

Continuous Measurements and Feedback Control of a Quantum Harmonic Oscillator

Max Eriksson

Bachelor Thesis, 15 ECTS, Spring 2025

Supervised by Kalle Kansanen and Peter Samuelsson
Division of Mathematical Physics | Department of Physics



Contents

1	Introduction	1
1.1	Open Quantum Systems	1
1.1.1	Lindblad Master Equation	1
1.2	Continuous Measurements	1
1.3	Feedback Control	1
1.4	Wigner Function	1
2	Method	1
3	Result	1
4	Discussion	1
	References	2
A	Appendix	3
B	EOM Calculation	3

1 Introduction

[1]

1.1 Open Quantum Systems

1.1.1 Lindblad Master Equation

1.2 Continuous Measurements

1.3 Feedback Control

1.4 Wigner Function

2 Method

3 Result

4 Discussion

References

- [1] B. Annby-Andersson, *Continuous measurements of small systems: Feedback control, thermodynamics, entanglement*. Doctoral thesis (compilation), Lund University, 4 2024.

A Appendix

B EOM Calculation

Consider a quantum harmonic oscillator described by the Hamiltonian

$$\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + 1/2). \quad (1)$$

Consider also that the quantum harmonic oscillator is subject to a temperature bath and is continuously measured. The system then evolves according to the master equation

$$\partial_t \hat{\rho} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}] + \gamma(\bar{n} + 1)\mathcal{D}[\hat{a}]\hat{\rho} + \gamma\bar{n}\mathcal{D}[\hat{a}^\dagger]\hat{\rho} + \lambda\mathcal{D}[\hat{A}]\hat{\rho} \quad (2)$$

where γ is the damping rate, \bar{n} is the thermal occupancy defined as

$$\bar{n} = \frac{1}{e^{\hbar\omega/k_B T} - 1}, \quad (3)$$

λ is the measurement rate, \hat{A} is the measurement operator, and $\mathcal{D}[\hat{O}]$ is the Lindblad super operator defined as

$$\mathcal{D}[\hat{O}]\hat{\rho} = \hat{O}\hat{\rho}\hat{O}^\dagger - \frac{1}{2}\{\hat{O}^\dagger\hat{O}, \hat{\rho}\}. \quad (4)$$

Then we also have the quadrature operators defined as

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger), \quad \hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a} - \hat{a}^\dagger). \quad (5)$$

We want to find the equations of motion when measuring the position quadrature

$$\partial_t \langle \hat{x}^2 \rangle = \text{tr}(\hat{x}^2 \partial_t \hat{\rho}) \quad (6)$$

$$\partial_t \langle \hat{p}^2 \rangle = \text{tr}(\hat{p}^2 \partial_t \hat{\rho}) \quad (7)$$

$$\partial_t \langle \{\hat{x}, \hat{p}\} \rangle = \text{tr}(\{\hat{x}, \hat{p}\} \partial_t \hat{\rho}). \quad (8)$$

We start by defining the Liouvillian super operator

$$\mathcal{L}\hat{\rho} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}] + \gamma(\bar{n} + 1)\mathcal{D}[\hat{a}]\hat{\rho} + \gamma\bar{n}\mathcal{D}[\hat{a}^\dagger]\hat{\rho} \quad (9)$$

which turns the master equation into

$$\partial_t \hat{\rho} = \mathcal{L} \hat{\rho} + \lambda \mathcal{D}[\hat{x}] \hat{\rho}. \quad (10)$$

We can then write Eq. (6) as

$$\partial_t \langle \hat{x}^2 \rangle = \text{tr}(\hat{x}^2 \mathcal{L} \hat{\rho}) + \text{tr}(\hat{x}^2 \lambda \mathcal{D}[\hat{x}] \hat{\rho}) \quad (11)$$

and then rewriting the left term using the creation and annihilation operators we get

$$\text{tr}(\hat{x}^2 \mathcal{L} \hat{\rho}) = \frac{\hbar}{2m\omega} \text{tr}((\hat{a}^2 + (\hat{a}^\dagger)^2 + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}) \mathcal{L} \hat{\rho}) \quad (12)$$

$$= \frac{\hbar}{2m\omega} [\text{tr}(\hat{a}^2 \mathcal{L} \hat{\rho}) + \text{tr}((\hat{a}^\dagger)^2 \mathcal{L} \hat{\rho}) + \text{tr}(\hat{a}\hat{a}^\dagger \mathcal{L} \hat{\rho}) + \text{tr}(\hat{a}^\dagger\hat{a} \mathcal{L} \hat{\rho})] \quad (13)$$

