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Continuous Measurements and Feedback Control of a Quantum Harmonic Oscillator

Max Eriksson

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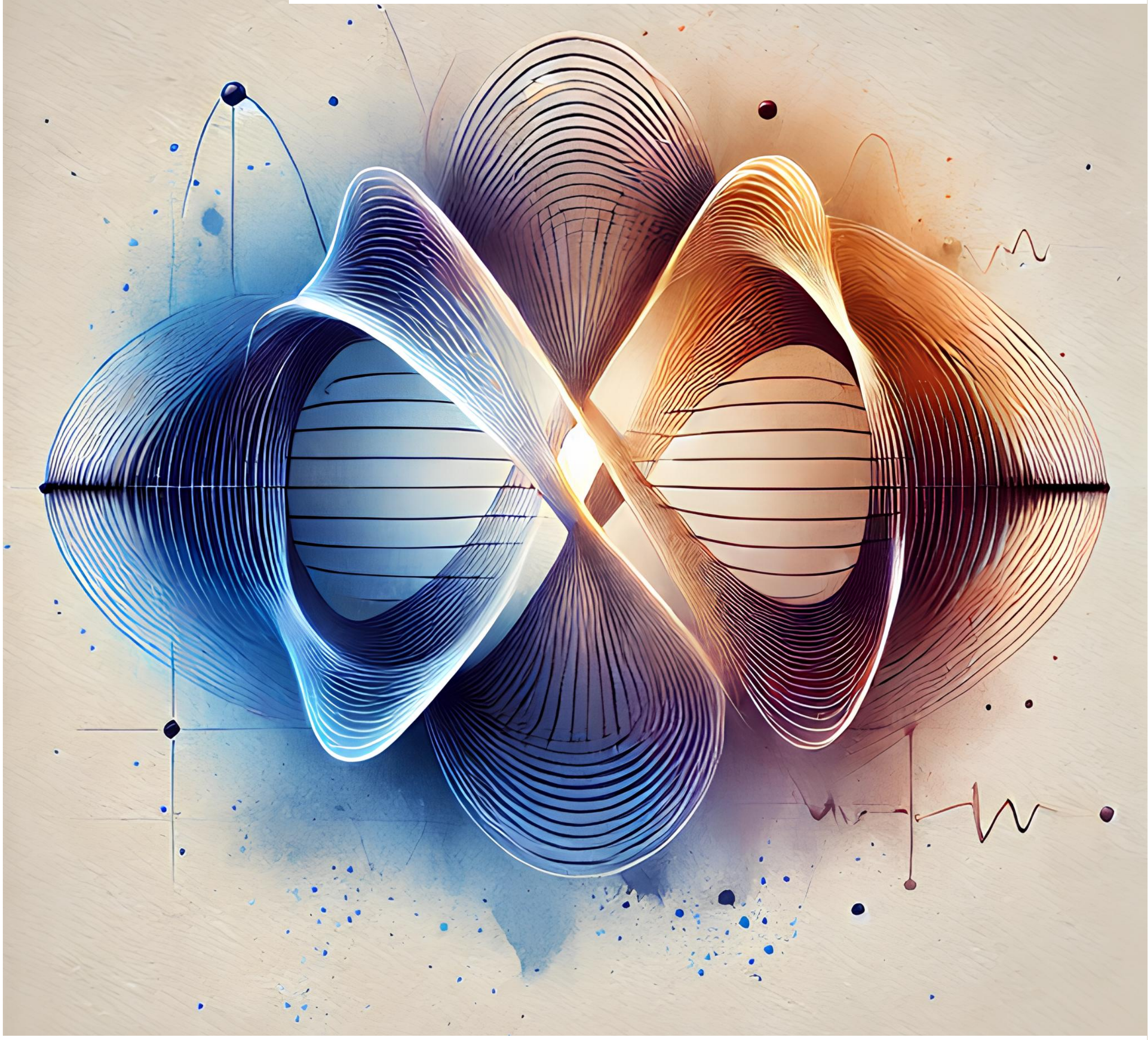
Supervised by Kalle Kansanen and Peter Samuelsson
Division of Mathematical Physics | Department of Physics



List of Abbreviations

EOM Equations of Motion. 7

POVM Positive Operator-Valued Measurement. 5

QHO Quantum Harmonic Oscillator. 2

Acknowledgements

Abstract

This is the abstract

Popular Science Summary

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1 Introduction

As our society is becoming increasingly dependent on technology, the demand for better and more efficient technologies is growing. One kind of technology which has been really important since its conception during the second world war by Alan Turing is computers [1]. The field of computer science has been researched a lot during the second half of the 20th century but has hit a fundamental problem. Our current classical computers have reached a limit where the transistors cannot get any smaller without quantum mechanics causing issues [1]. This has prompted research into quantum technologies such as quantum computers and quantum simulations [1].

To properly understand how quantum mechanical systems work, it is important to understand what happens to them when they are interacted with, during for example a measurement [2]. It is also this interaction with a quantum system, which has prompted many interpretations of quantum mechanics and given rise to what is known as the measurement problem [2]. The measurement problem is a fundamental philosophical problem in quantum mechanics, which arises since a quantum mechanical state evolves deterministically according to the Schrödinger equation, but collapses probabilistically when measured or interacted with [2].

It is also interesting to see how such a system can be manipulated to create a certain state which can be used for a specific purpose. This could include, but is not limited to, creating a qubit state to be used in a quantum computer or a state which can simulate a certain physical system [1]. Here it is important to understand the effect feedback has on a measured system, and how to utilize this to create a desired state [3].

This thesis will look at a quantum harmonic oscillator which is coupled to an environment. Thus creating an open quantum system whose state is temperature dependent. The system will be measured continuously using weak measurements with a feedback loop to control the system. The goal is to see how the system evolves under these conditions, and how the feedback loop can be changed and manipulated to observe different behaviours.

1.1 Outline

This text will start in section 2 by introducing the theoretical framework central in this thesis by first defining what we mean by a quantum harmonic oscillator as well as shortly introducing the density matrix formalism of quantum mechanics. Then, we will move on to discuss open quantum systems and the mathematical framework for the evolution of this type of system, and here we will define a Markovian master equation in Lindblad form. We then move on to discussing measurements on open systems as well as feedback control, and how these concepts can be introduced in the master equation to allow for a mathematical description of the evolution of the system under these effects. In section 3 we will use what has been discussed to derive equations of motion for the QHO. In section 4 there will be a discussion of the relevancy of the results obtained in section 3.

2 Theoretical Framework

2.1 Quantum Harmonic Oscillator

The Quantum Harmonic Oscillator (QHO) is a quantum mechanical system useful for many applications. This stems from the fact that many systems can be approximated as harmonic, that is quadratic, close to their equilibrium position, and since the QHO is a simple system, which is possible to solve analytically it is a good starting approximation. Let us start by assuming that a bosonic particle with mass m is confined in a harmonic potential, then the system has the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right), \tag{1}$$

where $\hat{p}$ and $\hat{x}$ are the momentum and position operators, m is the mass of the particle, and ω is the angular frequency of the oscillator. The operators $\hat{a}^\dagger$ and $\hat{a}$ are the creation and annihilation operators, collectively referred to as the ladder operators, which are defined as

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right) \quad \text{and} \quad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right). \tag{2}$$

These operators can be used to define the number operator $\hat{n} = \hat{a}^\dagger\hat{a}$ which has the number states, or Fock states, $|n\rangle$ as its eigenstates with eigenvalue n [4]. The ladder operators have a useful the commutation relation $\left[\hat{a}, \hat{a}^\dagger\right] = \mathbb{1}$. The Fock states are also eigenstates to the ladder operators with properties

$$\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle\,, \tag{3}$$

$$\hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle\,, \tag{4}$$

which means that they change the excitation level of the QHO [4].

2.2 Open Quantum Systems

Before introducing what open quantum systems we will shortly introduce the language of density matrices. A density matrix describes an ensemble of states defined as

$$\hat{\rho} = \sum_i p_i \left|\psi_i\right\rangle \left\langle\psi_i\right|, \tag{5}$$

where there is probability p_i for the system to be prepared in the state $|\psi_i\rangle$ [1]. Two conditions imposed on a density matrix is that it I) has unit trace, and II) is positive semi-definite [1]. These conditions ensure that the probabilities are $0 \leq p_i \leq 1$ and $\sum_i p_i = 1$ and that the density matrix is hermitian. With this we can reformulate the postulates of quantum mechanics using density matrices [1].

With an open quantum system we mean a quantum system which in some ways interact with an environment. This interaction could be described as a thermal coupling between the main system and some temperature bath. This will cause the system to be in a thermal equilibrium with the environment if left alone, and therefore it will be dependent on the temperature. Unless the temperature of the bath is zero, the system will be in a mixed state, described by a density matrix $\hat{\rho}$. Notably, if the temperature is zero, the system is purely dissipative since the coupling is thermal. [3]

The thermal coupling to the environment will lead to dissipation of quantum information from the system to the environment. During this process, the system loses coherence. That is, the quantum mechanical properties of the system are lost and a classical description of the state becomes more appropriate. The coherence of the system is manifested in the off-diagonal elements of the density matrix. If the off-diagonal elements are zero, either by dissipation to the environment or by other means of decoherence, the system will exist in a classical probabilistic state, and any superposition of states will be lost. [3]

The combination of the system and environment can be considered a closed system, though more complicated than the main system itself. Then, by performing a partial trace over the environment, a description of the system alone arises at the cost of losing information about the correlation between the two parts [3]. This introduces an uncertainty in the state, and it is therefore necessary to treat the resulting system to be in a mixed state. To describe the evolution of this system with a master equation two approximations about the coupling need to be performed. Firstly, we need to consider the Born approximation, which says that the coupling between the system and environment is such that only negligible excitations appear in the environment [5]. The other approximation is the Markov approximation saying that the excitations that do appear in the environment will decay much faster than the timescale that the system varies on, and that the system's time evolution is only affected by the current state of the system and not previous states [5]. Together these approximations allow us to write the total density matrix as

$$\rho_{\mathrm{SE}} = \rho_{\mathrm{S}} \otimes \rho_{\mathrm{E}}, \tag{6}$$

and derive a Markovian master equation.

