Problem Sheet 3

FYSC22

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9 Two-State Mixing

(a)

A diagonal matrix has the eigenvalues in the diagonal, thus by finding the eigenvalues of the matrix

$$\mathcal{M} = \begin{pmatrix} E_1 & V_{12} \\ V_{12} & E_2 \end{pmatrix} \tag{1}$$

we get the diagonal matrix. Thus we solve

$$\det(\mathcal{M} - \lambda I) = 0 \iff \begin{vmatrix} E_1 - \lambda & V_{12} \\ V_{12} & E_2 - \lambda \end{vmatrix} = 0 \iff (E_1 - \lambda)(E_2 - \lambda) - V_{12}^2 = 0 \tag{2}$$

$$\iff \lambda^2 - \lambda(E_1 + E_2) + E_1 E_2 - V_{12}^2 = 0 \tag{3}$$

$$\iff \lambda = \frac{E_1 + E_2}{2} \pm \sqrt{\frac{E_1^2 - 2E_1E_2 + E_2^2 + 4V_{12}^2}{4}} \tag{4}$$

$$\iff \lambda = \frac{E_1 + E_2}{2} \pm \frac{\sqrt{(E_1 - E_2)^2 + 4V_{12}^2}}{2}. \tag{5}$$

And the eigenvalues are E_I and E_{II} while the eigenvectors are the wavefunctions. Thus we have

$$E_I = \frac{E_1 + E_2}{2} - \frac{\sqrt{(E_1 - E_2)^2 + 4V_{12}^2}}{2} \tag{6}$$

$$E_{II} = \frac{E_1 + E_2}{2} + \frac{\sqrt{(E_1 - E_2)^2 + 4V_{12}^2}}{2} \tag{7}$$

Now we get

$$\Delta E_{I;II} = E_{II} - E_I = \sqrt{(E_1 - E_2)^2 + 4V_{12}^2} = \sqrt{(\Delta E_{12})^2 + 4\left(\frac{\Delta E_{12}}{R}\right)^2} \tag{8}$$

$$= \Delta E_{12}\sqrt{1 + \frac{4}{R^2}} \tag{9}$$

QED.

(b)

For $\Delta \boldsymbol{E}_{\mathbf{12}} = \mathbf{40}\,\mathbf{keV}$ and $\boldsymbol{V}_{\mathbf{12}} = \mathbf{30}\,\mathbf{keV}$:

$$\Delta E_{I;II} = 40\,\mathrm{keV}\sqrt{1 + 4\frac{(30\,\mathrm{keV})^2}{(40\,\mathrm{keV})^2}} = 72\,\mathrm{keV} \tag{10}$$

and $\Delta E_{I;II}/\Delta E_{12} = 1.8$

For $\Delta \boldsymbol{E}_{\mathbf{12}} = \mathbf{40}\,\mathbf{keV}$ and $\boldsymbol{V}_{\mathbf{12}} = \mathbf{100}\,\mathbf{keV}$:

$$\Delta E_{I;II} = 40\,\mathrm{keV}\sqrt{1 + 4\frac{(100\,\mathrm{keV})^2}{(40\,\mathrm{keV})^2}} = 204\,\mathrm{keV} \tag{11}$$

and $\Delta E_{I;II}/\Delta E_{12} = 5.1$

For $\Delta \boldsymbol{E}_{\mathbf{12}} = \mathbf{150}\,\mathbf{keV}$ and $\boldsymbol{V}_{\mathbf{12}} = \mathbf{30}\,\mathbf{keV}$:

$$\Delta E_{I;II} = 150\,\mathrm{keV}\sqrt{1 + 4\frac{(30\,\mathrm{keV})^2}{(150\,\mathrm{keV})^2}} = 162\,\mathrm{keV} \tag{12}$$

and $\Delta E_{I;II}/\Delta E_{12} = 1.08$

For $\Delta \boldsymbol{E}_{\mathbf{12}} = \mathbf{150}\,\mathbf{keV}$ and $\boldsymbol{V}_{\mathbf{12}} = \mathbf{100}\,\mathbf{keV}$:

$$\Delta E_{I;II} = 150\,\mathrm{keV}\sqrt{1 + 4\frac{(100\,\mathrm{keV})^2}{(150\,\mathrm{keV})^2}} = 250\,\mathrm{keV} \tag{13}$$

and $\Delta E_{I;II}/\Delta E_{12} = 1.67$

Discussion: The more energetic the interaction the larger the energy gap is afterwards. When the energy gap is about the same size as the interaction energy the energy difference afterwards is about double.

