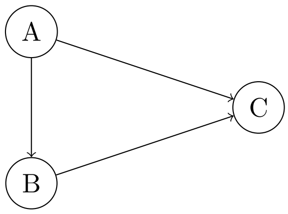
1 Trivial example 1

Consider the graph



where A is located at $(0,1)$, B is located at $(0,-1)$ and C is located at $(h,0)$. Denote the distance from A or B to C by l. The supplies are $s_A = (1,0)$, $s_B = (0,1)$ and $s_C = (-1,-1)$. Consider the trivial cost function $c(x,y) = |x+y|$ where x, y are the amounts of (signed) flow. Then

$$\partial c(0,0) = \operatorname{conv}\{(1,1),(-1,-1),(1,-1),(-1,1)\}$$

or with inequalities $\partial c(0,0) = \{z : |e_i^T z| \leq 1 \, \forall i\}$. Obviously, an optimal flow is

$$f = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

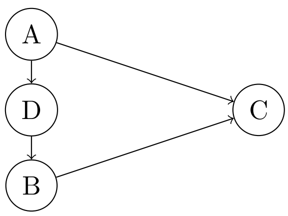
An optimal dual solution is $\phi_A = (l,l)$, $\phi_B = (l,l)$ and $\phi_C = (0,0)$. It can be easily checked that this satisfies all constraints. The derivative of ϕ is

$$D\phi = \begin{pmatrix} -l/h & 0 \\ -l/h & 0 \end{pmatrix}.$$

We now check whether $D\phi$ meets the constraints, i.e. if $\|e_i^T D\phi\| \leq 1$. We see that $\|e_i^T D\phi\| = l/h > 1$ for all i. This is maybe surprising, as the flow f is globally optimal (with respect to all possible graph topologies).

Observation 1 *Even if the global optimum is found, the dual constraints might not be satisfied.*

Since $e_i^T D\phi = (-l/h, 0)$, this suggests that we should add an edge parallel to $(1,0)$ to the graph. We get:



The optimal flow stays the same

$$f = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

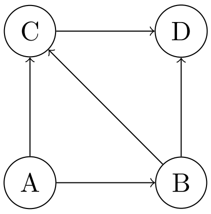
and we get optimal duals $\phi_A = (l, l)$, $\phi_B = (l, l)$, $\phi_C = (0, 0)$, $\phi_D = (h, h)$. One can again directly check that all constraints are satisfied. To see if $D\phi$ is feasible, by symmetry it suffices to consider the triangle DCA. Here,

$$D\phi = \begin{pmatrix} -1 & l-h \\ -1 & l-h \end{pmatrix}.$$

Then $\|e_i^T D\phi\|^2 = \|(-1, l-h)\|^2 = 1 + (l-h)^2 > 1$, so the dual solution is still not feasible.

2 Shortest path

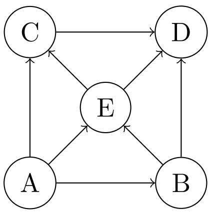
Consider



With supply 1 at A und -1 at D. An optimal solution is

$$f = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

(where edges are ordered lexicographically). An optimal dual solution is $\phi_A = 2$, $\phi_B = 1$, $\phi_C = 1$, $\phi_D = 0$. In the triangle BDC, $D\phi = (-1 - 1)$ which suggests an edge in the $(1, 1)$-direction. In the triangle ABC, $D\phi = (-1 - 1)$ as well. Hence, we refine the traingulation and get



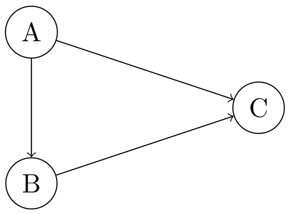
The optimal flow now is

$$f_{AE} = f_{ED} = 1$$

with duals $\phi_A = \sqrt{2}$, $\phi_B = 1/\sqrt{2}$, $\phi_C = 1/\sqrt{2}$, $\phi_D = 0$, $\phi_E = 1/\sqrt{2}$. This is obviously the "correct" optimal solution. Consider now $D\phi$ on all 4 triangles. In ABE, $D\phi = (-1/\sqrt{2}, -1/\sqrt{2})$ which has norm 1. By symmetry, this also holds for all other triangles. Hence in this example we indeed terminate with the optimal solution after one refinement.

3 Steiner tree

Consider



where the distance from A to B is 1 and the distance from AB to C is $h = 3$, so the distance from A to C is $l = \sqrt{10}$. The supplies are $s_A = (1, 0)$, $s_B = (0, 1)$ and $s_C = (-1, -1)$. We use the cost function

$$c(x, y) = \max(x, y, 0) + \max(-x, -y, 0),$$

so we consider the Steiner tree problem. An optimal flow is given by $f_{AB} = (1, 0)$, $f_{BC} = (1, 1)$. Choosing $\phi_C = (0, 0)$, ϕ_B must lie in $l\partial c(1, 1)$, so $\phi_B = (l\lambda, l(1 - \lambda))$ for some $\lambda \in [0, 1]$. Additionally, $\phi_A - \phi_B \in 2\partial c(1, 0)$, so $\phi_A = (l\lambda, l(1 - \lambda)) + (2, 2\mu)$ for a $\mu \in [-1, 1]$. We choose $\mu = -1$ and

$$\lambda = 0,$$

so $\phi_B = (0, l)$ and $\phi_A = (2, l - 2)$. One can check that $\phi_A \in l\partial c(0, 0)$, so we indeed constructed an optimal dual solution. The differential $D\phi$ is given by

$$D\phi = \begin{pmatrix} -1/h & 1 \\ (1 - l)/h & -1 \end{pmatrix}.$$

In terms of inequalities, $\partial c(0, 0)$ can be written as

$$\partial c(0, 0) = \{x : |e_i^T x| \leq 1 \text{ and } |(1, 1)x| \leq 1\}.$$

We have $\|e_1^T D\phi\|^2 = 1 + 1/h^2 = 10/9$, $\|e_2^T D\phi\| = (1 - l)^2/h^2 + 1 \approx 1.5$ and $\|(1, 1)D\phi\| = l^2/h^2 = 10/9$. The largest relative and absolute error comes from e_2, and $e_2^T D\phi = ((1 - l)/h, -1)$. Note that the angle between an edge in this direction and AB is approximately $62°$, which is quite close to the correct edge direction.