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Continuous Measurements and Feedback Control of a Quantum Harmonic Oscillator

Max Eriksson

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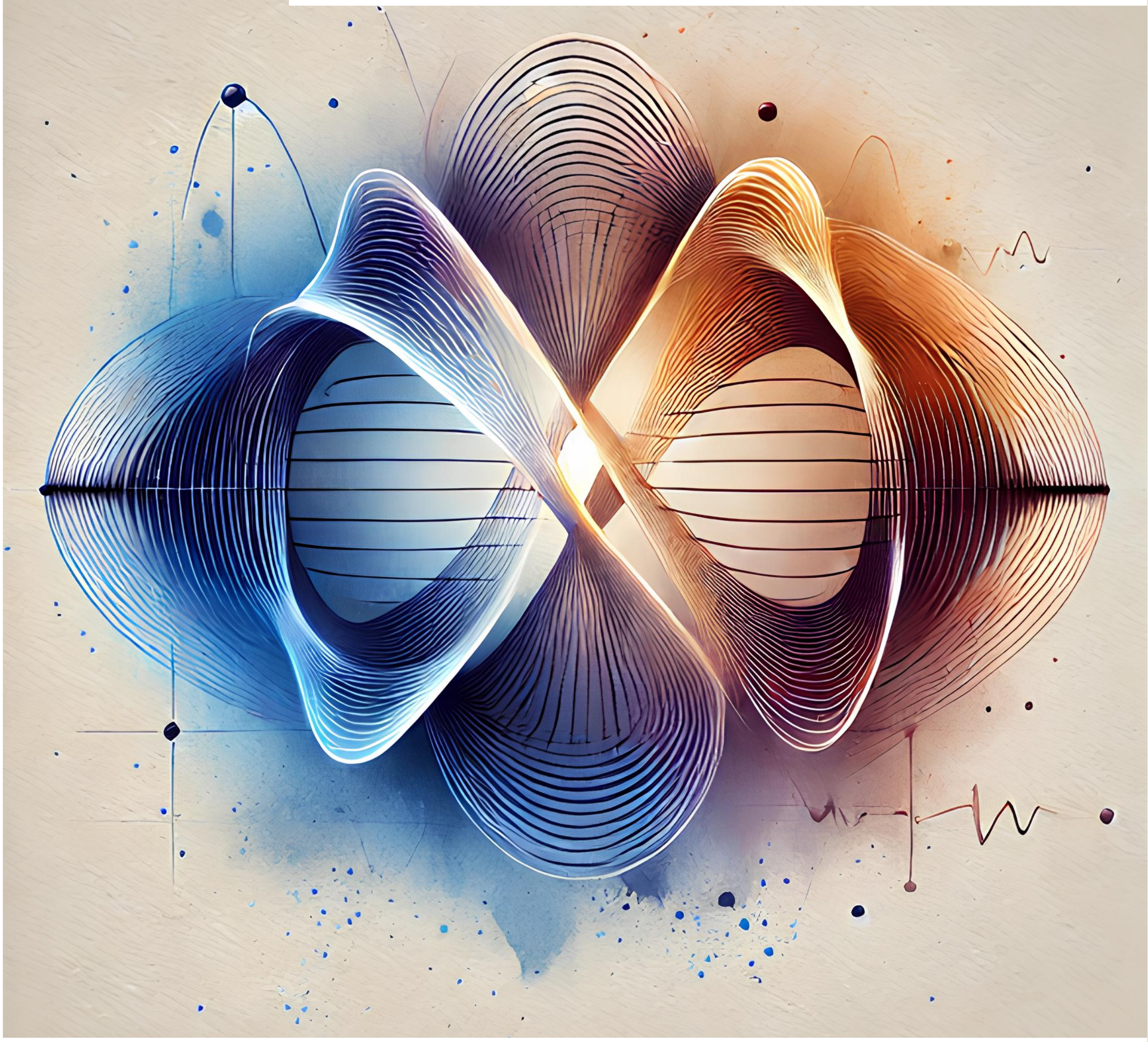
Supervised by Kalle Kansanen and Peter Samuelsson
Division of Mathematical Physics | Department of Physics



List of Abbreviations

Acknowledgements

Abstract

This is the abstract

Populärvetenskaplig sammanfattning

Kvantmekaniken har sedan början av 1900-talet revolutionerat vår försåelse av fysik, framförallt hur ljus och partiklar fungerar och interagerar. En unik egenskap av kvantmekaniken är att den i grunden är en probibalistisk teori, vilket låter oss beskriva system som befinner sig i flera olika tillstånd samtidigt. Detta fenomenet kallas för superposition och är vad som gör kvantmekanik unikt från klassisk fysik.

Som konsekvens av att kvantmekaniska system kan befinna sig i superposition, gör det att mätningar av systemet ger olika resultat fördelade enligt en sannolikhetsfördelning. Detta öppnar också upp för frågan vad en kvantmekanisk mätning är för något. I grund och botten kan vi säga att en mätning är en interaktion mellan vårt system och ett yttre system, en detektor. Det är denna interaktion som gör att systemet kollapsar från en superposition till det observerade tillståndet. En mätning, eller interaktion, påverkar alltså vårt kvantmekaniska system. I detta arbete undersöker vi hur man kan använda mätningar och återkoppling för att kyla ett kvantmekaniskt system.

Det system som vi studerar är ett av de simplaste kvantmekaniska systemen, en kvantharmonisk oscillator. Det är ett av få systm som kan lösas analytiskt, vilket är en av anledningarna till att vi väljer att studera det. En annan fördel med en en kvantharmonisk oscillator är att det är en bra approximation av många system som befinner sig nära sitt jämviktsläge. I modellen som används betraktar vi även ett värmebad som systemet är kopplat till, vilket är ett sätt att modellera en omgivning som påverkar systemet och ger oss en mer realistisk bild.

Vi undersöker sedan hur svaga mätningar påverkar systemet. Med en svag mätning menar vi att istället för att kollapsa systemet helt till ett tilstånd och förstöra superpositionen så får vi ut en begränsad mängd information om systemet, samtidigt som systemet fortfarande är i superposition. Vi matar sedan in denna information i ett återkopplingssystem, som i sin tur påverkar systemet. Genom att justera styrkan och fasen av vår återkopplingsparameter kan vi kyla systemet, genom att minimera dessa fluktuationer. Det intressanta resultatet visar att det är möjligt att kyla systemet till en temperatur som är lägre än den i värmebadet. Det vill säga att systemet blir kallare än sin omgivning. Vi visar även i arbetet att systemet är stabilt under återkopplingen som kan kyla systemet. Detta är viktigt för att kunna vara prakiskt tillämpbart då ett instabilt system hade varit väldigt känsligt för störningar, något som alltid finns i verkligheten.

En anledning att vi vill kunna kyla kvantystem är för att kunna observera och utnyttja kvantmekaniska fenomen. Exempelvis kan höga temperaturer excitera systemet till högre energinivåer, och om energinivåer är ett av de tillstånden som är viktiga för vår applikation så kan det leda till brus och informationsförluster. Ett exemepel skulle kunna vara kvantdatorer där vi utnyttjar superposition för att utföra beräkningar. Detta bygger på användandet av kvantbitar, som är en superposition av två tillstånd. Ofta kan den fysiska realiseringen av en kvantbit vara två energinivåer i ett system, där grundtillståndet motsvarar 0 och det första exciterade tillståndet motsvarar 1. Här är det lätt att se att om systemet exciteras av vår omgivning så förlorar vi den informationen som är lagrad i systemet.

Contents

1 Introduction

As our society is becoming increasingly dependent on technology, the demand for better and more efficient technologies is growing. One kind of technology which has been really important since its conception during the second world war by Alan Turing is computers [1]. The field of computer science has been researched a lot during the second half of the 20th century but has hit a fundamental problem. Our current classical computers have reached a limit where the transistors cannot get any smaller without quantum mechanics causing issues [1]. This has prompted research into quantum technologies such as quantum computers and quantum simulations [1].

To properly understand how quantum mechanical systems work, it is important to understand what happens to them when they are interacted with, during for example a measurement [2]. It is also this interaction with a quantum system, which has prompted many interpretations of quantum mechanics and given rise to what is known as the measurement problem [2]. The measurement problem is a fundamental philosophical problem in quantum mechanics, which arises since a quantum mechanical state evolves deterministically according to the Schrödinger equation, but collapses probabilistically when measured or interacted with [2].

It is also interesting to see how such a system can be manipulated to create a certain state which can be used for a specific purpose. This could include, but is not limited to, creating a qubit state to be used in a quantum computer or a state which can simulate a certain physical system [1]. Here it is important to understand the effect feedback has on a measured system, and how to utilize this to create a desired state [3].

This thesis will look at a quantum harmonic oscillator which is coupled to an environment. Thus creating an open quantum system whose state is temperature dependent. The system will be measured continuously using weak measurements with a feedback loop to control the system. The goal is to see how the system evolves under these conditions, and how the feedback loop can be changed and manipulated to observe different behaviours.

1.1 Outline

The remaining is organized as follows: Section 2 starts by introducing the theoretical framework central to this thesis by first defining what we mean by a quantum harmonic oscillator as well as shortly introducing the density matrix formalism of quantum mechanics. Then, we move on to discuss open quantum systems and the mathematical framework for the evolution of this type of system, and here we define a Markovian master equation in Lindblad form. We then move on to discussing measurements on open systems as well as feedback control, in the same mathematical formalism. In section 3, we use what has been discussed to derive equations of motion for the QHO as well as then computing the energy of the system. These results also allow analysing the stability of the system both with and without feedback. In section 4, there is a discussion of the relevancy of the results obtained in section 3, as well as mentions of a few applications.

2 Theoretical Framework

2.1 Quantum Harmonic Oscillator

The Quantum Harmonic Oscillator (QHO) is a quantum mechanical system useful for many applications. This stems from the fact that many systems can be approximated as harmonic, that is quadratic close to their equilibrium position, and since the QHO is a simple system, which is possible to solve analytically it is a good starting approximation. Let us start by assuming that a particle with mass m is confined in a harmonic potential, then the system has the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right), \tag{1}$$

where $\hat{p}$ and $\hat{x}$ are the momentum and position operators, m is the mass of the particle, and ω is the angular frequency of the oscillator. The operators $\hat{a}^\dagger$ and $\hat{a}$ are the creation and annihilation operators, collectively referred to as the ladder operators, which are defined as

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right) \quad \text{and} \quad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right). \tag{2}$$

These operators can be used to define the number operator $\hat{n} = \hat{a}^\dagger\hat{a}$ which has the number states, or Fock states, $|n\rangle$ as its eigenstates with eigenvalue n [4]. The ladder operators have a useful the commutation relation $\left[\hat{a}, \hat{a}^\dagger\right] = \mathbb{1}$, that follows from the canonical commutation relation $[\hat{x}, \hat{p}] = i\hbar$. The Fock states are also eigenstates to the ladder operators with properties

$$\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle\,, \tag{3}$$

$$\hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle\,, \tag{4}$$

which means that they change the excitation level of the QHO [4].

