Stratified hyperkähler spaces

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Abstract

We describe the local topology of singular hyperkähler quotients. They are shown to be stratified spaces whose strata are smooth hyperkähler manifolds. We also give examples from Lie theory and describe their stratification in terms of combinatorial data associated to root systems.

Stratified spaces

Roughly speaking, a stratified space is a topological space X that can be partitioned into manifolds which “fit together nicely”. The partition $\mathcal{P}$ of X has a natural partial order: $S \leq T \iff S \subseteq \overline{T}$.

Examples. Manifolds with corners, graphs, real or complex algebraic varieties, cones over stratified spaces.

Definition. A *stratified space* is a second countable Hausdorff space X together with a partition $\mathcal{P}$ such that every point has a neighbourhood isomorphic (as a partitioned topological space) to $\mathbb{R}^k \times C(L)$ for some k and some compact stratified space L, where $C(L)$ is the open cone over L. (This is an inductive definition.)

Stratified symplectic spaces

Sjamaar–Lerman [6] showed that symplectic reductions (the natural quotient construction in symplectic geometry) by non-free actions of compact Lie groups are stratified spaces whose strata are symplectic manifolds.

More precisely, when a compact Lie group G acts on a symplectic manifold M with moment map $\mu : M \to \mathfrak{g}^*$, the **symplectic reduction** is $\mu^{-1}(0)/G$ and its partition into symplectic manifolds is the **orbit type partition** $\mathcal{P} := \{\mu^{-1}(0)_{(H)}/G : H \subseteq G\}$ where $\mu^{-1}(0)_{(H)}$ is the set of points in $\mu^{-1}(0)$ whose stabilizer is conjugate to H.

Hyperkähler quotients

A **hyperkähler manifold** is a Riemannian manifold (M, g) with three complex structures I, J, K that are Kähler with respect to g and satisfy $\mathsf{IJK} = -1$.

The hyperkähler quotient construction [2] is the analogue of symplectic reduction in hyperkähler geometry and is the most important tool for constructing these special manifolds. If a compact Lie group G acts on a hyperkähler manifold M with hyperkähler moment map $\mu : M \to \mathfrak{g}^* \otimes \mathbb{R}^3$, the **hyperkähler quotient** is

$$M /\!\!/\!\!/ G := \mu^{-1}(0)/G.$$

It is a hyperkähler manifold when G acts freely.

Main result

Let M, G and μ be as above. Dancer–Swann [1] showed that when the G-action is not free, the hyperkähler quotient $M /\!\!/\!\!/ G$ can be partitioned into smooth hyperkähler manifolds (by orbit types).

Theorem [5]**.** *If the action of G extends to a holomorphic action of $G_{\mathbb{C}}$, then the partition of $M /\!\!/\!\!/ G$ into hyperkähler manifolds is a stratification.*

Remark. The assumption holds, for example, when M is compact or M is a complex affine variety and the G-action is real algebraic.

Idea of proof. We first prove a hyperkähler analogue of the local normal form for the moment map in order to reduce the problem to complex-symplectic quotients of $\mathbb{C}^{2n}$ by linear actions. More precisely, $\mu_{\mathbb{C}} := \mu_2 + i\mu_3$ is a moment map for the $G_{\mathbb{C}}$-action with respect to the complex-symplectic form $\omega_{\mathbb{C}} := \omega_2 + i\omega_3$, and we have:

Theorem [5]**.** *The complex-Hamiltonian $G_{\mathbb{C}}$-manifold $(M, \mathsf{I}, \omega_{\mathbb{C}}, \mu_{\mathbb{C}})$ is completely determined in a neighbourhood of a point $p \in \mu^{-1}(0)$ by the representation of $(G_p)_{\mathbb{C}}$ on $T_p(G_{\mathbb{C}} \cdot p)^{\omega_{\mathbb{C}}}/T_p(G_{\mathbb{C}} \cdot p)$.*

