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Continuous Measurements and Feedback Control of a Quantum Harmonic Oscillator

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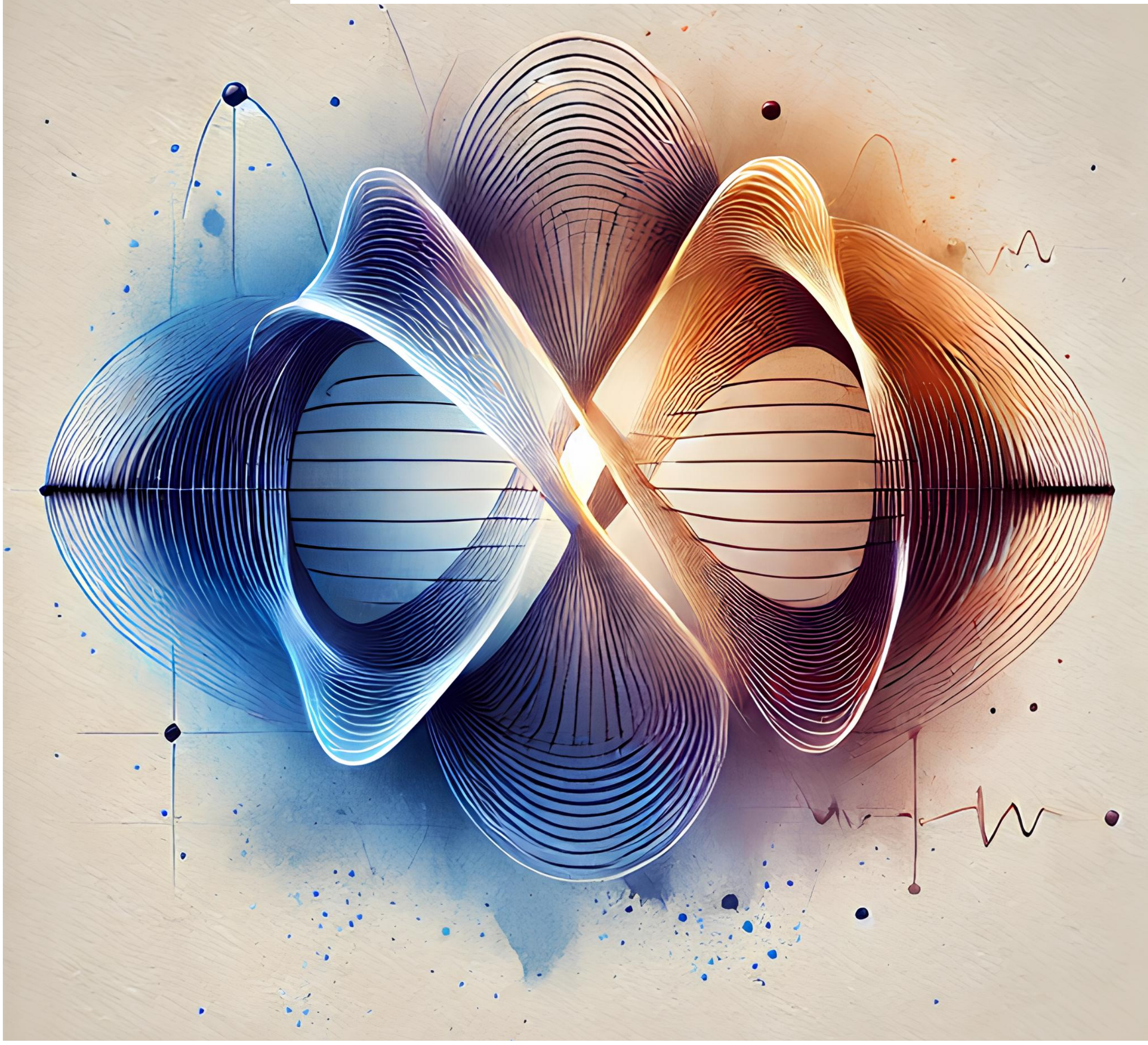


Contents

1 Introduction

As our society is becoming increasingly dependent on technology, the demand for better and more efficient technologies is growing. One kind of technology which has been really important since its conception during the second world war by Alan Turing is computers [1]. The field of computer science has been researched a lot during the second half of the 20th century but has hit a fundamental problem. Our current classical computers have reached a limit where the transistors cannot get any smaller without quantum mechanics causing issues [1]. This has prompted research into quantum technologies such as quantum computers and quantum simulations [1].

To properly understand how quantum mechanical systems work, it is important to understand what happens to them when they are interacted with, during for example a measurement [2]. It is also this interaction with a quantum system, which has prompted many interpretations of quantum mechanics and given rise to what is colloquially known as the measurement problem [2]. The measurement problem is a fundamental philosophical problem in quantum mechanics, which arises since a quantum mechanical state evolves deterministically according to the Schrödinger equation, but collapses probabilistically when measured or interacted with [2].

It is also interesting to see how such a system can be manipulated to create a certain state which can be used for a specific purpose. This could include, but is not limited to, creating a qubit state to be used in a quantum computer or a state which can simulate a certain physical system [1]. Here it is important to understand the effect feedback has on a measured system, and how to utilize this to create a desired state [3].

1.1 Outline

This thesis will look at a quantum harmonic oscillator which is coupled to an environment. Thus creating an open quantum system whose state is temperature dependent. The system will be measured continuously using weak measurements with a feedback loop to control the system. The goal is to see how the system evolves under these conditions, and how the feedback loop can be changed and manipulated to observe different behaviours.

