

2/7/25

- Exam 1 content up until (including) elementary matrices

Recall: A $n \times n$, A^{-1} $n \times n$, $A^{-1}A = I = A^{-1}A$

useful:

$A\vec{x} = \vec{b}$, if a is invertible \downarrow is a solution

$$\hookrightarrow A^{-1}A\vec{x} = A^{-1}\vec{b} \Rightarrow \vec{x} = A^{-1}\vec{b}$$
$$\parallel$$
$$I\vec{x}$$

Prop: If A $n \times n$ w/ $AB = I$, then A can be row reduced to I .

Pf. Since $AB = I$, $A\vec{x} = \vec{b}$ always has the solution $\vec{x} = B\vec{b}$. Thus, A must have a pivot in every row/col $\Rightarrow A$ row reduces to I

Check: $\vec{x} = B\vec{b}$
 $A\vec{x} = \vec{b}$ \swarrow plugin
 $AB\vec{b} = \vec{b}$ \checkmark
 $I\vec{b} = \vec{b}$

□

Thm If A and B are $n \times n$, then

$$AB = I \iff BA = I$$

Pf. Suppose $AB = I$. Then A can be row reduced to I , so there is a matrix C (a product of elementary matrices) w/ $CA = I$

$$B = IB = CAB = CI = C$$

$$\text{so } B = C, \text{ so } BA = I \quad \square$$

Corollary: If A is $n \times n$, $AB = I \iff A$ is invertible with $B = A^{-1}$

Example: Every elementary matrix is invertible with an inverse that is also an elementary matrix

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ has inverse } \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_2 \leftrightarrow R_1 \quad (\text{undo actions})$$

$$E_2 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ has inverse } \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \frac{1}{3} R_1$$

$$E_3 = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ has inverse } \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_1 - 4R_3$$

Thm $(AB)^{-1} = B^{-1}A^{-1}$

Pf: $(AB)(B^{-1}A^{-1})$
(associative) $= A(BB^{-1})A^{-1}$
 $= AIA^{-1} = AA^{-1} = I. \quad \square$

To find A^{-1}

Find a matrix B whose columns are solutions to $A\vec{x} = \vec{e}_i$. If such a matrix exists, then A is invertible with $A^{-1} = B$. Otherwise, A is not invertible.

Pf. $A \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix} = \begin{bmatrix} A\vec{v}_1 & \cdots & A\vec{v}_n \end{bmatrix}$

If $A\vec{v}_i = \vec{e}_i$ for all i , then $A[\vec{v}_1 \cdots \vec{v}_n] = I$.

Otherwise, $A\vec{x} = \vec{e}_i$ does not have a solution for some i ,
so A does not have a pivot in every column,
so not invertible.

↓
Practically Speaking: Given A ,

① put in form $[A|I]$

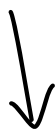
② row reduce to $[I|A^{-1}]$ (if possible; if not, A is not invertible)

Ex: $A = \begin{bmatrix} 2 & 7 \\ 1 & 3 \end{bmatrix}$ Find A^{-1}

Soln. $\begin{bmatrix} 2 & 7 & | & 1 & 0 \\ 1 & 3 & | & 0 & 1 \end{bmatrix} \xrightarrow{R_2 - \frac{1}{2}R_1} \begin{bmatrix} 2 & 7 & | & 1 & 0 \\ 0 & -\frac{1}{2} & | & -\frac{1}{2} & 1 \end{bmatrix}$

$\xrightarrow{\frac{1}{2}R_1, -2R_2} \begin{bmatrix} 1 & \frac{7}{2} & | & \frac{1}{2} & 0 \\ 0 & 1 & | & 1 & -2 \end{bmatrix} \xrightarrow{R_1 - \frac{7}{2}R_2} \begin{bmatrix} 1 & 0 & | & -3 & 7 \\ 0 & 1 & | & 1 & -2 \end{bmatrix}$

$$A^{-1} = \begin{bmatrix} -3 & 7 \\ 1 & -2 \end{bmatrix}$$



Thm The following are equivalent for an $n \times n$ matrix A :

i) A is invertible

ii) $\text{RREF}(A) = I$

iii) $A\vec{x} = \vec{b}$ has a solution for all \vec{b}

iv) A is a product of elementary matrices

v) The span of the columns of A is \mathbb{R}^n