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Continuous Measurements and Feedback Control of a Quantum Harmonic Oscillator

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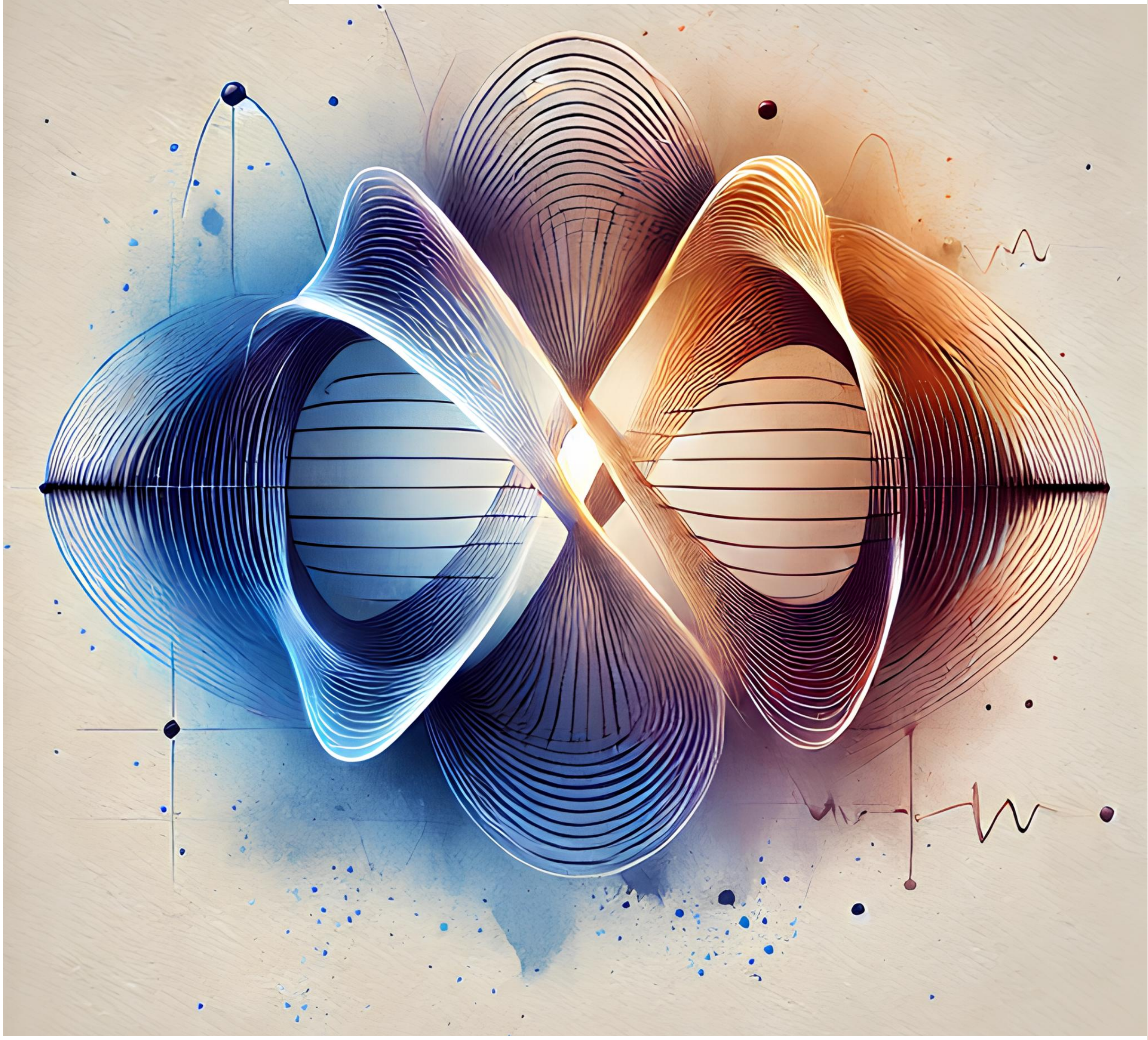