2 Theoretical Framework

2.1 Quantum Harmonic Oscillator

The quantum harmonic oscillator (QHO) is a quantum mechanical system useful for many applications. This stems from the fact that many systems can be approximated as harmonic close to their equilibrium position, and since the QHO is a simple system, which is possible to solve analytically it is a good starting approximation. The system has the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right), \tag{1}$$

where $\hat{p}$ and $\hat{x}$ are the momentum and position operators, m is the mass of the particle, and ω is the angular frequency of the oscillator. The operators $\hat{a}^\dagger$ and $\hat{a}$ are the creation and annihilation operators, which are defined as

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right) \quad \text{and} \quad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right). \tag{2}$$

It is also reasonable to mention the number operator $\hat{n} = \hat{a}^\dagger\hat{a}$ which has the number states, or Fock states, $|n\rangle$ as its eigenstates.

2.2 Open Quantum Systems

With an open quantum system we mean a quantum system which in some ways interact with an environment. This interaction could be described as a thermal coupling between the main system and some temperature bath. This will cause the system to be in a thermal equilibrium with the environment if left alone, and therefore it will be dependent on the temperature. Unless the temperature of the bath is zero, the system will be in a mixed state, described by a density matrix $\hat{\rho}$. Notably, if the temperature is zero, it is equivalent of considering a closed quantum system, since the coupling is thermal. [3]

The thermal coupling will lead to dissipation of quantum information from the system to the environment. During this process, the system loses coherence. That is, the quantum mechanical properties of the system are lost and a classical description of the state becomes more appropriate. The coherence of the system is manifested in the off-diagonal elements of the density matrix. If the off-diagonal elements are zero, either by dissipation to the environment or by other means of decoherence, the system will exist in a classical probabilistic state, and any superposition of states will be lost. [3]

The combination of the system and environment can be considered a closed system, though more complicated than the main system itself. Then by performing a partial trace over the environment a description of the system alone arises at the cost of losing information about the correlation between the two parts [3]. This introduces an uncertainty in the state, and it is therefore necessary to treat the resulting system to be in a mixed state. If the coupling is also weak the system, after the environment has been traced over, will follow a Markovian master equation [3].

2.2.1 Master Equation

The evolution of an open quantum system can be described by a master equation, which is a differential equation and generalization of the Schrödinger equation to involve open quantum systems instead of pure states [3]. By introducing the super operator

$$\mathcal{D}[\hat{L}_k]\hat{\rho} = \hat{L}_k\hat{\rho}\hat{L}_k^\dagger - \frac{1}{2}\left\{\hat{L}_k^\dagger\hat{L}_k, \hat{\rho}\right\}, \tag{3}$$

where $\hat{L}_k$ are called Lindblad operators, the master equation on Lindblad form can be written as

$$\partial_t\hat{\rho} = -\frac{i}{\hbar}\left[\hat{H}, \hat{\rho}\right] + \sum_k \gamma_k \mathcal{D}[\hat{L}_k]\hat{\rho}, \tag{4}$$

where $\hat{H}$ is the hamiltonian of the system, and γ_k are the decay rates of the system, relating the decoherence to the environment depending on the coupling to the system [3]. If $\gamma_k = 0$ for all k the equation reduces to the von Neumann equation for a closed quantum system and the coupling to the bath is removed. The remaining term thus describes the unitary time evolution of the system and is the analogue of the Schrödinger equation for the density matrix formalism [3]. At this stage one might also introduce the Liouvillian super operator and write the master equation more compactly as

$$\partial_t\hat{\rho} = \mathcal{L}\hat{\rho}. \tag{5}$$

This compactness will be useful when considering other types of perturbing effects on the system such as measurements and feedback [3].

For the special case considered in this thesis where the system is a QHO, the Lindblad operators can be chosen as $\hat{L}_1 = \hat{a}$ and $\hat{L}_2 = \hat{a}^\dagger$ with coefficients $\gamma_1 = \gamma(\bar{n}+1)$ and $\gamma_2 = \gamma\bar{n}$, where γ is a decay rate and $\bar{n}$ is the thermal occupation number which can be written as

$$\bar{n} = \frac{1}{e^{\hbar\omega/k_{\mathrm{B}}T} - 1}, \tag{6}$$

where T is the temperature of the bath and k_{B} is the Boltzmann constant [4]. We can also note that $\hat{L}_1$ and γ_1 refer to the spontaneous emission from the system to the environment while $\hat{L}_2$ refer to spontaneous absorption from the environment to the system [4]. Notably, for $T = 0$, the thermal occupation is $\bar{n} = 0$ and the system will only exhibit emission and will decay.

2.3 Continuous Measurements

Measurement is a process which introduces decoherence in the system, and it is therefore interesting to look at its effects [2]. The simplest view on measurements takes the form of von Neumann measurements. This type of measurement is described by a set of measurement operators which projects the system onto the eigenstates of the observable [3]. This essentially means that all quantum information in the system is lost and full decoherence has happened. By generalizing the measurement theory one can derive what is called positive operator-valued measurement (POVM) [3].