2.2 Open Quantum Systems

Before introducing open quantum systems, we shortly introduce the language of density matrices. A density matrix describes an ensemble of states defined as

$$\hat{\rho} = \sum_i p_i\,|\psi_i\rangle\,\langle\psi_i|\,, \tag{5}$$

where there is probability p_i for the system to be prepared in the state $|\psi_i\rangle$ [1]. Two conditions imposed on a density matrix is that it I) has unit trace, and II) is positive semi-definite [1]. These conditions ensure that the probabilities are $0 \leq p_i \leq 1$ and $\sum_i p_i = 1$ and that the density matrix is hermitian. With this definition of the density matrix object, we can reformulate the postulates of quantum mechanics allowing for open systems and mixed states [1].

With an open quantum system we mean a quantum system which in some ways interacts with an environment. This interaction could be described as a thermal coupling between the main system and some temperature bath. This will cause the system to be in a thermal equilibrium with the environment if left alone, and therefore it will be dependent on the temperature. Unless the temperature of the bath is zero, the system will be in a mixed state, described by a density matrix $\hat{\rho}$. Notably, if the temperature is zero, the system is purely dissipative since the bath cannot excite the system, and thus energy can only be transferred from the system to the bath. [3]

The thermal coupling to the environment will lead to dissipation of quantum information from the system to the environment. During this process, the system loses coherence. That is, the quantum mechanical properties of the system are lost and a classical description of the state becomes more appropriate. The coherence of the system is manifested in the off-diagonal elements of the density matrix. If the off-diagonal elements are zero, either by dissipation to the environment or by other means of decoherence, the system will exist in a classical probabilistic state, and any superposition of states will be lost. [3]

The combination of the system and environment can be considered a closed system, though more complicated than the main system itself. Then, by performing a partial trace over the environment, a description of the system alone arises at the cost of losing information about the correlation between the two parts [3]. This introduces an uncertainty in the state, and it is therefore necessary to treat the resulting system to be in a mixed state. To describe the evolution of this system with a master equation two approximations about the coupling need to be performed. First, we consider the Born approximation, which says that the coupling between the system and environment is weak enough that only negligible excitations appear in the environment [5]. The other approximation is the Markov approximation saying that the excitations that do appear in the environment will decay much faster than the timescale that the system varies on, and that the system's time evolution is only affected by the current state of the system and not previous states [5]. Together these approximations allow us to write the total density matrix as

$$\rho_{\mathrm{SE}} = \rho_{\mathrm{S}} \otimes \rho_{\mathrm{E}}, \tag{6}$$

where $\otimes$ is the tensor product, and derive a Markovian master equation. For ease of notation we will drop the subscript S and always consider the system unless otherwise stated.

2.2.1 Master Equation

The evolution of an open quantum system can be described by a master equation, which is a differential equation and generalization of the Schrödinger equation to involve open quantum systems instead of pure states [3]. By introducing the superoperator

$$\mathcal{D}[\hat{L}_k]\hat{\rho} = \hat{L}_k\hat{\rho}\hat{L}_k^\dagger - \frac{1}{2}\Big\{\hat{L}_k^\dagger\hat{L}_k, \hat{\rho}\Big\}, \tag{7}$$

where $\hat{L}_k$ are called Lindblad jump operators, the master equation on Lindblad form can be written as

$$\partial_t\hat{\rho} = -\frac{i}{\hbar}\Big[\hat{H}, \hat{\rho}\Big] + \sum_k \gamma_k \mathcal{D}[\hat{L}_k]\hat{\rho}, \tag{8}$$

where $\hat{H}$ is the Hamiltonian of the system, and γ_k are the decay rates of the system, relating the decoherence to the environment depending on the coupling to the system [3]. If $\gamma_k = 0$ for all k, and the coupling to the bath is removed, the equation reduces to the von Neumann equation for a closed quantum system. The remaining term thus describes the unitary time evolution of the system and is the analogue of the Schrödinger equation for the density matrix formalism [3]. At this stage, one might also introduce the Liouvillian superoperator $\mathcal{L}$, and write the master equation more compactly as

$$\partial_t\hat{\rho} = \mathcal{L}\hat{\rho}. \tag{9}$$

This compactness will be useful when considering other types of perturbing effects on the system such as measurements and feedback [3].

In the case considered in this thesis with a QHO coupled to a thermal reservoir we can imagine that we have two types of Lindblad jump operators, on of which tranfer particles into the system and one transfer particles out of the system [4]. As mentioned in Sec. 2.1 the ladder operators can be used to excite or deexcite a system. It is also reasonable to assume that the decay, or the amount of particles flowing between the systems and the environment, is proportional to the thermal occupation $\bar{n}$ defined by

$$\bar{n} = \frac{1}{e^{\hbar\omega/k_{\mathrm{B}}T} - 1}, \tag{10}$$

where T is the temperature of the bath and k_{B} is the Boltzmann constant [4]. One can show that for a QHO coupled to a thermal bath, the Lindblad jump operators can be chosen as $\hat{L}_1 = \hat{a}$ and $\hat{L}_2 = \hat{a}^\dagger$ with coefficients $\gamma_1 = \gamma(\bar{n} + 1)$ and $\gamma_2 = \gamma\bar{n}$, wher γ is a decay rate. We can also note that $\hat{L}_1$ and γ_1 refer to the spontaneous emission from the system to the environment while $\hat{L}_2$ refer to spontaneous absorption from the environment to the system, consistent with what we know about ladded operators from Sec. 2.1 [4]. Notably, for $T = 0$, the thermal occupation is $\bar{n} = 0$ and the system will only exhibit emission and will decay.

2.3 Continuous Measurements

Measurement is a process which allow us to know the state of the system and has the consequence of introduces decoherence, and it is therefore interesting to look at its effects [2]. The simplest view on measurements takes the form of von Neumann measurements. This type of measurement is described by a set of measurement operators which projects the system onto the eigenstates of the observable [3]. This essentially means that all quantum information in the system is lost and full decoherence has happened. By generalizing the measurement theory one can derive what is called Positive Operator-Valued Measurement (POVM) [3].

Since the POVM is not necessarily a projective von Neumann measurement all coherence need not be lost after the measurement. Thus, this opens up for the possibility of performing time-continuous weak measurement [3]. To describe this type of POVM, we first consider a Gaussian measurement operator

$$\hat{K}(z) = \left(\frac{2\bar{\lambda}}{\pi}\right)^{1/4} e^{-\bar{\lambda}(z-\hat{A})^2}, \tag{11}$$

where $\bar{\lambda}$ represents the strength of the measurement, z is a continuous outcome of the measurement, and $\hat{A}$ is the measured observable [3]. We note that the post measurement state of such a measurement is described by

$$\hat{\rho}_{\text{post}} = \frac{\hat{K}(z)\hat{\rho}\hat{K}^{\dagger}(z)}{p(z)}, \tag{12}$$

where the probability is defined as $p(z) = \mathrm{tr}\Big(\hat{K}^{\dagger}(z)\hat{K}(z)\hat{\rho}\Big)$ [3]. In this thesis we will only consider the case where $\hat{A} = \hat{x}$, that is we only measure the position quadrature.

Then by discretizing the time interval to segments of $\mathrm{d}t$ and defining $\bar{\lambda} = \lambda \mathrm{d}t$ we approach a situation where in the limit $\mathrm{d}t \to 0$ all measurements will be weak, and the coherence of the system is minimally affected [3]. Considering the stochastic nature of the process and averaging the possible trajectories one can derive the master equation [3] in Lindblad form to be

$$\partial_t \hat{\rho} = \mathcal{L}\hat{\rho} + \lambda \mathcal{D}[\hat{A}]\hat{\rho}. \tag{13}$$

2.4 Feedback Control

Until this point we have only considered measurements where we omit the information about the measurement outcome. That is, we interact with the system and look at how it evolves due to this interaction on average, instead of looking at the specific outcome of any given measurement [3]. However, now we want to consider feedback control of the system, and thus we will need to include the information about the measurement outcome [3]. By feedback control we mean a process by which we manipulate the evolution of a system using a measurement outcome [6]. Since we are dealing specifically with quantum systems, we can further talk about quantum feedback control, where quantum mechanical effects play a role in the modelling of the feedback mechanism's effect on the system [6]. However, it is worth noting that the physical realization of the feedback mechanism does not necessarily need to be entirely quantum mechanical, but at least part of the mechanism need to incorporate quantum mechanics in its description [6]. Specifically, for a measurement outcome z we will consider a linear feedback modification of $\mathcal{L}$ such that

$$\mathcal{L} \to \mathcal{L} + z\mathcal{K}, \tag{14}$$

where $\mathcal{K}$ is a superoperator describing the feedback on the system [3] which takes the form

$$\mathcal{K}\hat{\rho} = -\frac{i}{\hbar}\left[\hat{H}_{\mathrm{c}}, \hat{\rho}\right], \tag{15}$$

where $\hat{H}_{\mathrm{c}}$ is the control Hamiltonian of the system. We will consider a control Hamiltonian which is linear and has the form

$$\hat{H}_{\mathrm{c}} = f^*\hat{a} + f\hat{a}^\dagger, \tag{16}$$

where f is the feedback amplitude, which essentially changes the system Hamiltonian to

$$\hat{H}_{\mathrm{tot}} = \hat{H}_{\mathrm{QHO}} + \hat{H}_{\mathrm{c}} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + z(f^*\hat{a} + f\hat{a}^\dagger). \tag{17}$$

Thus, for a real f, $\mathfrak{I}\{f\} = 0$, we have $\hat{H}_{\mathrm{c}} \propto \hat{x}$ and for an imaginary f, $\mathfrak{R}\{f\} = 0$, we have $\hat{H}_{\mathrm{c}} \propto -\hat{p}$. If the physical realization of the feedback is for example a laser, then $|f|$ represent the power while $\arg f$ represent the relative phase difference between the oscillator's measured quadrature and the feedback field. Starting from the same place as one derives Eq. (13) we can derive a master equation including feedback [3] to be

$$\partial_t\hat{\rho} = \mathcal{L}\hat{\rho} + \lambda\mathcal{D}[\hat{A}]\hat{\rho} + \frac{1}{2}\mathcal{K}\left\{\hat{A}, \hat{\rho}\right\} + \frac{1}{8\lambda}\mathcal{K}^2\hat{\rho}, \tag{18}$$

where the square on $\mathcal{K}$ means $\mathcal{K}^2\hat{\rho} = \mathcal{K}(\mathcal{K}\hat{\rho})$. The first additional term introduces is the drive arising from the feedback on the system, while the second term is attributed to the noise and fluctuations of the feedback [3]. We can see that as the measurement strength, $\lambda \to \infty$, the noise term vanishes. However, this would also make the measurement term diverge and leave us with a von Neumann measurement.