Contents

1 Introduction

As our society is becoming increasingly dependent on technology, the demand for better and more efficient technologies is growing. One kind of technology which has been really important since its conception during the second world war by Alan Turing is computers [1]. The field of computer science has been researched a lot during the second half of the 20th century but has hit a fundamental problem. Our current classical computers have reached a limit where the transistors cannot get any smaller without quantum mechanics causing issues [1]. This has prompted research into quantum technologies such as quantum computers and quantum simulations [1]. To properly understand how quantum mechanical systems work, it is important to understand what happens to them when they are interacted with, during for example a measurement [2]. It is also interesting to see how such a system can be manipulated to create a certain state which can be used for a specific purpose.

1.1 Outline

This thesis will look at a quantum harmonic oscillator which is coupled to an environment. Thus creating an open quantum system whose state is temperature dependent. The system will be measured continuously using weak measurements with a feedback loop to control the system. The goal is to see how the system evolves under these conditions, and how the feedback loop can be changed and manipulated to observe different behaviours.

[3]

2 Theoretical Framework

2.1 Open Quantum Systems

2.2 Lindblad Master Equation

2.3 Continuous Measurements

2.4 Feedback Control

2.5 Wigner Function

3 Result

4 Discussion

References

[1] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information: 10th Anniversary Edition*. Cambridge: Cambridge University Press, 2010.

[2] A. N. Jordan and I. A. Siddiqi, *Quantum Measurement: Theory and Practice*. Cambridge: Cambridge University Press, 2024.

[3] B. Annby-Andersson, *Continuous measurements of small systems: Feedback control, thermodynamics, entanglement*. Doctoral thesis (compilation), Lund University, 4 2024.

A Appendix

B EOM Calculation

Consider a quantum harmonic oscillator described by the Hamiltonian

$$\hat{H} = \hbar\omega(\hat{a}^\dagger \hat{a} + 1/2). \tag{1}$$

Consider also that the quantum harmonic oscillator is subject to a temperature bath and is continuously measured. The system then evolves according to the master equation

$$\partial_t \hat{\rho} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}] + \gamma(\bar{n}+1)\mathcal{D}[\hat{a}]\hat{\rho} + \gamma\bar{n}\mathcal{D}[\hat{a}^\dagger]\hat{\rho} + \lambda\mathcal{D}[\hat{A}]\hat{\rho} \tag{2}$$

where γ is the damping rate, $\bar{n}$ is the thermal occupancy defined as

$$\bar{n} = \frac{1}{e^{\hbar\omega/k_B T} - 1}, \tag{3}$$

λ is the measurement rate, $\hat{A}$ is the measurement operator, and $\mathcal{D}[\hat{O}]$ is the Lindblad super operator defined as

$$\mathcal{D}[\hat{O}]\hat{\rho} = \hat{O}\hat{\rho}\hat{O}^\dagger - \frac{1}{2}\{\hat{O}^\dagger\hat{O}, \hat{\rho}\}. \tag{4}$$

Then we also have the quadrature operators defined as

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger), \quad \hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a} - \hat{a}^\dagger). \tag{5}$$

We want to find the equations of motion when measuring the position quadrature

$$\partial_t \left\langle \hat{x}^2 \right\rangle = \mathrm{tr}\big(\hat{x}^2 \partial_t \hat{\rho}\big) \tag{6}$$

$$\partial_t \left\langle \hat{p}^2 \right\rangle = \mathrm{tr}\big(\hat{p}^2 \partial_t \hat{\rho}\big) \tag{7}$$

$$\partial_t \left\langle \{\hat{x}, \hat{p}\} \right\rangle = \mathrm{tr}(\{\hat{x}, \hat{p}\} \partial_t \hat{\rho}). \tag{8}$$

