max-gurewitz-sutton-barto-reinforcement-exercises-ch-2

December 2, 2018

2.1:

When using an ϵ-greedy action selection criterion, in the case of two actions and $\epsilon = 0.5$, actions would be selected at random half of the time. Of the times when actions are selected at random, the greedy action will be select half of the time. Meaning the greedy action will be selected a total of 75% of the time.

2.2:

Given that all of the reward estimates are initialized to 0, the first action will be selected arbitrarily, possible at random.

The following actions must have been selected at random as a result of the ϵ case: A_2, A_5.

This can be demonstrated by observing that $Q_1(1) = 1 > Q_1(2) = 0$ and $Q_4(2) = \frac{R_2+R_3+R_4}{3} = \frac{5}{3} > Q_4(3) = 0$.

Regarding the question of when random ϵ selection could have occured, it could have occured at any time step after the first.

2.3:

In the long run $\epsilon = 0.01$ will achieve the highest cumulative reward and probability of selecting the correct answer. The disadvantage of $\epsilon = 0.01$ relative to $\epsilon = 0.1$ is that it will cause our agent to take longer to find the optimal action, but it will select that action more consistently once it has been found.

The optimal action has a reward of 1.55. The expected reward per action once the agent has found the best possible action is equal to the sum of the greedy case and the ϵ random case: $1.55(1-\epsilon)+\epsilon\sum_{i=1}^{10}\frac{q_*(i)}{10}$.

So the difference between the expected long term reward per action in the $\epsilon = .01$ and $\epsilon = .1$ cases is $1.55(.1 - .01) = 0.1395$.

2.4:

If α is not constant, but instead varies with with each update step n, then our incremental update rule utilizing α_n can be expressed as follows:

$$\begin{aligned}
Q_{n+1} &= Q_n + \alpha_n[R_n - Q_n] && (1)\\
&= \alpha_n R_n + (1-\alpha_n)Q_n && (2)\\
&= \alpha_n R_n + (1-\alpha_n)(\alpha_{n-1}R_{n-1} + (1-\alpha_{n-1})Q_{n-1}) && (3)\\
&= \alpha_n R_n + (1-\alpha_n)\alpha_{n-1}R_{n-1} + (1-\alpha_n)(1-\alpha_{n-1})(Q_{n-2} + \alpha_{n-2}[R_{n-2} - Q_{n-2}]) && (4)\\
&= \alpha_n R_n + (1-\alpha_n)\alpha_{n-1}R_{n-1} + (1-\alpha_n)(1-\alpha_{n-1})((1-\alpha_{n-2})Q_{n-2} + \alpha_{n-2}R_{n-2})) && (5)\\
&= \alpha_n R_n + (1-\alpha_n)\alpha_{n-1}R_{n-1} + (1-\alpha_n)(1-\alpha_{n-1})\alpha_{n-2}R_{n-2} + && (6)\\
&... + Q_1\prod_{i=1}^{n}(1-\alpha_i) + \alpha_1 R_1\prod_{j=1}^{n-1}(1-\alpha_{n-j+1}) && (7)\\
&= Q_1\prod_{i=1}^{n}(1-\alpha_i) + \sum_{i=1}^{n}\alpha_i R_i\prod_{j=1}^{n-i}(1-\alpha_{n-j+1}) && (8)
\end{aligned}$$

2.5:

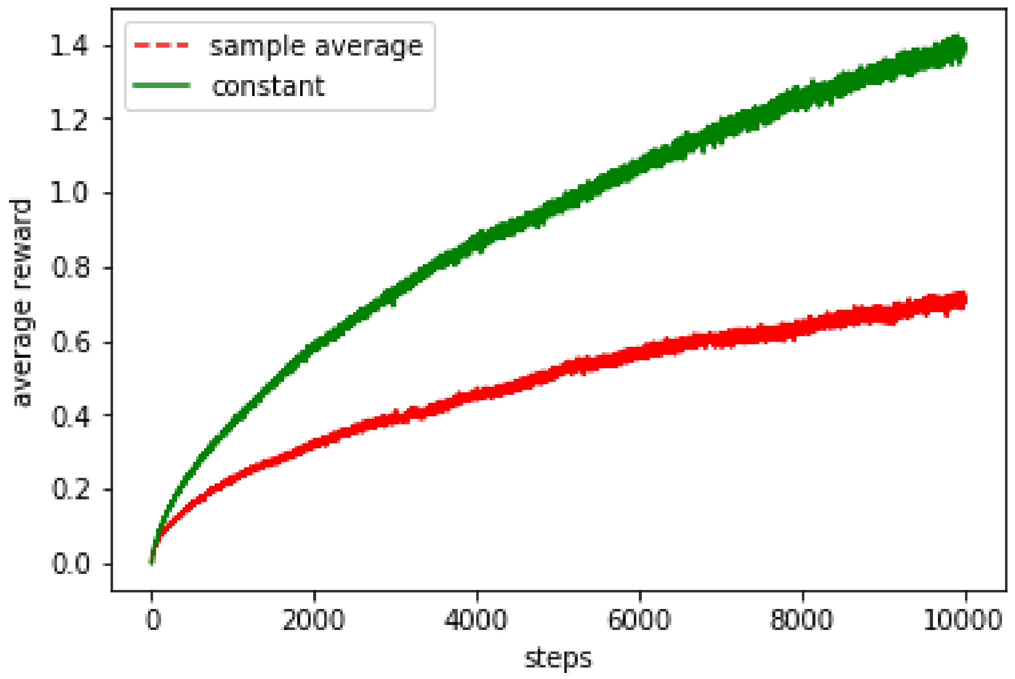
```
In [108]: %matplotlib inline

          import numpy as np
          import time
          import random
          import math
          import numpy.random
          from enum import Enum
          import matplotlib.pyplot as plt

          # EXERCISE 2.5

          runs = 2000

          methods = Enum("BanditMethod", "SAMPLE_AVERAGE EPSILON_GREEDY UCB GRADIENT_BANDIT
          OPTIMISTIC_GREEDY")

          """
          returns data in the format

          {
              "rewards": np.array([1.5, .2, ...]),
              "optimal": np.array([1, 0, ...])
```

```
}
"""
def ten_armed_bandit(steps, method = methods.EPSILON_GREEDY, constant_alpha = .1, Q0 =
0.0, c = 0, epsilon = .1):
    q = np.zeros(10, dtype = np.dtype("float64"))

    if method == methods.GRADIENT_BANDIT:
        average_reward = 0
        H = np.zeros(10, dtype = np.dtype("float64"))

    else:
        Q = np.array(list(map(lambda x : Q0, range(10))), dtype = np.dtype("float64"))

    N_a = np.zeros(10)

    rewards = []
    optimal = []

    for i in range(steps):
        arg_max_reward = None
        arg_max_actions = []

        if method == methods.GRADIENT_BANDIT:
            N_total = 0

            for H_a in H:
                N_total = N_total + math.e ** H_a

            prob_A = []

            for H_a in H:
                prob_A.append((math.e ** H_a) / N_total)

        for j in range(len(q)):
            if method == methods.UCB:
                if N_a[j] == 0:
                    arg_max_actions = [j]
                    break
                else:
                    estimated_reward = Q[j] + c * ((math.log(i + 1)/N_a[j])**.5)

            elif method == methods.GRADIENT_BANDIT:
                estimated_reward = prob_A[j]
            else:
                estimated_reward = Q[j]

            if arg_max_reward == None or estimated_reward > arg_max_reward:
                arg_max_reward = estimated_reward
                arg_max_actions = [j]

            elif estimated_reward == arg_max_reward:
                arg_max_actions.append(j)

        if (method in [methods.SAMPLE_AVERAGE, methods.EPSILON_GREEDY]) and
random.uniform(0, 1) < epsilon:
            selected_action = random.choice(range(10))
        else:
            selected_action = random.choice(arg_max_actions)

        N_a[selected_action] = N_a[selected_action] + 1
        actual_reward = q[selected_action]
        was_action_optimal = 1 if actual_reward == max(q) else 0

        rewards.append(actual_reward)
        optimal.append(was_action_optimal)

        if method == methods.GRADIENT_BANDIT:
            average_reward = average_reward + (1/(i + 2))*(actual_reward -
average_reward)

            for j in range(10):
                if j == selected_action:
                    H[j] = H[j] + constant_alpha * (actual_reward - average_reward) * (1
- prob_A[j])
                else:
                    H[j] = H[j] - constant_alpha * (actual_reward - average_reward) *
prob_A[j]


        else:
            alpha = 1/(i + 1) if (method in [methods.SAMPLE_AVERAGE, methods.UCB]) else
constant_alpha
            Q[selected_action] = Q[selected_action] + alpha * (actual_reward -
Q[selected_action])

        random_walk_values = np.random.normal(0, .01, len(q))
        q = q + random_walk_values

    return {
        "rewards": np.array(rewards),
        "optimal": np.array(optimal)
    }


def average_bandit_results(sample_average = False, steps = 10000):
    rewards = np.zeros(steps)
    optimal = np.zeros(steps)
    method = methods.SAMPLE_AVERAGE if sample_average else methods.EPSILON_GREEDY

    for i in range(runs):
        results = ten_armed_bandit(steps, method = method)
        rewards = rewards + (results["rewards"] / runs)
        optimal = optimal + (results["optimal"] / runs)

    return {
        "rewards": rewards,
        "optimal": optimal
    }

def plot_average_rewards(with_sample_average, without_sample_average):
    plt.plot(with_sample_average, "r--", label = "sample average")
    plt.plot(without_sample_average, "g", label = "constant")
    plt.xlabel("steps")
    plt.ylabel("average reward")
    plt.legend()
    plt.show()

def plot_average_optimality(with_sample_average, without_sample_average):
    plt.plot(with_sample_average, "r--", label = "sample average")
    plt.plot(without_sample_average, "g", label = "constant")
    plt.xlabel("steps")
    plt.ylabel("percent optimal")
    plt.legend()
    plt.show()
```

