Hand in 2
FYST85

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Exercise 2.70

Problem: Suppose E is any positive operator acting on Alice's qubit. Show that $\langle\psi| E \otimes \mathbb{I} |\psi\rangle$ *takes the same value* when $|\psi\rangle$ is any of the four Bell states. Suppose some malevolent third party ('Eve') intercepts Alice's qubit on the way to Bob in the superdense coding protocol. Can Eve infer anything about which of the four possible bit strings 00, 01, 10, 11 Alice is trying to send? If so, how, or if not, why not?

Solution: Let's number the Bell states as

$$|\psi_1\rangle = \frac{|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B}{\sqrt{2}} \tag{1}$$
$$|\psi_2\rangle = \frac{|0\rangle_A \otimes |0\rangle_B - |1\rangle_A \otimes |1\rangle_B}{\sqrt{2}} \tag{2}$$
$$|\psi_3\rangle = \frac{|0\rangle_A \otimes |1\rangle_B + |1\rangle_A \otimes |0\rangle_B}{\sqrt{2}} \tag{3}$$
$$|\psi_4\rangle = \frac{|0\rangle_A \otimes |1\rangle_B - |1\rangle_A \otimes |0\rangle_B}{\sqrt{2}}. \tag{4}$$

I will not write the subscript A or B further, but infer that the first ket is Alice and second Bob. Applying the operator $E \otimes \mathbb{I}$ on the states we get the following

$$\langle\psi_1| E \otimes \mathbb{I} |\psi_1\rangle = \frac{1}{2}\left(\langle 0| \otimes \langle 0| + \langle 1| \otimes \langle 1|\right) E \otimes \mathbb{I} \left(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle\right) \tag{5}$$
$$= \frac{1}{2}\left(\langle 0| \otimes \langle 0| + \langle 1| \otimes \langle 1|\right)\left(E|0\rangle \otimes |0\rangle + E|1\rangle \otimes |1\rangle\right) \tag{6}$$
$$= \frac{1}{2}(\langle 0| E |0\rangle \otimes \langle 0|0\rangle + \langle 0| E |1\rangle \otimes \langle 0|1\rangle \tag{7}$$
$$+ \langle 1| E |0\rangle \otimes \langle 1|0\rangle + \langle 1| E |1\rangle \otimes \langle 1|1\rangle) \tag{8}$$
$$= \frac{\langle 0| E |0\rangle + \langle 1| E |1\rangle}{2} \tag{9}$$

Writing this for the other states with fewer steps

$$\begin{aligned}
\langle\psi_2|\, E\otimes\mathbb{I}\,|\psi_2\rangle &= \frac{1}{2}(\langle 0|\otimes\langle 0| - \langle 1|\otimes\langle 1|)(E\,|0\rangle\otimes|0\rangle - E\,|1\rangle\otimes|1\rangle) && (10)\\
&= \frac{1}{2}(\langle 0|\, E\,|0\rangle\otimes\langle 0|0\rangle - 0 - 0 + \langle 1|\, E\,|1\rangle\otimes\langle 1|1\rangle) && (11)\\
&= \frac{\langle 0|\, E\,|0\rangle + \langle 1|\, E\,|1\rangle}{2} && (12)
\end{aligned}$$

$$\begin{aligned}
\langle\psi_3|\, E\otimes\mathbb{I}\,|\psi_3\rangle &= \frac{1}{2}(\langle 0|\otimes\langle 1| + \langle 1|\otimes\langle 0|)(E\,|0\rangle\otimes|1\rangle + E\,|1\rangle\otimes|0\rangle) && (13)\\
&= \frac{1}{2}(\langle 0|\, E\,|0\rangle\otimes\langle 1|1\rangle + 0 + 0 + \langle 1|\, E\,|1\rangle\otimes\langle 0|0\rangle) && (14)\\
&= \frac{\langle 0|\, E\,|0\rangle + \langle 1|\, E\,|1\rangle}{2} && (15)
\end{aligned}$$

$$\begin{aligned}
\langle\psi_4|\, E\otimes\mathbb{I}\,|\psi_4\rangle &= \frac{1}{2}(\langle 0|\otimes\langle 1| - \langle 1|\otimes\langle 0|)(E\,|0\rangle\otimes|1\rangle - E\,|1\rangle\otimes|0\rangle) && (16)\\
&= \frac{1}{2}(\langle 0|\, E\,|0\rangle\otimes\langle 1|1\rangle - 0 - 0 + \langle 1|\, E\,|1\rangle\otimes\langle 0|0\rangle) && (17)\\
&= \frac{\langle 0|\, E\,|0\rangle + \langle 1|\, E\,|1\rangle}{2} && (18)
\end{aligned}$$

