Math for 3D/Games Programmers

4. Geometrical Objects Equations

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Line

- A **straight line** can be represented mathematically in multiple ways
- The most common is the **linear equation** (**linear function**):

$$y = ax + b$$

- Another way is to use the **implicit equation**:

$$ax + by + c = 0$$

- Finally there is the **parametric equation**:

$$p(t) = p_0 + \vec{v}t$$

Line

- Each of those forms has its own unique advantages
- The linear equation and implicit equation allow only for representation of a 2D line.
  The parametric equation can represent both a 2D and 3D line
- In this subchapter we will find a straight line equation in all those three forms, given points $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$
- We will find equations of two distinct lines and find their intersection point

Line (2D) – Line Equation

- The classical equation, a function of one variable $y = f(x)$, which we all know from school:

$$y = ax + b$$

- a is called **slope of the straight line**
- b is called **y-intercept**

Line (2D) – Line Equation

- Given points p_1 and p_2 let's find the line equation that passes via those points
- The points satisfy the following system of equations:

$$\begin{cases} y_1 = \boldsymbol{a}x_1 + \boldsymbol{b} \\ y_2 = \boldsymbol{a}x_2 + \boldsymbol{b} \end{cases}$$

- After solving the above system we get a and b
- The solution is:

$$\begin{cases} a = \dfrac{y_2 - y_1}{x_2 - x_1} \\ b = y_1 - ax_1 \end{cases}$$

Line (2D) – Line Equation

- We have equations of two lines:

$$y = a_1 x + b_1$$

$$y = a_2 x + b_2$$

- We're looking for their intersection point (p_x, p_y)
- Intersection point satisfies both equations at the same time, so:

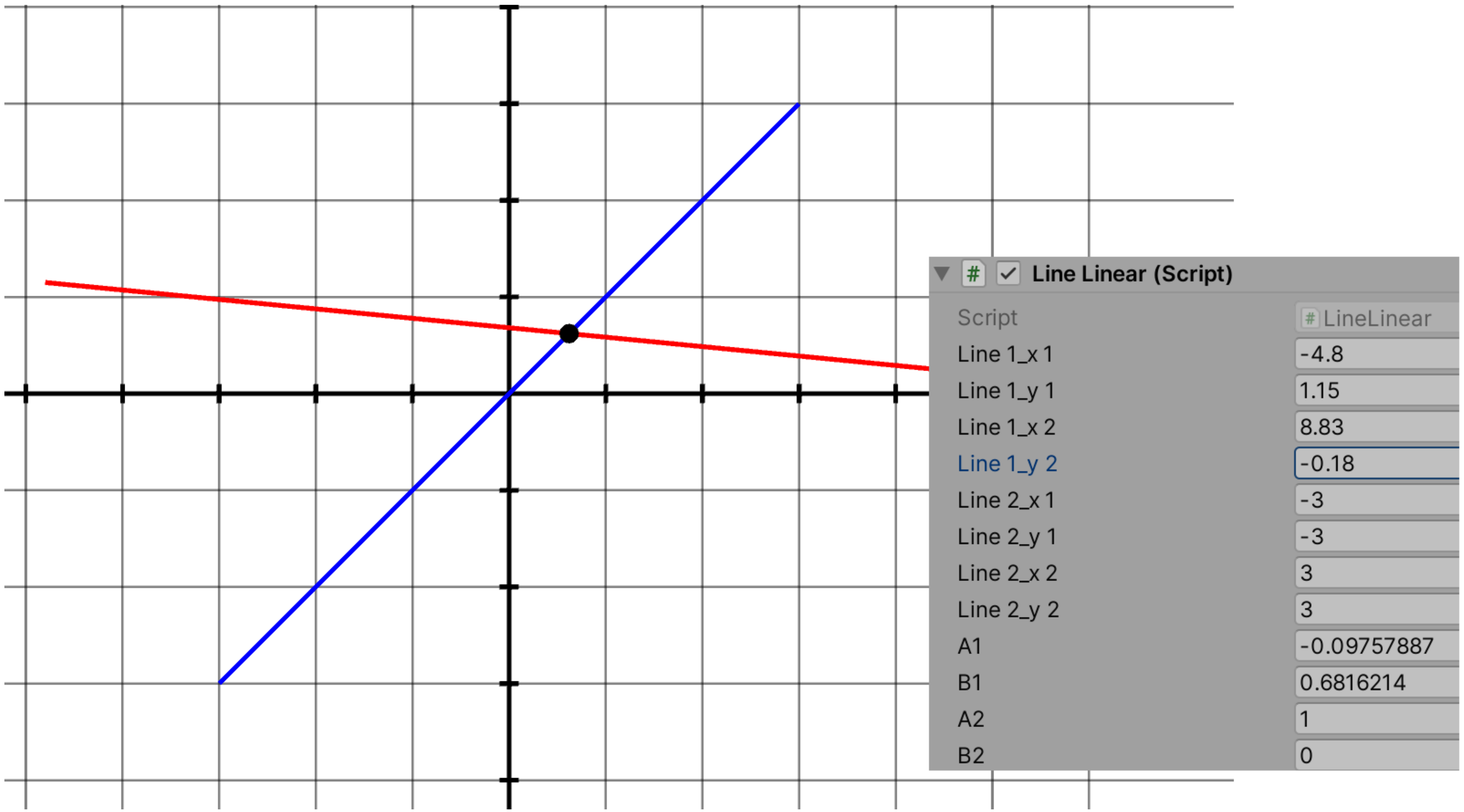
$$\begin{cases} \boldsymbol{p_y} = a_1 \boldsymbol{p_x} + b_1 \\ \boldsymbol{p_y} = a_2 \boldsymbol{p_x} + b_2 \end{cases}$$

Line (2D) – Line Equation

- A solution to this system is:

$$\begin{cases} p_x = \dfrac{b_2 - b_1}{a_1 - a_2} \\ p_y = a_1 p_x + b_1 \end{cases}$$

Line (2D) – Line Equation



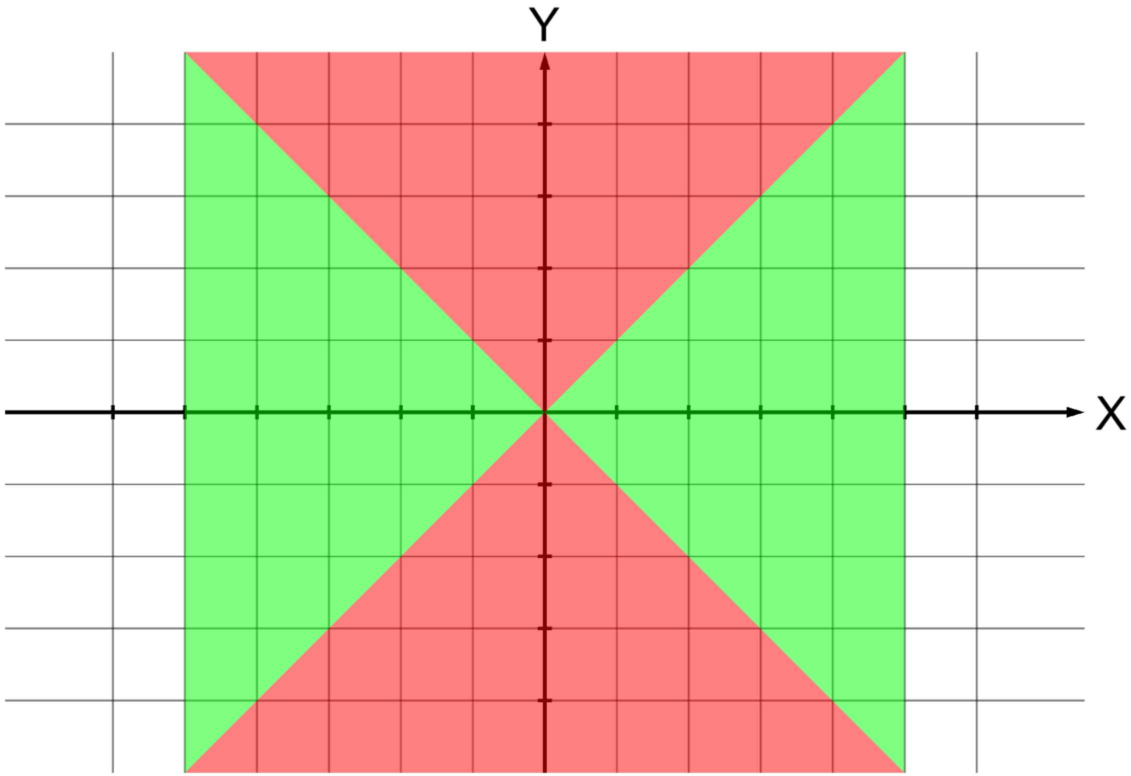
Line (2D) – Line Equation

- The biggest disadvantage of this straight line representation is the inability to represent a vertical line
- Additionally, the more vertical a line gets, the less precise are calculations that use it
- A constant change in argument x leads to a „step” of varying distance on a line, that depends on the parameter a
- To represent a line we only need two values

Line (2D) – Line Equation – Vertical Line

- A solution to the vertical line problem may be an alternative equation:

$$x = ay + b$$



Line (2D) – Line Equation – Tangent

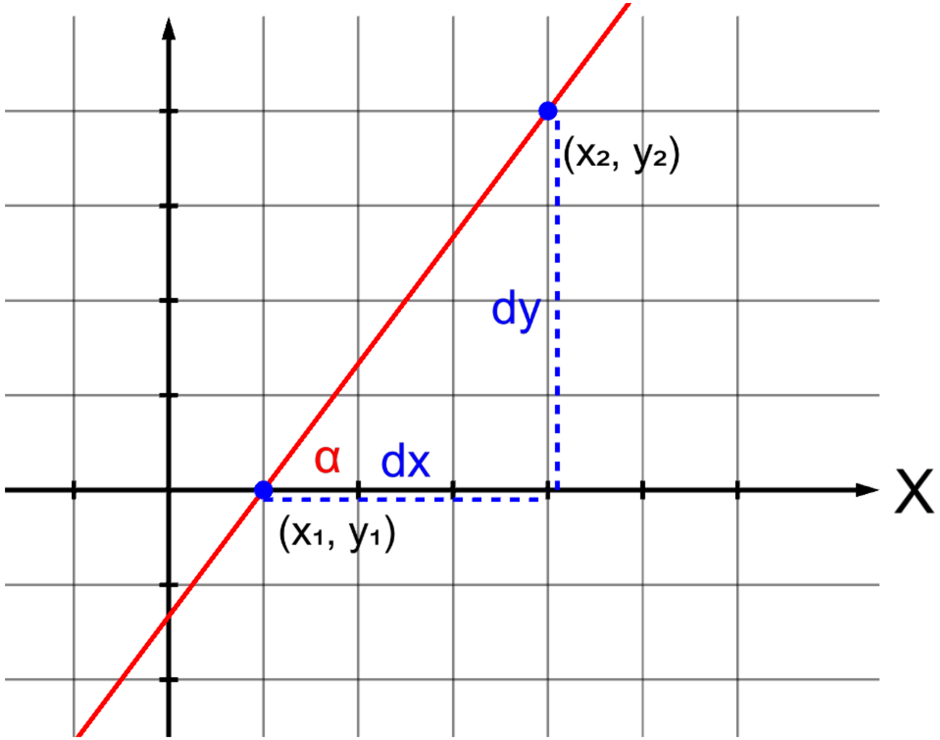
- The following formula is true:

$$a = \tan(\alpha) \qquad \alpha - \text{angle between the line and the X axis}$$

$$dx = x_2 - x_1$$

$$dy = y_2 - y_1$$

$$a = \frac{dy}{dx} = \frac{y_2 - y_1}{x_2 - x_1}$$



Line (2D) – Line Equation – Linear Mapping

- The linear equation’s meaning extends beyond its geometric interpretation
- The linear equation, in general, can represent any „linear mapping” – a function which linearly maps one interval $[a, b]$ onto another $[c, d]$

Line (2D) – Line Equation – Linear Mapping

- As an example, with the linear function we can easily find a formula that „maps" the interval $[10, 50]$ onto $[0, 1]$
- Value of 10 is to be mapped to 0, while 50 to 1.
  Values in-between are mapped linearly, so for example 30 is mapped to 0.5
- Such a mapping can be used, for example, to implement how a light's or sound's intensity changes with distance from the source

Line (2D) – Line Equation – Linear Mapping

- Let's find a linear function which determines the aforementioned mapping
- The function is of form:

$$y \;=\; f(x) \;=\; ax + b$$

- We know that:

$$f(10) = 0$$
$$f(50) = 1$$

- Therefore:

$$0 = a * 10 + b$$
$$1 = a * 50 + b$$

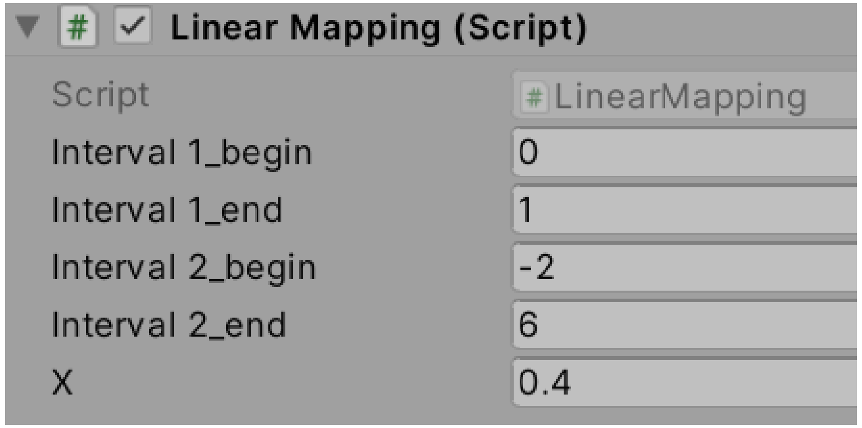
Line (2D) – Line Equation – Linear Mapping