2.2.1 Master Equation

The evolution of an open quantum system can be described by a master equation, which is a differential equation and generalization of the Schrödinger equation to involve open quantum systems instead of pure states [3]. By introducing the super operator

$$\mathcal{D}[\hat{L}_k]\hat{\rho} = \hat{L}_k\hat{\rho}\hat{L}_k^\dagger - \frac{1}{2}\left\{\hat{L}_k^\dagger\hat{L}_k, \hat{\rho}\right\}, \tag{7}$$

where $\hat{L}_k$ are called Lindblad operators, the master equation on Lindblad form can be written as

$$\partial_t\hat{\rho} = -\frac{i}{\hbar}\left[\hat{H}, \hat{\rho}\right] + \sum_k \gamma_k \mathcal{D}[\hat{L}_k]\hat{\rho}, \tag{8}$$

where $\hat{H}$ is the Hamiltonian of the system, and γ_k are the decay rates of the system, relating the decoherence to the environment depending on the coupling to the system [3]. If $\gamma_k = 0$ for all k the equation reduces to the von Neumann equation for a closed quantum system and the coupling to the bath is removed. The remaining term thus describes the unitary time evolution of the system and is the analogue of the Schrödinger equation for the density matrix formalism [3]. At this stage one might also introduce the Liouvillian superoperator and write the master equation more compactly as

$$\partial_t\hat{\rho} = \mathcal{L}\hat{\rho}. \tag{9}$$

This compactness will be useful when considering other types of perturbing effects on the system such as measurements and feedback [3].

In the case considered in this thesis with a QHO coupled to a thermal reservoir we can imagine that we have two types of decay. One decay of particles into the system and one decay of particles out of the system [4]. As mentioned in Sec. 2.1 the ladder operators can be used to excite or deexcite a system. It is also reasonable to assume that the decay, or the amount of particles flowing between the systems and the environment, is proportional to the thermal excitation $\bar{n}$ defined by

$$\bar{n} = \frac{1}{e^{\hbar\omega/k_{\mathrm{B}}T} - 1}, \tag{10}$$

where T is the temperature of the bath and k_{B} is the Boltzmann constant [4]. If one goes through the mathematical proof one can show that the Lindblad operators can be chosen as $\hat{L}_1 = \hat{a}$ and $\hat{L}_2 = \hat{a}^\dagger$ with coefficients $\gamma_1 = \gamma(\bar{n}+1)$ and $\gamma_2 = \gamma\bar{n}$, wher γ is a decay rate. We can also note that $\hat{L}_1$ and γ_1 refer to the spontaneous emission from the system to the environment while $\hat{L}_2$ refer to spontaneous absorption from the environment to the system, consistent with what we know about ladded operators from Sec. 2.1 [4]. Notably, for $T = 0$, the thermal occupation is $\bar{n} = 0$ and the system will only exhibit emission and will decay.

2.3 Continuous Measurements

Measurement is a process which introduces decoherence in the system, and it is therefore interesting to look at its effects [2]. The simplest view on measurements takes the form of von Neumann measurements. This type of measurement is described by a set of measurement operators which projects the system onto the eigenstates of the observable [3]. This essentially means that all quantum information in the system is lost and full decoherence has happened. By generalizing the measurement theory one can derive what is called Positive Operator-Valued Measurement (POVM) [3].

Since the POVM is not necessarily a projective von Neumann measurement all coherence need not be lost after the measurement. Thus, this opens up for the possibility of performing time continuous weak measurement [3]. To describe this type of POVM we first consider a Gaussian measurement operator

$$\hat{K}(z) = \left(\frac{2\bar{\lambda}}{\pi}\right)^{1/4} e^{-\bar{\lambda}(z-\hat{A})^2}, \tag{11}$$

where $\bar{\lambda}$ represents the strength of the measurement, z is a continuous outcome of the measurement, and $\hat{A}$ is the measured observable [3]. We note that the post measurement state of such a measurement is described by

$$\hat{\rho}_{\text{post}} = \frac{\hat{K}(z)\hat{\rho}\hat{K}^{\dagger}(z)}{p(z)}, \tag{12}$$

where the probability is defined as $p(z) = \mathrm{tr}\left(\hat{K}^{\dagger}(z)\hat{K}(z)\hat{\rho}\right)$ [3].

Then by discretizing the time interval to segments of $\mathrm{d}t$ and defining $\bar{\lambda} = \lambda \mathrm{d}t$ we approach a situation where in the limit $\mathrm{d}t \to 0$ all measurements will be weak, and the coherence of the system is minimally affected [3]. Considering the stochastic nature of the process and averaging the possible trajectories one can derive the master equation [3] in Lindblad form to be

$$\partial_t \hat{\rho} = \mathcal{L}\hat{\rho} + \lambda \mathcal{D}[\hat{A}]\hat{\rho}. \tag{13}$$

2.4 Feedback Control

Until this point we have only considered measurements where we omit the information about the measurement outcome. That is, we interact with the system and look at how it evolves due to this interaction on average, instead of looking at the specific outcome of any given measurement [3]. However, now we want to consider feedback control of the system, and thus we will need to include the information about the measurement outcome [3]. By feedback control we mean a process by which we manipulate the evolution of a system due to a measurement outcome [6]. Since we are dealing specifically with quantum systems, we can further talk about quantum feedback control, where quantum mechanical effects of the system play a role in the modelling of the feedback mechanisms effect on the system [6]. However, worth noting is that the physical realization of the feedback mechanism does not necessarily need to be entirely quantum mechanical, but at least part of the mechanism need to incorporate quantum mechanics. [6]. Specifically, for a measurement outcome z we will consider a linear feedback modification of $\mathcal{L}$ such that

$$\mathcal{L} \to \mathcal{L} + z\mathcal{K}, \tag{14}$$

where $\mathcal{K}$ is a superoperator describing the feedback on the system [3] which takes the form

$$\mathcal{K}\hat{\rho} = -\frac{i}{\hbar}\left[\hat{H}_{\text{c}}, \hat{\rho}\right], \tag{15}$$

where $\hat{H}_{\mathrm{c}}$ is the control Hamiltonian of the system. We will consider a control Hamiltonian which is linear on the form

$$\hat{H}_{\mathrm{c}} = f^* \hat{a} + f \hat{a}^\dagger, \tag{16}$$

where f is the feedback amplitude [6]. Thus, for $\mathfrak{I}\{f\} = 0$ we have $\hat{H}_{\mathrm{c}} \propto \hat{x}$ and for $\mathfrak{R}\{f\} = 0$ we have $\hat{H}_{\mathrm{c}} \propto \hat{p}$. Starting from the same place as one derives Eq. (13) we can derive a master equation including feedback [3] to be

$$\partial_t \hat{\rho} = \mathcal{L}\hat{\rho} + \lambda \mathcal{D}[\hat{A}]\hat{\rho} + \frac{1}{2}\mathcal{K}\left\{\hat{A}, \hat{\rho}\right\} + \frac{1}{8\lambda}\mathcal{K}^2\hat{\rho}, \tag{17}$$

where the square on $\mathcal{K}$ means $\mathcal{K}^2\hat{\rho} = \mathcal{K}(\mathcal{K}\hat{\rho})$.

3 Result

This section will look at the results in the form of calculated equations of motion and steady state solutions. Both with and without feedback. The first subsection will deal with a QHO which is measured continuously and without feedback, while the second subsection will add feedback into the scheme.

3.1 Measurement Without Feedback

Consider a QHO described by the Hamiltonian in Eq. (1) which is coupled to a thermal bath with temperature T. If the oscillator's position quadrature is also continuously measured the evolution of the system can be described by the master equation in Eq. (13) using the Lindblad operators mentioned in Sec. 2.2.1. We want to solve for Equations of Motion (EOM) when measuring for the position quadrature

$$\partial_t \left\langle \hat{x}^2 \right\rangle = \mathrm{tr}\left(\hat{x}^2 \partial_t \hat{\rho}\right), \tag{18}$$

$$\partial_t \left\langle \hat{p}^2 \right\rangle = \mathrm{tr}\left(\hat{p}^2 \partial_t \hat{\rho}\right), \tag{19}$$

$$\partial_t \left\langle \{\hat{x}, \hat{p}\} \right\rangle = \mathrm{tr}(\{\hat{x}, \hat{p}\}\partial_t \hat{\rho}). \tag{20}$$

