Hand In 1

FYST85

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Problem: Consider the following two-qubit state:

$$|\psi\rangle = \frac{1}{2}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle + \frac{1}{2}|11\rangle$$

A. Show that the state is normalized.

B. If you make a measurement on the first qubit and obtain the result $|0\rangle$, what is now the two-qubit state after this measurement?

C. Suppose that you instead obtained the result $|1\rangle$, what is now the two-qubit state after this measurement? (Remember to answer as a normalized state)

D. Is the original state an entangled state?

A. For a normalized state the coefficients c_n have the property

$$\sum_n |c_n|^2 = 1,$$

thus for the state $|\psi\rangle$ we have the coefficients $c_1 = 1/2$, $c_2 = 1/\sqrt{2}$, $c_3 = 1/2$. Thus

$$\sum_n |c_n|^2 = \left|\frac{1}{2}\right|^2 + \left|\frac{1}{\sqrt{2}}\right|^2 + \left|\frac{1}{2}\right|^2 = \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1$$

$\square$

B. Let the measurement be described by the set $\{M_0, M_1\}$ where M_m are measurement operators defined as

$$M_m = |m\rangle\langle m| \quad \text{for } m = 0, 1$$

where m correspond to the measured value. Since we obtain the result $|0\rangle$ we apply M_0 on the first qubit we have

$$\begin{aligned}
M_0 \otimes \mathbb{1}(|\psi\rangle) &= M_0 \otimes \mathbb{1}\left(\frac{1}{2}|0\rangle \otimes |0\rangle + \frac{1}{\sqrt{2}}|0\rangle \otimes |1\rangle + \frac{1}{2}|1\rangle \otimes |1\rangle\right) \\
&= \frac{1}{2}M_0|0\rangle \otimes \mathbb{1}|0\rangle + \frac{1}{\sqrt{2}}M_0|1\rangle \otimes \mathbb{1}|0\rangle + \frac{1}{2}M_0|1\rangle \otimes \mathbb{1}|1\rangle \\
&= \frac{1}{2}|0\rangle\langle 0|0\rangle \otimes |0\rangle + \frac{1}{\sqrt{2}}|0\rangle\langle 0|1\rangle \otimes |0\rangle + \frac{1}{2}|0\rangle\langle 0|1\rangle \otimes |1\rangle \\
&= \frac{1}{2}|00\rangle .
\end{aligned}$$

Normalizing this by dividing by the square root of the probability of this measurement outcome occurring we get the state after measurement. First we calculate the probability $p(0)$

$$\begin{aligned}
p(0) &= \langle\psi|(M_0^\dagger \otimes \mathbb{1})(M_0 \otimes \mathbb{1})|\psi\rangle = \langle\psi|(M_0 \otimes I)|\psi\rangle \\
&= \frac{1}{2}\langle\psi|00\rangle = \frac{1}{2}\left(\frac{1}{2}\langle 00|00\rangle + \frac{1}{\sqrt{2}}\langle 10|00\rangle + \frac{1}{2}\langle 11|00\rangle\right) = \frac{1}{4}\langle 00|00\rangle = \frac{1}{4}
\end{aligned}$$

Thus we get the final state

$$|\psi\rangle' = \frac{(M_0 \otimes \mathbb{1})|\psi\rangle}{\sqrt{p(0)}} = |00\rangle$$

C. Since we obtain the value $|1\rangle$ we apply M_1 to the first qubit. We get

$$\begin{aligned}
M_1 \otimes \mathbb{1}(|\psi\rangle) &= M_1 \otimes \mathbb{1}\left(\frac{1}{2}|0\rangle \otimes |0\rangle + \frac{1}{\sqrt{2}}|0\rangle \otimes |1\rangle + \frac{1}{2}|1\rangle \otimes |1\rangle\right) \\
&= \frac{1}{2}M_0|0\rangle \otimes \mathbb{1}|0\rangle + \frac{1}{\sqrt{2}}M_0|1\rangle \otimes \mathbb{1}|0\rangle + \frac{1}{2}M_0|1\rangle \otimes \mathbb{1}|1\rangle \\
&= \frac{1}{2}|1\rangle\langle 1|0\rangle \otimes |0\rangle + \frac{1}{\sqrt{2}}|1\rangle\langle 1|1\rangle \otimes |0\rangle + \frac{1}{2}|1\rangle\langle 1|1\rangle \otimes |1\rangle \\
&= \frac{1}{\sqrt{2}}|10\rangle + \frac{1}{2}|11\rangle .
\end{aligned}$$

Since we must have $\sum_m p(m) = 1$ we get from **B.** that $p(1) = 3/4$. Thus we get the final state

$$|\psi\rangle' = \frac{(M_1 \otimes \mathbb{1})\,|\psi\rangle}{\sqrt{p(1)}} = \frac{2}{\sqrt{3}}\Big(\frac{1}{\sqrt{2}}\,|10\rangle + \frac{1}{2}\,|11\rangle\Big) = \frac{\sqrt{2}}{\sqrt{3}}\,|10\rangle + \frac{1}{\sqrt{3}}\,|11\rangle$$

D. If the state is separable and not entangled we can write it as $|A\rangle \otimes |B\rangle$ where

$$\begin{aligned} |A\rangle &= a_0\,|0\rangle + a_1\,|1\rangle \\ |B\rangle &= b_0\,|0\rangle + b_1\,|1\rangle\,. \end{aligned}$$

Performing the product we obtain

$$\begin{aligned} |A\rangle \otimes |B\rangle &= (a_0\,|0\rangle + a_1\,|1\rangle) \otimes (b_0\,|0\rangle + b_1\,|1\rangle) \\ &= a_0b_0\,|00\rangle + a_0b_1\,|01\rangle + a_1b_0\,|01\rangle + a_1b_1\,|11\rangle\,. \end{aligned}$$

So if the state is separable there is a unique solution to the system

$$\begin{cases} a_0b_0 = \dfrac{1}{2} \\ a_0b_1 = 0 \\ a_1b_0 = \dfrac{1}{\sqrt{2}} \\ a_1b_1 = \dfrac{1}{2} \end{cases}.$$

For the second equation we must have either $a_0 = 0$ or $b_1 = 0$. However, $a_0 = 0$ contradicts the first equation and $b_1 = 0$ contradicts the last equation. Thus $|\psi\rangle$ is entangled.