LUND UNIVERSITY

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Continuous Measurements and Feedback Control of a Quantum Harmonic Oscillator

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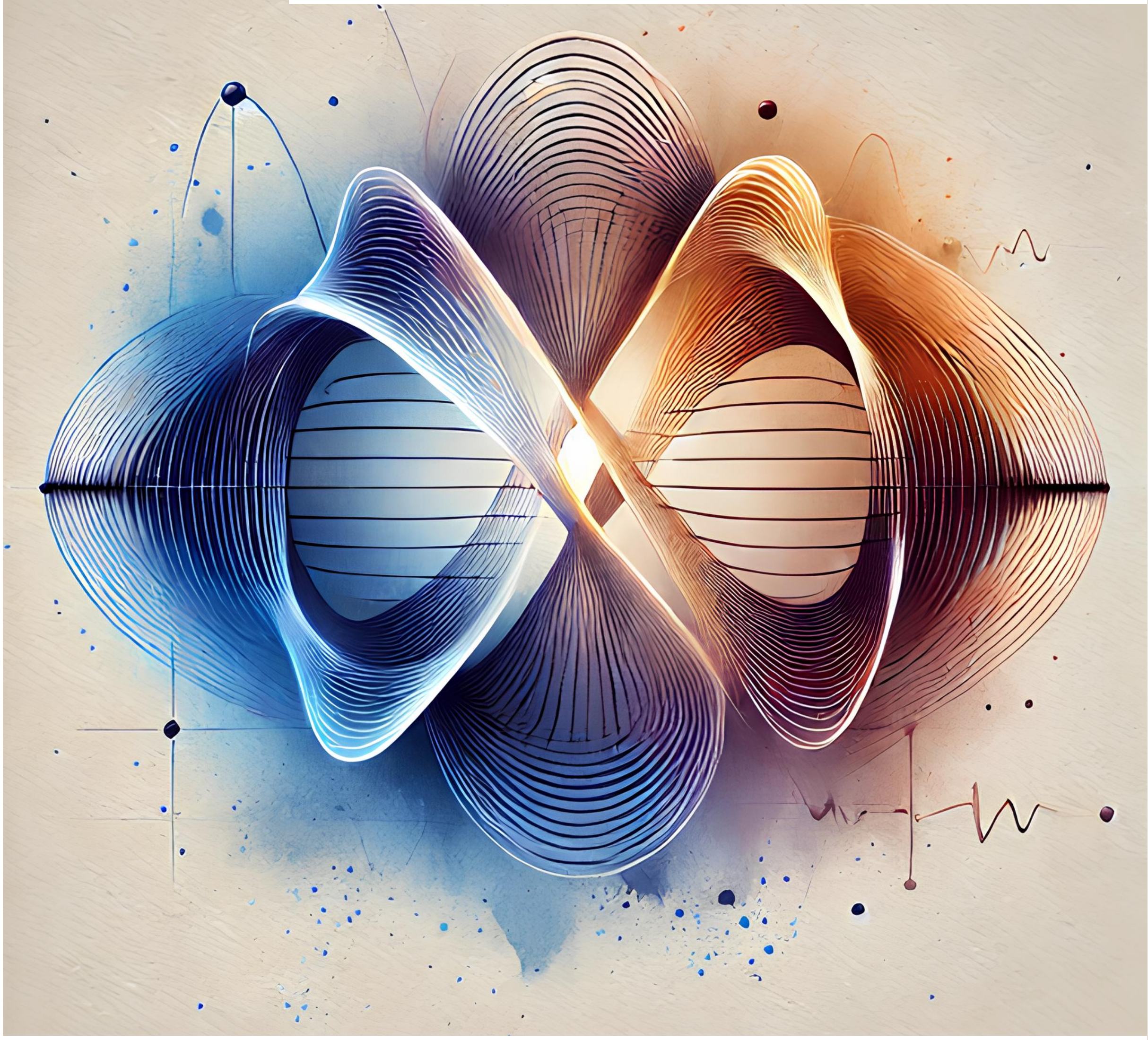

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References

[1] B. Annby-Andersson, *Continuous measurements of small systems: Feedback control, thermodynamics, entanglement*. Doctoral thesis (compilation), Lund University, 4 2024.

