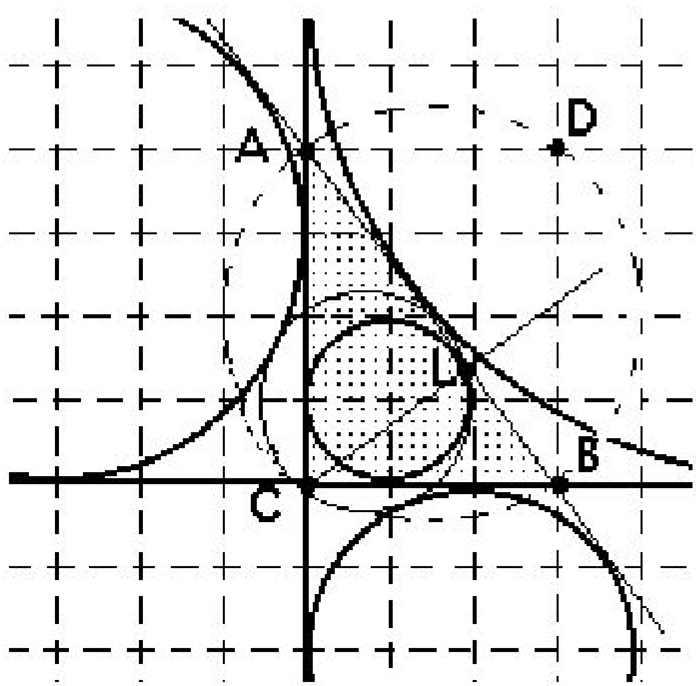
Introduction

As an avid computer programmer, perfection and generalization have always been topics that have interested me wildly. The idea of creating efficient programs and generalized methods for functionality in my code amazes me and is entertaining to no end. As such, the idea of a generalized formula for a nine-point circle was just as appealing. Nine-point circles are “perfect” and “beautiful” in their own sense, as they are present in all triangles, and have special properties and connections related to other triangular circles. These are numerous and make the nine-point circles special and quite unique.

A few of these properties are quite amazing, such as the fact that it is always tangent to the in-circle and the ex-circles, no matter the angles or side lengths of the triangle:



Taken from “Heron’s”

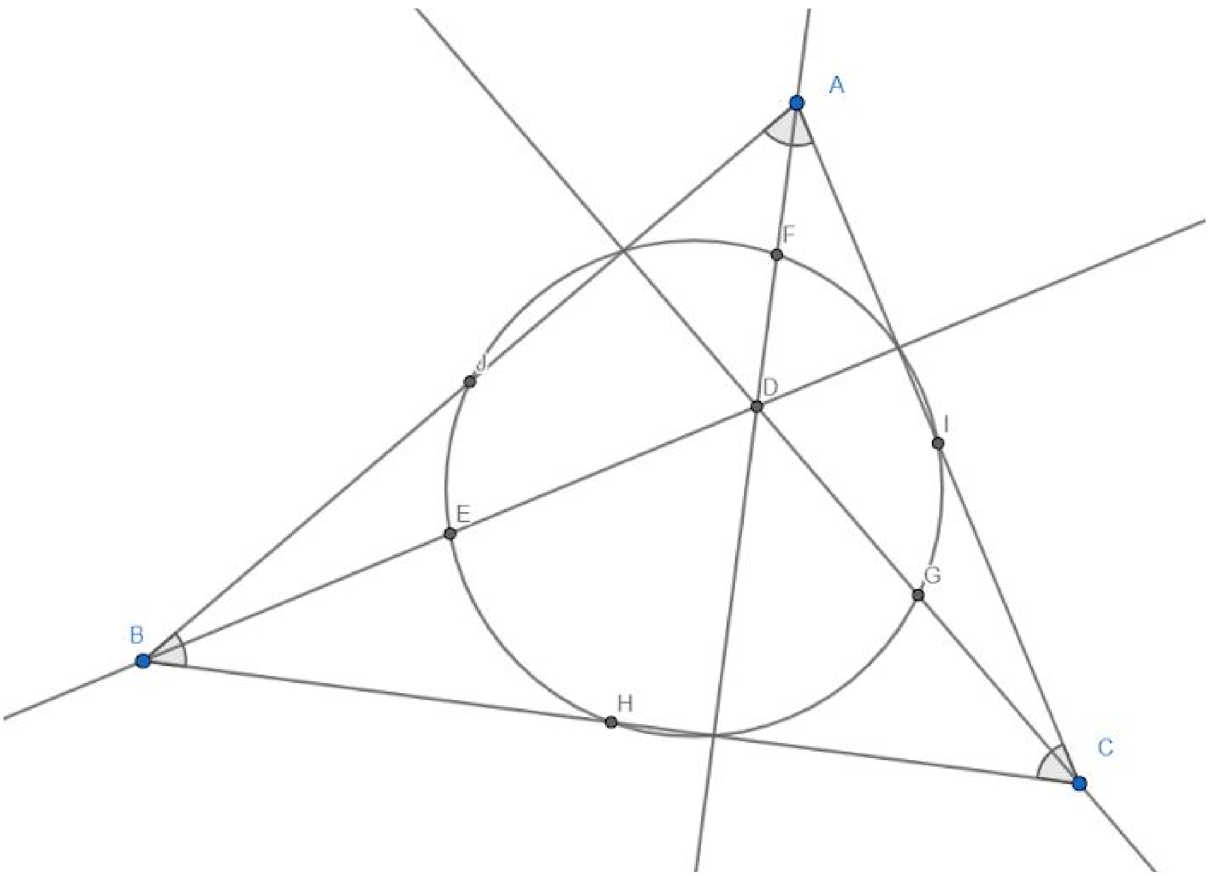
However, my research on the subject revolved around the fact that all triangles have nine-point circles, and thus, a search to find a generalized formula for the radius based off of the side lengths of the triangles. This closely relates to my love of efficiency and generalization that I’ve gained through ym coding experience.

