1 Metrics

Definitions

Cauchy: $\sum |x_i y_i| \leq ||x_i|| ||y_i||$
Minkowski: $||x+y|| \leq ||x|| + ||y||$
metric: i) $\sigma(x,y) = 0 \iff x = y$
ii) $\sigma(x,y) = \sigma(y,x) \forall x,y \in \mathbf{X}$
iii) $\sigma(x,y) \leq \sigma(x,z) + \sigma(z,y)$
Open ball: $B_\sigma(x,\delta) = \{y \in \mathbf{X} : \sigma(x,y) < \delta\}$
σ-open: $\forall x \in A, \exists \delta_x > 0 : B_\sigma(x,\delta_x) \subseteq A$
Cont. at x_0: $\forall \epsilon > 0, \exists \delta > 0 : d(x,y) < \delta \implies \sigma(f(x),f(y)) < \epsilon, x \in \mathbf{X}, y \in \mathbf{Y}$. cont $\forall x \in \mathbf{X}$

Theorems & Lemmas

$(\mathbf{X},\sigma) \in |\text{MET}|$, $B_\sigma(x,s) \subseteq \mathbf{X}$ is σ-open $\square$
f is continuous $\iff \forall \sigma$-open $V \subseteq \mathbf{Y}, f^{-1}(v)$ is d-open $\square$

2 Topologies

Definitions

Topology: i) $\emptyset, \mathbf{X} \in \tau$ ii) $A,B \to A \cap B$ iii)$A_j \to \cup A_j$. $(\mathbf{X},\tau) \in |\text{TOP}|$
Cont. at x_0: $\forall V \in \sigma, f(x_0) \in V, \implies \exists U \in \tau, x_0 \in U : f(U) \subseteq V$. cont $\forall x \in \mathbf{X}$

Theorems & Lemmas

f cont. $\iff \forall V \in \sigma, f^{-1}(V) \in \tau \square$

Special

Indiscrete :$\tau = \{\emptyset, \mathbf{X}\}$

3 Bases & Subbases

Definitions

Base ($\mathbf{B}$): $\forall V \in \tau, x \in V, \exists B \in \mathbf{B} : x \in B \subseteq V$. $V = \cup\{B_i\}$
First countable: countable basis at each x_0
Second countable: $|\mathbf{B}|$ countable OR τ has a countable basis, i.e. $\exists \mathbf{B} \subseteq \tau : \forall V \in \tau, x \in V, \exists B \in \mathbf{B} : x \in B \subseteq V$ and $\mathbf{B}$ is countable
Subbase ($\mathcal{S}$): $\mathcal{S} \subseteq \mathcal{P}(\mathbf{X})$ and the set of finite intersections of $\mathcal{S}$ forms a basis for τ
Local basis at x_0 ($\mathbf{B}_{\mathbf{x_0}}$): i) $x_0 \in B \ \forall B \in \mathbf{B}_{\mathbf{x_0}}$ ii) $V \in \tau, x_0 \in V \implies \exists B \in \mathbf{B}_{\mathbf{x_0}} : B \subseteq V$

Theorems & Lemmas

If $\mathbf{B} \subseteq \mathcal{P}(\mathbf{X}) \wedge$ i) $\cup \mathbf{B} = \mathbf{X}$ ii) $x \in B_1 \cap B_2, \exists B_3 \in \mathbf{B} : x \in B_3 \subseteq B_1 \cap B_2$. Then $\mathbf{B}$ is a basis for $\tau = \{A \subseteq \mathbf{X} : A = \cup\{B_i \in \mathbf{B}\}\}\square$
$\mathcal{S} \subseteq \mathcal{P}(\mathbf{X}) \wedge \cup \mathcal{S} = \mathbf{X} \implies \mathcal{S}$ is a subbasis for a top. τ on $\mathbf{X}\square$
Second countable $\implies$ first countable $\square$

4 Closure, Interior & Boundary

Definitions

Closure ($\overline{A}$): $\{x \in \mathbf{X} : V \cap A \neq \emptyset, \forall V \in \tau : x \in V\}$
Interior ($\mathring{A}$): $\{x \in A : \exists V \in \tau : x \in V \subseteq A\}$
Boundary (∂A): $\{x \in \mathbf{X} : V \cap A \neq \emptyset$ and $V \cap A^c \neq \emptyset, \forall V \in \tau : x \in V\}$. $\partial A = \overline{A} \cap \overline{A^c}$

