고급양자물리 과제 5

2023-19002 경영학과 강기택

Question 1.

$$
\begin{aligned}
e^{ikz} &= e^{ikr\cos\theta} \\
\text{Let } f(x) &= e^{ikrx}, \quad x = \cos\theta \\
\text{Expand } f(x) &= \sum_{l=0}^{\infty} C_l P_l(x) \\
\text{Using orthogonality } C_l &= \frac{2l+1}{2}\int_{-1}^{1} f(x) P_l(x)\, dx \\
&= \frac{2l+1}{2}\int_{-1}^{1} e^{ikrx} P_l(x)\, dx \\
\text{Use Bauer's formula: } j_l(\rho) &= \frac{1}{2i^l}\int_{-1}^{1} e^{i\rho x} P_l(x)\, dx \\
\implies \int_{-1}^{1} e^{i\rho x} P_l(x)\, dx &= 2i^l j_l(\rho) \\
\text{Let } \rho = kr: \quad \int_{-1}^{1} e^{ikrx} P_l(x)\, dx &= 2i^l j_l(kr) \\
\text{Substitute back into } C_l: \quad C_l &= \frac{2l+1}{2}\left[2i^l j_l(kr)\right] \\
&= i^l (2l+1) j_l(kr) \\
\text{Therefore: } e^{ikrx} &= \sum_{l=0}^{\infty} i^l (2l+1) j_l(kr) P_l(x) \\
e^{ikz} &= \sum_{l=0}^{\infty} i^l (2l+1) j_l(kr) P_l(\cos\theta)
\end{aligned}
$$

Question 2.

2-(a)

Region I($x < -a$): $\psi_I(x) = Ae^{ikx} + Be^{-ikx}$ (Incident + Reflected)
Region II($-a \leq x \leq 0$): $\psi_{II}(x) = Ce^{ik'x} + De^{-ik'x}$
Region III($x > 0$): $\psi_{III}(x) = 0$ (due to $V = \infty$)

1. Boundary Condition at $x = 0$

Continuity of ψ: $\psi_{II}(0) = \psi_{III}(0) \implies C + D = 0$

$\psi_{II}(x) = C(e^{ik'x} - e^{-ik'x}) = 2iC\sin(k'x)$
$\psi_{II}(0) = 0$ and $\psi_{III}(0) = 0$ (since ψ_{III} is zero)

2. Boundary Conditions at $x = -a$

Continuity of ψ ($\psi_I(-a) = \psi_{II}(-a)$):
$Ae^{-ika} + Be^{ika} = 2iC\sin(-k'a) = -2iC\sin(k'a)$
Continuity of $d\psi/dx$ ($\psi'_I(-a) = \psi'_{II}(-a)$):
$ikAe^{-ika} - ikBe^{ika} = 2iCk'\cos(-k'a) = 2iCk'\cos(k'a)$
$kAe^{-ika} - kBe^{ika} = 2Ck'\cos(k'a)$

3. Solving B/A

$$\frac{kAe^{-ika} - kBe^{ika}}{Ae^{-ika} + Be^{ika}} = \frac{2Ck'\cos(k'a)}{-2iC\sin(k'a)} = ik'\cot(k'a)$$

Let $R = B/A$. Dividing by Ae^{ika}:

$$k\frac{Ae^{-ika}/Ae^{ika} - Be^{ika}/Ae^{ika}}{Ae^{-ika}/Ae^{ika} + Be^{ika}/Ae^{ika}} = ik'\cot(k'a)$$
$$k\frac{e^{-2ika} - R}{e^{-2ika} + R} = ik'\cot(k'a)$$

Solving for R:

$$\begin{aligned}
k(e^{-2ika} - R) &= ik'\cot(k'a)(e^{-2ika} + R) \\
ke^{-2ika} - kR &= ik'\cot(k'a)e^{-2ika} + ik'\cot(k'a)R \\
-R\left[k + ik'\cot(k'a)\right] &= e^{-2ika}\left[ik'\cot(k'a) - k\right] \\
\frac{B}{A} &= -e^{-2ika}\frac{ik'\cot(k'a) - k}{k + ik'\cot(k'a)} \\
\frac{B}{A} &= e^{-2ika}\frac{k - ik'\cot(k'a)}{k + ik'\cot(k'a)}
\end{aligned}$$

The reflected wave is Be^{-ikx}, substituting $B = A \cdot (B/A)$:

$$\text{Reflected Wave} = Ae^{-2ika}\frac{k - ik' \cot(k'a)}{k + ik' \cot(k'a)}e^{-ikx}$$

The reflected wave is given by:

$$Ae^{-2ika}\frac{k - ik' \cot(k'a)}{k + ik' \cot(k'a)}e^{-ikx}$$

2-(b)

For two waves to have the same amplitude, $||\psi||^2$ must be equal.

$$\begin{aligned}
\psi_1 &= Ae^{ikx}, \\
\psi_2 &= Ae^{-2ika}[\frac{k - ik'cot(k'a)}{k + ikcot(ka)}]e^{-ikx}
\end{aligned}$$

Then,

$$\begin{aligned}
||\psi_1||^2 &= |A|^2, \\
||\psi_2||^2 &= A^* e^{2ika}[\frac{k + ik'cot(k'a)}{k - ik'cot(ka)}]e^{ikx}Ae^{-2ika}[\frac{k - ik'cot(k'a)}{k + ik'cot(ka)}] \\
&= |A|^2\frac{k^2 + k'^2cot^2(k'a)}{k^2 + k'^2cot^2(k'a)} \\
&= |A|^2
\end{aligned}$$

Therefore, the amplitude of two waves are the same.