The choice of looking at the second momenta has to do with their relation to the variance and thus the fluctuations of the system. These fluctuations are then related to the energy contained in the oscillator. When looking at feedback control and interesting application is to minimize the energy in the oscillator and, which is thus also an argument for looking at the second momenta. Solving Eqs. (18) to (20) yields the following EOM

$$\partial_t \left\langle \hat{x}^2 \right\rangle = -\gamma \left\langle \hat{x}^2 \right\rangle - \frac{1}{m} \left\langle \{\hat{x},\hat{p}\} \right\rangle + \frac{\gamma\hbar}{m\omega}(\bar{n}+1/2), \tag{21}$$

$$\partial_t \left\langle \hat{p}^2 \right\rangle = -\gamma \left\langle \hat{p}^2 \right\rangle + m\omega^2 \left\langle \{\hat{x},\hat{p}\} \right\rangle + \gamma m\omega\hbar(\bar{n}+1/2) + \lambda\hbar^2, \tag{22}$$

$$\partial_t \left\langle \{\hat{x},\hat{p}\} \right\rangle = -\gamma \left\langle \{\hat{x},\hat{p}\} \right\rangle + 2m\omega^2 \left\langle \hat{x}^2 \right\rangle - \frac{2}{m} \left\langle \hat{p}^2 \right\rangle \tag{23}$$

for more detailed calculations see App. B. Performing a change of variable to make the equations dimensionless

$$\tilde{x} = \sqrt{\frac{m\omega}{\hbar}}x \quad \text{and} \quad \tilde{p} = \sqrt{\frac{1}{m\omega\hbar}}p, \tag{24}$$

we can solve for the steady state. By introducing the quality factor $Q = \omega/\gamma$ we obtain the steady state solutions

$$\left\langle \tilde{x}^2 \right\rangle_{\text{ss}} = (\bar{n}+1/2) + \frac{\lambda\hbar}{m\omega^2}\frac{2Q^3}{4Q^2+1}, \tag{25}$$

$$\left\langle \tilde{p}^2 \right\rangle_{\text{ss}} = (\bar{n}+1/2) + \frac{\lambda\hbar}{m\omega^2}\left(Q - \frac{2Q^3}{4Q^2+1}\right), \tag{26}$$

$$\tag{27}$$

One interesting aspects of the equations is that both equations have identical terms capturing the thermal aspect of the fluctuations. Another interesting thing is that without measurement the equations are only thermal, which is to be expected. Interestingly, when calculating the energy of the steady state the complicated fraction disappears and we are left with a purely linear term in the quality factor.

$$E_{\text{ss}} = \left\langle \tilde{H} \right\rangle_{\text{ss}} = \frac{\hbar\omega}{2}\left(\left\langle \tilde{p}^2 \right\rangle + \left\langle \tilde{x}^2 \right\rangle\right) = \hbar\omega(\bar{n}+1/2) + \frac{\lambda\hbar^2}{2m\omega}Q \tag{28}$$

Looking at panels **a** and **b** in Fig. 1 one can see that a stronger measurement correlates to the system steady state increasing in energy, as does it for an increasing quality factor. Both also affect the system linearly. Thus, by continuously measuring the system we add energy into it, which make the steady state higher in energy than what the thermal effects from the bath would otherwise place it. That is, if we do not perform any measurement the system would be stable at around $\bar{n}+1/2$ which for the parameters used here would give $\left\langle \tilde{E} \right\rangle \approx 10\hbar\omega$

In panels **d** and **f** we can see that there is some non-linear behaviour near $Q = 0$. However,

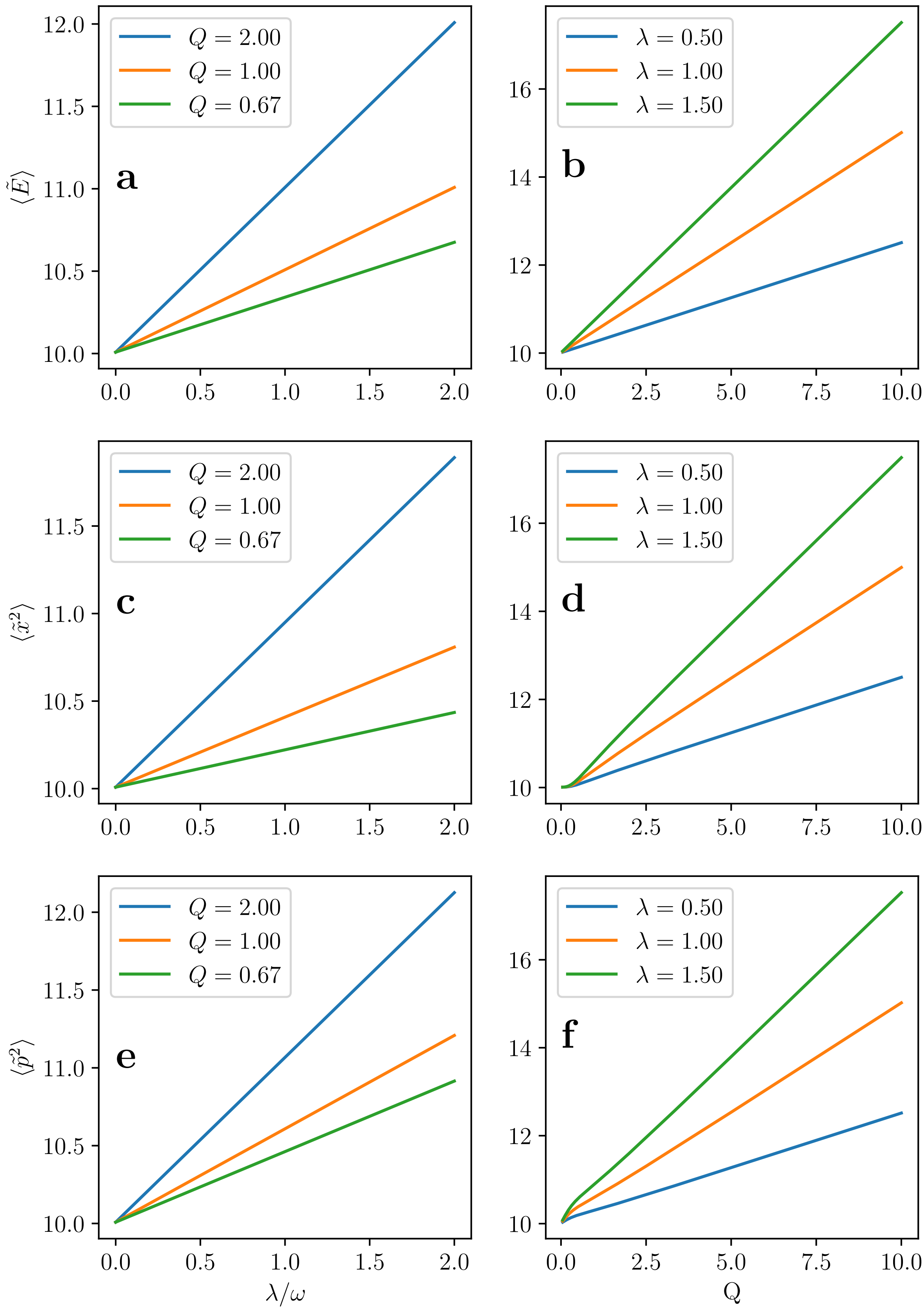

Figure 1: The top panels show Eq. (28), the middle panels show Eq. (25) and the bottom panels show Eq. (26). All plots use the parameters $k_\mathrm{B}T = 10$ and $\omega = \hbar = 1$. The left panels are plotted against λ/ω and with three different values for Q, while the right panels are plotted against Q and three different values of λ.

due to the approximations made in Sec. 2.2 we cannot trust the results in the region of a low quality factor. Due to this regime having a relatively high coupling to the environment, and thus the excitations in the environment might not decay fast enough and therefore affect the oscillator. Panels **c** and **e** show the linear dependence on λ for the second momenta.

Using the same master equation as above we can also solve for the first momenta

$$\partial_t \langle \hat{x} \rangle = \mathrm{tr}(\hat{x}\partial_t \hat{\rho}), \tag{29}$$

$$\partial_t \langle \hat{p} \rangle = \mathrm{tr}(\hat{p}\partial_t \hat{\rho}). \tag{30}$$

Solving these equations we find

$$\partial_t \langle \hat{x} \rangle = -\frac{\gamma}{2} \langle \hat{x} \rangle - \frac{1}{m} \langle \hat{p} \rangle , \tag{31}$$

$$\partial_t \langle \hat{p} \rangle = -\frac{\gamma}{2} \langle \hat{p} \rangle + m\omega^2 \langle x \rangle . \tag{32}$$

Then when solving for the steady state we find

$$\langle \hat{x} \rangle = \langle \hat{p} \rangle = 0. \tag{33}$$

This result also confirms the intuition that for a harmonic oscillator, the system has a steady state around the origin. To check for stability we can rewrite the equation as an eigenvalue problem and solve for the eigenvalues. That is, for the matrix

$$\mathcal{M} = \begin{pmatrix} -\gamma/2 & -1/m \\ m\omega^2 & -\gamma/2 \end{pmatrix} \tag{34}$$

the eigenvalues are

$$\lambda_1 = -\frac{\gamma}{2} - i\omega, \tag{35}$$

$$\lambda_2 = -\frac{\gamma}{2} + i\omega. \tag{36}$$

Since the real part of the eigenvalues are negative the system is stable.

Eq. (33) also shows that the variance of the system is only dependent on the first momenta, since

$$\sigma^2_{\hat{A}} = \left\langle \hat{A}^2 \right\rangle - \left\langle \hat{A} \right\rangle^2 \tag{37}$$

for an operator $\hat{A}$. It is also an easy calculation then to show that for temperature $T = 0$ and without measurement we have equality in the Heisenberg uncertainty relation, justifying the accuracy of the results, and the approximations made in the derivation of the master equation.

