Hand in 5

FYSC20

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Problem: A uniform surface current density $\mathbf{K} = K\hat{\mathbf{z}}$ fills the entire xz-plane.

(a) Calculate the magnetic field. Argue and explain your steps carefully.

(b) Calculate the vector potential in Coulomb gauge. Convince yourself that you have found the correct result.

Solution (a): First step is to sketch the problem.

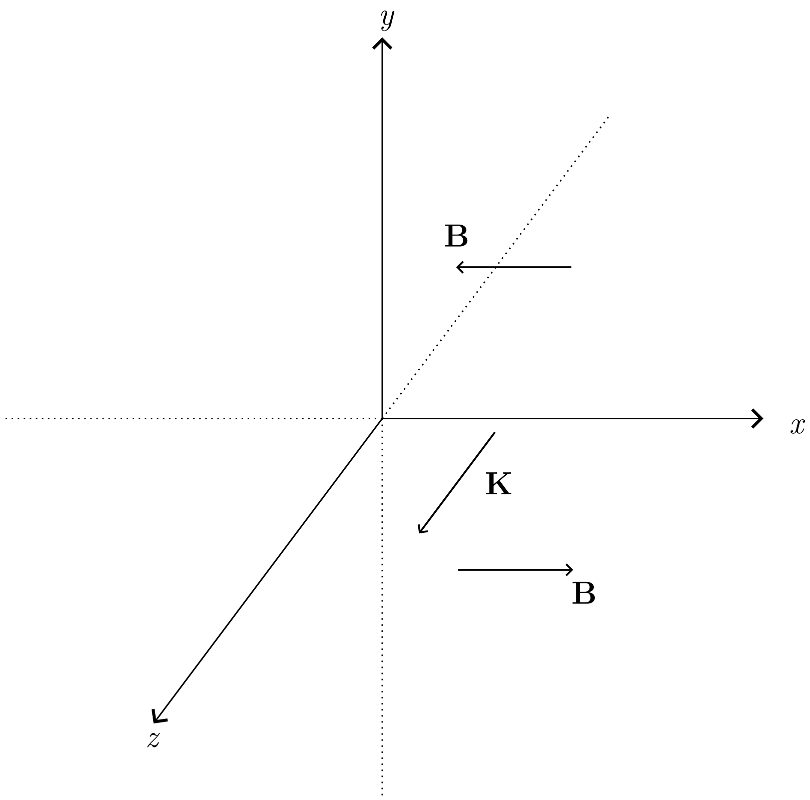

Figure 1: Sketch of the problem. The direction of the **B**-field is given by the right-hand rule.

As per the right-hand rule and the sketch seen above and the fact that the charge is extending on the entire xz-field we can write

$$\mathbf{B} = B(y)\hat{\mathbf{x}}. \tag{1}$$

Then use Ampere's law in integral form

$$\oint_{\partial S} \mathbf{B} \cdot \mathrm{d}\mathbf{l} = \mu_0 I_{\mathrm{enc}}. \tag{2}$$

Integrating along the border of a rectangular surface S, which can be seen in Fig. 2. The normal of the surface is parallel to $\mathbf{K}$, and half of the surface is above the xz-plane. Let the width be l and the height $2y$ with corners (coordinates in (x, y)) $a = (0, -y), b = (l, -y), c = (l, y), d = (0, y)$.