2-(c)

Let the phase shift be δ. From

$$\psi_2 = Ae^{-2ika}[\frac{k - ik'cot(k'a)}{k + ikcot(ka)}]e^{-ikx},$$

ψ_2 can also be written as the following:

$$\psi_2 = Ae^{i2\delta}e^{-2ika}e^{-ikx}$$

Then,

$$e^{i2\delta} = \frac{B}{A}e^{i2ka} = \frac{k - ik' \cot(k'a)}{k + ik' \cot(k'a)}$$

Since $V_0 \gg E$, k is much smaller than k', therefore it is ignorable compared to k'. $e^{2i\delta}$ simplifies to,

$$e^{i2\delta} \approx \frac{-ik' \cot(k'a)}{+ik' \cot(k'a)}$$

$$e^{i2\delta} \approx -1$$

$$2\delta = \pi \pm 2n\pi, \quad n = 0, 1, 2, \ldots$$

Choose δ as π

$$\delta \approx \frac{\pi}{2}$$

Question 3.

For the wavefunction withen the soft sphere, solve the Schrodinger Equation with potential V_0.

$$-\frac{\hbar^2}{2m}\frac{d^2u_0}{dr^2} + V_0u_0(r) = Eu_0(r)$$

$$\frac{d^2u_0}{dr^2} + \frac{2m(E-V_0)}{\hbar^2}u_0(r) = 0$$

Solve the Schrodinger equation, and get the following:

$$u_0(r) = Asin(k_0r) + Bcos(k_0r)$$

With the Boundary condition of $\lim_{r\to 0} u(r) = 0$, $B = 0$. $u(r) = Asin(k_0r)$, $k_0 = \sqrt{2m(E-V_0)}/\hbar$

Similarly, for the free particle ouside the soft sphere, solve the Schrodinger equation with zero potential.

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} = Eu(r)$$

$$\frac{d^2u}{dr^2} + k^2u(r) = 0$$

Solve, the Schrodinger equation, and get the following:

$$u(r) = Csin(kr + \delta_0),\ k = \sqrt{2mE}/\hbar$$

Use the boundary condition at $r = a$:

$$Asin(k_0a) = Csin(ka + \delta_0)$$

Use the continuity of the derivation of the wave function:

$$Ak_0cos(k_0a) = Ckcos(ka + \delta_0)$$

From these two equations,

$$Ak_0cot(k_0a) = Ckcot(ka + \delta_0)$$

Since $ka \ll 1$ and the phase shift is small enough,

$$\frac{k_0}{k}cot(k_0a) \approx \frac{1}{ka + \delta_0},$$

$$\begin{aligned}
ka + \delta_0 &\approx \frac{k}{k_0 \cot(k_0 a)} \\
&= \frac{k \tan(k_0 a)}{k_0} \\
\delta_0 &\approx \frac{k \tan k_0 a)}{k_0} - ka
\end{aligned}$$

In the low-energy limit (δ_0 is small), $\sin \delta_0 \approx \delta_0$:

$$f(\theta) \approx \frac{\delta_0}{k}$$

Substituting for δ_0:

$$f(\theta) \approx \frac{ka}{k} \left(\frac{\tan(K_0 a)}{K_0 a} - 1 \right) = a \left(\frac{\tan(K_0 a)}{K_0 a} - 1 \right)$$

$$f(\theta) \approx f_0 = \frac{e^{i\delta_0} \sin \delta_0}{k}$$

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2$$

$$\frac{d\sigma}{d\Omega} \approx a^2 \left(1 - \frac{\tan(K_0 a)}{K_0 a} \right)^2$$

Total Cross-Section (σ_{total}):The total cross-section is $\sigma_{total} = \int \frac{d\sigma}{d\Omega} d\Omega = 4\pi \frac{d\sigma}{d\Omega}$ (since $f(\theta)$ is isotropic).

$$\sigma_{total} \approx 4\pi a^2 \left(1 - \frac{\tan(K_0 a)}{K_0 a} \right)^2$$

Case 1: Repulsive ($V_0 > 0$)

$k_0^2 = -2mV_0/\hbar^2$ is negative. Let $k_0 = i\kappa_0$, where $\kappa_0 = \sqrt{2mV_0/\hbar^2}$ is real.

$$\tan(ix) = i \tanh(x)$$

.

$$\frac{\tan(k_0 a)}{k_0 a} = \frac{\tan(i\kappa_0 a)}{i\kappa_0 a} = \frac{i \tanh(\kappa_0 a)}{i\kappa_0 a} = \frac{\tanh(\kappa_0 a)}{\kappa_0 a}$$

Phase Shift: $\delta_0 \approx ka\left(\frac{\tanh(\kappa_0 a)}{\kappa_0 a} - 1\right)$

Total Cross-Section: $\sigma_{total} \approx 4\pi a^2 \left(1 - \frac{\tanh(\kappa_0 a)}{\kappa_0 a}\right)^2$

Case 2: Attractive ($V_0 < 0$)

$V_0 = -|V_0|$. $k_0^2 = -2m(-|V_0|)/\hbar^2 = 2m|V_0|/\hbar^2$ is positive.

$k_0 = \sqrt{2m|V_0|/\hbar^2}$ is real.

Phase Shift: $\delta_0 \approx ka\left(\frac{\tan(K_0 a)}{K_0 a} - 1\right)$

Total Cross-Section: $\sigma_{total} \approx 4\pi a^2 \left(1 - \frac{\tan(K_0 a)}{K_0 a}\right)^2$

Question 4.

$$-\frac{\hbar^2}{2m}\frac{d^2 u_l}{dr^2} + \left[V(r) + \frac{\hbar^2 l(l+1)}{2mr^2}\right] u_l(r) = E u_l(r)$$

Substitute V_0:

$$\frac{d^2 u_0}{dr^2} + \left[k^2 - \frac{2mV_0 a}{\hbar^2}\delta(r-a)\right] u_0(r) = 0$$

$k = \sqrt{2mE}/\hbar$.