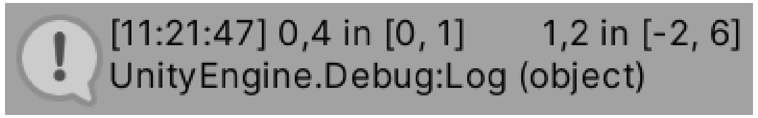
- A solution is the formula that we've seen before:

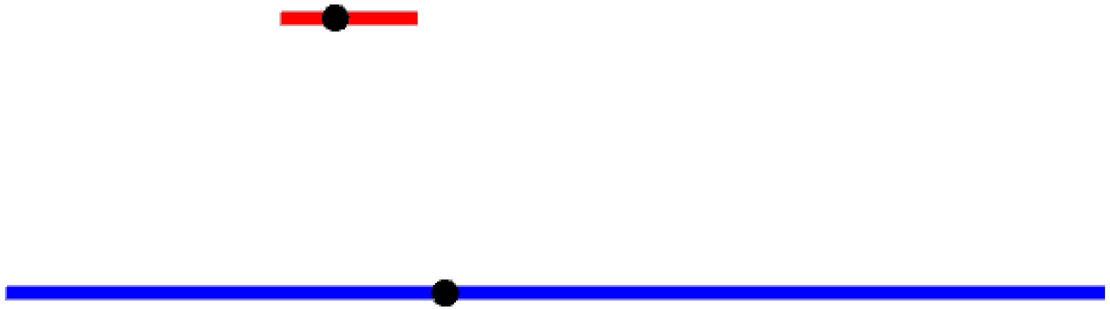
$$\begin{cases} a = \dfrac{y_2 - y_1}{x_2 - x_1} \\ b = y_1 - ax_1 \end{cases}$$

$$\begin{cases} a = \dfrac{1 - 0}{50 - 10} \\ b = 0 - a * 10 \end{cases}$$

Line (2D) – Line Equation – Linear Mapping







Line (2D) – Implicit Equation

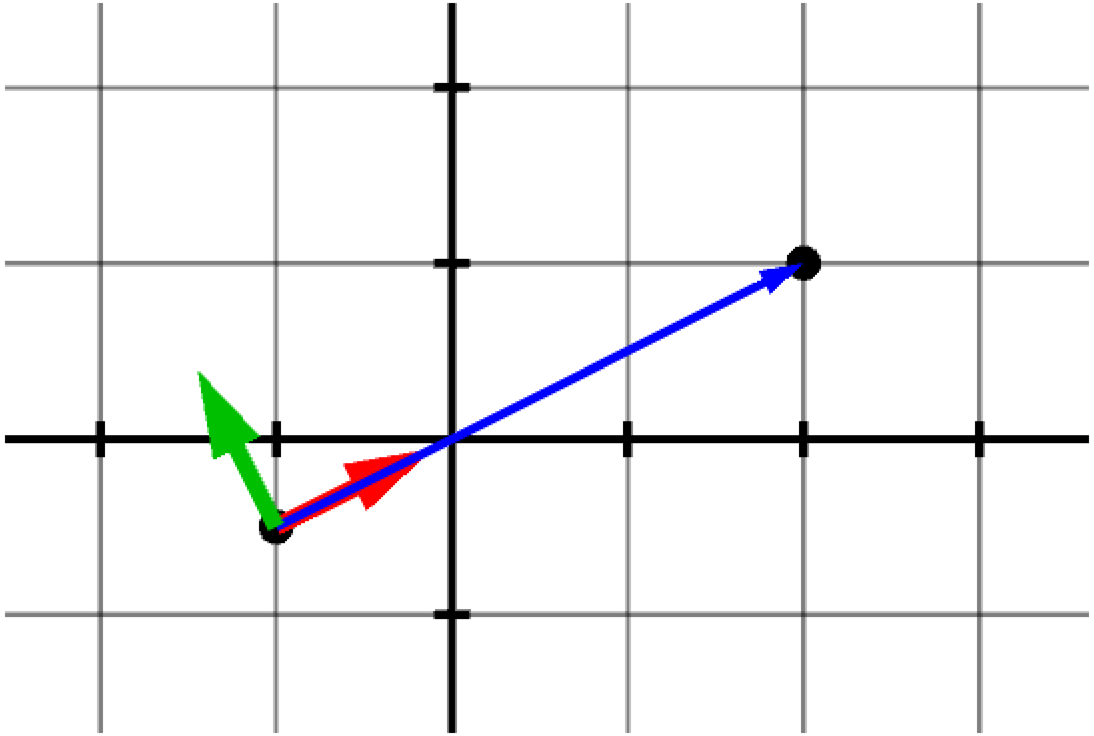
- The equation of form:

$$ax + by + c = 0$$

- Implicit equations do not give as „explicit solutions" to variables/functions. Solutions (in this case points that belong on the line) are „embedded" in the equation (x and y coordinates)
- $[a, b]$ is the normal vector of the line.
  It doesn't have to be normalized, although it usually is
- c is the distance of the line from the origin
- (x, y) are the coordinates of any point that lies on the line

Line (2D) – Implicit Equation

- To determine the equation of the implicit line we start by determining a and b:



Line (2D) – Implicit Equation

- We find $\vec{v} = p_2 - p_1$. We normalize it
- We find the normal $[a, b] = [-\vec{v}_y, \vec{v}_x]$
- Now we can find c:

$$ax + by + c = 0$$

$$c = -(ax + by)$$

$$c = -(ax_1 + by_1)$$

- If $[a, b]$ is normalized, then the expression $ax + by + c$ gives us the **„signed"** distance of point (x, y) from the line

Line (2D) – Implicit Equation

- We have two distinct lines:

$$a_1 x + b_1 y + c_1 = 0$$

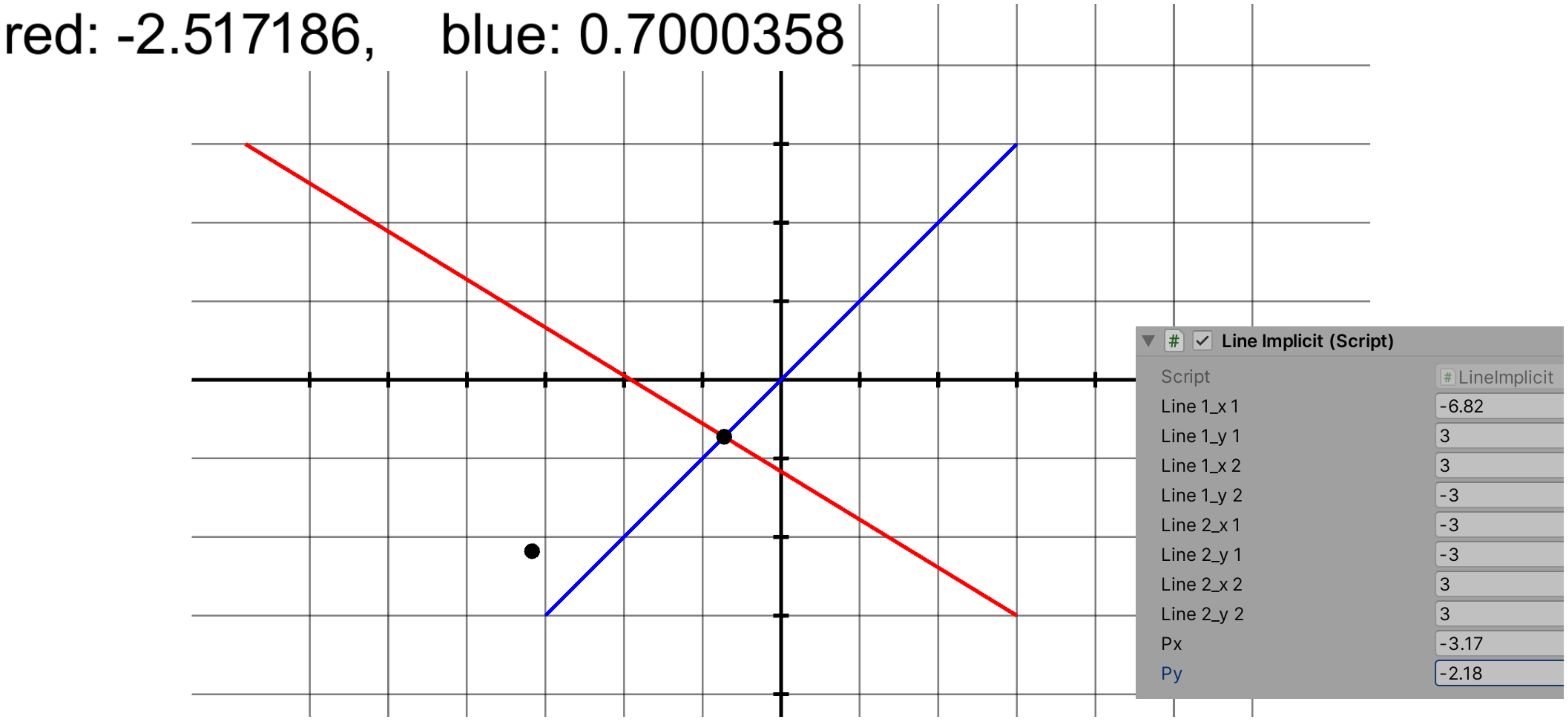
$$a_2 x + b_2 y + c_2 = 0$$

- Finding (x, y) in this system gives us the intersection point of those two lines:

$$\begin{cases} a_1 \boldsymbol{x} + b_1 \boldsymbol{y} + c_1 = 0 \\ a_2 \boldsymbol{x} + b_2 \boldsymbol{y} + c_2 = 0 \end{cases}$$

- [Wolfram]

Line (2D) – Implicit Equation



Line (2D) – Implicit Equation

- We've just discussed the implicit line equation:

$$ax + by + c = 0$$

- It's worth knowing though that quite often we can find this equation written as:

$$Ax + By + C = 0$$

Line (2D) – Implicit Equation

- The implicit equation is a generalization of the linear equation:

$$Ax + By + C = 0$$

$$By = -(Ax + C)$$

$$y = -\frac{Ax + C}{B}$$

$$y = -\frac{A}{B}x + \frac{C}{B}$$

$$a = -\frac{A}{B} \qquad b = \frac{C}{B}$$

Line (2D) – Implicit Equation

- As an example, let's take an implicit equation the represents a certain line:

$$3x - 5y + 1 = 0$$

- The linear equation, which represents the exact same line is:

$$a = -\frac{A}{B} \qquad b = \frac{C}{B}$$

$$a = -\frac{3}{-5} \qquad b = \frac{1}{-5}$$

$$y \;=\; ax + b \;=\; \frac{3}{5}x - \frac{1}{5}$$

Line (2D) – Implicit Equation

- It’s easy to determine on which side of a plane a points is
- It’s easy to find the distance of a point from a line
- Certain formulas/applications require that a geometric object be in implicit form
- Points on a line can be generated by solving the equation for one of the coordinates
- It’s possible to represent a vertical line
- Needs three values to represent a line

Line (2D/3D) – Parametric Equation

- The equation of form:

$$p(t) = p_0 + \vec{v}t$$

- p are coordinates of a point on the line
- p_0 are coordinates of some fixed (arbitrary) point on the line
- $\vec{v}$ is the line's **direction vector**. Depending on our needs it's normalized or not
- t is a **parameter**, whose different values generate us different points on the line. For $t \geq 0$ the line reduces to a **ray**
- Allows us to represnt a line in both 2D and 3D

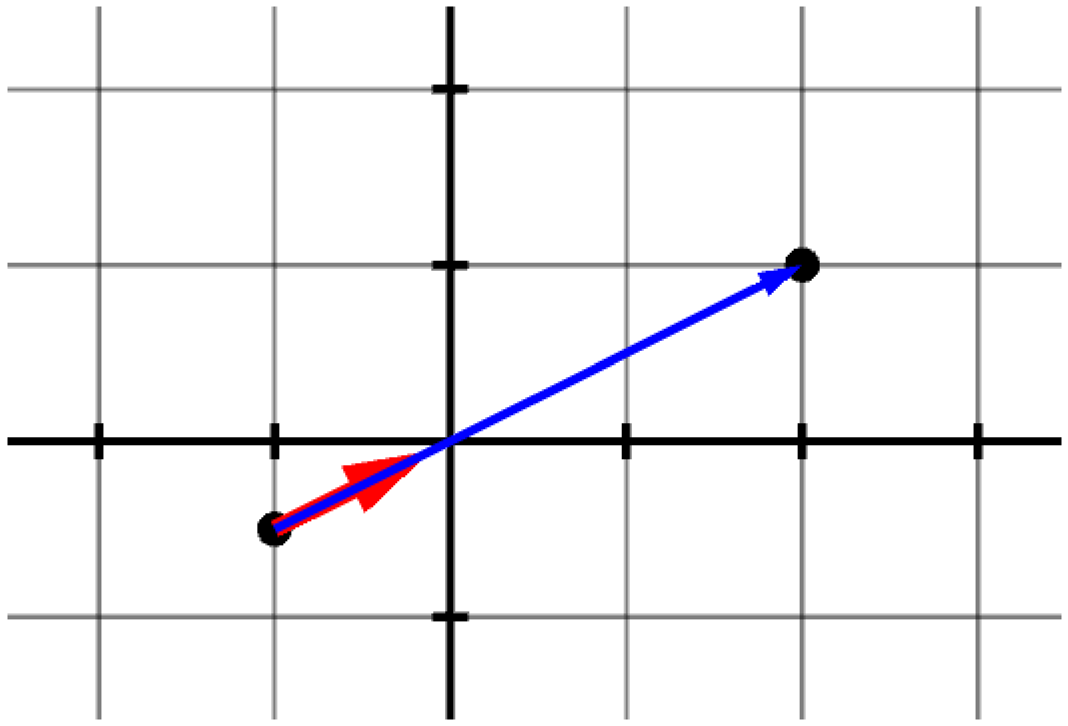
Line (2D/3D) – Parametric Equation

- We can also write the equation for each coordinate separately (2D case):

$$\begin{cases} x(t) = x_0 + \vec{v}_x t \\ y(t) = y_0 + \vec{v}_y t \end{cases}$$