Theorems & Lemmas

Let $(\mathbf{X},\tau) \in |\text{TOP}|$, $A \subseteq \mathbf{X}, B \subseteq \mathbf{X}$:
i) $A \subseteq \overline{A}$ ii) $A \subseteq B \implies \overline{A} \subseteq \overline{B}$
iii) $\overline{A \cup B} = \overline{A} \cup \overline{B}$ iv) $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$
v) $\overline{(\overline{A})} = \overline{A}$ vi) $\overline{A}$ is closed vii) $\overline{A}$ is the smallest closed set containing A
viii) $\overline{A} = A \iff A^c \in \tau\square$
Let $(\mathbf{X},\tau) \in |\text{TOP}|$, $A \subseteq \mathbf{X}, B \subseteq \mathbf{X}$:
i) $\mathring{A} \subseteq A$ ii) $A \subseteq B \implies \mathring{A} \subseteq \mathring{B}$
iii) $(A \mathring{\cap} B) = \mathring{A} \cap \mathring{B}$ iv) $\mathring{A} \cup \mathring{B} \subseteq (A \mathring{\cup} B)$
v) $(\mathring{\mathring{A}}) = \mathring{A}$ vi) $\mathring{A}$ is open vii) $\mathring{A}$ is the largest closed set contained in A
viii) $\mathring{A} = A \iff A \in \tau\square$

5 Separability, Convergence & Compactness

Definitions

Separable: $(\mathbf{X},\tau) \in |\text{TOP}|$: $\exists$ a countable $A \subseteq \mathbf{X} : \overline{A} = \mathbf{X}$. A is *dense* in $\mathbf{X}$
Converging sequence ($\{x_m\}$): $\forall V \in \tau, x \in V \implies x_m \in V \forall m \geq N, \exists N$, i.e., $\{x_m\}$ belongs to V eventually
Cauchy sequence ($\{x_m\}$): $\forall \epsilon > 0, \exists N : d(x_a,x_b) < \epsilon \forall a,b \geq N$
Complete $M \in |MET|$: Each Cauchy sequence converges
Hausdorff: $\forall x \neq y \in \mathbf{X}, \exists U,V \in \tau : x \in U, y \in V, U \cap V = \emptyset$

Theorems & Lemmas

Second countable $\implies$ separable $\square$
$(\mathbf{X},d) \in |\text{MET}|$ is separable $\iff$ second countable $\square$
Let $(\mathbf{X},\tau) \in |\text{TOP}|$ be first countable, and $A \subseteq \mathbf{X} \implies x \in \overline{A} \iff \exists$ a sequence in A which converges to x

6 Continuity

Theorems & Lemmas

TFAE: i) f is continuous ii) $f^{-1}(V) \in \tau \forall V \in \sigma$ iii) $f^{-1}(F)$ is closed $\forall F \subseteq Y$ that is closed iv) $f(\overline{A}) \subseteq \overline{f(A)} \forall A \subseteq \mathbf{X}$ v) $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B}) \forall B \subseteq Y \ \square$
f:(X,τ) $\mapsto (\mathbf{Y},\sigma)$ and $\mathcal{S}$ be a subbase for σ. $f^{-1}(S) \in \tau \forall S \in \mathcal{S} \implies f$ is continuous $\square$

7 Homeomorphisms

Definitions

Homeomorphism: i) bijection ii) continuous iii) inverse is continuous
Open (*Closed*) map: $f(V)(f(F))$ is open (closed) for every open (closed) V (F)
Topological property: property invariant under homeomorphisms

Theorems & Lemmas

Homeomorphisms are open & closed $\square$

8 Subspaces

Definitions

Subspace topology on A (τ_A): $\{B \subseteq A : B = V \cap A \exists V \in \tau\}$. Also: $\{V \cap A : V \in \tau\}$
Hereditary property: Invariant through subspaces
Embedding: $f : (\mathbf{X},\tau) \mapsto (\mathbf{Y},\sigma)$. If $F : (\mathbf{X},\tau) \mapsto (f(\mathbf{X}), \tau_{f(\mathbf{X})})$ is a homeomorphism, then f is an embedding.
Embedded: $(\mathbf{X},\tau)$ is embedded in $(\mathbf{Y},\sigma)$

