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Continuous Measurements and Feedback Control of a Quantum Harmonic Oscillator

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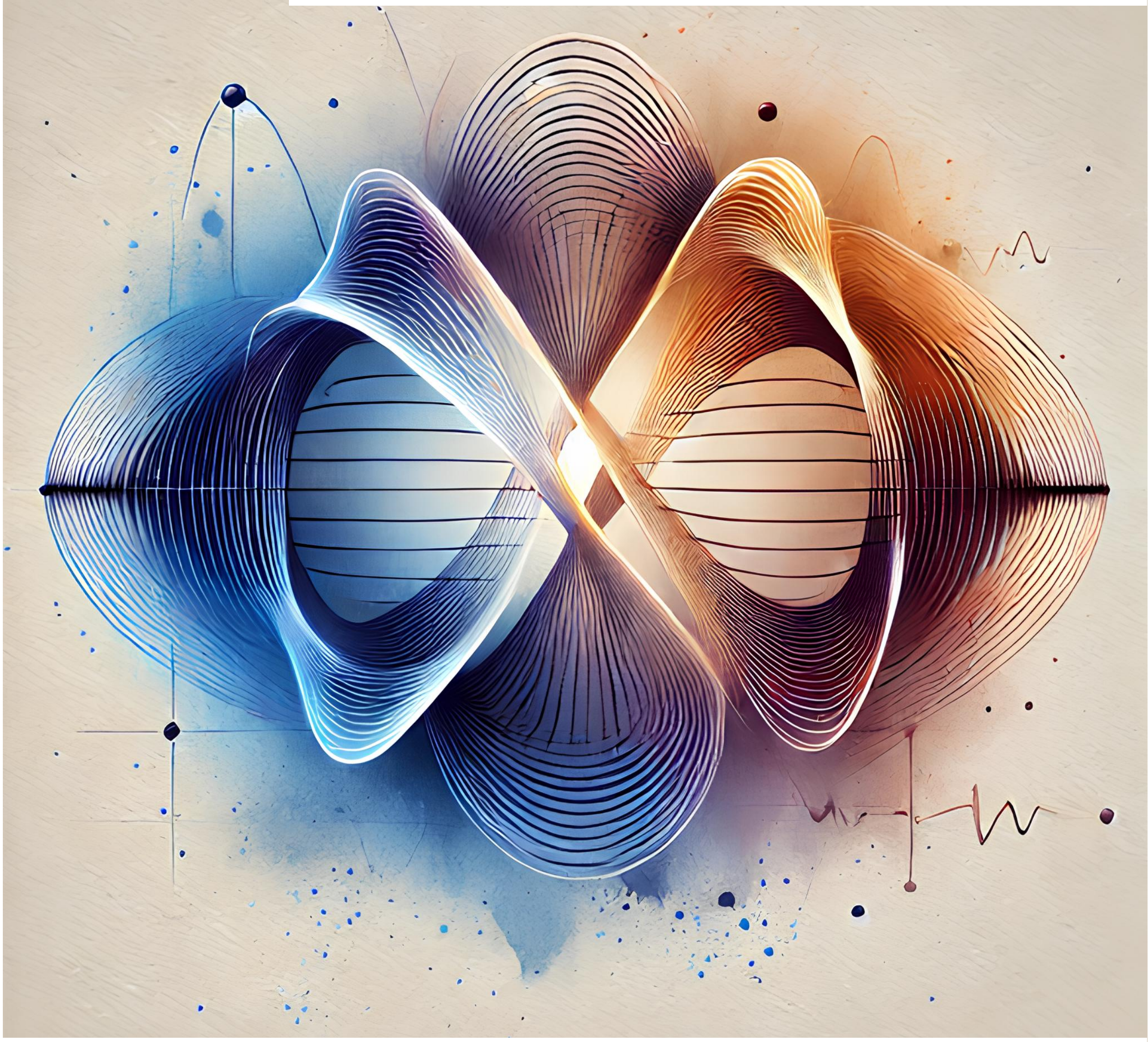

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References

[1] B. Annby-Andersson, *Continuous measurements of small systems: Feedback control, thermodynamics, entanglement*. Doctoral thesis (compilation), Lund University, 4 2024.