We start by defining the Liouvillian super operator

$$\mathcal{L}\hat{\rho} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}] + \gamma(\bar{n}+1)\mathcal{D}[\hat{a}]\hat{\rho} + \gamma\bar{n}\mathcal{D}[\hat{a}^\dagger]\hat{\rho} \tag{9}$$

which turns the master equation into

$$\partial_t \hat{\rho} = \mathcal{L}\hat{\rho} + \lambda \mathcal{D}[\hat{x}]\hat{\rho}. \tag{10}$$

B.1 Solving for $\partial_t \langle \hat{x}^2 \rangle$

We can write Eq. (6) as

$$\partial_t \langle \hat{x}^2 \rangle = \operatorname{tr}\big(\hat{x}^2 \mathcal{L}\hat{\rho}\big) + \operatorname{tr}\big(\hat{x}^2 \lambda \mathcal{D}[\hat{x}]\hat{\rho}\big) \tag{11}$$

and then rewriting the left term using the creation and annihilation operators we get

$$\operatorname{tr}\big(\hat{x}^2 \mathcal{L}\hat{\rho}\big) = \frac{\hbar}{2m\omega} \operatorname{tr}\Big((\hat{a}^2 + (\hat{a}^\dagger)^2 + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a})\mathcal{L}\hat{\rho}\Big) \tag{12}$$

$$= \frac{\hbar}{2m\omega} \Big[\operatorname{tr}\big(\hat{a}^2 \mathcal{L}\hat{\rho}\big) + \operatorname{tr}\Big((\hat{a}^\dagger)^2 \mathcal{L}\hat{\rho}\Big) + \operatorname{tr}\Big(\hat{a}\hat{a}^\dagger \mathcal{L}\hat{\rho}\Big) + \operatorname{tr}\Big(\hat{a}^\dagger\hat{a}\mathcal{L}\hat{\rho}\Big)\Big] \tag{13}$$

Solving these separately

$$\operatorname{tr}\big(\hat{a}^2 \mathcal{L}\hat{\rho}\big) = -\frac{i}{\hbar} \operatorname{tr}\Big(\hat{a}^2 \Big[\hat{H}, \hat{\rho}\Big]\Big) + \gamma(\bar{n}+1) \operatorname{tr}\big(\hat{a}^2 \mathcal{D}[\hat{a}]\hat{\rho}\big) + \gamma\bar{n} \operatorname{tr}\Big(\hat{a}^2 \mathcal{D}[\hat{a}^\dagger]\hat{\rho}\Big) \tag{14}$$

$$= -i\omega \operatorname{tr}\Big(\hat{a}^2\hat{a}^\dagger\hat{a}\hat{\rho}\Big) + i\omega \operatorname{tr}\Big(\hat{a}^2\hat{\rho}\hat{a}^\dagger\hat{a}\Big) \tag{15}$$

$$+ \gamma(\bar{n}+1) \operatorname{tr}\Big(\hat{a}^3\hat{\rho}\hat{a}^\dagger\Big) - \frac{\gamma(\bar{n}+1)}{2} \operatorname{tr}\Big(\hat{a}^2\hat{a}^\dagger\hat{a}\hat{\rho}\Big) - \frac{\gamma(\bar{n}+1)}{2} \operatorname{tr}\Big(\hat{a}^2\hat{\rho}\hat{a}^\dagger\hat{a}\Big) \tag{16}$$

$$+ \gamma\bar{n} \operatorname{tr}\Big(\hat{a}^2\hat{a}^\dagger\hat{\rho}\hat{a}\Big) - \frac{\gamma\bar{n}}{2} \operatorname{tr}\Big(\hat{a}^3\hat{a}^\dagger\hat{\rho}\Big) - \frac{\gamma\bar{n}}{2} \operatorname{tr}\Big(\hat{a}^2\hat{\rho}\hat{a}\hat{a}^\dagger\Big) \tag{17}$$

