
Name of Hand In

Course code

Author

Max Eriksson

maxerikss@gmail.com

Lund University

Department of Physics



LUND
UNIVERSITY

November 30, 2024

1 First Exercise

Problem: A sphere of radius R carries a “frozen-in” polarisation $\mathbf{P}(\mathbf{r}) = kr^2\hat{\mathbf{r}}$, where k is a constant and \mathbf{r} is the vector from the center.

- (a) Calculate the bound charges σ_b and ρ_b .
- (b) Find the \mathbf{E} field inside and outside the sphere from the charge densities found at (a).
- (c) Alternatively, compute the \mathbf{E} field by first determining the \mathbf{D} field using Gauss’s law and then use that $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$.

(a) Solution: Start by drawing a sketch of the problem.

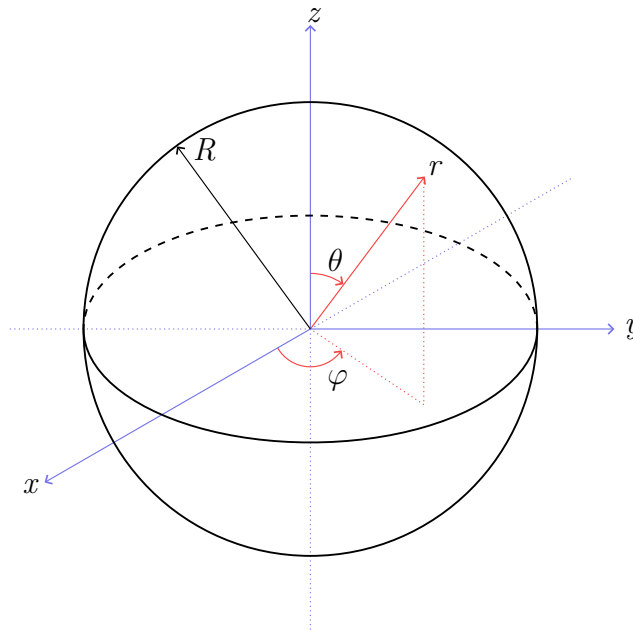


Figure 1: The problem sketch with cartesian and spherical coordinates drawn.

We can calculate the bound charge densities from the polarisation \mathbf{P} .

$$\sigma_b = \hat{\mathbf{n}} \cdot \mathbf{P}, \quad \rho_b = -\nabla \cdot \mathbf{P}. \quad (1)$$

Since we have a sphere the normal vector is $\hat{\mathbf{n}} = \hat{\mathbf{r}}$, the surface charge density is also only

on $r = R$, and \mathbf{P} only have a radial component. Thus

$$\sigma_b = \hat{\mathbf{r}} \cdot \mathbf{P} = \hat{\mathbf{r}} \cdot (kR^2 \hat{\mathbf{r}}) = kR^2 \quad (2)$$

$$\rho_b = -\nabla \cdot \mathbf{P} = -\frac{1}{r^2} \frac{\partial}{\partial r} (kr^4) = -4kr \quad (3)$$

The charge densities are only non zero on or in the sphere and zero elsewhere.

(a) Answer: The charge densities are

$$\sigma_b = kR^2, \quad \rho_b = -4kr. \quad (4)$$

(b) Solution: To calculate the \mathbf{E} field we can use Gauss's law. Adding a gaussian surface to the sketch

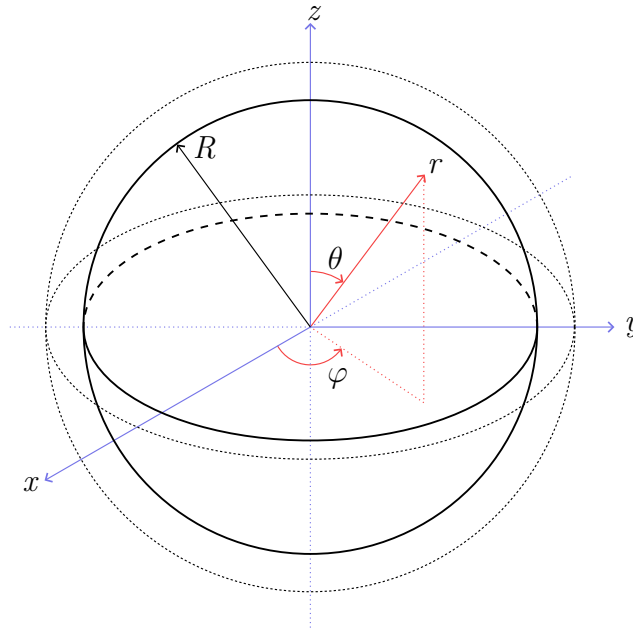


Figure 2: The sketch with and added gaussian surface.