Since the POVM is not necessarily a projective von Neumann measurement all coherence need not be lost after the measurement. Thus, this opens up for the possibility of performing time continuous weak measurement [3]. To describe this type of POVM we first consider a Gaussian measurement operator

$$\hat{K} = \left(\frac{2\bar{\lambda}}{\pi}\right)^{1/4} e^{-\bar{\lambda}(z-\hat{A})^2}, \tag{7}$$

where $\bar{\lambda}$ represents the strength of the measurement, z is a continuous outcome of the measurement, and $\hat{A}$ is the measured observable [3]. Then by discretizing the time interval to segments of $\mathrm{d}t$ and defining $\bar{\lambda} = \lambda \mathrm{d}t$ we approach a situation where in the limit $\mathrm{d}t \to 0$ all measurements will be weak, and the coherence of the system is minimally affected [3]. Considering the stochastic nature of the process and averaging the possible trajectories one can derive the master equation [3] in Lindblad form to be

$$\partial_t \hat{\rho} = \mathcal{L}\hat{\rho} + \lambda \mathcal{D}[\hat{A}]\hat{\rho}. \tag{8}$$

2.4 Feedback Control

2.5 Wigner Function

3 Result

Consider a QHO described by the Hamiltonian in Eq. (1) which is coupled to a thermal bath with temperature T. If the system is also continuously measured the evolution of the system can be described by the master equation in Eq. (8) using the Lindblad operators mentioned in Sec. 2.2.1. We want to solve for the equations of motions when measuring for the position quadrature

$$\partial_t \left\langle \hat{x}^2 \right\rangle = \mathrm{tr}\left(\hat{x}^2 \partial_t \hat{\rho}\right) \tag{9}$$

$$\partial_t \left\langle \hat{p}^2 \right\rangle = \mathrm{tr}\left(\hat{p}^2 \partial_t \hat{\rho}\right) \tag{10}$$

$$\partial_t \left\langle \{\hat{x}, \hat{p}\} \right\rangle = \mathrm{tr}(\{\hat{x}, \hat{p}\} \partial_t \hat{\rho}). \tag{11}$$

Solving these equations yields the following EOM

$$\partial_t \left\langle \hat{x}^2 \right\rangle = -\gamma \left\langle \hat{x}^2 \right\rangle + \frac{1}{m} \left\langle \{\hat{x}, \hat{p}\} \right\rangle + \frac{\gamma \hbar}{m\omega}(\bar{n} + 1/2) \tag{12}$$

$$\partial_t \left\langle \hat{p}^2 \right\rangle = -\gamma \left\langle \hat{p}^2 \right\rangle - m\omega^2 \left\langle \{\hat{x}, \hat{p}\} \right\rangle + \gamma m\omega\hbar(\bar{n} + 1/2) + \lambda\hbar^2 \tag{13}$$

$$\partial_t \left\langle \{\hat{x}, \hat{p}\} \right\rangle = -\gamma \left\langle \{\hat{x}, \hat{p}\} \right\rangle + 2m\omega^2 \left\langle \hat{x}^2 \right\rangle - \frac{2}{m} \left\langle \hat{p}^2 \right\rangle \tag{14}$$

for more detailed calculations see App. B. Performing a change of variable to make the equations dimensionless

$$\tilde{x} = \sqrt{\frac{m\omega}{\hbar}} x \quad \text{and} \quad \tilde{p} = \sqrt{\frac{1}{m\omega\hbar}} p \tag{15}$$

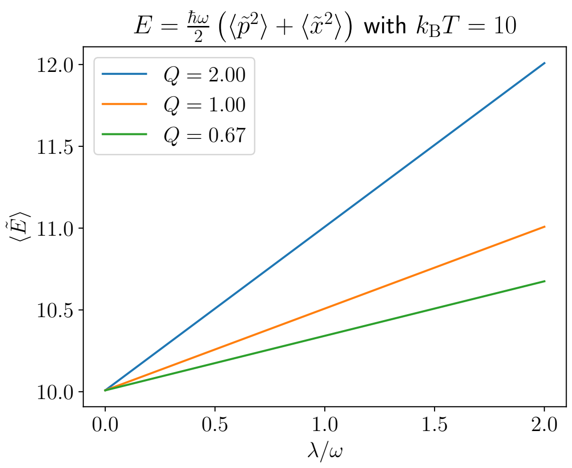
we can solve for the steady state. By introducing the quality factor $Q = \omega/\gamma$ we obtain the steady state solutions

$$\left\langle \tilde{x}^2 \right\rangle = (\bar{n} + 1/2) + \frac{\lambda\hbar}{m\omega^2} \frac{2Q^3}{4Q^2 - 1} \tag{16}$$

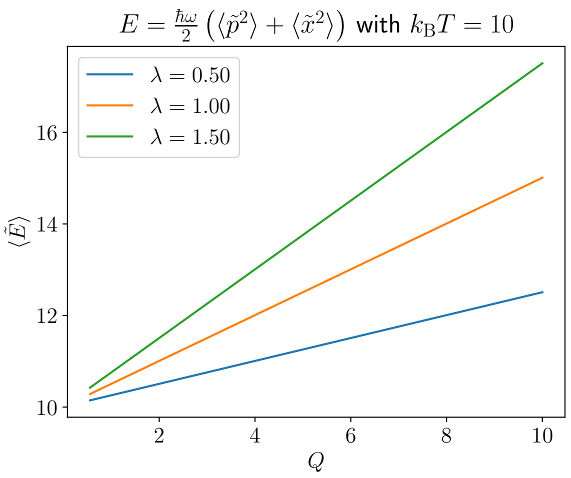
$$\left\langle \tilde{p}^2 \right\rangle = (\bar{n} + 1/2) + \frac{\lambda\hbar}{m\omega^2} \left(Q - \frac{2Q^3}{4Q^2 - 1} \right) \tag{17}$$

$$E = \left\langle \tilde{H} \right\rangle = \frac{\hbar\omega}{2} \left(\left\langle \tilde{p}^2 \right\rangle + \left\langle \tilde{x}^2 \right\rangle \right). \tag{18}$$

Plotting the energy of the system



(a) The energy plotted against λ/ω with varying Q.



