PDEs and Functional Analysis
Lecture 1

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1 Overview

1.1 Outline

This course will cover PDEs and Functional Analysis. In particular:

1. PDEs

    (a) First Order PDEs (Linear and Nonlinear)
    (b) Second Order PDEs
        i. Elliptic (Laplace's Equation)
        ii. Parabolic (Heat Equation)
        iii. Hyperbolic (Wave Equation)

2. Functional Analysis

    (a) Hilbert Spaces
    (b) Banach Spaces
    (c) Fourier Transforms

1.2 Resources

Books:
Partial Differential Equations (Evans)
Functional Analysis and Partial Differential Equations (Brezis)

2 First Order PDEs

2.1 The Transport Equation

Let $b \in \mathbb{R}^n$ and consider

$$u_t + b \cdot \nabla u = 0 \tag{1}$$

Where:

- $u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$

- $u(t, x) \in \mathbb{R} \forall t \geq 0$

- $\nabla = (\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n})$

- $u_t = \frac{\partial}{\partial t} u$

Concretely, the equation becomes:

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} = 0 \tag{2}$$

Similarly, it can be written as:

$$(u_t, \nabla u) \cdot (1, b) = 0 \tag{3}$$

(1), (2) and (3) are equivalent, and are called *Transport Equations* (of the first order). The equations are clearly homogeneous and of constant coefficients. To solve the transport equation, we are looking for a $u \in \mathcal{C}^1$ which satisfies the partial differential equation. The equation is said to be free, and has no boundary condition.

The standard way to proceed is to:

1. Look for special solutions (explicit)

2. Review the existence of a solution in some class (the larger the class, the better)

3. Review uniqueness of a solution (the smaller the class, the better)

4. Review the regularity[1] of a solution (typically the difficult point)

In order to review the existence of a solution, functional analysis is typically employed. In particular, Sobolov spaces and distributions are of importance.

Consider (1) once again. The equation is saying that the derivative along the $(1, b)$ vector is 0; u has a directional derivative which is constant in time-space as follows:

$$z(s) = u(t + s, x + sb) = u((t, x) + s(1, b)), s \in \mathbb{R} \tag{4}$$

Taking it's derivative, we see z is constant:

$$z'(s) = u_t((t, x) + s(1, b)) + \nabla u((t, x) + s(1, b)) = 0$$

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[1]Typically means high integrability or high differentiability, but generally serves as a measure of having desirable qualities

Let Σ be the hyperplane $\{t = 0\} \subset \mathbb{R}^{n+1}$, $\overline{u} \in \mathcal{C}^1(\Sigma)$, and π be a projection map along the vector $(1, b)$. From these components we create the following system:

$$\begin{cases} u_t + b \cdot \nabla u = 0 & (t, x) \in (0, \infty) \times \mathbb{R}^n \\ u \in \mathcal{C}^1([0, \infty) \times \mathbb{R}^n) \\ u = \overline{u} & (t, x) \in \Sigma \end{cases} \tag{5}$$

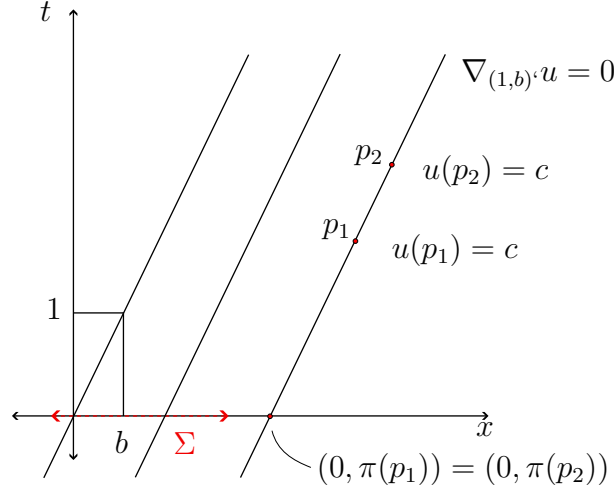Which is now an *initial value problem* with a boundary condition.