3.2 Feedback

We now consider a feedback mechanism on the oscillator described by Eq. (17) which is a linear feedback scheme. Solving for the first momenta's EOM we find

$$\partial_t \langle \hat{x} \rangle = -\left(\frac{\gamma}{2} + \frac{2\Im\{f\}}{\sqrt{2m\omega\hbar}}\right) \langle \hat{x} \rangle - \frac{1}{m} \langle \hat{p} \rangle , \tag{38}$$

$$\partial_t \langle \hat{p} \rangle = -\frac{\gamma}{2} \langle \hat{p} \rangle + \left(\Re\{f\} \sqrt{\frac{2m\omega}{\hbar}} + m\omega^2\right) \langle \hat{x} \rangle . \tag{39}$$

Choosing f such that

$$\Re\{f\} = -\sqrt{\frac{m\omega^3\hbar}{2}} \quad \text{and} \quad \Im\{f\} = -\frac{\gamma\sqrt{m\omega\hbar}}{2\sqrt{2}} \tag{40}$$

The system of equations reduces to

$$\partial_t \langle \hat{x} \rangle = -\frac{1}{m} \langle \hat{p} \rangle , \tag{41}$$

$$\partial_t \langle \hat{p} \rangle = -\frac{\gamma}{2} \langle \hat{p} \rangle , \tag{42}$$

which has a steady state solution for $\langle \hat{p} \rangle = 0$ and any $\langle \hat{x} \rangle$. To check for stability we can again rewrite the equations as an eigenvalue problem with matrix

$$\mathcal{M} = \begin{pmatrix} -\left(\frac{\gamma}{2} + \frac{2\Im\{f\}}{\sqrt{2m\omega\hbar}}\right) & -\frac{1}{m} \\ \Re\{f\} \sqrt{\frac{2m\omega}{\hbar}} + m\omega^2 & -\frac{\gamma}{2} & , \end{pmatrix} \tag{43}$$

which has eigenvalues

$$\varepsilon\pm = \frac{\pm\sqrt{2}\sqrt{-m^2\left(2\sqrt{2}\Re\{f\}\omega\hbar\sqrt{\frac{m\omega}{\hbar}} + 2m\omega^3\hbar - \Im\{f\}^2\right)} - \sqrt{2}m\Im\{f\} - \gamma m\sqrt{m\omega\hbar}}{2m\sqrt{m\omega\hbar}}. \tag{44}$$

Inserting the values of $\Re\{f\}$ and $\Im\{f\}$ from Eq. (40) we find the eigenvalues to be $\varepsilon_- = -\gamma/2$ and $\varepsilon_+ = 0$. It is interesting that this choice of f removes all the oscillatory behaviour from the system, which can be seen by the eigenvalues being real. Since one of the eigenvalues are zero

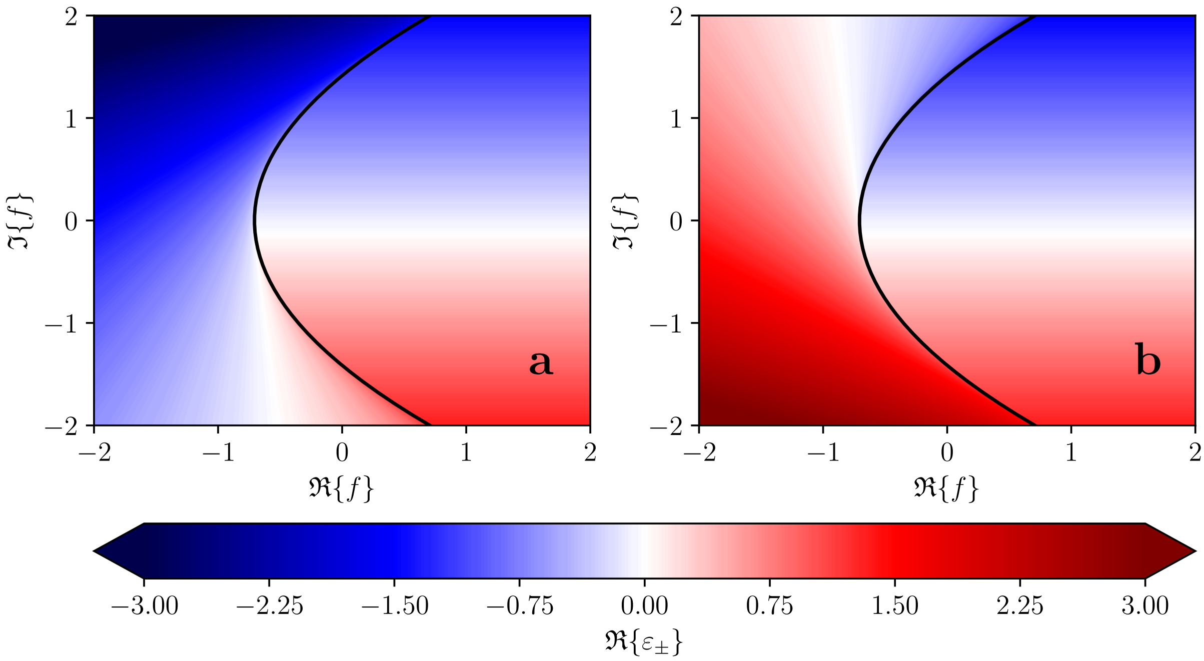

Figure 2: Eq. (44) plotted as a contour plot against the real and imaginary part of f. The parameters used are $m = \omega = \hbar = 1$ and $\gamma = 0.2$. Panel **a** shows $\Re\{\varepsilon_-\}$ and panel **b** shows $\Re\{\varepsilon_+\}$. The parabola that can be seen in both panels is the points where the first square root in Eq. (44) is zero. Thus, points to the left of this parabola are real, while points to the right are complex, only the real part is plotted, however.

the system is also reduced to a one dimensional problem, which is reasonable as the choice of f is such that the system is unaffected by the position quadrature.

Looking at Fig. 2 we can see that $\Re\{\varepsilon_-\}$ is mostly negative while $\Re\{\varepsilon_+\}$ is mostly positive for the parameters used in the region closest to the origin. The only part where both eigenvalues are negative is in top right region of the figures. That is, the region where $\Im\{f\} > 0$ and $\Re\{f\} \gtrapprox -1/\sqrt{2}$. The reason for the approximate is that looking at panel **b** we can see that line where the eigenvalue is zero has a slant and is not vertical.

Since we can write the solutions as an exponential with the eigenvalues while the coefficients are chosen by the initial condition the value of the eignvalue will determine the behaviour of the system. The consequence of the eigenvalue being negative is that it will make the function decay with time and is thus considered stable. If both eigenvalues are negative the system will decay no matter what the initial conditions are since both terms in the solution will decay. Positive eigenvalues on the other hand will correspond to the system growing with time and will thus be unstable. When one of the eigenvalues is positive and the other is negative the stability of the system will be determined by the initial conditions. That is, the initial condition will determine which eigenvalue will dominate the solution.

Another thing to note is that the eigenvalues are comples to the right of the black parabola in the figure, and thus the solution will have oscillatory motion in this region.

Solving for the EOM for the second momenta we obtain

$$\partial_t \left\langle x^2 \right\rangle = -\left(\gamma + \frac{4\Im\left\{f\right\}}{\sqrt{2m\omega\hbar}}\right)\left\langle x^2 \right\rangle - \frac{1}{m}\left\langle \{x,p\} \right\rangle + \frac{\gamma\hbar}{m\omega}(\bar{n}+1/2) - \frac{1}{4\lambda m\omega\hbar}\Re\left\{f\right\}^2, \tag{45}$$

$$\partial_t \left\langle p^2 \right\rangle = -\gamma\left\langle p^2 \right\rangle + \left(m\omega^2 + \Re\left\{f\right\}\sqrt{\frac{2m\omega}{\hbar}}\right)\left\langle \{x,p\} \right\rangle + \gamma m\omega\hbar(\bar{n}+1/2) + \lambda\hbar^2 + \frac{m\omega}{4\lambda\hbar}\Re\left\{f\right\}^2, \tag{46}$$

$$\partial_t \left\langle \{x,p\} \right\rangle = \left(\gamma + \frac{2\Im\left\{f\right\}}{\sqrt{2m\omega\hbar}}\right)\left\langle \{x,p\} \right\rangle + \left(2m\omega^2 + 2\sqrt{\frac{2m\omega}{\hbar}}\Re\left\{f\right\}\right)\left\langle x^2 \right\rangle - \frac{2}{m}\left\langle p^2 \right\rangle - \frac{\Re\left\{f\right\}\Im\left\{f\right\}}{2\lambda\hbar}. \tag{47}$$

We can solve for the steady state by first rewriting the system of equations to a matrix equation $\partial_t X = \mathcal{A}X + \mathcal{B}$ where

$$X = \begin{pmatrix} \left\langle \hat{x}^2 \right\rangle \\ \left\langle \hat{p}^2 \right\rangle \\ \left\langle \{\hat{x},\hat{p}\} \right\rangle \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} -\left(\gamma + \frac{4\Im\{f\}}{\sqrt{2m\omega\hbar}}\right) & 0 & -\frac{1}{m} \\ 0 & -\gamma & m\omega^2 + \Re\left\{f\right\}\sqrt{\frac{2m\omega}{\hbar}} \\ 2m\omega^2 + 2\sqrt{\frac{2m\omega}{\hbar}}\Re\left\{f\right\} & -\frac{2}{m} & -\left(\gamma + \frac{2\Im\{f\}}{\sqrt{2m\omega\hbar}}\right) \end{pmatrix} \tag{48}$$

and

$$\mathcal{B} = \begin{pmatrix} \frac{\gamma\hbar}{m\omega}(\bar{n}+1/2) - \frac{\Re\{f\}^2}{4\lambda m\omega\hbar} \\ \gamma m\omega\hbar(\bar{n}+1/2) + \lambda\hbar^2 + \frac{m\omega\Re\{f\}^2}{4\lambda\hbar} \\ -\frac{\Re\{f\}\Im\{f\}}{2\lambda\hbar} \end{pmatrix}. \tag{49}$$

To then obtain the steady state solutions is then to solve $X_{\text{ss}} = -\mathcal{A}^{-1}\mathcal{B}$. To do this is we perform the same change of variables as before, in addition to

$$\tilde{f} = \frac{f}{\omega\sqrt{m\omega\hbar}}, \quad \Lambda = \frac{\lambda\hbar}{m\omega^2} \tag{50}$$

The full analytical solutions are not shown here due to the length of the equations.

