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Homework: Assignment 4
STAT/CS 387

Problem 1

Counterfactuals. Recall the example population datasets from class:

X	Y	C_0	C_1
0	0	0	0*
0	0	0	0*
0	0	0	0*
0	0	0	0*
1	1	1*	1
1	1	1*	1
1	1	1*	1
1	1	1*	1

Where asterisks denote unobserved values. In class we showed for this example that the average causal effect $\theta = E[C_1] - E[C_0] = 0$ and that the association $\alpha = E[Y|X=1]$ - $E[Y|X=0] = 1$,i.e., $\theta \neq \alpha$.

Create an example like this one in which $\alpha > 0$ and $\theta < 0$. (include the computation of α and θ for your example.) What is the "intuition" of your example?

1.1 Solution

In this problem, we are examining how subjects were given or not given a treatment and subsequently elicited or did not elicit a response. We symbolize the potential treatments(X) and responses(Y) as :

$$X = \begin{cases} 1 & \text{if treated} \\ 0 & \text{if not treated} \end{cases} \quad Y = \begin{cases} 1 & \text{if responded} \\ 0 & \text{if not responded} \end{cases}$$

We also introduce the notion of counterfactuals to help distinguish the difference between an "association between X and Y and a causal relationship. Counterfactuals are the unobservable alternate potential outcomes. For example, in reality, if patient i gets the treatment and has a response, to know the effect of the treatment, we would like to know the response of the same patient if they were never given the treatment. The counterfactual represents this unknown response from a patient. This would be possible only if we had access to a parallel universe. In this example there are two random variables that belong to response distributions. C_0 in which the subject is not treated, $x = 0$ and C_1 the opposite case. Thees have direct relationships with response Y.

$$C_1 = \begin{cases} 1 & \text{response observed} \\ 0 & \text{no response} \end{cases} \quad C_2 = \begin{cases} 1 & \text{response observed} \\ 0 & \text{no response} \end{cases}$$

This may seem useless because we will never know one of the counterfactuals. However it allows us to illustrate how association cannot be used to estimate casual effect. For example, if we have the following treatment response data set:

X	Y	C_0	C_1
1	0	0*	0
1	0	0*	0
1	1	0*	1
1	1	0*	1
-	-	-	-
0	0	0	1*
0	0	0	1*
0	0	0	1*
0	0	0	1*

By only seeing the true data, we would think that the effect of the treatment is unclear likely deny any causal relationship between treatment and response. There is not a strong association out front. However, by including the counterfactual in this analysis, we can see one example situation in which the predisposition of the patients masked the outcome response and that there is fact a slight causal relationship present.

$$\theta = E[C_1] - E[C_0] = \frac{1}{8}(6) - (0) = \frac{3}{4} \tag{1}$$

$$\alpha = E[Y|X = 1] - E[Y|X = 0] = \frac{1}{2} - 0 = \frac{1}{2} \tag{2}$$

In this example, we see that the only people that showed no response to the treatment were "doomed" to begin with and wouldn't be helped in either case. This is sort of the opposite case of the example in class, we are possibly falsely rejecting the effects of the treatment.

2 Problem 2

Suppose the variables X,Y, and Z have the following joint distribution:

	Z = 0		Z = 1	
	Y = 0	Y = 1	Y = 0	Y = 1
X = 0	0.405	0.045	0.125	0.125
X = 1	0.045	0.005	0.125	0.125

- Find the conditional distribution of X and Y given Z = 0 and conditional distribution of X and Y given Z = 1.
- Show that X $\perp\!\!\!\perp Y|Z$.
- Find the marginal distribution of X and Y.
- Show that X and Y are not mutually independent.