□

Superdense coding relies on the Bell basis which consists of the Bell states seen above. The bit string 00 corresponds to $|\psi_1\rangle$, 01 corresponds to $|\psi_2\rangle$, 10 corresponds to $|\psi_3\rangle$, and 11 corresponds to $|\psi_4\rangle$. So if Eve intercepts Alice's qubit she will either intercept $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ or $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. This assumes that Eve knows that we transmit using the Bell basis. However, she cannot get any information out of this since $|+\rangle$ applies to both 00 and 10, and $|-\rangle$ applies to both 01 and 11. Thus since Eve does not know about Bob's state she cannot get any information.

Exercise 2.72

Problem: (Bloch sphere for mixed states) The Bloch sphere picture for pure states of a single qubit was introduced in Section 1.2. This description has an important generalization to mixed states as follows.

(1) Show that an arbitrary density matrix for a mixed state qubit may be written as

$$\rho = \frac{\mathbb{I} + \vec{r} \cdot \vec{\sigma}}{2}, \tag{19}$$

where $\vec{r}$ is ta real three-dimensional vector such that $\|\vec{r}\| \leq 1$. This vector is known as the *Bloch vector* for the state ρ.

(2) What is the Bloch vector representation for the state $\rho = \mathbb{I}/2$?

(3) Show that a state ρ is pure if and only if $\|\vec{r}\| = 1$.

(4) Show that for pure states the description of the Bloch vectors we have given coincides with that in Section 1.2.

(1) Solution: We can start by creating a general density matrix for a qubit, which will be in the form

$$\rho = \begin{bmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{bmatrix}, \tag{20}$$

where $c_{ij} \in \mathbb{C}$ for $i, j = 0, 1$. Then we have three conditions on a density matrix

i. $\operatorname{tr} \rho = 1$

ii. ρ is unitary

iii. ρ is positive

The first condition gives that $c_{00} + c_{11} = 1$.Unitarity gives that $c_{00} = c_{00}^*$, $c_{11} = c_{11}^*$, $c_{01} = c_{10}^*$. This gives us a possible parametrization

$$c_{00} = \frac{1+z}{2}, \quad c_{11} = \frac{1-z}{2}, \quad c_{01} = \frac{x-iy}{2}, \quad c_{10} = \frac{x+iy}{2}. \tag{21}$$

Rewriting ρ

$$\rho = \frac{1}{2}\begin{bmatrix} 1+z & x-iy \\ x+iy & 1-z \end{bmatrix}. \tag{22}$$

Now we can separate this matrix such that

$$\begin{bmatrix} 1+z & x+iy \\ x-iy & 1-z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + z\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + x\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + y\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}. \tag{23}$$

We can identify these as the Pauli matrices and thus

$$\rho = \frac{1}{2}(\mathbb{I} + x\sigma_x + y\sigma_y + z\sigma_z) = \frac{\mathbb{I} + \vec{r}\cdot\vec{\sigma}}{2}, \tag{24}$$

where $\vec{r} = (x, y, z)$ and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$. To verify $\|\vec{r}\| \leq 1$ we use the positivity condition which is equivalent to say that all eigenvalues λ of ρ must be positive. Thus, using Eq. (22) to calculate the eigenvalues we obtain

$$0 = \det(\rho - \lambda\mathbb{I}) = \begin{vmatrix} \frac{1+z}{2} - \lambda & \frac{x-iy}{2} \\ \frac{x+iy}{2} & \frac{1-z}{2} - \lambda \end{vmatrix} \tag{25}$$