Solving these separately

$$\text{tr}(\hat{a}^2 \mathcal{L} \hat{\rho}) = -\frac{i}{\hbar} \text{tr}(\hat{a}^2 [\hat{H}, \hat{\rho}]) + \gamma(\bar{n} + 1) \text{tr}(\hat{a}^2 \mathcal{D}[\hat{a}] \hat{\rho}) + \gamma\bar{n} \text{tr}(\hat{a}^2 \mathcal{D}[\hat{a}^\dagger] \hat{\rho}) \quad (14)$$

$$= -i\omega \text{tr}(\hat{a}^2 \hat{a}^\dagger \hat{a} \hat{\rho}) + i\omega \text{tr}(\hat{a}^2 \hat{\rho} \hat{a}^\dagger \hat{a}) \quad (15)$$

$$+ \gamma(\bar{n} + 1) \text{tr}(\hat{a}^3 \hat{\rho} \hat{a}^\dagger) - \frac{\gamma(\bar{n} + 1)}{2} \text{tr}(\hat{a}^2 \hat{a}^\dagger \hat{a} \hat{\rho}) - \frac{\gamma(\bar{n} + 1)}{2} \text{tr}(\hat{a}^2 \hat{\rho} \hat{a}^\dagger \hat{a}) \quad (16)$$

$$+ \gamma\bar{n} \text{tr}(\hat{a}^2 \hat{a}^\dagger \hat{\rho} \hat{a}) - \frac{\gamma\bar{n}}{2} \text{tr}(\hat{a}^3 \hat{a}^\dagger \hat{\rho}) - \frac{\gamma\bar{n}}{2} \text{tr}(\hat{a}^2 \hat{\rho} \hat{a} \hat{a}^\dagger) \quad (17)$$

Using cyclic permutations of the trace and combining terms we get

$$\text{tr}(\hat{a}^2 \mathcal{L} \hat{\rho}) = i\omega \text{tr}([\hat{a}^\dagger \hat{a}, \hat{a}^2] \hat{\rho}) + \frac{\gamma(\bar{n} + 1)}{2} \text{tr}([\hat{a}^\dagger \hat{a}, \hat{a}^2] \hat{\rho}) + \frac{\gamma\bar{n}}{2} \text{tr}([\hat{a}^2, \hat{a} \hat{a}^\dagger] \hat{\rho}) \quad (18)$$

The commutators are

$$[\hat{a}^\dagger \hat{a}, \hat{a}^2] = [\hat{a}^\dagger, \hat{a}^2] \hat{a} = \hat{a} [\hat{a}^\dagger, \hat{a}] \hat{a} + [\hat{a}^\dagger, \hat{a}] \hat{a}^2 = -2\hat{a}^2 \quad (19)$$

$$[\hat{a}^2, \hat{a} \hat{a}^\dagger] = \hat{a} [\hat{a}^2, \hat{a}^\dagger] = \hat{a}^2 [\hat{a}, \hat{a}^\dagger] + \hat{a} [\hat{a}, \hat{a}^\dagger] \hat{a} = 2\hat{a}^2 \quad (20)$$

Thus

$$\text{tr}(\hat{a}^2 \mathcal{L} \hat{\rho}) = -2i\omega \text{tr}(\hat{a}^2 \hat{\rho}) - \gamma(\bar{n} + 1) \text{tr}(\hat{a}^2 \hat{\rho}) + \gamma\bar{n} \text{tr}(\hat{a}^2 \hat{\rho}) = -(\gamma + 2i\omega) \text{tr}(\hat{a}^2 \hat{\rho}) \quad (21)$$

$$= -(\gamma + 2i\omega) \langle \hat{a}^2 \rangle. \quad (22)$$

By taking the hermition conjugate of the above we get

$$\text{tr}\left((\hat{a}^\dagger)^2 \mathcal{L}\hat{\rho}\right) = (\gamma - 2i\omega) \text{tr}\left((\hat{a}^\dagger)^2 \hat{\rho}\right) = -(\gamma - 2i\omega) \left\langle (\hat{a}^\dagger)^2 \right\rangle. \quad (23)$$

Now we do the same for $\text{tr}(\hat{a}^\dagger \hat{a} \mathcal{L}\hat{\rho})$, so

$$\text{tr}(\hat{a}^\dagger \hat{a} \mathcal{L}\hat{\rho}) = -\frac{i}{\hbar} \text{tr}\left(\hat{a}^\dagger \hat{a} [\hat{H}, \hat{\rho}]\right) + \gamma(\bar{n} + 1) \text{tr}\left(\hat{a}^\dagger \hat{a} \mathcal{D}[\hat{a}]\hat{\rho}\right) + \gamma\bar{n} \text{tr}\left(\hat{a}^\dagger \hat{a} \mathcal{D}[\hat{a}^\dagger]\hat{\rho}\right) \quad (24)$$

$$= -i\omega \text{tr}\left(\hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a} \hat{\rho}\right) + i\omega \text{tr}\left(\hat{a}^\dagger \hat{a} \hat{\rho} \hat{a}^\dagger \hat{a}\right) \quad (25)$$

$$+ \gamma(\bar{n} + 1) \text{tr}\left(\hat{a}^\dagger \hat{a}^2 \hat{\rho} \hat{a}^\dagger\right) - \frac{\gamma(\bar{n} + 1)}{2} \text{tr}\left(\hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a} \hat{\rho}\right) - \frac{\gamma(\bar{n} + 1)}{2} \text{tr}\left(\hat{a}^\dagger \hat{a} \hat{\rho} \hat{a}^\dagger \hat{a}\right) \quad (26)$$