Using cyclic permutations of the trace and combining terms we get

$$\operatorname{tr}\big(\hat{a}^2 \mathcal{L}\hat{\rho}\big) = i\omega \operatorname{tr}\Big(\Big[\hat{a}^\dagger\hat{a}, \hat{a}^2\Big]\hat{\rho}\Big) + \frac{\gamma(\bar{n}+1)}{2} \operatorname{tr}\Big(\Big[\hat{a}^\dagger\hat{a}, \hat{a}^2\Big]\hat{\rho}\Big) + \frac{\gamma\bar{n}}{2} \operatorname{tr}\Big(\Big[\hat{a}^2, \hat{a}\hat{a}^\dagger\Big]\hat{\rho}\Big) \tag{18}$$

The commutators are

$$\Big[\hat{a}^\dagger\hat{a}, \hat{a}^2\Big] = \Big[\hat{a}^\dagger, \hat{a}^2\Big]\hat{a} = \hat{a}\Big[\hat{a}^\dagger, \hat{a}\Big]\hat{a} + \Big[\hat{a}^\dagger, \hat{a}\Big]\hat{a}^2 = -2\hat{a}^2 \tag{19}$$

$$\Big[\hat{a}^2, \hat{a}\hat{a}^\dagger\Big] = \hat{a}\Big[\hat{a}^2, \hat{a}^\dagger\Big] = \hat{a}^2\Big[\hat{a}, \hat{a}^\dagger\Big] + \hat{a}\Big[\hat{a}, \hat{a}^\dagger\Big]\hat{a} = 2\hat{a}^2 \tag{20}$$

Thus

$$\operatorname{tr}\big(\hat{a}^2 \mathcal{L}\hat{\rho}\big) = -2i\omega \operatorname{tr}\big(\hat{a}^2\hat{\rho}\big) - \gamma(\bar{n}+1) \operatorname{tr}\big(\hat{a}^2\hat{\rho}\big) + \gamma\bar{n} \operatorname{tr}\big(\hat{a}^2\hat{\rho}\big) = -(\gamma + 2i\omega) \operatorname{tr}\big(\hat{a}^2\hat{\rho}\big) \tag{21}$$

$$= -(\gamma + 2i\omega) \langle \hat{a}^2 \rangle . \tag{22}$$

By taking the hermition conjugate of the above we get

$$\operatorname{tr}\left((\hat{a}^\dagger)^2\mathcal{L}\hat{\rho}\right) = (\gamma - 2i\omega)\operatorname{tr}\left((\hat{a}^\dagger)^2\hat{\rho}\right) = -(\gamma - 2i\omega)\left\langle(\hat{a}^\dagger)^2\right\rangle. \tag{23}$$

Now we do the same for $\operatorname{tr}(\hat{a}^\dagger\hat{a}\mathcal{L}\hat{\rho})$, so

$$\operatorname{tr}\left(\hat{a}^\dagger\hat{a}\mathcal{L}\hat{\rho}\right) = -\frac{i}{\hbar}\operatorname{tr}\left(\hat{a}^\dagger\hat{a}\left[\hat{H},\hat{\rho}\right]\right) + \gamma(\bar{n}+1)\operatorname{tr}\left(\hat{a}^\dagger\hat{a}\mathcal{D}[\hat{a}]\hat{\rho}\right) + \gamma\bar{n}\operatorname{tr}\left(\hat{a}^\dagger\hat{a}\mathcal{D}[\hat{a}^\dagger]\hat{\rho}\right) \tag{24}$$

$$= -i\omega\operatorname{tr}\left(\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}\hat{\rho}\right) + i\omega\operatorname{tr}\left(\hat{a}^\dagger\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a}\right) \tag{25}$$

$$+\gamma(\bar{n}+1)\operatorname{tr}\left(\hat{a}^\dagger\hat{a}^2\hat{\rho}\hat{a}^\dagger\right) - \frac{\gamma(\bar{n}+1)}{2}\operatorname{tr}\left(\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}\hat{\rho}\right) - \frac{\gamma(\bar{n}+1)}{2}\operatorname{tr}\left(\hat{a}^\dagger\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a}\right) \tag{26}$$