A Appendix

B EOM Calculation

Consider a quantum harmonic oscillator described by the Hamiltonian

$$\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + 1/2). \tag{1}$$

Consider also that the quantum harmonic oscillator is subject to a temperature bath and is continuously measured. The system then evolves according to the master equation

$$\partial_t\hat{\rho} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}] + \gamma(\bar{n}+1)\mathcal{D}[\hat{a}]\hat{\rho} + \gamma\bar{n}\mathcal{D}[\hat{a}^\dagger]\hat{\rho} + \lambda\mathcal{D}[\hat{A}]\hat{\rho} \tag{2}$$

where γ is the damping rate, $\bar{n}$ is the thermal occupancy defined as

$$\bar{n} = \frac{1}{e^{\hbar\omega/k_BT} - 1}, \tag{3}$$

λ is the measurement rate, $\hat{A}$ is the measurement operator, and $\mathcal{D}[\hat{O}]$ is the Lindblad super operator defined as

$$\mathcal{D}[\hat{O}]\hat{\rho} = \hat{O}\hat{\rho}\hat{O}^\dagger - \frac{1}{2}\{\hat{O}^\dagger\hat{O}, \hat{\rho}\}. \tag{4}$$

Then we also have the quadrature operators defined as

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger), \quad \hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a} - \hat{a}^\dagger). \tag{5}$$

We want to find the equations of motion when measuring the position quadrature

$$\partial_t\left\langle\hat{x}^2\right\rangle = \mathrm{tr}\big(\hat{x}^2\partial_t\hat{\rho}\big) \tag{6}$$

$$\partial_t\left\langle\hat{p}^2\right\rangle = \mathrm{tr}\big(\hat{p}^2\partial_t\hat{\rho}\big) \tag{7}$$

$$\partial_t\left\langle\{\hat{x}, \hat{p}\}\right\rangle = \mathrm{tr}(\{\hat{x}, \hat{p}\}\partial_t\hat{\rho}). \tag{8}$$

We start by defining the Liouvillian super operator

$$\mathcal{L}\hat{\rho} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}] + \gamma(\bar{n}+1)\mathcal{D}[\hat{a}]\hat{\rho} + \gamma\bar{n}\mathcal{D}[\hat{a}^\dagger]\hat{\rho} \tag{9}$$

which turns the master equation into

$$\partial_t \hat{\rho} = \mathcal{L}\hat{\rho} + \lambda \mathcal{D}[\hat{x}]\hat{\rho}. \tag{10}$$

B.1 Solving for $\partial_t \left\langle \hat{x}^2 \right\rangle$

We can write Eq. (6) as

$$\partial_t \left\langle \hat{x}^2 \right\rangle = \operatorname{tr}\left(\hat{x}^2 \mathcal{L}\hat{\rho}\right) + \operatorname{tr}\left(\hat{x}^2 \lambda \mathcal{D}[\hat{x}]\hat{\rho}\right) \tag{11}$$

and then rewriting the left term using the creation and annihilation operators we get

$$\operatorname{tr}\left(\hat{x}^2 \mathcal{L}\hat{\rho}\right) = \frac{\hbar}{2m\omega} \operatorname{tr}\Big((\hat{a}^2 + (\hat{a}^\dagger)^2 + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a})\mathcal{L}\hat{\rho}\Big) \tag{12}$$

$$= \frac{\hbar}{2m\omega} \left[\operatorname{tr}\left(\hat{a}^2 \mathcal{L}\hat{\rho}\right) + \operatorname{tr}\Big((\hat{a}^\dagger)^2 \mathcal{L}\hat{\rho}\Big) + \operatorname{tr}\Big(\hat{a}\hat{a}^\dagger \mathcal{L}\hat{\rho}\Big) + \operatorname{tr}\Big(\hat{a}^\dagger\hat{a}\mathcal{L}\hat{\rho}\Big)\right] \tag{13}$$

Solving these separately

$$\operatorname{tr}\left(\hat{a}^2 \mathcal{L}\hat{\rho}\right) = -\frac{i}{\hbar} \operatorname{tr}\Big(\hat{a}^2 \Big[\hat{H}, \hat{\rho}\Big]\Big) + \gamma(\bar{n}+1) \operatorname{tr}\left(\hat{a}^2 \mathcal{D}[\hat{a}]\hat{\rho}\right) + \gamma\bar{n} \operatorname{tr}\Big(\hat{a}^2 \mathcal{D}[\hat{a}^\dagger]\hat{\rho}\Big) \tag{14}$$

$$= -i\omega \operatorname{tr}\Big(\hat{a}^2\hat{a}^\dagger\hat{a}\hat{\rho}\Big) + i\omega \operatorname{tr}\Big(\hat{a}^2\hat{\rho}\hat{a}^\dagger\hat{a}\Big) \tag{15}$$

$$+ \gamma(\bar{n}+1) \operatorname{tr}\Big(\hat{a}^3\hat{\rho}\hat{a}^\dagger\Big) - \frac{\gamma(\bar{n}+1)}{2} \operatorname{tr}\Big(\hat{a}^2\hat{a}^\dagger\hat{a}\hat{\rho}\Big) - \frac{\gamma(\bar{n}+1)}{2} \operatorname{tr}\Big(\hat{a}^2\hat{\rho}\hat{a}^\dagger\hat{a}\Big) \tag{16}$$

$$+ \gamma\bar{n} \operatorname{tr}\Big(\hat{a}^2\hat{a}^\dagger\hat{\rho}\hat{a}\Big) - \frac{\gamma\bar{n}}{2} \operatorname{tr}\Big(\hat{a}^3\hat{a}^\dagger\hat{\rho}\Big) - \frac{\gamma\bar{n}}{2} \operatorname{tr}\Big(\hat{a}^2\hat{\rho}\hat{a}\hat{a}^\dagger\Big) \tag{17}$$

Using cyclic permutations of the trace and combining terms we get

$$\operatorname{tr}\left(\hat{a}^2 \mathcal{L}\hat{\rho}\right) = i\omega \operatorname{tr}\Big(\Big[\hat{a}^\dagger\hat{a}, \hat{a}^2\Big]\hat{\rho}\Big) + \frac{\gamma(\bar{n}+1)}{2} \operatorname{tr}\Big(\Big[\hat{a}^\dagger\hat{a}, \hat{a}^2\Big]\hat{\rho}\Big) + \frac{\gamma\bar{n}}{2} \operatorname{tr}\Big(\Big[\hat{a}^2, \hat{a}\hat{a}^\dagger\Big]\hat{\rho}\Big) \tag{18}$$

The commutators are

$$\Big[\hat{a}^\dagger\hat{a}, \hat{a}^2\Big] = \Big[\hat{a}^\dagger, \hat{a}^2\Big]\hat{a} = \hat{a}\Big[\hat{a}^\dagger, \hat{a}\Big]\hat{a} + \Big[\hat{a}^\dagger, \hat{a}\Big]\hat{a}^2 = -2\hat{a}^2 \tag{19}$$

$$\Big[\hat{a}^2, \hat{a}\hat{a}^\dagger\Big] = \hat{a}\Big[\hat{a}^2, \hat{a}^\dagger\Big] = \hat{a}^2\Big[\hat{a}, \hat{a}^\dagger\Big] + \hat{a}\Big[\hat{a}, \hat{a}^\dagger\Big]\hat{a} = 2\hat{a}^2 \tag{20}$$

Thus

$$\operatorname{tr}\left(\hat{a}^2 \mathcal{L}\hat{\rho}\right) = -2i\omega \operatorname{tr}\left(\hat{a}^2\hat{\rho}\right) - \gamma(\bar{n}+1) \operatorname{tr}\left(\hat{a}^2\hat{\rho}\right) + \gamma\bar{n} \operatorname{tr}\left(\hat{a}^2\hat{\rho}\right) = -(\gamma + 2i\omega) \operatorname{tr}\left(\hat{a}^2\hat{\rho}\right) \tag{21}$$

$$= -(\gamma + 2i\omega) \left\langle \hat{a}^2 \right\rangle. \tag{22}$$

By taking the hermition conjugate of the above we get

$$\operatorname{tr}\Big((\hat{a}^\dagger)^2\mathcal{L}\hat{\rho}\Big) = (\gamma - 2i\omega)\operatorname{tr}\Big((\hat{a}^\dagger)^2\hat{\rho}\Big) = -(\gamma - 2i\omega)\Big\langle(\hat{a}^\dagger)^2\Big\rangle. \tag{23}$$