3 Results

This section looks at the equations of motion and steady-state solutions of a QHO, both with and without feedback. The first subsection deals with a QHO which is measured continuously and without feedback, while the second subsection adds feedback into the scheme.

3.1 Measurement Without Feedback

Consider a QHO described by the Hamiltonian in Eq. (1) which is coupled to a thermal bath with temperature T. If the oscillator's position quadrature is also continuously measured, the evolution of the system can be described by the master equation in Eq. (13) using the Lindblad operators mentioned in Sec. 2.2.1. We want to solve for the Equations of Motion (EOM) when measuring for the position quadrature $\hat{x}$, which are derived from

$$\partial_t \left\langle \hat{x}^2 \right\rangle = \mathrm{tr}\big(\hat{x}^2 \partial_t \hat{\rho}\big), \tag{19}$$

$$\partial_t \left\langle \hat{p}^2 \right\rangle = \mathrm{tr}\big(\hat{p}^2 \partial_t \hat{\rho}\big), \tag{20}$$

$$\partial_t \left\langle \{\hat{x}, \hat{p}\} \right\rangle = \mathrm{tr}(\{\hat{x}, \hat{p}\} \partial_t \hat{\rho}). \tag{21}$$

The choice of looking at the second moments has to do with their relation to the variance and thus the fluctuations of the system. These fluctuations are then related to the energy contained in the oscillator. When looking at feedback control and interesting application is to minimize the energy in the oscillator and, which is thus also an argument for looking at the second moments. Inserting the quantum master equation in Eq. (13) and solving Eqs. (19) to (21) yields the following EOM

$$\partial_t \left\langle \hat{x}^2 \right\rangle = -\gamma \left\langle \hat{x}^2 \right\rangle - \frac{1}{m} \left\langle \{\hat{x}, \hat{p}\} \right\rangle + \frac{\gamma \hbar}{m\omega}(\bar{n} + 1/2), \tag{22}$$

$$\partial_t \left\langle \hat{p}^2 \right\rangle = -\gamma \left\langle \hat{p}^2 \right\rangle + m\omega^2 \left\langle \{\hat{x}, \hat{p}\} \right\rangle + \gamma m\omega\hbar(\bar{n} + 1/2) + \lambda\hbar^2, \tag{23}$$

$$\partial_t \left\langle \{\hat{x}, \hat{p}\} \right\rangle = -\gamma \left\langle \{\hat{x}, \hat{p}\} \right\rangle + 2m\omega^2 \left\langle \hat{x}^2 \right\rangle - \frac{2}{m} \left\langle \hat{p}^2 \right\rangle. \tag{24}$$

For more detailed calculations, see App. B. Performing a change of variable to make the equations dimensionless

$$\tilde{x} = \sqrt{\frac{m\omega}{\hbar}}x \quad \text{and} \quad \tilde{p} = \sqrt{\frac{1}{m\omega\hbar}}p, \tag{25}$$

we can solve for the steady-state, by setting the time derivative to zero. Introducing the quality factor $Q = \omega/\gamma$ we obtain the steady-state solutions

$$\left\langle \tilde{x}^2 \right\rangle_{\mathrm{ss}} = (\bar{n} + 1/2) + \frac{\lambda\hbar}{m\omega^2}\frac{2Q^3}{4Q^2+1}, \tag{26}$$

$$\left\langle \tilde{p}^2 \right\rangle_{\mathrm{ss}} = (\bar{n} + 1/2) + \frac{\lambda\hbar}{m\omega^2}\left(Q - \frac{2Q^3}{4Q^2+1}\right), \tag{27}$$

One interesting aspects of the steady-state solutions is that both equations have identical terms capturing the thermal aspect of the fluctuations. Without measurement, the steady-states are only thermal, which is to be expected. Interestingly, when calculating the energy of the steady state the complicated fraction above cancels, and we are left with a purely linear term in the quality factor.

$$E_{\mathrm{ss}} = \left\langle \tilde{H} \right\rangle_{\mathrm{ss}} = \frac{\hbar\omega}{2}\left(\left\langle \tilde{p}^2 \right\rangle + \left\langle \tilde{x}^2 \right\rangle\right) = \hbar\omega(\bar{n} + 1/2) + \frac{\lambda\hbar^2}{2m\omega}Q \tag{28}$$

Looking at panels **a** and **b** in Fig. 1 one can see that a stronger measurement correlates to the system steady state increasing in energy, as does it for an increasing quality factor. Both also affect the system linearly. Thus, by continuously measuring the system we add energy into it, which make the steady state higher in energy than what the thermal effects from the bath would otherwise place it. That is, if we do not perform any measurement the system would be stable at around $\bar{n} + 1/2$ which for the parameters used here would give $\left\langle \tilde{E} \right\rangle \approx 10\hbar\omega$

In panels **d** and **f** we can see that there is some non-linear behaviour near $Q = 0$. However, due to the approximations made in Sec. 2.2 we cannot trust the results in the region of a low quality factor. This is because this regime having a relatively high coupling to the environment, and thus the excitations in the environment might not decay

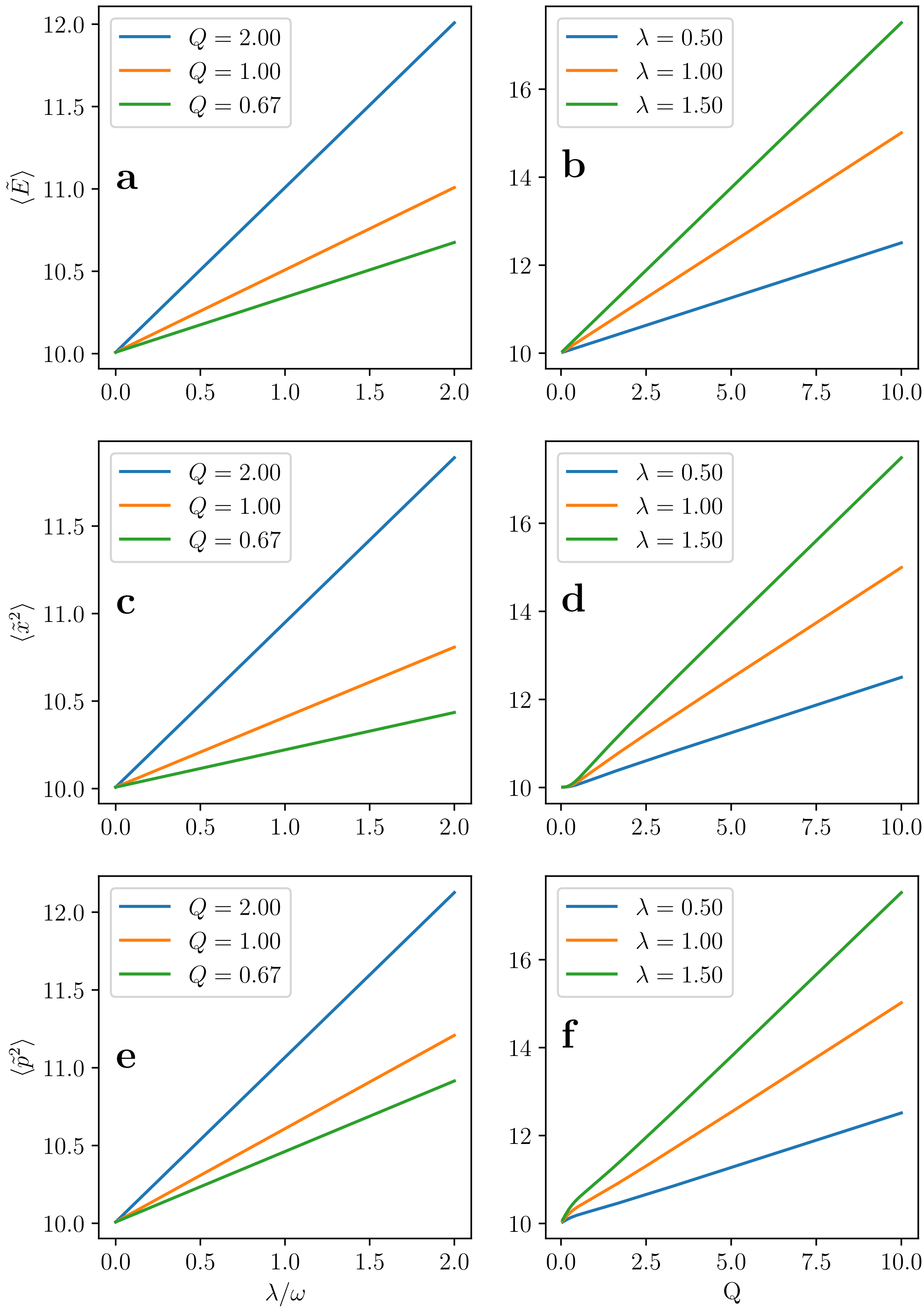


Figure 1: The top panels show Eq. (28), the middle panels show Eq. (26) and the bottom panels show Eq. (27). All plots use the parameters $k_{\mathrm{B}}T = 10$ and $\omega = \hbar = 1$. The left panels are plotted against λ/ω and with three different values for Q, while the right panels are plotted against Q and three different values of λ.

fast enough and therefore affect the oscillator. Panels **c** and **e** show the linear dependence on λ for the second moments.

Using the same approach, inserting the master equation as above we can also solve for the first moments

$$\partial_t \langle \hat{x} \rangle = \mathrm{tr}(\hat{x}\partial_t\hat{\rho}), \tag{29}$$

$$\partial_t \langle \hat{p} \rangle = \mathrm{tr}(\hat{p}\partial_t\hat{\rho}). \tag{30}$$

Solving these equations we find

$$\partial_t \langle \hat{x} \rangle = -\frac{\gamma}{2} \langle \hat{x} \rangle - \frac{1}{m} \langle \hat{p} \rangle, \tag{31}$$

$$\partial_t \langle \hat{p} \rangle = -\frac{\gamma}{2} \langle \hat{p} \rangle + m\omega^2 \langle x \rangle. \tag{32}$$

Then when solving for the steady-state we find

$$\langle \hat{x} \rangle = \langle \hat{p} \rangle = 0. \tag{33}$$

This result also confirms the intuition that for a harmonic oscillator, the system has a steady state around the origin.

It is also important to investigate the stability of the system. That is, if and when the system, in the long time limit, will end up in a steady state, and that the steady-state are stable to small perturbations. To check for stability we can rewrite the equation as an eigenvalue problem and solve for the eigenvalues. That is, for the matrix

$$\mathcal{M} = \begin{pmatrix} -\gamma/2 & -1/m \\ m\omega^2 & -\gamma/2 \end{pmatrix} \tag{34}$$

the eigenvalues are

$$\lambda_1 = -\frac{\gamma}{2} - i\omega, \tag{35}$$

$$\lambda_2 = -\frac{\gamma}{2} + i\omega. \tag{36}$$

Since the real part of the eigenvalues are negative the system is stable. This is because we can write the time-dependant solutions of the dynamical equations as an exponential with the eigenvalues while the coefficients are chosen by the initial condition, the value of the eigenvalues will determine the behaviour of the system. The consequence of the real part of the eigenvalue being negative is that it will make the function decay with time and is thus considered stable. If both eigenvalues are negative the system will decay no matter what the initial conditions are since both terms in the solution will decay. Positive eigenvalues on the other hand will correspond to the system growing with time and will thus be unstable. When one of the eigenvalues is positive and the other is negative the stability of the system will be determined by the initial conditions. That is, the initial condition will determine which eigenvalue will dominate the solution.