In Fig. 3 we can see the affect that the feedback has on the energy of the system. Looking at panel **b** we can see areas which are negative. Since the energy of the system without measurement is constant, as those parameters are frozen, it must be the feedback energy that becomes negative. In the way we have set up the model a negative energy is unreasonable, since a temperature of $T = 0$ would correspond to $E = \hbar\omega(1/2 + \Lambda Q/2)$, which without measurement would mean an

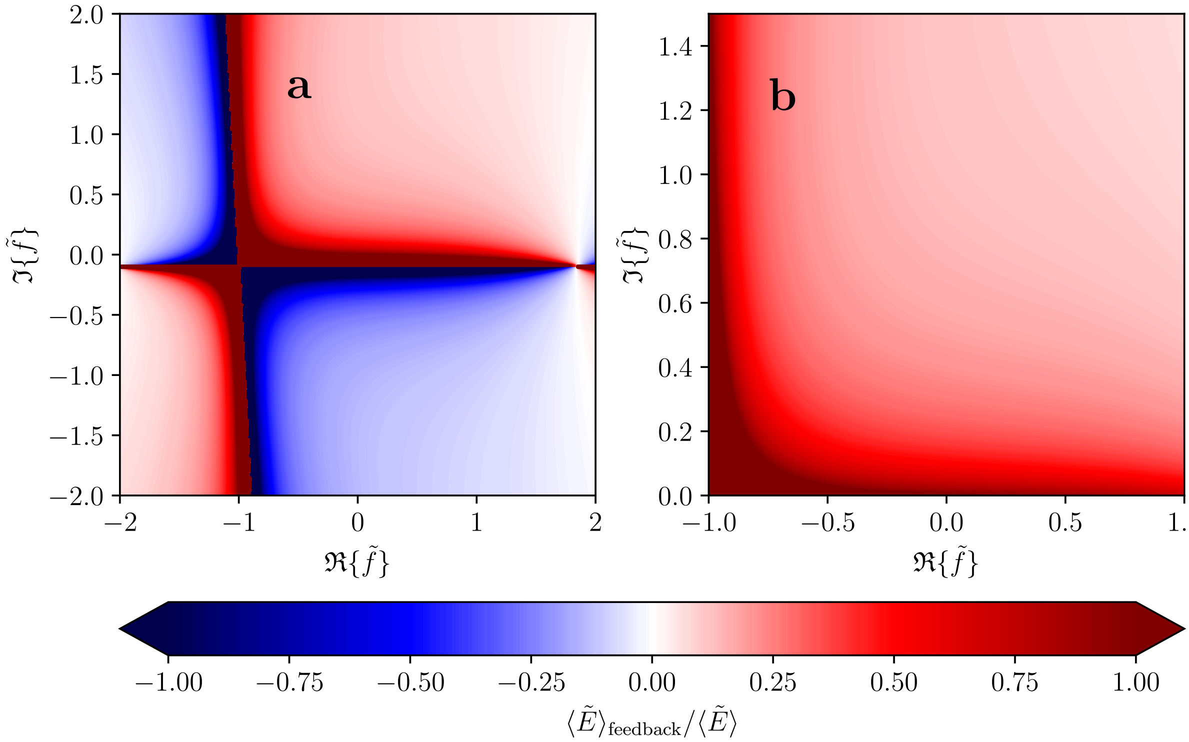

Figure 3: The ratio between the energy with feedback and without plotted as a contour plot against $\Re\left\{\tilde{f}\right\}$ and $\Im\left\{\tilde{f}\right\}$. The parameters used are $k_{\mathrm{B}}T = 10$, $Q = 10$, $\Lambda = 2$. Panel **a** shows a large variation of the parameters. There are divergences in the plot which almost follows a $\left(\Re\left\{\tilde{f}\right\} + 1\right)\Im\left\{\tilde{f}\right\} = 1$ curve. Panel **b** is a zoomed in version of panel **a** and shows the behaviour of the system in a region where the ratio always is positive.

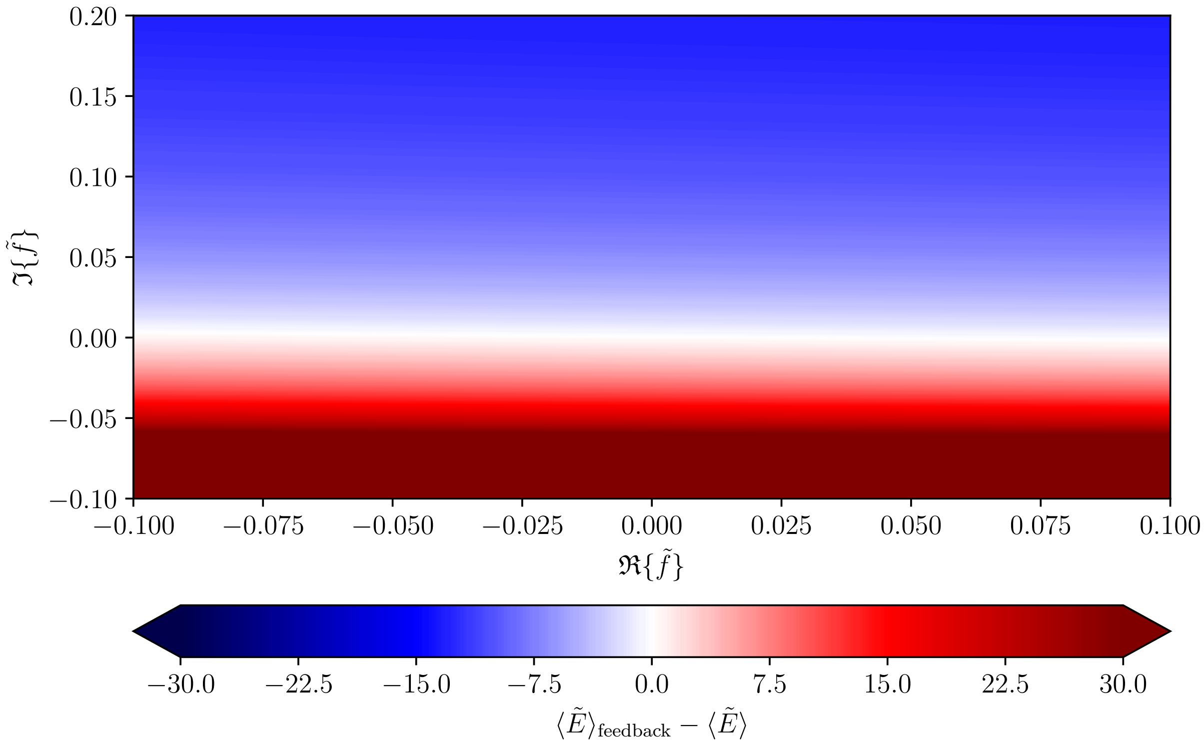

Figure 4: The difference between the energy of the system with feedback plotted and the energy without feedback plotteda against $\Re\left\{\tilde{f}\right\}$ and $\Im\left\{\tilde{f}\right\}$. The parameters used are $k_{\mathrm{B}}T = 10$, $Q = 10$, $\Lambda = 2$. The plot is done in a small region around the origin, thus showing the effect a small feedback has on the system.

energy of $E = \hbar\omega/2$. A possible explantation for the negative energy is that the specific feedback that give rise to this is affecting the system in such a way that at least one of our assumptions is no longer accurate. This could for example be that the system no longer have a physical steady state. Another reason might be that the Markovian approximation no longer holds.

Another interesing aspect of Fig. 3 is that when looking at panel **b** in conjunction with Eq. (28) is that it is possible to cool the system to a lower energy than the thermal energy. Using the same parameters in Eq. (28) as are used in the figure we find that the thermal energy is half of the total energy in the system without feedback, and looking at the figure we see that there exist a region which has a ratio of less than 0.5.

Using the parameters listed in Fig. 4 the energy without feedback is $\left\langle\tilde{E}\right\rangle \approx 20$ with the thermal part accounting for around 10 of that. Looking at Fig. 4 it is possible to see that even with a relatively low amount of feedback the system will be cooled to some degree. However, depending on how the feedback is applied it can also make the energy in the system explode which can be seen for $\Im\left\{\tilde{f}\right\} < 0$.

