Hand In 3

FYST85

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1 First Exercise

The matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 0 & 0 & c \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & b & 0 & 0 & d \end{pmatrix} \tag{1}$$

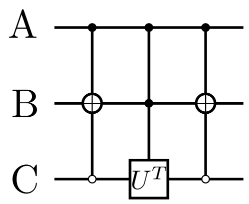
acts non-trivially on the states $|100\rangle$ and $|111\rangle$. Thus we can write a Gray code

$$\begin{matrix} A & B & C \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{matrix} \tag{2}$$

That is, we need a three-qubit operation switching the second qubit before applying the unitary operator U^T where

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{3}$$

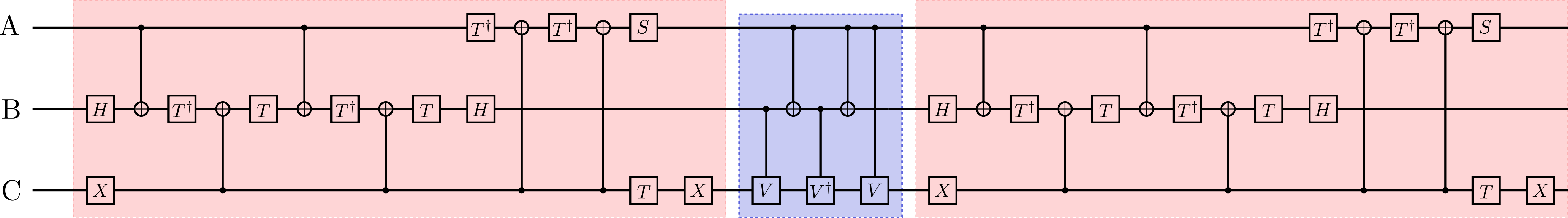
which would look like

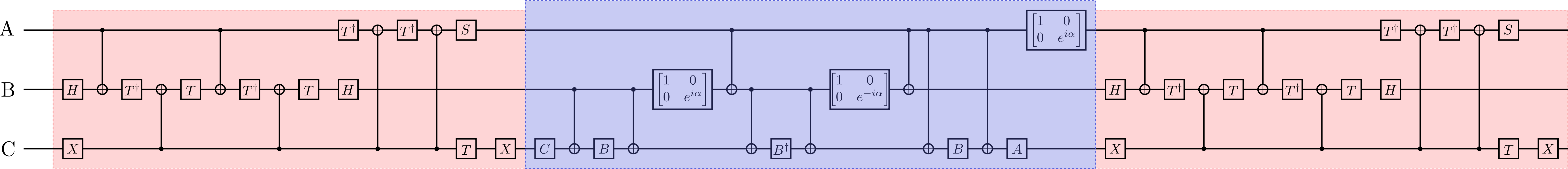


Showing this with only single-qubit operations and two-qubit operations. This is shown in the next page. It is very wide so can be a bit hard to see. The red parts are the CCNOT gates while the blue part is the double controlled U^T part, where V is a unitary operator such that $V^2 = U^T$. However, we want to only use CNOT as two qubit operations. We know that we can rewrite any controlled O operator as

$$O = e^{i\alpha} AXBXC, \quad \text{where } ABC = \mathbb{1}. \tag{4}$$

Since $O^\dagger = C^\dagger X B^\dagger X A^\dagger$ the circuit becomes what is seen on the pages after the last circuit.


A
B
C
H
X
T†
T
H
T
S
X
V
V†
V


A
B
C
H
X
T†
T
S
C
B
B†
A
$\begin{bmatrix}1 & 0\\0 & e^{i\alpha}\end{bmatrix}$
$\begin{bmatrix}1 & 0\\0 & e^{-i\alpha}\end{bmatrix}$

2 Second Exercise

We have the identities

$$X = HZH, \quad Y = SHZHS^{\dagger}. \tag{5}$$

Then we have the hamiltonian

$$\mathcal{H} = X_1 \otimes Y_2 \otimes Z_3 \tag{6}$$

rewriting the Hamiltonian using the identities above we get

$$\mathcal{H} = (H_1 Z_1 H_1) \otimes (S_2 H_2 Z_2 H_2 S_2^{\dagger}) \otimes Z_3 \tag{7}$$

which can be factored as

$$\mathcal{H} = (H_1 \otimes S_2 H_2 \otimes \mathbb{1})(Z_1 \otimes Z_2 \otimes Z_3)(H_1 \otimes H_2 S_2^{\dagger} \otimes \mathbb{1}). \tag{8}$$

Now using Taylor expansion

$$e^{-i\Delta t \mathcal{H}} = \sum_{n}^{\infty} \frac{(-i\Delta t \mathcal{H})^n}{n!} \tag{9}$$

Since all operations are unitary and H is hermitian we can use that for a unitary operator U we have that

$$(UAU^{\dagger})^n = (UAU^{\dagger})(UAU^{\dagger})(UAU^{\dagger}) \cdots = UA(U^{\dagger}U)A(U^{\dagger}U)AU^{\dagger} \cdots = UA^nU^{\dagger} \tag{10}$$

That is, we can write $U = H_1 \otimes S_2 H_2 \otimes \mathbb{1}$ and $\mathcal{Z} = Z_1 \otimes Z_2 \otimes Z_3$. Rewriting Eq. (9) we get

$$e^{-i\Delta t \mathcal{H}} = \sum_{n}^{\infty} \frac{(-i\Delta t U \mathcal{Z} U^{\dagger})^n}{n!} = U \left(\sum_{n}^{\infty} \frac{(-i\Delta t \mathcal{Z})^n}{n!} \right) U^{\dagger} = U e^{-i\Delta t \mathcal{Z}} U^{\dagger} \tag{11}$$

Thus simulating the time evolution of the hamiltonian means implementing

$$e^{-i\Delta t \mathcal{H}} = (H_1 \otimes S_2 H_2 \otimes \mathbb{1}) e^{-i\Delta t \mathcal{Z}} (H_1 \otimes H_2 S_2^{\dagger} \otimes \mathbb{1}) \tag{12}$$

From Nielsen & Chuang we know that the quantum circuit implementing $e^{-i\Delta t \mathcal{Z}}$ corresponds to Figure 4.19 in the book. Thus we need to apply the corresponding gates before and after this and we get