Line (2D/3D) – Parametric Equation

- The direction vector $\vec{v}$ can be normalized or not:



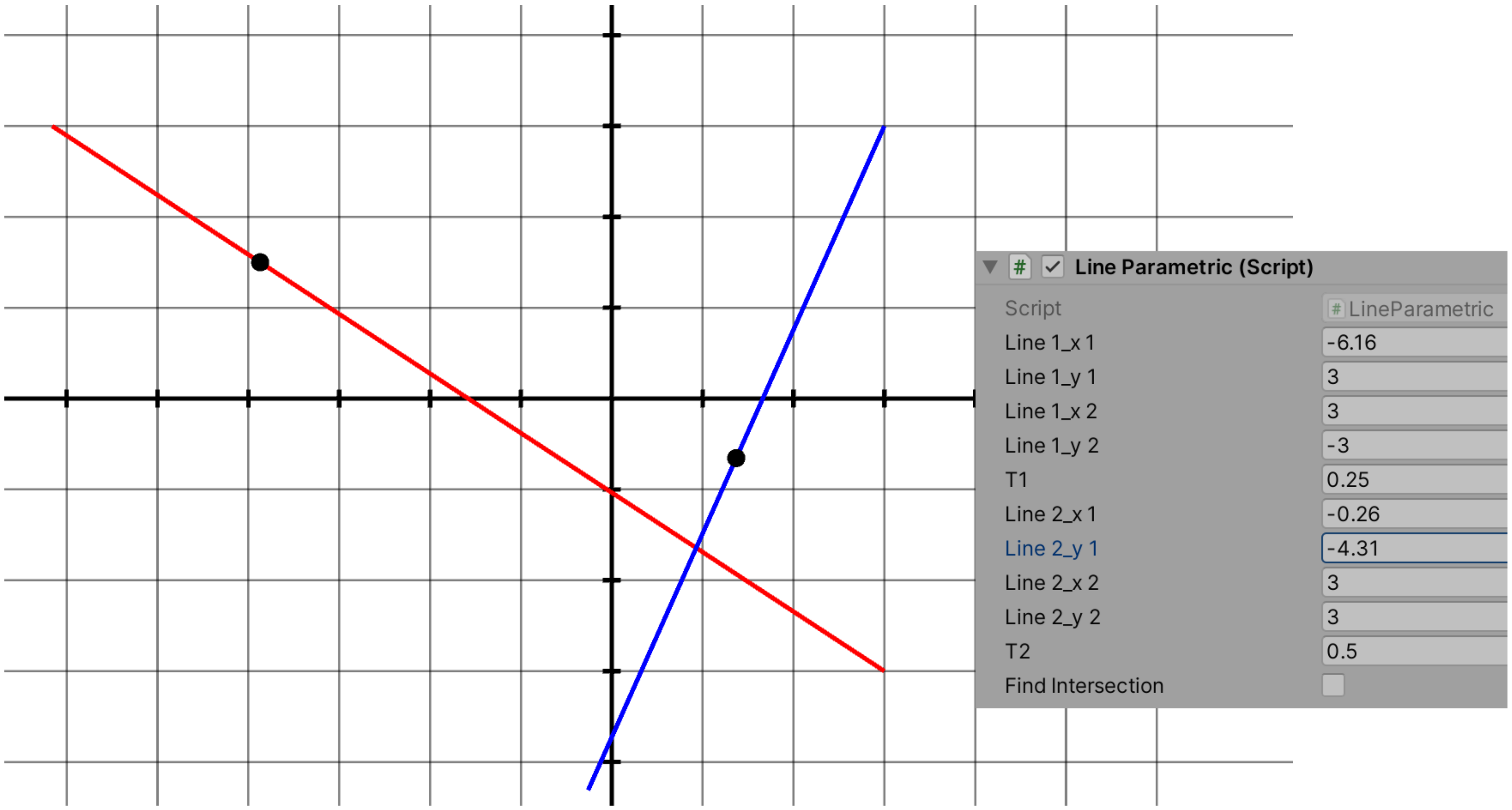
Line (2D/3D) – Parametric Equation

- Let's take two points p_1 and p_2 and find a parametric line equation for them
- For p_0 we will take p_1
- Let's now find $\vec{v} = p_2 - p_1$. We decide not to normalize
- Now we have:

$$p(t) = p_1 + (p_2 - p_1)t \qquad\qquad t \in [0, 1]$$

- If $\vec{v}$ is normalized, then t (for a line) is in range from 0 to distance between p_1 and p_2

Line (2D/3D) – Parametric Equation



Line (2D/3D) – Parametric Equation

- Let's find the intersection point
- We have two lines k and l:

$$k_p = k_{p_0} + k_{\vec{v}}\boldsymbol{k_t}$$
$$l_p = l_{p_0} + l_{\vec{v}}\boldsymbol{l_t}$$

$$k_{p_0} + k_{\vec{v}}\boldsymbol{k_t} = l_{p_0} + l_{\vec{v}}\boldsymbol{l_t}$$

- One equation and two unknowns $\boldsymbol{k_t}$ i $\boldsymbol{l_t}$?

Line (2D/3D) – Parametric Equation

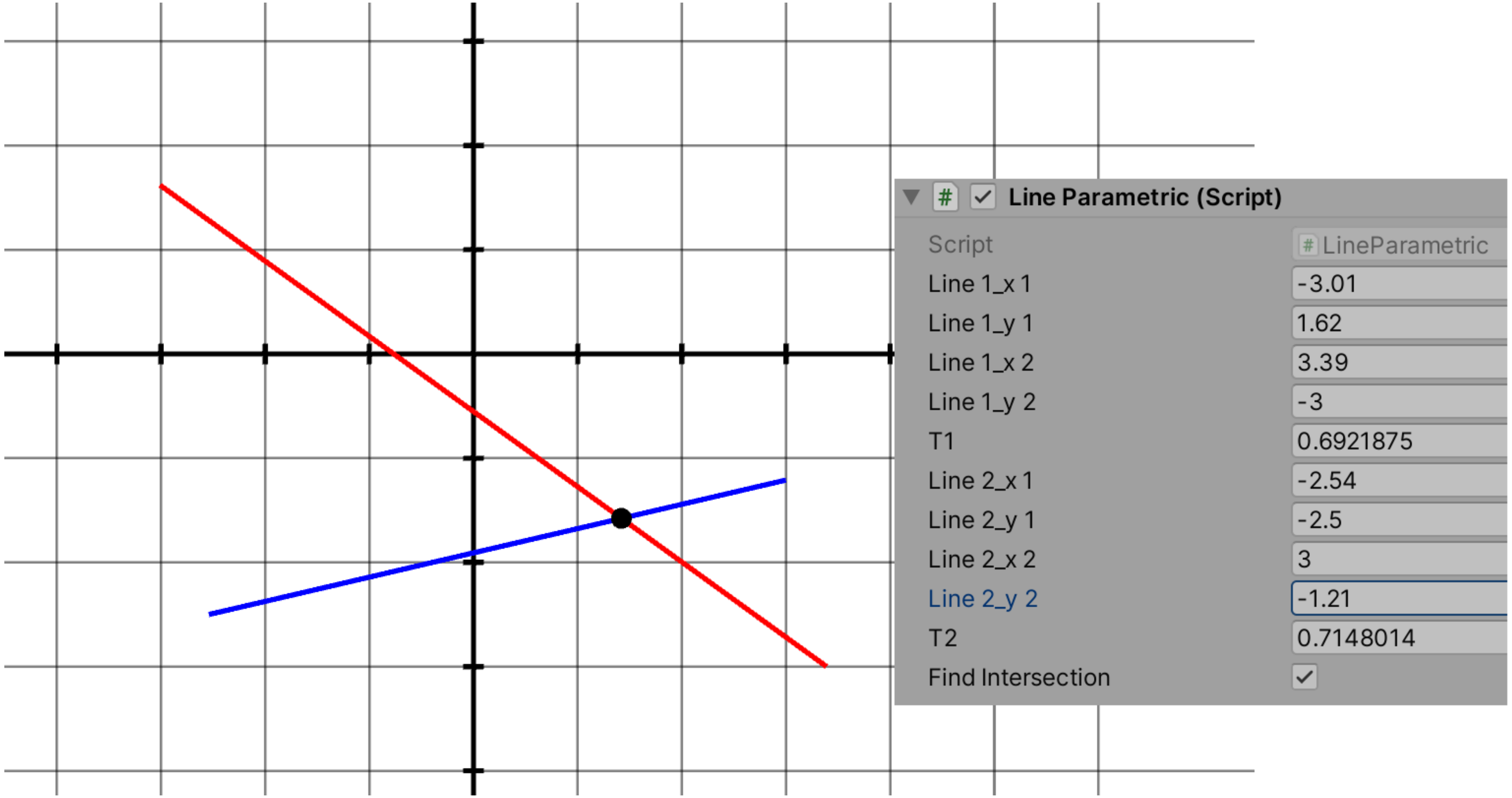
- There are actually two equations (one for x, and one for y):

$$\begin{cases} k_{p_{0x}} + k_{\vec{v}_x}\boldsymbol{k_t} = l_{p_{0x}} + l_{\vec{v}_x}\boldsymbol{l_t} \\ k_{p_{0y}} + k_{\vec{v}_y}\boldsymbol{k_t} = l_{p_{0y}} + l_{\vec{v}_y}\boldsymbol{l_t} \end{cases}$$

- Wolfram

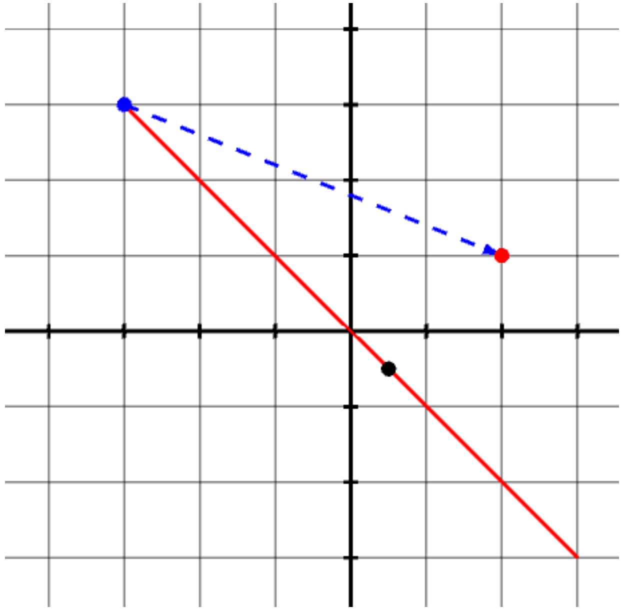
```
float a = line1_p1.x;
float b = line1_v.x;
float c = line2_p1.x;
float d = line2_v.x;
float e = line1_p1.y;
float f = line1_v.y;
float g = line2_p1.y;
float h = line2_v.y;
```

Line (2D/3D) – Parametric Equation



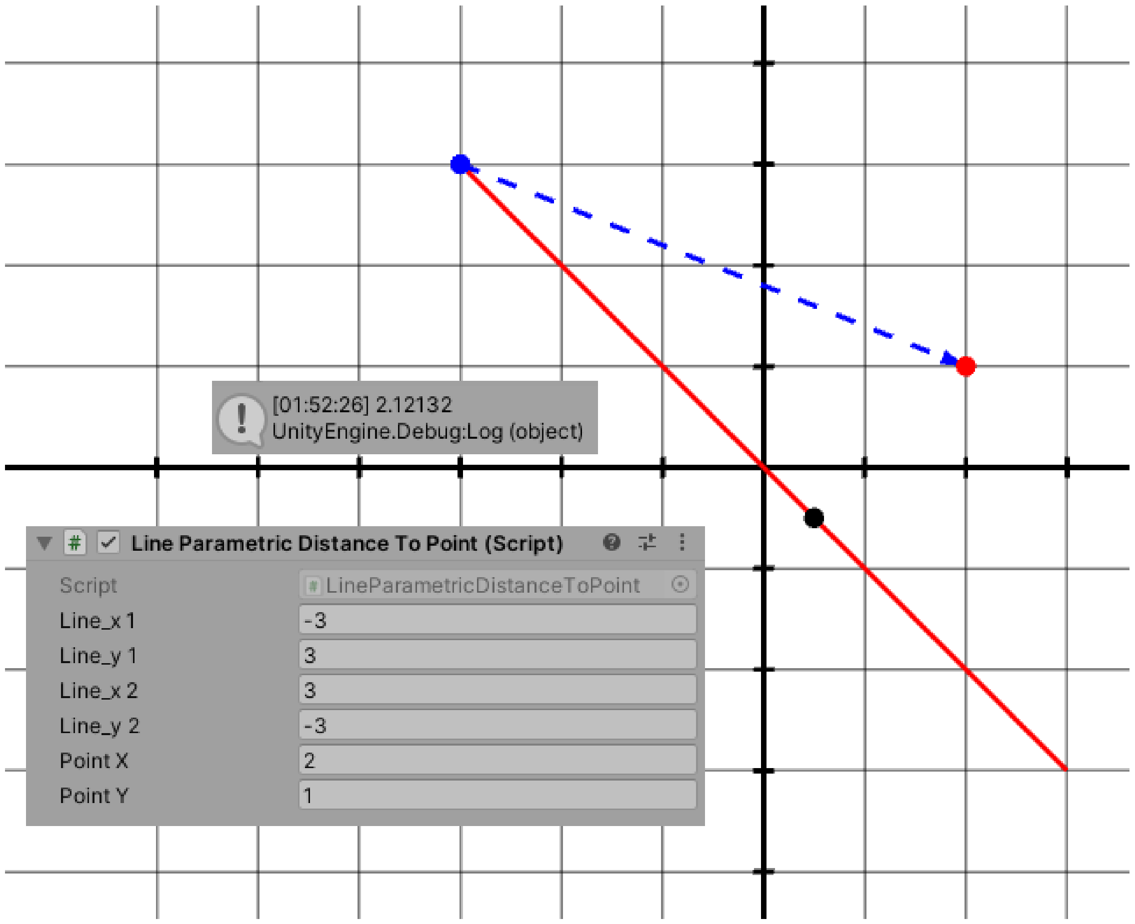
Line (2D/3D) – Parametric Equation

- We will now find the distance of a point from a line (via vector projection):