(b) The energy plotted against Q with varying λ.

Figure 1: The energy plotted against λ/ω and Q with the parameters $k_BT = 10$ and $\omega = 1$. This gives $\bar{n} + 1/2 \approx 10$

Looking at Fig. 1 one can see that a stronger measurement correlates to the system steady state increasing in energy. It is also noteworthy that the same applies for the quality factor. Thus, by continuously measuring the system we add energy into it, which make the steady state higher in energy than what the thermal effects from the bath would otherwise place it.

4 Discussion

5 Outlook

References

[1] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information: 10th Anniversary Edition.* Cambridge: Cambridge University Press, 2010.

[2] A. N. Jordan and I. A. Siddiqi, *Quantum Measurement: Theory and Practice.* Cambridge: Cambridge University Press, 2024.

[3] B. Annby-Andersson, *Continuous measurements of small systems: Feedback control, thermodynamics, entanglement.* Doctoral thesis (compilation), Lund University, 4 2024.

[4] P. Meystre, *Quantum Optics: Taming the Quantum.* Springer Charm, 2021.

A Appendix

B EOM Calculation

Consider a quantum harmonic oscillator described by the Hamiltonian

$$\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + 1/2). \tag{19}$$

Consider also that the quantum harmonic oscillator is subject to a temperature bath and is continuously measured. The system then evolves according to the master equation

$$\partial_t\hat{\rho} = -\frac{i}{\hbar}[\hat{H},\hat{\rho}] + \gamma(\bar{n}+1)\mathcal{D}[\hat{a}]\hat{\rho} + \gamma\bar{n}\mathcal{D}[\hat{a}^\dagger]\hat{\rho} + \lambda\mathcal{D}[\hat{A}]\hat{\rho} \tag{20}$$

where γ is the damping rate, $\bar{n}$ is the thermal occupancy defined as

$$\bar{n} = \frac{1}{e^{\hbar\omega/k_BT}-1}, \tag{21}$$

λ is the measurement rate, $\hat{A}$ is the measurement operator, and $\mathcal{D}[\hat{O}]$ is the Lindblad super operator defined as

$$\mathcal{D}[\hat{O}]\hat{\rho} = \hat{O}\hat{\rho}\hat{O}^\dagger - \frac{1}{2}\{\hat{O}^\dagger\hat{O},\hat{\rho}\}. \tag{22}$$

Then we also have the quadrature operators defined as

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}+\hat{a}^\dagger), \quad \hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a}-\hat{a}^\dagger). \tag{23}$$

We want to find the equations of motion when measuring the position quadrature

$$\partial_t\left\langle\hat{x}^2\right\rangle = \mathrm{tr}\big(\hat{x}^2\partial_t\hat{\rho}\big) \tag{24}$$

$$\partial_t\left\langle\hat{p}^2\right\rangle = \mathrm{tr}\big(\hat{p}^2\partial_t\hat{\rho}\big) \tag{25}$$

$$\partial_t\left\langle\{\hat{x},\hat{p}\}\right\rangle = \mathrm{tr}(\{\hat{x},\hat{p}\}\partial_t\hat{\rho}). \tag{26}$$

We start by defining the Liouvillian super operator

$$\mathcal{L}\hat{\rho} = -\frac{i}{\hbar}[\hat{H},\hat{\rho}] + \gamma(\bar{n}+1)\mathcal{D}[\hat{a}]\hat{\rho} + \gamma\bar{n}\mathcal{D}[\hat{a}^\dagger]\hat{\rho} \tag{27}$$

which turns the master equation into

$$\partial_t \hat{\rho} = \mathcal{L}\hat{\rho} + \lambda \mathcal{D}[\hat{x}]\hat{\rho}. \tag{28}$$

B.1 Solving for $\partial_t \langle \hat{x}^2 \rangle$

We can write Eq. (24) as

$$\partial_t \langle \hat{x}^2 \rangle = \mathrm{tr}\big(\hat{x}^2 \mathcal{L}\hat{\rho}\big) + \mathrm{tr}\big(\hat{x}^2 \lambda \mathcal{D}[\hat{x}]\hat{\rho}\big) \tag{29}$$

and then rewriting the left term using the creation and annihilation operators we get

$$\begin{aligned}
\mathrm{tr}\big(\hat{x}^2 \mathcal{L}\hat{\rho}\big) &= \frac{\hbar}{2m\omega}\,\mathrm{tr}\Big((\hat{a}^2 + (\hat{a}^\dagger)^2 + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a})\mathcal{L}\hat{\rho}\Big) && (30)\\
&= \frac{\hbar}{2m\omega}\Big[\mathrm{tr}\big(\hat{a}^2\mathcal{L}\hat{\rho}\big) + \mathrm{tr}\Big((\hat{a}^\dagger)^2\mathcal{L}\hat{\rho}\Big) + \mathrm{tr}\Big(\hat{a}\hat{a}^\dagger\mathcal{L}\hat{\rho}\Big) + \mathrm{tr}\Big(\hat{a}^\dagger\hat{a}\mathcal{L}\hat{\rho}\Big)\Big] && (31)
\end{aligned}$$