2.1 Solutions

(2.1) The conditional distributions are shown in the following tables:

Table 1: Given Z = 0 and Z = 1

	Y = 0	Y = 1
X = 0	0.81	0.09
X = 1	0.09	0.01

	Y = 0	Y = 1
X = 0	0.25	0.25
X = 1	0.25	0.25

(2.2) To prove that X and Y are independent given Z, we exhaustively show that $f((x,y)|z) = f(x|z)f(y|z)$.

$$\begin{aligned}
&\text{Conditioning on Z = 0}\\
f_{x,y|z=0}(0,0) = 0.81 \text{ and } f_{x|z=0}(0) * f_{y|z=0}(0) &= (0.9)(0.9) = 0.81\\
f_{x,y|z=0}(0,1) = 0.09 \text{ and } f_{x|z=0}(0) * f_{y|z=0}(1) &= (0.9)(0.1) = 0.09\\
f_{x,y|z=0}(1,0) = 0.09 \text{ and } f_{x|z=0}(1) * f_{y|z=0}(0) &= (0.1)(0.9) = 0.09\\
f_{x,y|z=0}(1,1) = 0.01 \text{ and } f_{x|z=0}(1) * f_{y|z=0}(1) &= (0.1)(0.1) = 0.01\\
&\text{Conditioning on Z = 1}\\
f_{x,y|z=1}(0,0) = 0.25 \text{ and } f_{x|z=1}(0) * f_{y|z=1}(0) &= (0.5)(0.5) = 0.25\\
f_{x,y|z=1}(0,1) = 0.25 \text{ and } f_{x|z=1}(0) * f_{y|z=1}(1) &= (0.5)(0.5) = 0.25\\
f_{x,y|z=1}(1,0) = 0.25 \text{ and } f_{x|z=1}(1) * f_{y|z=1}(0) &= (0.5)(0.5) = 0.25\\
f_{x,y|z=1}(1,1) = 0.25 \text{ and } f_{x|z=1}(1) * f_{y|z=1}(1) &= (0.5)(0.5) = 0.25
\end{aligned}$$

(2.3) The marginal distribution of X and Y is attained by summing X and Y between the two conditional tables.

Table 2: Marginal Distribution of X and Y

	Y = 0	Y = 1
X = 0	0.53	0.17
X = 1	0.17	0.13

(2.4) To prove that X and Y are not marginally independent, we will show one contradiction in the assumption that they are in fact independent.

$$f_{x,y}(1,1) = 0.13 \text{ and } f_x(1) * f_y(1) = (0.3)(0.3) \neq 0.13$$

3 Problem 3

Consider the following DAG, called a collider:

$$X \rightarrow Y \leftarrow Z$$

Prove that $X \perp\!\!\!\perp Z$ and that X and Z are dependent given Y. Use these results to interpret the meaning of a collider.

3.1 Solution

We want to show that $f(x,z) = f(x)f(z)$.

$$f(x,y) = \sum_y f(x,y,z) = f(x)f(z)\sum_y f(y|x,z) \tag{3}$$

$$f(x,z) = f(x)f(z) \tag{4}$$

Now we want to show that X and Z are dependent given Y. We show this by proving that $P((x,z)|y) \neq p(x|y)p(z|y)$.

$$P((x,z)|y) \neq p(x|y)p(z|y)$$

$$\frac{p(x,y,z)}{p(y)} \neq p(x|y)p(z|y)$$

Substituting in the Dag to LHS

$$\frac{p(z)p(x)p(y|(x,z))}{p(y)} \neq p(x|y)p(z|y)$$

Now, assuming that X and Z are independent given Y, the above equality would resolve

$$p((x,z)|y)(\frac{p(y|(x,z))}{p(y)}) \neq p((x,z)|y)$$

However, the equality does not resolve, thus X and Z are dependent given Y. This DAG represents a variable Y that is influenced by two other variables X and Z. It is said that these influences collide with one another.

4 Problem 4

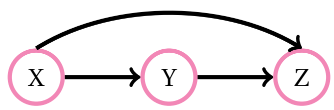
Let V = (X,Y,X) be distributed as follows:

$$X \sim \text{Bernoulli}(\frac{1}{2}) \tag{5}$$

$$Y|X = x \sim \text{Bernoulli}(\frac{^{4x-2}}{1+^{4x-2}}) \tag{6}$$

$$Z|X = x, Y = y \sim \text{Bernoulli}(\frac{e^{2(x+y)-2}}{1+e^{2(x+y)-2}}) \tag{7}$$

(1) Make a diagram showing the DAG corresponding to this model.



(2) Derive a mathematical expression for Pr(Z=z|Y=y). What is Pr(Z = 1|Y = 1)?

$$\begin{aligned}
Pr(Z = z|Y = y) &= \frac{Pr(Y = y, Z = z)}{Pr(Y = y)} \\
&= \frac{f(y,z)}{f(y)} \\
&= \frac{\sum_x f(x,y,z)}{f(y)} \\
&= \frac{\sum_x f(x)f(y|x)f(z|x,y)}{f(y)}
\end{aligned}$$

From the DAG we know that $\frac{f(x)f(Y|X)}{f(y)} = \frac{f(x,y)}{f(y)} = f(x|y)$

$$= \sum_x f(z|x,y)f(x|y)$$

To find Pr(Z = 1 | Y = 1), we perform the summations as such:

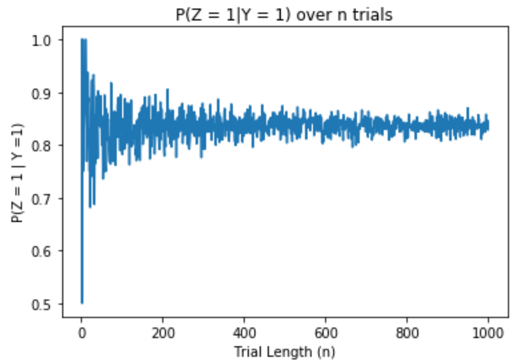
$$\frac{f_{y|x}(1|0)f_{z|x,y}(1|0,1) + f_{y|x}(1|1)f_{z|x,y}(1|1,1)}{f_{y|x}(1|0) + f_{y|y}(1|1)}$$

$$\frac{\frac{e^{-2}}{1+e^{-2}}\frac{1}{2} + \frac{e^2}{1+e^2}\frac{e^2}{1+e^2}}{\frac{e^{-2}}{1+e^{-2}} + \frac{e^2}{1+e^2}}$$

$$= 0.83$$

(3) Write a program to simulate this model. Conduct simulations to compute Pr(Z = 1|Y = 1) empirically. Plot this probability as a function of simulation size N and show that it converges to the theoretical value you derived in (4).

Figure 1: Convergence of Conditional Probability



We see that this converges to the expected value of 0.83.

(4) Derive a mathematical expression for Pr(Z = 1 | Y := y). What is Pr(Z = 1| Y: = 1)?

$$Pr(Z=1|Y:=y) = \sum_x f_x f_{z|x,y=y} \tag{8}$$

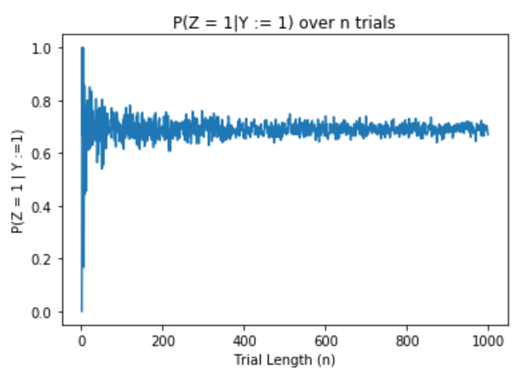
$$Pr(Z=1|Y:=1) = \sum_x f_x f_{z|x,y=1} \tag{9}$$

$$= f_x(0) f_{z|x,y}(1|0,1) + f_x(1) f_{z|x,y}(1|1,1) \tag{10}$$

$$= \frac{1}{2}(\frac{1}{2} + \frac{e^2}{1+e^2}) \tag{11}$$

$$= 0.69 \tag{12}$$

Figure 2: Convergence of Conditional Probability with Intervention



Again, like magic, our empirical results match the predicted value of 0.69!