A Appendix

B EOM Calculation

Consider a quantum harmonic oscillator described by the Hamiltonian

$$\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + 1/2). \tag{1}$$

Consider also that the quantum harmonic oscillator is subject to a temperature bath and is continuously measured. The system then evolves according to the master equation

$$\partial_t\hat{\rho} = -\frac{i}{\hbar}[\hat{H},\hat{\rho}] + \gamma(\bar{n}+1)\mathcal{D}[\hat{a}]\hat{\rho} + \gamma\bar{n}\mathcal{D}[\hat{a}^\dagger]\hat{\rho} + \lambda\mathcal{D}[\hat{A}]\hat{\rho} \tag{2}$$

where γ is the damping rate, $\bar{n}$ is the thermal occupancy defined as

$$\bar{n} = \frac{1}{e^{\hbar\omega/k_BT} - 1}, \tag{3}$$

λ is the measurement rate, $\hat{A}$ is the measurement operator, and $\mathcal{D}[\hat{O}]$ is the Lindblad super operator defined as

$$\mathcal{D}[\hat{O}]\hat{\rho} = \hat{O}\hat{\rho}\hat{O}^\dagger - \frac{1}{2}\{\hat{O}^\dagger\hat{O},\hat{\rho}\}. \tag{4}$$

Then we also have the quadrature operators defined as

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}+\hat{a}^\dagger), \quad \hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a}-\hat{a}^\dagger). \tag{5}$$

We want to find the equations of motion when measuring the position quadrature

$$\partial_t\left\langle\hat{x}^2\right\rangle = \mathrm{tr}\big(\hat{x}^2\partial_t\hat{\rho}\big) \tag{6}$$

$$\partial_t\left\langle\hat{p}^2\right\rangle = \mathrm{tr}\big(\hat{p}^2\partial_t\hat{\rho}\big) \tag{7}$$

$$\partial_t\left\langle\{\hat{x},\hat{p}\}\right\rangle = \mathrm{tr}(\{\hat{x},\hat{p}\}\partial_t\hat{\rho}). \tag{8}$$

We start by defining the Liouvillian super operator

$$\mathcal{L}\hat{\rho} = -\frac{i}{\hbar}[\hat{H},\hat{\rho}] + \gamma(\bar{n}+1)\mathcal{D}[\hat{a}]\hat{\rho} + \gamma\bar{n}\mathcal{D}[\hat{a}^\dagger]\hat{\rho} \tag{9}$$

which turns the master equation into

$$\partial_t \hat{\rho} = \mathcal{L}\hat{\rho} + \lambda \mathcal{D}[\hat{x}]\hat{\rho}. \tag{10}$$

We can then write Eq. (6) as

$$\partial_t \left\langle \hat{x}^2 \right\rangle = \operatorname{tr}\big(\hat{x}^2 \mathcal{L}\hat{\rho}\big) + \operatorname{tr}\big(\hat{x}^2 \lambda \mathcal{D}[\hat{x}]\hat{\rho}\big) \tag{11}$$

and then rewriting the left term using the creation and annihilation operators we get

$$\begin{aligned}
\operatorname{tr}\big(\hat{x}^2 \mathcal{L}\hat{\rho}\big) &= \frac{\hbar}{2m\omega} \operatorname{tr}\Big((\hat{a}^2 + (\hat{a}^\dagger)^2 + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a})\mathcal{L}\hat{\rho}\Big) & (12)\\
&= \frac{\hbar}{2m\omega} \Big[\operatorname{tr}\big(\hat{a}^2 \mathcal{L}\hat{\rho}\big) + \operatorname{tr}\Big((\hat{a}^\dagger)^2 \mathcal{L}\hat{\rho}\Big) + \operatorname{tr}\Big(\hat{a}\hat{a}^\dagger \mathcal{L}\hat{\rho}\Big) + \operatorname{tr}\Big(\hat{a}^\dagger\hat{a}\mathcal{L}\hat{\rho}\Big)\Big] & (13)
\end{aligned}$$