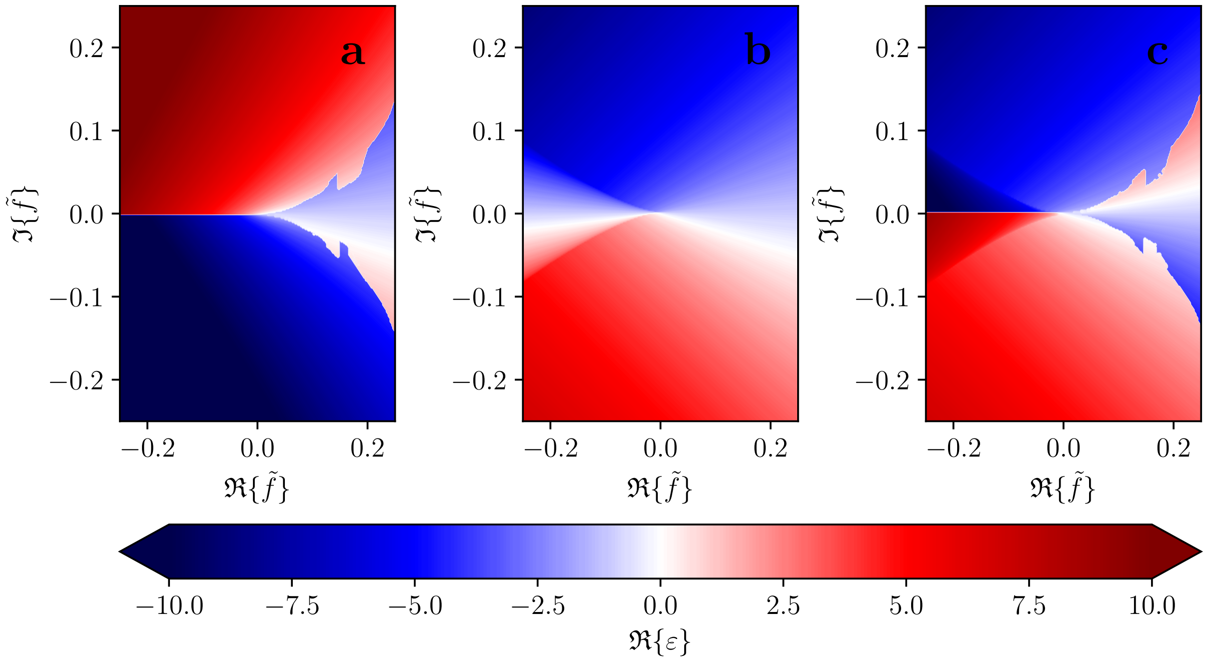

Figure 5: The eigenvalues of $\mathcal{A}$ in Eq. (48) plotted after doing the change of variables as a contour plot against $\Re\left\{\tilde{f}\right\}$ and $\Im\left\{\tilde{f}\right\}$ using the parameter $Q = 10$. Panel **a**, **b**, and **c** show the real part of eigenvalue ε_1, ε_2, and ε_3, respectively.

4 Discussion

4.1 Cooling of an Harmonic Oscillator

4.2 Stability with Feedback

5 Outlook

References

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A Appendix

B EOM Calculation

Consider a quantum harmonic oscillator described by the Hamiltonian

$$\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + 1/2). \tag{51}$$

Consider also that the quantum harmonic oscillator is subject to a temperature bath and is continuously measured. The system then evolves according to the master equation

$$\partial_t\hat{\rho} = -\frac{i}{\hbar}[\hat{H},\hat{\rho}] + \gamma(\bar{n}+1)\mathcal{D}[\hat{a}]\hat{\rho} + \gamma\bar{n}\mathcal{D}[\hat{a}^\dagger]\hat{\rho} + \lambda\mathcal{D}[\hat{A}]\hat{\rho} \tag{52}$$

where γ is the damping rate, $\bar{n}$ is the thermal occupancy defined as

$$\bar{n} = \frac{1}{e^{\hbar\omega/k_BT} - 1}, \tag{53}$$

λ is the measurement rate, $\hat{A}$ is the measurement operator, and $\mathcal{D}[\hat{O}]$ is the Lindblad super operator defined as

$$\mathcal{D}[\hat{O}]\hat{\rho} = \hat{O}\hat{\rho}\hat{O}^\dagger - \frac{1}{2}\{\hat{O}^\dagger\hat{O},\hat{\rho}\}. \tag{54}$$

Then we also have the quadrature operators defined as

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}+\hat{a}^\dagger), \quad \hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a}-\hat{a}^\dagger). \tag{55}$$

We want to find the equations of motion when measuring the position quadrature

$$\partial_t\left\langle\hat{x}^2\right\rangle = \mathrm{tr}\big(\hat{x}^2\partial_t\hat{\rho}\big) \tag{56}$$

$$\partial_t\left\langle\hat{p}^2\right\rangle = \mathrm{tr}\big(\hat{p}^2\partial_t\hat{\rho}\big) \tag{57}$$

$$\partial_t\left\langle\{\hat{x},\hat{p}\}\right\rangle = \mathrm{tr}(\{\hat{x},\hat{p}\}\partial_t\hat{\rho}). \tag{58}$$

We start by defining the Liouvillian super operator

$$\mathcal{L}\hat{\rho} = -\frac{i}{\hbar}[\hat{H},\hat{\rho}] + \gamma(\bar{n}+1)\mathcal{D}[\hat{a}]\hat{\rho} + \gamma\bar{n}\mathcal{D}[\hat{a}^\dagger]\hat{\rho} \tag{59}$$

which turns the master equation into

$$\partial_t \hat{\rho} = \mathcal{L}\hat{\rho} + \lambda \mathcal{D}[\hat{x}]\hat{\rho}. \tag{60}$$

B.1 Solving for $\partial_t \langle \hat{x}^2 \rangle$

We can write Eq. (56) as

$$\partial_t \langle \hat{x}^2 \rangle = \mathrm{tr}\big(\hat{x}^2 \mathcal{L}\hat{\rho}\big) + \mathrm{tr}\big(\hat{x}^2 \lambda \mathcal{D}[\hat{x}]\hat{\rho}\big) \tag{61}$$

and then rewriting the left term using the creation and annihilation operators we get

$$\begin{aligned}
\mathrm{tr}\big(\hat{x}^2 \mathcal{L}\hat{\rho}\big) &= \frac{\hbar}{2m\omega}\,\mathrm{tr}\Big((\hat{a}^2 + (\hat{a}^\dagger)^2 + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a})\mathcal{L}\hat{\rho}\Big) && (62)\\
&= \frac{\hbar}{2m\omega}\left[\mathrm{tr}\big(\hat{a}^2\mathcal{L}\hat{\rho}\big) + \mathrm{tr}\Big((\hat{a}^\dagger)^2\mathcal{L}\hat{\rho}\Big) + \mathrm{tr}\Big(\hat{a}\hat{a}^\dagger\mathcal{L}\hat{\rho}\Big) + \mathrm{tr}\Big(\hat{a}^\dagger\hat{a}\mathcal{L}\hat{\rho}\Big)\right] && (63)
\end{aligned}$$

Solving these separately

$$\begin{aligned}
\mathrm{tr}\big(\hat{a}^2\mathcal{L}\hat{\rho}\big) &= -\frac{i}{\hbar}\,\mathrm{tr}\Big(\hat{a}^2\Big[\hat{H},\hat{\rho}\Big]\Big) + \gamma(\bar{n}+1)\,\mathrm{tr}\big(\hat{a}^2\mathcal{D}[\hat{a}]\hat{\rho}\big) + \gamma\bar{n}\,\mathrm{tr}\Big(\hat{a}^2\mathcal{D}[\hat{a}^\dagger]\hat{\rho}\Big) && (64)\\
&= -i\omega\,\mathrm{tr}\Big(\hat{a}^2\hat{a}^\dagger\hat{a}\hat{\rho}\Big) + i\omega\,\mathrm{tr}\Big(\hat{a}^2\hat{\rho}\hat{a}^\dagger\hat{a}\Big) && (65)\\
&+ \gamma(\bar{n}+1)\,\mathrm{tr}\Big(\hat{a}^3\hat{\rho}\hat{a}^\dagger\Big) - \frac{\gamma(\bar{n}+1)}{2}\,\mathrm{tr}\Big(\hat{a}^2\hat{a}^\dagger\hat{a}\hat{\rho}\Big) - \frac{\gamma(\bar{n}+1)}{2}\,\mathrm{tr}\Big(\hat{a}^2\hat{\rho}\hat{a}^\dagger\hat{a}\Big) && (66)\\
&+ \gamma\bar{n}\,\mathrm{tr}\Big(\hat{a}^2\hat{a}^\dagger\hat{\rho}\hat{a}\Big) - \frac{\gamma\bar{n}}{2}\,\mathrm{tr}\Big(\hat{a}^3\hat{a}^\dagger\hat{\rho}\Big) - \frac{\gamma\bar{n}}{2}\,\mathrm{tr}\Big(\hat{a}^2\hat{\rho}\hat{a}\hat{a}^\dagger\Big) && (67)
\end{aligned}$$

Using cyclic permutations of the trace and combining terms we get

$$\mathrm{tr}\big(\hat{a}^2\mathcal{L}\hat{\rho}\big) = i\omega\,\mathrm{tr}\Big(\Big[\hat{a}^\dagger\hat{a},\hat{a}^2\Big]\hat{\rho}\Big) + \frac{\gamma(\bar{n}+1)}{2}\,\mathrm{tr}\Big(\Big[\hat{a}^\dagger\hat{a},\hat{a}^2\Big]\hat{\rho}\Big) + \frac{\gamma\bar{n}}{2}\,\mathrm{tr}\Big(\Big[\hat{a}^2,\hat{a}\hat{a}^\dagger\Big]\hat{\rho}\Big) \tag{68}$$