Integrate the equation at $r \in (a-\epsilon, a+\epsilon)$:

$$\frac{du_0}{dr}|_{r=a-}^{r=a+} - \frac{2mV_0 a}{\hbar} u_0(a) = 0$$

$$u_>(r) = Csin(ka + \delta_0),\ u_<(r) = Asin(ka)$$

$$Ckcos(ka + \delta_0) - Akcos(ka) = \frac{2mV_0 a}{\hbar} Asin(ka)$$

Use the continuity of the wave function.

$$Asin(ka) = Csin(ka + \delta_0)$$

From those two equations,

$$Ckcos(ka+\delta_0) = A(kcos(ka) + \frac{2mV_0a}{\hbar}sin(ka))$$

$$Csin(ka+\delta_0) = Asin(ka)$$

Divide both sides,

$$k\cot(ka+\delta_0) = kcot(ka) + \frac{2mV_0a}{\hbar}$$

$$\cot(ka+\delta_0) = cot(ka) + \frac{2mV_0a}{\hbar k}$$

Since $1 \gg ka, \delta_0$ is small enough,

$$\frac{1}{ka+\delta_0} \approx \frac{1}{ka} + \frac{\gamma}{k}$$

$$\delta_0 \approx -\frac{(\gamma a)ka}{1+\gamma a}$$

, Where $\gamma = \frac{2mV_0a}{\hbar}$

$$f(\theta) \approx f_0 = \frac{e^{i\delta_0}\sin\delta_0}{k}$$

$$f(\theta) \approx \frac{\delta_0}{k}$$

$$f(\theta) \approx \frac{1}{k}\left[-\frac{(\gamma a)ka}{1+\gamma a}\right] = -\frac{(\gamma a)a}{1+\gamma a}$$

$$f(\theta) \approx -\frac{2mV_0a^3}{\hbar^2 + 2mV_0a^2}$$

$$\frac{d\sigma}{d\Omega} \approx \left(-\frac{2mV_0a^3}{\hbar^2 + 2mV_0a^2}\right)^2 = \left(\frac{2mV_0a^3}{\hbar^2 + 2mV_0a^2}\right)^2$$

$$\sigma_{total} = \int \frac{d\sigma}{d\Omega}d\Omega = 4\pi\left(\frac{d\sigma}{d\Omega}\right)$$

$$\sigma_{total} \approx 4\pi\left(\frac{2mV_0a^3}{\hbar^2 + 2mV_0a^2}\right)^2$$

Question 5.

5-(a)

$$\int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{e^{iqr}}{q^2 + \lambda^2} = \int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{e^{iqr}}{q + i\lambda} \frac{1}{q - i\lambda}$$

Use the calculus of residues in the theory of complex variables:

$$\int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{e^{iqr}}{q + i\lambda} \frac{1}{q - i\lambda} = 2\pi i \times \frac{e^{iq(i\lambda)}}{4\pi i\lambda} = \frac{1}{2\lambda} e^{-q\lambda}$$

5-(b)

$$\int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{e^{iqr}}{q^2 - k^2 \mp i\eta}$$

If $\pm$ is $+$, then the singular point is included in the upper semicircle. Then we can perform integral on the semicircle counter-clockwise.

$$\int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{e^{iqr}}{q^2 - k^2 \mp i\eta} = \int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{e^{iqr}}{q^2 - k^2} = \int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{e^{iqr}}{q - k} \frac{1}{q + k}$$

Note that although not included in the integral itself, but given $i\eta$ makes the singular point be included in the semicircle, giving the singular point just right infinitesimally above the real line.

$$\int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{e^{iqr}}{q - k} \frac{1}{q + k} = 2\pi i \frac{e^{ikr}}{4\pi k} = \frac{ie^{ikr}}{2k}$$

If $\pm$ is $-$, then the singular point is included in the lower semicircle. Then we can perform integral on the semicircle clockwise.

$$\int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{e^{iqr}}{q^2 - k^2 \mp i\eta} = \int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{e^{iqr}}{q^2 - k^2} = \int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{e^{iqr}}{q + k} \frac{1}{q - k}$$

$$= \int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{e^{iqr}}{q + k} \frac{1}{q - k} = 2\pi i \frac{e^{-ikr}}{-4\pi k} = \frac{-ie^{-ikr}}{2k}$$

Therefore,

$$\int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{e^{iqr}}{q^2 - k^2 \mp i\eta} = \pm \frac{e^{\pm ikr}}{2k}$$

5-(c)

(I)

$$\int \frac{d^3q}{(2\pi)^3}\frac{e^{iq\cdot r}}{q^2+\lambda^2} = \frac{1}{(2\pi)^3}\int_0^\infty q^2dq\int_{-1}^1 dcos\theta\int_0^{2\pi} d\varphi\frac{e^{iq\cdot r}}{q+i\lambda}\frac{1}{q-i\lambda}$$
$$= \frac{1}{8\pi^2}\int_{-\infty}^\infty q^2dq\int_{-1}^1 dcos\theta\frac{e^{iqrcos\theta}}{q+i\lambda}\frac{1}{q-i\lambda}$$