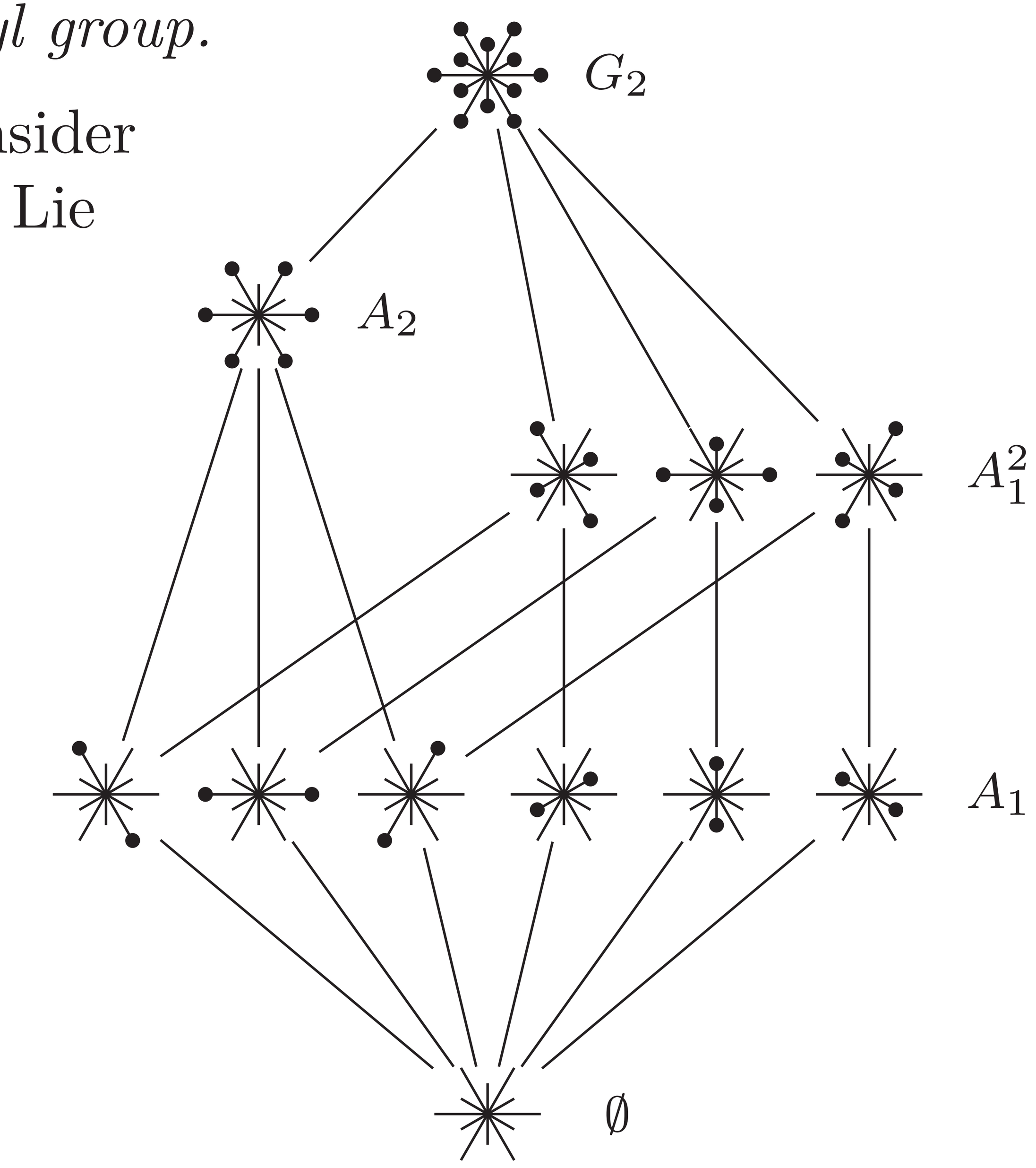
This can be thought of as a complex-Hamiltonian version of the Darboux Theorem. We then use a Kempf–Ness type theorem which says that $\mu^{-1}(0)/G \cong \mu_{\mathbb{C}}^{-1}(0)^{\mathrm{ss}} /\!\!/ G_{\mathbb{C}}$.

Examples from Lie theory

Let G be a compact Lie group. Kronheimer [3] showed that $T^*G_{\mathbb{C}}$ has a $G \times G$-invariant hyperkähler structure and there is a hyperkähler moment map. Let $T \subseteq G$ be a maximal torus and let $\mathcal{D}(G) := T^*G_{\mathbb{C}} /\!\!/\!\!/ (T \times T)$.

Theorem [4]**.** *The stratification poset of $\mathcal{D}(G)$ is isomorphic to the poset of root subsystems of G. The strata corresponding to a root subsystem Ψ is a disjoint union of $|W_G|/|W_{G_\Psi}|$ copies of the smooth locus of $\mathcal{D}(G_\Psi)$, where $G_\Psi \subseteq G$ is a compact Lie group with root system Ψ and W_{G_Ψ} is its Weyl group.*

Example. Consider the exceptional Lie group G_2. The stratification poset of $\mathcal{D}(G_2)$ is:



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