Figure 1: Visualizing the IVP

Theorem 1. *The solution to (5) is*

$$u(t, x) = \overline{u}(0, x - tb)$$

Proof. We assume there's a generic point (t, x), for which we want to find a solution $u(t, x)$. From the diagram above, we can conclude that $u(t, x) = \overline{u}(0, \pi(t, x))$.

Our goal is to find $\pi(t, x) \in \mathbb{R}^n, s(t, x) \in \mathbb{R}$ such that $(t, x) = (0, \pi(t, x)) + s(t, x)(1, b)$:

$$\begin{aligned} (t, x) &= (s(t, x), \pi(t, x) + s(t, x)b) \\ t &= s(t, x), x = \pi(t, x) + tb \\ \pi(t, x) &= x - tb \end{aligned}$$

So a solution to the initial value problem (5) is $u(t, x) = \overline{u}(0, x - tb)$. $\qquad \square$

This can be checked by substituting u into the initial value problem.

Remark. *$u \in \mathcal{C}^1$, the same smoothness as $\overline{u}$. However, there is no regularizing effect, the smoothness of the solution is equal to the smoothness of our $\overline{u}$. This follows necessarily as our solution to (5) is $u(t, x) = \overline{u}(0, x - tb)$.*

The importance of this is that linear, first order PDEs will not regularize the equation. Some PDEs are "worse" in the sense that they will not regularize the initial condition. On the other hand, consider the heat equation (a second order PDE), it will regularize any initial non-smooth item, it will immediately become smooth.

Remark. *In the proof, there is an assumption that $(1, b)$ is transverse to Σ. That is, $(1, b)$ is not tangent to Σ.*

Remark. *This is a linear transport equation in the sense that we transport the initial condition, we are "moving" the value of $\overline{u}$ to different points.*

Remark. *If $\overline{u}$ is non-smooth, e.g. $\overline{u}$ discontinuous with constant values, is it reasonable that $\overline{u}(0, x - tb)$ solves (5) in some weak sense? Later it will possibly be shown why.*

Remark. *The lines parallel to $(1, b)$ solve the following system:*

$$\begin{cases} X'(s) = (1, b) \in \mathbb{R}^{1+n} \\ X(0) = (0, \overline{x}) \end{cases}$$

Called the System of Characteristics Where $X = (X_0, X_1, ..., X_n)$, and $X_0(s) = s, (X_1, ..., X_n)(s) = \overline{x} + sb$.
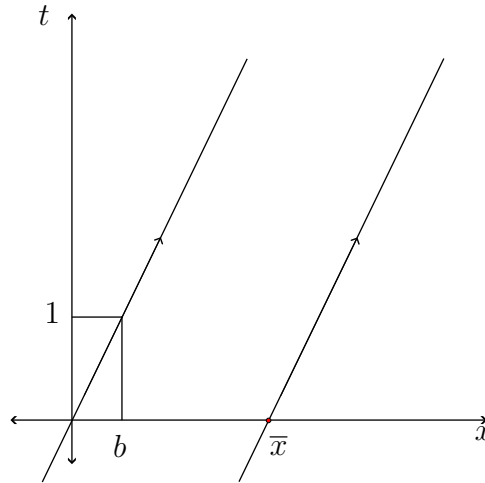

Figure 2: Visualizing the System of Characteristics

2.2 Nonhomogeneous Transport Equation

Consider[2] $f \in \mathcal{C}^1([0, \infty) \times \mathbb{R}^n)$ and the initial value problem:

$$\begin{cases} u_t + b \cdot \nabla u = f & (t, x) \in (0, \infty) \times \mathbb{R}^n \\ u \in \mathcal{C}^1([0, \infty) \times \mathbb{R}^n) \\ u = \overline{u} & (t, x) \in \Sigma = \{0\} \times \mathbb{R}^n \end{cases} \qquad (6)$$

This is a linear, first order PDE with constant coefficients, but nonhomogeneous.