- Please note that with the implicit equation that distance could be calculated directly from the equation

Line (2D/3D) – Parametric Equation



Line (2D/3D) – Parametric Equation

- It’s easy to generate points on a line/ray/segment
- Fairly easy to find intersection points of a line with different geometric objects
- Support for both 2D and 3D (higher dimensions as well)
- A constant change in argument t leads to a „step” of the same distance on a line
- (2D) If the direction vector is not normalized, a line requires four values for representation. If normalized then three is enough

Circle

- A **circle** is usually represented in two ways
- First is the **implicit equation**:

$$(x-a)^2+(y-b)^2=r^2$$

- Second is the system of two **parametric equations**:

$$\begin{cases} x(\theta) = a + r\cos(\theta) \\ y(\theta) = b + r\sin(\theta) \end{cases}$$

Circle – Implicit Equation

- The equation of form:

$$(x-a)^2+(y-b)^2=r^2$$

- (a,b) are the coordinates of the center of the circle, whose radius is r
- (x,y) are the coordinates of any point that belongs to the circle
- Alternatively we can write this as:

$$(x-a)^2+(y-b)^2-r^2=0$$

Circle – Implicit Equation

- Let’s find the intersecion point of a cricle and a line
- The circle equation:

$$(x-a)^2+(y-b)^2=r^2$$

- The parametric line equation:

$$\begin{cases} x(t) = x_0 + \vec{v}_x t \\ y(t) = y_0 + \vec{v}_y t \end{cases}$$

- The intersection point of the line and circle is such a point, which (obviously) lies on both the line and circle

Circle – Implicit Equation

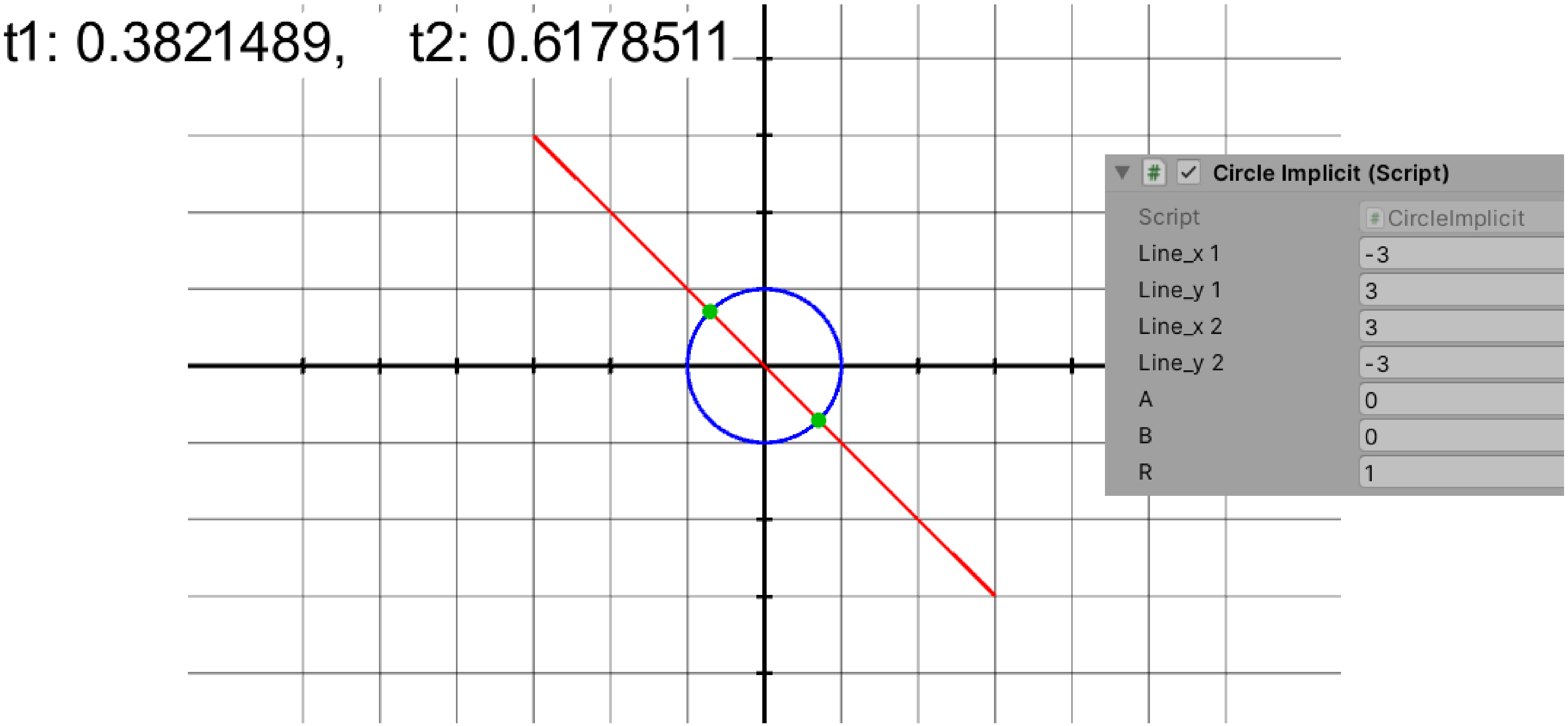
- We now substitute the coordinates of a point that lies on the line into the circle equation:

$$(x - a)^2 + (y - b)^2 = r^2$$

$$((x_0 + \vec{v}_x \boldsymbol{t}) - a)^2 + ((y_0 + \vec{v}_y \boldsymbol{t}) - b)^2 = r^2$$

- We need to solve the above for t
- Wolfram. The result is not particularly pleasant...

Circle – Implicit Equation



Circle – Implicit Equation

- It’s easy to determine if a point is inside a circle or outside
- It’s not possible to use this form to calculate the distance of point from a circle!
- Points on a circle can be generated by solving the equation for one of the coordinates

Circle – Implicit Equation

- To generate points on a circle we can solve the implicit equation for y
- We then end up with two functions $f(x)$, one of which represents the upper hemi-circle and the second one represents the bottom hemi-circle
- Wolfram
- We can also do the reverse – solve for x as a function of y

$$x = f(y)$$

Circle – Parametric Equation

- The equation of form:

$$\begin{cases} x(\theta) = a + r\cos(\theta) \\ y(\theta) = b + r\sin(\theta) \end{cases}$$

- The **parameter** is θ.
  This value in range $[0, 2\pi)$ gives us different coordinates (x, y) of points lying on a circle
- (a, b) is the center of a circle, and r is its radius
- This equation originates from polar coordinates

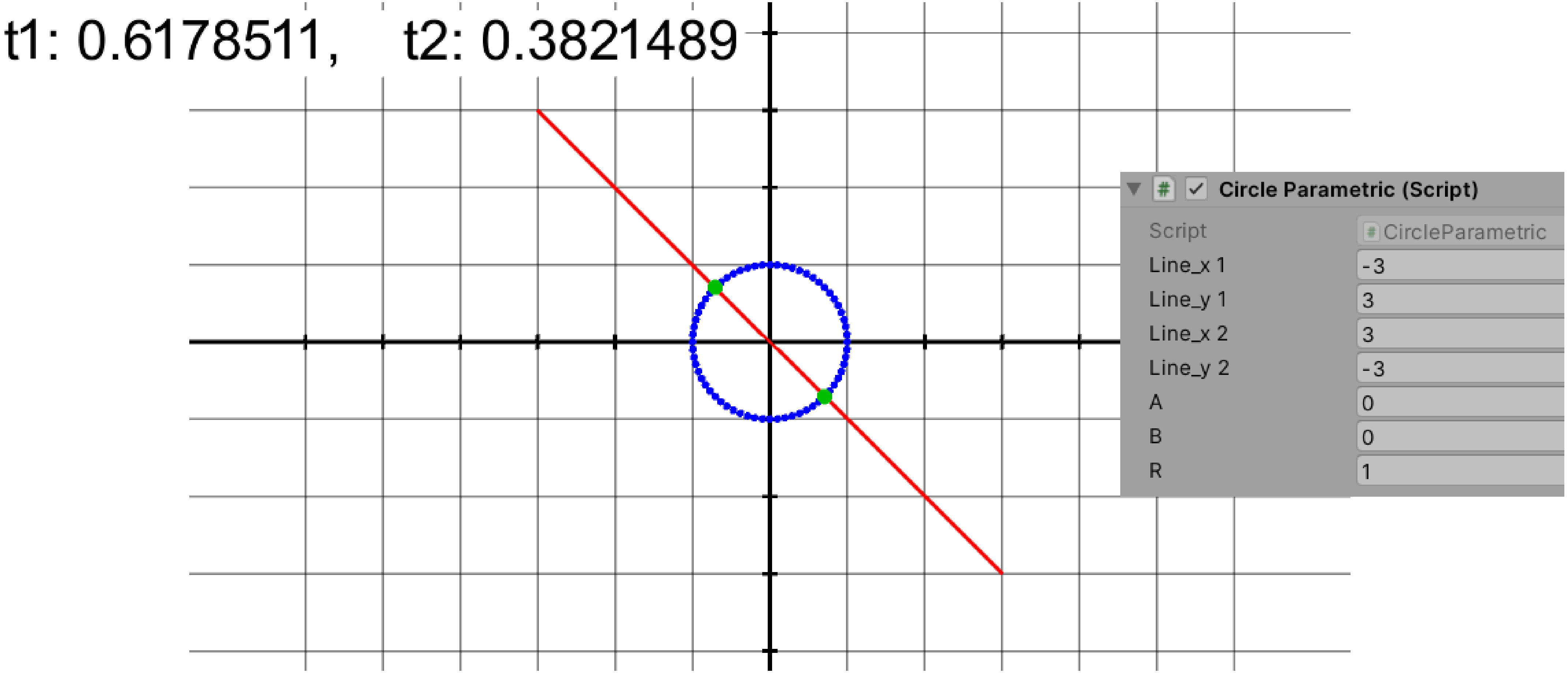
Circle – Parametric Equation

- We will now find the intersection point of a line with a parametric circle, by equating the parametric line equation to the circle parametric equation:

$$\begin{cases} x_0 + \vec{v}_x \boldsymbol{t} = a + r\cos(\boldsymbol{\theta}) \\ y_0 + \vec{v}_y \boldsymbol{t} = b + r\sin(\boldsymbol{\theta}) \end{cases}$$

- In this system the unknowns are t and θ
- Wolfram. Let’s limit ourselves to t...

Circle – Parametric Equation



Circle – Parametric Equation

- It’s easy to generate points on a circle
- Calculation of intersection points with other objects can be involving
- A constant change in argument θ leads to a „step” of the same distance on a circle

Sphere

- A **sphere** is a „circle in 3D"
- The implicit equation is a natural extension of the circle equation:

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2$$

- We already know the parametric equation – it's the spherical coordinates:

$$\begin{cases} x(\theta,\varphi) = a + r\sin(\theta)\cos(\varphi) \\ y(\theta,\varphi) = b + r\sin(\theta)\sin(\varphi) \\ z(\theta,\varphi) = c + r\cos(\theta) \end{cases}$$

- The intersection point is found in the same way as it is with a circle and a 2D line

Plane

- Just like a circle/sphere, a **plane** can be represented in two different ways
- The first one is the **implicit equation**:

$$ax + by + cz + d = 0$$

- The second one is the **parametric equation**:

$$p(u, v) = o + u\vec{t} + v\vec{b}$$

Plane – Implicit Equation

- As a reminder this is the implicit equation of line:

$$ax + by + c = 0$$

- Implicit equation of plane:

$$ax + by + cz + d = 0$$

- A plane is a 3D object. In „some sense” it is an extension of line into 3D
- $[a, b, c]$ is the plane’s normal vector
  It doesn’t have to be normalized, although it usually is
- d is the distance of the plane from the origin

Plane – Implicit Equation

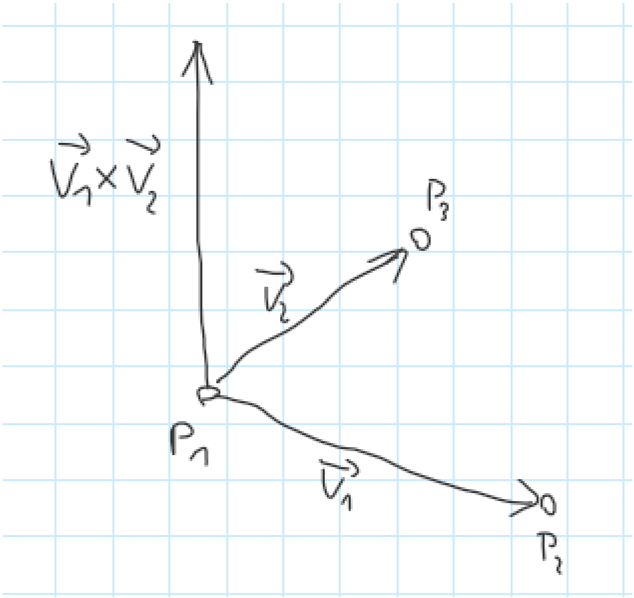
- Usually we have coordinates of a point belonging to a plane $\left(p_x, p_y, p_z\right)$ and the plane's normal vector $[a, b, c]$
- Let's find d:

$$d = -\left(ap_x + bp_y + cp_z\right)$$

- If $[a, b, c]$ is normalized then $ax + by + cz + d$ gives us **signed distance** of point (x, y, z) from the plane