More correctly we might write the charge densities

$$\sigma_b(\mathbf{r}) = \begin{cases} kR^2 & r = R \\ 0 & \text{elsewhere} \end{cases}, \quad \text{and} \quad \rho_b(\mathbf{r}) = \begin{cases} -4kr & 0 \leq r < R \\ 0 & \text{elsewhere} \end{cases} \quad (5)$$

Now we can define

$$\rho(\mathbf{r}) = \rho_b(\mathbf{r}) + \sigma_b(\mathbf{r}) \quad (6)$$

Due to spherical symmetry we can write

$$\mathbf{E}(\mathbf{r}) = E(r)\hat{\mathbf{r}}. \quad (7)$$

Using Gauss's law

$$\nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{\rho(\mathbf{r})}{\epsilon_0} \iff E(r) \oint_{\partial V} da_r = \frac{1}{\epsilon_0} \int_V \rho(\mathbf{r}') d\tau' \quad (8)$$

where V is the volume we integrate over, that is the sphere. Thus

$$E(r) \int_0^\pi \int_0^{2\pi} r^2 \sin \theta d\theta d\varphi = \frac{1}{\epsilon_0} \int_0^\pi \int_0^{2\pi} \int_0^r (\rho_b(\mathbf{r}') + \sigma_b(\mathbf{r}')) (dr')(r' d\theta')(r' \sin \theta' d\varphi') \quad (9)$$

$$E(r)r^2 4\pi = \frac{4\pi}{\epsilon_0} \int_0^r (\rho_b(\mathbf{r}') + \sigma_b(\mathbf{r}')) r'^2 dr' \quad (10)$$

$$E(r) = \frac{1}{\epsilon_0 r^2} \int_0^r (\rho_b(\mathbf{r}') + \sigma_b(\mathbf{r}')) r'^2 dr' \quad (11)$$

Now we get two cases: $r < R$ and $r > R$

Case $r < R$: In this case $\sigma_b(\mathbf{r})$ will always be zero by Eq. (5). Thus

$$E(r) = \frac{1}{\epsilon_0 r^2} \int_0^r \rho_b(\mathbf{r}') r'^2 dr' = -\frac{4k}{\epsilon_0 r^2} \int_0^r r'^3 dr' \quad (12)$$

$$E(r) = -\frac{4k}{\epsilon_0 r^2} \left[\frac{r'^4}{4} \right]_0^r = -\frac{k}{\epsilon_0} r^2 \quad (13)$$

And thus we get that

$$\mathbf{E}(\mathbf{r})_{\text{inside}} = -\frac{k}{\epsilon_0} r^2 \hat{\mathbf{r}} \quad (14)$$

Case $r > R$: Rewriting $\sigma_b(\mathbf{r})$ using a delta function and using that $\rho_b(\mathbf{r})$ is zero outside the sphere we get

$$E(r) = \frac{1}{\epsilon_0 r^2} \int_0^r (\rho_b(\mathbf{r}) + \delta(r' - R)kR^2)r'^2 dr' \quad (15)$$

$$E(r) = \frac{1}{\epsilon_0 r^2} \left(-4k \int_0^R r'^3 dr' + kR^2 \int_0^r \delta(r' - R)r'^2 dr' \right) \quad (16)$$

$$E(r) = \frac{k}{\epsilon_0 r^2} \left(-4 \left[\frac{r'^4}{4} \right]_0^R + R^4 \right) = \frac{k}{\epsilon_0 r^2} (-R^4 + R^4) = 0 \quad (17)$$

Therefore the field outside is

$$\mathbf{E}(\mathbf{r})_{\text{outside}} = \mathbf{0} \quad (18)$$

(b) Answer: We get the \mathbf{E} field

$$\mathbf{E}(\mathbf{r}) = \begin{cases} -\frac{k}{\epsilon_0} r^2 \hat{\mathbf{r}} & r < R \\ \mathbf{0} & r > R \end{cases} \quad (19)$$

(c) Solution: We use the same gaussian surface as seen in Fig. 2. We know have

$$\nabla \cdot \mathbf{D} = \rho_f \iff \oint_{\partial V} \mathbf{D} \cdot d\mathbf{a} = \int_V \rho_f d\tau. \quad (20)$$

However, we don't have any free charges so $\rho_f = 0$. Since we have a spherical geometry and the polarisation is spherically symmetrical we can write

$$\mathbf{D}(\mathbf{r}) = D(r) \hat{\mathbf{r}} \quad (21)$$

And thus the equation simplifies to

$$D(r)r^2 4\pi = 0 \iff D(r) = 0 \quad (22)$$

Now using that

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \quad (23)$$

and since \mathbf{P} is zero outside the sphere we have two cases: $r < R$ and $r > R$.

Case $r < R$: Here we have

$$\mathbf{0} = \epsilon_0 \mathbf{E}_{\text{inside}} + \mathbf{P} \iff \mathbf{E}_{\text{inside}} = -\frac{k}{\epsilon_0} r^2 \hat{\mathbf{r}} \quad (24)$$

Case $r > R$: Since \mathbf{P} is zero in this region we have

$$\mathbf{E}_{\text{outside}} = \mathbf{0} \quad (25)$$

(c) Answer: We see that the answer is the exact same as in (b) but with a lot less calculations. We need to be careful however, as we see in Exercise F.4, the polarisation must have spherical symmetry to be able to write $\mathbf{D}(\mathbf{r}) = D(r)\hat{\mathbf{r}}$.