Solving these separately

$$\begin{aligned}
\mathrm{tr}\big(\hat{a}^2\mathcal{L}\hat{\rho}\big) &= -\frac{i}{\hbar}\,\mathrm{tr}\Big(\hat{a}^2\Big[\hat{H},\hat{\rho}\Big]\Big) + \gamma(\bar{n}+1)\,\mathrm{tr}\big(\hat{a}^2\mathcal{D}[\hat{a}]\hat{\rho}\big) + \gamma\bar{n}\,\mathrm{tr}\Big(\hat{a}^2\mathcal{D}[\hat{a}^\dagger]\hat{\rho}\Big) && (32)\\
&= -i\omega\,\mathrm{tr}\Big(\hat{a}^2\hat{a}^\dagger\hat{a}\hat{\rho}\Big) + i\omega\,\mathrm{tr}\Big(\hat{a}^2\hat{\rho}\hat{a}^\dagger\hat{a}\Big) && (33)\\
&+ \gamma(\bar{n}+1)\,\mathrm{tr}\Big(\hat{a}^3\hat{\rho}\hat{a}^\dagger\Big) - \frac{\gamma(\bar{n}+1)}{2}\,\mathrm{tr}\Big(\hat{a}^2\hat{a}^\dagger\hat{a}\hat{\rho}\Big) - \frac{\gamma(\bar{n}+1)}{2}\,\mathrm{tr}\Big(\hat{a}^2\hat{\rho}\hat{a}^\dagger\hat{a}\Big) && (34)\\
&+ \gamma\bar{n}\,\mathrm{tr}\Big(\hat{a}^2\hat{a}^\dagger\hat{\rho}\hat{a}\Big) - \frac{\gamma\bar{n}}{2}\,\mathrm{tr}\Big(\hat{a}^3\hat{a}^\dagger\hat{\rho}\Big) - \frac{\gamma\bar{n}}{2}\,\mathrm{tr}\Big(\hat{a}^2\hat{\rho}\hat{a}\hat{a}^\dagger\Big) && (35)
\end{aligned}$$

Using cyclic permutations of the trace and combining terms we get

$$\mathrm{tr}\big(\hat{a}^2\mathcal{L}\hat{\rho}\big) = i\omega\,\mathrm{tr}\Big(\Big[\hat{a}^\dagger\hat{a},\hat{a}^2\Big]\hat{\rho}\Big) + \frac{\gamma(\bar{n}+1)}{2}\,\mathrm{tr}\Big(\Big[\hat{a}^\dagger\hat{a},\hat{a}^2\Big]\hat{\rho}\Big) + \frac{\gamma\bar{n}}{2}\,\mathrm{tr}\Big(\Big[\hat{a}^2,\hat{a}\hat{a}^\dagger\Big]\hat{\rho}\Big) \tag{36}$$

The commutators are

$$\begin{aligned}
\Big[\hat{a}^\dagger\hat{a},\hat{a}^2\Big] &= \Big[\hat{a}^\dagger,\hat{a}^2\Big]\hat{a} = \hat{a}\Big[\hat{a}^\dagger,\hat{a}\Big]\hat{a} + \Big[\hat{a}^\dagger,\hat{a}\Big]\hat{a}^2 = -2\hat{a}^2 && (37)\\
\Big[\hat{a}^2,\hat{a}\hat{a}^\dagger\Big] &= \hat{a}\Big[\hat{a}^2,\hat{a}^\dagger\Big] = \hat{a}^2\Big[\hat{a},\hat{a}^\dagger\Big] + \hat{a}\Big[\hat{a},\hat{a}^\dagger\Big]\hat{a} = 2\hat{a}^2 && (38)
\end{aligned}$$

Thus

$$\begin{aligned}
\mathrm{tr}\big(\hat{a}^2\mathcal{L}\hat{\rho}\big) &= -2i\omega\,\mathrm{tr}\big(\hat{a}^2\hat{\rho}\big) - \gamma(\bar{n}+1)\,\mathrm{tr}\big(\hat{a}^2\hat{\rho}\big) + \gamma\bar{n}\,\mathrm{tr}\big(\hat{a}^2\hat{\rho}\big) = -(\gamma+2i\omega)\,\mathrm{tr}\big(\hat{a}^2\hat{\rho}\big) && (39)\\
&= -(\gamma+2i\omega)\,\langle\hat{a}^2\rangle\,. && (40)
\end{aligned}$$

By taking the hermition conjugate of the above we get

$$\mathrm{tr}\Big((\hat{a}^\dagger)^2\mathcal{L}\hat{\rho}\Big) = (\gamma - 2i\omega)\,\mathrm{tr}\Big((\hat{a}^\dagger)^2\hat{\rho}\Big) = -(\gamma - 2i\omega)\left\langle(\hat{a}^\dagger)^2\right\rangle. \tag{41}$$

Now we do the same for $\mathrm{tr}\big(\hat{a}^\dagger\hat{a}\mathcal{L}\hat{\rho}\big)$, so

