# Name of Hand In
## Course code

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# 1　First Exercise

**Problem:**　Three identical springs are connected in a row and then attached between a pair of walls. Two identical masses are attached at the connection points between the central spring and the two others. Using Lagrangian Mechanics, determine the frequencies for horizontal longitudinal vibrations.

**Solution:**　Assuming the springs are massless and of length $l$ with a spring constant $k$, we can describe the system by setting the coordinate for the left mass to $x_1$ and the right mass $x_2$. Then by putting the origin at the left wall and the the distance $L$ between the walls we obtain the figure below.
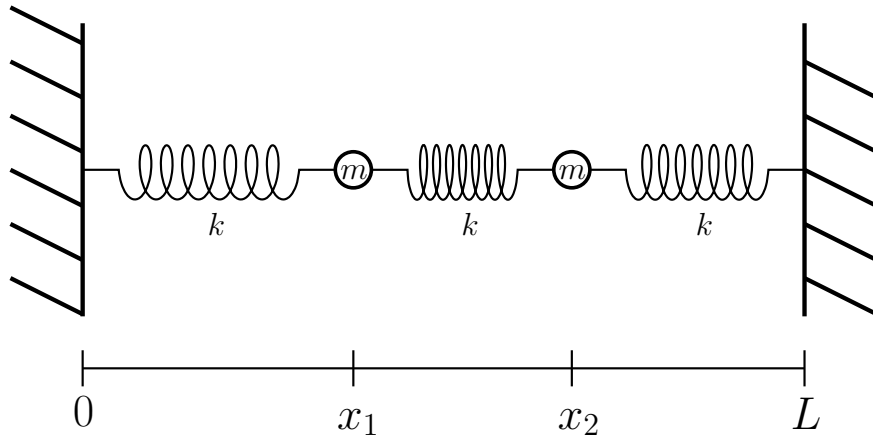


Figure 1: Problem setup

The kinetic energy of the system is the sum of the kinetic energy of the two masses.

$$T = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2)$$

The potential is the sum of the potentials for the different springs

$$V_{\text{left}} = \frac{1}{2}k(x_1 - l)^2$$

$$V_{\text{middle}} = \frac{1}{2}k(x_2 - x_1 - l)^2$$

$$V_{\text{right}} = \frac{1}{2}k(L - x_1 - l)^2$$

Thus the Lagrangian is

$$\mathcal{L} = T - V_{\text{left}} - V_{\text{middle}} - V_{\text{right}} = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) - \frac{1}{2}k[(x_1 - l)^2 + (x_2 - x_1 - l)^2 + (L - x_2 - k)^2]$$

Setting up the Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial x_i} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_i}\right) \quad \text{for } i = 1, 2$$

we obtain the system of equations

$$\begin{cases} k(2x_1 - x_2) = m\ddot{x}_1 \\ k(2x_2 - x_1 - L) = m\ddot{x}_2 \end{cases}$$

The equilibrium for the masses occur when $\ddot{x}_1 = \ddot{x}_2 = 0$ and since $k \neq 0$ we get the equilibrium points $x_1^0$ and $x_2^0$ when

$$\begin{cases} 2x_1^0 - x_2^0 = 0 \\ 2x_2^0 - x_1^0 - L = 0 \end{cases} \iff \begin{cases} x_2^0 = \dfrac{2L}{3} \\ 3x_1^0 = \dfrac{L}{3} \end{cases}$$

Doing a change of coordinates

$$\begin{cases} x_1 \to u_1 = x_1 - x_1^0 = x_1 - \dfrac{L}{3} \\ x_2 \to u_2 = x_2 - x_2^0 = x_2 - \dfrac{2L}{3} \end{cases} \iff \begin{cases} \ddot{x}_1 = \ddot{u}_1 \\ \ddot{x}_2 = \ddot{u}_2 \end{cases}$$

gives us the system

$$\begin{cases} k(2u_1 - u_2) = m\ddot{u}_1 \\ k(2u_2 - u_1) = m\ddot{u}_2 \end{cases} \iff \begin{cases} \ddot{u}_1 = \dfrac{k}{m}(2u_1 - u_2) \\ \ddot{u}_2 = \dfrac{k}{m}(2u_2 - u_1) \end{cases}$$

Doing the ansatz $u_i = C_i e^{i\omega t}$ we can write the system as a matrix equation

$$-\omega^2 \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \frac{k}{m} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \iff \left( \frac{k}{m} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} + \omega^2 \mathbb{I} \right) \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = 0.$$

It then follows that

$$0 = \begin{vmatrix} 2k/m + \omega^2 & -k/m \\ -k/m & 2k/m + \omega^2 \end{vmatrix} = \left( \frac{2k}{m} + \lambda \right)^2 - \frac{k^2}{m^2} = \omega^4 - \frac{4k}{m}\omega^2 + \frac{3k^2}{m^2}.$$

The pq-formula then gives

$$\omega^2 = \frac{2k}{m} \pm \frac{k}{m} \iff \omega_1 = \sqrt{\frac{k}{m}}, \ \omega_2 = \sqrt{\frac{3k}{m}}$$

**Answer:**   The vibrational frequencies are

$$\omega_1 = \sqrt{\frac{k}{m}}$$

$$\omega_2 = \sqrt{\frac{3k}{m}}$$