PDEs and Functional Analysis
Lecture 1

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1 Overview

1.1 Outline

This course will cover PDEs and Functional Analysis. In particular:

1. PDEs
   (a) First Order PDEs (Linear and Nonlinear)
   (b) Second Order PDEs
      i. Elliptic (Laplace's Equation)
      ii. Parabolic (Heat Equation)
      iii. Hyperbolic (Wave Equation)
2. Functional Analysis
   (a) Hilbert Spaces
   (b) Banach Spaces
   (c) Fourier Transforms

1.2 Resources

Books:
Partial Differential Equations (Evans)
Functional Analysis and Partial Differential Equations (Brezis)

2 First Order PDEs

2.1 Introduction

Let $b \in \mathbb{R}^n$ and consider

$$u_t + b \cdot \nabla u = 0 \tag{1}$$

Where:

- $u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$
- $u(t, x) \in \mathbb{R} \forall\, t \geq 0$
- $\nabla = (\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n})$
- $u_t = \frac{\partial}{\partial t} u$

Concretely, the equation becomes:

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} = 0 \tag{2}$$

Similarly, it can be written as:

$$(u_t, \nabla u) \cdot (1, b) = 0 \tag{3}$$

(1), (2) and (3) are equivalent, and are called *Transport Equations* (of the first order). The equations are clearly homogeneous and of constant coefficients. To solve the transport equation, we are looking for a $u \in \mathcal{C}^1$ which satisfies the partial differential equation. The equation is said to be free, and has no boundary condition.

The standard way to proceed is to:

1. Look for special solutions (explicit)
2. Review the existence of a solution in some class (the larger the class, the better)
3. Review uniqueness of a solution (the smaller the class, the better)
4. Review the regularity[1] of a solution (typically the difficult point)

In order to review the existence of a solution, functional analysis is typically employed. In particular, Sobolov spaces and distributions are of importance.

Consider (1) once again. The equation is saying that the derivative along the $(1, b)$ vector is 0:

[1]Typically means high integrability or high differentiability, but generally serves as a measure of having desirable qualities