$$\begin{aligned}
\mathrm{tr}\Big(\hat{a}^\dagger\hat{a}\mathcal{L}\hat{\rho}\Big) &= -\frac{i}{\hbar}\,\mathrm{tr}\Big(\hat{a}^\dagger\hat{a}\Big[\hat{H},\hat{\rho}\Big]\Big) + \gamma(\bar{n}+1)\,\mathrm{tr}\Big(\hat{a}^\dagger\hat{a}\mathcal{D}[\hat{a}]\hat{\rho}\Big) + \gamma\bar{n}\,\mathrm{tr}\Big(\hat{a}^\dagger\hat{a}\mathcal{D}[\hat{a}^\dagger]\hat{\rho}\Big) && (42)\\
&= -i\omega\,\mathrm{tr}\Big(\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}\hat{\rho}\Big) + i\omega\,\mathrm{tr}\Big(\hat{a}^\dagger\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a}\Big) && (43)\\
&+ \gamma(\bar{n}+1)\,\mathrm{tr}\Big(\hat{a}^\dagger\hat{a}^2\hat{\rho}\hat{a}^\dagger\Big) - \frac{\gamma(\bar{n}+1)}{2}\,\mathrm{tr}\Big(\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}\hat{\rho}\Big) - \frac{\gamma(\bar{n}+1)}{2}\,\mathrm{tr}\Big(\hat{a}^\dagger\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a}\Big) && (44)\\
&+ \gamma\bar{n}\,\mathrm{tr}\Big(\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{\rho}\hat{a}\Big) - \frac{\gamma\bar{n}}{2}\,\mathrm{tr}\Big(\hat{a}^\dagger\hat{a}^2\hat{a}^\dagger\hat{\rho}\Big) - \frac{\gamma\bar{n}}{2}\,\mathrm{tr}\Big(\hat{a}^\dagger\hat{a}\hat{\rho}\hat{a}\hat{a}^\dagger\Big) && (45)
\end{aligned}$$

using cyclic permutations of the trace and combining terms we get

$$\begin{aligned}
\mathrm{tr}\Big(\hat{a}^\dagger\hat{a}\mathcal{L}\hat{\rho}\Big) &= i\omega\,\mathrm{tr}\Big(\Big[\hat{a}^\dagger\hat{a},\hat{a}^\dagger\hat{a}\Big]\hat{\rho}\Big) + \gamma(\bar{n}+1)\,\mathrm{tr}\Big((\hat{a}^\dagger)^2\hat{a}^2\hat{\rho}\Big) - \gamma(\bar{n}+1)\,\mathrm{tr}\Big(\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}\hat{\rho}\Big) && (46)\\
&+ \gamma\bar{n}\,\mathrm{tr}\Big(\hat{a}\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{\rho}\Big) - \frac{\gamma\bar{n}}{2}\,\mathrm{tr}\Big(\hat{a}^\dagger\hat{a}^2\hat{a}^\dagger\hat{\rho}\Big) - \frac{\gamma\bar{n}}{2}\,\mathrm{tr}\Big(\hat{a}(\hat{a}^\dagger)^2\hat{a}\hat{\rho}\Big) && (47)
\end{aligned}$$

Commuting to simplify we get

$$\mathrm{tr}\Big(\hat{a}^\dagger\hat{a}\mathcal{L}\hat{\rho}\Big) = -\gamma(\bar{n}+1)\,\mathrm{tr}\Big(\hat{a}^\dagger\hat{a}\hat{\rho}\Big) - \gamma\bar{n}\,\mathrm{tr}\Big(\Big[\hat{a}\hat{a}^\dagger,\hat{a}^\dagger\hat{a}\Big]\hat{\rho}\Big) + \gamma\bar{n}\,\mathrm{tr}\Big(\hat{a}\hat{a}^\dagger\hat{\rho}\Big) \tag{48}$$

The commutator is equal to 0, and thus we get

$$\mathrm{tr}\Big(\hat{a}^\dagger\hat{a}\mathcal{L}\hat{\rho}\Big) = -\gamma(\bar{n}+1)\,\mathrm{tr}\Big(\hat{a}^\dagger\hat{a}\hat{\rho}\Big)+\gamma\bar{n}+\gamma\bar{n}\,\mathrm{tr}\Big(\hat{a}^\dagger\hat{a}\hat{\rho}\Big) = -\gamma\,\mathrm{tr}\Big(\hat{a}^\dagger\hat{a}\hat{\rho}\Big)+\gamma\bar{n} = -\gamma\left\langle\hat{a}^\dagger\hat{a}\right\rangle+\gamma\bar{n}. \tag{49}$$

Similarly we can find that

$$\mathrm{tr}\Big(\hat{a}\hat{a}^\dagger\mathcal{L}\hat{\rho}\Big) = -\gamma\,\mathrm{tr}\Big(\hat{a}\hat{a}^\dagger\hat{\rho}\Big) + \gamma(\bar{n}+1) = -\gamma\left\langle\hat{a}\hat{a}^\dagger\right\rangle + \gamma(\bar{n}+1). \tag{50}$$

Then since $\hat{x}^2$ commutes with $\hat{x}$ (the measurement operator) we can write Eq. (29) the equation of motion

$$\partial_t \left\langle \hat{x}^2 \right\rangle = \mathrm{tr}\big(\hat{x}^2 \mathcal{L}\hat{\rho}\big) \tag{51}$$

$$= \frac{\hbar}{2m\omega}\Big(-(\gamma+2i\omega)\left\langle \hat{a}^2\right\rangle - (\gamma-2i\omega)\left\langle (\hat{a}^\dagger)^2\right\rangle - \gamma\left\langle \hat{a}^\dagger\hat{a}\right\rangle + \gamma\bar{n} - \gamma\left\langle \hat{a}\hat{a}^\dagger\right\rangle + \gamma(\bar{n}+1)\Big) \tag{52}$$

$$= -\gamma\frac{\hbar}{2m\omega}\Big(\left\langle \hat{a}^2\right\rangle + \left\langle (\hat{a}^\dagger)^2\right\rangle + \left\langle \hat{a}^\dagger\hat{a}\right\rangle + \left\langle \hat{a}\hat{a}^\dagger\right\rangle\Big) + \frac{i\hbar}{m}\Big(\left\langle (\hat{a}^\dagger)^2\right\rangle - \left\langle \hat{a}^2\right\rangle\Big) + \frac{\gamma\hbar}{m\omega}(\bar{n}+1/2) \tag{53}$$