$$+\gamma\bar{n}\operatorname{tr}\left(\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{\rho}\hat{a}\right) - \frac{\gamma\bar{n}}{2}\operatorname{tr}\left(\hat{a}^\dagger\hat{a}^2\hat{a}^\dagger\hat{\rho}\right) - \frac{\gamma\bar{n}}{2}\operatorname{tr}\left(\hat{a}^\dagger\hat{a}\hat{\rho}\hat{a}\hat{a}^\dagger\right) \tag{27}$$

using cyclic permutations of the trace and combining terms we get

$$\operatorname{tr}\left(\hat{a}^\dagger\hat{a}\mathcal{L}\hat{\rho}\right) = i\omega\operatorname{tr}\left(\left[\hat{a}^\dagger\hat{a},\hat{a}^\dagger\hat{a}\right]\hat{\rho}\right) + \gamma(\bar{n}+1)\operatorname{tr}\left((\hat{a}^\dagger)^2\hat{a}^2\hat{\rho}\right) - \gamma(\bar{n}+1)\operatorname{tr}\left(\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}\hat{\rho}\right) \tag{28}$$

$$+\gamma\bar{n}\operatorname{tr}\left(\hat{a}\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{\rho}\right) - \frac{\gamma\bar{n}}{2}\operatorname{tr}\left(\hat{a}^\dagger\hat{a}^2\hat{a}^\dagger\hat{\rho}\right) - \frac{\gamma\bar{n}}{2}\operatorname{tr}\left(\hat{a}(\hat{a}^\dagger)^2\hat{a}\hat{\rho}\right) \tag{29}$$

Commuting to simplify we get

$$\operatorname{tr}\left(\hat{a}^\dagger\hat{a}\mathcal{L}\hat{\rho}\right) = -\gamma(\bar{n}+1)\operatorname{tr}\left(\hat{a}^\dagger\hat{a}\hat{\rho}\right) - \gamma\bar{n}\operatorname{tr}\left(\left[\hat{a}\hat{a}^\dagger,\hat{a}^\dagger\hat{a}\right]\hat{\rho}\right) + \gamma\bar{n}\operatorname{tr}\left(\hat{a}\hat{a}^\dagger\hat{\rho}\right) \tag{30}$$

The commutator is equal to 0, and thus we get

$$\operatorname{tr}\left(\hat{a}^\dagger\hat{a}\mathcal{L}\hat{\rho}\right) = -\gamma(\bar{n}+1)\operatorname{tr}\left(\hat{a}^\dagger\hat{a}\hat{\rho}\right) + \gamma\bar{n} + \gamma\bar{n}\operatorname{tr}\left(\hat{a}^\dagger\hat{a}\hat{\rho}\right) = -\gamma\operatorname{tr}\left(\hat{a}^\dagger\hat{a}\hat{\rho}\right) + \gamma\bar{n} = -\gamma\left\langle\hat{a}^\dagger\hat{a}\right\rangle + \gamma\bar{n}. \tag{31}$$

Similarly we can find that

$$\operatorname{tr}\left(\hat{a}\hat{a}^\dagger\mathcal{L}\hat{\rho}\right) = -\gamma\operatorname{tr}\left(\hat{a}\hat{a}^\dagger\hat{\rho}\right) + \gamma(\bar{n}+1) = -\gamma\left\langle\hat{a}\hat{a}^\dagger\right\rangle + \gamma(\bar{n}+1). \tag{32}$$