Plane – Implicit Equation

- Alternatively, instead of having coords of a point and the normal vector we might have coords of three (or more) points belonging to the plane
- In that case we can easily find the normal vector using the cross product




```
static public Vector3 GetNormal(Vector3 p1, Vector3 p2, Vector3 p3)
{
    return Vector3.Cross(p2 - p1, p3 - p1).normalized;
}
```

Plane – Implicit Equation

- We will find the intersection point of a plane and a line in 3D
- A 3D line is described using the following parametric equation:

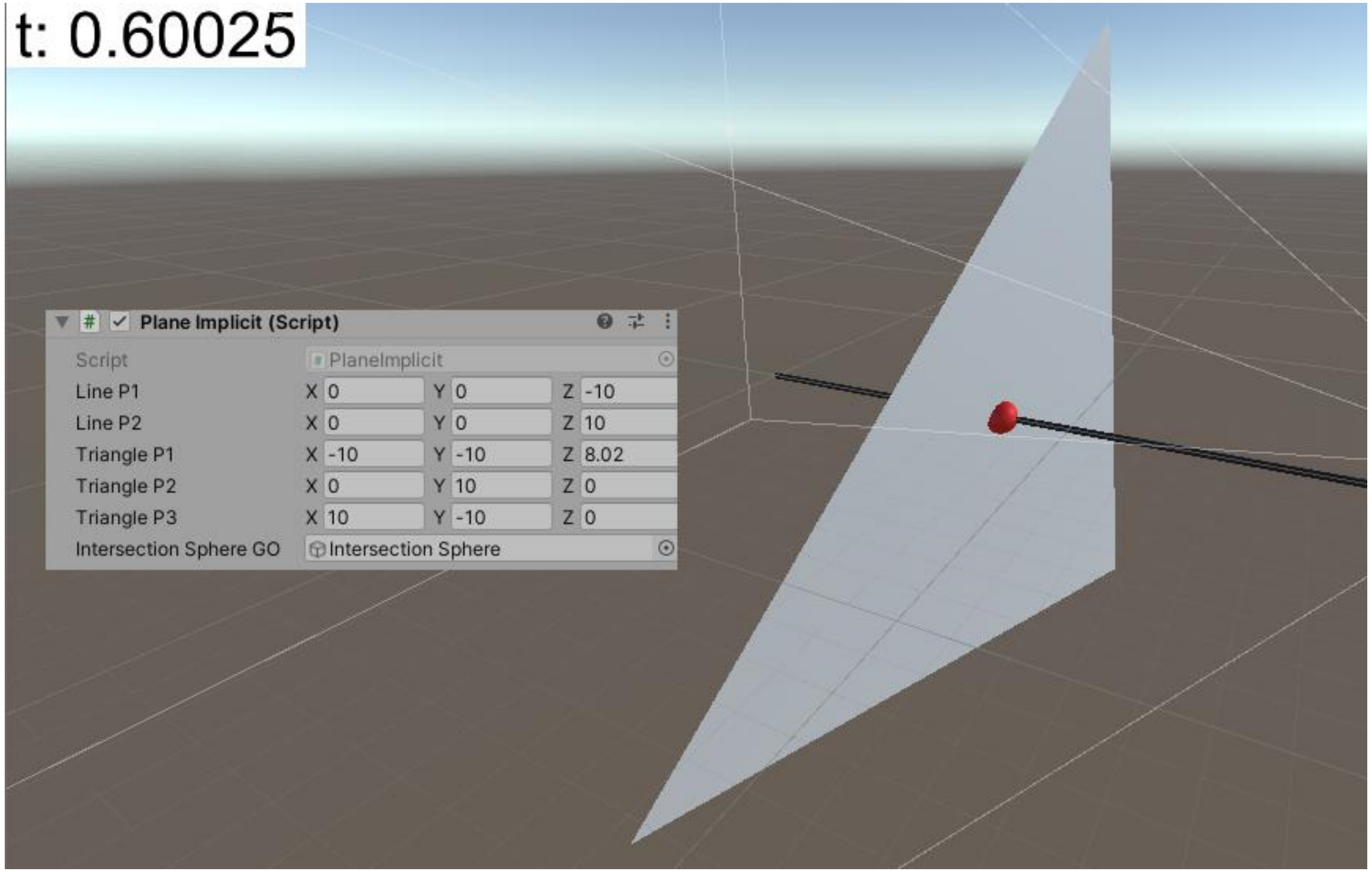
$$\begin{cases} x = x_0 + \vec{v}_x t \\ y = y_0 + \vec{v}_y t \\ z = z_0 + \vec{v}_z t \end{cases}$$

- We plug these into the plane equation:

$$a(x_0 + \vec{v}_x \boldsymbol{t}) + b\left(y_0 + \vec{v}_y \boldsymbol{t}\right) + c(z_0 + \vec{v}_z \boldsymbol{t}) + d = 0$$

- [Wolfram]

Plane – Implicit Equation



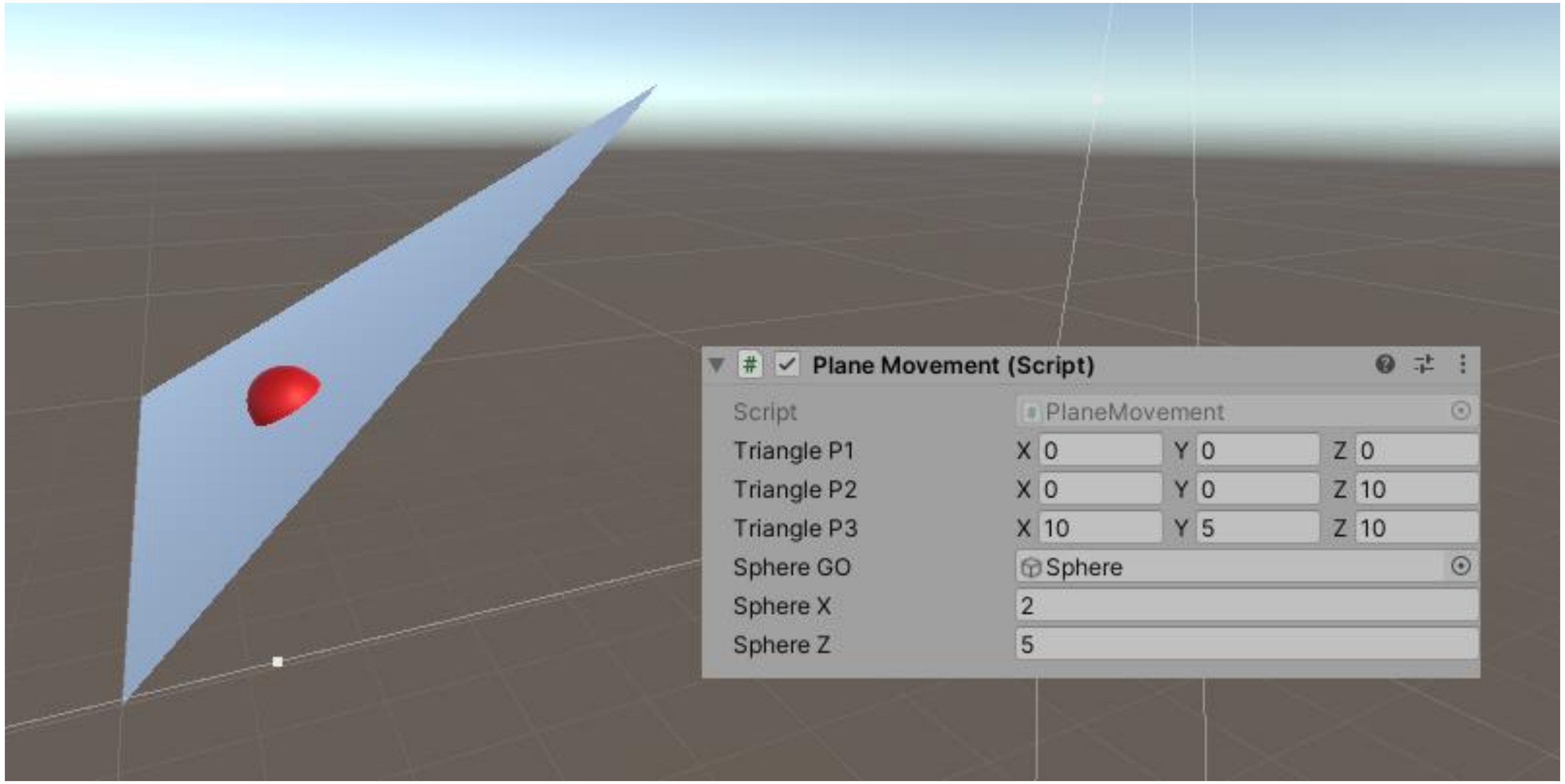
Plane – Implicit Equation

- The plane equation makes it easy to implement movement on plane
- Let's say we have (x, z) coords of a point and we want to find the height y on a plane
- In order to do that all we need to do is to solve for y the plane equation:

$$ax + by + cz + d = 0$$

$$y = -\frac{(ax + cz + d)}{b}$$

Plane – Implicit Equation



Plane – Implicit Equation

- We've discussed the implicit plane equation in the form:

$$ax + by + cz + d = 0$$

- It's worth knowing though that quite often we can find this equation written as:

$$Ax + By + Cz + D = 0$$

Plane – Implicit Equation

- It’s easy to determine on which side of a plane a point is
- It’s easy to find the distance of a point from a plane
- Points on a plane can be generated by solving the equation for one of the coordinates

Plane – Parametric Equation

- As a reminder here is the parametric line equation:

$$p(t) = p_0 + \vec{v}t$$

- Again, a plane is in some sense an extension of line into 3D
- The parametric plane equation:

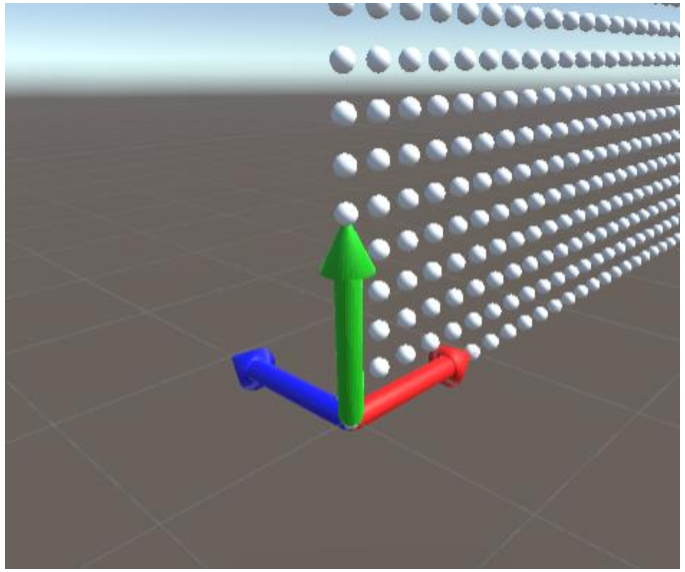
$$p(u, v) = o + u\vec{t} + v\vec{b}$$

- o are coordinates of some fixed (arbitrary) point on the plane
- $\vec{t}$ and $\vec{b}$ to are the plane's **tangent vectors**
- u and v are parameters, which we use to generate points on the plane

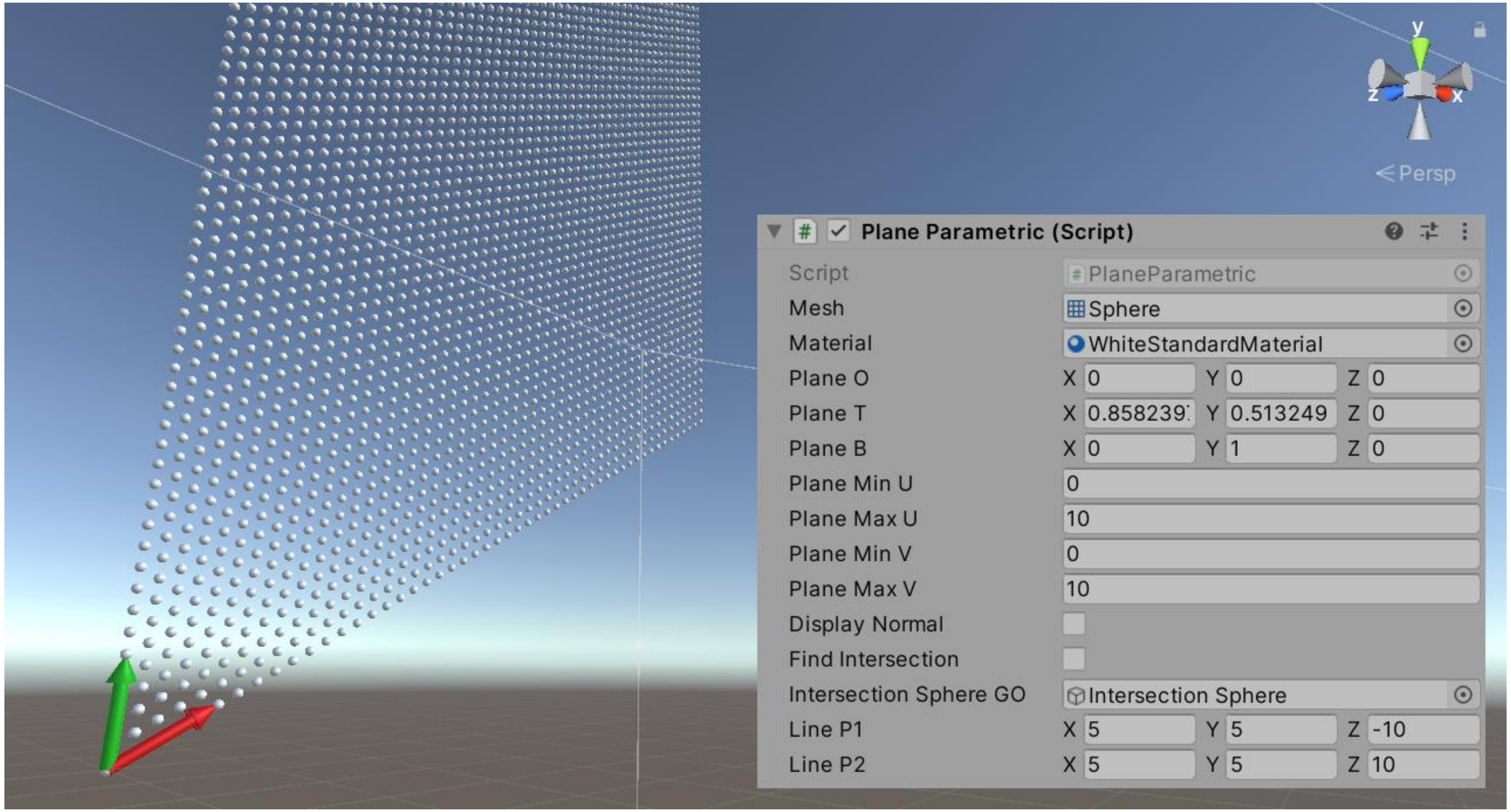
Plane – Parametric Equation

- The parametric equation is defined using two tangent vectors: $\vec{t}$ (tangent vector) and $\vec{b}$ (bitangent vector, sometimes mistakenly called „binormal”)
- The normal vector can be determined by taking the cross product of the tangent vectors:

$$\vec{n} = \vec{t} \times \vec{b}$$



Plane – Parametric Equation



Plane – Parametric Equation

- Let's find the intersection point of a plane and a line in 3D
- A line in 3D is described with the parametric equation:

$$\begin{cases} x = x_0 + \vec{v}_x t \\ y = y_0 + \vec{v}_y t \\ z = z_0 + \vec{v}_z t \end{cases}$$

