Hand in 3

FYSC20

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November 30, 2024

Problem: A sphere of radius R carries a "frozen-in" polarisation $\mathbf{P}(\mathbf{r}) = kr^2\hat{\mathbf{r}}$, where k is a constant and $\mathbf{r}$ is the vector from the center.

(a) Calculate the bound charges σ_b and ρ_b.

(b) Find the $\mathbf{E}$ field inside and outside the sphere from the charge densities found at (a).

(c) Alternatively, compute the $\mathbf{E}$ field by first determining the $\mathbf{D}$ field using Gauss's law and then use that $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$.

(a) Solution: Start by drawing a sketch of the problem.

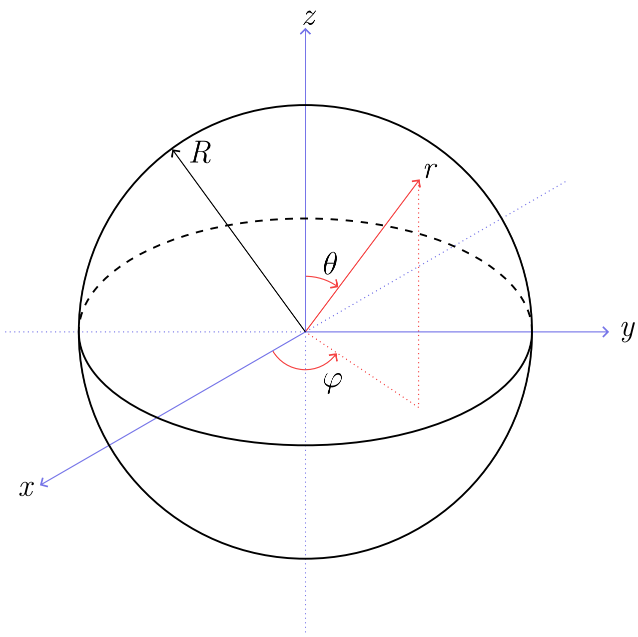

Figure 1: The problem sketch with cartesian and spherical coordinates drawn.

We can calculate the bound charge densities from the polarisation $\mathbf{P}$.

$$\sigma_b = \hat{\mathbf{n}} \cdot \mathbf{P}, \quad \rho_b = -\nabla \cdot \mathbf{P}. \tag{1}$$

Since we have a sphere the normal vector is $\hat{\mathbf{n}} = \hat{\mathbf{r}}$, the surface charge density is also only on $r = R$, and $\mathbf{P}$ only have a radial component. Thus

$$\sigma_b = \hat{\mathbf{r}} \cdot \mathbf{P} = \hat{\mathbf{r}} \cdot (kR^2\hat{\mathbf{r}}) = kR^2 \tag{2}$$

$$\rho_b = -\nabla \cdot \mathbf{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}(kr^4) = -4kr \tag{3}$$

The charge densities are only non zero on or in the sphere and zero elsewhere.

(a) Answer: The charge densities are

$$\sigma_b = kR^2, \quad \rho_b = -4kr. \tag{4}$$

(b) Solution: To calculate the **E** field we can use Gauss's law. Adding a gaussian surface to the sketch

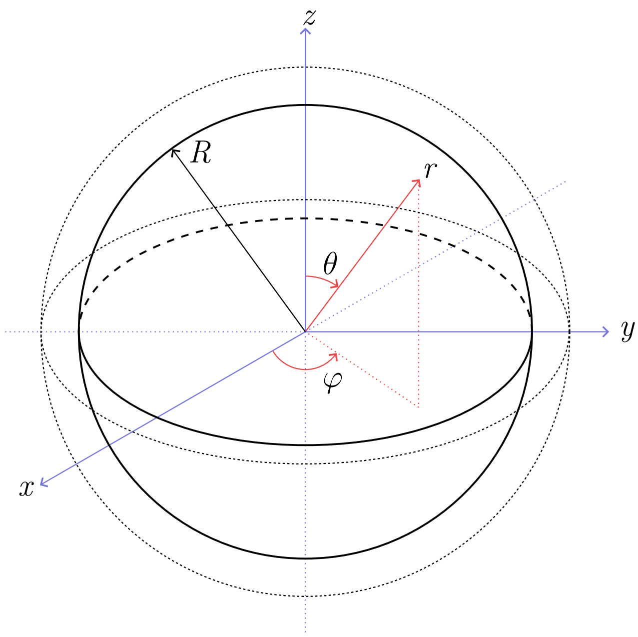

Figure 2: The sketch with and added gaussian surface.