Then since $\hat{x}^2$ commutes with $\hat{x}$ (the measurement operator) we can write Eq. (11) the equation of motion

$$\partial_t \left\langle \hat{x}^2 \right\rangle = \mathrm{tr}\big(\hat{x}^2 \mathcal{L}\hat{\rho}\big) \tag{33}$$

$$= \frac{\hbar}{2m\omega}\Big(-(\gamma+2i\omega)\left\langle \hat{a}^2\right\rangle - (\gamma-2i\omega)\left\langle (\hat{a}^\dagger)^2\right\rangle - \gamma\left\langle \hat{a}^\dagger\hat{a}\right\rangle + \gamma\bar{n} - \gamma\left\langle \hat{a}\hat{a}^\dagger\right\rangle + \gamma(\bar{n}+1)\Big) \tag{34}$$

$$= -\gamma\frac{\hbar}{2m\omega}\Big(\left\langle \hat{a}^2\right\rangle + \left\langle (\hat{a}^\dagger)^2\right\rangle + \left\langle \hat{a}^\dagger\hat{a}\right\rangle + \left\langle \hat{a}\hat{a}^\dagger\right\rangle\Big) + \frac{i\hbar}{m}\Big(\left\langle (\hat{a}^\dagger)^2\right\rangle - \left\langle \hat{a}^2\right\rangle\Big) + \frac{\gamma\hbar}{m\omega}(\bar{n}+1/2) \tag{35}$$

Since we have

$$\{\hat{x},\hat{p}\} = i\hbar((\hat{a}^\dagger)^2 - \hat{a}^2) \tag{36}$$

We can write the equation of motion as

$$\partial_t \left\langle \hat{x}^2 \right\rangle = -\gamma \left\langle \hat{x}^2 \right\rangle + \frac{1}{m}\left\langle \{\hat{x},\hat{p}\}\right\rangle + \frac{\gamma\hbar}{m\omega}(\bar{n}+1/2) \tag{37}$$

B.2 Solving for $\partial_t \left\langle \hat{p}^2 \right\rangle$

Writing Eq. (7) as

$$\partial_t \left\langle \hat{p}^2 \right\rangle = \mathrm{tr}\big(\hat{p}^2 \mathcal{L}\hat{\rho}\big) + \mathrm{tr}\big(\hat{p}^2 \lambda \mathcal{D}[\hat{x}]\hat{\rho}\big) \tag{38}$$

Rewriting the left terms using $\hat{a}, \hat{a}^\dagger$ we get

$$\mathrm{tr}\big(\hat{p}^2 \mathcal{L}\hat{\rho}\big) = -\frac{m\omega\hbar}{2}\,\mathrm{tr}\Big((\hat{a}^2 + (\hat{a}^\dagger)^2 - \hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger)\mathcal{L}\hat{\rho}\Big) \tag{39}$$

$$= -\frac{m\omega\hbar}{2}\Big(\mathrm{tr}\big(\hat{a}^2\mathcal{L}\hat{\rho}\big) + \mathrm{tr}\Big((\hat{a}^\dagger)^2\mathcal{L}\hat{\rho}\Big) - \mathrm{tr}\Big(\hat{a}^\dagger\hat{a}\mathcal{L}\hat{\rho}\Big) - \mathrm{tr}\Big(\hat{a}\hat{a}^\dagger\mathcal{L}\hat{\rho}\Big)\Big) \tag{40}$$

We have solved for these before, so we can write

$$\mathrm{tr}\big(\hat{p}^2 \mathcal{L}\hat{\rho}\big) = -\frac{m\omega\hbar}{2}\Big(-(\gamma+2i\omega)\left\langle \hat{a}^2\right\rangle - (\gamma-2i\omega)\left\langle (\hat{a}^\dagger)^2\right\rangle + \gamma\left\langle \hat{a}^\dagger\hat{a}\right\rangle - \gamma\bar{n} + \gamma\left\langle \hat{a}\hat{a}^\dagger\right\rangle - \gamma(\bar{n}+1)\Big) \tag{41}$$

$$= -\gamma\left\langle \hat{p}^2\right\rangle + mi\omega^2\hbar\Big(\left\langle \hat{a}^2\right\rangle - \left\langle (\hat{a}^\dagger)^2\right\rangle\Big) + \gamma m\omega\hbar(\bar{n}+1/2) \tag{42}$$

$$= -\gamma\left\langle \hat{p}^2\right\rangle - m\omega^2\left\langle \{\hat{x},\hat{p}\}\right\rangle + \gamma m\omega\hbar(\bar{n}+1/2) \tag{43}$$