Now we do the same for $\operatorname{tr}(\hat{a}^\dagger\hat{a}\mathcal{L}\hat{\rho})$, so

$$\operatorname{tr}\Big(\hat{a}^\dagger\hat{a}\mathcal{L}\hat{\rho}\Big) = -\frac{i}{\hbar}\operatorname{tr}\Big(\hat{a}^\dagger\hat{a}\Big[\hat{H},\hat{\rho}\Big]\Big) + \gamma(\bar{n}+1)\operatorname{tr}\Big(\hat{a}^\dagger\hat{a}\mathcal{D}[\hat{a}]\hat{\rho}\Big) + \gamma\bar{n}\operatorname{tr}\Big(\hat{a}^\dagger\hat{a}\mathcal{D}[\hat{a}^\dagger]\hat{\rho}\Big) \tag{24}$$

$$= -i\omega\operatorname{tr}\Big(\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}\hat{\rho}\Big) + i\omega\operatorname{tr}\Big(\hat{a}^\dagger\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a}\Big) \tag{25}$$

$$+\gamma(\bar{n}+1)\operatorname{tr}\Big(\hat{a}^\dagger\hat{a}^2\hat{\rho}\hat{a}^\dagger\Big) - \frac{\gamma(\bar{n}+1)}{2}\operatorname{tr}\Big(\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}\hat{\rho}\Big) - \frac{\gamma(\bar{n}+1)}{2}\operatorname{tr}\Big(\hat{a}^\dagger\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a}\Big) \tag{26}$$

$$+\gamma\bar{n}\operatorname{tr}\Big(\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{\rho}\hat{a}\Big) - \frac{\gamma\bar{n}}{2}\operatorname{tr}\Big(\hat{a}^\dagger\hat{a}^2\hat{a}^\dagger\hat{\rho}\Big) - \frac{\gamma\bar{n}}{2}\operatorname{tr}\Big(\hat{a}^\dagger\hat{a}\hat{\rho}\hat{a}\hat{a}^\dagger\Big) \tag{27}$$

using cyclic permutations of the trace and combining terms we get

$$\operatorname{tr}\Big(\hat{a}^\dagger\hat{a}\mathcal{L}\hat{\rho}\Big) = i\omega\operatorname{tr}\Big(\Big[\hat{a}^\dagger\hat{a},\hat{a}^\dagger\hat{a}\Big]\hat{\rho}\Big) + \gamma(\bar{n}+1)\operatorname{tr}\Big((\hat{a}^\dagger)^2\hat{a}^2\hat{\rho}\Big) - \gamma(\bar{n}+1)\operatorname{tr}\Big(\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}\hat{\rho}\Big) \tag{28}$$

$$+\gamma\bar{n}\operatorname{tr}\Big(\hat{a}\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{\rho}\Big) - \frac{\gamma\bar{n}}{2}\operatorname{tr}\Big(\hat{a}^\dagger\hat{a}^2\hat{a}^\dagger\hat{\rho}\Big) - \frac{\gamma\bar{n}}{2}\operatorname{tr}\Big(\hat{a}(\hat{a}^\dagger)^2\hat{a}\hat{\rho}\Big) \tag{29}$$

Commuting to simplify we get

$$\operatorname{tr}\Big(\hat{a}^\dagger\hat{a}\mathcal{L}\hat{\rho}\Big) = -\gamma(\bar{n}+1)\operatorname{tr}\Big(\hat{a}^\dagger\hat{a}\hat{\rho}\Big) - \gamma\bar{n}\operatorname{tr}\Big(\Big[\hat{a}\hat{a}^\dagger,\hat{a}^\dagger\hat{a}\Big]\hat{\rho}\Big) + \gamma\bar{n}\operatorname{tr}\Big(\hat{a}\hat{a}^\dagger\hat{\rho}\Big) \tag{30}$$

The commutator is equal to 0, and thus we get

$$\operatorname{tr}\Big(\hat{a}^\dagger\hat{a}\mathcal{L}\hat{\rho}\Big) = -\gamma(\bar{n}+1)\operatorname{tr}\Big(\hat{a}^\dagger\hat{a}\hat{\rho}\Big) + \gamma\bar{n} + \gamma\bar{n}\operatorname{tr}\Big(\hat{a}^\dagger\hat{a}\hat{\rho}\Big) = -\gamma\operatorname{tr}\Big(\hat{a}^\dagger\hat{a}\hat{\rho}\Big) + \gamma\bar{n} = -\gamma\Big\langle\hat{a}^\dagger\hat{a}\Big\rangle + \gamma\bar{n}. \tag{31}$$

Similarly we can find that

$$\operatorname{tr}\Big(\hat{a}\hat{a}^\dagger\mathcal{L}\hat{\rho}\Big) = -\gamma\operatorname{tr}\Big(\hat{a}\hat{a}^\dagger\hat{\rho}\Big) + \gamma(\bar{n}+1) = -\gamma\Big\langle\hat{a}\hat{a}^\dagger\Big\rangle + \gamma(\bar{n}+1). \tag{32}$$

Then since $\hat{x}^2$ commutes with $\hat{x}$ (the measurement operator) we can write Eq. (11) the equation of motion

$$\partial_t \left\langle \hat{x}^2 \right\rangle = \mathrm{tr}\big(\hat{x}^2 \mathcal{L}\hat{\rho}\big) \tag{33}$$

$$= \frac{\hbar}{2m\omega}\Big(-(\gamma+2i\omega)\left\langle \hat{a}^2\right\rangle - (\gamma-2i\omega)\left\langle (\hat{a}^\dagger)^2\right\rangle - \gamma\left\langle \hat{a}^\dagger\hat{a}\right\rangle + \gamma\bar{n} - \gamma\left\langle \hat{a}\hat{a}^\dagger\right\rangle + \gamma(\bar{n}+1)\Big) \tag{34}$$

$$= -\gamma\frac{\hbar}{2m\omega}\Big(\left\langle \hat{a}^2\right\rangle + \left\langle (\hat{a}^\dagger)^2\right\rangle + \left\langle \hat{a}^\dagger\hat{a}\right\rangle + \left\langle \hat{a}\hat{a}^\dagger\right\rangle\Big) + \frac{i\hbar}{m}\Big(\left\langle (\hat{a}^\dagger)^2\right\rangle - \left\langle \hat{a}^2\right\rangle\Big) + \frac{\gamma\hbar}{m\omega}(\bar{n}+1/2) \tag{35}$$

Since we have

$$\{\hat{x},\hat{p}\} = i\hbar((\hat{a}^\dagger)^2 - \hat{a}^2) \tag{36}$$

We can write the equation of motion as

$$\partial_t \left\langle \hat{x}^2\right\rangle = -\gamma\left\langle \hat{x}^2\right\rangle + \frac{1}{m}\left\langle \{\hat{x},\hat{p}\}\right\rangle + \frac{\gamma\hbar}{m\omega}(\bar{n}+1/2) \tag{37}$$

B.2 Solving for $\partial_t \left\langle \hat{p}^2 \right\rangle$

Writing Eq. (7) as

$$\partial_t \left\langle \hat{p}^2\right\rangle = \mathrm{tr}\big(\hat{p}^2\mathcal{L}\hat{\rho}\big) + \mathrm{tr}\big(\hat{p}^2\lambda\mathcal{D}[\hat{x}]\hat{\rho}\big) \tag{38}$$

Rewriting the left terms using $\hat{a}, \hat{a}^\dagger$ we get

$$\mathrm{tr}\big(\hat{p}^2\mathcal{L}\hat{\rho}\big) = -\frac{m\omega\hbar}{2}\,\mathrm{tr}\Big((\hat{a}^2 + (\hat{a}^\dagger)^2 - \hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger)\mathcal{L}\hat{\rho}\Big) \tag{39}$$

$$= -\frac{m\omega\hbar}{2}\Big(\mathrm{tr}\big(\hat{a}^2\mathcal{L}\hat{\rho}\big) + \mathrm{tr}\Big((\hat{a}^\dagger)^2\mathcal{L}\hat{\rho}\Big) - \mathrm{tr}\Big(\hat{a}^\dagger\hat{a}\mathcal{L}\hat{\rho}\Big) - \mathrm{tr}\Big(\hat{a}\hat{a}^\dagger\mathcal{L}\hat{\rho}\Big)\Big) \tag{40}$$

We have solved for these before, so we can write

$$\mathrm{tr}\big(\hat{p}^2\mathcal{L}\hat{\rho}\big) = -\frac{m\omega\hbar}{2}\Big(-(\gamma+2i\omega)\left\langle \hat{a}^2\right\rangle - (\gamma-2i\omega)\left\langle (\hat{a}^\dagger)^2\right\rangle + \gamma\left\langle \hat{a}^\dagger\hat{a}\right\rangle - \gamma\bar{n} + \gamma\left\langle \hat{a}\hat{a}^\dagger\right\rangle - \gamma(\bar{n}+1)\Big) \tag{41}$$

$$= -\gamma\left\langle \hat{p}^2\right\rangle + mi\omega^2\hbar\Big(\left\langle \hat{a}^2\right\rangle - \left\langle (\hat{a}^\dagger)^2\right\rangle\Big) + \gamma m\omega\hbar(\bar{n}+1/2) \tag{42}$$

$$= -\gamma\left\langle \hat{p}^2\right\rangle - m\omega^2\left\langle \{\hat{x},\hat{p}\}\right\rangle + \gamma m\omega\hbar(\bar{n}+1/2) \tag{43}$$