The commutators are

$$\begin{aligned}
\Big[\hat{a}^\dagger\hat{a},\hat{a}^2\Big] &= \Big[\hat{a}^\dagger,\hat{a}^2\Big]\hat{a} = \hat{a}\Big[\hat{a}^\dagger,\hat{a}\Big]\hat{a} + \Big[\hat{a}^\dagger,\hat{a}\Big]\hat{a}^2 = -2\hat{a}^2 && (69)\\
\Big[\hat{a}^2,\hat{a}\hat{a}^\dagger\Big] &= \hat{a}\Big[\hat{a}^2,\hat{a}^\dagger\Big] = \hat{a}^2\Big[\hat{a},\hat{a}^\dagger\Big] + \hat{a}\Big[\hat{a},\hat{a}^\dagger\Big]\hat{a} = 2\hat{a}^2 && (70)
\end{aligned}$$

Thus

$$\begin{aligned}
\mathrm{tr}\big(\hat{a}^2\mathcal{L}\hat{\rho}\big) &= -2i\omega\,\mathrm{tr}\big(\hat{a}^2\hat{\rho}\big) - \gamma(\bar{n}+1)\,\mathrm{tr}\big(\hat{a}^2\hat{\rho}\big) + \gamma\bar{n}\,\mathrm{tr}\big(\hat{a}^2\hat{\rho}\big) = -(\gamma+2i\omega)\,\mathrm{tr}\big(\hat{a}^2\hat{\rho}\big) && (71)\\
&= -(\gamma+2i\omega)\left\langle \hat{a}^2 \right\rangle. && (72)
\end{aligned}$$

By taking the hermition conjugate of the above we get

$$\mathrm{tr}\Big((\hat{a}^\dagger)^2\mathcal{L}\hat{\rho}\Big) = (\gamma - 2i\omega)\,\mathrm{tr}\Big((\hat{a}^\dagger)^2\hat{\rho}\Big) = -(\gamma - 2i\omega)\left\langle(\hat{a}^\dagger)^2\right\rangle. \tag{73}$$

Now we do the same for $\mathrm{tr}(\hat{a}^\dagger\hat{a}\mathcal{L}\hat{\rho})$, so

$$\begin{aligned}
\mathrm{tr}\Big(\hat{a}^\dagger\hat{a}\mathcal{L}\hat{\rho}\Big) &= -\frac{i}{\hbar}\,\mathrm{tr}\Big(\hat{a}^\dagger\hat{a}\Big[\hat{H},\hat{\rho}\Big]\Big) + \gamma(\bar{n}+1)\,\mathrm{tr}\Big(\hat{a}^\dagger\hat{a}\mathcal{D}[\hat{a}]\hat{\rho}\Big) + \gamma\bar{n}\,\mathrm{tr}\Big(\hat{a}^\dagger\hat{a}\mathcal{D}[\hat{a}^\dagger]\hat{\rho}\Big) && (74)\\
&= -i\omega\,\mathrm{tr}\Big(\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}\hat{\rho}\Big) + i\omega\,\mathrm{tr}\Big(\hat{a}^\dagger\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a}\Big) && (75)\\
&+ \gamma(\bar{n}+1)\,\mathrm{tr}\Big(\hat{a}^\dagger\hat{a}^2\hat{\rho}\hat{a}^\dagger\Big) - \frac{\gamma(\bar{n}+1)}{2}\,\mathrm{tr}\Big(\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}\hat{\rho}\Big) - \frac{\gamma(\bar{n}+1)}{2}\,\mathrm{tr}\Big(\hat{a}^\dagger\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a}\Big) && (76)\\
&+ \gamma\bar{n}\,\mathrm{tr}\Big(\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{\rho}\hat{a}\Big) - \frac{\gamma\bar{n}}{2}\,\mathrm{tr}\Big(\hat{a}^\dagger\hat{a}^2\hat{a}^\dagger\hat{\rho}\Big) - \frac{\gamma\bar{n}}{2}\,\mathrm{tr}\Big(\hat{a}^\dagger\hat{a}\hat{\rho}\hat{a}\hat{a}^\dagger\Big) && (77)
\end{aligned}$$

using cyclic permutations of the trace and combining terms we get

$$\begin{aligned}
\mathrm{tr}\Big(\hat{a}^\dagger\hat{a}\mathcal{L}\hat{\rho}\Big) &= i\omega\,\mathrm{tr}\Big(\Big[\hat{a}^\dagger\hat{a},\hat{a}^\dagger\hat{a}\Big]\hat{\rho}\Big) + \gamma(\bar{n}+1)\,\mathrm{tr}\Big((\hat{a}^\dagger)^2\hat{a}^2\hat{\rho}\Big) - \gamma(\bar{n}+1)\,\mathrm{tr}\Big(\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}\hat{\rho}\Big) && (78)\\
&+ \gamma\bar{n}\,\mathrm{tr}\Big(\hat{a}\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{\rho}\Big) - \frac{\gamma\bar{n}}{2}\,\mathrm{tr}\Big(\hat{a}^\dagger\hat{a}^2\hat{a}^\dagger\hat{\rho}\Big) - \frac{\gamma\bar{n}}{2}\,\mathrm{tr}\Big(\hat{a}(\hat{a}^\dagger)^2\hat{a}\hat{\rho}\Big) && (79)
\end{aligned}$$

Commuting to simplify we get

$$\mathrm{tr}\Big(\hat{a}^\dagger\hat{a}\mathcal{L}\hat{\rho}\Big) = -\gamma(\bar{n}+1)\,\mathrm{tr}\Big(\hat{a}^\dagger\hat{a}\hat{\rho}\Big) - \gamma\bar{n}\,\mathrm{tr}\Big(\Big[\hat{a}\hat{a}^\dagger,\hat{a}^\dagger\hat{a}\Big]\hat{\rho}\Big) + \gamma\bar{n}\,\mathrm{tr}\Big(\hat{a}\hat{a}^\dagger\hat{\rho}\Big) \tag{80}$$

The commutator is equal to 0, and thus we get

$$\mathrm{tr}\Big(\hat{a}^\dagger\hat{a}\mathcal{L}\hat{\rho}\Big) = -\gamma(\bar{n}+1)\,\mathrm{tr}\Big(\hat{a}^\dagger\hat{a}\hat{\rho}\Big) + \gamma\bar{n} + \gamma\bar{n}\,\mathrm{tr}\Big(\hat{a}^\dagger\hat{a}\hat{\rho}\Big) = -\gamma\,\mathrm{tr}\Big(\hat{a}^\dagger\hat{a}\hat{\rho}\Big) + \gamma\bar{n} = -\gamma\left\langle\hat{a}^\dagger\hat{a}\right\rangle + \gamma\bar{n}. \tag{81}$$

Similarly we can find that

$$\mathrm{tr}\Big(\hat{a}\hat{a}^\dagger\mathcal{L}\hat{\rho}\Big) = -\gamma\,\mathrm{tr}\Big(\hat{a}\hat{a}^\dagger\hat{\rho}\Big) + \gamma(\bar{n}+1) = -\gamma\left\langle\hat{a}\hat{a}^\dagger\right\rangle + \gamma(\bar{n}+1). \tag{82}$$

Then since $\hat{x}^2$ commutes with $\hat{x}$ (the measurement operator) we can write Eq. (61) the equation of motion

$$\partial_t \left\langle \hat{x}^2 \right\rangle = \mathrm{tr}\big(\hat{x}^2 \mathcal{L}\hat{\rho}\big) \tag{83}$$

$$= \frac{\hbar}{2m\omega} \Big(-(\gamma + 2i\omega) \left\langle \hat{a}^2 \right\rangle - (\gamma - 2i\omega) \left\langle (\hat{a}^\dagger)^2 \right\rangle - \gamma \left\langle \hat{a}^\dagger \hat{a} \right\rangle + \gamma\bar{n} - \gamma \left\langle \hat{a}\hat{a}^\dagger \right\rangle + \gamma(\bar{n}+1) \Big) \tag{84}$$

$$= -\gamma \frac{\hbar}{2m\omega} \Big(\left\langle \hat{a}^2 \right\rangle + \left\langle (\hat{a}^\dagger)^2 \right\rangle + \left\langle \hat{a}^\dagger \hat{a} \right\rangle + \left\langle \hat{a}\hat{a}^\dagger \right\rangle \Big) + \frac{i\hbar}{m} \Big(\left\langle (\hat{a}^\dagger)^2 \right\rangle - \left\langle \hat{a}^2 \right\rangle \Big) + \frac{\gamma\hbar}{m\omega}(\bar{n} + 1/2) \tag{85}$$

Since we have

$$\{\hat{x}, \hat{p}\} = i\hbar(\hat{a}^2 - (\hat{a}^\dagger)^2) \tag{86}$$

We can write the equation of motion as

$$\partial_t \left\langle \hat{x}^2 \right\rangle = -\gamma \left\langle \hat{x}^2 \right\rangle - \frac{1}{m} \left\langle \{\hat{x}, \hat{p}\} \right\rangle + \frac{\gamma\hbar}{m\omega}(\bar{n} + 1/2) \tag{87}$$

B.2 Solving for $\partial_t \langle \hat{p}^2 \rangle$

Writing Eq. (57) as

$$\partial_t \left\langle \hat{p}^2 \right\rangle = \mathrm{tr}\big(\hat{p}^2 \mathcal{L}\hat{\rho}\big) + \mathrm{tr}\big(\hat{p}^2 \lambda \mathcal{D}[\hat{x}]\hat{\rho}\big) \tag{88}$$