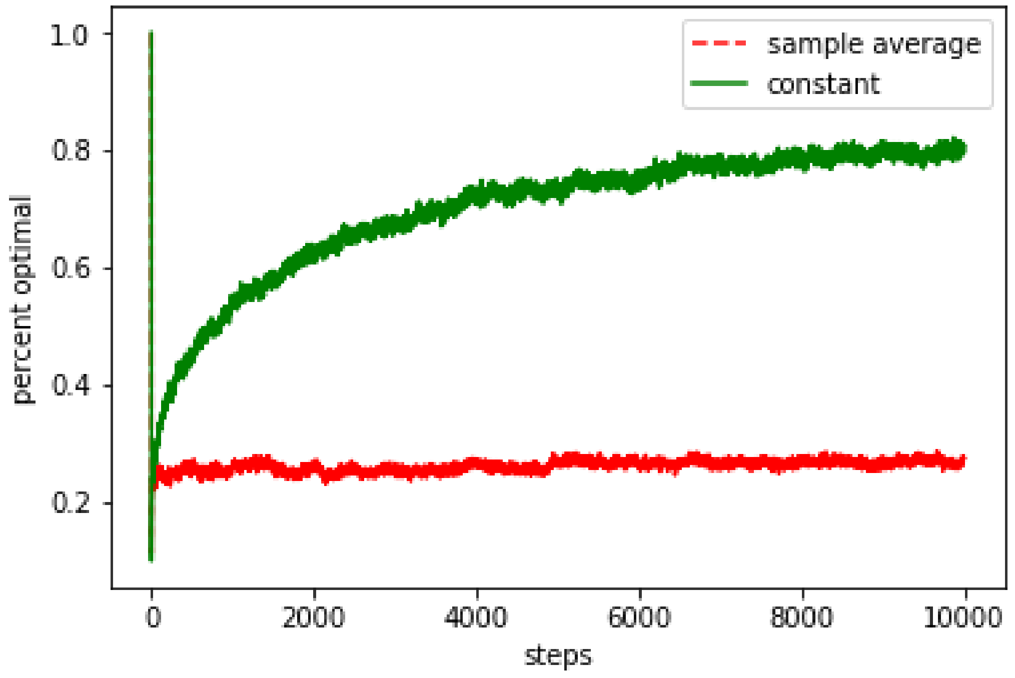
```
In [ ]: with_sample_average = average_bandit_results(True)
```

```
In [148]: without_sample_average = average_bandit_results()
```

```
In [149]: plot_average_rewards(with_sample_average["rewards"], without_sample_average["rewards"])
```



```
In [150]: plot_average_optimality(with_sample_average["optimal"],
          without_sample_average["optimal"])
```