(II)

$$\int_{-1}^1 dcos\theta\frac{e^{iqrcos\theta}}{q+i\lambda}\frac{1}{q-i\lambda} = \frac{1}{(q+i\lambda)(q-i\lambda)}\int_{-1}^1 dcos\theta e^{iqrcos\theta}$$
$$= \frac{1}{(q+i\lambda)(q-i\lambda)}\frac{e^{iqr}-e^{-iqr}}{iqr}$$
$$= \frac{2}{q-i\lambda}\frac{1}{q+i\lambda}\frac{sin(qr)}{qr}$$

(III)

$$\int_{-\infty}^\infty \frac{d^3q}{(2\pi)^3}\frac{e^{iq\cdot r}}{q^2+\lambda^2} = \frac{1}{8\pi^2}\int_{-\infty}^\infty \frac{2q^2}{q-i\lambda}\frac{1}{q+i\lambda}\frac{sin(qr)}{qr}$$
$$= \frac{1}{4\pi^2}\int_{-\infty}^\infty \frac{qsin(qr)}{r(q+i\lambda)}\frac{1}{q-i\lambda}$$

Use $sin(qr) = Im(e^{iqr})$ and residue theorem. Take $q = i\lambda$ as the pole and integrate the upper semicircle.

$$\frac{1}{4\pi^2}\int_{-\infty}^\infty \frac{qsin(qr)}{r(q+i\lambda)}\frac{1}{q-i\lambda} \to \frac{1}{4\pi^2}\int_{-\infty}^\infty \frac{qe^{iqr}}{r(q+i\lambda)}\frac{1}{q-i\lambda},$$
$$\to 2\pi i \times \frac{1}{4\pi^2}\times Im(\frac{i\lambda e^{-\lambda r}}{r})\times\frac{1}{2i\lambda}$$
$$= \frac{e^{-\lambda r}}{4\pi}$$

5-(d)

$$\int \frac{d^3q}{(2\pi)^3}\frac{e^{i\mathbf{q}\cdot\mathbf{r}}}{q^2-k^2-i\eta} = \frac{1}{(2\pi)^3}\int_0^\infty dq\frac{q^2}{q^2-k^2-i\eta}\int_0^{2\pi} d\phi\int_{-1}^1 d(\cos\theta)e^{iqr\cos\theta}$$

Similar to 5-(c),

$$\int_0^{2\pi} d\phi\int_{-1}^1 d(\cos\theta)e^{iqr\cos\theta} = 4\pi\frac{\sin(qr)}{qr}$$

Then,

$$\int \frac{d^3q}{(2\pi)^3} \frac{e^{iq\cdot r}}{q^2-k^2-i\eta} = \frac{1}{2\pi^2 r}\int_0^\infty dq \frac{q\sin(qr)}{q^2-k^2-i\eta}$$

By the properties of even function and expanding the interval to $-\infty \to \infty$,

$$\begin{aligned}\frac{1}{2\pi^2 r}\int_0^\infty \frac{q}{q^2-k^2-i\eta}\left(\frac{e^{iqr}-e^{-iqr}}{2i}\right)dq &= \frac{1}{4\pi^2 ir}\int_0^\infty \frac{q}{q^2-k^2-i\eta}(e^{iqr}-e^{-iqr})dq \\ &= \frac{1}{4\pi^2 ir}\int_{-\infty}^\infty dq \frac{qe^{iqr}}{q^2-k^2-i\eta}\end{aligned}$$

The poles are $q = \pm\sqrt{k^2+i\eta}$, finding this is the same for 5-(b).
Take $q_1 = \sqrt{k^2+i\eta} \approx k+i\delta$. q_1 is infinitesimally right above the real line. By integrating the upper semicircle which includes q_1 counter-clockwise, we can get the result.

$$J = \int_{-\infty}^\infty f(q)dq = 2\pi i \times \mathrm{Res}(f, q_1),$$

$$\begin{aligned}Res(f,q_1) = \lim_{q\to q_1}(q-q_1)\frac{qe^{iqr}}{(q-q_1)(q-q_2)} &= \frac{q_1 e^{iq_1 r}}{q_1-q_2} = \frac{\sqrt{k^2+i\eta}\cdot e^{i(\sqrt{k^2+i\eta})r}}{(\sqrt{k^2+i\eta})-(-\sqrt{k^2+i\eta})} \\ &= \frac{\sqrt{k^2+i\eta}\cdot e^{i(\sqrt{k^2+i\eta})r}}{2\sqrt{k^2+i\eta}} = \frac{e^{i(\sqrt{k^2+i\eta})r}}{2}\end{aligned}$$

Take $\eta \to 0^+$,

$$J = 2\pi i \times \left(\frac{e^{ikr}}{2}\right) = \pi i e^{ikr}$$

Therefore,

$$\int \frac{d^3q}{(2\pi)^3}\frac{e^{iq\cdot r}}{q^2-k^2-i\eta} = \frac{1}{4\pi^2 ir}\cdot J = \frac{1}{4\pi^2 ir}(\pi i e^{ikr}) = \frac{e^{ikr}}{4\pi r}$$

Similarly, if $\pm$ is $-$, take q_2 as $\sqrt{k^2-i\eta}$ and integrate the lower semicircle clockwise. Then, we can get acheive $\frac{e^{-ikr}}{4\pi r}$.