Modeling and Generalizing Nine Point Circles

A nine-point circle is a circle that can be constructed from all triangles. This circle passes through the nine concyclic points of the triangle, thus displaying the concyclic properties. These nine points can be identified as follows:

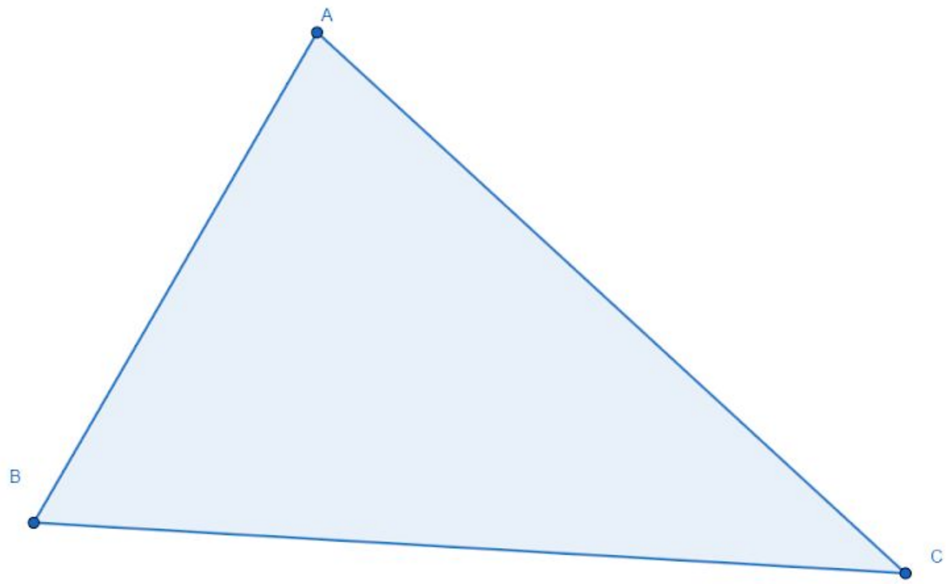
- The midpoint of each side (3)
- The point of intersection of the altitude with the opposite side (3)
- The midpoint of the line segment between each vertex and the orthocenter (3)

An example of a nine-point circle can be seen below.



