \varphi_0(q)$ and $g(1) = \varphi_1(q)$. Then, consider

$$q(g \cdot A)q^{-1},$$

where $q$ and $q^{-1}$ act using the natural $\mathbb{H}$-bi-module structure of $C^0(I, \mathfrak{g}) \otimes \mathbb{H}$. Using the identity

$$qiq^{-1} \otimes qiq^{-1} + qjq^{-1} \otimes qjq^{-1} + qkq^{-1} \otimes qkq^{-1} = i \otimes i + j \otimes j + k \otimes k$$

in $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H}$, we see that $q(g \cdot A)q^{-1}$ has the same poles at the boundary points as $A$. Hence, we get another element of $\mathcal{A}_{\mathbb{H}}(\rho_0, \rho_1)$ and this descends to a well-defined action of $\mathrm{Sp}(1)$ on $\mathcal{M}(\rho_0, \rho_1)$, which is a hyperkähler rotation.

## 5.9 Examples

### 5.9.1 SU(2)-monopole space

The most famous example is when $G = U(n)$ and $\rho_0, \rho_1$ are both equal to the irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$ on $\mathbb{C}^n$. In other words, the $\mathfrak{sl}_2$-triples in $\mathfrak{gl}(n, \mathbb{C})$ are

$$e = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 2 & & \\ & & 0 & \ddots & \\ & & & \ddots & n-1 \\ & & & & 0 \end{pmatrix}$$

$$h = \begin{pmatrix} n-1 & & & & \\ & n-3 & & & \\ & & n-5 & & \\ & & & \ddots & \\ & & & & -(n-1) \end{pmatrix}$$

$$f = \begin{pmatrix} 0 & & & & & \\ n-1 & 0 & & & & \\ & n-2 & 0 & & & \\ & & \ddots & \ddots & & \\ & & & 2 & 0 & \\ & & & & 1 & 0 \end{pmatrix}.$$

The space $\mathcal{M}(\rho_0, \rho_1)$ is also isomorphic to the framed moduli space of SU(2)-monopoles of charge $n$ on $\mathbb{R}^3$ [28, 16, 50].

### 5.9.2 A simple bow variety

More generally, if $\rho_0$ and $\rho_1$ are not necessarily irreducible, we can study $\mathcal{M}(\rho_0, \rho_1)$ by the decomposition $(\mathcal{M}_{\rho_0} \times \mathcal{M}_{\rho_1}) /\!\!/\!\!/ G$ as explained in §**??**, so it suffices to describe $\mathcal{M}_\rho$ for a general $\rho$. We focus on the case where the $\rho$ comes from the irreducible representation on $\{0\} \times \mathbb{C}^{n-m} \subseteq \mathbb{C}^n$ for some $0 \le m \le n$ and denote the resulting moduli space by $F_n(m)$, following Bielawski [5] and Cherkis [10].

**Proposition 5.5.** *We have* $F_n(m) = \mathrm{GL}(n, \mathbb{C}) \times \mathfrak{gl}(m, \mathbb{C}) \times \mathbb{C}^{m+n}$ *if* $m < n$ *and* $F_n(n) = \mathrm{GL}(n, \mathbb{C}) \times \mathfrak{gl}(n, \mathbb{C})$.

*Proof.* This amounts to show that $\mathcal{S}_\rho = \mathfrak{gl}(m, \mathbb{C}) \times \mathbb{C}^{m+n}$ for $m < n$. It is easy to see that a general element of $\mathcal{S}_\rho$ is of the form

$$\left(\begin{array}{ccc|cccc} * & \cdots & * & 0 & \cdots & 0 & * \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ * & \cdots & * & 0 & \cdots & 0 & * \\ \hline * & \cdots & * & & & & \\ 0 & \cdots & 0 & & & & \\ \vdots & & \vdots & & f+x & & \\ 0 & \cdots & 0 & & & & \end{array}\right)$$

where $x \in Z_{\mathfrak{gl}(n-m,\mathbb{C})}(e) \cong \mathbb{C}^{n-m}$ and $e, h, f$ are viewed as elements of $\mathfrak{gl}(n-m, \mathbb{C})$. ☐

We will now discuss a very simple bow variety, which combines several Nahm's moduli spaces and generalizes the example in §5.9.1. We enrich our notation and denote by $\mathcal{M}_I^G(\rho_0, \rho_1)$ the moduli space of solutions to Nahm's equation with gauge group $G$ on the interval $I$ with poles of residues $\rho_0$ and $\rho_1$ at the boundary points of $I$.