Question 6

6-(a)

By the Born Approximation,

$$f(k',k) = \frac{-m}{2\pi\hbar^2}\int_0^\infty dx' e^{-ik'\cdot x}V(x')\psi(x'),$$

where

$$V(q) = \frac{4\pi}{q}\int_0^\infty dr sin(qr)V(r),\ q = |k' - k| = 2ksin(\frac{\theta}{2})$$

From this equation,

$$V(q) = \frac{4\pi}{q}\int_0^\infty dr sin(qr)e^{-\mu r^2},$$

$$\int_0^\infty re^{-\mu r^2}\sin(qr)\,dr = \frac{q\sqrt{\pi}}{4\mu^{3/2}}e^{-q^2/4\mu}$$

$$f(\theta) = -\frac{\sqrt{\pi}mA}{2\hbar^2\mu^{3/2}}e^{-q^2/4\mu}$$

$$\frac{q^2}{4\mu} = \frac{(2k\sin(\theta/2))^2}{4\mu} = \frac{4k^2\sin^2(\theta/2)}{4\mu} = \frac{k^2}{\mu}\sin^2\left(\frac{\theta}{2}\right)$$

$$f(\theta) = -\frac{\sqrt{\pi}mA}{2\hbar^2\mu^{3/2}}\exp\left(-\frac{k^2}{\mu}\sin^2\left(\frac{\theta}{2}\right)\right)$$

6-(b)

$$C = \frac{\sqrt{\pi}mA}{2\hbar^2\mu^{3/2}}$$

Substitute $A = \frac{\hbar^2}{2ma_B^2}$ and $\mu = (a_B^{-2})$:

$$C = \frac{\sqrt{\pi}m}{2\hbar^2(a_B^{-2})^{3/2}}\cdot\left(\frac{\hbar^2}{2ma_B^2}\right)$$

$$C = \frac{\sqrt{\pi}m\hbar^2}{4m\hbar^2a_B^2(a_B^{-3})}$$

Cancel m and $\hbar^2$:

$$C = \frac{\sqrt{\pi}}{4a_B^2\cdot a_B^{-3}} = \frac{\sqrt{\pi}}{4a_B^{-1}} = \frac{\sqrt{\pi}}{4}a_B$$

$$f(\theta) = -\frac{\sqrt{\pi}}{4}a_B\exp\left(-\sin^2\left(\frac{\theta}{2}\right)\right)$$

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \left(-\frac{\sqrt{\pi}}{4}a_B\right)^2\left[\exp\left(-\sin^2\left(\frac{\theta}{2}\right)\right)\right]^2$$

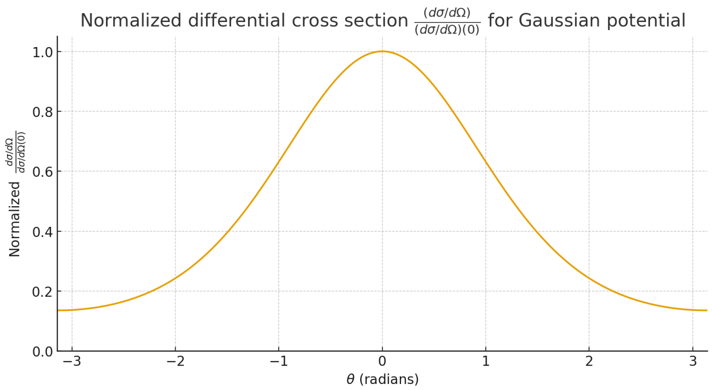
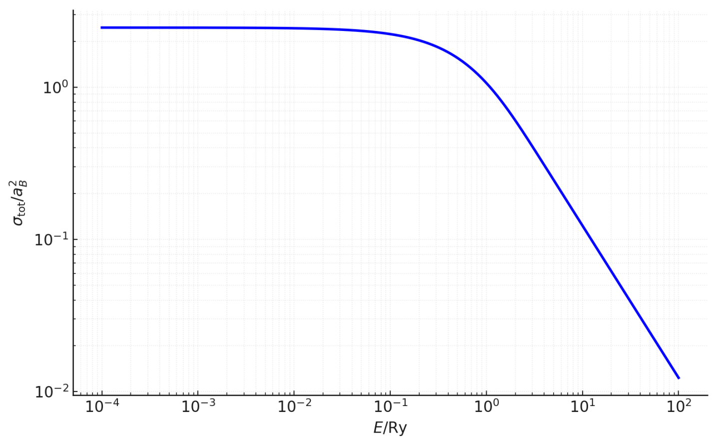


Figure 1: Numerical plot of $\frac{\frac{d\sigma}{d\Omega}}{\frac{d\sigma}{d\Omega}|_{\theta=0}}$

$$\frac{d\sigma}{d\Omega} = \frac{\pi}{16} a_B^2 \exp\left(-2\sin^2\left(\frac{\theta}{2}\right)\right)$$
$$\frac{\frac{d\sigma}{d\Omega}}{\frac{d\sigma}{d\Omega}|_{\theta=0}} = \exp\left(-2\sin^2\left(\frac{\theta}{2}\right)\right)$$

6-(c)

$$\sigma_{\text{tot}} = 2\pi \int_0^\pi \sin\theta \, |f(\theta)|^2 \, d\theta = \frac{m^2 A^2 \pi^2}{2\hbar^4 \mu^2 k^2} \left(1 - e^{-2k^2/\mu}\right), \qquad k = \frac{\sqrt{2mE}}{\hbar}.$$



Question 7.