Solving the measurement term

$$\operatorname{tr}\left(\hat{p}^2\lambda\mathcal{D}[\hat{x}]\hat{\rho}\right) = \lambda\left(\operatorname{tr}\left(\hat{p}^2\hat{x}\hat{\rho}\hat{x}\right) - \frac{1}{2}\operatorname{tr}\left(\hat{p}^2\hat{x}^2\hat{\rho}\right) - \frac{1}{2}\operatorname{tr}\left(\hat{p}^2\hat{\rho}\hat{x}^2\right)\right) \tag{44}$$

$$= \lambda\left(\operatorname{tr}\left(\hat{x}\hat{p}^2\hat{x}\hat{\rho}\right) - \frac{1}{2}\operatorname{tr}\left(\hat{p}^2\hat{x}^2\hat{\rho}\right) - \frac{1}{2}\operatorname{tr}\left(\hat{x}^2\hat{p}^2\hat{\rho}\right)\right) \tag{45}$$

Commuting the operators

$$\hat{x}\hat{p}^2\hat{x} = \frac{1}{2}(\hat{p}\hat{x} + i\hbar)\hat{p}\hat{x} + \frac{1}{2}\hat{x}\hat{p}(\hat{x}\hat{p} - i\hbar) = \frac{1}{2}\hat{p}\hat{x}\hat{p}\hat{x} + \frac{1}{2}\hat{x}\hat{p}\hat{x}\hat{p} + \frac{i\hbar}{2}[\hat{p},\hat{x}] \tag{46}$$

$$= \frac{1}{2}\hat{p}\hat{x}\hat{p}\hat{x} + \frac{1}{2}\hat{x}\hat{p}\hat{x}\hat{p} + \frac{\hbar^2}{2} \tag{47}$$

$$\hat{p}^2\hat{x}^2 = \hat{p}(\hat{x}\hat{p} - i\hbar)\hat{x} = \hat{p}\hat{x}\hat{p}\hat{x} - i\hbar\hat{p}\hat{x} \tag{48}$$

$$\hat{x}^2\hat{p}^2 = \hat{x}(\hat{p}\hat{x} + i\hbar)\hat{p} = \hat{x}\hat{p}\hat{x}\hat{p} + i\hbar\hat{x}\hat{p} \tag{49}$$

This gives

$$\operatorname{tr}\left(\hat{p}^2\lambda\mathcal{D}[\hat{x}]\hat{\rho}\right) = \lambda\left(\frac{1}{2}\operatorname{tr}(\hat{p}\hat{x}\hat{p}\hat{x}\hat{\rho}) + \frac{1}{2}\operatorname{tr}(\hat{x}\hat{p}\hat{x}\hat{p}\hat{\rho}) + \frac{\hbar^2}{2} - \frac{1}{2}\operatorname{tr}(\hat{p}\hat{x}\hat{p}\hat{x}\hat{\rho}) - \frac{1}{2}\operatorname{tr}(\hat{x}\hat{p}\hat{x}\hat{p}\hat{\rho}) + \frac{i\hbar}{2}\operatorname{tr}([\hat{p},\hat{x}])\right) \tag{50}$$

$$= \frac{\lambda\hbar^2}{2} + \frac{\lambda i\hbar}{2}\operatorname{tr}([\hat{p},\hat{x}]) = \lambda\hbar^2 \tag{51}$$

The final equation of motion then becomes

$$\partial_t\left\langle\hat{p}^2\right\rangle = -\gamma\left\langle\hat{p}^2\right\rangle - m\omega^2\left\langle\{\hat{x},\hat{p}\}\right\rangle + \gamma m\omega\hbar(\bar{n} + 1/2) + \lambda\hbar^2 \tag{52}$$