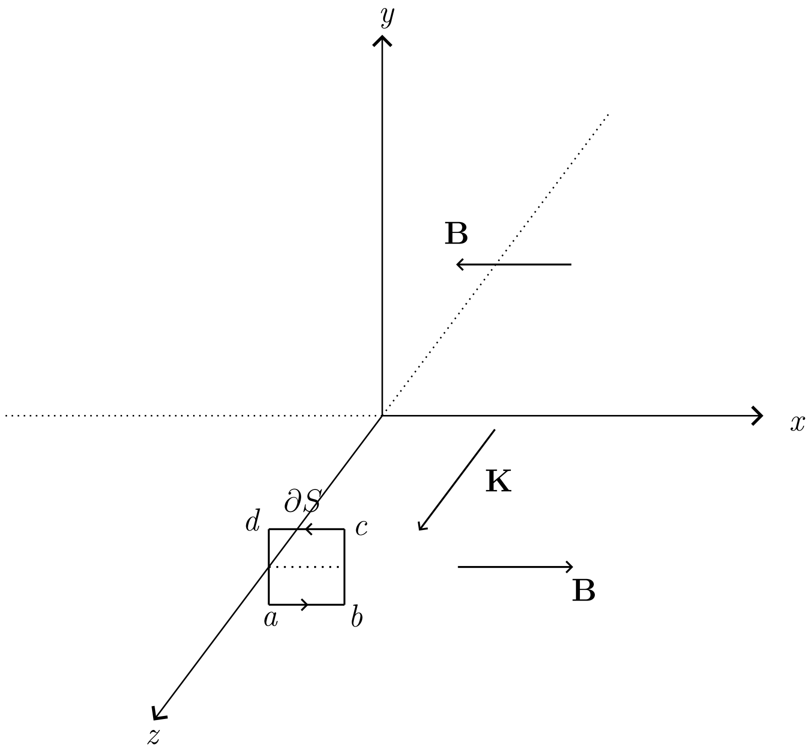

Figure 2: Integration line inserted to the sketch.

Due to the symmetry we can write

$$B(-y) = -B(y). \tag{3}$$

Combining Eq. (2) and (1) we get

$$\oint_{\partial S} \mathbf{B} \cdot \mathrm{d}\mathbf{l} = \int_a^b B(-y)\mathrm{d}x + \int_c^d B(y)\mathrm{d}x = \mu_0 K l \tag{4}$$

Here we have used that the line segments $b-c$ and $d-a$ are perpendicular to the $\mathbf{B}$-field, so those parts don't contribute. The x-coordinates of the corners are $a_x = 0$, $b_x = l$,

$c_x = l$ and $d_x = 0$ Thus

$$\int_a^b B(-y)\mathrm{d}x + \int_c^d B(y)\mathrm{d}x = B(-y)\int_0^l \mathrm{d}x + B(y)\int_l^0 \mathrm{d}x = \mu_0 K l \tag{5}$$

Now using Eq. (3) we get

$$2B(y)\int_l^0 \mathrm{d}x = -2B(y)l = \mu_0 K l \iff B(y) = -\frac{\mu_0 K}{2} \tag{6}$$

However, we need to consider both cases $y < 0$ and $y > 0$. Inspecting the sketch and using the right-hand rule we get

$$\mathbf{B} = \begin{cases} -\frac{\mu_0 K}{2}\hat{\mathbf{x}} & y > 0 \\ \frac{\mu_0 K}{2}\hat{\mathbf{x}} & y < 0 \end{cases}. \tag{7}$$

Solution (b): We let $\mathbf{A} = a_x\hat{\mathbf{x}} + a_y\hat{\mathbf{y}} + a_z\hat{\mathbf{z}}$. Then we use

$$\mathbf{B} = \nabla \times \mathbf{A} = \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)\hat{\mathbf{z}} \tag{8}$$

We can conclude that

$$\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z} = \begin{cases} -\frac{\mu_0 K}{2} & y > 0 \\ \frac{\mu_0 K}{2} & y < 0 \end{cases}. \tag{9}$$

We also have that in Coulomb gauge

$$\nabla \cdot A = 0 \iff \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} = 0. \tag{10}$$

We can solve this if a_x and a_y is constant and a_z is only dependent on y. Thus

$$\frac{\partial a_z}{\partial y} = \begin{cases} -\frac{\mu_0 K}{2} & y > 0 \\ \frac{\mu_0 K}{2} & y < 0 \end{cases} \iff a_z = \begin{cases} -\frac{\mu_0 K y}{2} + C & y > 0 \\ \frac{\mu_0 K y}{2} + C & y < 0 \end{cases} \tag{11}$$

where C is some constant. Then let $\mathbf{c} = a_x\hat{\mathbf{x}} + a_y\hat{\mathbf{y}} + C\hat{\mathbf{z}}$ be a constant vector. Thus

$$\mathbf{A} = \begin{cases} -\dfrac{\mu_0 K y}{2}\hat{\mathbf{z}} + \mathbf{c} & y > 0 \\ \dfrac{\mu_0 K y}{2}\hat{\mathbf{z}} + \mathbf{c} & y < 0 \end{cases} \tag{12}$$

Answer: The magnetic field and vector potential in Coulomb gauge is

$$\mathbf{B} = \begin{cases} -\dfrac{\mu_0 K}{2}\hat{\mathbf{x}} & y > 0 \\ \dfrac{\mu_0 K}{2}\hat{\mathbf{x}} & y < 0 \end{cases} \quad \text{and} \quad \mathbf{A} = \begin{cases} -\dfrac{\mu_0 K y}{2}\hat{\mathbf{z}} + \mathbf{c} & y > 0 \\ \dfrac{\mu_0 K y}{2}\hat{\mathbf{z}} + \mathbf{c} & y < 0 \end{cases} \tag{13}$$