7-(a)

$$(E - H_0)|\psi\rangle = V|\psi\rangle,$$

$$\Rightarrow |\psi^{(\pm)}\rangle = |\psi_0\rangle + \frac{1}{E - H_0 \pm i\eta}|\psi^{(\pm)}\rangle$$

$$\langle x|\psi^{(\pm)}\rangle = \langle x|\psi_0\rangle + \langle x|\frac{1}{E - H_0 \pm i\eta}|\psi^{(\pm)}\rangle$$

$$\psi^{(\pm)}(x) = \psi(x_0) + \int dx' \langle x|\frac{1}{E - H_0 \pm i\eta}|x'\rangle\langle x'|\psi^{(\pm)}\rangle$$

Since

$$G_\pm(x, x') = \frac{\hbar^2}{2m}\int dx' \langle x|\frac{1}{E - H_0 \pm i\eta}|x'\rangle\langle x'|\psi^{(\pm)}\rangle = -\frac{e^{\pm ik\cdot|x-x\|}}{4\pi|x-x\|}$$

,

$$(\nabla^2 + k^2 \pm i\eta)G_\pm(x, x) = \delta(x),$$

At large r with an incident wave of

$$\psi_0(x) = e^{ik\cdot x},$$

$$\psi^{(\pm)}(x) \xrightarrow{kr>>1} e^{ik\cdot x} - \frac{m}{2\pi^2\hbar}\frac{e^{ikr}}{r}\int dx' e^{-ik'\cdot x}V(x')\psi^{(\pm)}(x')$$

7-(b)

From $|\psi^{(\pm)}(x)$, Substitute $|\psi_s\rangle$

Then,

$$\begin{aligned}
|\psi_s\rangle &= |\psi_0\rangle + \frac{1}{E - H_0 + i\eta} V |\psi_s\rangle \\
&= |\psi_0\rangle + \frac{1}{E - H_0 + i\eta} V |\psi_0\rangle + \frac{1}{E - H_0 + i\eta} V \frac{1}{E - H_0 + i\eta} V |\psi_s\rangle
\end{aligned}$$

Continue substituting,

$$\begin{aligned}
|\psi_s\rangle &= \left(I + \frac{1}{E - H_0 + i\eta} V + \frac{1}{E - H_0 + i\eta} V \frac{1}{E - H_0 + i\eta} V + \dots \right) |\psi_0\rangle \\
V|\psi_s\rangle &= \left(V + V \frac{1}{E - H_0 + i\eta} V + V \frac{1}{E - H_0 + i\eta} V \frac{1}{E - H_0 + i\eta} V + \dots \right) |\psi_0\rangle \\
\Rightarrow V|\psi_s\rangle &= T|\psi_0\rangle
\end{aligned}$$

Therefore $T = V + V \frac{1}{E - H_0 + i\eta} V + \dots$

7-(c)

$$\begin{aligned}
f(k', k) &= -\frac{m}{2\pi\hbar^2} \langle k' | V | \psi_k^{(+)} \rangle \\
&= -\frac{m}{2\pi\hbar^2} \langle k' | T | k \rangle \\
&= \sum_{n=1}^{\infty} f^{(n)}(k', k)
\end{aligned}$$

since $V|\psi^{(\pm)}\rangle = T|\psi_0\rangle$ substitute from the Lippmann-Schwinger Equation.

$$\Psi^{(+)}(x) = e^{ik\cdot x} - \frac{m}{2\pi\hbar^2}\int d^3x' \frac{e^{ik|x-x'|}}{|x-x'|}V(x')\Psi^{(+)}(x')$$

for $|x| \gg |x'|$

$$|x-x'| \approx x - \hat{r}\cdot x'$$

$$\frac{1}{|x-x'|} \approx \frac{1}{x}$$

substitute,

$$\Psi^{(+)}(x) \approx e^{ik\cdot x} - \frac{m}{2\pi\hbar^2}\frac{e^{ikr}}{r}\int d^3x' e^{-ik'\cdot x'}V(x')\Psi^{(+)}(x')$$

7-(d)

$$\begin{aligned} f(k',k) &= -\frac{m}{2\pi\hbar^2}\langle k'|V|\psi_k^{(+)}\rangle \\ &= -\frac{m}{2\pi\hbar^2}\langle k'|T|k\rangle \\ &= \sum_{n=1}^{\infty} f^{(n)}(k',k) \end{aligned}$$

$$f^{(2)}(\mathbf{k}',\mathbf{k}) = -\frac{m}{2\pi\hbar^2}\int d^3x \int d^3x' \langle \mathbf{k}'|V|\mathbf{x}\rangle\langle \mathbf{x}|(E-H_0+i\eta)^{-1}|\mathbf{x}'\rangle\langle \mathbf{x}'|V|\mathbf{k}\rangle$$

$$f^{(2)}(\mathbf{k}',\mathbf{k}) = -\frac{m}{2\pi\hbar^2}\int d^3x \int d^3x' \left(e^{-i\mathbf{k}'\cdot\mathbf{x}}V(\mathbf{x})\right)\langle \mathbf{x}|(E-H_0+i\eta)^{-1}|\mathbf{x}'\rangle \left(V(\mathbf{x}')e^{i\mathbf{k}\cdot\mathbf{x}'}\right)$$

$$f^{(2)}(\mathbf{k}',\mathbf{k}) = -\frac{m}{2\pi\hbar^2}\int d^3x \int d^3x' e^{-i\mathbf{k}'\cdot\mathbf{x}}V(\mathbf{x})\langle \mathbf{x}|(E-H_0+i\eta)^{-1}|\mathbf{x}'\rangle V(\mathbf{x}')e^{i\mathbf{k}\cdot\mathbf{x}'}$$

$$f^{(3)}(\mathbf{k}',\mathbf{k}) = -\frac{m}{2\pi\hbar^2}\langle \mathbf{k}'|V(E-H_0+i\eta)^{-1}V(E-H_0+i\eta)^{-1}V|\mathbf{k}\rangle$$

$$f^{(3)} = -\frac{m}{2\pi\hbar^2}\int d^3x_1 \int d^3x_2 \int d^3x_3 \langle \mathbf{k}'|\mathbf{x}_1\rangle\langle \mathbf{x}_1|V(E-H_0+i\eta)^{-1}V(E-H_0+i\eta)^{-1}V|\mathbf{x}_3\rangle\langle \mathbf{x}_3|\mathbf{k}\rangle$$

$$\begin{aligned} f^{(3)}(\mathbf{k}',\mathbf{k}) = -\frac{m}{2\pi\hbar^2}\int d^3x_1 \int d^3x_2 \int d^3x_3 e^{-i\mathbf{k}'\cdot\mathbf{x}_1}V(\mathbf{x}_1)\langle \mathbf{x}_1|(E-H_0+i\eta)^{-1}|\mathbf{x}_2\rangle V(\mathbf{x}_2) \\ \times\langle \mathbf{x}_2|(E-H_0+i\eta)^{-1}|\mathbf{x}_3\rangle V(\mathbf{x}_3)e^{i\mathbf{k}\cdot\mathbf{x}_3} \end{aligned}$$

Question 8.

