Linearization Gaussian Influence Function

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1 Introduction

This is a small paper explaining the the forward model and the inverse model for the kinematic model. The kinematic model is based on the Gaussian Influence Function. The Gaussian Influence Function is, in turn, based on functions described in Kratzsch 1983 and Ren, Reddish, and Whittaker 1987.
The forward model is used for creating detectability maps. The question that needs to be answered is as follows; "If a sinkhole of a magnitude S and radius R would happen at a position (x_0, y_0), would we be able to measure it"?
The inverse model is used to determine the sinkhole parameters from measured sinkhole subsidence lines retrieved from papers.

2 Defining Variables and Equations

$$z = S \cdot \frac{1}{R^2} e^{-\pi \frac{r^2}{R^2}} \tag{1}$$

$$r = \sqrt{(x - x_0)^2 + (y - y_0)^2} \tag{2}$$

$$S = at + b \tag{3}$$

The Kinematic Model equation is shown as equation 1. Two variable will be computed using the forward and inverse model. These two parameters are the scaling factor S and the radius of influence R. The scaling factor can be specified more precise by making it a linear function as shown in equation 3. The reason for making it linear and dependent on t is to introduce the time aspect into the function. The scaling factor can also be a polynomial or other type of function. This is not shown here. The parameter r is the radius from the surface center point towards a position on the surface. r is a function of x and y. The relationship is shown in equation 2. x_0 and y_0 are the center points on the surface for the sinkhole.

3 Forward Model

There are two variables estimated in the forward model. The variables are S and R. However, equation 1 is not a linear model thus it needs to be linearized. The linearization of the equation takes the derivative of equation 1 with respect of the two variables that need to be determined.

$$\Delta z = \begin{bmatrix} \frac{\delta z}{\delta S} & \frac{\delta z}{\delta R} \end{bmatrix} \begin{bmatrix} \Delta S \\ \Delta R \end{bmatrix} \tag{4}$$

$$\frac{\delta z}{\delta S} = \frac{1}{R^2} e^{\frac{-\pi r^2}{R^2}} \tag{5}$$

The derivative of equation 1 w.r.t. S is very straightforward and shown in equation 5.

$$\frac{\delta z}{\delta R} = S \cdot \frac{2\pi r^2 e^{\frac{-\pi r^2}{R^2}}}{R^5} - \frac{2e^{\frac{-\pi r^2}{R^2}}}{R^3} = \frac{-2S(R^2 - \pi r^2)e^{\frac{-\pi r^2}{R^2}}}{R^5} \tag{6}$$

The derivative w.r.t. R however, is slightly complicated and is shown in equation 6.
Combining both equation 5 and 6 into equation 4 gives equation 7.

$$\Delta z = \begin{bmatrix} \frac{1}{R^2} e^{\frac{-\pi r^2}{R^2}} & \frac{-2S(R^2 - \pi r^2)e^{\frac{-\pi r^2}{R^2}}}{R^5} \end{bmatrix} \begin{bmatrix} \Delta S \\ \Delta R \end{bmatrix} \tag{7}$$

The variable r is defined as the spatial variable. Equation 7 can incorporate more points and the system of equations will look as follows:

$$\begin{bmatrix} \Delta z_1 \\ \Delta z_2 \\ \vdots \\ \Delta z_n \end{bmatrix} = \begin{bmatrix} \frac{1}{R^2} e^{\frac{-\pi r_1^2}{R^2}} & \frac{-2S(R^2 - \pi r_1^2)e^{\frac{-\pi r_1^2}{R^2}}}{R^5} \\ \frac{1}{R^2} e^{\frac{-\pi r_2^2}{R^2}} & \frac{-2S(R^2 - \pi r_2^2)e^{\frac{-\pi r_2^2}{R^2}}}{R^5} \\ \vdots & \vdots \\ \frac{1}{R^2} e^{\frac{-\pi r_n^2}{R^2}} & \frac{-2S(R^2 - \pi r_n^2)e^{\frac{-\pi r_n^2}{R^2}}}{R^5} \end{bmatrix} \begin{bmatrix} \Delta S \\ \Delta R \end{bmatrix} \tag{8}$$

The forward model can be adapter to incorporate the time factor by using the relationship of equation 3.

$$\Delta z = \begin{bmatrix} \frac{t}{R^2} e^{\frac{-\pi r^2}{R^2}} & \frac{1}{R^2} e^{\frac{-\pi r^2}{R^2}} & \frac{-2(at+b)(R^2 - \pi r^2)e^{\frac{-\pi r^2}{R^2}}}{R^5} \end{bmatrix} \begin{bmatrix} \Delta a \\ \Delta b \\ \Delta R \end{bmatrix} \tag{9}$$

Equation 9 shows the forward model incorporated with a time factor.

4 Inverse Model

The inverse model.

References

Kratzsch, Helmut (1983). *Mining subsidence engineering*. Springer Science & Business Media. DOI: 10.1007/978-3-642-81923-0.

Ren, G., D.J. Reddish, and B.N. Whittaker (May 1987). "Mining subsidence and displacement prediction using influence function methods". In: *Mining Science and Technology* 5.1, pp. 89–104. DOI: 10.1016/s0167-9031(87)90966-2.