10 Spins and Parities of States in Odd-Odd Nuclei

(a)

(b)

(c)

(d)

11 Thermal Neutron Flux

The reaction rate $n + {}^{115}\mathrm{In} \rightarrow {}^{116}\mathrm{In}$ can be described with

$$r = nV\sigma\Phi, \tag{14}$$

where r is the reaction rate in reactions per second, n is the number of particles per unit volume, V is the volume of the target, σ is the capture cross-section, and Φ is the neutron flux. We are given $\sigma = 160\,\mathrm{b}$ and can easily calculate $V = \text{thickness} \times \text{area} = 9 \times 10^{-6}\,\mathrm{cm}^3$. The number of ${}^{115}\mathrm{In}$ per unit volume can be calculated from the density as

$$n = \frac{\rho}{m} \tag{15}$$

and the density is $\rho_{\mathrm{In}} = 7.3\,\mathrm{g\,cm^{-3}}$ and the atomic mass is $m_{\mathrm{In}} = 1.9 \times 10^{22}\,\mathrm{g}$. Thus $n_{\mathrm{In}} = 3.8 \times 10^{22}\,\mathrm{cm^{-3}}$.

To calculate the reaction rate we can use the total number of counts as well as the decay formula. The total number of counts is

$$\text{Counts} = \epsilon \int_{15\,\mathrm{min}}^{75\,\mathrm{min}} N_0 e^{-(\ln 2)t/T_{1/2}} \mathrm{d}t. \tag{16}$$

Since N_0 is the total number of radioactive isotopes, we can assume that it equals the number of $n + {}^{115}\mathrm{In} \rightarrow {}^{116}\mathrm{In}$ reactions. Solving the equation for N_0 gives

$$N_0 = \frac{\text{Counts}}{\epsilon \int_{15\,\mathrm{min}}^{75\,\mathrm{min}} e^{-(\ln 2)t/T_{1/2}} \mathrm{d}t} \tag{17}$$

using the half-life $T_{1/2} = 54\,\mathrm{min}$ and calculating with python gives about $N_0 = 4.8 \times 10^6$. That gives a reaction rate of $r = N_0/1\,\mathrm{min} = 80\,000\,\mathrm{s^{-1}}$. Solving Eq. (14) for Φ we get

$$\Phi = \frac{r}{nV\sigma} = \frac{80\,000\,\mathrm{s^{-1}}}{3.8 \times 10^{22}\,\mathrm{cm^{-3}} \times 9 \times 10^{-6}\,\mathrm{cm^3} \times 160\,\mathrm{b}} = 1.5 \times 10^9\,\mathrm{cm^{-2}\,s^{-1}} \tag{18}$$

12 Coulomb Scattering

(a)

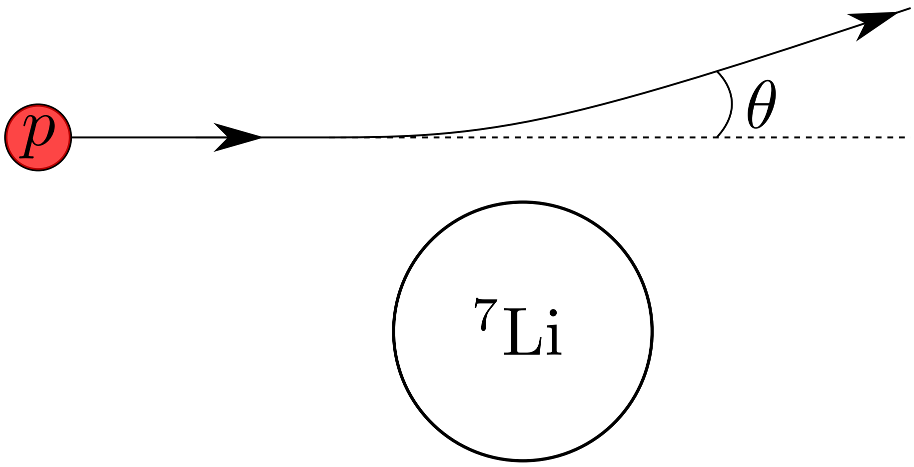

Figure 1: A schematic view of Coulomb scattering of a proton in vicinity of ^{7}Li.

From the lecture we know that

$$\sqrt{T_{p'}} = r \pm \sqrt{r^2 + s} \quad \text{and} \begin{cases} r = \dfrac{\sqrt{m_p^2 T_p}}{m_p + m_{\text{Li}}} \cos\theta \\ s = \dfrac{m_{\text{Li}} Q + T_p(m_{\text{Li}} - m_p)}{m_p + m_{\text{Li}}} \end{cases} . \tag{19}$$

Since the reaction is $^7\text{Li}(p, p')\,^7\text{Li}$, the masses are identical before and after the reaction and therefore $Q = 0$. Using python to compute we get that $T_{p'}(45°) = 8.27\,\text{MeV}$, $T_{p'}(90°) = 6.74\,\text{MeV}$, and $T_{p'}(135°) = 5.49\,\text{MeV}$

(b)

Since the reaction is $^7\text{Li}(p, p')\,^{7*}\text{Li}$ and is inelastic we get $Q = -E_x = -0.477\,\text{MeV}$. Calculating with python we get that $T^*_{p'}(90°) = 6.32\,\text{MeV}$

(c)

If we think of the reaction as the proton just overcoming the Coulomb potential, that is doing the work

$$E_p^{\min} = \frac{Z_{\mathrm{Au}} Z_p e^2}{4\pi\epsilon_0 (r_{\mathrm{Au}} + r_p)}. \tag{20}$$

The radius for nuclei around $20A$ and above can be calculated with $r = r_0 A^{1/3}$ where $r_0 = 1.2\,\mathrm{fm}$. The radius of gold is $r_{\mathrm{Au}} = 6.98\,\mathrm{fm}$ and the proton radius is measured to about $r_p = 0.833\,\mathrm{fm}$, and the charges are $Z_{\mathrm{Au}} = 79$ and $Z_p = 1$. Calculating with python, the energy needed is $E_p^{\min} = 14.6\,\mathrm{MeV}$.