$$= \left(\frac{1+z}{2} - \lambda\right)\left(\frac{1-z}{2} - \lambda\right) - \frac{x-iy}{2}\cdot\frac{x+iy}{2} \tag{26}$$

$$= \frac{1-z^2}{4} - \left(\frac{1+z}{2} + \frac{1-z}{2}\right)\lambda + \lambda^2 - \frac{x^2+y^2}{4} \tag{27}$$

$$= \lambda^2 - \lambda - \frac{x^2+y^2+z^2-1}{4}. \tag{28}$$

Using the pq-formula with

$$p = -1, \quad q = -\frac{x^2+y^2+z^2-1}{4} \tag{29}$$

we get

$$\lambda = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q} = \frac{1}{2} \pm \sqrt{\frac{x^2+y^2+z^2}{4}} = \frac{1}{2}\left(1 \pm \sqrt{\|\vec{r}\|^2}\right). \tag{30}$$

From the definition of a density matrix we know that for the spectral decomposition the eigenvalues are probabilities. Thus $0 \leq \lambda \leq 1$, and it then follows that $\|\vec{r}\| \leq 1$.

$\square$

(2) Solution: For the density matrix to equal $\rho = \mathbb{I}/2$ we must have that $\vec{r} \cdot \vec{\sigma} = \vec{0}$. Since $\vec{\sigma}$ is non-zero we must have that $\vec{r} = \vec{0}$.

$\square$

(3) Solution: The density matrix of a pure state $|\psi\rangle$ can be written $\rho = |\psi\rangle \langle\psi|$. That is, the probability of obtaining this state is 1. So if we do a spectral decomposition of the density matrix the eigenvalues must be $\lambda = 0, 1$. And by Eq. (30) this happens if and only if $\|\vec{r}\| = 1$

(4) Solution: According to section 1.2 we can write a pure state as

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|0\rangle. \tag{31}$$

Since all pure states lie on the Bloch sphere, which has a radius of 1. We can switch to spherical coordinates with

$$x = \sin\theta\cos\varphi, \quad y = \sin\theta\sin\varphi, \quad z = \cos\theta. \tag{32}$$

Rewriting ρ in Eq. (22) we obtain

$$\rho = \frac{1}{2}\begin{bmatrix} 1+\cos\theta & \sin\theta\cos\varphi - i\sin\theta\sin\varphi \\ \sin\theta\cos\varphi + i\sin\theta\sin\varphi & 1-\cos\theta \end{bmatrix} \tag{33}$$

$$= \begin{bmatrix} \dfrac{1+\cos\theta}{2} & (\cos\varphi - i\sin\varphi)\dfrac{\sin\theta}{2} \\ (\cos\varphi + i\sin\varphi)\dfrac{\sin\theta}{2} & \dfrac{1-\cos\theta}{2} \end{bmatrix} \tag{34}$$

Using Euler's formula we can rewrite $\cos\varphi \pm i\sin\varphi = e^{\pm i\varphi}$. Then using the trigonometric identities $2\cos^2\theta = 1+\cos 2\theta$, $2\sin^2\theta = 1-\cos 2\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$, we rewrite the matrix as

$$\rho = \begin{bmatrix} \cos^2\frac{\theta}{2} & e^{-i\varphi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} & \sin^2\frac{\theta}{2} \end{bmatrix}. \tag{35}$$

Changing to state vector representation we get

$$\begin{aligned} \rho &\doteq \cos^2\frac{\theta}{2}\left|0\right\rangle\left\langle 0\right| + e^{-i\varphi}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\left|0\right\rangle\left\langle 1\right| + e^{i\varphi}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\left|1\right\rangle\left\langle 0\right| + \sin^2\frac{\theta}{2}\left|1\right\rangle\left\langle 1\right| && (36) \\ &= \left(\cos\frac{\theta}{2}\left|0\right\rangle + e^{i\varphi}\sin\frac{\theta}{2}\left|1\right\rangle\right)\cdot\left(\cos\frac{\theta}{2}\left\langle 0\right| + e^{-i\varphi}\sin\frac{\theta}{2}\left\langle 1\right|\right) && (37) \\ &= \left|\psi\right\rangle\left\langle\psi\right|. && (38) \end{aligned}$$

$\square$