Theorems & Lemmas

$(\mathbf{X},\tau) \in |\text{TOP}|$ and $A \subseteq \mathbf{X} \implies (A,\tau_A) \in |\text{TOP}| \square$
$(\mathbf{X},\tau) \in |\text{TOP}|, A \subseteq \mathbf{X} \implies$ i) $B \subseteq A, B^c \in \tau_A \iff B = C \cap A, \exists C \subseteq \mathbf{X}$ ii) $B \subseteq A \implies \overline{B}^{\tau_A} = \overline{B}^{\tau} \cap A$ iii) $B^c \in \tau, B \subseteq A, \implies B^c \in \tau_A$ iv) $A^c \in \tau, B \subseteq A, B^c \in \tau_A \implies B^c \in \tau\square$

9 Initial and Final Topologies

Definitions

Source: $\mathbf{X} \in |\text{SET}|$, $(\mathbf{Y}_j,\sigma_j) \in |\text{TOP}|$, $j \in J$.
Suppose $f_j : \mathbf{X} \mapsto \mathbf{Y}_j$. $\mathbf{X} \xmapsto{f_j} (\mathbf{Y},\sigma_j)$ is a source.
Sink: $(\mathbf{X}_j,\tau_j) \xmapsto{f_j} \mathbf{Y}$

Theorems & Lemmas

Consider the source $\mathbf{X} \xmapsto{f_j} (\mathbf{Y}_j,\tau_j)$:
i) $\exists$ a coarsest (smallest) topology τ_I on $\mathbf{X} : f_j : (\mathbf{X},\tau_I) \mapsto (\mathbf{Y}_j,\sigma_j)$ is continuous
ii) $g : (\mathbf{Z},\delta) \mapsto (\mathbf{X},\tau_I)$ is continuous $\iff f_j \circ g : (\mathbf{Z},\delta \mapsto (\mathbf{Y}_j,\sigma_j)$ is cont. $\forall j$
iii) τ_I is the unique top. on $\mathbf{X}$ which obeys (ii)
iv) if $\{x_m\} \xrightarrow{\tau_I} x \implies f_j(x_m) \xrightarrow{\sigma_j} f_j(x), \forall j\square$
Consider the sink $(\mathbf{X}_j,\tau_j) \xmapsto{f_j} \mathbf{Y}$:
i) $\exists$ a finest (largest) top. τ_F on $\mathbf{Y}$ such that $f_j : (\mathbf{X},\tau_j) \mapsto (\mathbf{Y},\tau_F)$ is cont. $\forall j$
ii) $\forall g : (\mathbf{Y},\tau_F) \mapsto (\mathbf{Z},\delta), g$ is cont. $\iff g \circ f_j : (\mathbf{X}_j,\tau_j \mapsto (\mathbf{Z},\delta)$ is cont. $\forall j$
iii) τ_F is the unique top. on $\mathbf{Y}$ obeying (ii), called the final topology $\square$

Remarks

Basis for τ_I: $\mathcal{S} = \{f_j^{-1}(V) : V \in \sigma_j, \forall j\}\square$
Basis for τ_I with $\mathcal{B}_j$ basis of σ_j: $\mathcal{S} = \{f_j^{-1}(B) : B \in \mathcal{B}_j, \forall j\}\square$

10 Product Spaces

Definitions

jth projection map: $\pi_j : \mathbf{X} \mapsto \mathbf{X}_j, \pi_j((x_i)_j) = x_j$
Product topology (τ_p): τ_I such that $\mathbf{X} \xmapsto{\pi_j} (X_j,\tau_j)$ is continuous $\forall j, \mathbf{X} = \prod_{j \in J} \mathbf{X}_j$
Subbase for τ_p: $\{\pi_j^{-1}(V) : V \in \tau_j\}$ or $\{\pi_j^{-1}(B) : B \in \mathbf{B}_j\}, \mathbf{B}_j$ a basis of τ_j
Typical basis member for τ_p: $\prod_{j \in J} B_j : B_j = \mathbf{X}_j$ except for finitely many j which are members of $\mathbf{B}_j$
Box topology: $\tau_B = \{\prod_{j \in J} B_j : B_j \in \mathbf{B}_j\}$
Productive property: $(\mathbf{X},\tau_p)$ has the property provided each $(\mathbf{X}_j,\tau_j)$ has it