Eq. (33) also shows that the variance of the system is only dependent on the first momenta, since

$$\sigma_{\hat{A}}^2 = \left\langle \hat{A}^2 \right\rangle - \left\langle \hat{A} \right\rangle^2 \tag{37}$$

for an operator $\hat{A}$. It is also an easy calculation then to show that for temperature $T = 0$ and without measurement we have equality in the Heisenberg uncertainty relation, $\sigma_{\hat{x}}\sigma_{\hat{p}} = \hbar/2$, justifying the accuracy of the results, and the approximations made in the derivation of the master equation.

3.2 Feedback

We now consider a feedback mechanism on the oscillator described by Eq. (18) which is a linear feedback scheme. Solving for the first moments' EOM we find

$$\partial_t \langle \hat{x} \rangle = -\left(\frac{\gamma}{2} + \frac{2\Im\{f\}}{\sqrt{2m\omega\hbar}} \right) \langle \hat{x} \rangle - \frac{1}{m} \langle \hat{p} \rangle, \tag{38}$$

$$\partial_t \langle \hat{p} \rangle = -\frac{\gamma}{2} \langle \hat{p} \rangle + \left(\Re\{f\} \sqrt{\frac{2m\omega}{\hbar}} + m\omega^2 \right) \langle \hat{x} \rangle. \tag{39}$$

We see that when measuring $\hat{x}$ the feedback terms enter the equations on the position quadrature. A real f, $\Im\{f\} = 0$, removes the feedback for the equation of the position, while an imaginary f, $\Re\{f\} = 0$, removes the feedback for the equation of the momentum. The equations are still coupled however, since the position is dependent on the momentum and vice versa. Setting $f = 0$, yields the previous result.

Choosing f such that

$$\Re\{f\} = -\sqrt{\frac{m\omega^3\hbar}{2}} \quad \text{and} \quad \Im\{f\} = -\frac{\gamma\sqrt{m\omega\hbar}}{2\sqrt{2}} \tag{40}$$

The system of equations reduces to

$$\partial_t \langle \hat{x} \rangle = -\frac{1}{m} \langle \hat{p} \rangle, \tag{41}$$

$$\partial_t \langle \hat{p} \rangle = -\frac{\gamma}{2} \langle \hat{p} \rangle, \tag{42}$$

which has a steady state solution for $\langle \hat{p} \rangle = 0$ and any $\langle \hat{x} \rangle$. To check for stability we can again rewrite the equations as an eigenvalue problem with matrix

$$\mathcal{M} = \begin{pmatrix} -\left(\frac{\gamma}{2} + \frac{2\Im\{f\}}{\sqrt{2m\omega\hbar}} \right) & -\frac{1}{m} \\ \Re\{f\} \sqrt{\frac{2m\omega}{\hbar}} + m\omega^2 & -\frac{\gamma}{2} \end{pmatrix}, \tag{43}$$

which has eigenvalues

$$\varepsilon\pm = \frac{\pm\sqrt{2}\sqrt{\Im\{f\}^2 - 2\sqrt{2}\Re\{f\}\,\omega\hbar\sqrt{\frac{m\omega}{\hbar}} - 2m\omega^3\hbar} - \sqrt{2}\Im\{f\} - \gamma\sqrt{m\omega\hbar}}{2\sqrt{m\omega\hbar}}. \tag{44}$$

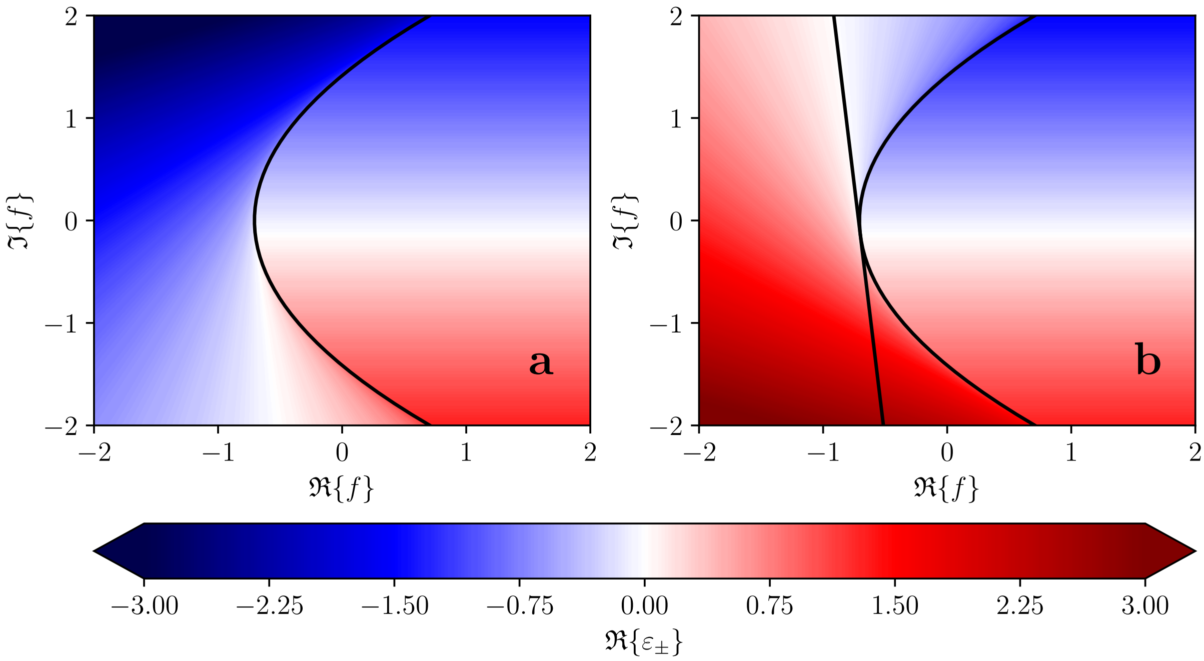


Figure 2: Eq. (44) plotted as a contour plot against the real and imaginary part of f. The parameters used are $m = \omega = \hbar = 1$ and $\gamma = 0.2$. Panel **a** shows $\Re\{\varepsilon_-\}$ and panel **b** shows $\Re\{\varepsilon_+\}$. The parabola that can be seen in both panels is the points where the first square root in Eq. (44) is zero. Thus, points to the left of this parabola are real, while points to the right are complex, only the real part is plotted, however.

Inserting the values of $\Re\{f\}$ and $\Im\{f\}$ from Eq. (40) we find the eigenvalues to be $\varepsilon_- = -\gamma/2$ and $\varepsilon_+ = 0$. It is interesting that this choice of f removes all the oscillatory behaviour from the system, which can be seen by the eigenvalues being real. This puts it in the region of being overdamped by the choice of f. Since one of the eigenvalues are zero, the system is also one dimensional, which is reasonable as the choice of f is such that the first moments' dynamics are independent of the position average.

Looking at Fig. 2 we can see that $\Re\{\varepsilon_-\}$ is mostly negative while $\Re\{\varepsilon_+\}$ is mostly positive for the parameters used in the region closest to the origin. The only part where both eigenvalues are negative is in top right region of the figures. That is, the region where $\Im\{f\} > 0$ and $\Re\{f\} \gtrapprox -0.7$. The reason for the approximation is because it is not an anaytical result, but instead apparent from looking at panel **b**, where we can see that line where the eigenvalue is zero, and has a slant and is not vertical.

Another thing to note is that the eigenvalues are couple to the right of the black parabola in the figure, and thus the solution will have oscillatory motion in this region.

Solving for the EOM for the second moments we obtain

$$\begin{aligned}
\partial_t \left\langle x^2 \right\rangle &= -\left(\gamma + \frac{4\Im\{f\}}{\sqrt{2m\omega\hbar}}\right)\left\langle x^2\right\rangle - \frac{1}{m}\left\langle \{x,p\}\right\rangle + \frac{\gamma\hbar}{m\omega}(\bar{n}+1/2) \\
&\quad - \frac{1}{4\lambda m\omega\hbar}\Re\{f\}^2, \qquad (45)\\
\partial_t \left\langle p^2 \right\rangle &= -\gamma\left\langle p^2\right\rangle + \left(m\omega^2 + \Re\{f\}\sqrt{\frac{2m\omega}{\hbar}}\right)\left\langle \{x,p\}\right\rangle + \gamma m\omega\hbar(\bar{n}+1/2) \\
&\quad + \lambda\hbar^2 + \frac{m\omega}{4\lambda\hbar}\Re\{f\}^2, \qquad (46)\\
\partial_t \left\langle \{x,p\}\right\rangle &= -\left(\gamma + \frac{2\Im\{f\}}{\sqrt{2m\omega\hbar}}\right)\left\langle \{x,p\}\right\rangle + \left(2m\omega^2 + 2\sqrt{\frac{2m\omega}{\hbar}}\Re\{f\}\right)\left\langle x^2\right\rangle - \frac{2}{m}\left\langle p^2\right\rangle \\
&\quad - \frac{\Re\{f\}\Im\{f\}}{2\lambda\hbar}. \qquad (47)
\end{aligned}$$

We can solve for the steady state by first rewriting the system of equations to a matrix equation $\partial_t X = \mathcal{A}X + \mathcal{B}$ where

$$X = \begin{pmatrix} \langle \hat{x}^2\rangle \\ \langle \hat{p}^2\rangle \\ \langle\{\hat{x},\hat{p}\}\rangle \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} -\left(\gamma + \frac{4\Im\{f\}}{\sqrt{2m\omega\hbar}}\right) & 0 & -\frac{1}{m} \\ 0 & -\gamma & m\omega^2 + \Re\{f\}\sqrt{\frac{2m\omega}{\hbar}} \\ 2m\omega^2 + 2\sqrt{\frac{2m\omega}{\hbar}}\Re\{f\} & -\frac{2}{m} & -\left(\gamma + \frac{2\Im\{f\}}{\sqrt{2m\omega\hbar}}\right) \end{pmatrix} \qquad (48)$$

and

$$\mathcal{B} = \begin{pmatrix} \frac{\gamma\hbar}{m\omega}(\bar{n}+1/2) - \frac{\Re\{f\}^2}{4\lambda m\omega\hbar} \\ \gamma m\omega\hbar(\bar{n}+1/2) + \lambda\hbar^2 + \frac{m\omega\Re\{f\}^2}{4\lambda\hbar} \\ -\frac{\Re\{f\}\Im\{f\}}{2\lambda\hbar} \end{pmatrix}. \qquad (49)$$

We notice the addition of a constant term in all three equations corresponding to the noise of the feedback, that is, all these terms comes from the second feedback term in Eq. (18). Furthermore, we see that a driving feedback term has entered all three equations on the position quadrature and the anticommutator, which contains $\hat{x}$. It's worth noting the similarities of the terms between equations.

To obtain the steady-state solutions we solve $X_{\text{ss}} = -\mathcal{A}^{-1}\mathcal{B}$. To simplify this we

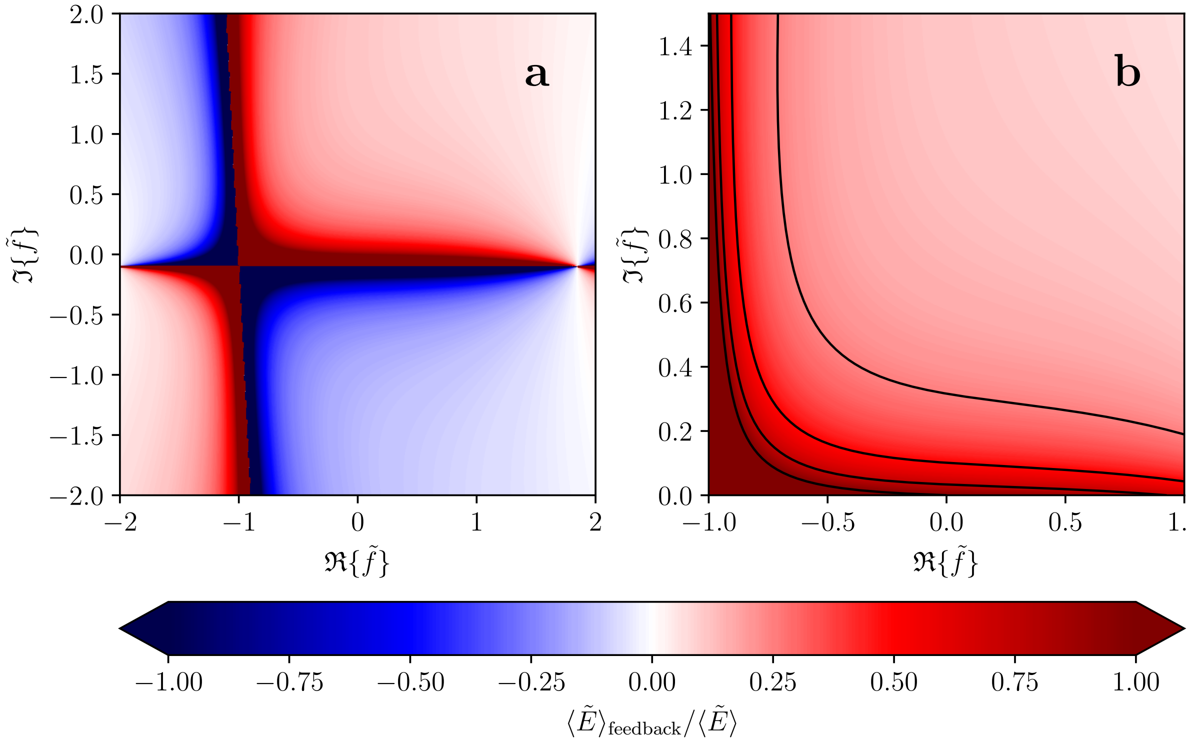


Figure 3: The ratio between the energy with feedback and without plotted as a contour plot against $\Re\left\{\tilde{f}\right\}$ and $\Im\left\{\tilde{f}\right\}$. The parameters used are $k_\mathrm{B}T = 10\hbar\omega$, $Q = 10$, $\Lambda = 2$. Panel **a** shows a large variation of the parameters. There are divergences in the plot which almost follow a $\left(\Re\left\{\tilde{f}\right\}+1\right)\Im\left\{\tilde{f}\right\} = 1$ curve. Panel **b** is a zoomed in version of panel **a** and shows the behaviour of the system in a region where the ratio always is positive. The contour lines in panel **b** are placed at the values of the tick marks in the colourbar.

perform the same change of variables as before, in addition to

$$\tilde{f} = \frac{f}{\omega\sqrt{m\omega\hbar}}, \quad \Lambda = \frac{\lambda\hbar}{m\omega^2}. \tag{50}$$

The full analytical solutions are not shown here due to the length of the equations. It would also be possible to solve the matrix equation numerically after the change of variables. To get an understanding of the steady state solution we look at figures generated by the solution.

In Fig. 3 we can see the effect the feedback has on the energy of the system. Looking at panel **a** we can see areas which are negative. If the energy of the system without measurement is constant, it must be the feedback energy that becomes negative. In the way we have set up the model, a negative energy is unreasonable, since a temperature

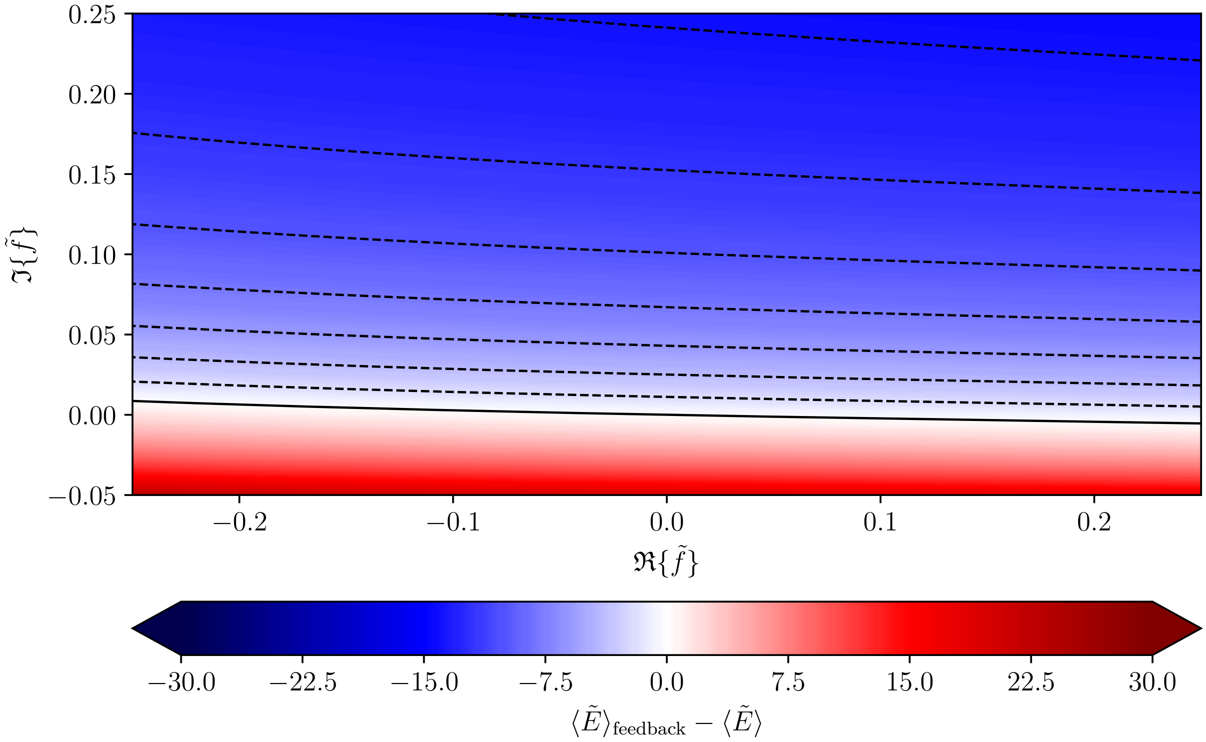


Figure 4: The difference between the energy of the system with feedback plotted and the energy without feedback plotteda against $\Re\left\{\tilde{f}\right\}$ and $\Im\left\{\tilde{f}\right\}$. The parameters used are $k_{\mathrm{B}}T = 10\hbar\omega$, $Q = 10$, $\Lambda = 2$. The plot is done in a small region around the origin, thus showing the effect a small feedback has on the system. The spacing between the contour lines is 2 and the solid lines is at 0.

of $T = 0$ would correspond to $E = \hbar\omega(1/2 + \Lambda Q/2)$, which without measurement would mean an energy of $E = \hbar\omega/2$. A possible explantation for the negative energy is that the specific feedback that give rise to this is affecting the system in such a way that at least one of our assumptions is no longer accurate. This could, for example, be that the system no longer have a physical steady state. Another reason might be that the Markovian approximation no longer holds.

Another interesing aspect of Fig. 3 is that when looking at panel **b** in conjunction with Eq. (28) is that it is possible to cool the system to a lower energy than the thermal energy. Using the same parameters in Eq. (28) as are used in the figure we find that the thermal energy is half of the total energy in the system without feedback, and looking at the figure we see that there exist a region which has a ratio of less than 0.5.

Using the parameters as in Fig. 4 the energy without feedback is $\left\langle\tilde{E}\right\rangle \approx 20$ with the thermal part accounting for around 10 of that. Looking at Fig. 4 it is possible to see that even with a relatively low amount of feedback the system will be cooled to some degree. However, depending on how the feedback is applied it can also make the energy in the system diverge which can be seen for $\Im\left\{\tilde{f}\right\} < 0$.

The stability for the steady states of the system can be checked by rewriting Eqs. (45) to (47) and using the Lyapunov equation

$$\mathcal{M}\partial_t S + \partial_t S \mathcal{M}^T + \mathcal{C} = 0 \tag{51}$$

where

$$S = \begin{pmatrix} \langle \hat{x}^2 \rangle & \langle \{\hat{x}, \hat{p}\} \rangle /2 \\ \langle \{\hat{x}, \hat{p}\} \rangle /2 & \langle \hat{p}^2 \rangle \end{pmatrix}, \tag{52}$$

$$\mathcal{C} = \begin{pmatrix} \frac{\gamma\hbar}{m\omega}(\bar{n} + 1/2) - \frac{\Re\{f\}^2}{4\lambda m\omega\hbar} & -\frac{\Re\{f\}\Im\{f\}}{4\lambda\hbar} \\ -\frac{\Re\{f\}\Im\{f\}}{4\lambda\hbar} & \gamma m\omega\hbar(\bar{n} + 1/2) + \lambda\hbar^2 + \frac{m\omega\Re\{f\}^2}{4\lambda\hbar} \end{pmatrix}, \tag{53}$$

and $\mathcal{M}$ is the same matrix as in Eq. (43). S has unique steady state solutions if the real part of the eigenvalues of $\mathcal{M}$ are negative [7]. We can therefore refer back to Eq. (44) and Fig. 2 to see the stability of the system. In conjunction with Fig. 3 and 4 we can see that the system is stable in the region where we cool the system. Thus, it is not only possible to cool the system but the system is also stable. We can therefore see that this type of feedback mechanism is a valid way to cool a QHO.