Solving these separately

$$\begin{aligned}
\operatorname{tr}\big(\hat{a}^2 \mathcal{L}\hat{\rho}\big) &= -\frac{i}{\hbar} \operatorname{tr}\Big(\hat{a}^2 \Big[\hat{H}, \hat{\rho}\Big]\Big) + \gamma(\bar{n}+1) \operatorname{tr}\big(\hat{a}^2 \mathcal{D}[\hat{a}]\hat{\rho}\big) + \gamma\bar{n} \operatorname{tr}\Big(\hat{a}^2 \mathcal{D}[\hat{a}^\dagger]\hat{\rho}\Big) & (14)\\
&= -i\omega \operatorname{tr}\Big(\hat{a}^2\hat{a}^\dagger\hat{a}\hat{\rho}\Big) + i\omega \operatorname{tr}\Big(\hat{a}^2\hat{\rho}\hat{a}^\dagger\hat{a}\Big) & (15)\\
&+ \gamma(\bar{n}+1) \operatorname{tr}\Big(\hat{a}^3\hat{\rho}\hat{a}^\dagger\Big) - \frac{\gamma(\bar{n}+1)}{2} \operatorname{tr}\Big(\hat{a}^2\hat{a}^\dagger\hat{a}\hat{\rho}\Big) - \frac{\gamma(\bar{n}+1)}{2} \operatorname{tr}\Big(\hat{a}^2\hat{\rho}\hat{a}^\dagger\hat{a}\Big) & (16)\\
&+ \gamma\bar{n} \operatorname{tr}\Big(\hat{a}^2\hat{a}^\dagger\hat{\rho}\hat{a}\Big) - \frac{\gamma\bar{n}}{2} \operatorname{tr}\Big(\hat{a}^3\hat{a}^\dagger\hat{\rho}\Big) - \frac{\gamma\bar{n}}{2} \operatorname{tr}\Big(\hat{a}^2\hat{\rho}\hat{a}\hat{a}^\dagger\Big) & (17)
\end{aligned}$$

Using cyclic permutations of the trace and combining terms we get

$$\operatorname{tr}\big(\hat{a}^2 \mathcal{L}\hat{\rho}\big) = i\omega \operatorname{tr}\Big(\Big[\hat{a}^\dagger\hat{a}, \hat{a}^2\Big]\hat{\rho}\Big) + \frac{\gamma(\bar{n}+1)}{2} \operatorname{tr}\Big(\Big[\hat{a}^\dagger\hat{a}, \hat{a}^2\Big]\hat{\rho}\Big) + \frac{\gamma\bar{n}}{2} \operatorname{tr}\Big(\Big[\hat{a}^2, \hat{a}\hat{a}^\dagger\Big]\hat{\rho}\Big) \tag{18}$$

The commutators are

$$\begin{aligned}
\Big[\hat{a}^\dagger\hat{a}, \hat{a}^2\Big] &= \Big[\hat{a}^\dagger, \hat{a}^2\Big]\hat{a} = \hat{a}\Big[\hat{a}^\dagger, \hat{a}\Big]\hat{a} + \Big[\hat{a}^\dagger, \hat{a}\Big]\hat{a}^2 = -2\hat{a}^2 & (19)\\
\Big[\hat{a}^2, \hat{a}\hat{a}^\dagger\Big] &= \hat{a}\Big[\hat{a}^2, \hat{a}^\dagger\Big] = \hat{a}^2\Big[\hat{a}, \hat{a}^\dagger\Big] + \hat{a}\Big[\hat{a}, \hat{a}^\dagger\Big]\hat{a} = 2\hat{a}^2 & (20)
\end{aligned}$$

Thus

$$\operatorname{tr}\big(\hat{a}^2 \mathcal{L}\hat{\rho}\big) = -2i\omega \operatorname{tr}\big(\hat{a}^2\hat{\rho}\big) - \gamma(\bar{n}+1) \operatorname{tr}\big(\hat{a}^2\hat{\rho}\big) + \gamma\bar{n} \operatorname{tr}\big(\hat{a}^2\hat{\rho}\big) = -(\gamma + 2i\omega) \operatorname{tr}\big(\hat{a}^2\hat{\rho}\big) \tag{21}$$

By taking the hermition conjugate of the above we get

$$\operatorname{tr}\Big((\hat{a}^\dagger)^2\mathcal{L}\hat{\rho}\Big) = (\gamma - 2i\omega)\operatorname{tr}\Big((\hat{a}^\dagger)^2\hat{\rho}\Big). \tag{22}$$