Theorems & Lemmas

$\mathcal{S}$ is a subbase for τ. If $\{x_m\} \xrightarrow{\tau} x \iff \forall S \in \mathcal{S}$ with $x \in S; x_m \in S \forall m \geq N\square$
Consider $(\mathbf{X},\tau_I) \xmapsto{f_j} (\mathbf{Y}_j,\sigma_j)$. $\{x_m\} \xrightarrow{\tau_I} x \iff \forall j, f_j(x_m) \xrightarrow{\sigma_j} f_j(x), m \to \infty\square$ Let $(\mathbf{X}_\kappa,\tau_\kappa), \kappa \in \mathcal{K}$ be a collection of top. spaces. Fix $j \in \mathcal{K}$, then $(\mathbf{X}_j,\tau_j)$, can be embedded in $(\mathbf{X},\tau_p), \mathbf{X} = \prod_{\kappa \in \mathcal{K}} \mathbf{X}_\kappa\square$
$(\mathbf{X},\tau) \in |\text{TOP}|$ is Hausdorff $\iff \Delta = \{(x,x) : x \in \mathbf{X}\}$ is closed in $(\mathbf{X} \times \mathbf{X},\tau_p)$

Remarks

$\tau_p \subseteq \tau_B$, thus $\pi_j : (\mathbf{X},\tau_B) \mapsto (\mathbf{X}_j,\tau_j)$ is cont.

11 Quotient Spaces

Definitions

Quotient topology: Final topology where $(\mathbf{X},\tau) \xmapsto{f} \mathbf{Y}, f$ surjective (onto). $\tau_Q = \{V \subseteq Y : f^{-1}(V) \in \tau\}$
Quotient map: $f : (\mathbf{X},\tau) \mapsto (\mathbf{Y},\tau_Q)$
Open map: $f : (\mathbf{X},\tau) \mapsto (\mathbf{Y},\sigma)$, where $V \in \tau \implies f(V) \in \sigma$
Closed map: $f : (\mathbf{X},\tau) \mapsto (\mathbf{Y},\sigma)$, where $V^c \in \tau \implies (f(V))^c \in \sigma$

Theorems & Lemmas

f cont. surjection $\wedge$ (f is open $\vee$ f is closed) $\implies$ f is a quotient map $\square$

12 Connectedness

Definitions

Disconnected: $\mathbf{X} = V \cup W, V,W \in \tau, V \neq \emptyset \neq W, V \cap W = \emptyset$
Connected: Not disconnected
Connected subset: $A \subseteq \mathbf{X}, (A,\tau_A)$ is connected
Path (arcwise) connected: $\forall a,b \in \mathbf{X}, \exists f^{cont} : [0,1] \mapsto \mathbf{X} : f(0) = a, f(1) = b$. f is a path from a to b

Theorems & Lemmas

$(\mathbb{R},\tau)$ is connected $\square$
$f : (\mathbf{X},\tau) \mapsto (\mathbf{Y},\sigma)$ cont. and surj., then $(\mathbf{X},\tau)$ connected $\implies (\mathbf{Y},\sigma)$ connected $\square$
Suppose $(\mathbf{X},\tau) \in |\text{TOP}|$: i) $A \subseteq \mathbf{X}$ conn. $\implies B$ conn., $A \subseteq B \subseteq \overline{A}$ ii) A_j conn. and $\exists \kappa \in J : A_\kappa \cap A_j \neq \emptyset \forall j \neq \kappa \implies \bigcup_{j \in J} A_j$ is conn. $\square$
A subset of $(\mathbb{R},\tau)$ is conn $\iff$ it is an interval $\square$
IVT: Assume $f : (\mathbf{X},\tau) \mapsto \mathbb{R}$ is cont. and $(\mathbf{X},\tau)$ is conn. If $a,b \in \mathbf{X}$ and $f(a) < f(b)$ then $[f(a),f(b)] \subseteq f(\mathbf{X})\square$
i) A path conn. space is conn.
ii) $\mathbb{R}^\kappa$ is path conn. $\square$
$(\mathbf{X},\tau_p)$ conn. $\iff (\mathbf{X}_j,\tau_j)$ is conn.$\forall j$

13 Compactness

Definitions

Open cover: $\mathbf{e} \subseteq \tau, \cup\mathbf{e} = \mathbf{X}$ *Open cover of A*: $A \subseteq \cup\mathbf{e}$

Compact: Every open cover has finite subcover. $\mathbf{e} \subseteq \mathbf{X}$ and $A \subseteq \cup\mathbf{e}, \implies \exists V \subseteq \mathbf{e} : A \subseteq \cup V \wedge$

Neighborhood of x: $B \subseteq \mathbf{X} : x \in V \subseteq B, \exists V \in \tau$

Locally compact: $\forall x \in \mathbf{X}$, x has a compact neighborhood.

Theorems & Lemmas

Let $(\mathbf{X}, \tau) \in |\mathrm{TOP}|$:

i) if $\mathbf{X}$ compact, so is each closed subset

ii) if $\mathbf{X}$ Hausdorff, each compact subset is closed

iii) if $\mathbf{X}$ is a metric space, each copmact subset is closed and bounded

iv) if $f : (\mathbf{X}, \tau) \mapsto (\mathbf{Y}, \sigma)$ is cont., $\mathbf{X}$ is compact, $\mathbf{Y}$ is Hausdorff, then $f(\mathbf{X})$ is a compact subset of $\mathbf{Y}$ and is closed □

Let $I = [a, b] \subseteq (\mathbb{R}, \tau)$. I is compact □

$(\mathbf{X}, \tau)$ compact $\iff (\mathbf{X}_j, \tau_j)$ compact □

$A \subseteq \mathbb{R}$ compact $\iff$ A closed and bounded □

Let $f : (\mathbf{X}, \tau) \mapsto \mathbb{R}$ be cont.:

i) $\mathbf{X}$ comp. $\implies \inf f(\mathbf{X}) \in f(\mathbf{X})$ and $\sup f(\mathbf{X}) \in f(\mathbf{X})$

ii) $\mathbf{X}$ comp. and conn. $\implies f(\mathbf{X}) = [f(x_0), f(x_1)]$□

Let $f : (\mathbf{X}, \tau) \mapsto (\mathbf{Y}, \sigma)$ be a cont., open surjection. $(\mathbf{X}, \tau)$ loc. comp. $\implies (\mathbf{Y}, \sigma)$ loc. comp. □

Let $(\mathbf{X}, \tau)$ be loc. comp. and Hausdorff. If $A \subseteq \mathbf{X}$ is closed, (A, τ_A) is also loc. comp. □

Let $(\mathbf{X}, \tau)$ be Hausdorff. Let compacts subsets $A, B \subseteq \mathbf{X}, A \cap B = \emptyset$:

i) if $x \notin A, \exists V_1, V_2 \in \tau : x \in V_1, A \subseteq V_2, V_1 \cap V_2 = \emptyset$

ii) $\exists V_1, V_2 \in \tau : A \subseteq V_1, B \subseteq V_2, V_1 \cap V_2 = \emptyset$□

Let $(\mathbf{X}, \tau)$ be loc. comp. and Hausdorff. $x \in V \in \tau \implies \exists W \in \tau : x \in W \subseteq \overline{W} \subseteq V, \overline{W}$ compact □

Remarks

If $\mathbf{e}$ is a τ-open cover of A, then $\mathcal{D} = \{V \cap A : V \in \mathbf{e}\}$ is a τ_A-open cover of

A *Compactification* $(((\mathbf{Y}, \sigma), f))$: i) $(\mathbf{Y}, \sigma)$ comp. ii) $f : (\mathbf{X}, \tau) \mapsto (\mathbf{Y}, \sigma)$ is an into homeomorphism iii) $\overline{f(\mathbf{X})} = \mathbf{Y}$

One-point compactification: above, and $Y - f(\mathbf{X})$ is a singleton

Theorems & Lemmas

Assume $(\mathbf{X}, \tau)$ is Hausdorff. Then it has a Hausdorff one-point compactif. iff $(\mathbf{X}, \tau)$ is locally compact. Assume $(\mathbf{X}, \tau)$ is not compact □

14 Topological Groups

Definitions

Group $((\mathbf{G}, \cdot))$: $\mathbf{G} \in |\mathrm{SET}|, \cdot : \mathbf{G} \times \mathbf{G} \mapsto \mathbf{G}$ s.t.

i) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ (associativity)

ii) $\exists e \in \mathbf{G} : ae = ea = a \forall a \in \mathbf{G}$ (identity)

iii) $\forall a \in \mathbf{G} \exists v \in \mathbf{G} : ab = ba = e$ (inverse)

$\theta : \mathbf{G} \times \mathbf{G} \mapsto \mathbf{G}, \theta(x, y) = xy$

$\psi : \mathbf{G} \mapsto \mathbf{G}, (x) = x^{-1}$

$\delta : \mathbf{G} \times \mathbf{G} \mapsto \mathbf{G}, \delta(x, y) = xy^{-1}$

Homomorphism: $h : \mathbf{G} \mapsto \mathbf{H}, h(x, y) = h(x)h(y)$

Topological Group: $(\mathbf{G}, \cdot, \tau)$ where

i) θ is cont.

ii) ψ is cont.