(d)

The same equation can be used but with other values, the charge for the beam and target are $Z_{\mathrm{Ca}} = 20$ and $Z_{\mathrm{Am}} = 95$ respectively. The radius is calculated the same way as the radius of gold was calculated. The radius of $^{48}\mathrm{Ca}$ is $r_{\mathrm{Ca}} = 4.36\,\mathrm{fm}$ and the radius of $^{243}\mathrm{Am}$ is $r_{\mathrm{Am}} = 7.47\,\mathrm{fm}$. Doing the same calculation in python the energy of the beam must be $E_{\mathrm{Ca}}^{\min} = 230.9\,\mathrm{MeV}$.

(e)

It would be the difference between the beam energy and the minimum energy needed to overcome the Coulomb potential which was calculated in (d). Thus

$$E_{\mathrm{excitation}} = E_{\mathrm{Ca}} - E_{\mathrm{Ca}}^{\min} = 245.0\,\mathrm{MeV} - 231.3\,\mathrm{MeV} = 14.1\,\mathrm{MeV} \tag{21}$$

A Code

A.1 Thermal Neutron Flux

```python
#%%
import numpy as np
import scipy.constants as c
from uncertainties import ufloat
from scipy.integrate import quad


#%%

rho_In = 7.3
m_In = ufloat(114.903878773, 0.000000012) * c.u*1000
n_In = rho_In/m_In

#%%
def decay(t, half_life=54):
  return np.exp(- np.log(2) * t / half_life)

counts = 5e4
efficiency = 3e-4
I = quad(decay, 15, 75)[0]
N_0 = counts / (efficiency * I)
r = N_0 / 60
print(f"There have been a total of {int(N_0)} reactions and a reaction
    rate of {r}")

#%%

flux = r / (n_In * 9e-6 * 160e-24)
print(f"The neutron flux is {flux}")
```

A.2 Coulomb Scattering

```python
#%%
import numpy as np
import scipy.constants as c

#%%
# (a)

def r(theta, Ta , ma, mX):
  nom = np.sqrt(ma**2 * Ta)
  denom = ma + mX
  return nom/denom * np.cos(theta*np.pi/180)

def s(Q, Ta, ma,  mX):
  nom = mX*Q + Ta*(mX - ma)
  denom = ma + mX
  return nom/denom

def Tb(theta, Ta = 9, Q = 0,
    ma = c.physical_constants['proton mass energy equivalent in MeV'
    ][0],
    mX = 7.016003434*c.physical_constants['atomic mass constant energy
    equivalent in MeV'][0]):
```

```
  T = []
  pm = np.sqrt(r(theta, Ta, ma, mX)**2 + s(Q, Ta, ma, mX))
  sqrtTb_pos = r(theta, Ta, ma, mX) + pm
  sqrtTb_neg = r(theta, Ta, ma, mX) - pm
  T.append(sqrtTb_pos**2)
  if sqrtTb_neg >= 0:
    T.append(sqrtTb_neg**2)
  else:
    T.append(np.NaN)
  return T

#%%

print(f"The energy of the proton after scattering with theta = 45 deg, is T = {np.round(Tb(45)[0], 2)} or T = {np.round(Tb(45)[1], 2)}")
print(f"The energy of the proton after scattering with theta = 90 deg, is T = {np.round(Tb(90)[0], 2)} or T = {np.round(Tb(90)[1], 2)}")
print(f"The energy of the proton after scattering with theta = 135 deg, is T = {np.round(Tb(135)[0], 2)} or T = {np.round(Tb(135)[1], 2)}")

#%%
# (b)
print(f"The energy of the proton after scattering with theta = 90 deg and Q = -0.477 MeV, is T = {np.round(Tb(90, Q=-0.477)[0], 2)} or T = {np.round(Tb(90, Q=-0.477)[1], 2)}")

#%%
# (c)

def coulombWork(Za, ZX, ra, rX):
  vacuum = 1/(4*np.pi*c.epsilon_0)
  e = c.elementary_charge
  distance = (ra + rX)*1e-15
  return vacuum * Za * ZX * e**2 / (distance * c.electron_volt*1e6)

E_p_Au = coulombWork(1, 79, 0.833, 6.98)
print(f"The energy of the proton must be E_p = {E_p_Au} MeV")

#%%
# (d)

E_Ca_Am = coulombWork(20, 95, 4.36, 7.49)
print(f"The energy of the Ca-48 beam must be E_p = {E_Ca_Am} MeV")
```

References