Now we do the same for $\operatorname{tr}\big(\hat{a}^\dagger\hat{a}\mathcal{L}\hat{\rho}\big)$, so

$$\begin{aligned}
\operatorname{tr}\Big(\hat{a}^\dagger\hat{a}\mathcal{L}\hat{\rho}\Big) &= -\frac{i}{\hbar}\operatorname{tr}\Big(\hat{a}^\dagger\hat{a}\Big[\hat{H},\hat{\rho}\Big]\Big) + \gamma(\bar{n}+1)\operatorname{tr}\Big(\hat{a}^\dagger\hat{a}\mathcal{D}[\hat{a}]\hat{\rho}\Big) + \gamma\bar{n}\operatorname{tr}\Big(\hat{a}^\dagger\hat{a}\mathcal{D}[\hat{a}^\dagger]\hat{\rho}\Big) && (23)\\
&= -i\omega\operatorname{tr}\Big(\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}\hat{\rho}\Big) + i\omega\operatorname{tr}\Big(\hat{a}^\dagger\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a}\Big) && (24)\\
&+ \gamma(\bar{n}+1)\operatorname{tr}\Big(\hat{a}^\dagger\hat{a}^2\hat{\rho}\hat{a}^\dagger\Big) - \frac{\gamma(\bar{n}+1)}{2}\operatorname{tr}\Big(\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}\hat{\rho}\Big) - \frac{\gamma(\bar{n}+1)}{2}\operatorname{tr}\Big(\hat{a}^\dagger\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a}\Big) && (25)\\
&+ \gamma\bar{n}\operatorname{tr}\Big(\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{\rho}\hat{a}\Big) - \frac{\gamma\bar{n}}{2}\operatorname{tr}\Big(\hat{a}^\dagger\hat{a}^2\hat{a}^\dagger\hat{\rho}\Big) - \frac{\gamma\bar{n}}{2}\operatorname{tr}\Big(\hat{a}^\dagger\hat{a}\hat{\rho}\hat{a}\hat{a}^\dagger\Big) && (26)
\end{aligned}$$

using cyclic permutations of the trace and combining terms we get

$$\begin{aligned}
\operatorname{tr}\Big(\hat{a}^\dagger\hat{a}\mathcal{L}\hat{\rho}\Big) &= i\omega\operatorname{tr}\Big(\Big[\hat{a}^\dagger\hat{a},\hat{a}^\dagger\hat{a}\Big]\hat{\rho}\Big) + \gamma(\bar{n}+1)\operatorname{tr}\Big((\hat{a}^\dagger)^2\hat{a}^2\hat{\rho}\Big) - \gamma(\bar{n}+1)\operatorname{tr}\Big(\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}\hat{\rho}\Big) && (27)\\
&+ \gamma\bar{n}\operatorname{tr}\Big(\hat{a}\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{\rho}\Big) - \frac{\gamma\bar{n}}{2}\operatorname{tr}\Big(\hat{a}^\dagger\hat{a}^2\hat{a}^\dagger\hat{\rho}\Big) - \frac{\gamma\bar{n}}{2}\operatorname{tr}\Big(\hat{a}(\hat{a}^\dagger)^2\hat{a}\hat{\rho}\Big) && (28)
\end{aligned}$$

Commuting to simplify we get

$$\operatorname{tr}\Big(\hat{a}^\dagger\hat{a}\mathcal{L}\hat{\rho}\Big) = -\gamma(\bar{n}+1)\operatorname{tr}\Big(\hat{a}^\dagger\hat{a}\hat{\rho}\Big) - \gamma\bar{n}\operatorname{tr}\Big(\Big[\hat{a}\hat{a}^\dagger,\hat{a}^\dagger\hat{a}\Big]\hat{\rho}\Big) + \gamma\bar{n}\operatorname{tr}\Big(\hat{a}\hat{a}^\dagger\hat{\rho}\Big) \tag{29}$$

The commutator is equal to 0, and thus we get

$$\operatorname{tr}\Big(\hat{a}^\dagger\hat{a}\mathcal{L}\hat{\rho}\Big) = -\gamma(\bar{n}+1)\operatorname{tr}\Big(\hat{a}^\dagger\hat{a}\hat{\rho}\Big) + \gamma\bar{n} + \gamma\bar{n}\operatorname{tr}\Big(\hat{a}^\dagger\hat{a}\hat{\rho}\Big) = -\gamma\operatorname{tr}\Big(\hat{a}^\dagger\hat{a}\hat{\rho}\Big) + \gamma\bar{n} \tag{30}$$

Similarly we can find that

$$\operatorname{tr}\Big(\hat{a}\hat{a}^\dagger\mathcal{L}\hat{\rho}\Big) = -\gamma\operatorname{tr}\Big(\hat{a}\hat{a}^\dagger\hat{\rho}\Big) + \gamma(\bar{n}+1). \tag{31}$$