- The parametric plane equation broken down into components is:

$$\begin{cases} x = o_x + u\vec{t}_x + v\vec{b}_x \\ y = o_y + u\vec{t}_y + v\vec{b}_y \\ z = o_z + u\vec{t}_z + v\vec{b}_z \end{cases}$$

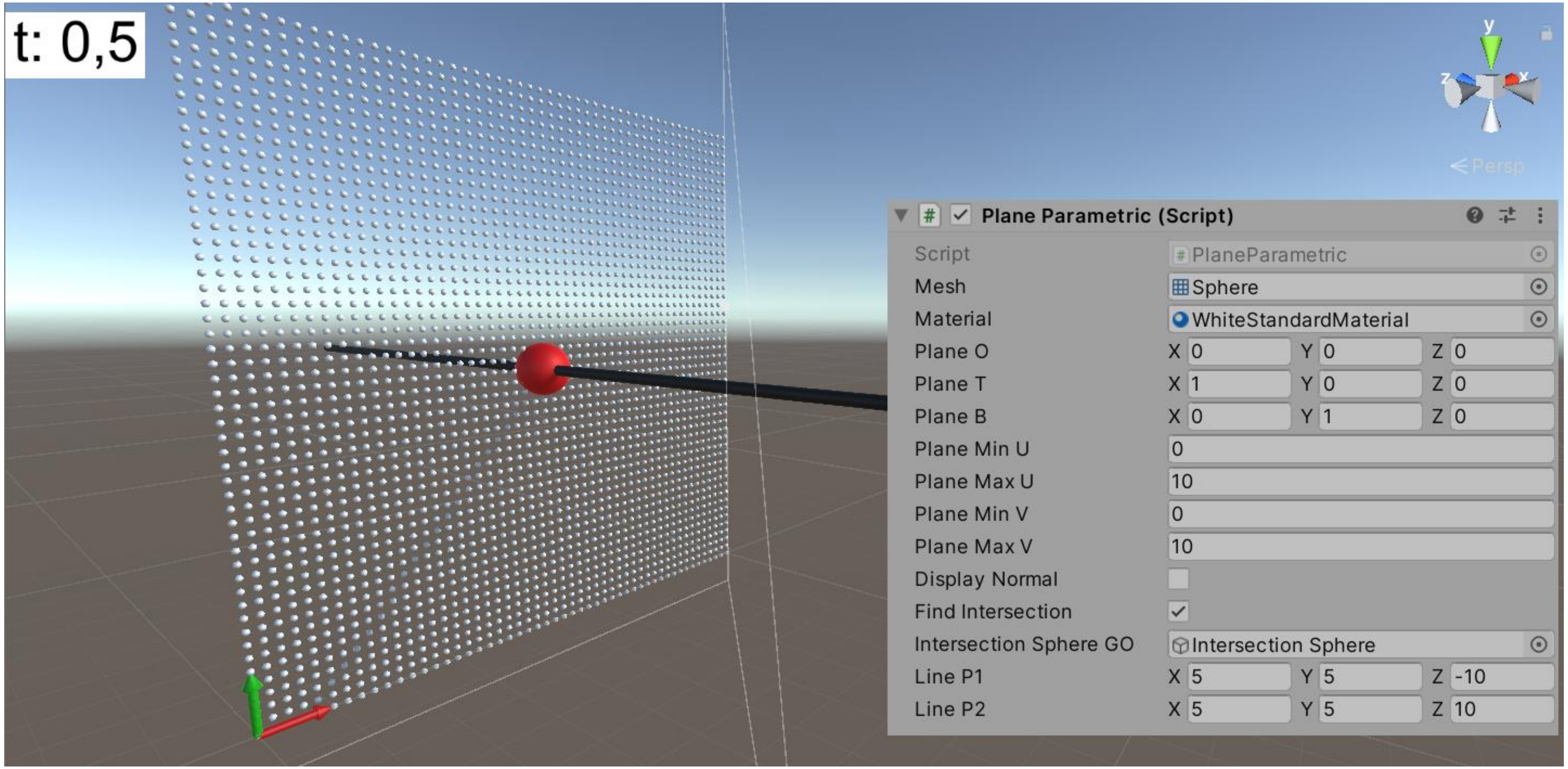
Plane – Parametric Equation

- We equate the coordinates, coming up with a system:

$$\begin{cases} x_0 + \vec{V}_x \boldsymbol{t} = o_x + \boldsymbol{u}\vec{T}_x + \boldsymbol{v}\vec{B}_x \\ y_0 + \vec{V}_y \boldsymbol{t} = o_y + \boldsymbol{u}\vec{T}_y + \boldsymbol{v}\vec{B}_y \\ z_0 + \vec{V}_z \boldsymbol{t} = o_z + \boldsymbol{u}\vec{T}_z + \boldsymbol{v}\vec{B}_z \end{cases}$$

- Wolfram. We only need t

Plane – Parametric Equation



Plane – Parametric Equation

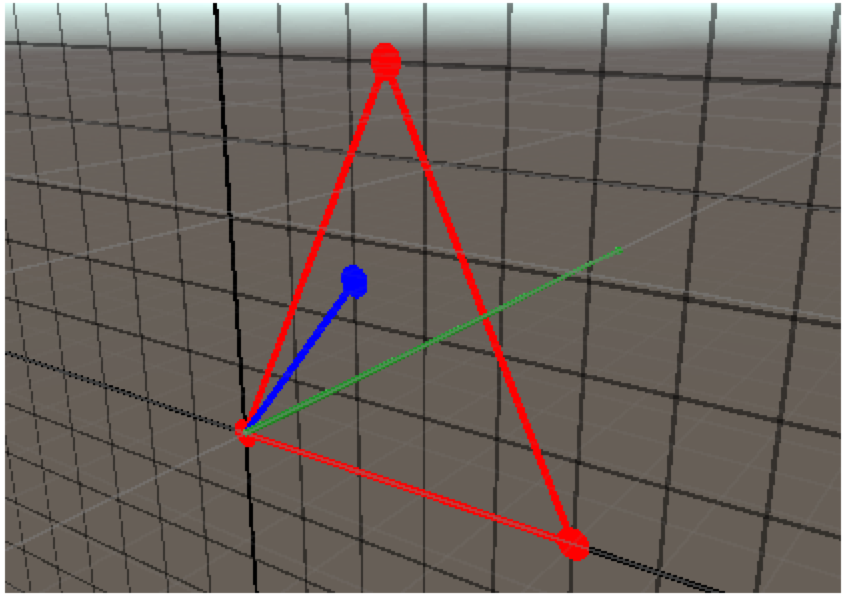
- Easy to generate points on a plane
- Calculating intersection points with other objects can be involving
- A constant change in arguments u and v leads to a „step" of the same distance along the tangent vectors

Triangle

- So far we have learned to find intersection points of a line with various 3D primitives
- We have not discussed though how to find the intersection point between a line and a **triangle**, one of the most commonly used 3D primitives
- A triangle always lies in one plane. So we construct a plane equation (from the coords of the points that make up the triangle) and we are looking for the intersection with the line
- When we've found the intersection all we need to do is check if the intersection point lies within the triangle, i.e. if it is bounded by all three sides of the triangle

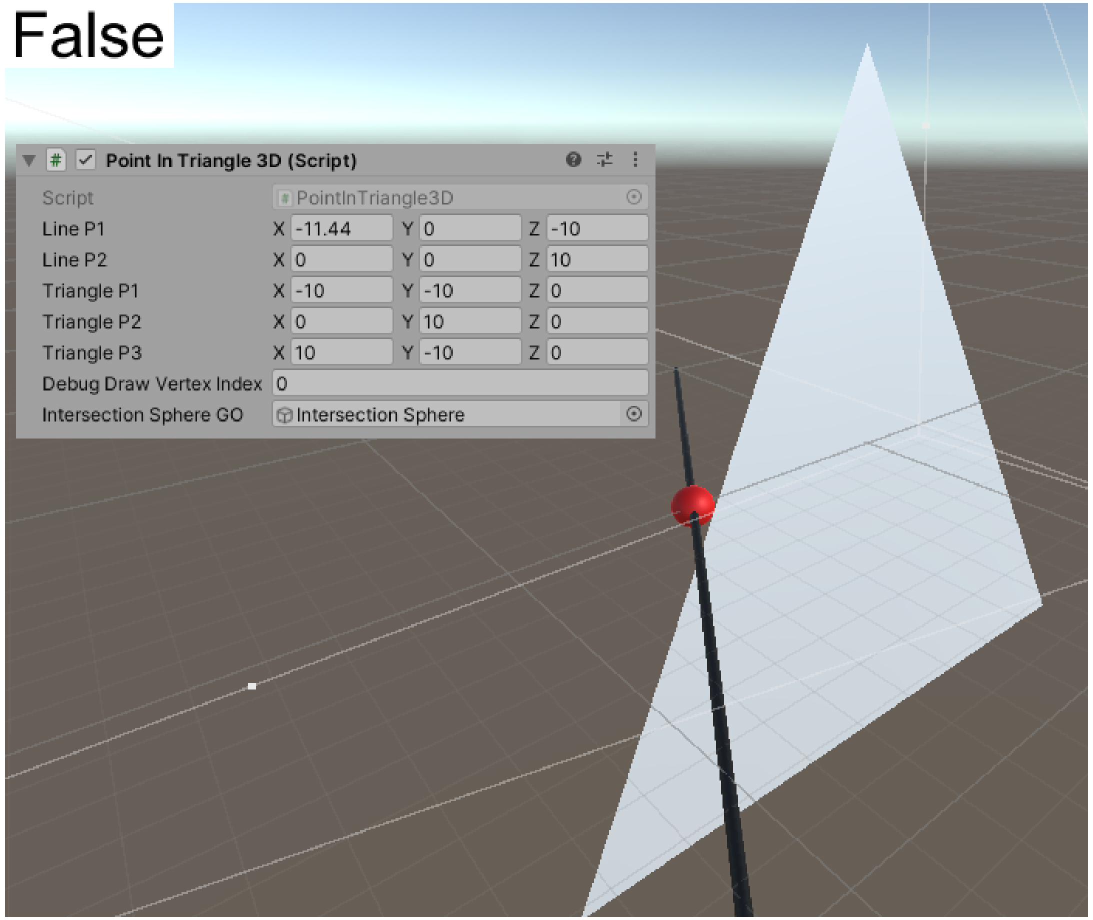
Triangle

- In chapter 3 we discussed a program where we checked if a point is inside a triangle in 2D:



- We can check the 3D case in exactly the same way

Triangle


False
Point In Triangle 3D (Script)
Script
PointInTriangle3D
Line P1
X -11.44 Y 0 Z -10
Line P2
X 0 Y 0 Z 10
Triangle P1
X -10 Y -10 Z 0
Triangle P2
X 0 Y 10 Z 0
Triangle P3
X 10 Y -10 Z 0
Debug Draw Vertex Index
0
Intersection Sphere GO
Intersection Sphere

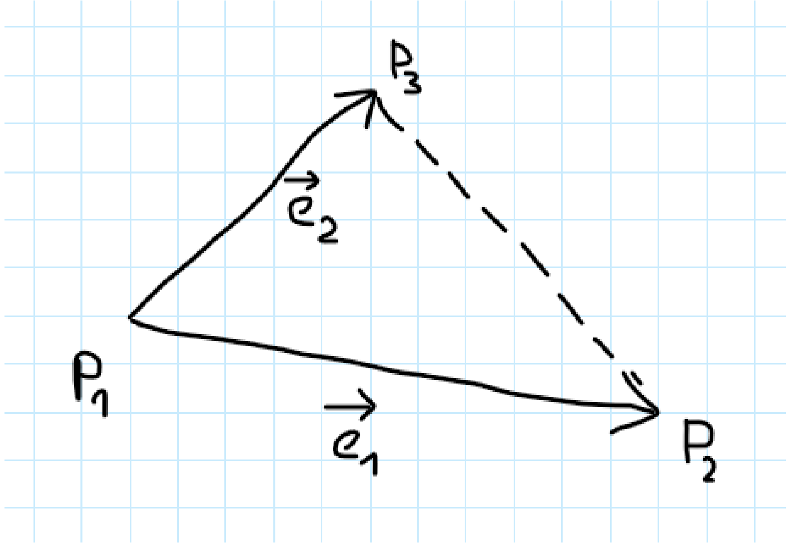
Triangle – Barycentric Coordinates

- A concept that is worth discussing while we are talking about triangles are **barycentric coordinates**
- Let's say we have a triangle with points p_1, p_2 and p_3. Barycentric coordinates (b_1, b_2) let us calculate any point p that lies in the triangle's plane:

$$p(b_1, b_2) \quad = \quad p_1 + b_1(p_2 - p_1) + b_2(p_3 - p_1) \quad = \quad p_1 + b_1\vec{e}_1 + b_2\vec{e}_2$$

$$\vec{e}_1 = p_2 - p_1$$
$$\vec{e}_2 = p_3 - p_1$$

- Vectors $\vec{e}_1$ and $\vec{e}_2$ are so called **edge vectors**



Triangle – Barycentric Coordinates

- The definition of barycentric coordinates is very similar to that of the parametric plane equation:

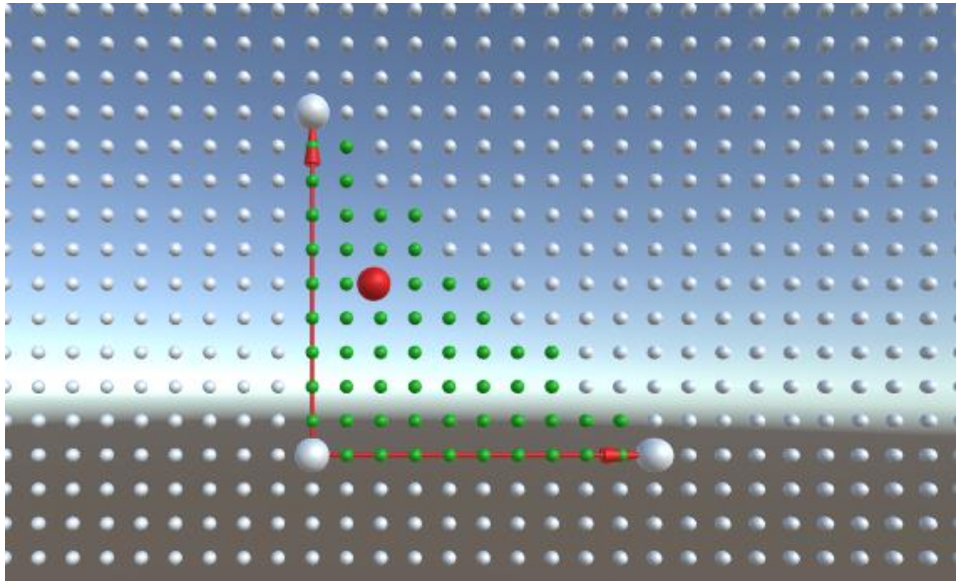
$$p(u, v) = o + u\vec{t} + v\vec{b}$$

- The main difference comes from the fact that vectors $\vec{t}$ and $\vec{b}$ are usually normalized and the only point of reference is the plane’s fixed point o.
  On the other hand with barycentric coordinates the vectors $\vec{e}_1$ and $\vec{e}_2$ additionally, implicitly denote points p_2 oraz p_3
- In some sense barycentric coordinates (b_1, b_2) are also „normalized”

Triangle – Barycentric Coordinates

- The fact that barycentrics are „normalized" lets us specify generic constraints, whose meeting can determine if a point lies inside a triangle
- Point p with barycentric coordinates (b_1, b_2) is inside a triangle if:

$$b_1 \geq 0 \qquad b_2 \geq 0 \qquad b_1 + b_2 \leq 1$$



Triangle – Barycentric Coordinates – Solution in 2D

- We will now derive a formula for calculating barycentric coordinates (b_1, b_2) of a given point p, what will let us check if the point is inside a triangle.
  We will start with the 2D case
- Our goal is to solve the following system and find b_1 and b_2:

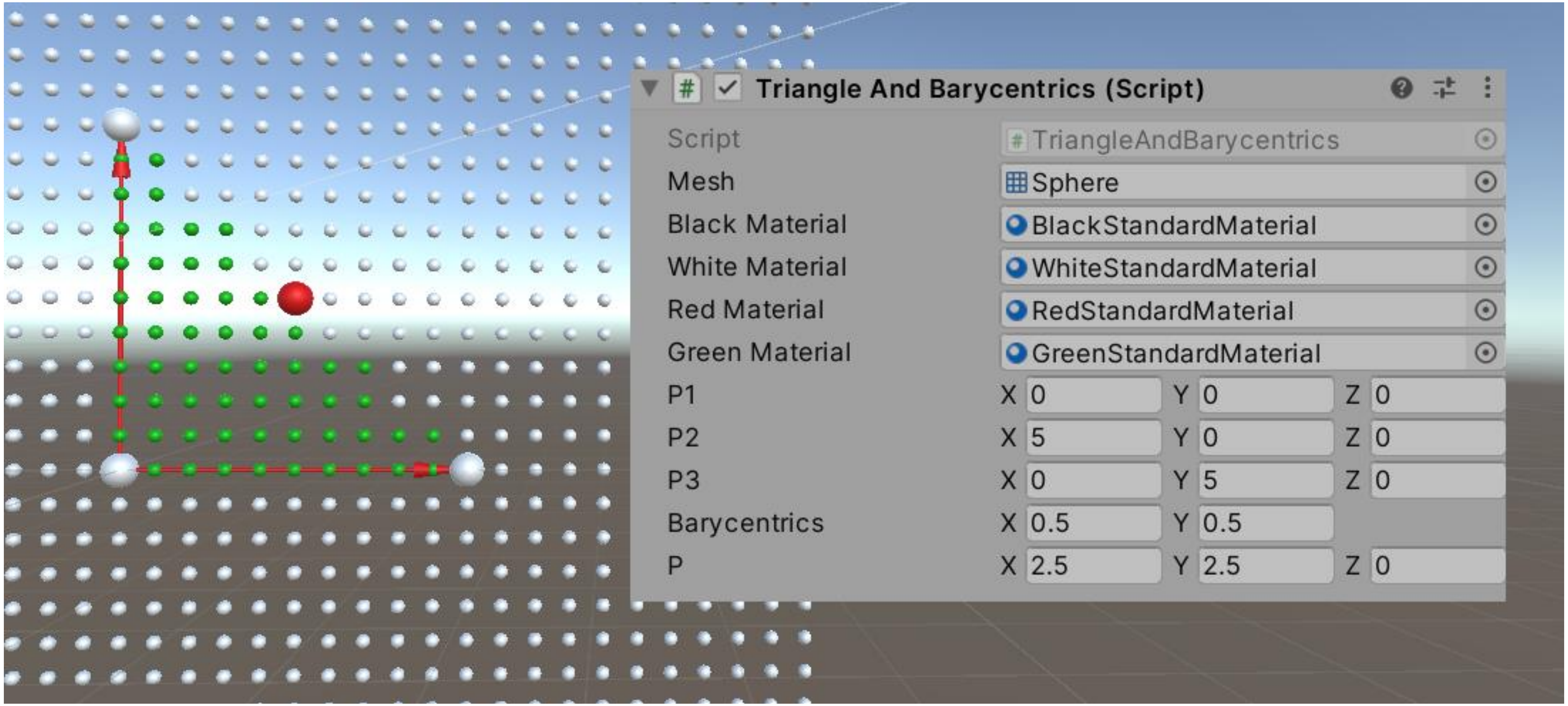
$$\begin{cases} p_x = p_{1x} + \boldsymbol{b_1}(p_{2x} - p_{1x}) + \boldsymbol{b_2}(p_{3x} - p_{1x}) \\ p_y = p_{1y} + \boldsymbol{b_1}(p_{2y} - p_{1y}) + \boldsymbol{b_2}(p_{3y} - p_{1y}) \end{cases}$$

- Wolfram

```
private Vector2 BarycentricsXY(Vector3 p1, Vector3 p2, Vector3 p3, Vector3 p)
{
    // https://www.wolframalpha.com/input?i=a_1+%2B+u*%28b_1-a_1%29+%2B+v*%28c_1-a_1%29+%3D+d_1%2C+a_2+%2B+u*%28b_2-a_2%29+%
    float a1 = p1.x;
    float b1 = p2.x;
    float c1 = p3.x;
    float d1 = p.x;
    float a2 = p1.y;
    float b2 = p2.y;
    float c2 = p3.y;
    float d2 = p.y;
    //
    float u = (a2 * (c1 - d1) + a1 * (d2 - c2) + c2 * d1 - c1 * d2) / (a2 * (c1 - b1) + a1 * (b2 - c2) - b2 * c1 + b1 * c2);
    float v = (a2 * (d1 - b1) + a1 * (b2 - d2) - b2 * d1 + b1 * d2) / (a2 * (c1 - b1) + a1 * (b2 - c2) - b2 * c1 + b1 * c2);

    return new Vector2(u, v);
}
```

Triangle – Barycentric Coordinates



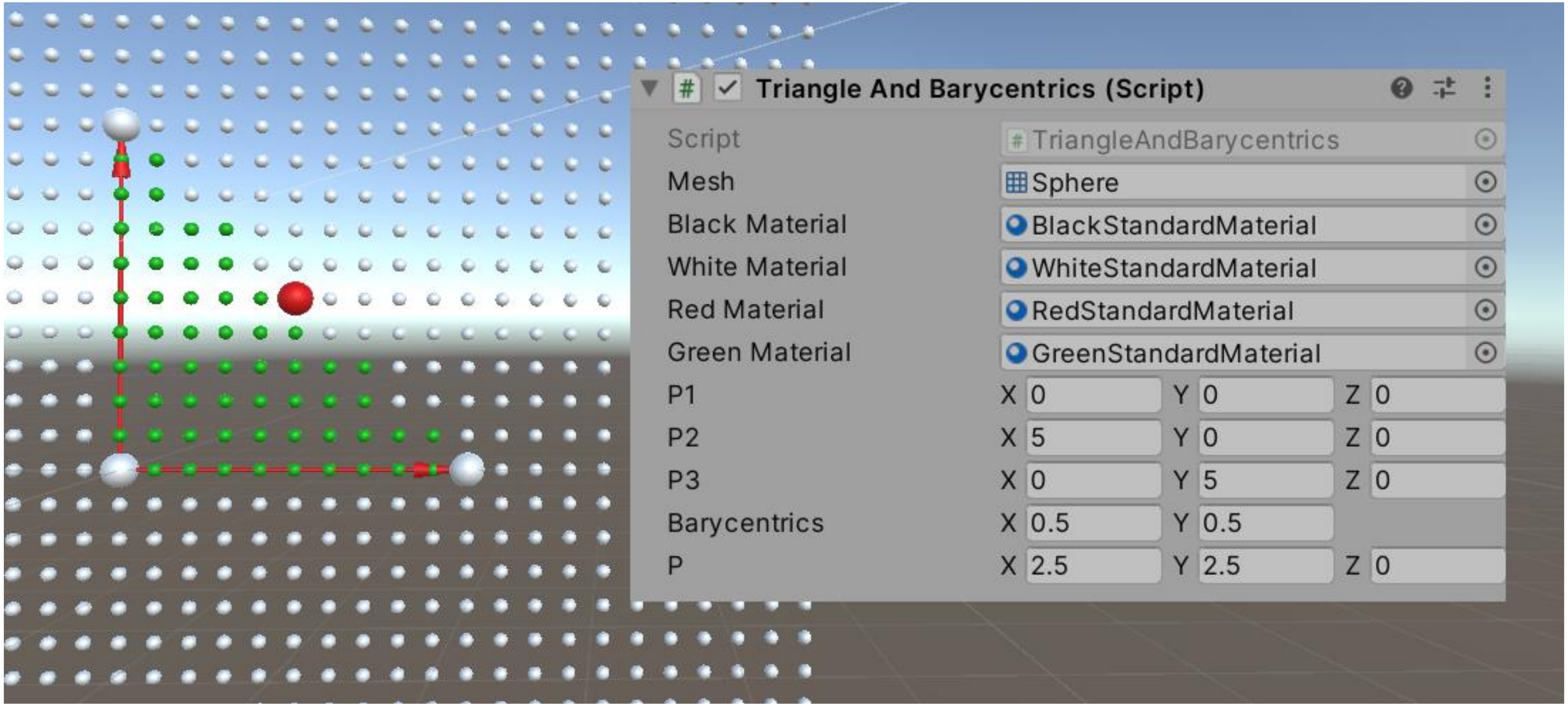
Triangle – Barycentric Coordinates – Solution in 3D

- Let's now solve the same problem in 3D:

$$\begin{cases} p_x = p_{1x} + \boldsymbol{b_1}(p_{2x} - p_{1x}) + \boldsymbol{b_2}(p_{3x} - p_{1x}) \\ p_y = p_{1y} + \boldsymbol{b_1}(p_{2y} - p_{1y}) + \boldsymbol{b_2}(p_{3y} - p_{1y}) \\ p_z = p_{1z} + \boldsymbol{b_1}(p_{2z} - p_{1z}) + \boldsymbol{b_2}(p_{3z} - p_{1z}) \end{cases}$$

- Here we have 3 equations and 2 unknowns
- This „incompatibility" results from the fact that barycentrics „live in 2D" – they determine coords of points that all lie on one plane in 3D. Hence the right hand side of the above equations can only express points lying in that plane
- On the left hand side of the equations above is any point in „full" 3D
- In practice this means that to find (b_1, b_2) we can use any 2 equations of the 3, but we have to pick them with consideration...