Created with GeoGebra

To begin the modeling and generalization, a triangle ABC will be constructed:



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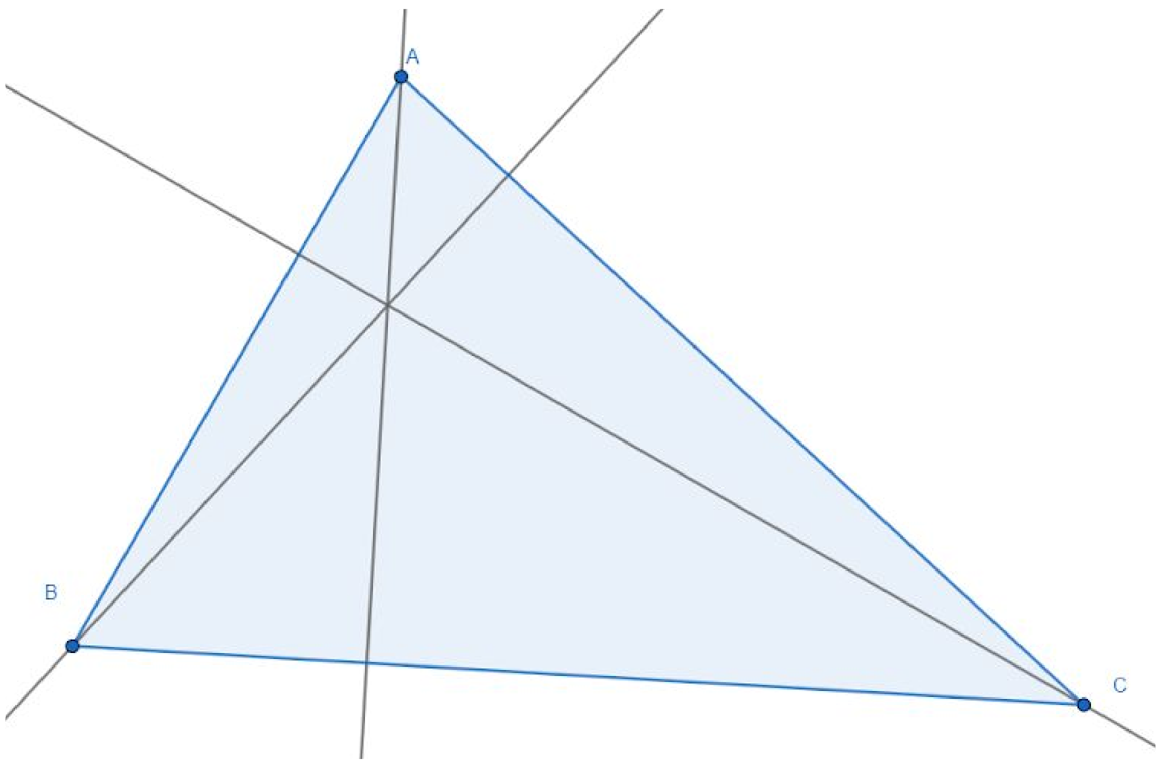
Then the altitudes of ABC are drawn in:



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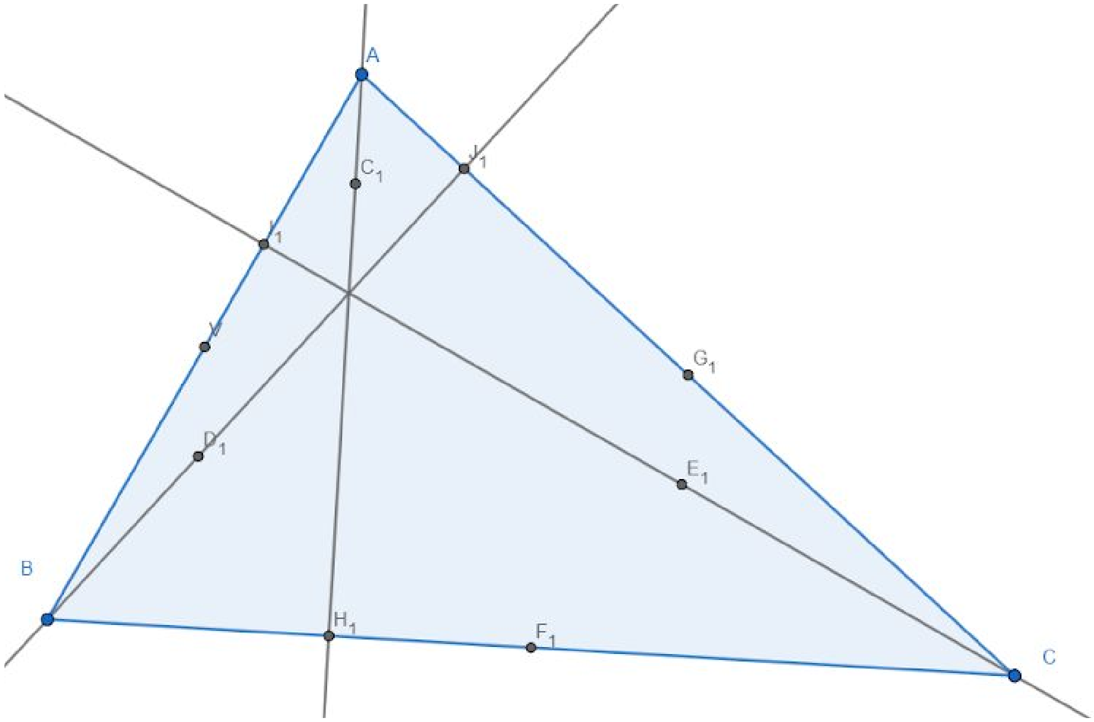
Once the altitudes have been identified, the midpoints of the three sides, along with the midpoints between the orthocenter and the vertices will be identified. In addition, the intersection between the altitudes and the corresponding bases are identified as well:



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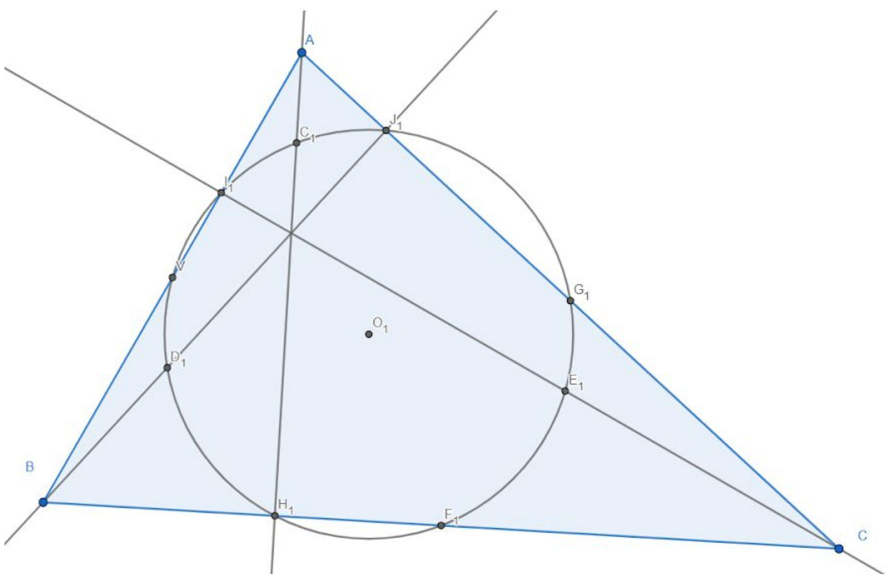
Below the completed diagram with the nine-point circle can be seen:



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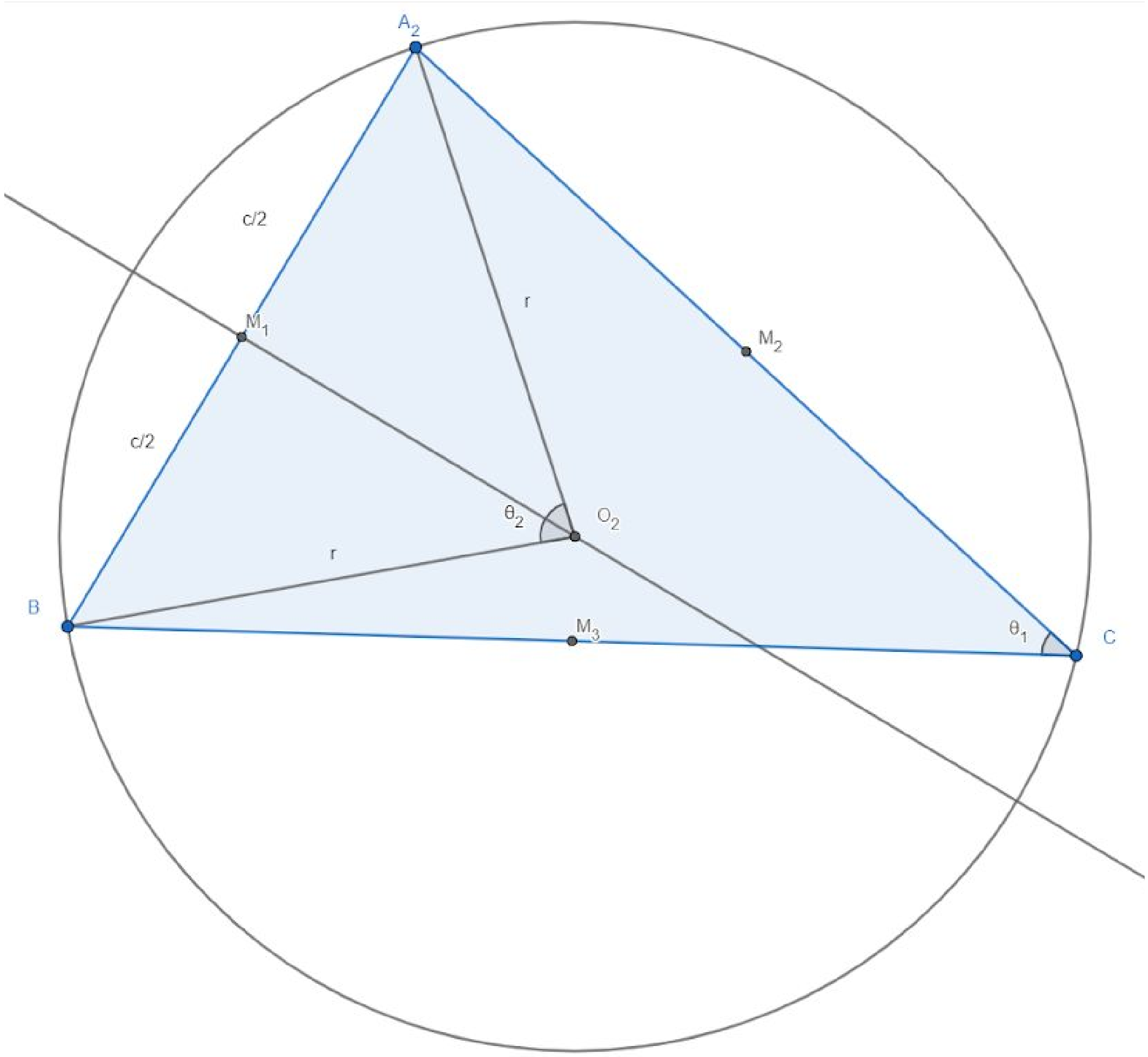
In order to generalize the circle, giving a general formula for its radius, area, and circumference, we need to utilize the circumcircle created by the triangle:



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The radius for the larger circle can be calculated with the use of inscribed angles:

$$\angle AO_2B = 2\angle ACB$$

$$sin(\theta_1) = \frac{c}{2r}$$

Let A_t be the area of the triangle:

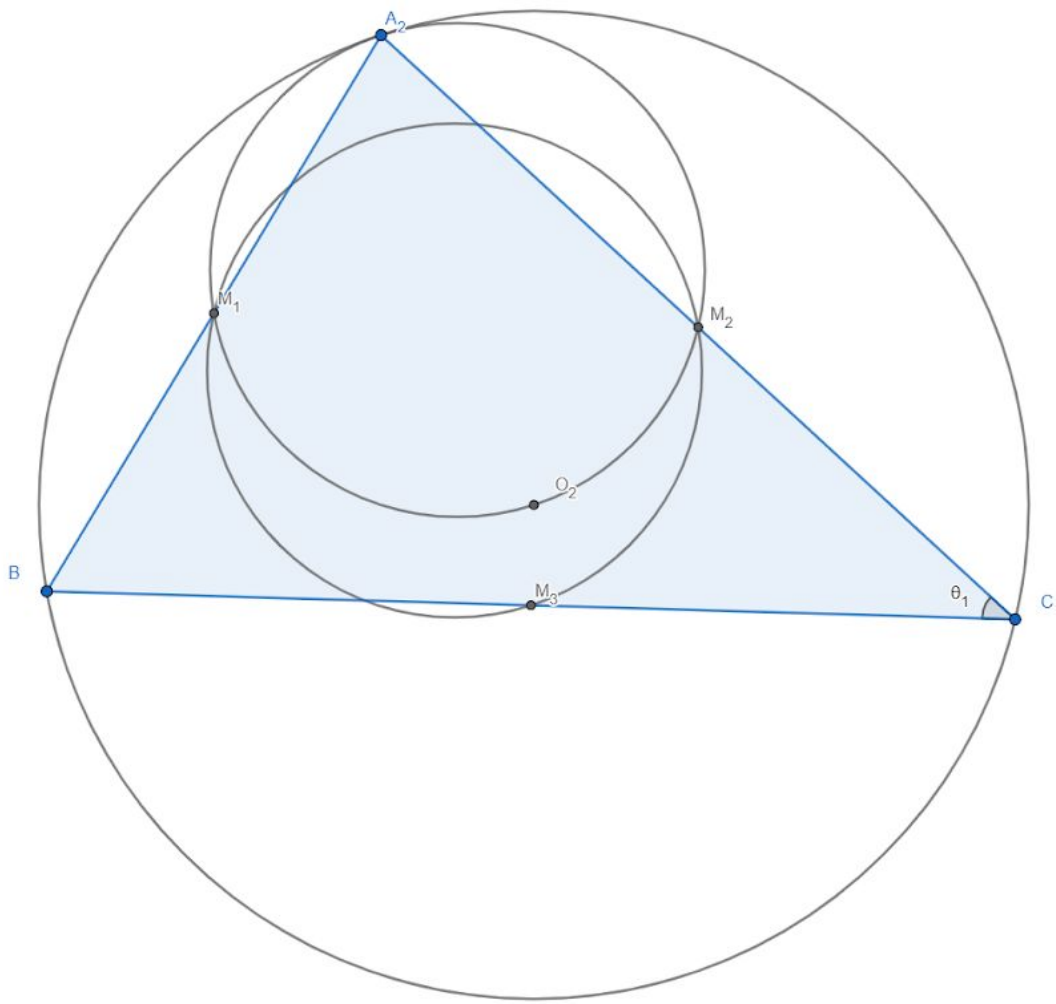
$$A_t = \frac{1}{2}absin(\theta_1)$$

$$A_t = \frac{1}{2}ab(\frac{c}{2r})$$

$$A_t = \frac{abc}{4r}$$

$$r = \frac{abc}{4A_t}$$

Now that a general radius for the circumcircle has been identified, the general formula for the nine-point circle radius, $r_n = \frac{r_c}{2}$, can be applied. This formula can be proved by utilizing the similar circle created by points $A,\ M_1,$ and M_2 . This circle has the same radius as the one created by points $M_1,\ M_2,$and M_3 and has a diameter that's half of the circumcircle's diameter:



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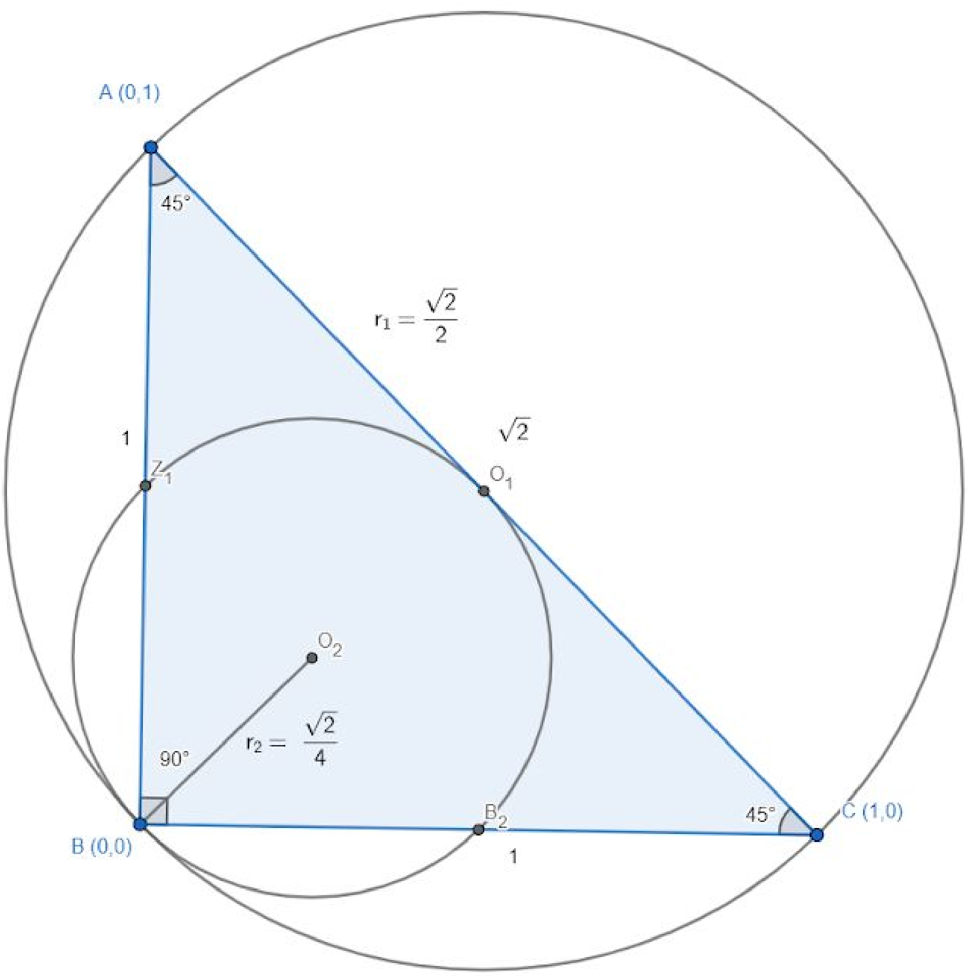
Therefore, a new generalized formula for the radius of the nine-point circle is given as $r = \frac{abc}{8A_t}$. This newly found formula can be utilized to find the area or circumference of the nine-point circle given any three points A, B, C .

In order to further generalize this formula, A_t can be expanded and defined for all triangles. In order to do this, the sides will be expressed as vectors, allowing for the use of the formula $sin(C) = \frac{AC \times BC}{|AC||BC|}$. This formula allows for the solution of through $A_t = \frac{1}{2}ab \times sin(C)$.

So by substituting, the final formula for the radius of the circle would be $r = \frac{c|AC||BC|}{4(AC \times BC)}$.

An example of the use of this generalization can be done with a 45, 45, 90 triangle:



Created with GeoGebra

By observation, the radius of the nine-point circle is $\frac{\sqrt{2}}{4}$, however if we solve it using the generalization, it results in the same number:

$$r = \frac{1(\sqrt{2})(1)}{4(AC\times BC)}$$

$$AC \times BC = [(1)(0)] - [(1)(-1)] = 0 + 1 = 1$$

$$\therefore r = \frac{\sqrt{2}}{4(1)} = \frac{\sqrt{2}}{4}$$

This generalization can also be used to find the smallest circle possible given two fixed points and a freely moving point. To do so, we first define the equation in terms of constants and a single variable point p. The two fixed points will be O (0,0) and N (n,0), where n is a constant. This is due to the fact that any triangle can be translated and rotated freely and it will retain its original side lengths as well as its initial angles. Hence, any vertex of the triangle can be placed at (0,0) with a second vertex being placed at (n,0).

The final point will be named point p with coordinates (x_p, y_p).

This leads to the formula $r = \frac{|OP||PN||ON|}{4(PN\times ON)}$. This can be defined in terms of constants and the point p:

$$r = \frac{\sqrt{x_p^2+y_p^2}\sqrt{(n-x_p)^2+y_p^2}(n)}{4[(n-x_p)(0)-(n)(-y_p)]}$$

$$r = \frac{\sqrt{x_p^2+y_p^2}\sqrt{n^2-2nx_p+x_p^2+y_p^2}}{4y_p}$$

In order to find the smallest circle, partial derivatives would need to be used, so in this scenario we will set point p as being on the line $y = x$. This means that y_p can now be substituted for x_p. To simplify, all x_p will be substituted for x as it is the only variable.

