Elliptic Partial Differential Equations
Lecture 1

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1 Classical Form

Consider the following problem: Given $\Omega \subseteq \mathbb{R}^n, n \geq 1, \Omega$ open, with

$$f : \Omega \to \mathbb{R}, \text{ continuous}$$

$$g : \partial\Omega \to \mathbb{R}, \text{ continuous}$$

Solve the following system:

$$\begin{cases} u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^0(\overline{\Omega}) & \\ -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

Where Δ is the laplace operator: $\Delta u(x) := \operatorname{div}(\nabla u(x))$. That is:

u is twice differentiable on Ω, continuous on it's boundary, and

$$\begin{cases} -\Delta u(x) = f(x) & \forall\, x \in \Omega \\ u(x) = g(x) & \forall\, x \in \partial\Omega \end{cases}$$

where the first condition is called the *Poisson Equation*, and the second condition is called the *Dirichlet Boundary Condition.*

Our interests are in the existence of u, the uniqueness of u, and the regularity of u. These characteristics are found through our initial data Ω, f, and g.

1.1 Laplace's Equation

Consider the above problem with the initial condition $f = 0$:

$$\begin{cases} u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^0(\overline{\Omega}) & \\ -\Delta u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

The new PDE, $-\Delta u = 0$, is called *Laplace's Equation.* Regularity is especially of interest with Laplace's Equation, as smoothness can be much higher than simply $u \in \mathcal{C}^2(\Omega)$

2 Semiclassical Form

Consider Ω bounded, $g \in \text{Lip}(\partial\Omega)$, that is, g is Lipschitz continuous. Now let

$$f(\xi) = \frac{|\xi|^2}{2} \quad \forall \xi \in \mathcal{R}$$

And consider the *Dirichlet Functional*

$$F(u) = \int_\Omega f(\nabla u)\, dx = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx \quad \forall u \in \text{Lip}(\Omega)$$

Remark. *$u \in Lip(\Omega) \implies u$ is differentiable almost everywhere in Ω, so $|\nabla u| \in L^\infty(\Omega)$, implying the Dirichlet Functional is well defined.*

The Semiclassical Approach is to minimize the Dirichlet Functional:

$$\inf\{F(u) | u \in \text{Lip}(\Omega), u = g \text{ on } \partial\Omega\}.$$

Lemma 2.1. *Suppose that u is a solution of the Semiclassical Approach, that is:*

$$\frac{1}{2}\int_\Omega |\nabla u|^2\, dx = \inf\{F(u) | u \in Lip(\Omega), u = g \text{ on } \partial\Omega\}$$

And suppose $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$, then u solves the Classical Form with $f = 0$ (Laplace's Equation).

Proof. Take $\phi \in \text{Lip}(\Omega)$ and suppose ϕ has compact support in Ω, that is, ϕ is 0 in the boundary of Ω[1]: $\phi \in \text{Lip}_c(\Omega) \implies f(\partial\Omega) = \{0\}$. Further, let $\lambda \in \mathbb{R}$, and consider $u + \lambda\phi$:

$$u + \lambda\phi \in \text{Lip}(\Omega) \text{ and } u + \lambda\phi = g \text{ on } \partial\Omega$$

We can also compare $u + \lambda\phi$ to u since they are both Lipschitz. Since u solves the Semiclassical Approach:

$$F(u) \leq F(u + \lambda\phi) \implies h(\lambda) := F(u + \lambda\phi) \text{ has minimum } \lambda = 0$$

Thus, $h'(0) = 0$. Computing h':

$$h'(\lambda) = \frac{d}{d\lambda}F(u + \lambda\phi) = \frac{d}{d\lambda}\frac{1}{2}\int_\Omega |\nabla u + \lambda\nabla\phi|^2\, dx$$

$$\frac{1}{2}|\nabla u + \lambda\nabla\phi|^2 = \frac{1}{2}\frac{d}{d\lambda}\langle \nabla u + \lambda\nabla\phi, \nabla u + \lambda\nabla\phi\rangle = \langle \nabla u + \lambda\nabla\phi, \nabla\phi\rangle$$

$$\frac{d}{d\lambda}\frac{1}{2}\int_\Omega |\nabla u + \lambda\nabla\phi|^2\, dx = \int_\Omega \langle \nabla u + \lambda\nabla\phi, \nabla\phi\rangle\, dx$$

$$h'(0) = \int_\Omega \langle \nabla u, \nabla\phi\rangle\, dx$$

Thus, $\int_\Omega \langle \nabla u, \nabla\phi\rangle\, dx = 0 \quad \forall\phi \in \text{Lip}_c(\Omega)$. This is a weak formulation of the Classical Form. Integrating $h'(0)$ by parts[2]:

$$\int_\Omega \langle \nabla u, \nabla\phi\rangle\, dx = -\int_\Omega \Delta u\phi\, dx = 0 \quad \forall\phi \in \text{Lip}_c(\Omega)$$

Sinc this is true for any ϕ, Δu must be 0 point-wise everywhere. That is:

$$\Delta u = 0 \qquad \text{(Laplace's Equation!)}$$

□

[1] Why? Look up compact support later.

[2] No clue how this works but I'll write an appendix to explain it eventually