Triangle – Barycentric Coordinates



Triangle – Barycentric Coordinates

- Let's now plug, on the left hand side, coords of a point that belongs to a parametric line:

$$\begin{cases} x_0 + \vec{v}_x \boldsymbol{t} = p_{1x} + \boldsymbol{b_1}(p_{2x} - p_{1x}) + \boldsymbol{b_2}(p_{3x} - p_{1x}) \\ y_0 + \vec{v}_y \boldsymbol{t} = p_{1y} + \boldsymbol{b_1}(p_{2y} - p_{1y}) + \boldsymbol{b_2}(p_{3y} - p_{1y}) \\ z_0 + \vec{v}_z \boldsymbol{t} = p_{1z} + \boldsymbol{b_1}(p_{2z} - p_{1z}) + \boldsymbol{b_2}(p_{3z} - p_{1z}) \end{cases}$$

- Now we have exactly 3 equations and 3 unknowns because both on the right and left hand side the point is „constrained"
- On the left hand side we have some point on a line, and on the right hand side we have some point on a plane
- This system solves simultaneously for the intersection point and barycentric coordinates of that point (b_1, b_2)

Triangle – Barycentric Coordinates

- We had an identical problem when we discussed the parametric plane equation:

$$\begin{cases} x_0 + \vec{V}_x \boldsymbol{t} = o_x + \boldsymbol{u}\vec{T}_x + \boldsymbol{v}\vec{B}_x \\ y_0 + \vec{V}_y \boldsymbol{t} = o_y + \boldsymbol{u}\vec{T}_y + \boldsymbol{v}\vec{B}_y \\ z_0 + \vec{V}_z \boldsymbol{t} = o_z + \boldsymbol{u}\vec{T}_z + \boldsymbol{v}\vec{B}_z \end{cases}$$

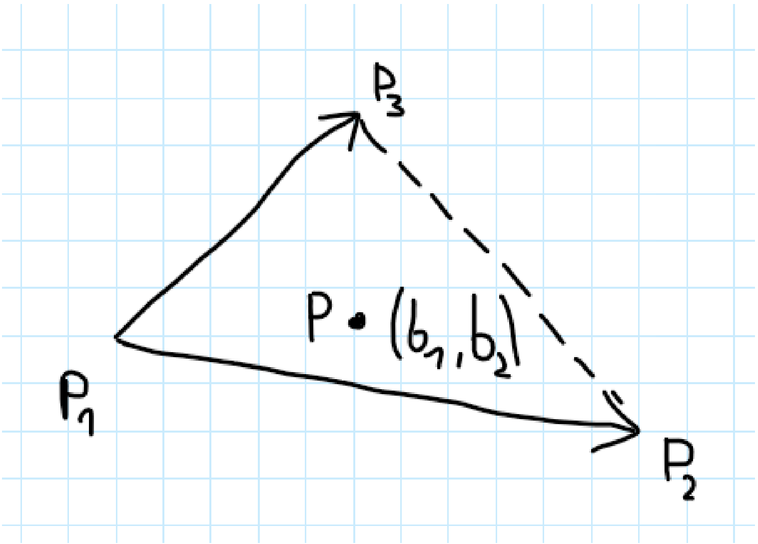
  The difference here is that vectors $\vec{T}$ and $\vec{B}$ are (usually) normalized

- Möller-Trumbore Algorithm is an efficient implementation of the solution to the above system

Triangle – Barycentric Coordinates – Attributes Interpolation

- An important merit that comes from using barycentric coordinates is that we can easily calculate/interpolate various parameters/attributes within a triangle
- We've got barycentric coords (b_1, b_2) of some point p
- We may also have values of some attributes (like color or normal vector) denoted as a_1, a_2, a_3 at vertices p_1, p_2 and p_3
- We can now calculate the value of attribute a at point p as:

$$a = a_1 + b_1(a_2 - a_1) + b_2(a_3 - a_1)$$



Triangle – Barycentric Coordinates

- It's worth knowing that sometimes barycentric coords are derived from the following:

$$p(b_0, b_1, b_2) = b_0 p_1 + b_1 p_2 + b_2 p_3 \qquad\qquad b_0 + b_1 + b_2 = 1$$

- Point p is simply a weighted average of points p_1, p_2, p_3
- This definition is equivalent with the previous one:

$$p(b_0, b_1, b_2) = (1 - b_1 - b_2)p_1 + b_1 p_2 + b_2 p_3$$

$$p(b_0, b_1, b_2) = p_1 - p_1 b_1 - p_1 b_2 + b_1 p_2 + b_2 p_3$$

$$p(b_0, b_1, b_2) = p_1 + b_1(p_2 - p_1) + b_2(p_3 - p_1)$$

Triangle – Barycentric Coordinates

- There are other interesting algorithms which can be used to derive barycentrics
- One of them can be found in Mathematics for 3D Game Programming and Computer Graphics and its implementation is included in the examle program (`Barycentrics_Lengyel` function)
- Another algorithm that involves calculation of triangles areas can be found in 3D Math Primer for Graphics and Game Development

Equations Optimization

- Some formulas in this chapter were quite complicated
- For example, to find the intersection point of a parametric line with an implicit circle required us to solve the following:

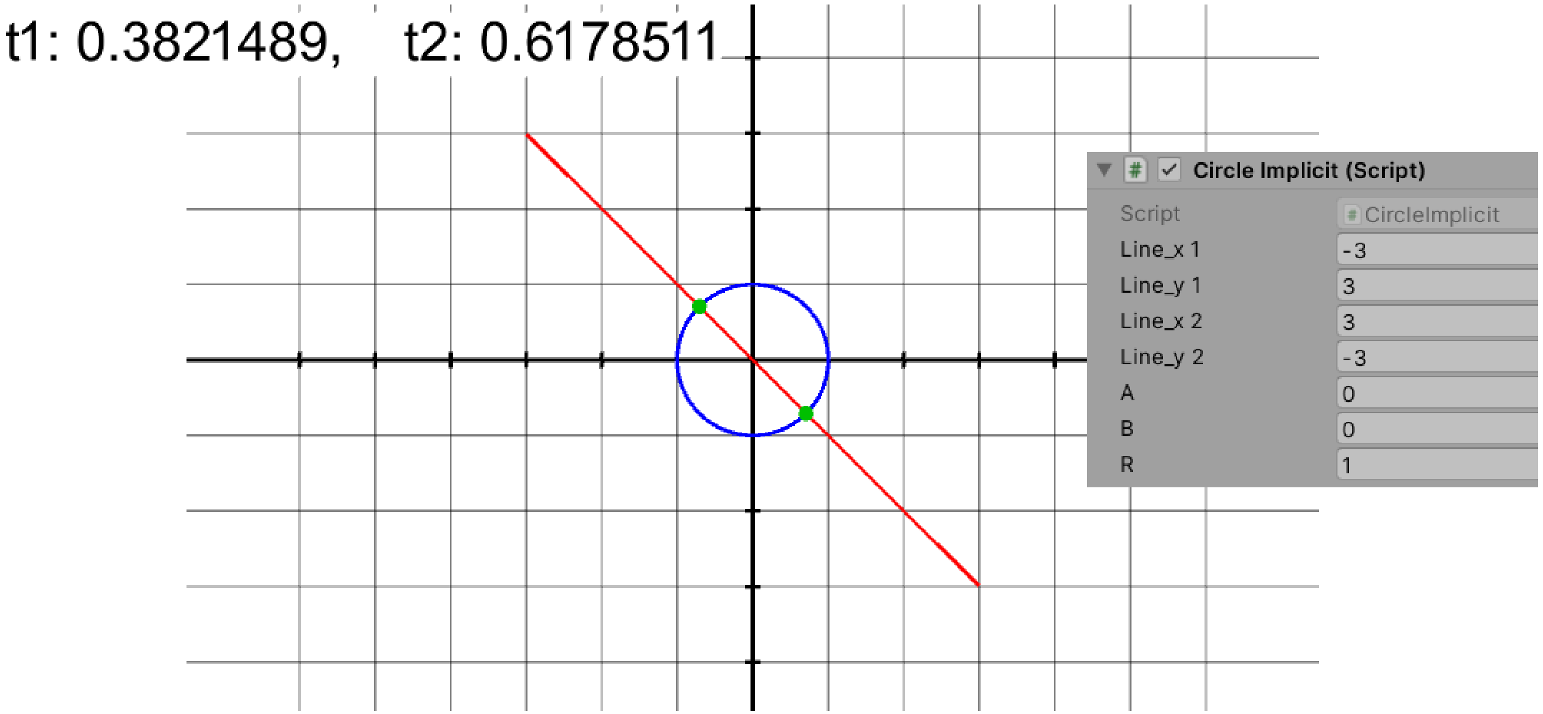
$$((x_0 + \vec{v}_x \boldsymbol{t}) - a)^2 + ((y_0 + \vec{v}_y \boldsymbol{t}) - b)^2 = r^2$$

  for t
- [Wolfram]

```
float delta = 8.0f*b*X*(Y*(a - x0) + X*y0) - 4.0f*(Y*(a - x0) + X*y0)*(Y*(a - x0) + X*y0) - 4.0f*b*b*X*X + 4.0f*r*r*(X*X + Y*Y);
float t1 = 1.0f / (X*X + Y*Y) * (-0.5f*Mathf.Sqrt(delta) + a*X + b*Y - x0*X - y0*Y);
float t2 = 1.0f / (X*X + Y*Y) * (0.5f*Mathf.Sqrt(delta) + a*X + b*Y - x0*X - y0*Y);
```

Equations Optimization

- Let's examine this problem once again, but with an assumption that the circle is at the origin



Equations Optimization

- Assuming that $a = 0$ and $b = 0$ the equation at hand simplifies greatly:

$$((x_0 + \vec{v}_x \boldsymbol{t}) - a)^2 + ((y_0 + \vec{v}_y \boldsymbol{t}) - b)^2 = r^2$$

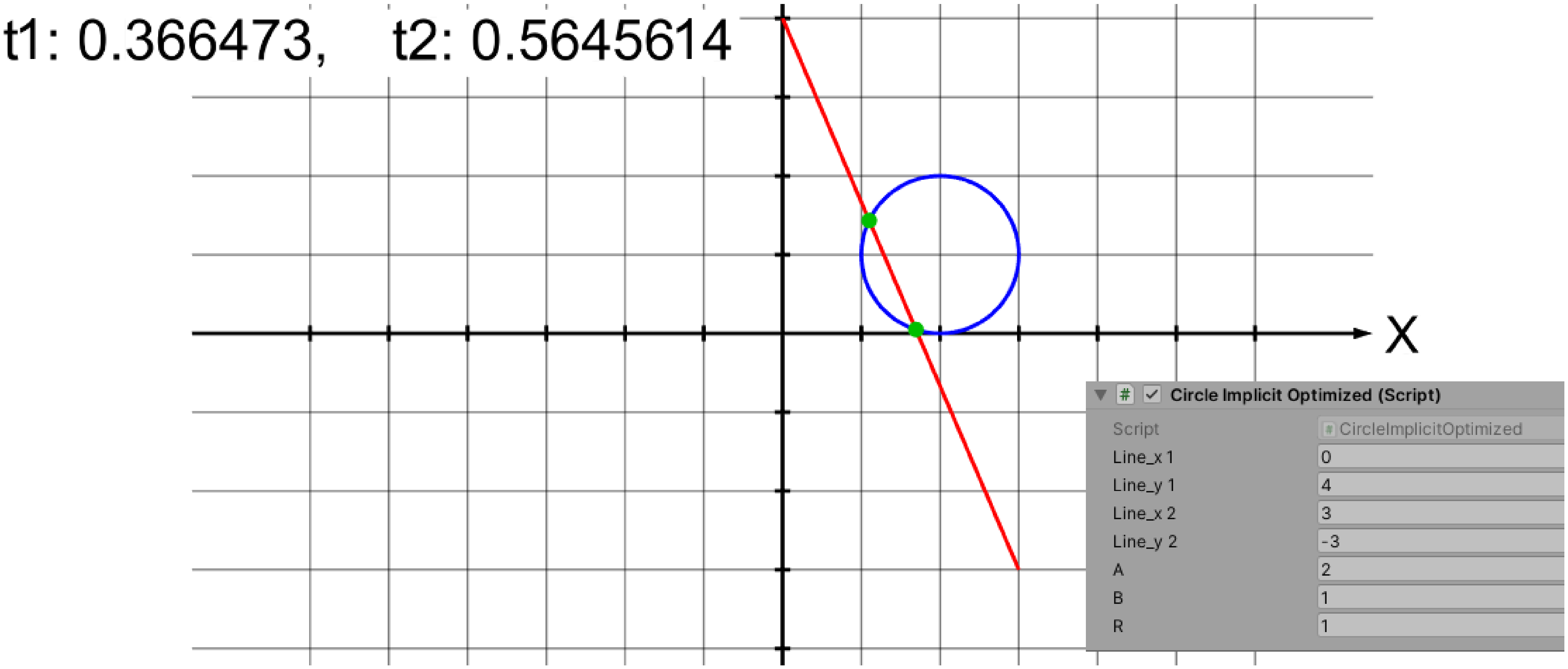
$$(x_0 + \vec{v}_x \boldsymbol{t})^2 + (y_0 + \vec{v}_y \boldsymbol{t})^2 = r^2$$

- Wolfram

```
float delta = r*r*(X*X + Y*Y) + 2.0f*x0*X*y0*Y - x0*x0*Y*Y - X*X*y0*y0;
float t1 = -(Mathf.Sqrt(delta) + x0*X + y0*Y) / (X*X + Y*Y);
float t2 = (Mathf.Sqrt(delta) - x0*X - y0*Y) / (X*X + Y*Y);
```

- We can first „offset” (just for the calculations) all objects so that the center of the circle is at the origin, and after the calculations we can „cancel” that offset

Equations Optimization



Equations Optimization

- In geometrical problems like this one it's always worth to look for ways of „offseting" calculations so that they take place around the origin (we will revisit that in chapter 6)

Exercises

1. Write a program that finds the distance of a point from a circle (slide 38)
2. Write a program that generates a few points on a circle described with parameters a, b and r.
   Do not use the parametric equation (slide 44)
3. Write a program that calculates the intersection point between a line and a sphere (slide 49)
4. Write a program that finds the intersection point of a line with a triangle in 3D.
   Find the barycentric coordinates of the intersection point (slide 78)