4 Discussion and Conclusions

The results presented in this thesis show the possibility of cooling a QHO using a measurement and feedback loop. The discussion analyses the results, comparing to previous work as well as discussing different applications of cooled QHOs. The stability of the system under feedback is also discussed, as well as mentioning other ways to affect the system using feedback.

4.1 Cooling of a Quantum Harmonic Oscillator

Technologies relying on quantum phonemona such as quantum computers, generally have a need for a low temperature system [1]. It is therefore an interesting and active research topic to find ways to cool a quantum system. Currently there exist multiple types of cooling, and the work perfomed in this thesis is in the form of measurement and feedback based cooling, similar to [8].

As briefely mentioned in Sec. 2.4, the physical realisation of the feedback scheme could be a driving laser acting on the system. The successfullness of the cooling would then also depend on the possibilites to physically control the parameters. Especially the feedback strength and phase. It is especially apparent from the results in Fig. 3 that the phase plays a large role in if the feedback is successful, which can be seen from quadrant 2 and 4 having negative energy, which is not physical. The explanation for this is that the system most likely breaks down and for a steady state to exist given these parameters the temperature would have to be negative. Thus, the only physically relevant solutions is in quadrant 1 and 3. We can then also conclude that in general a functionate feedback scheme has a control hamiltonian which is either proportional to $\hat{x} - \hat{p}$ or $-\hat{x} + \hat{p}$. It is possible to have a feedback hamiltonian proportional to $-\hat{x} - \hat{p}$ but this will not be as effective at cooling the system. In panel **b** in Fig. 3 this area can be seen for $\Re\left\{\tilde{f}\right\} < 0$. So even if it is possible to cool in this area, the same amplitude but smaller phase would yield a better result.

The feedback employed in this thesis takes the form of an external force acting on the system, for example a laser interacting with the field inside a cavity, which linearly changes the system hamiltonian. A different way to cool the system would be to modify the frequency of the oscillator [9], which has applications in quantum optomechanics. Quantum optomechanics has the ability to use light to prepare macroscopic objects in quantum states [10], allowing for quantum mechanical control over macroscopic systems, which have application in for example LIGO. Changing the frequence could prove more difficult to model since the feedback would be non-linear. It's therefore interesting to ask if the results in this thesis, and the feedback scheme used, could be realized in optomechanical systems.

We only consider the case of infinite detector bandwidth and instantaneous feedback in this thesis. In reality, this idealized picture would not hold, and the effectivness of the feedback scheme might decrease [3], affecting the efficiency of the cooling. The addition of finite bandwidth is a new addition in the field of quantum feedback control, with [8] being one of the first to consider this.

Increasing the quality factor of the system has a positive impact on the effectivness of the cooling, allowing smaller feedback parameters to have greater control of the system. A consequence however, is that the system will also be more sensitive to the feedback and measurement. Thus, if the parameters used are prone to noise, the system will vary more than with a lower quality factor.

To get an accurate conclusion about the usefulness of the feedback scheme, it is important to conisder the stability of the system in the physical regions. Thus, it is not certain that just because there exist a steady state solution, we have a stable solution at that point.

4.2 Stability with Feedback

The stability of a quantum system is of importance when considering applications based on said system. That is, a stable system can be used and its properties exploited, while an unstable system might behave unpredicatbly, and cause issues. Specifically, we call a system stable, if in the long time limit it decays to a specific state, and does not diverge from this if affected by a small disturbance, barring any external factors which may affect the stability. In the case of the system modeled in this thesis it means that the expectation value of the variance in the sytem decays to a finite value.

We can see from the results in Fig. 2 that the stability of the system is highly dependent on the feedback parameter. It is also important to note that when looking at Fig. 3 only quadrant 1 is in the region of stability of the system. Thus, even if quadrant 3 cools the system, the state will not be stable and it would be difficult to isolate the system from external fluctuations and keep it in this state. This essentially solidifies the fact that the only relevant feedback parameters are those which are in quadrant 1.

4.3 Applications

One interesting application of QHOs is in the field of quantum computation. A particular application is the use of QHOs in conjunction with qubits to create higher dimensional systems, qudits, to be used for calculations [11]. A problem noted in [11] is that when coupled to a bath, which all physical system are, the qudit will experience noise due to the fluctuations of the bath, which would require quantum error correction to combat [11]. However, with too much noise these protocols could fail, and thus cooling the system may help by reducing the noise.

Another interesting use for QHOs is as thermal baths themselves inside quantum circuits as described by [12]. In order to realize this thermalization an ensamble of QHOs would be needed with variable energies [12]. The results in this thesis show that a possible protocol to control the temperature of a QHO which therefore would allow QHOs to be used in this way. Quantum circuits themselves are interesting since they are a way to physically realize a quantum computer [1].

Quantum sensing

5 Outlook

The field of measurement and feedback based cooling seems to be a promising way to cool quantum systems. As mentioned in the discussion, implementing detectors with finite bandwidth is an important consideration for future research as this would more accurately model a real system.

Furthermore, this thesis only consider one type of feedback scheme, looking at the feedback hamiltonian as a linear modification of the position and momentum quadratures. There is promising work to be done either considering modifying the position of the potential akin to [8], or modifying the frequency of the oscillator similar to [9]. The latter would be interesting as it seems to have a more direct application in quantum optomechanics.

References

[1] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information: 10th Anniversary Edition*. Cambridge: Cambridge University Press, 2010.

[2] A. N. Jordan and I. A. Siddiqi, *Quantum Measurement: Theory and Practice*. Cambridge: Cambridge University Press, 2024.

[3] B. Annby-Andersson, *Continuous measurements of small systems: Feedback control, thermodynamics, entanglement*. Doctoral thesis (compilation), Lund University, 4 2024.

[4] P. Meystre, *Quantum Optics: Taming the Quantum*. Cham: Springer, 2021.

[5] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems*. Oxford: Oxford University Press, 2007.

[6] H. M. Wiseman and G. J. Milburn, *Quantum Measurement and Control*. Cambridge: Cambridge University Press, 2009.

[7] A. Purkayastha, "Lyapunov equation in open quantum systems and non-hermitian physics," *Physical Review A*, vol. 105, pp. 062204–, 06 2022.

[8] G. De Sousa, P. Bakhshinezhad, B. Annby-Andersson, P. Samuelsson, P. P. Potts, and C. Jarzynski, "Continuous feedback protocols for cooling and trapping a quantum harmonic oscillator," *Physical Review E*, vol. 111, pp. 014152–, 01 2025.

[9] H. Habibi, E. Zeuthen, M. Ghanaatshoar, and K. Hammerer, "Quantum feedback cooling of a mechanical oscillator using variational measurements: tweaking heisenberg's microscope," *Journal of Optics*, vol. 18, no. 8, p. 084004, 2016.

[10] Y. Chen, "Macroscopic quantum mechanics: theory and experimental concepts of optomechanics," *Journal of Physics B: Atomic, Molecular and Optical Physics*, vol. 46, no. 10, p. 104001, 2013.

[11] Y. Liu, J. Sinanan-Singh, M. T. Kearney, G. Mintzer, and I. L. Chuang, "Constructing qudits from infinite-dimensional oscillators by coupling to qubits," *Physical Review A*, vol. 104, pp. 032605–, 09 2021.

[12] J. P. Pekola and B. Karimi, "Heat bath in a quantum circuit," *Entropy*, vol. 26, no. 5, 2024.

A Appendix

B EOM Calculation Without Feedback

Consider a quantum harmonic oscillator described by the Hamiltonian

$$\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + 1/2). \tag{54}$$

Consider also that the quantum harmonic oscillator is subject to a temperature bath and is continuously measured. The system then evolves according to the master equation

$$\partial_t\hat{\rho} = -\frac{i}{\hbar}[\hat{H},\hat{\rho}] + \gamma(\bar{n}+1)\mathcal{D}[\hat{a}]\hat{\rho} + \gamma\bar{n}\mathcal{D}[\hat{a}^\dagger]\hat{\rho} + \lambda\mathcal{D}[\hat{A}]\hat{\rho} \tag{55}$$

where γ is the damping rate, $\bar{n}$ is the thermal occupancy defined as

$$\bar{n} = \frac{1}{e^{\hbar\omega/k_BT} - 1}, \tag{56}$$

λ is the measurement rate, $\hat{A}$ is the measurement operator, and $\mathcal{D}[\hat{O}]$ is the Lindblad super operator defined as

$$\mathcal{D}[\hat{O}]\hat{\rho} = \hat{O}\hat{\rho}\hat{O}^\dagger - \frac{1}{2}\{\hat{O}^\dagger\hat{O},\hat{\rho}\}. \tag{57}$$

Then we also have the quadrature operators defined as

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}+\hat{a}^\dagger), \quad \hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a}-\hat{a}^\dagger). \tag{58}$$

We want to find the equations of motion when measuring the position quadrature

$$\partial_t\left\langle\hat{x}^2\right\rangle = \mathrm{tr}\big(\hat{x}^2\partial_t\hat{\rho}\big) \tag{59}$$

$$\partial_t\left\langle\hat{p}^2\right\rangle = \mathrm{tr}\big(\hat{p}^2\partial_t\hat{\rho}\big) \tag{60}$$

$$\partial_t\left\langle\{\hat{x},\hat{p}\}\right\rangle = \mathrm{tr}(\{\hat{x},\hat{p}\}\partial_t\hat{\rho}). \tag{61}$$

We start by defining the Liouvillian super operator

$$\mathcal{L}\hat{\rho} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}] + \gamma(\bar{n}+1)\mathcal{D}[\hat{a}]\hat{\rho} + \gamma\bar{n}\mathcal{D}[\hat{a}^\dagger]\hat{\rho} \tag{62}$$

which turns the master equation into

$$\partial_t\hat{\rho} = \mathcal{L}\hat{\rho} + \lambda\mathcal{D}[\hat{x}]\hat{\rho}. \tag{63}$$

B.1 Solving for $\partial_t \left\langle \hat{x}^2 \right\rangle$

We can write Eq. (59) as

$$\partial_t \left\langle \hat{x}^2 \right\rangle = \mathrm{tr}\big(\hat{x}^2\mathcal{L}\hat{\rho}\big) + \mathrm{tr}\big(\hat{x}^2\lambda\mathcal{D}[\hat{x}]\hat{\rho}\big) \tag{64}$$

and then rewriting the left term using the creation and annihilation operators we get

$$\begin{aligned}
\mathrm{tr}\big(\hat{x}^2\mathcal{L}\hat{\rho}\big) &= \frac{\hbar}{2m\omega}\,\mathrm{tr}\big((\hat{a}^2 + (\hat{a}^\dagger)^2 + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a})\mathcal{L}\hat{\rho}\big) && (65)\\
&= \frac{\hbar}{2m\omega}\left[\mathrm{tr}\big(\hat{a}^2\mathcal{L}\hat{\rho}\big) + \mathrm{tr}\big((\hat{a}^\dagger)^2\mathcal{L}\hat{\rho}\big) + \mathrm{tr}\big(\hat{a}\hat{a}^\dagger\mathcal{L}\hat{\rho}\big) + \mathrm{tr}\big(\hat{a}^\dagger\hat{a}\mathcal{L}\hat{\rho}\big)\right] && (66)
\end{aligned}$$

