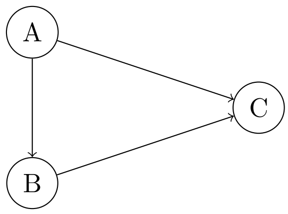
1 Trivial example 1

Consider the graph



where A is located at $(0,1)$, B is located at $(0,-1)$ and C is located at $(h,0)$. Denote the distance from A or B to C by l. The supplies are $s_A = (1,0)$, $s_B = (0,1)$ and $s_C = (-1,-1)$. Consider the trivial cost function $c(x,y) = |x+y|$ where x, y are the amounts of (signed) flow. Then

$$\partial c(0,0) = \operatorname{conv}\{(1,1),(-1,-1),(1,-1),(-1,1)\}$$

or with inequalities $\partial c(0,0) = \{z : |e_i^T z| \leq 1 \, \forall i\}$. Obviously, an optimal flow is

$$f = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

An optimal dual solution is $\phi_A = (l,l)$, $\phi_B = (l,l)$ and $\phi_C = (0,0)$. It can be easily checked that this satisfies all constraints. The derivative of ϕ is

$$D\phi = \begin{pmatrix} -l/h & 0 \\ -l/h & 0 \end{pmatrix}.$$

We now check whether $D\phi$ meets the constraints, i.e. if $\|e_i^T D\phi\| \leq 1$. We see that $\|e_i^T D\phi\| = l/h > 1$ for all i. This is maybe surprising, as the flow f is globally optimal (with respect to all possible graph topologies).

Observation 1 *Even if the global optimum is found, the dual constraints might not be satisfied.*

Since $e_i^T D\phi = (-l/h, 0)$, this suggests that we should add an edge parallel to $(1,0)$ to the graph.