Rewriting the left terms using $\hat{a}, \hat{a}^\dagger$ we get

$$\mathrm{tr}\big(\hat{p}^2 \mathcal{L}\hat{\rho}\big) = -\frac{m\omega\hbar}{2} \mathrm{tr}\Big((\hat{a}^2 + (\hat{a}^\dagger)^2 - \hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger)\mathcal{L}\hat{\rho} \Big) \tag{89}$$

$$= -\frac{m\omega\hbar}{2} \Big(\mathrm{tr}\big(\hat{a}^2 \mathcal{L}\hat{\rho}\big) + \mathrm{tr}\Big((\hat{a}^\dagger)^2 \mathcal{L}\hat{\rho}\Big) - \mathrm{tr}\Big(\hat{a}^\dagger\hat{a}\mathcal{L}\hat{\rho}\Big) - \mathrm{tr}\Big(\hat{a}\hat{a}^\dagger \mathcal{L}\hat{\rho}\Big) \Big) \tag{90}$$

We have solved for these before, so we can write

$$\mathrm{tr}\big(\hat{p}^2 \mathcal{L}\hat{\rho}\big) = -\frac{m\omega\hbar}{2} \Big(-(\gamma + 2i\omega) \left\langle \hat{a}^2 \right\rangle - (\gamma - 2i\omega) \left\langle (\hat{a}^\dagger)^2 \right\rangle + \gamma \left\langle \hat{a}^\dagger \hat{a} \right\rangle - \gamma\bar{n} + \gamma \left\langle \hat{a}\hat{a}^\dagger \right\rangle - \gamma(\bar{n}+1) \Big) \tag{91}$$

$$= -\gamma \left\langle \hat{p}^2 \right\rangle + m i\omega^2\hbar \Big(\left\langle \hat{a}^2 \right\rangle - \left\langle (\hat{a}^\dagger)^2 \right\rangle \Big) + \gamma m\omega\hbar(\bar{n} + 1/2) \tag{92}$$

$$= -\gamma \left\langle \hat{p}^2 \right\rangle + m\omega^2 \left\langle \{\hat{x}, \hat{p}\} \right\rangle + \gamma m\omega\hbar(\bar{n} + 1/2) \tag{93}$$

Solving the measurement term

$$\mathrm{tr}\big(\hat{p}^2\lambda\mathcal{D}[\hat{x}]\hat{\rho}\big) = \lambda\left(\mathrm{tr}\big(\hat{p}^2\hat{x}\hat{\rho}\hat{x}\big) - \frac{1}{2}\,\mathrm{tr}\big(\hat{p}^2\hat{x}^2\hat{\rho}\big) - \frac{1}{2}\,\mathrm{tr}\big(\hat{p}^2\hat{\rho}\hat{x}^2\big)\right) \tag{94}$$

$$= \lambda\left(\mathrm{tr}\big(\hat{x}\hat{p}^2\hat{x}\hat{\rho}\big) - \frac{1}{2}\,\mathrm{tr}\big(\hat{p}^2\hat{x}^2\hat{\rho}\big) - \frac{1}{2}\,\mathrm{tr}\big(\hat{x}^2\hat{p}^2\hat{\rho}\big)\right) \tag{95}$$

Commuting the operators

$$\hat{x}\hat{p}^2\hat{x} = \frac{1}{2}(\hat{p}\hat{x}+i\hbar)\hat{p}\hat{x} + \frac{1}{2}\hat{x}\hat{p}(\hat{x}\hat{p}-i\hbar) = \frac{1}{2}\hat{p}\hat{x}\hat{p}\hat{x} + \frac{1}{2}\hat{x}\hat{p}\hat{x}\hat{p} + \frac{i\hbar}{2}[\hat{p},\hat{x}] \tag{96}$$

$$= \frac{1}{2}\hat{p}\hat{x}\hat{p}\hat{x} + \frac{1}{2}\hat{x}\hat{p}\hat{x}\hat{p} + \frac{\hbar^2}{2} \tag{97}$$

$$\hat{p}^2\hat{x}^2 = \hat{p}(\hat{x}\hat{p}-i\hbar)\hat{x} = \hat{p}\hat{x}\hat{p}\hat{x} - i\hbar\hat{p}\hat{x} \tag{98}$$

$$\hat{x}^2\hat{p}^2 = \hat{x}(\hat{p}\hat{x}+i\hbar)\hat{p} = \hat{x}\hat{p}\hat{x}\hat{p} + i\hbar\hat{x}\hat{p} \tag{99}$$

This gives

$$\mathrm{tr}\big(\hat{p}^2\lambda\mathcal{D}[\hat{x}]\hat{\rho}\big) = \lambda\left(\frac{1}{2}\,\mathrm{tr}(\hat{p}\hat{x}\hat{p}\hat{x}\hat{\rho}) + \frac{1}{2}\,\mathrm{tr}(\hat{x}\hat{p}\hat{x}\hat{p}\hat{\rho}) + \frac{\hbar^2}{2} - \frac{1}{2}\,\mathrm{tr}(\hat{p}\hat{x}\hat{p}\hat{x}\hat{\rho}) - \frac{1}{2}\,\mathrm{tr}(\hat{x}\hat{p}\hat{x}\hat{p}\hat{\rho}) + \frac{i\hbar}{2}\,\mathrm{tr}([\hat{p},\hat{x}])\right) \tag{100}$$

$$= \frac{\lambda\hbar^2}{2} + \frac{\lambda i\hbar}{2}\,\mathrm{tr}([\hat{p},\hat{x}]) = \lambda\hbar^2 \tag{101}$$

The final equation of motion then becomes

$$\partial_t\left\langle\hat{p}^2\right\rangle = -\gamma\left\langle\hat{p}^2\right\rangle + m\omega^2\left\langle\{\hat{x},\hat{p}\}\right\rangle + \gamma m\omega\hbar(\bar{n}+1/2) + \lambda\hbar^2 \tag{102}$$

B.3 Solving for $\partial_t\left\langle\{\hat{x},\hat{p}\}\right\rangle$

Writing Eq. (58) as

$$\partial_t\left\langle\{\hat{x},\hat{p}\}\right\rangle = \mathrm{tr}(\{\hat{x},\hat{p}\}\mathcal{L}\hat{\rho}) + \mathrm{tr}(\{\hat{x},\hat{p}\}\lambda\mathcal{D}[\hat{x}]\hat{\rho}) \tag{103}$$

Rewriting the left term as

$$\mathrm{tr}(\{\hat{x},\hat{p}\}\mathcal{L}\hat{\rho}) = i\hbar\,\mathrm{tr}\Big((\hat{a}^2-(\hat{a}^\dagger)^2)\mathcal{L}\hat{\rho}\Big) = i\hbar\,\mathrm{tr}\big(\hat{a}^2\mathcal{L}\hat{\rho}\big) - i\hbar\,\mathrm{tr}\Big((\hat{a}^\dagger)^2\mathcal{L}\hat{\rho}\Big) \tag{104}$$

We have calculated these terms before, so we can write

$$\operatorname{tr}(\{\hat{x},\hat{p}\}\mathcal{L}\hat{\rho}) = -i\hbar\left((\gamma+2i\omega)\left\langle\hat{a}^2\right\rangle - (\gamma-2i\omega)\left\langle(\hat{a}^\dagger)^2\right\rangle\right) \tag{105}$$

$$= -\gamma i\hbar\left(\left\langle\hat{a}^2\right\rangle - \left\langle(\hat{a}^\dagger)^2\right\rangle\right) + 2\hbar\omega\left(\left\langle\hat{a}^2\right\rangle + \left\langle(\hat{a}^\dagger)^2\right\rangle\right) \tag{106}$$

We also have that

$$\hat{a}^2 + (\hat{a}^\dagger)^2 = \frac{m\omega}{\hbar}\hat{x}^2 - \frac{1}{m\omega\hbar}\hat{p}^2 \tag{107}$$

Thus we get

$$\operatorname{tr}(\{\hat{x},\hat{p}\}\mathcal{L}\hat{\rho}) = -\gamma\left\langle\{\hat{x},\hat{p}\}\right\rangle + 2m\omega^2\left\langle\hat{x}^2\right\rangle - \frac{2}{m}\left\langle\hat{p}^2\right\rangle \tag{108}$$

Solving for the measurement term

$$\lambda\operatorname{tr}(\{\hat{x},\hat{p}\}\mathcal{D}[\hat{x}]\hat{\rho}) = \frac{\lambda}{2}\left(\operatorname{tr}\left(\left[\hat{x}^2,\hat{p}\hat{x}\right]\hat{\rho}\right) + \operatorname{tr}\left(\left[\hat{x}\hat{p},\hat{x}^2\right]\right)\right) \tag{109}$$

Solving the commutators shows that this will become zero.

B.4 Results

The final equations of motion are

$$\partial_t\left\langle\hat{x}^2\right\rangle = -\gamma\left\langle\hat{x}^2\right\rangle - \frac{1}{m}\left\langle\{\hat{x},\hat{p}\}\right\rangle + \frac{\gamma\hbar}{m\omega}(\bar{n}+1/2) \tag{110}$$

$$\partial_t\left\langle\hat{p}^2\right\rangle = -\gamma\left\langle\hat{p}^2\right\rangle + m\omega^2\left\langle\{\hat{x},\hat{p}\}\right\rangle + \gamma m\omega\hbar(\bar{n}+1/2) + \lambda\hbar^2 \tag{111}$$

$$\partial_t\left\langle\{\hat{x},\hat{p}\}\right\rangle = -\gamma\left\langle\{\hat{x},\hat{p}\}\right\rangle + 2m\omega^2\left\langle\hat{x}^2\right\rangle - \frac{2}{m}\left\langle\hat{p}^2\right\rangle \tag{112}$$