More correctly we might write the charge densities

$$\sigma_b(\mathbf{r}) = \begin{cases} kR^2 & r = R \\ 0 & \text{elsewhere} \end{cases}, \quad \text{and} \quad \rho_b(\mathbf{r}) = \begin{cases} -4kr & 0 \le r < R \\ 0 & \text{elsewhere} \end{cases} \tag{5}$$

Now we can define

$$\rho(\mathbf{r}) = \rho_b(\mathbf{r}) + \sigma_b(\mathbf{r}) \tag{6}$$

Due to spherical symmetry we can write

$$\mathbf{E}(\mathbf{r}) = E(r)\hat{\mathbf{r}}. \tag{7}$$

Using Gauss's law

$$\nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{\rho(\mathbf{r})}{\epsilon_0} \iff E(r) \oint_{\partial V} \mathrm{d}a_r = \frac{1}{\epsilon_0} \int_V \rho(\mathbf{r}')\mathrm{d}\tau' \tag{8}$$

where V is the volume we integrate over, that is the sphere. Thus

$$E(r) \int_0^\pi \int_0^{2\pi} r^2 \sin\theta \mathrm{d}\theta \mathrm{d}\varphi = \frac{1}{\epsilon_0} \int_0^\pi \int_0^{2\pi} \int_0^r (\rho_b(\mathbf{r}') + \sigma_b(\mathbf{r}'))(\mathrm{d}r')(r'\mathrm{d}\theta')(r' \sin\theta' \mathrm{d}\varphi') \tag{9}$$

$$E(r) r^2 4\pi = \frac{4\pi}{\epsilon_0} \int_0^r (\rho_b(\mathbf{r}') + \sigma_b(\mathbf{r}')) r'^2 \mathrm{d}r' \tag{10}$$

$$E(r) = \frac{1}{\epsilon_0 r^2} \int_0^r (\rho_b(\mathbf{r}') + \sigma_b(\mathbf{r}')) r'^2 \mathrm{d}r' \tag{11}$$

Now we get two cases: $r < R$ and $r > R$

Case $r < R$: In this case $\sigma_b(\mathbf{r})$ will always be zero by Eq. (5). Thus

$$E(r) = \frac{1}{\epsilon_0 r^2} \int_0^r \rho_b(\mathbf{r}') r'^2 \mathrm{d}r' = -\frac{4k}{\epsilon_0 r^2} \int_0^r r'^3 \mathrm{d}r' \tag{12}$$

$$E(r) = -\frac{4k}{\epsilon_0 r^2} \left[\frac{r'^4}{4}\right]_0^r = -\frac{k}{\epsilon_0} r^2 \tag{13}$$

And thus we get that

$$\mathbf{E}(\mathbf{r})_{\text{inside}} = -\frac{k}{\epsilon_0} r^2 \hat{\mathbf{r}} \tag{14}$$

Case $r > R$: Rewriting $\sigma_b(\mathbf{r})$ using a delta function and using that $\rho_b(\mathbf{r})$ is zero outside the sphere we get

$$E(r) = \frac{1}{\epsilon_0 r^2}\int_0^r (\rho_b(\mathbf{r}) + \delta(r' - R)kR^2)r'^2 \mathrm{d}r' \tag{15}$$

$$E(r) = \frac{1}{\epsilon_0 r^2}\left(-4k\int_0^R r'^3 \mathrm{d}r' + kR^2\int_0^r \delta(r' - R)r'^2 \mathrm{d}r'\right) \tag{16}$$

$$E(r) = \frac{k}{\epsilon_0 r^2}\left(-4\left[\frac{r'^4}{4}\right]_0^R + R^4\right) = \frac{k}{\epsilon_0 r^2}(-R^4 + R^4) = 0 \tag{17}$$

Therefore the field outside is

$$\mathbf{E}(\mathbf{r})_{\text{outside}} = \mathbf{0} \tag{18}$$

(b) Answer: We get the $\mathbf{E}$ field

$$\mathbf{E}(\mathbf{r}) = \begin{cases} -\dfrac{k}{\epsilon_0}r^2\hat{\mathbf{r}} & r < R \\ \mathbf{0} & r > R \end{cases} \tag{19}$$

(c) Solution: We use the same gaussian surface as seen in Fig. 2. We know have

$$\nabla \cdot \mathbf{D} = \rho_f \iff \oint_{\partial V} \mathbf{D} \cdot \mathrm{d}\mathbf{a} = \int_V \rho_f \mathrm{d}\tau. \tag{20}$$

However, we don't have any free charges so $\rho_f = 0$. Since we have a spherical geometry and the polarisation is spherically symmetrical we can write

$$\mathbf{D}(\mathbf{r}) = D(r)\hat{\mathbf{r}} \tag{21}$$

And thus the equation simplifies to

$$D(r)r^2 4\pi = 0 \iff D(r) = 0 \tag{22}$$

Now using that

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} \tag{23}$$

and since $\mathbf{P}$ is zero outside the sphere we have two cases: $r < R$ and $r > R$.

Case $r < R$: Here we have

$$\mathbf{0} = \epsilon_0 \mathbf{E}_{\text{inside}} + \mathbf{P} \iff \mathbf{E}_{\text{inside}} = -\frac{k}{\epsilon_0} r^2 \hat{\mathbf{r}} \tag{24}$$

Case $r > R$: Since $\mathbf{P}$ is zero in this region we have

$$\mathbf{E}_{\text{outside}} = \mathbf{0} \tag{25}$$

(c) Answer: We see that the answer is the exact same as in (b) but with a lot less calculations. We need to be careful however, as we see in Exercise F.4, the polarisation must have spherical symmetry to be able to write $\mathbf{D}(\mathbf{r}) = D(r)\hat{\mathbf{r}}$, i.e. it doesn't hold for a uniformly polarized sphere.