Solving these separately

$$\begin{aligned}
\mathrm{tr}\big(\hat{a}^2\mathcal{L}\hat{\rho}\big) &= -\frac{i}{\hbar}\,\mathrm{tr}\Big(\hat{a}^2\Big[\hat{H}, \hat{\rho}\Big]\Big) + \gamma(\bar{n}+1)\,\mathrm{tr}\big(\hat{a}^2\mathcal{D}[\hat{a}]\hat{\rho}\big) + \gamma\bar{n}\,\mathrm{tr}\big(\hat{a}^2\mathcal{D}[\hat{a}^\dagger]\hat{\rho}\big) && (67)\\
&= -i\omega\,\mathrm{tr}\big(\hat{a}^2\hat{a}^\dagger\hat{a}\hat{\rho}\big) + i\omega\,\mathrm{tr}\big(\hat{a}^2\hat{\rho}\hat{a}^\dagger\hat{a}\big) && (68)\\
&+ \gamma(\bar{n}+1)\,\mathrm{tr}\big(\hat{a}^3\hat{\rho}\hat{a}^\dagger\big) - \frac{\gamma(\bar{n}+1)}{2}\,\mathrm{tr}\big(\hat{a}^2\hat{a}^\dagger\hat{a}\hat{\rho}\big) - \frac{\gamma(\bar{n}+1)}{2}\,\mathrm{tr}\big(\hat{a}^2\hat{\rho}\hat{a}^\dagger\hat{a}\big) && (69)\\
&+ \gamma\bar{n}\,\mathrm{tr}\big(\hat{a}^2\hat{a}^\dagger\hat{\rho}\hat{a}\big) - \frac{\gamma\bar{n}}{2}\,\mathrm{tr}\big(\hat{a}^3\hat{a}^\dagger\hat{\rho}\big) - \frac{\gamma\bar{n}}{2}\,\mathrm{tr}\big(\hat{a}^2\hat{\rho}\hat{a}\hat{a}^\dagger\big) && (70)
\end{aligned}$$

Using cyclic permutations of the trace and combining terms we get

$$\mathrm{tr}\big(\hat{a}^2\mathcal{L}\hat{\rho}\big) = i\omega\,\mathrm{tr}\big(\big[\hat{a}^\dagger\hat{a}, \hat{a}^2\big]\hat{\rho}\big) + \frac{\gamma(\bar{n}+1)}{2}\,\mathrm{tr}\big(\big[\hat{a}^\dagger\hat{a}, \hat{a}^2\big]\hat{\rho}\big) + \frac{\gamma\bar{n}}{2}\,\mathrm{tr}\big(\big[\hat{a}^2, \hat{a}\hat{a}^\dagger\big]\hat{\rho}\big) \tag{71}$$

The commutators are

$$\left[\hat{a}^\dagger\hat{a}, \hat{a}^2\right] = \left[\hat{a}^\dagger, \hat{a}^2\right]\hat{a} = \hat{a}\left[\hat{a}^\dagger, \hat{a}\right]\hat{a} + \left[\hat{a}^\dagger, \hat{a}\right]\hat{a}^2 = -2\hat{a}^2 \tag{72}$$

$$\left[\hat{a}^2, \hat{a}\hat{a}^\dagger\right] = \hat{a}\left[\hat{a}^2, \hat{a}^\dagger\right] = \hat{a}^2\left[\hat{a}, \hat{a}^\dagger\right] + \hat{a}\left[\hat{a}, \hat{a}^\dagger\right]\hat{a} = 2\hat{a}^2 \tag{73}$$

Thus

$$\operatorname{tr}\left(\hat{a}^2\mathcal{L}\hat{\rho}\right) = -2i\omega\operatorname{tr}\left(\hat{a}^2\hat{\rho}\right) - \gamma(\bar{n}+1)\operatorname{tr}\left(\hat{a}^2\hat{\rho}\right) + \gamma\bar{n}\operatorname{tr}\left(\hat{a}^2\hat{\rho}\right) = -(\gamma + 2i\omega)\operatorname{tr}\left(\hat{a}^2\hat{\rho}\right) \tag{74}$$

$$= -(\gamma + 2i\omega)\left\langle\hat{a}^2\right\rangle. \tag{75}$$

By taking the hermition conjugate of the above we get

$$\operatorname{tr}\left((\hat{a}^\dagger)^2\mathcal{L}\hat{\rho}\right) = (\gamma - 2i\omega)\operatorname{tr}\left((\hat{a}^\dagger)^2\hat{\rho}\right) = -(\gamma - 2i\omega)\left\langle(\hat{a}^\dagger)^2\right\rangle. \tag{76}$$

Now we do the same for $\operatorname{tr}\left(\hat{a}^\dagger\hat{a}\mathcal{L}\hat{\rho}\right)$, so

$$\operatorname{tr}\left(\hat{a}^\dagger\hat{a}\mathcal{L}\hat{\rho}\right) = -\frac{i}{\hbar}\operatorname{tr}\left(\hat{a}^\dagger\hat{a}\left[\hat{H}, \hat{\rho}\right]\right) + \gamma(\bar{n}+1)\operatorname{tr}\left(\hat{a}^\dagger\hat{a}\mathcal{D}[\hat{a}]\hat{\rho}\right) + \gamma\bar{n}\operatorname{tr}\left(\hat{a}^\dagger\hat{a}\mathcal{D}[\hat{a}^\dagger]\hat{\rho}\right) \tag{77}$$

$$= -i\omega\operatorname{tr}\left(\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}\hat{\rho}\right) + i\omega\operatorname{tr}\left(\hat{a}^\dagger\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a}\right) \tag{78}$$

$$+ \gamma(\bar{n}+1)\operatorname{tr}\left(\hat{a}^\dagger\hat{a}^2\hat{\rho}\hat{a}^\dagger\right) - \frac{\gamma(\bar{n}+1)}{2}\operatorname{tr}\left(\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}\hat{\rho}\right) - \frac{\gamma(\bar{n}+1)}{2}\operatorname{tr}\left(\hat{a}^\dagger\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a}\right) \tag{79}$$

$$+ \gamma\bar{n}\operatorname{tr}\left(\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{\rho}\hat{a}\right) - \frac{\gamma\bar{n}}{2}\operatorname{tr}\left(\hat{a}^\dagger\hat{a}^2\hat{a}^\dagger\hat{\rho}\right) - \frac{\gamma\bar{n}}{2}\operatorname{tr}\left(\hat{a}^\dagger\hat{a}\hat{\rho}\hat{a}\hat{a}^\dagger\right) \tag{80}$$

using cyclic permutations of the trace and combining terms we get

$$\operatorname{tr}\left(\hat{a}^\dagger\hat{a}\mathcal{L}\hat{\rho}\right) = i\omega\operatorname{tr}\left(\left[\hat{a}^\dagger\hat{a}, \hat{a}^\dagger\hat{a}\right]\hat{\rho}\right) + \gamma(\bar{n}+1)\operatorname{tr}\left((\hat{a}^\dagger)^2\hat{a}^2\hat{\rho}\right) - \gamma(\bar{n}+1)\operatorname{tr}\left(\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}\hat{\rho}\right) \tag{81}$$

$$+ \gamma\bar{n}\operatorname{tr}\left(\hat{a}\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{\rho}\right) - \frac{\gamma\bar{n}}{2}\operatorname{tr}\left(\hat{a}^\dagger\hat{a}^2\hat{a}^\dagger\hat{\rho}\right) - \frac{\gamma\bar{n}}{2}\operatorname{tr}\left(\hat{a}(\hat{a}^\dagger)^2\hat{a}\hat{\rho}\right) \tag{82}$$

Commuting to simplify we get

$$\operatorname{tr}\left(\hat{a}^\dagger\hat{a}\mathcal{L}\hat{\rho}\right) = -\gamma(\bar{n}+1)\operatorname{tr}\left(\hat{a}^\dagger\hat{a}\hat{\rho}\right) - \gamma\bar{n}\operatorname{tr}\left(\left[\hat{a}\hat{a}^\dagger, \hat{a}^\dagger\hat{a}\right]\hat{\rho}\right) + \gamma\bar{n}\operatorname{tr}\left(\hat{a}\hat{a}^\dagger\hat{\rho}\right) \tag{83}$$

The commutator is equal to 0, and thus we get

$$\operatorname{tr}\left(\hat{a}^\dagger \hat{a} \mathcal{L} \hat{\rho}\right) = -\gamma(\bar{n}+1)\operatorname{tr}\left(\hat{a}^\dagger \hat{a} \hat{\rho}\right) + \gamma\bar{n} + \gamma\bar{n}\operatorname{tr}\left(\hat{a}^\dagger \hat{a} \hat{\rho}\right) = -\gamma\operatorname{tr}\left(\hat{a}^\dagger \hat{a} \hat{\rho}\right) + \gamma\bar{n} = -\gamma\left\langle \hat{a}^\dagger \hat{a}\right\rangle + \gamma\bar{n}. \tag{84}$$

Similarly we can find that

$$\operatorname{tr}\left(\hat{a}\hat{a}^\dagger \mathcal{L} \hat{\rho}\right) = -\gamma\operatorname{tr}\left(\hat{a}\hat{a}^\dagger \hat{\rho}\right) + \gamma(\bar{n}+1) = -\gamma\left\langle \hat{a}\hat{a}^\dagger\right\rangle + \gamma(\bar{n}+1). \tag{85}$$

Then since $\hat{x}^2$ commutes with $\hat{x}$ (the measurement operator) we can write Eq. (64) the equation of motion

$$\begin{aligned}
\partial_t\left\langle \hat{x}^2\right\rangle &= \operatorname{tr}\left(\hat{x}^2 \mathcal{L}\hat{\rho}\right) &(86)\\
&= \frac{\hbar}{2m\omega}\left(-(\gamma+2i\omega)\left\langle \hat{a}^2\right\rangle - (\gamma-2i\omega)\left\langle (\hat{a}^\dagger)^2\right\rangle - \gamma\left\langle \hat{a}^\dagger\hat{a}\right\rangle + \gamma\bar{n} - \gamma\left\langle \hat{a}\hat{a}^\dagger\right\rangle + \gamma(\bar{n}+1)\right) &(87)\\
&= -\gamma\frac{\hbar}{2m\omega}\left(\left\langle \hat{a}^2\right\rangle + \left\langle (\hat{a}^\dagger)^2\right\rangle + \left\langle \hat{a}^\dagger\hat{a}\right\rangle + \left\langle \hat{a}\hat{a}^\dagger\right\rangle\right) + \frac{i\hbar}{m}\left(\left\langle (\hat{a}^\dagger)^2\right\rangle - \left\langle \hat{a}^2\right\rangle\right) + \frac{\gamma\hbar}{m\omega}(\bar{n}+1/2) &(88)
\end{aligned}$$