Subgroup: $\mathbf{H} \subseteq \mathbf{G} : \forall a, b \in \mathbf{H}, ab^{-1} \in \mathbf{H}. \mathbf{H} \cdot \mathbf{H}^{-1} \subseteq \mathbf{H}$

Normal Subgroup: Subgroup $\mathbf{H}$ where $\forall a \in \mathbf{G}, a\mathbf{H}a^{-1} \subseteq \mathbf{H}$

Theorems & Lemmas

If h is a homomorphism, then $h(e_{\mathbf{G}}) = e_{\mathbf{H}}, h(x^{-1}) = (h(x))^{-1}$□

Assume $f_i : \mathbf{X}_i \mapsto \mathbf{Y}_i, i = 1, 2$ are cont. Define $f_1 \times f_2 : \mathbf{X}_1 \times \mathbf{X}_2 \mapsto \mathbf{Y}_1 \times \mathbf{Y}_2, (f_1 \times f_2)(x_1, x_2) = (f_1(x_1), f_2(x_2))$ then $f_1 \times f_2$ is cont. □

$(\mathbf{G}, \cdot, \tau) \in |\mathrm{TG}| \iff \delta$ is cont. □

Fix $a \in \mathbf{G}$. The following are homeomorphisms:

i) $x \mapsto ax$ (left transl.)

ii) $x \mapsto xa$ (right trans.)

iii) $x \mapsto axa^{-1}$□

$e \in V \in \tau \iff a \in aV \in \tau$□

$a \in W \in \tau \implies \exists e \in V \in \tau : W = aV$□

$(\mathbf{G}, \tau)$ Hausdorff $\iff \{e\}$ closed □

$\mathbf{H}$ (normal) subgroup of $\mathbf{G} \implies \overline{\mathbf{H}}$ (normal) subgroup of $\mathbf{G}$. In particular, $(\mathbf{H}, \cdot, \tau_{\mathbf{H}}), (\overline{\mathbf{H}}, \cdot, \tau_{\overline{\mathbf{H}}}) \in |\mathrm{TG}|$ □

Let $h : (\mathbf{G}, \cdot, \tau) \mapsto (\mathbf{H}, \cdot, \sigma)$ be a homomorphism. h cont. at $x = a \implies h$ is cont. □

$(\mathbf{G}, \cdot) \overset{h_j}{\longmapsto} (\mathbf{H}_j, \cdot, \sigma_j), \tau$ the initial top. on $\mathbf{G}$. Then $(\mathbf{G}, \cdot, \tau) \in |\mathrm{TOP}|$ provided $(\mathbf{H}_j, \cdot, \sigma_j) \in |\mathrm{TG}|\ \forall j$□

Let h be a homomorphism. If h maps open sets containing $e_{\mathbf{G}}$ to open sets containing $e_{\mathbf{H}}$, then h is an open map. Further, if h is a cont. surj., then h is a quotient map □

15 Actions

Definitions

Action (λ), $\mathbf{G}$ *acts on* $\mathbf{X}$: $\lambda : \mathbf{G} \times \mathbf{X} \mapsto \mathbf{X}$ such that

i) $\lambda(e, x) = x \forall x \in \mathbf{X}$

ii) $\lambda(g, \lambda(h, x)) = \lambda(gh, x) \forall g, h \in \mathbf{G}, x \in \mathbf{X}$

iii) λ is cont. when $\mathbf{G} \times \mathbf{X}$ has the product topology

Theorems & Lemmas

Suppose $(\mathbf{G}, \cdot, \sigma)$ acts cont. on $(\mathbf{X}, \tau)$ with action . Fix $g \in \mathbf{G}$ and define $\theta_g : \mathbf{X} \mapsto \mathbf{X}, \theta_g(x) = \lambda(g, x), x \in \mathbf{X}$. θ_g is a homeomorphism □

16 More on Metric Spaces

Definitions

Countably compact: Every countable open cover of $\mathbf{X}$ has a finite subcover