$$r(x) = \frac{\sqrt{x^2+x^2}\sqrt{n^2-2nx+x^2+x^2}}{4x}$$

$$r(x) = \frac{\sqrt{2x^2}\sqrt{n^2-2nx+2x^2}}{4x}$$

$$r(x) = \frac{\sqrt{n^2-2nx+2x^2}}{2\sqrt{2}}$$

Now that r(x) is in terms of constants and a single variable x, it can be derived and solved for the x when the derivative equals zero in order to find the shortest radius for a nine-point circle in a triangle with a side length of n:

$$r(x) = \frac{\sqrt{n^2 - 2nx + 2x^2}}{2\sqrt{2}}$$

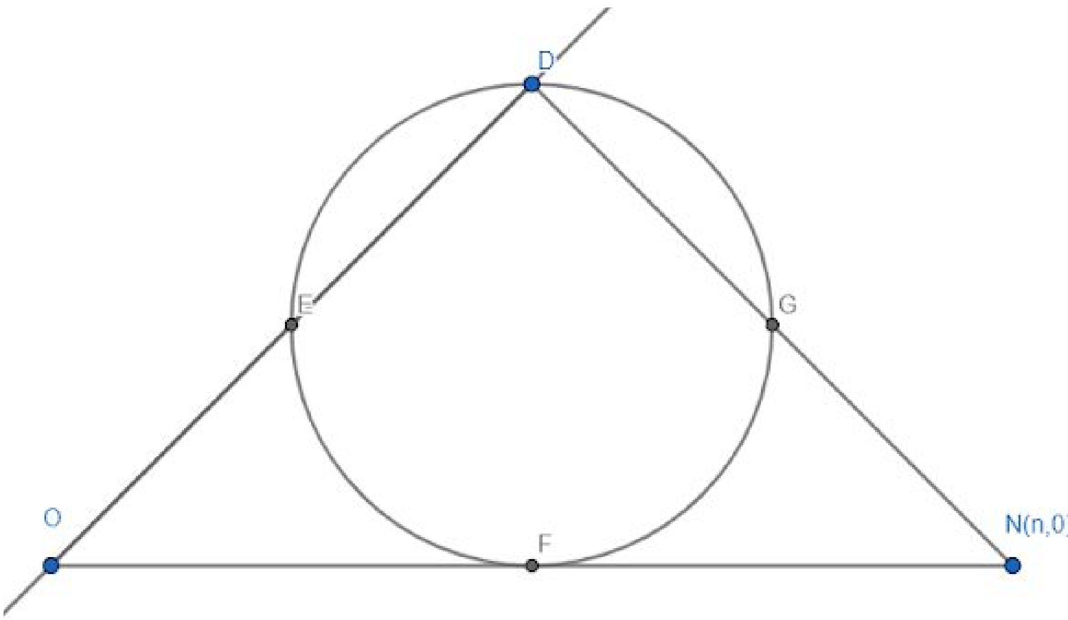
$$r'(x) = \frac{2x - n}{\sqrt{2x^2 - 2nx + n^2}} \times \frac{1}{2\sqrt{2}}$$

$$0 = \frac{2x - n}{\sqrt{2x^2 - 2nx + n^2}} \times \frac{1}{2\sqrt{2}}$$

$$0 = 2x - n$$

$$2x = n$$

$$x = n/2 \ \Rightarrow p(\tfrac{n}{2}, \tfrac{n}{2})$$



Example nine-point circle for n=1. Created with GeoGebra

The general formula for the radius can also be applied for any function of x, leading to finding the smallest possible nine-point circle for a triangle with vertices at (0,0), (n,0), and (x,f(x)), for n εR:

$$r = \frac{\sqrt{x^2 + f(x)^2}\sqrt{n^2 - 2nx + f(x)^2 + x_p^2}}{4f(x)}$$

Works Cited

"Calculadora De Derivadas." Calculadora De Derivadas • ¡Con Pasos!, www.calculadora-de-derivadas.com/.

"Geometry." GeoGebra, www.geogebra.org/geometry.

Heron's Formula, Descartes Circles, and Pythagorean Triangles - Scientific Figure on ResearchGate. Available from: https://www.researchgate.net/figure/In-circle-ex-circles-all-tangent-to-nine-point-circle-N-between-the-circum-circle-and_fig6_2135021