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[2]One could also consider $f \in \mathcal{C}^0([0, \infty) \times \mathbb{R}^n)$

> **Theorem 2.** *The solution to the nonhomogeneous case of the transport equation,*
> *(6), is*
> $$u(t,x) = \bar{u}(0, x - tb) + \int_0^t f(s, x + (s-t)b)\, ds$$

We can introduce the same $z(s) = u(t+s, x+sb)$, $t \geq 0$, $x \in \mathbb{R}^n$. Differentiating:

$$z'(s) = u_t(t+s, x+sb) + b \cdot \nabla u(t+s, x+sb)$$

This implies that $z'(s) = f(t+s, x+sb)$. We can integrate this equation from $-t$ to 0:

$$\int_{-t}^0 z'(s)\, ds = \int_{-t}^0 f(t+s, x+sb)\, ds$$

$$z(0) - z(t) = u(t,x) - u(0, x - tb) = \int_{-t}^0 f(t+s, x+sb)\, ds$$

$$u(t,x) = \bar{u}(0, x - tb) + \int_{-t}^0 f(t+s, x+sb)\, ds$$

We make the variable change $t + s = \sigma$:

$$u(t,x) = \bar{u}(0, x - tb) + \int_0^t f(\sigma, x + (\sigma - t)b)\, d\sigma$$

And arrive at the solution ($s = \sigma$):

$$u(t,x) = \bar{u}(0, x - tb) + \int_0^t f(s, x + (s-t)b)\, ds \tag{7}$$

Note that when $f = 0$, which represents the homogeneous transport equation, this still holds. Consider the system of characteristics from before:

$$\begin{cases} X'(s) = (1, b) \in \mathbb{R}^{1+n} \\ X(0) = (0, \bar{x}) \end{cases}$$

And consider the new system:

$$\begin{cases} y' = f(X) \\ y(0) = \bar{u}(0, \bar{x}) \end{cases} \tag{8}$$

We can define X more implicitly by $X = X(s, \bar{x})$, and we can introduce $y(s) = u(X(s, \bar{x}))$.

> **Lemma 1.** $y(x) = u(X(s, \bar{x}))$ *satisfies the system (8)*

Proof. It suffices to show that $y(0) = \bar{u}(0, \bar{x})$ and $y' = f(X(s, \bar{x}))$:

$$y(0) = u(X(0, \bar{x})) = u(0, \bar{x}) = \bar{u}(0, \bar{x})$$

since $u = \bar{u}\, \forall (t,x) \in \Sigma$, and

$$y' = u_t(s, \bar{x} + sb) + b \cdot \nabla u(s, \bar{x} + sb) = f(s, \bar{x} + sb) = f(X(s, \bar{x}))$$

$\square$

For any u which solves (5), we can solve for X, and we can define y, a solution to (8). The importance of this remark is to rework y to find u. From X and y, we can attempt to define $u(t, x)$:

$$u(t, x) = y(s(t, x), \overline{x}(t, x)) \tag{9}$$

Lemma 2. *(9) is a solution to (6)*

Proof. Recall that $s(t, x) = t$ and that $\overline{x}(t, x) = x - tb$.

$$y(s(t, x), \overline{x}(t, x)) = \overline{u}(\overline{x}(t, x)) + \int_0^{s(t,x)} f(X(\sigma)) \, d\sigma$$

$$\int_0^{s(t,x)} f(X(\sigma)) \, d\sigma = \int_0^{s(t,x)} f(\sigma, \overline{x}(t, x) + \sigma b) \, d\sigma$$

$$u = \overline{u}(x - tb) + \int_0^t f(\sigma, x + (\sigma - t)b) \, d\sigma$$

which was shown to be a solution to (6). $\qquad\square$

We have shown one can construct a solution to (6) by constructing the system of characteristics, constructing (8), and solving (9). The solution to the PDE comes about from solving a system of ODEs. This is the *method of characteristics.*