8-(a)

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

$$b = \frac{q_1 q_2}{4\pi\epsilon_0}\frac{1}{2E}\cot\left(\frac{\theta}{2}\right) = \frac{q_1 q_2}{8\pi\epsilon_0 E}\cot\left(\frac{\theta}{2}\right)$$

$$\frac{db}{d\theta} = \frac{d}{d\theta}\left[\frac{q_1 q_2}{8\pi\epsilon_0 E}\cot\left(\frac{\theta}{2}\right)\right]$$

$$= \frac{q_1 q_2}{8\pi\epsilon_0 E}\left[-\csc^2\left(\frac{\theta}{2}\right)\cdot\frac{1}{2}\right]$$

$$= -\frac{q_1 q_2}{16\pi\epsilon_0 E\sin^2(\theta/2)}$$

Substitute,

$$\frac{d\sigma}{d\Omega} = \left(\frac{q_1 q_2}{8\pi\epsilon_0 E}\cot\left(\frac{\theta}{2}\right)\right)\cdot\frac{1}{\sin\theta}\cdot\left|-\frac{q_1 q_2}{16\pi\epsilon_0 E\sin^2(\theta/2)}\right|$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{q_1 q_2}{16\pi\epsilon_0 E\sin^2(\theta/2)}\right)\cdot\left(\frac{q_1 q_2}{16\pi\epsilon_0 E\sin^2(\theta/2)}\right)$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{q_1 q_2}{16\pi\epsilon_0 E\sin^2(\theta/2)}\right)^2$$

8-(b)

$$f(\theta) = -\frac{2m}{\hbar^2 q}\int_0^\infty rV(r)\sin(qr)\,dr$$

We must use a screened potential $V(r) = \frac{q_1 q_2}{4\pi\epsilon_0 r}e^{-\mu r}$ and take the limit $\mu \to 0$.

$$f(\theta) = -\frac{2m}{\hbar^2 q}\int_0^\infty r\left(\frac{q_1 q_2}{4\pi\epsilon_0 r}e^{-\mu r}\right)\sin(qr)\,dr$$

$$f(\theta) = -\frac{2m}{\hbar^2 q}\left(\frac{q_1 q_2}{4\pi\epsilon_0}\right)\int_0^\infty e^{-\mu r}\sin(qr)\,dr$$

$$\int_0^\infty e^{-\mu r}\sin(qr)\,dr = \frac{q}{\mu^2+q^2},$$

Taking the limit $\mu \to 0$:

$$f(\theta) = -\frac{2m}{\hbar^2 q^2}\left(\frac{q_1 q_2}{4\pi\epsilon_0}\right)$$

Express in terms of E and θ Use $q^2 = 4k^2\sin^2(\theta/2)$ and $E = \frac{\hbar^2 k^2}{2m}$

$$f(\theta) = -\frac{2m}{\hbar^2(4k^2\sin^2(\theta/2))}\left(\frac{q_1 q_2}{4\pi\epsilon_0}\right)$$

$$f(\theta) = -\frac{2m}{4(\hbar^2 k^2)\sin^2(\theta/2)}\left(\frac{q_1 q_2}{4\pi\epsilon_0}\right)$$

$$f(\theta) = -\frac{2m}{4(2mE)\sin^2(\theta/2)}\left(\frac{q_1 q_2}{4\pi\epsilon_0}\right)$$

$$f(\theta) = -\frac{1}{4E\sin^2(\theta/2)}\left(\frac{q_1 q_2}{4\pi\epsilon_0}\right) = -\frac{q_1 q_2}{16\pi\epsilon_0 E\sin^2(\theta/2)}$$

$$\frac{d\sigma}{d\Omega} = \left(-\frac{q_1 q_2}{16\pi\epsilon_0 E\sin^2(\theta/2)}\right)^2 = \left(\frac{q_1 q_2}{16\pi\epsilon_0 E\sin^2(\theta/2)}\right)^2$$

8-(c)

$$\sigma_{\text{tot}} = \int \frac{d\sigma}{d\Omega} d\Omega = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta \left(\frac{q_1 q_2}{16\pi\epsilon_0 E\sin^2(\theta/2)}\right)^2$$

$$\sigma_{\text{tot}} = 2\pi\left(\frac{q_1 q_2}{16\pi\epsilon_0 E}\right)^2 \int_0^{\pi} \frac{\sin\theta}{\sin^4(\theta/2)}\, d\theta$$

As $\theta \to 0$, $\sin\theta \approx \theta$ and $\sin(\theta/2) \approx \theta/2$.

$$\int_0^{\pi} \frac{\sin\theta}{\sin^4(\theta/2)}\, d\theta \approx \int_0^{\epsilon} \frac{\theta}{(\theta/2)^4}\, d\theta = 16\int_0^{\epsilon} \frac{1}{\theta^3}\, d\theta$$

Analysis: The total cross-section is infinite. every particle in the beam is scattered.