Fix real numbers

$$\lambda_0 < \lambda_1 < \cdots < \lambda_{r+1}, \quad \lambda_i \in \mathbb{R}$$

to define $r+1$ intervals

$$I_0 = [\lambda_0, \lambda_1], \ldots, I_r = [\lambda_r, \lambda_{r+1}]$$

and fix integers

$$n_0 < n_1 < \cdots < n_r, \quad n_i \in \mathbb{Z}_{\geq 0}$$

to define principal $\mathrm{U}(n_i)$-bundles over $I_i$ for each $i$.

For any integers $0 \leq k \leq n$, let $\rho^n_k : \mathfrak{su}(2) \to \mathfrak{u}(n, \mathbb{C})$ be the irreducible representation of $\mathfrak{su}(2)$ on $\{0\} \times \mathbb{C}^{n-k} \subseteq \mathbb{C}^n$. Note that that the stabilizer is $\mathrm{U}(n)_{\rho^n_k} = \mathrm{U}(k)$ for all $0 \leq k \leq n$. Also, $\rho^n_0$ is the irreducible representation of $\mathfrak{su}(2)$ on $\mathbb{C}^n$.

Now, consider

$$\mathcal{M}^{\mathrm{U}(n_0)}_{I_0}(\rho^{n_0}_0, 0) \times \mathcal{M}^{\mathrm{U}(n_1)}_{I_1}(\rho^{n_1}_{n_0}, 0) \times \cdots \times \mathcal{M}^{\mathrm{U}(n_r)}_{I_r}(\rho^{n_r}_{n_{r-1}}, \rho^{n_r}_0)$$

There is an action of $\mathrm{U}(n_0) \times \cdots \times \mathrm{U}(n_{r-1})$ and hence we can consider the hyperkähler quotient

$$\mathcal{M}(\lambda, n) \coloneqq (\mathcal{M}^{\mathrm{U}(n_0)}_{I_0}(\rho^{n_0}_0, 0) \times \cdots \times \mathcal{M}^{\mathrm{U}(n_r)}_{I_r}(\rho^{n_r}_{k_r}, \rho^{n_r}_0)) /\!/\!/ (\mathrm{U}(n_0) \times \cdots \times \mathrm{U}(n_{r-1})),$$

which is a smooth hyperkähler manifold canonically associated to $\lambda = (\lambda_0, \ldots, \lambda_{r+1})$ and $n = (n_0, \ldots, n_r)$.

There is another way to think of this manifold directly as an infinite-dimensional hyperkähler quotient, and this is how Cherkis bow varieties are originally defined. We take the point of view at the end of §2.5, i.e. we can think of the data of $\lambda$ and $n$ as a collection of hermitian vector bundles $E_i$ of rank $n_i$ over $I_i = [\lambda_i, \lambda_{i+1}]$ for each $i$. We think of a solution to Nahm's equations on $E_i$ as a unitary connection $\nabla$ together with three skew-hermitian sections $A_1, A_2, A_3$ of $\mathrm{End}(E_i)$ satisfying

$$\begin{aligned}
\nabla_{\partial_t} A_1 + [A_2, A_3] &= 0 \\
\nabla_{\partial_t} A_2 + [A_3, A_1] &= 0 \\
\nabla_{\partial_t} A_3 + [A_1, A_2] &= 0.
\end{aligned}$$

We fix injections $E_i|_{\lambda_{i+1}} \subseteq E_{i+1}|_{\lambda_{i+1}}$ for $i = 0, \ldots, r$, which is possible since $\operatorname{rk} E_i < \operatorname{rk} E_{i+1}$. We view $A_i$ as maps $I_i \to \mathfrak{u}(n_i, \mathbb{C})$. Then, we look for the space $\mathcal{A}_{\mathbb{H}}$ of tuples $(\nabla^i, A_1^i, A_2^i, A_3^i)$ where $A_j^i : I \to \mathfrak{u}(n_i, \mathbb{C})$ have regular limit on the left and on the right have the boundary conditions

$$A_j^i(t) = \left(\begin{array}{c|c} \frac{\sigma_i}{t-\lambda_i} + O(1) & O(t^{\frac{k-1}{2}}) \\ \hline O(t^{\frac{k-1}{2}}) & A_j^{i-1}(\lambda_i) + O(t) \end{array}\right).$$

Consider the group $\mathcal{G}$ of smooth maps $(g_0, \ldots, g_r)$ where $g_i : I_i = [\lambda_i, \lambda_{i+1}] \to \mathrm{U}(n_i)$ is such that

$$g_i(\lambda_i) = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & g_{i-1}(\lambda_i) \end{array}\right)$$

for $i = 1, \ldots, r$, $g_0(0) = 1$, and $g_r(\lambda_{r+1}) = 1$. Then, $\mathcal{G}$ acts on $\mathcal{A}_{\mathbb{H}}$ and the hyperkähler quotient is isomorphic to $\mathcal{M}(\lambda, n)$.

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