Solving the measurement term

$$\mathrm{tr}\big(\hat{p}^2\lambda\mathcal{D}[\hat{x}]\hat{\rho}\big) = \lambda\left(\mathrm{tr}\big(\hat{p}^2\hat{x}\hat{\rho}\hat{x}\big) - \frac{1}{2}\,\mathrm{tr}\big(\hat{p}^2\hat{x}^2\hat{\rho}\big) - \frac{1}{2}\,\mathrm{tr}\big(\hat{p}^2\hat{\rho}\hat{x}^2\big)\right) \tag{44}$$

$$= \lambda\left(\mathrm{tr}\big(\hat{x}\hat{p}^2\hat{x}\hat{\rho}\big) - \frac{1}{2}\,\mathrm{tr}\big(\hat{p}^2\hat{x}^2\hat{\rho}\big) - \frac{1}{2}\,\mathrm{tr}\big(\hat{x}^2\hat{p}^2\hat{\rho}\big)\right) \tag{45}$$

Commuting the operators

$$\hat{x}\hat{p}^2\hat{x} = \frac{1}{2}(\hat{p}\hat{x}+i\hbar)\hat{p}\hat{x} + \frac{1}{2}\hat{x}\hat{p}(\hat{x}\hat{p}-i\hbar) = \frac{1}{2}\hat{p}\hat{x}\hat{p}\hat{x} + \frac{1}{2}\hat{x}\hat{p}\hat{x}\hat{p} + \frac{i\hbar}{2}[\hat{p},\hat{x}] \tag{46}$$

$$= \frac{1}{2}\hat{p}\hat{x}\hat{p}\hat{x} + \frac{1}{2}\hat{x}\hat{p}\hat{x}\hat{p} + \frac{\hbar^2}{2} \tag{47}$$

$$\hat{p}^2\hat{x}^2 = \hat{p}(\hat{x}\hat{p}-i\hbar)\hat{x} = \hat{p}\hat{x}\hat{p}\hat{x} - i\hbar\hat{p}\hat{x} \tag{48}$$

$$\hat{x}^2\hat{p}^2 = \hat{x}(\hat{p}\hat{x}+i\hbar)\hat{p} = \hat{x}\hat{p}\hat{x}\hat{p} + i\hbar\hat{x}\hat{p} \tag{49}$$

This gives

$$\mathrm{tr}\big(\hat{p}^2\lambda\mathcal{D}[\hat{x}]\hat{\rho}\big) = \lambda\left(\frac{1}{2}\,\mathrm{tr}(\hat{p}\hat{x}\hat{p}\hat{x}\hat{\rho}) + \frac{1}{2}\,\mathrm{tr}(\hat{x}\hat{p}\hat{x}\hat{p}\hat{\rho}) + \frac{\hbar^2}{2} - \frac{1}{2}\,\mathrm{tr}(\hat{p}\hat{x}\hat{p}\hat{x}\hat{\rho}) - \frac{1}{2}\,\mathrm{tr}(\hat{x}\hat{p}\hat{x}\hat{p}\hat{\rho}) + \frac{i\hbar}{2}\,\mathrm{tr}([\hat{p},\hat{x}])\right) \tag{50}$$

$$= \frac{\lambda\hbar^2}{2} + \frac{\lambda i\hbar}{2}\,\mathrm{tr}([\hat{p},\hat{x}]) = \lambda\hbar^2 \tag{51}$$

The final equation of motion then becomes

$$\partial_t\left\langle\hat{p}^2\right\rangle = -\gamma\left\langle\hat{p}^2\right\rangle - m\omega^2\left\langle\{\hat{x},\hat{p}\}\right\rangle + \gamma m\omega\hbar(\bar{n}+1/2) + \lambda\hbar^2 \tag{52}$$

B.3 Solving for $\partial_t\left\langle\{\hat{x},\hat{p}\}\right\rangle$