B.3 Solving for $\partial_t\left\langle\{\hat{x},\hat{p}\}\right\rangle$

Writing Eq. (8) as

$$\partial_t\left\langle\{\hat{x},\hat{p}\}\right\rangle = \operatorname{tr}(\{\hat{x},\hat{p}\}\mathcal{L}\hat{\rho}) + \operatorname{tr}(\{\hat{x},\hat{p}\}\lambda\mathcal{D}[\hat{x}]\hat{\rho}) \tag{53}$$

Rewriting the left term as

$$\operatorname{tr}(\{\hat{x},\hat{p}\}\mathcal{L}\hat{\rho}) = i\hbar\operatorname{tr}\left((\hat{a}^2 - (\hat{a}^\dagger)^2)\mathcal{L}\hat{\rho}\right) = i\hbar\operatorname{tr}\left(\hat{a}^2\mathcal{L}\hat{\rho}\right) - i\hbar\operatorname{tr}\left((\hat{a}^\dagger)^2\mathcal{L}\hat{\rho}\right) \tag{54}$$

We have calculated these terms before, so we can write

$$\mathrm{tr}(\{\hat{x},\hat{p}\}\mathcal{L}\hat{\rho}) = -i\hbar\left((\gamma+2i\omega)\left\langle\hat{a}^2\right\rangle - (\gamma-2i\omega)\left\langle(\hat{a}^\dagger)^2\right\rangle\right) \tag{55}$$

$$= -\gamma i\hbar\left(\left\langle\hat{a}^2\right\rangle - \left\langle(\hat{a}^\dagger)^2\right\rangle\right) + 2\hbar\omega\left(\left\langle\hat{a}^2\right\rangle + \left\langle(\hat{a}^\dagger)^2\right\rangle\right) \tag{56}$$

We also have that

$$\hat{a}^2 + (\hat{a}^\dagger)^2 = \frac{m\omega}{\hbar}\hat{x}^2 - \frac{1}{m\omega\hbar}\hat{p}^2 \tag{57}$$

Thus we get

$$\mathrm{tr}(\{\hat{x},\hat{p}\}\mathcal{L}\hat{\rho}) = -\gamma\left\langle\{\hat{x},\hat{p}\}\right\rangle + 2m\omega^2\left\langle\hat{x}^2\right\rangle - \frac{2}{m}\left\langle\hat{p}^2\right\rangle \tag{58}$$

Solving for the measurement term

$$\lambda\,\mathrm{tr}(\{\hat{x},\hat{p}\}\mathcal{D}[\hat{x}]\hat{\rho}) = \frac{\lambda}{2}\left(\mathrm{tr}\big(\big[\hat{x}^2,\hat{p}\hat{x}\big]\hat{\rho}\big) + \mathrm{tr}\big(\big[\hat{x}\hat{p},\hat{x}^2\big]\big)\right) \tag{59}$$

Solving the commutators shows that this will become zero.

B.4 Results

The final equations of motion are

$$\partial_t\left\langle\hat{x}^2\right\rangle = -\gamma\left\langle\hat{x}^2\right\rangle + \frac{1}{m}\left\langle\{\hat{x},\hat{p}\}\right\rangle + \frac{\gamma\hbar}{m\omega}(\bar{n}+1/2) \tag{60}$$

$$\partial_t\left\langle\hat{p}^2\right\rangle = -\gamma\left\langle\hat{p}^2\right\rangle - m\omega^2\left\langle\{\hat{x},\hat{p}\}\right\rangle + \gamma m\omega\hbar(\bar{n}+1/2) + \lambda\hbar^2 \tag{61}$$

$$\partial_t\left\langle\{\hat{x},\hat{p}\}\right\rangle = -\gamma\left\langle\{\hat{x},\hat{p}\}\right\rangle + 2m\omega^2\left\langle\hat{x}^2\right\rangle - \frac{2}{m}\left\langle\hat{p}^2\right\rangle \tag{62}$$