Finite Intersection Property: $\bigcap_{j=1}^{m} \mathbf{D}_j \neq \emptyset$ for each finite collection $\{\mathbf{D}_j : 1 \leq j \leq m\} \subseteq \mathcal{D}$

Nonempty Intersection: $\bigcap\limits_{\mathbf{D} \in \mathcal{D}} \mathbf{D} \neq \emptyset$

Sequentially Compact: Each sequence in $\mathbf{X}$ has a convergent subsequence

Lindelof: Every open cover has countable subcover

Totally Bounded: $\forall \delta > 0, \exists x_1, x_2, \ldots, x_n \in \mathbf{X} : \bigcup\limits_{i \in I} \mathcal{B}(x_i, \delta) = \mathbf{X}$

Theorems & Lemmas

$(\mathbf{X}, d) \in |\mathrm{MET}|$, TFAE:

i) $(\mathbf{X}, d)$ is countably compact

ii) Each countable collection of closed subsets of $\mathbf{X}$ with f.i.p. has nonempty intersection

iii) $(\mathbf{X}, d)$ is sequentially compact □

$\mathbf{X}$ compact $\implies$ totally bounded □

Sequential compactness $\implies$ total bound. $\implies$ second countable $\implies$ Lindelof □

TFAE: i) $(\mathbf{X}, d)$ is countably compact

ii) $(\mathbf{X}, d)$ is sequentially compact

iii) $(\mathbf{X}, d)$ is compact □

17 Path Homotopy

Definitions

Loop: A *path* where $f(0) = f(1)$

Reverse of f: The path $\overline{f} : [0, 1] \mapsto \mathbf{X}, \overline{f}(s) = f(1 - s), 0 \leq s \leq 1$

Path Multiplication: Let f, g be paths in $\mathbf{X} : f(1) = g(0)$. Define $f * g$ to be the path $(f*g)(s) = \begin{cases} f(2s) & 0 \leq s \leq \frac{1}{2} \\ g(2s-1) & \frac{1}{2} \leq x \leq 1 \end{cases}$ *Homotopic Paths*: f, g paths in $\mathbf{X} : f(0) = g(0), f(1) = g(1)$. f, g are path homotopic provided $\exists$ a continuous map $F : I^2 \mapsto \mathbf{X}$ where

i) $F(0, t) = f(0) = g(0)$

ii) $F(1, t) = f(1) = g(1)$

iii) $F(s, 0) = f(s)$

iv) $F(s, 1) = g(s)$

F is a *Path Homotopy*

Homotopy Path Equivalence Class: $[f] = \{g : g \sim f$, g is a path in $\mathbf{X}\}$

Constant Path: $e_x : I \mapsto \mathbf{X}, e_x(s) = x \forall s \in I$. Note e_x is a loop

Straight Line Homotopy: $F(s, t) = (1 - t)f(s) + tg(s)$

Theorems & Lemmas

Parting Lemma: Assume $(\mathbf{X}, \tau) \in |\mathrm{TOP}|$, $\mathbf{X} = A \cup B$, A and B closed subsets of $\mathbf{X}$. Suppose $f : (A, \tau_A) \mapsto (\mathbf{Y}, \sigma)$ and $g : (B, \tau_B) \mapsto (\mathbf{Y}, \sigma)$ are each cont. Further, let $f(x) = g(x) \forall x \in A \cap B$. Define $h : \mathbf{X} \mapsto \mathbf{Y}$ by $h(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases}$. h is continuous □

Let $(\mathbf{X}, \tau) \in |\mathrm{TOP}|$. If f, g are paths in $\mathbf{X}$, define $f \sim g \iff f$ and g are path homotopic. $\sim$ is an equivalence relation on the set of all paths in $\mathbf{X}$□

All paths in $(\mathbf{X}, \tau)$:

i) $f * e_{x1} \sim f$ and $e_{x0} * f \sim f$ where $f(0) = x_0, f(1) = x_1$

ii) $f * \overline{f} \sim e_{x0}$ and $\overline{f} * f \sim e_{x1}$

iii) $f \sim g \implies \overline{f} \sim \overline{g}$

iv) $f \sim f_1, g \sim g_1, \exists f * g \implies f * g \sim f_1 * g_1$

v) If $f * g, g * h$ exist, then $(f * g) * h \sim f * (g * h)$□