Since we have

$$\{\hat{x},\hat{p}\} = i\hbar((\hat{a}^\dagger)^2 - \hat{a}^2) \tag{54}$$

We can write the equation of motion as

$$\partial_t \left\langle \hat{x}^2\right\rangle = -\gamma\left\langle \hat{x}^2\right\rangle + \frac{1}{m}\left\langle \{\hat{x},\hat{p}\}\right\rangle + \frac{\gamma\hbar}{m\omega}(\bar{n}+1/2) \tag{55}$$

B.2 Solving for $\partial_t \left\langle \hat{p}^2 \right\rangle$

Writing Eq. (25) as

$$\partial_t \left\langle \hat{p}^2\right\rangle = \mathrm{tr}\big(\hat{p}^2\mathcal{L}\hat{\rho}\big) + \mathrm{tr}\big(\hat{p}^2\lambda\mathcal{D}[\hat{x}]\hat{\rho}\big) \tag{56}$$

Rewriting the left terms using $\hat{a}, \hat{a}^\dagger$ we get

$$\mathrm{tr}\big(\hat{p}^2\mathcal{L}\hat{\rho}\big) = -\frac{m\omega\hbar}{2}\,\mathrm{tr}\Big((\hat{a}^2 + (\hat{a}^\dagger)^2 - \hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger)\mathcal{L}\hat{\rho}\Big) \tag{57}$$

$$= -\frac{m\omega\hbar}{2}\Big(\mathrm{tr}\big(\hat{a}^2\mathcal{L}\hat{\rho}\big) + \mathrm{tr}\Big((\hat{a}^\dagger)^2\mathcal{L}\hat{\rho}\Big) - \mathrm{tr}\Big(\hat{a}^\dagger\hat{a}\mathcal{L}\hat{\rho}\Big) - \mathrm{tr}\Big(\hat{a}\hat{a}^\dagger\mathcal{L}\hat{\rho}\Big)\Big) \tag{58}$$

We have solved for these before, so we can write

$$\mathrm{tr}\big(\hat{p}^2\mathcal{L}\hat{\rho}\big) = -\frac{m\omega\hbar}{2}\Big(-(\gamma+2i\omega)\left\langle \hat{a}^2\right\rangle - (\gamma-2i\omega)\left\langle (\hat{a}^\dagger)^2\right\rangle + \gamma\left\langle \hat{a}^\dagger\hat{a}\right\rangle - \gamma\bar{n} + \gamma\left\langle \hat{a}\hat{a}^\dagger\right\rangle - \gamma(\bar{n}+1)\Big) \tag{59}$$

$$= -\gamma\left\langle \hat{p}^2\right\rangle + mi\omega^2\hbar\Big(\left\langle \hat{a}^2\right\rangle - \left\langle (\hat{a}^\dagger)^2\right\rangle\Big) + \gamma m\omega\hbar(\bar{n}+1/2) \tag{60}$$

$$= -\gamma\left\langle \hat{p}^2\right\rangle - m\omega^2\left\langle \{\hat{x},\hat{p}\}\right\rangle + \gamma m\omega\hbar(\bar{n}+1/2) \tag{61}$$

Solving the measurement term

$$\operatorname{tr}\big(\hat{p}^2\lambda\mathcal{D}[\hat{x}]\hat{\rho}\big) = \lambda\left(\operatorname{tr}\big(\hat{p}^2\hat{x}\hat{\rho}\hat{x}\big) - \frac{1}{2}\operatorname{tr}\big(\hat{p}^2\hat{x}^2\hat{\rho}\big) - \frac{1}{2}\operatorname{tr}\big(\hat{p}^2\hat{\rho}\hat{x}^2\big)\right) \tag{62}$$

$$= \lambda\left(\operatorname{tr}\big(\hat{x}\hat{p}^2\hat{x}\hat{\rho}\big) - \frac{1}{2}\operatorname{tr}\big(\hat{p}^2\hat{x}^2\hat{\rho}\big) - \frac{1}{2}\operatorname{tr}\big(\hat{x}^2\hat{p}^2\hat{\rho}\big)\right) \tag{63}$$

Commuting the operators

$$\hat{x}\hat{p}^2\hat{x} = \frac{1}{2}(\hat{p}\hat{x}+i\hbar)\hat{p}\hat{x} + \frac{1}{2}\hat{x}\hat{p}(\hat{x}\hat{p}-i\hbar) = \frac{1}{2}\hat{p}\hat{x}\hat{p}\hat{x} + \frac{1}{2}\hat{x}\hat{p}\hat{x}\hat{p} + \frac{i\hbar}{2}[\hat{p},\hat{x}] \tag{64}$$

$$= \frac{1}{2}\hat{p}\hat{x}\hat{p}\hat{x} + \frac{1}{2}\hat{x}\hat{p}\hat{x}\hat{p} + \frac{\hbar^2}{2} \tag{65}$$

$$\hat{p}^2\hat{x}^2 = \hat{p}(\hat{x}\hat{p}-i\hbar)\hat{x} = \hat{p}\hat{x}\hat{p}\hat{x} - i\hbar\hat{p}\hat{x} \tag{66}$$

$$\hat{x}^2\hat{p}^2 = \hat{x}(\hat{p}\hat{x}+i\hbar)\hat{p} = \hat{x}\hat{p}\hat{x}\hat{p} + i\hbar\hat{x}\hat{p} \tag{67}$$

