Hand in 2

FYST85

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Exercise 2.70

Problem: Suppose E is any positive operator acting on Alice's qubit. Show that $\langle\psi| E \otimes \mathbb{I} |\psi\rangle$ *takes the same value* when $|\psi\rangle$ is any of the four Bell states. Suppose some malevolent third party ('Eve') intercepts Alice's qubit on the way to Bob in the superdense coding protocol. Can Eve infer anything about which of the four possible bit strings 00, 01, 10, 11 Alice is trying to send? If so, how, or if not, why not?

Solution: Let's number the Bell states as

$$\begin{aligned}
|\psi_1\rangle &= \frac{|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B}{\sqrt{2}} \qquad (1)\\
|\psi_2\rangle &= \frac{|0\rangle_A \otimes |0\rangle_B - |1\rangle_A \otimes |1\rangle_B}{\sqrt{2}} \qquad (2)\\
|\psi_3\rangle &= \frac{|0\rangle_A \otimes |1\rangle_B + |1\rangle_A \otimes |0\rangle_B}{\sqrt{2}} \qquad (3)\\
|\psi_4\rangle &= \frac{|0\rangle_A \otimes |1\rangle_B - |1\rangle_A \otimes |0\rangle_B}{\sqrt{2}}. \qquad (4)
\end{aligned}$$

I will not write the subscript A or B further, but infer that the first ket is Alice and second Bob. Applying the operator $E \otimes \mathbb{I}$ on the states we get the following

$$\begin{aligned}
\langle\psi_1| E \otimes \mathbb{I} |\psi_1\rangle &= \frac{1}{2}\left(\langle 0| \otimes \langle 0| + \langle 1| \otimes \langle 1|\right) E \otimes \mathbb{I} \left(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle\right) \qquad (5)\\
&= \frac{1}{2}\left(\langle 0| \otimes \langle 0| + \langle 1| \otimes \langle 1|\right)\left(E|0\rangle \otimes |0\rangle + E|1\rangle \otimes |1\rangle\right) \qquad (6)\\
&= \frac{1}{2}(\langle 0| E |0\rangle \otimes \langle 0|0\rangle + \langle 0| E |1\rangle \otimes \langle 0|1\rangle \qquad (7)\\
&\quad + \langle 1| E |0\rangle \otimes \langle 1|0\rangle + \langle 1| E |1\rangle \otimes \langle 1|1\rangle) \qquad (8)\\
&= \frac{\langle 0| E |0\rangle + \langle 1| E |1\rangle}{2} \qquad (9)
\end{aligned}$$

Writing this for the other states with fewer steps

$$\begin{aligned}
\langle\psi_2| E\otimes\mathbb{I}|\psi_2\rangle &= \frac{1}{2}((\langle 0|\otimes\langle 0| - \langle 1|\otimes\langle 1|)(E|0\rangle\otimes|0\rangle - E|1\rangle\otimes|1\rangle)) && (10)\\
&= \frac{1}{2}(\langle 0|E|0\rangle\otimes\langle 0|0\rangle - 0 - 0 + \langle 1|E|1\rangle\otimes\langle 1|1\rangle) && (11)\\
&= \frac{\langle 0|E|0\rangle + \langle 1|E|1\rangle}{2} && (12)
\end{aligned}$$

$$\begin{aligned}
\langle\psi_3| E\otimes\mathbb{I}|\psi_3\rangle &= \frac{1}{2}((\langle 0|\otimes\langle 1| + \langle 1|\otimes\langle 0|)(E|0\rangle\otimes|1\rangle + E|1\rangle\otimes|0\rangle)) && (13)\\
&= \frac{1}{2}(\langle 0|E|0\rangle\otimes\langle 1|1\rangle + 0 + 0 + \langle 1|E|1\rangle\otimes\langle 0|0\rangle) && (14)\\
&= \frac{\langle 0|E|0\rangle + \langle 1|E|1\rangle}{2} && (15)
\end{aligned}$$

$$\begin{aligned}
\langle\psi_4| E\otimes\mathbb{I}|\psi_4\rangle &= \frac{1}{2}((\langle 0|\otimes\langle 1| - \langle 1|\otimes\langle 0|)(E|0\rangle\otimes|1\rangle - E|1\rangle\otimes|0\rangle)) && (16)\\
&= \frac{1}{2}(\langle 0|E|0\rangle\otimes\langle 1|1\rangle - 0 - 0 + \langle 1|E|1\rangle) && (17)\\
&= \frac{\langle 0|E|0\rangle + \langle 1|E|1\rangle}{2} && (18)
\end{aligned}$$

$\square$

Exercise 2.72

Problem: (Bloch sphere for mixed states) The Bloch sphere picture for pure states of a single qubit was introduced in Section 1.2. This description has an important generalization to mixed states as follows.

(1) Show that an arbitrary density matrix for a mixed state qubit may be written as

$$\rho = \frac{\mathbb{I} + \vec{r} \cdot \vec{\sigma}}{2}, \tag{19}$$

where $\vec{r}$ is ta real three-dimensional vector such that $\|\vec{r}\| \leq 1$. This vector is known as the *Bloch vector* for the state ρ.

(2) What is the Bloch vector representation for the state $\rho = \mathbb{I}/2$?

(3) Show that a state ρ is pure if and only if $\|\vec{r}\| = 1$.

(4) Show that for pure states the description of the Bloch vectors we have given coincides with that in Section 1.2.

Solution: