Title: Inscribed Polynomials

Research question: Can an arc of a parabola inside a circle of radius 1 have a length greater than 4?

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Introduction

The William Lowell Putnam Mathematical Competition is a mathematics competition for undergraduate college students which is considered by many as the most difficult math exam world-wide. The Putnam is split into two three hour sections, each containing six questions. The exam is usually taken by university students specializing in mathematics, yet the median score for the competition is often zero out of a possible 120 points.

In my extended essay I will be analyzing, solving, and extending a question from the 2001 Putnam exam. The question is stated as follows in the competition: "Can an arc of a parabola inside a circle of radius 1 have a length greater than 4?"

This question has proven to be one of the most difficult problems present in one of the most difficult mathematics competitions in the world. From the top 200 performing contestants, only one student was able to score all ten possible points for the question, only one was able to score one point out of the ten, 57 attempted to answer the question but received no points, and 141 didn't attempt to solve it. The average score for the question within this group was a miniscule 0.055 out of ten.

Questions such as this are significant towards the development of mathematics in various fields including geometry, calculus, and multivariable calculus. The progression of mathematics is dependent on the creation of more complex and difficult problems and how we manage to solve them. New mathematical discoveries, new theorems, and new formulae are bound to come out of problems such as these as we approach problems from different viewpoints and attempt them in different ways. This is why I will be answering the problem and creating my own solution, as well as analyzing other people's solutions. I will also be solving the research questions for a third degree curve.

In order to answer the question, I've applied the arc length formula to a representation of the problem on the XY coordinate plane. If the circle is represented as the equation $y^2 + x^2 = 1$ (or $y_1 = \sqrt{1 - x_1^2}$ and $y_2 =- \sqrt{1 - x_2^2}$), and the parabola is split into two curves, namely $y_1 = \sqrt{\varepsilon(x_1 - 1)}$ and $y_2 =- \sqrt{\varepsilon(x_2 - 1)}$, we can calculate the values of ε for which the curves y_1 and y_2 have a length greater than 4 when added together. This will be done through the use of the arc length formula: $\int_a^b \sqrt{1 + (\frac{dy}{dx})^2}dx$, where a and b are the end points of the curve and $\frac{dy}{dx}$ is the derivative of the curve. This method is time consuming and therefore not the most efficient in a timed competition where the allotted time is around 30 minutes, however it will answer the question fully and provide necessary information towards the answer to the research question.

My first attempts at completing the Putnam problem were unsuccessful as I utilized the equation $y = -ax^2 + 1$ instead of the two square root equations. This seems like a more efficient solution at first glance, but once the equation is inputted into the arc length formula, the integration is very lengthy and complicated making it undesirable for the conditions of a math competition. This approach to the problem will be lengthy nevertheless, however it can be simplified with the use of square root equations. Attempting the problem with quadratic equations is a common mistake which can lead to tedious work and mistakes along the way, creating erroneous solutions. Therefore, as I've discussed the various steps I will be taking to answer the research question, and have analyzed possible solutions that others have done, I can conclude that it is possible to fit a parabola of length greater than four inside of a circle with a radius of one.

Setting Up the Problem

There are various significant definitions which need to be addressed in order to answer the research question fully and correctly. We will define the following terms:

Parabola - A mirror-symmetrical shape which can be modeled through the equation $y = ax^2 + bx + c;\ a, b, c \,\varepsilon\, R$. Can also be modeled through the equation $x = ay^2 + by + c;\ a, b, c \,\varepsilon\, R$.

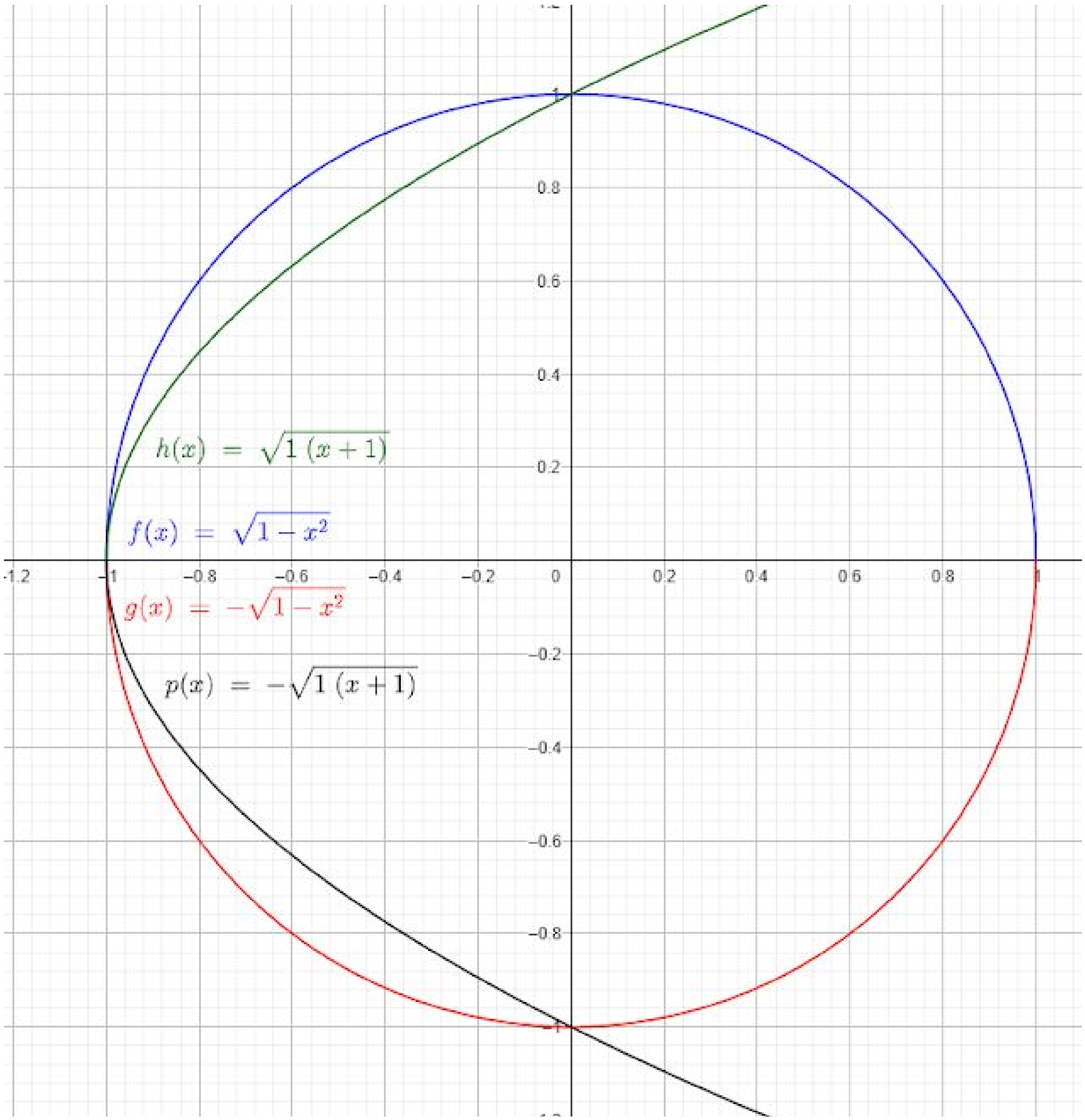
Circle - A closed figure where all points are equidistant from the center, this distance being denoted as the radius. Can be modeled through the equation $y^2 + x^2 = r;\ r \,\varepsilon\, R$, where r is the radius of the circle.

Unit circle - Circle of radius 1. (equation: $y^2 + x^2 = 1$)

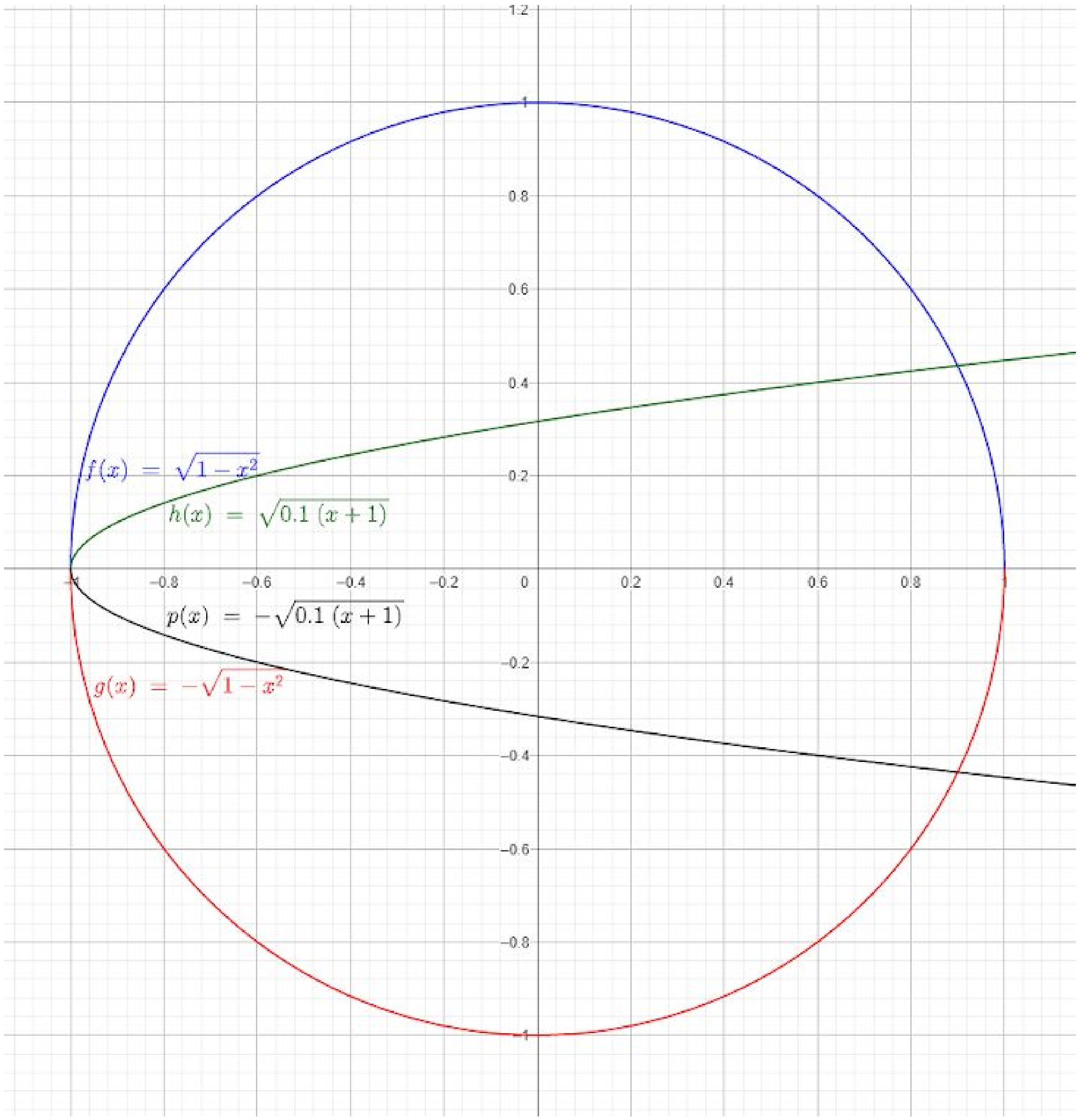
Arc of a curve - A section of a curve “cut” out through the selection of two points in a curve.

Arc length - The distance along an arc.

I will be solving the problem in the XY coordinate plane. The first equation that will be utilized is $y^2 + x^2 = 1$ and will be rewritten as $y = \pm\sqrt{1 - x^2}$. The curve can now be expressed as two separate functions, $f(x)$ and $g(x)$ where $f(x) = \sqrt{1 - x^2}$ and $g(x) = -f(x)$. Given that a parabola can be expressed in the form $x = ay^2 + by + c$, we can make a parabola $x = \frac{1}{\varepsilon}y^2 - 1$ and solve for y, $y = \pm\sqrt{\varepsilon(x + 1)}$. The curve for x_2 can now be expressed as two functions $h(x)$ and $p(x)$ where $h(x) = \sqrt{\varepsilon(x + 1)}$ and $p(x) = -h(x)$. For different values of ε, the graph would look as follows:



ε = 1 , *Graphed using Mathway*



ε = .1 , *Graphed using Mathway*

Solving the problem

The next step in finding an answer to the research question is calculating the x value for which a set of specific curves coincide. As it can be seen in the graph, no matter what the value of ε is, all curves will meet at point (-1,0). Functions $h(x)$ and $f(x)$ will meet above the x-axis at a point (x, y), while curves $p(x)$ and $g(x)$ will meet at point $(x, -y)$. In order to find what this point is for each set of functions I will set their expressions equal to each other and find the x value which completes the equation using the quadratic formula:

$$\sqrt{1-x^2} = \sqrt{\varepsilon(x+1)}$$

$$1 - x^2 = \varepsilon(x+1)$$

$$1 = x^2 + \varepsilon(x+1)$$

$$0 = x^2 + \varepsilon x + (\varepsilon - 1)$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(\varepsilon) \pm \sqrt{(\varepsilon)^2 - 4(1)(\varepsilon - 1)}}{2(1)}$$

$$x = \tfrac{1}{2}(-\varepsilon \pm \sqrt{\varepsilon^2 - 4\varepsilon - 4}))$$

$$x = \tfrac{1}{2}(-\varepsilon \pm \sqrt{(\varepsilon - 2)^2}$$

$$x = \tfrac{1}{2}(-\varepsilon \pm (\varepsilon - 2))$$

$$x = \tfrac{1}{2}(-\varepsilon + \varepsilon - 2),\ x = \tfrac{1}{2}(-\varepsilon - \varepsilon + 2)$$

$$x = \tfrac{1}{2}(-2),\ x = \tfrac{1}{2}(-2\varepsilon + 2)$$

$$x = -1,\ x = \tfrac{1}{2}(2(1 - \varepsilon))$$

$$x = -1,\ x = 1 - \varepsilon;\ \varepsilon\ \varepsilon\ ]0, 1]$$

This value is used to calculate the end points of the arc we will be finding the length of. The calculated value is our b in the formula $\int_{-1}^{1-\varepsilon} \sqrt{1 + (\frac{dy}{dx})^2} dx$ as it designates the point of contact between the parabola and the circle. Given that $p(x)$ is just a reflection of $h(x)$ along the x-axis, we can calculate the length of $h(x)$ from -1 to $1-\varepsilon$ and then double it in order to get the length of the parabola.

In order to calculate the arc length, the derivative of the parabolic curve has to be found in order to input it into the arc length formula. One can just use the power rule and chain rule in order to find $h'(x)$.

$$h(x) = \sqrt{\varepsilon(x+1)} = (\varepsilon x + \varepsilon)^{\frac{1}{2}}$$

$$h'(x) = \tfrac{1}{2}(\varepsilon x + \varepsilon)^{\frac{-1}{2}}(\varepsilon)$$

$$h'(x) = \frac{\varepsilon}{2\sqrt{\varepsilon(x+1)}}$$

$$\frac{dy}{dx} = \frac{\varepsilon}{2\sqrt{\varepsilon(x+1)}}$$

Once all this information is found, it can be plugged into the arc length formula:

$$\int_a^b \sqrt{1+\left(\tfrac{dy}{dx}\right)^2}\,dx$$

$$2\int_{-1}^{1-\varepsilon} \sqrt{1+\left(\frac{\varepsilon}{2\sqrt{\varepsilon(x+1)}}\right)^2}\,dx$$

$$I(\varepsilon) = 2\int_{-1}^{1-\varepsilon} \sqrt{1+\frac{\varepsilon}{4(x+1)}}\,dx$$

First the indefinite integral will be calculated, and then the definite integral will be calculated with the use of the indefinite integral using an integral calculator:

$$\text{Let } I_1 = 2\int \sqrt{1+\frac{\varepsilon}{4(x+1)}}\,dx$$

$$I_1 = \int \sqrt{4+\frac{\varepsilon}{(x+1)}}\,dx$$

$$\text{Let } u = -x-1,\ du = -dx$$

$$I_1 = -\int \sqrt{4 - \frac{\varepsilon}{u}}\,du$$

Let $v = \sqrt{4 - \frac{\varepsilon}{u}}$, $u = \frac{-\varepsilon}{v^2-4}$, $\frac{dv}{du} = \frac{1}{2}(4 - \frac{\varepsilon}{u})^{\frac{-1}{2}}(\frac{\varepsilon}{u^2})$

$$dv = \frac{\varepsilon}{2\sqrt{4-\frac{\varepsilon}{u}}u^2}du$$

$$du = \frac{2vu^2}{\varepsilon}dv = \frac{2v\varepsilon^2}{\varepsilon(v^2-4)^2}dv = \frac{2v\varepsilon}{(v^2-4)^2}dv$$

Then $I_1 = -\int \frac{2v^2\varepsilon}{(v^2-4)^2}dv = -2\varepsilon\int \frac{v^2}{(v^2-4)^2}dv$

Let $I_2 = \frac{I_1}{-2\varepsilon} = \int \frac{v^2}{(v^2-4)^2}dv$

$$I_2 = \int \frac{v^2+4-4}{(v^2-4)^2}dv = \int \frac{v^2-4}{(v^2-4)^2}dv + \int \frac{4}{(v^2-4)^2}dv = \int \frac{1}{v^2-4}dv + \int \frac{4}{(v^2-4)^2}dv$$

Let $I_3 = \int \frac{1}{v^2-4}dv$ and $I_4 = \int \frac{4}{(v^2-4)^2}dv$, $I_2 = I_3 + I_4$

$I_3 = \int \frac{1}{(v-2)(v+2)}dv$, which can be expanded using partial fraction expansion into,

$$I_3 = \int(\frac{1}{4(v-2)} - \frac{1}{4(v+2)})dv = \frac{1}{4}\int \frac{1}{v-2}dv - \frac{1}{4}\int \frac{1}{v+2}dv$$

$$I_3 = \frac{1}{4}(ln(v-2) - ln(v+2))$$

$$I_4 = 4\int \frac{1}{(v^2-4)^2}dv = 4\int \frac{1}{(v-2)^2(v+2)^2}dv$$

Once again, using partial fraction decomposition,

$$I_4 = 4\int(\frac{1}{32(v+2)} + \frac{1}{16(v+2)^2} - \frac{1}{32(v-2)} + \frac{1}{16(v-2)^2})dv$$

$$I_4 = \int \frac{4}{32(v+2)}dv + \int \frac{4}{16(v+2)^2}dv - \int \frac{4}{32(v-2)}dv + \int \frac{4}{16(v-2)^2}dv$$

$$I_4 = \frac{1}{4}(ln(v+2) - ln(v-2)) + \int \frac{4}{16(v+2)^2}dv + \int \frac{4}{16(v-2)^2}dv$$

$$I_4 = \frac{1}{4}ln(\frac{v+2}{v-2}) + \int \frac{4}{16(v+2)^2}dv + \int \frac{4}{16(v-2)^2}dv$$

where $\frac{1}{4}\int \frac{1}{(v+2)^2}dv$ and $\frac{1}{4}\int \frac{1}{(v-2)^2}dv$ will be defined as I_5 and I_6 respectively.

$$I_5 = \frac{1}{4}\int \frac{1}{(v+2)^2}dv \text{, let } w = v+2\text{, } \frac{dw}{dv} = 1\text{, } dw = dv$$

$$I_5 = \frac{1}{4}\int \frac{1}{w^2}dw = -\frac{1}{4w} = -\frac{1}{4(v+2)}$$

$$I_6 = \frac{1}{4}\int \frac{1}{(v-2)^2}dv\text{, let } w = v-2\text{, } \frac{dw}{dv} = 1\text{, } dw = dv$$

$$I_6 = \frac{1}{4}\int \frac{1}{w^2}dw = -\frac{1}{4w} = -\frac{1}{4(v-2)}$$

$$I_4 = \frac{1}{4}ln(\frac{v+2}{v-2}) - \frac{1}{4(v+2)} - \frac{1}{4(v-2)}$$

$$I_2 = I_3 + I_4 = \frac{1}{4}(ln(\frac{v-2}{v+2}) + ln(\frac{v+2}{v-2})) - \frac{1}{4(v+2)} - \frac{1}{4(v-2)}$$

$$I_1 = -2\varepsilon(\frac{1}{4}(ln(\frac{v-2}{v+2}) + ln(\frac{v+2}{v-2})) - \frac{1}{4(v+2)} - \frac{1}{4(v-2)})$$

$$I_1 = -2\varepsilon(\frac{1}{4}(ln(\frac{\sqrt{4+\frac{\varepsilon}{+x+1}}-2}{\sqrt{4+\frac{\varepsilon}{x+1}}+2}) + ln(\frac{\sqrt{4+\frac{\varepsilon}{x+1}}+2}{\sqrt{4+\frac{\varepsilon}{x+1}}-2})) - \frac{1}{4(\sqrt{4+\frac{\varepsilon}{x+1}}+2)} - \frac{1}{4(\sqrt{4+\frac{\varepsilon}{x+1}}-2)})$$

$$I(\varepsilon) = -2\varepsilon(\frac{1}{4}(ln(\frac{\sqrt{4+\frac{\varepsilon}{+x+1}}-2}{\sqrt{4+\frac{\varepsilon}{x+1}}+2}) + ln(\frac{\sqrt{4+\frac{\varepsilon}{x+1}}+2}{\sqrt{4+\frac{\varepsilon}{x+1}}-2})) - \frac{1}{4(\sqrt{4+\frac{\varepsilon}{x+1}}+2)} - \frac{1}{4(\sqrt{4+\frac{\varepsilon}{x+1}}-2)})$$

Now that $I(\varepsilon)$ has been properly defined, an ε that satisfies $I(\varepsilon) > 4$ can be found.

ε	$I(\varepsilon)$
0.02878645574	4.00000000001
0.02878645575	4.00000000001
0.02878645576	4

(Desmos Graphing Calculator)

From the table it can be concluded that for $\varepsilon = 0.02878645575$, the length of the curves

$\pm\sqrt{\varepsilon(x+1)}$ have a length greater than 4 contained inside the unit circle. Hence, it is possible to fit a parabola inside a unit circle with length greater than 4.

The Putnam Approach

The solution provided by the University of Hawaiʻi at Mānoa utilizes a distinct approach, one utilizing similar methods but much more fitting to an exam setting. The solution begins by describing the circle $x^2 + (y-1)^2 = 1$ and the parabola $y = \frac{1}{2}cx^2$ in order to solve the problem. After substituting $y = \frac{1}{2}cx^2$ into $x^2 + (y-1)^2 = 1$ and finding the points of intersection, the results are $x = 0,\ x = \pm\frac{2\sqrt{c-1}}{c}$ which are used for the previously defined integral $\int_a^b \sqrt{1 + (\frac{dy}{dx})^2}dx$.

The new arc length formula will look as follows:

$$\int_0^{\frac{2\sqrt{c-1}}{c}} \sqrt{1 + c^2x^2}dx$$

In order to solve this integral, the substitution $u = cx$ is used, implying $du = c \cdot dx$.

$$\frac{1}{c}\int_0^{2\sqrt{c-1}} \sqrt{1+u^2}du\text{, let } z = sinh(u)$$

$$[\frac{1}{c}(\frac{1}{2}arcsinh(u) + \frac{1}{2}u\sqrt{1+u^2}]_0^{2\sqrt{c-1}}$$

$$\frac{1}{2c}arcsinh(2\sqrt{c-1}) + 2\sqrt{1 - \frac{7}{4c} + \frac{3}{4c^2}}$$

$$2\sqrt{1 - \frac{7}{4c} + \frac{3}{4c^2}} > 2\sqrt{1 - \frac{7}{4c}} \approx 2(1 - \frac{1}{2}(\frac{7}{4c})) > 2 - \frac{2}{c} \text{ for } c \text{ sufficiently large.}$$

Proving $\frac{1}{2c}arcsinh(2\sqrt{c-1}) > \frac{2}{c}$ would then prove that there is a length greater than 4 as

$$\frac{1}{c}\int_0^{2\sqrt{c-1}} \sqrt{1+u^2}du = \frac{1}{2c}arcsinh(2\sqrt{c-1}) + 2\sqrt{1-\frac{7}{4c}+\frac{3}{4c^2}} > \frac{1}{2c}arcsinh(2\sqrt{c-1}) + 2 - \frac{2}{c}$$

Therefore if $\frac{1}{2c}arcsinh(2\sqrt{c-1}) + 2 - \frac{2}{c} > 2$ then $\frac{1}{c}\int_0^{2\sqrt{c-1}} \sqrt{1+u^2}du > 2$

$$\frac{1}{2c}arcsinh(2\sqrt{c-1}) - \frac{2}{c} > 0$$

$$\frac{1}{2c}arcsinh(2\sqrt{c-1}) > \frac{2}{c}$$

(Putnam 5)

Further simplifying the provided material, I can find possible values for c:

$$\sqrt{c-1} > \frac{sinh(4)}{2}$$

$$c-1 > \frac{sinh^2(4)}{4}$$

$$c > \frac{sinh^2(4)}{4} + 1$$

$$c > 187.184895157$$

So for $c = 188$, $\int_0^{2\sqrt{c-1}} \sqrt{1+c^2x^2}dx = 2.001335148211$

$$2\int_0^{2\sqrt{c-1}} \sqrt{1+c^2x^2}dx = 4.002670296421$$

Which proves that there can be a parabola with arc length greater than 4 inside a circle of radius 1.

This solution is clearly much faster and effective during the 30 minute period provided in the Putnam exam. Considering this alone makes it a significantly better solution to the one provided earlier in the paper. Anyhow, both attempts suffer from one large issue: the lack of calculators during the Putnam. Solving the original question either way would result in the need of scientific or graphing calculators in order to prove the overarching question; whether the arc can have a length greater than 4.

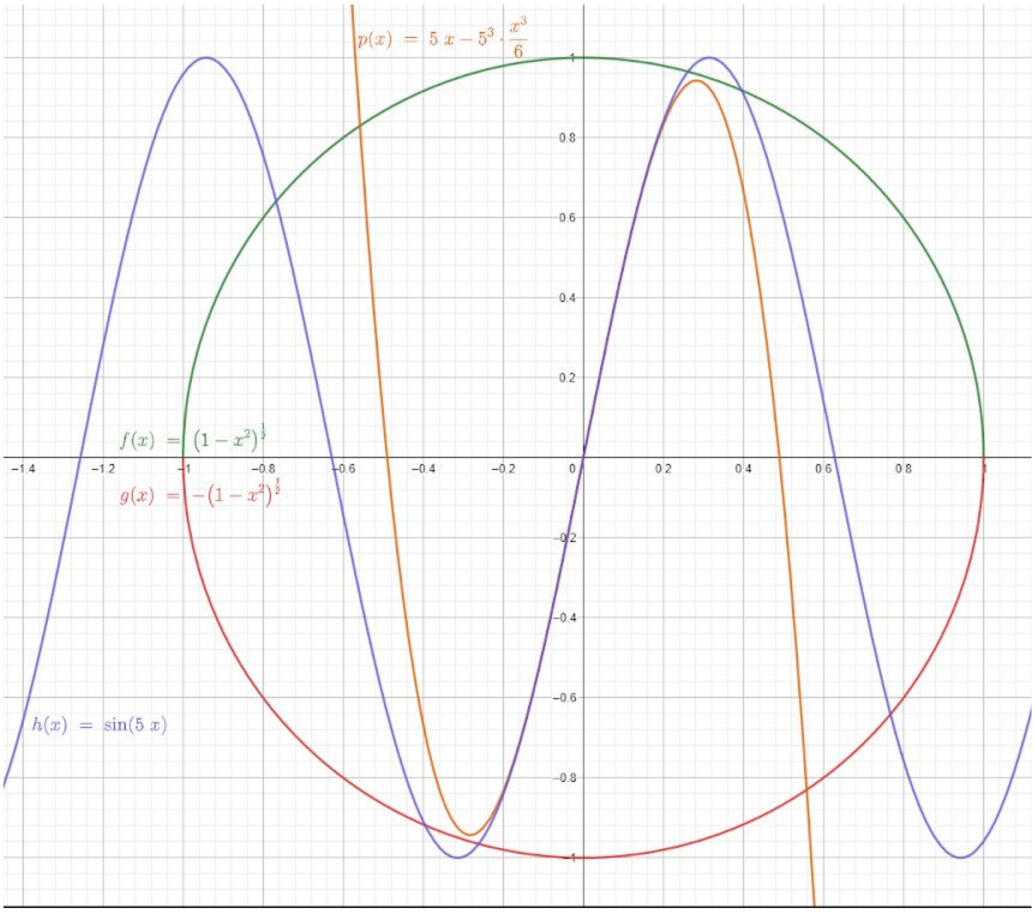
Expanding the Question

The next step in my research was analyzing this question with slightly altered parameters. This question can be stated as “Can an arc of a third degree curve inside a circle of radius 1 have a length greater than 6?” For this expansion in my research an additional term will be defined.

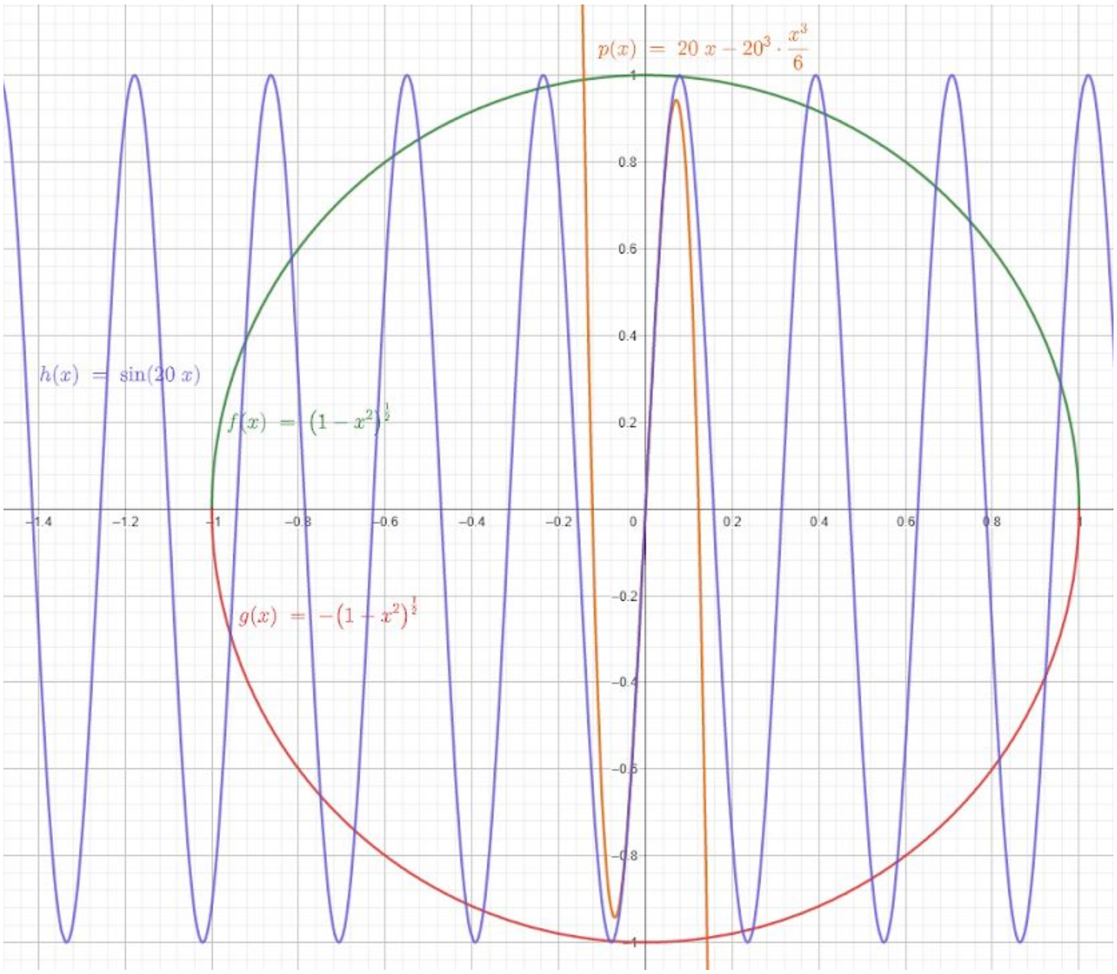
Third degree curve - A curve expressed in the XY plane with a polynomial of third degree: $y = ax^3 + bx^2 + cx^1 + d;\ a, b, c, d \ \varepsilon\ R$, or alternatively $x = ay^3 + by^2 + cy + d;\ a, b, c, d \ \varepsilon\ R$.

Solving the expansion

Creating a third degree curve which can fit inside of a unit circle can be done with the third degree maclaurin approximation for the sine curve. The curve $h(x) = sin(ax)$ contains a section between the first minimum and maximum of the function which resembles a third degree polynomial.



a = 5 *, Graphed using Mathway*



$a = 20$, *Graphed using Mathway*

Hence, the third degree maclaurin approximation for $h(x)$ will be used. The resulting function, $p(x) = ax - \frac{a^3x^3}{3!}$, is shown above for $a = 5$ and $a = 20$. As a tends to infinity, the space between the succeeding maximums of $f(x)$ will become infinitely small. This same property applies to the local maximum and minimum of $p(x)$. Hence, the value for solutions to the equation $p(x) = f(x),\ f(x) = \sqrt{1 - x^2}$ will tend towards (0,1) and (0,-1). However, the curve $p(x)$ has its local maximum and minimum "shortened" causing the local maximum and minimum y-values to tend towards the origin rather than (0,1) and (0,-1).

In order to avoid this, a new yet similar formula will be made for $p(x)$, $p(x) = ax - \frac{a^3x^3}{n(a)}$.

The objective of this new definition is finding a constant $n(a)$ which alters the function $p(x)$ in order for its local maximum and minimum to intersect the unit circle for all real a. This can be done by finding the maximum and minimum of $p(x)$

$$p'(x) = 0$$

$$a - \frac{3a^3x^2}{n(a)} = 0$$

$$x_{max} = \sqrt{\frac{n(a)}{3a^2}},\ x_{max}{}^2 = \frac{n(a)}{3a^2}$$

And then finding the intersection of $p(x)$ and $f(x)$ at $x = x_{max}$.

$$\sqrt{1-x^2} = ax - \frac{a^3x^3}{n(a)}$$

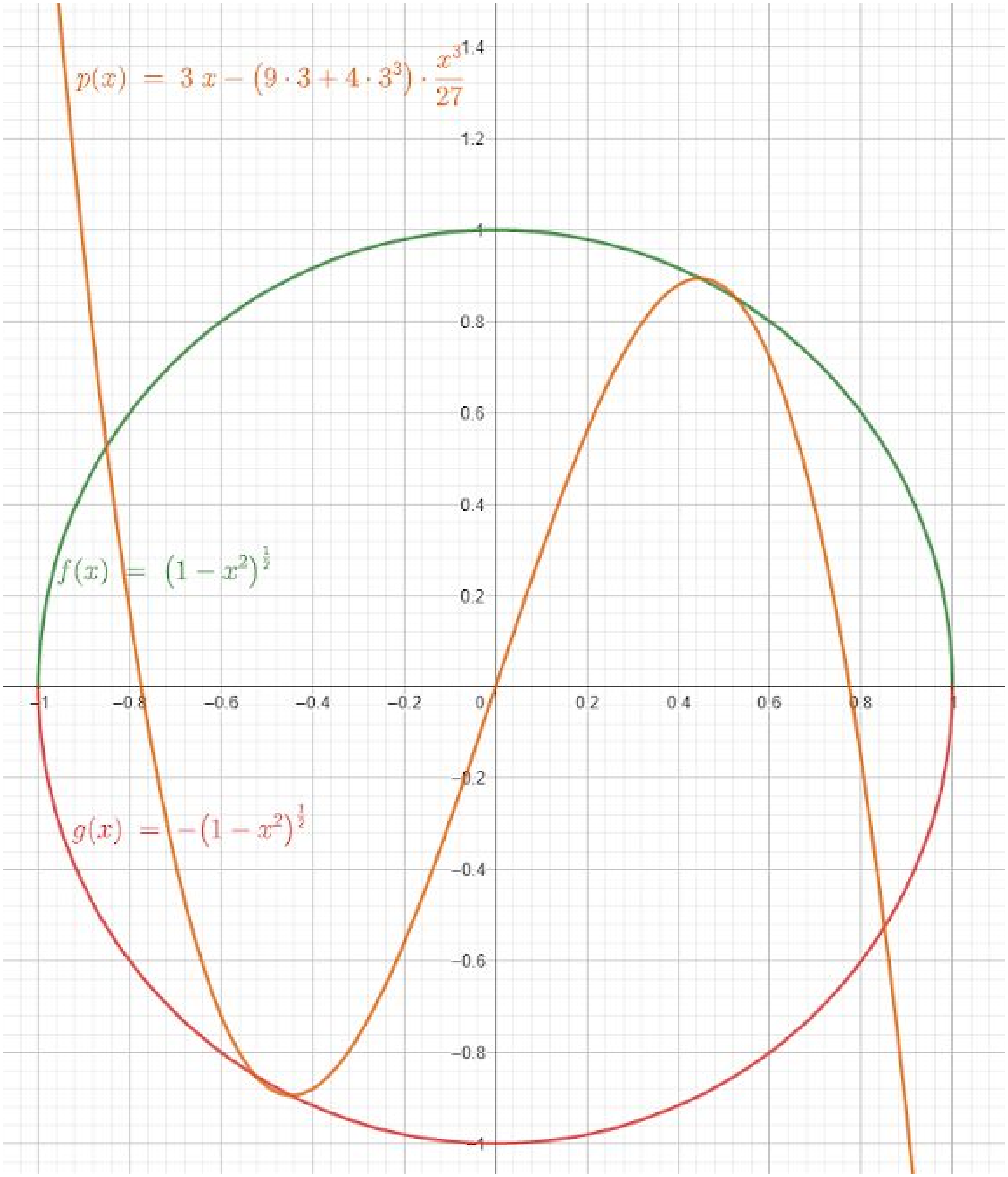
$$1 - x^2 = a^2x^2 - \frac{2a^4x^4}{n(a)} + \frac{a^6x^6}{n(a)^2}$$

$$1 - \frac{n(a)}{3a^2} = \frac{n(a)}{3} - \frac{2n(a)}{9} + \frac{n(a)}{27}$$

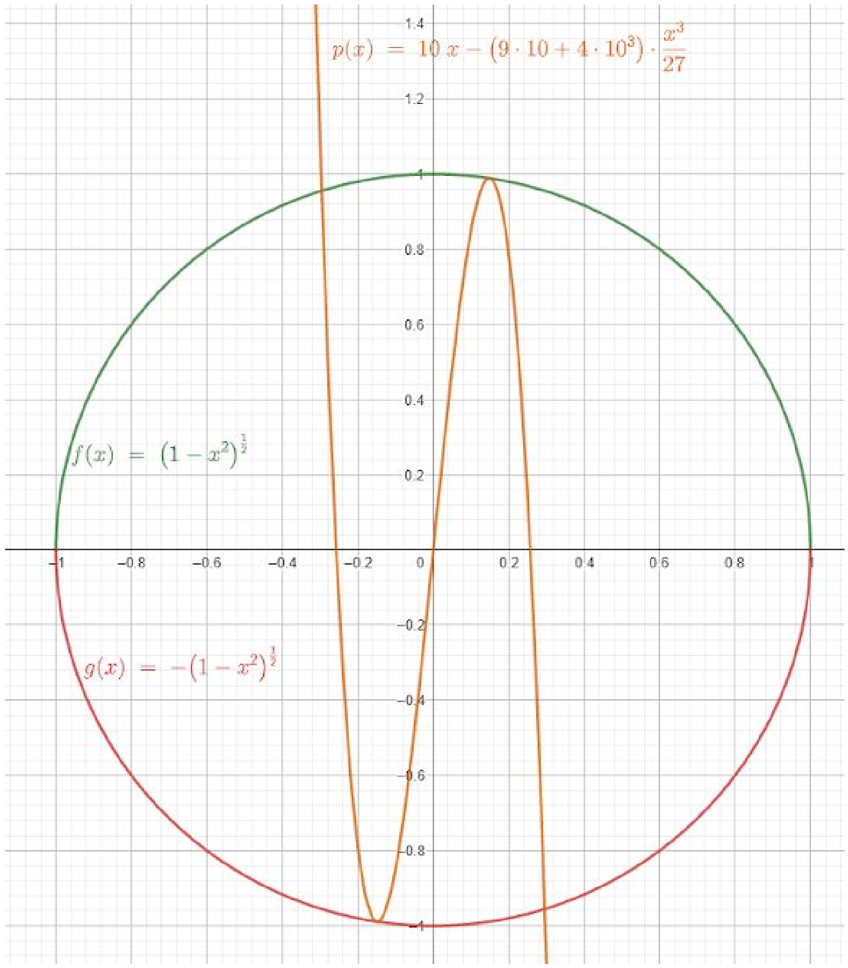
$$n(a) = \frac{27a^2}{9+4a^2}$$

$$\text{Therefore, } p(x) = ax - \frac{(9a+4a^3)x^3}{27}$$

Now p(x) behaves differently and has its local maximum and minimum intersecting the circle for $a \neq 0$:

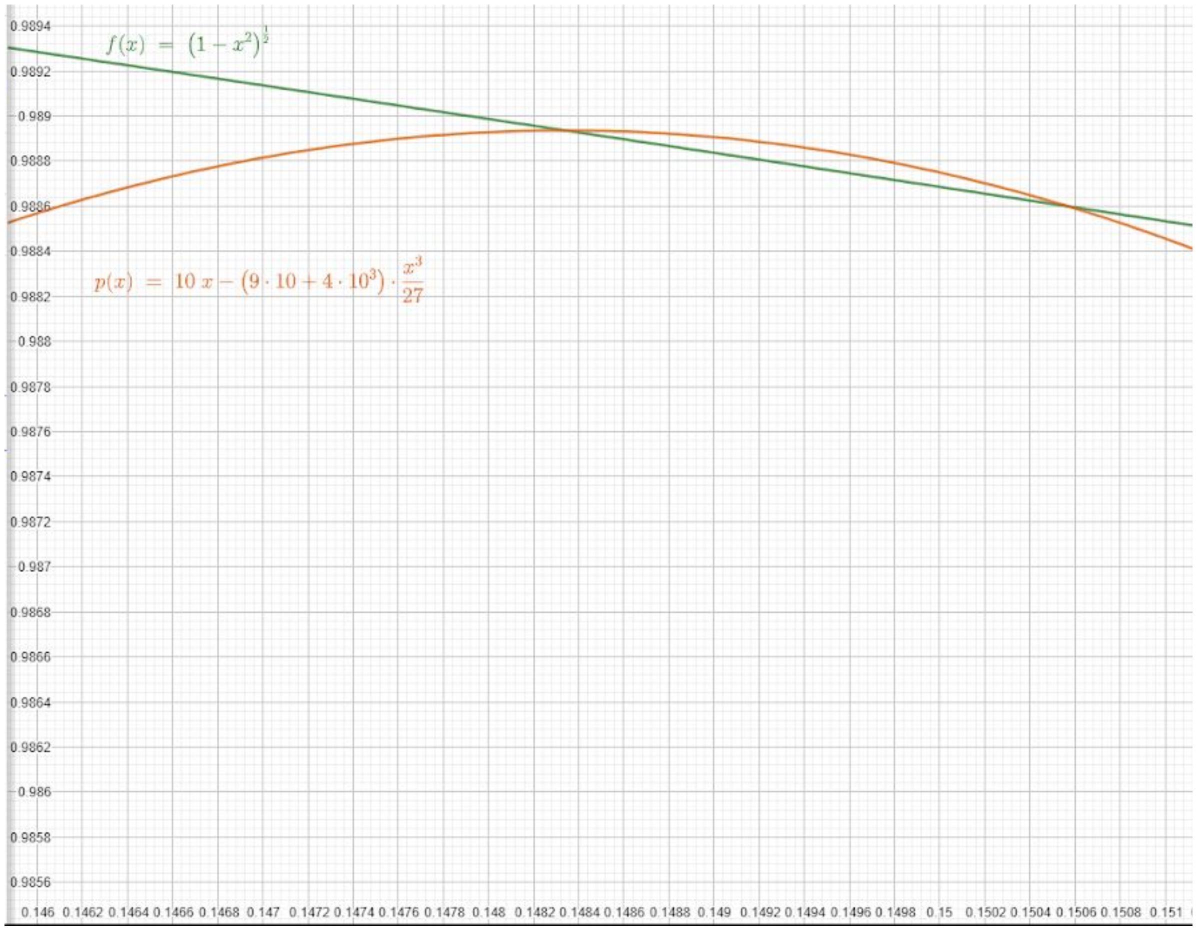


a = 3 *, Graphed using Mathway*



$a = 10$, Graphed using Mathway

One more issue that needs to be addressed before continuing is the fact that p(x) extends outside the circle for $a \neq 0$, $a \varepsilon R$ as seen below.