Since we have

$$\{\hat{x},\hat{p}\} = i\hbar(\hat{a}^2 - (\hat{a}^\dagger)^2) \tag{89}$$

We can write the equation of motion as

$$\partial_t\left\langle \hat{x}^2\right\rangle = -\gamma\left\langle \hat{x}^2\right\rangle - \frac{1}{m}\left\langle \{\hat{x},\hat{p}\}\right\rangle + \frac{\gamma\hbar}{m\omega}(\bar{n}+1/2) \tag{90}$$

B.2 Solving for $\partial_t\left\langle \hat{p}^2\right\rangle$

Writing Eq. (60) as

$$\partial_t\left\langle \hat{p}^2\right\rangle = \operatorname{tr}\left(\hat{p}^2\mathcal{L}\hat{\rho}\right) + \operatorname{tr}\left(\hat{p}^2\lambda\mathcal{D}[\hat{x}]\hat{\rho}\right) \tag{91}$$

Rewriting the left terms using $\hat{a}, \hat{a}^\dagger$ we get

$$\operatorname{tr}\big(\hat{p}^2\mathcal{L}\hat{\rho}\big) = -\frac{m\omega\hbar}{2}\operatorname{tr}\big((\hat{a}^2 + (\hat{a}^\dagger)^2 - \hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger)\mathcal{L}\hat{\rho}\big) \tag{92}$$

$$= -\frac{m\omega\hbar}{2}\left(\operatorname{tr}\big(\hat{a}^2\mathcal{L}\hat{\rho}\big) + \operatorname{tr}\big((\hat{a}^\dagger)^2\mathcal{L}\hat{\rho}\big) - \operatorname{tr}\big(\hat{a}^\dagger\hat{a}\mathcal{L}\hat{\rho}\big) - \operatorname{tr}\big(\hat{a}\hat{a}^\dagger\mathcal{L}\hat{\rho}\big)\right) \tag{93}$$

We have solved for these before, so we can write

$$\operatorname{tr}\big(\hat{p}^2\mathcal{L}\hat{\rho}\big) = -\frac{m\omega\hbar}{2}\left(-(\gamma + 2i\omega)\left\langle \hat{a}^2\right\rangle - (\gamma - 2i\omega)\left\langle (\hat{a}^\dagger)^2\right\rangle + \gamma\left\langle \hat{a}^\dagger\hat{a}\right\rangle - \gamma\bar{n} + \gamma\left\langle \hat{a}\hat{a}^\dagger\right\rangle - \gamma(\bar{n}+1)\right) \tag{94}$$

$$= -\gamma\left\langle \hat{p}^2\right\rangle + mi\omega^2\hbar\left(\left\langle \hat{a}^2\right\rangle - \left\langle (\hat{a}^\dagger)^2\right\rangle\right) + \gamma m\omega\hbar(\bar{n}+1/2) \tag{95}$$

$$= -\gamma\left\langle \hat{p}^2\right\rangle + m\omega^2\left\langle \{\hat{x},\hat{p}\}\right\rangle + \gamma m\omega\hbar(\bar{n}+1/2) \tag{96}$$

Solving the measurement term

$$\operatorname{tr}\big(\hat{p}^2\lambda\mathcal{D}[\hat{x}]\hat{\rho}\big) = \lambda\left(\operatorname{tr}\big(\hat{p}^2\hat{x}\hat{\rho}\hat{x}\big) - \frac{1}{2}\operatorname{tr}\big(\hat{p}^2\hat{x}^2\hat{\rho}\big) - \frac{1}{2}\operatorname{tr}\big(\hat{p}^2\hat{\rho}\hat{x}^2\big)\right) \tag{97}$$

$$= \lambda\left(\operatorname{tr}\big(\hat{x}\hat{p}^2\hat{x}\hat{\rho}\big) - \frac{1}{2}\operatorname{tr}\big(\hat{p}^2\hat{x}^2\hat{\rho}\big) - \frac{1}{2}\operatorname{tr}\big(\hat{x}^2\hat{p}^2\hat{\rho}\big)\right) \tag{98}$$

Commuting the operators

$$\hat{x}\hat{p}^2\hat{x} = \frac{1}{2}(\hat{p}\hat{x} + i\hbar)\hat{p}\hat{x} + \frac{1}{2}\hat{x}\hat{p}(\hat{x}\hat{p} - i\hbar) = \frac{1}{2}\hat{p}\hat{x}\hat{p}\hat{x} + \frac{1}{2}\hat{x}\hat{p}\hat{x}\hat{p} + \frac{i\hbar}{2}[\hat{p},\hat{x}] \tag{99}$$

$$= \frac{1}{2}\hat{p}\hat{x}\hat{p}\hat{x} + \frac{1}{2}\hat{x}\hat{p}\hat{x}\hat{p} + \frac{\hbar^2}{2} \tag{100}$$

$$\hat{p}^2\hat{x}^2 = \hat{p}(\hat{x}\hat{p} - i\hbar)\hat{x} = \hat{p}\hat{x}\hat{p}\hat{x} - i\hbar\hat{p}\hat{x} \tag{101}$$

$$\hat{x}^2\hat{p}^2 = \hat{x}(\hat{p}\hat{x} + i\hbar)\hat{p} = \hat{x}\hat{p}\hat{x}\hat{p} + i\hbar\hat{x}\hat{p} \tag{102}$$

This gives

$$\operatorname{tr}\big(\hat{p}^2\lambda\mathcal{D}[\hat{x}]\hat{\rho}\big) = \lambda\left(\frac{1}{2}\operatorname{tr}(\hat{p}\hat{x}\hat{p}\hat{x}\hat{\rho}) + \frac{1}{2}\operatorname{tr}(\hat{x}\hat{p}\hat{x}\hat{p}\hat{\rho}) + \frac{\hbar^2}{2} - \frac{1}{2}\operatorname{tr}(\hat{p}\hat{x}\hat{p}\hat{x}\hat{\rho}) - \frac{1}{2}\operatorname{tr}(\hat{x}\hat{p}\hat{x}\hat{p}\hat{\rho}) + \frac{i\hbar}{2}\operatorname{tr}([\hat{p},\hat{x}])\right) \tag{103}$$

$$= \frac{\lambda\hbar^2}{2} + \frac{\lambda i\hbar}{2}\operatorname{tr}([\hat{p},\hat{x}]) = \lambda\hbar^2 \tag{104}$$

The final equation of motion then becomes

$$\partial_t \left\langle \hat{p}^2 \right\rangle = -\gamma \left\langle \hat{p}^2 \right\rangle + m\omega^2 \left\langle \{\hat{x}, \hat{p}\} \right\rangle + \gamma m\omega\hbar(\bar{n} + 1/2) + \lambda\hbar^2 \tag{105}$$

B.3 Solving for $\partial_t \left\langle \{\hat{x}, \hat{p}\} \right\rangle$

Writing Eq. (61) as

$$\partial_t \left\langle \{\hat{x}, \hat{p}\} \right\rangle = \mathrm{tr}(\{\hat{x}, \hat{p}\}\mathcal{L}\hat{\rho}) + \mathrm{tr}(\{\hat{x}, \hat{p}\}\lambda\mathcal{D}[\hat{x}]\hat{\rho}) \tag{106}$$

Rewriting the left term as

$$\mathrm{tr}(\{\hat{x}, \hat{p}\}\mathcal{L}\hat{\rho}) = i\hbar\,\mathrm{tr}\big((\hat{a}^2 - (\hat{a}^\dagger)^2)\mathcal{L}\hat{\rho}\big) = i\hbar\,\mathrm{tr}\big(\hat{a}^2\mathcal{L}\hat{\rho}\big) - i\hbar\,\mathrm{tr}\big((\hat{a}^\dagger)^2\mathcal{L}\hat{\rho}\big) \tag{107}$$

We have calculated these terms before, so we can write

$$\begin{aligned}
\mathrm{tr}(\{\hat{x}, \hat{p}\}\mathcal{L}\hat{\rho}) &= -i\hbar\left((\gamma + 2i\omega)\left\langle \hat{a}^2 \right\rangle - (\gamma - 2i\omega)\left\langle (\hat{a}^\dagger)^2 \right\rangle\right) && (108)\\
&= -\gamma i\hbar\left(\left\langle \hat{a}^2 \right\rangle - \left\langle (\hat{a}^\dagger)^2 \right\rangle\right) + 2\hbar\omega\left(\left\langle \hat{a}^2 \right\rangle + \left\langle (\hat{a}^\dagger)^2 \right\rangle\right) && (109)
\end{aligned}$$

We also have that

$$\hat{a}^2 + (\hat{a}^\dagger)^2 = \frac{m\omega}{\hbar}\hat{x}^2 - \frac{1}{m\omega\hbar}\hat{p}^2 \tag{110}$$

Thus we get

$$\mathrm{tr}(\{\hat{x}, \hat{p}\}\mathcal{L}\hat{\rho}) = -\gamma \left\langle \{\hat{x}, \hat{p}\} \right\rangle + 2m\omega^2 \left\langle \hat{x}^2 \right\rangle - \frac{2}{m}\left\langle \hat{p}^2 \right\rangle \tag{111}$$

Solving for the measurement term

$$\lambda\,\mathrm{tr}(\{\hat{x}, \hat{p}\}\mathcal{D}[\hat{x}]\hat{\rho}) = \frac{\lambda}{2}\left(\mathrm{tr}\big(\big[\hat{x}^2, \hat{p}\hat{x}\big]\hat{\rho}\big) + \mathrm{tr}\big(\big[\hat{x}\hat{p}, \hat{x}^2\big]\big)\right) \tag{112}$$

Solving the commutators shows that this will become zero.

B.4 Results

The final equations of motion are

$$\partial_t \left\langle \hat{x}^2 \right\rangle = -\gamma \left\langle \hat{x}^2 \right\rangle - \frac{1}{m} \left\langle \{\hat{x}, \hat{p}\} \right\rangle + \frac{\gamma\hbar}{m\omega}(\bar{n} + 1/2) \tag{113}$$

$$\partial_t \left\langle \hat{p}^2 \right\rangle = -\gamma \left\langle \hat{p}^2 \right\rangle + m\omega^2 \left\langle \{\hat{x}, \hat{p}\} \right\rangle + \gamma m\omega\hbar(\bar{n} + 1/2) + \lambda\hbar^2 \tag{114}$$

$$\partial_t \left\langle \{\hat{x}, \hat{p}\} \right\rangle = -\gamma \left\langle \{\hat{x}, \hat{p}\} \right\rangle + 2m\omega^2 \left\langle \hat{x}^2 \right\rangle - \frac{2}{m} \left\langle \hat{p}^2 \right\rangle \tag{115}$$