Quantum Physics 2: Homework #5

[Total 120 points]

Due: 2025.11.15 (Sat)

Last update: 2025.11.3

Exercises: 2025.11.10(Mon)/11.11(Tue) 7 pm (Online)

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1. (10pts) Prove the Rayleigh's formula of Eq.10.28 (Eq.11.28 in the 2nd ed.):

$$e^{ikz} = \sum_{l=0}^{\infty} i^l (2l+1) j_l(kr) P_l(\cos\theta)$$

※ Referring to mathematical physics textbooks, prove the above identity.

2. (10pts) A particle of mass m and energy E is incident from the left on the potential

$$V(x) = \begin{cases} 0, & (x < -a). \\ -V_0, & (-a \le x \le 0). \\ \infty, & (x > 0). \end{cases}$$

(a) (5 pts) If the incoming wave is Ae^{ikx} where $k = \sqrt{2mE}/\hbar$, show that the reflected wave is given by $Ae^{-2ika}\left[\frac{k - ik'\cot(k'a)}{k + ik'\cot(k'a)}\right]e^{-ikx}$ where $k' = \sqrt{2m(E+V_0)}/\hbar$.

(b) (2 pts) Confirm that the reflected wave has the same amplitude as the incident wave.

(c) (3 pts) Find the phase shift δ for a very deep well ($E \ll V_0$).

※ Construct the wave functions for each region and use proper boundary conditions.

3. (15pts) Consider scattering by a soft-sphere potential $V(r) = V_0\Theta(a - r)$, where Θ is the step function. Using the partial wave analysis, obtain the phase shift, differential cross-section and total cross-section for low-energy scattering with $ka \ll 1$. Consider both repulsive $(V_0 > 0)$ and attractive $(V_0 < 0)$ cases, respectively.

4. (10pts) Consider scattering by a spherical delta-function well $V(r) = V_0 a\delta(r - a)$ for $V_0 > 0$. Using the partial wave analysis, obtain the scattering amplitude, differential cross-section and total cross-section for low-energy scattering with $ka \ll 1$.

5. (10pts) Prove the following formula. (Assume $r, k, \lambda > 0$.)

(a) (2pts) $\int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{e^{iqr}}{q^2+\lambda^2} = \frac{e^{-\lambda r}}{2\lambda}$

(b) (2pts) $\int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{e^{iqr}}{q^2-k^2 \mp i\eta} = \pm \frac{ie^{\pm ikr}}{2k}$

(c) (3pts) $\int \frac{d^3q}{(2\pi)^3} \frac{e^{i\mathbf{q}\cdot\mathbf{r}}}{q^2+\lambda^2} = \frac{e^{-\lambda r}}{4\pi r}$

(d) (3pts) $\int \frac{d^3q}{(2\pi)^3} \frac{e^{i\mathbf{q}\cdot\mathbf{r}}}{q^2-k^2 \mp i\eta} = \frac{e^{\pm ikr}}{4\pi r}$

Here, η is a positive infinitesimal number and $\boldsymbol{q} \cdot \boldsymbol{r} = qr\cos\theta$.

※ Obtain the above integrals using a calculus of residues in the theory of complex variables. Note that in (b) and (d), sign $\mp$ in front of $i\eta$ factor determines whether the wave is outgoing or incoming; in other words, it determines the boundary condition.

6. (15pts) Scattering from a Gaussian potential

(a) (5pts) Using the Born approximation, obtain the scattering amplitude $f(\theta)$ for a Gaussian potential $V(\boldsymbol{r}) = Ae^{-\mu r^2}$. Express your answer in terms of the constants A, μ, m and $k = \sqrt{2mE}/\hbar$, where m is the mass of the incident particle and E is the incident energy.

(b) (5pts) Draw $\frac{d\sigma}{d\Omega} \Big/ \frac{d\sigma}{d\Omega}\Big|_{\theta=0}$ as a function of θ for $-\pi < \theta < \pi$ *numerically* for $A = \text{Ry}$, $\mu = a_{\mathrm{B}}^{-2}$, $m = m_e$ and $k = a_{\mathrm{B}}^{-1}$. (Ry: Rydberg energy, a_{B}: Bohr radius, m_e: electron mass)

(c) (5 pts) Obtain the total cross section σ_{tot} and express it in terms of A, μ, m and k. Draw $\sigma_{\text{tot}}/a_{\mathrm{B}}^2$ as a function of E/Ry *numerically* for $A = \text{Ry}$, $\mu = a_{\mathrm{B}}^{-2}$ and $m = m_e$.

※ The differential cross-section depends on the form of interaction, thus by measuring it, we can obtain information on the interaction.

7. (15pts) Lippmann-Schwinger equation and *T*-matrix method

For an interaction potential V, consider a scattering problem for a particle with mass m and energy $E = \frac{\hbar^2 k^2}{2m}$:

$$(E - H_0)|\Psi\rangle = V|\Psi\rangle.$$

Then the formal solution for the outgoing wave $|\Psi^{(+)}\rangle$ is given by the Lippmann-Schwinger equation

$$|\Psi^{(+)}\rangle = |\Psi_0\rangle + \frac{1}{E - H_0 + i\eta} V |\Psi^{(+)}\rangle$$

where $|\Psi_0\rangle$ is the free-particle solution with the same energy E and η is a positive infinitesimal number. (See the discussion in the above Problem 5 and note the meaning of the

$\pm i\eta$. Also refer to Sakurai Ch.7.1.)

(a) (4pts) Show that $\Psi^{(+)}(\boldsymbol{x}) = \Psi_0(\boldsymbol{x}) - \frac{m}{2\pi\hbar^2}\int d^3x' \frac{e^{ik|x-x'|}}{|x-x'|} V(\boldsymbol{x}')\Psi^{(+)}(\boldsymbol{x}')$ where $\Psi(\boldsymbol{x}) \equiv \langle \boldsymbol{x}|\Psi\rangle$. Here we assume that V is local such that $\langle \boldsymbol{x}|V|\boldsymbol{x}'\rangle = V(\boldsymbol{x})\delta(\boldsymbol{x}-\boldsymbol{x}')$.

(b) (4pts) Define the transition operator T such that $V\left|\Psi^{(+)}\right\rangle = T|\Psi_0\rangle$. Show that

$$T = V + V\frac{1}{E - H_0 + i\eta}T = V + V\frac{1}{E - H_0 + i\eta}V + V\frac{1}{E - H_0 + i\eta}V\frac{1}{E - H_0 + i\eta}V + \cdots$$

(c) (4pts) Evaluate $\Psi^{(+)}(\boldsymbol{x})$ for $|\boldsymbol{x}| \gg |\boldsymbol{x}'|$ and show that the scattering amplitude from state $\boldsymbol{k}$ to $\boldsymbol{k}'$ is given by

$$f(\boldsymbol{k}',\boldsymbol{k}) = -\frac{m}{2\pi\hbar^2}\left\langle \boldsymbol{k}'\middle|V\middle|\Psi^{(+)}\right\rangle = -\frac{m}{2\pi\hbar^2}\langle \boldsymbol{k}'|T|\boldsymbol{k}\rangle.$$

Here $\langle \boldsymbol{x}|\boldsymbol{k}\rangle = e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$ normalization convention and $\Psi_0(\boldsymbol{x}) = e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$ were used.

(d) (3pts) From (b) and (c), we can express the scattering amplitude as $f(\boldsymbol{k}',\boldsymbol{k}) = \sum_{n=1}^{\infty} f^{(n)}(\boldsymbol{k}',\boldsymbol{k})$. Express explicitly $f^{(n)}(\boldsymbol{k}',\boldsymbol{k})$ using the $\boldsymbol{x}$-integral for $n = 1,2,3$. Referring to Fig.10.13 (or Fig.11.13 in the 2nd ed.), interpret the results.

※ Note that if we take only $n = 1$, it is called the first-order Born approximation. Using the T-matrix method, we can systematically obtain higher-order scattering amplitudes. The diagrammatic interpretation of the Born series is closely related to the Feynman diagrams you will learn about in the field theory course.

8. (15pts) Consider the Rutherford scattering in Problem 10.1 (or Problem 11.1 in the 2nd ed.) and obtain the differential cross-section using the following methods (a) and (b):

(a) (4pts) Classical theory using the equation of motion and impact parameter

(b) (7pts) First Born approximation

(c) (4pts) Calculate the total cross-section and interpret the result.

9. (10pts) Consider scattering by a soft-sphere potential $V(r) = V_0\Theta(a-r)$, where Θ is the step function.

(a) (7pts) Using the first Born approximation, obtain the differential cross-section. In the low-energy limit, find the total cross-section. Consider both repulsive $(V_0 > 0)$ and attractive $(V_0 < 0)$ cases, respectively.

(b) (3pts) Check that the results in (a) and Prob. 3 are consistent with each other in appropriate limits.

10. (10pts) Consider scattering by a spherical delta-function well $V(r) = V_0 a\delta(r-a)$ for $V_0 > 0$.

(a) (7pts) Using the first Born approximation, obtain the differential cross-section. In the low-energy limit, find the total cross-section.

(b) (3pts) Check that the results in (a) and Prob. 4 are consistent with each other in appropriate limits.

※ Problems 8, 9, and 10 compare scattering amplitudes obtained from different methods for a given potential. Check that the obtained results are consistent with each other.