$a = 10$, *Graphed using Mathway*

Thus, $j(x)$ will be defined as $j(x) = \frac{a^5}{(a^5+97)}p(x)$. This slight modification scales the function down, however as a tends to infinity, the scaling factor tends to 1, creating a scalar which only slightly shortens the function at large a. The values 5 and 97 were acquired through trial and error.

Now that the function $j(x)$ has been defined, the length can be determined with the function mentioned before:

$$\int_a^b \sqrt{1+(\frac{dy}{dx})^2}dx$$

As the point of intersection is not known, and as finding it would require solving a sextic equation, a different approach is going to be taken. The integral

$$I(b) = 2\int_0^b \sqrt{1 + j'(x)^2}\,dx$$

is greater than 6 for a $b \geq 0.03332801$, $a = 90$. This value was found through trial and error and could have a slightly lesser bound if more accurate solutions are found. Below is a table showing certain b values along with the approximate integral value.

b	$I(b)$
0.03332798	5.99998449983
0.03332799	5.99998989956
0.03332800	5.99999529929
0.03332801	6.00000069902
0.03332802	6.00000609876
0.03332803	6.0000114985

(Desmos Graphing Calculator)

These values were picked by approximating the intersection between $j(x)$ and the lower half of the unit circle.

Anyhow, for the functions $j(x) = \frac{a^{10}}{(a^{10}+1)}p(x)$ and $f(x) = \sqrt{1 - x^2}$, $j(b) < f(b)$ for $b = 0.002999996515531$. Hence, as $j(x)$ has not yet intersected $f(x)$, and as the value for the calculated integral is greater than 6, it can be concluded that a cubic function inside of a unit circle can be of length greater than 6.

Evaluation

The original question and the extension have been answered successfully with a "yes" in both accounts. A further extension was going to be applied regarding the question "Can an arc of an nth degree polynomial of length greater than 2n fit inside of a circle of radius one?" The original work for this question was fruitless due to the extensive nature of the problem. Generalizing the answer given for the parabola in order to acquire a concrete answer for polynomials of degree n, $n \varepsilon Z^{+} > 1$ is extremely difficult and requires exact mathematics beyond the scope of this research paper and beyond the scope of our current mathematical axioms. The Abel-Ruffini theorem, or Abel's impossibility theorem, proves the insolvability of quintic equations (and higher) through the use of radicals (Brown 7). This causes the extensions to fall apart quickly as they require the solving of sextic, septic, and further polynomials. The issue arises even when trying to solve the extended question "Can an arc of a third degree polynomial of length greater than six fit inside a circle of radius 1?" in a more efficient manner.

The original solution to the problem relied on setting the derivatives of $p(x) = ax - \frac{a^3x^3}{n(a)}$ and $h(x) = \sqrt{1-x^2}$ equal to each other:

$$p'(x) = a - \frac{3a^3x^2}{n(a)},\ h'(x) = -\frac{x}{\sqrt{1-x^2}}$$

$$a - \frac{3a^3x^2}{n(a)} = -\frac{x}{\sqrt{1-x^2}}$$

And then solving for x in order to find the appropriate $n(a)$ for the original equation (in order to have $p(x)$ and $h(x)$ tangent to each other when their slopes are equal for every real a).

$$a\sqrt{1-x^2} + x - \frac{3a^3x^2\sqrt{1-x^2}}{n(a)} = 0$$

Which is unsolvable through radicals as proven by Abel and Ruffini.

Applying the same methodology for curves of higher degrees becomes increasingly challenging, topping an already nearly impossible problem. Other approaches were thought of anyhow, especially for curves of degree $2n,\ n\varepsilon Z^{+}$. The underlying idea behind solving even degree polynomials is to create a curve which has two seemingly straight lines down the center of the circle (creating a length significantly close to two) and then the original curves from the parabola found in question 1. The extra length gained from the curvature of the parabolas should (ideally) make up from the difference between the pseudo-straight arcs and true diameter chords. The issue with this solution arises when defining the formula for the quartic, sextic, and higher power polynomials. Fourth degree polynomial curves have equi-distant maxima and minima. This can be seen by simply attempting to calculate the solutions to the derivative of such a polynomial (which would result in a third degree curve) which has equidistant roots (for curves lacking an x^2 term, which would deem them asymmetrical). The proof lies within the roots of a cubic equation, such as the one provided by Professor Schechter (Schechter 1).

$$\begin{aligned} x \quad = \quad & \sqrt[3]{\left(\frac{-b^3}{27a^3}+\frac{bc}{6a^2}-\frac{d}{2a}\right)+\sqrt{\left(\frac{-b^3}{27a^3}+\frac{bc}{6a^2}-\frac{d}{2a}\right)^2+\left(\frac{c}{3a}-\frac{b^2}{9a^2}\right)^3}} \\ + \quad & \sqrt[3]{\left(\frac{-b^3}{27a^3}+\frac{bc}{6a^2}-\frac{d}{2a}\right)-\sqrt{\left(\frac{-b^3}{27a^3}+\frac{bc}{6a^2}-\frac{d}{2a}\right)^2+\left(\frac{c}{3a}-\frac{b^2}{9a^2}\right)^3}} \quad - \quad \frac{b}{3a}. \end{aligned}$$

Roots of a cubic equation

The three possible roots can be found by switching the + and - within the cubic roots which separate the square root term and the terms inside the parentheses. In our case, b and d are both equal to zero, meaning the equation tremendously simplifies to

$$x = \sqrt[3]{\pm\sqrt{(\tfrac{c}{3a})^3}} + \sqrt[3]{\pm\sqrt{(\tfrac{c}{3a})^3}}$$

Giving the roots

$$x = -2\sqrt[2]{\tfrac{c}{3a}}, x = 0, x = 2\sqrt[2]{\tfrac{c}{3a}}$$

This identity of our specific cubic equations shows the distance between the maxima and minima to be constant for any a and c in the equation $y = ax^3 - cx$. Such a property makes it impossible to shift the slopes of the function without shifting the local extrema themselves. This results in an equation incapable of containing two "straight" diameters and the curvature of the parabola when expressed as a cubic function.

The difficulty of the question does not warrant its solution, however it displays a certain importance to its discovery and the work placed upon it. Bewildering questions such as this one, and their numerous extensions and applications, display the nature of mathematics and its ability to constantly develop and expand its ever growing boundaries. The mere fact that a single question can be seen through many different mathematical fields, such as multivariable calculus or algebraic geometry, act as a testimony for this very idea, how questions and problems are the origin and inevitable impetus on mathematical advancements.

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