$$+ \gamma\bar{n} \text{tr}\left(\hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{\rho} \hat{a}\right) - \frac{\gamma\bar{n}}{2} \text{tr}\left(\hat{a}^\dagger \hat{a}^2 \hat{a}^\dagger \hat{\rho}\right) - \frac{\gamma\bar{n}}{2} \text{tr}\left(\hat{a}^\dagger \hat{a} \hat{\rho} \hat{a} \hat{a}^\dagger\right) \quad (27)$$

using cyclic permutations of the trace and combining terms we get

$$\text{tr}(\hat{a}^\dagger \hat{a} \mathcal{L}\hat{\rho}) = i\omega \text{tr}\left([\hat{a}^\dagger \hat{a}, \hat{a}^\dagger \hat{a}]\hat{\rho}\right) + \gamma(\bar{n} + 1) \text{tr}\left((\hat{a}^\dagger)^2 \hat{a}^2 \hat{\rho}\right) - \gamma(\bar{n} + 1) \text{tr}\left(\hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a} \hat{\rho}\right) \quad (28)$$

$$+ \gamma\bar{n} \text{tr}\left(\hat{a} \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{\rho}\right) - \frac{\gamma\bar{n}}{2} \text{tr}\left(\hat{a}^\dagger \hat{a}^2 \hat{a}^\dagger \hat{\rho}\right) - \frac{\gamma\bar{n}}{2} \text{tr}\left(\hat{a} (\hat{a}^\dagger)^2 \hat{a} \hat{\rho}\right) \quad (29)$$

Commuting to simplify we get

$$\text{tr}(\hat{a}^\dagger \hat{a} \mathcal{L}\hat{\rho}) = -\gamma(\bar{n} + 1) \text{tr}\left(\hat{a}^\dagger \hat{a} \hat{\rho}\right) - \gamma\bar{n} \text{tr}\left([\hat{a} \hat{a}^\dagger, \hat{a}^\dagger \hat{a}]\hat{\rho}\right) + \gamma\bar{n} \text{tr}\left(\hat{a} \hat{a}^\dagger \hat{\rho}\right) \quad (30)$$

The commutator is equal to 0, and thus we get

$$\text{tr}(\hat{a}^\dagger \hat{a} \mathcal{L}\hat{\rho}) = -\gamma(\bar{n} + 1) \text{tr}\left(\hat{a}^\dagger \hat{a} \hat{\rho}\right) + \gamma\bar{n} + \gamma\bar{n} \text{tr}\left(\hat{a}^\dagger \hat{a} \hat{\rho}\right) = -\gamma \text{tr}\left(\hat{a}^\dagger \hat{a} \hat{\rho}\right) + \gamma\bar{n} = -\gamma \left\langle \hat{a}^\dagger \hat{a} \right\rangle + \gamma\bar{n}. \quad (31)$$

Similarly we can find that

$$\text{tr}(\hat{a} \hat{a}^\dagger \mathcal{L}\hat{\rho}) = -\gamma \text{tr}\left(\hat{a} \hat{a}^\dagger \hat{\rho}\right) + \gamma(\bar{n} + 1) = -\gamma \left\langle \hat{a} \hat{a}^\dagger \right\rangle + \gamma(\bar{n} + 1). \quad (32)$$

Then since \hat{x}^2 commutes with \hat{x} (the measurement operator) we can write Eq. (11) the equation

of motion

$$\partial_t \langle \hat{x}^2 \rangle = \text{tr}(\hat{x}^2 \mathcal{L} \hat{\rho}) \quad (33)$$

$$= \frac{\hbar}{2m\omega} \left(-(\gamma + 2i\omega) \langle \hat{a}^2 \rangle - (\gamma - 2i\omega) \langle (\hat{a}^\dagger)^2 \rangle - \gamma \langle \hat{a}^\dagger \hat{a} \rangle + \gamma \bar{n} - \gamma \langle \hat{a} \hat{a}^\dagger \rangle + \gamma(\bar{n} + 1) \right) \quad (34)$$

$$= -\gamma \frac{\hbar}{2m\omega} \left(\langle \hat{a}^2 \rangle + \langle (\hat{a}^\dagger)^2 \rangle + \langle \hat{a}^\dagger \hat{a} \rangle + \langle \hat{a} \hat{a}^\dagger \rangle \right) + \frac{i\hbar}{m} \left(\langle (\hat{a}^\dagger)^2 \rangle - \langle \hat{a}^2 \rangle \right) + \frac{\gamma\hbar}{m\omega} (\bar{n} + 1/2) \quad (35)$$

Since we have

$$\{\hat{x}, \hat{p}\} = i\hbar((\hat{a}^\dagger)^2 - \hat{a}^2) \quad (36)$$

We can write the equation of motion as

$$\partial_t \langle \hat{x}^2 \rangle = -\gamma \langle \hat{x}^2 \rangle - \frac{1}{m} \langle \{\hat{x}, \hat{p}\} \rangle + \frac{\gamma\hbar}{m\omega} (\bar{n} + 1/2) \quad (37)$$