Problem 9.

9-(a)

$$f(\theta) = -\frac{2m}{\hbar^2 q}\int_0^{\infty} rV(r)\sin(qr)\, dr$$

The given "soft-sphere" potential is $V(r) = V_0\Theta(a-r)$, which means $V(r) = V_0$ for $r < a$ and $V(r) = 0$ for $r > a$.Substitute the potential into the formula:

$$f(\theta) = -\frac{2mV_0}{\hbar^2 q}\int_0^a r\sin(qr)\,dr$$

$$\int_0^a r\sin(qr)\,dr = \left[r\left(-\frac{1}{q}\cos(qr)\right)\right]_0^a - \int_0^a \left(-\frac{1}{q}\cos(qr)\right)\,dr$$

$$= \left(-\frac{a}{q}\cos(qa) - 0\right) + \frac{1}{q}\int_0^a \cos(qr)\,dr$$

$$= -\frac{a}{q}\cos(qa) + \frac{1}{q}\left[\frac{1}{q}\sin(qr)\right]_0^a$$

$$= -\frac{a}{q}\cos(qa) + \frac{1}{q^2}(\sin(qa) - 0)$$

$$= \frac{\sin(qa) - qa\cos(qa)}{q^2}$$

$$f(\theta) = -\frac{2mV_0}{\hbar^2 q}\left[\frac{\sin(qa) - qa\cos(qa)}{q^2}\right]$$

$$f(\theta) = -\frac{2mV_0}{\hbar^2 q^3}\left[\sin(qa) - qa\cos(qa)\right]$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{2mV_0}{\hbar^2}\right)^2\frac{[\sin(qa) - qa\cos(qa)]^2}{q^6}$$

From $\frac{d\sigma}{d\Omega} = \left(\frac{2mV_0}{\hbar^2}\right)^2\frac{[\sin(qa)-qa\cos(qa)]^2}{q^6}$, $\sin(qa) = (qa) - \frac{(qa)^3}{6} + \mathcal{O}(q^5)\cos(qa) = 1 - \frac{(qa)^2}{2} + \mathcal{O}(q^4)$

$$\sin(qa) - qa\cos(qa) \approx= \frac{q^3a^3}{3}$$

$$\lim_{q\to 0} f(\theta) \approx \lim_{q\to 0}\left[-\frac{2mV_0}{\hbar^2 q^3}\left(\frac{q^3a^3}{3}\right)\right]$$

$$f = -\frac{2mV_0a^3}{3\hbar^2}$$

$$\sigma_{\text{tot}} = 4\pi\left|-\frac{2mV_0a^3}{3\hbar^2}\right|^2$$

$$\sigma_{\text{tot}} = \frac{16\pi m^2V_0^2a^6}{9\hbar^4}$$

9-(b)

In the low-energy limit, the scattering amplitude: $f(\theta) \approx -a$.

$$f_{9-(a)} = -\frac{2mV_0 a^3}{3\hbar^2}$$

By comparing the general form $f_{9-(a)} = -a$ with our result from part (a), we can identify the scattering length for this potential in the Born approximation:

$$a = \frac{2mV_0 a^3}{3\hbar^2}$$

$$\sigma_{\text{tot}} = 4\pi \left(\frac{2mV_0 a^3}{3\hbar^2}\right)^2 = \frac{16\pi m^2 V_0^2 a^6}{9\hbar^4}$$

Conclusion: The results are consistent. Both results yields the same.

Problem 10

10-1

$$f(\theta) = -\frac{2m}{\hbar^2 q}\int_0^\infty rV(r)\sin(qr)\,dr$$

$$f(\theta) = -\frac{2m}{\hbar^2 q}\int_0^\infty r(V_0 a\delta(r-a))\sin(qr)\,dr$$

$$f(\theta) = -\frac{2m}{\hbar^2 q}\left[V_0 a^2\sin(qa)\right] = -\frac{2mV_0 a^2\sin(qa)}{\hbar^2 q}$$

$$\frac{d\sigma}{d\Omega} = \left|-\frac{2mV_0 a^2\sin(qa)}{\hbar^2 q}\right|^2 = \left(\frac{2mV_0 a^2}{\hbar^2}\right)^2\frac{\sin^2(qa)}{q^2}$$

$$f_{\text{low-energy}} = \lim_{q\to 0}\left[-\frac{2mV_0 a^2\sin(qa)}{\hbar^2 q}\right]$$

$$f_{\text{low-energy}} = \lim_{q\to 0}\left[-\frac{2mV_0 a^2(qa)}{\hbar^2 q}\right]$$

$$f_{\text{low-energy}} = -\frac{2mV_0 a^2(a)}{\hbar^2} = -\frac{2mV_0 a^3}{\hbar^2}$$

$$\sigma_{\text{tot}} = 4\pi\left|-\frac{2mV_0 a^3}{\hbar^2}\right|^2$$

$$\sigma_{\text{tot}} = \frac{16\pi m^2 V_0^2 a^6}{\hbar^4}$$

10-(b)

From problem 4,

$$\sigma_{\text{tot}} = 4\pi a^2$$

From 10-(a),

$$f_{\text{low-energy}} = -\frac{2mV_0a^3}{\hbar^2}$$

$$f_{\text{low-energy}} = -a$$

$$\sigma_{\text{tot}} = 4\pi\left(\frac{2mV_0a^3}{\hbar^2}\right)^2 = \frac{16\pi m^2V_0^2a^6}{\hbar^4}$$

Conclusion: The results are consistent. The total cross-section derived from the low-energy limit of the Born approximation matches the general formula for s-wave scattering when the scattering length is identified from the same approximation.