This gives

$$\operatorname{tr}\big(\hat{p}^2\lambda\mathcal{D}[\hat{x}]\hat{\rho}\big) = \lambda\left(\frac{1}{2}\operatorname{tr}(\hat{p}\hat{x}\hat{p}\hat{x}\hat{\rho}) + \frac{1}{2}\operatorname{tr}(\hat{x}\hat{p}\hat{x}\hat{p}\hat{\rho}) + \frac{\hbar^2}{2} - \frac{1}{2}\operatorname{tr}(\hat{p}\hat{x}\hat{p}\hat{x}\hat{\rho}) - \frac{1}{2}\operatorname{tr}(\hat{x}\hat{p}\hat{x}\hat{p}\hat{\rho}) + \frac{i\hbar}{2}\operatorname{tr}([\hat{p},\hat{x}])\right) \tag{68}$$

$$= \frac{\lambda\hbar^2}{2} + \frac{\lambda i\hbar}{2}\operatorname{tr}([\hat{p},\hat{x}]) = \lambda\hbar^2 \tag{69}$$

The final equation of motion then becomes

$$\partial_t\left\langle\hat{p}^2\right\rangle = -\gamma\left\langle\hat{p}^2\right\rangle - m\omega^2\left\langle\{\hat{x},\hat{p}\}\right\rangle + \gamma m\omega\hbar(\bar{n}+1/2) + \lambda\hbar^2 \tag{70}$$

B.3 Solving for $\partial_t\left\langle\{\hat{x},\hat{p}\}\right\rangle$

Writing Eq. (26) as

$$\partial_t\left\langle\{\hat{x},\hat{p}\}\right\rangle = \operatorname{tr}(\{\hat{x},\hat{p}\}\mathcal{L}\hat{\rho}) + \operatorname{tr}(\{\hat{x},\hat{p}\}\lambda\mathcal{D}[\hat{x}]\hat{\rho}) \tag{71}$$

Rewriting the left term as

$$\operatorname{tr}(\{\hat{x},\hat{p}\}\mathcal{L}\hat{\rho}) = i\hbar\operatorname{tr}\Big((\hat{a}^2-(\hat{a}^\dagger)^2)\mathcal{L}\hat{\rho}\Big) = i\hbar\operatorname{tr}\big(\hat{a}^2\mathcal{L}\hat{\rho}\big) - i\hbar\operatorname{tr}\Big((\hat{a}^\dagger)^2\mathcal{L}\hat{\rho}\Big) \tag{72}$$

We have calculated these terms before, so we can write

$$\operatorname{tr}(\{\hat{x},\hat{p}\}\mathcal{L}\hat{\rho}) = -i\hbar\left((\gamma+2i\omega)\left\langle \hat{a}^2\right\rangle - (\gamma-2i\omega)\left\langle (\hat{a}^\dagger)^2\right\rangle\right) \tag{73}$$

$$= -\gamma i\hbar\left(\left\langle \hat{a}^2\right\rangle - \left\langle (\hat{a}^\dagger)^2\right\rangle\right) + 2\hbar\omega\left(\left\langle \hat{a}^2\right\rangle + \left\langle (\hat{a}^\dagger)^2\right\rangle\right) \tag{74}$$

We also have that

$$\hat{a}^2 + (\hat{a}^\dagger)^2 = \frac{m\omega}{\hbar}\hat{x}^2 - \frac{1}{m\omega\hbar}\hat{p}^2 \tag{75}$$

Thus we get

$$\operatorname{tr}(\{\hat{x},\hat{p}\}\mathcal{L}\hat{\rho}) = -\gamma\left\langle\{\hat{x},\hat{p}\}\right\rangle + 2m\omega^2\left\langle \hat{x}^2\right\rangle - \frac{2}{m}\left\langle \hat{p}^2\right\rangle \tag{76}$$

Solving for the measurement term

$$\lambda\operatorname{tr}(\{\hat{x},\hat{p}\}\mathcal{D}[\hat{x}]\hat{\rho}) = \frac{\lambda}{2}\left(\operatorname{tr}\big(\big[\hat{x}^2,\hat{p}\hat{x}\big]\hat{\rho}\big) + \operatorname{tr}\big(\big[\hat{x}\hat{p},\hat{x}^2\big]\big)\right) \tag{77}$$

Solving the commutators shows that this will become zero.

B.4 Results

The final equations of motion are

$$\partial_t\left\langle \hat{x}^2\right\rangle = -\gamma\left\langle \hat{x}^2\right\rangle + \frac{1}{m}\left\langle\{\hat{x},\hat{p}\}\right\rangle + \frac{\gamma\hbar}{m\omega}(\bar{n}+1/2) \tag{78}$$

$$\partial_t\left\langle \hat{p}^2\right\rangle = -\gamma\left\langle \hat{p}^2\right\rangle - m\omega^2\left\langle\{\hat{x},\hat{p}\}\right\rangle + \gamma m\omega\hbar(\bar{n}+1/2) + \lambda\hbar^2 \tag{79}$$

$$\partial_t\left\langle\{\hat{x},\hat{p}\}\right\rangle = -\gamma\left\langle\{\hat{x},\hat{p}\}\right\rangle + 2m\omega^2\left\langle \hat{x}^2\right\rangle - \frac{2}{m}\left\langle \hat{p}^2\right\rangle \tag{80}$$