We can observe above that the sample average method performs poorly for non-stationary reward distributions.

2.6:

The optimistic greedy agent's initial spike in rewards can be explained as follows.

$q_*(a)$ is normally distributed with a mean of 0 and a variance of 1. This means $Q_1 > max\{q_*(a)\}$ in the vast majority of cases.

For the reward distributions where the above is true, agents with the optimistic greedy action-value method will attempt every action at least once at the onset.

The spike we observe is a product of agents selecting the optimal action once during this preliminary period.

2.7:

The following demonstrates that Q_1, the initial reward estimates, does not impact the step size Q_n when using a step size coefficient,

$$\beta_n = \frac{\alpha}{\bar{o}_n}$$

where,

$$\bar{o}_n = \bar{o}_{n-1} + \alpha(1 - \bar{o}_{n-1})$$

for \$ n > 0\$. If you expand $\bar{o}_n$ you arrive at,

$$\bar{o}_n = \alpha \sum_{i=1}^{n} (1-\alpha)^{i-1}.$$

So,

$$\beta_n = \frac{1}{\sum_{i=0}^{n} (1-\alpha)^{i-1}}.$$

If we plug this coefficient into our solution for exercise 2.4,

$$
\begin{aligned}
Q_n &= Q_1 \prod_{i=1}^{n}(1-\beta_i) + \sum_{i=1}^{n} \beta_i R_i \prod_{j=1}^{n-i}(1-\beta_{n-j+1}) && (9)\\
&= Q_1 \prod_{i=1}^{n}(1 - \frac{1}{\sum_{j=1}^{i}(1-\alpha)^{j-1}}) + ... && (10)\\
&= Q_1(1 - \frac{1}{(1-\alpha)^{1-1}}) \prod_{i=2}^{n}(1 - \frac{1}{\sum_{j=1}^{i}(1-\alpha)^{j-1}}) + ... && (11)\\
&= Q_1(0) + ... && (12)\\
&= \sum_{i=1}^{n} \beta_i R_i \prod_{j=1}^{n-i}(1-\beta_{n-j+1}) && (13)
\end{aligned}
$$

So Q_n is not biased by Q_1.
2.8:

The spike observed on step 11 in figure 2.4 is a consequence of the $N_t(a)$ term in the UCB action selection criterion. Up until $N_t(a) > 0$ for each action, UCB will behave similarly to the optimistic greedy criterion, selecting each action once before resuming its standard behavior. This is a consequence of UCB treating each eaction as reward maximizing until it has been selected once.

2.9:

The softmax distribution with two actions consist of,

$$\pi_t(a) = \frac{e^{H_t(a)}}{e^{H_t(1)} + e^{H_t(2)}}.$$

Logistic classifiers are binary classifiers. In this case let $L(a)$ represent the probability assigned by a logistic classifier of selecting a, where $a \in \{1, 2\}$. $L(2)$ is formulated as,

$$L(2) = \frac{1}{1 + e^{-f(2)}}$$

where $f(a)$ is a function which accepts an action and outputs a preference score. In the case of the softmax distribution, the probability of selection action $a = 2$ is,

$$
\begin{aligned}
\pi_t(2) &= \frac{e^{H_t(2)}}{e^{H_t(1)} + e^{H_t(2)}} && (14)\\
&= \frac{1}{1 + e^{H_t(1) - H_t(2)}}. && (15)
\end{aligned}
$$

Now let $f(a)$ be a function which outputs a preference score such that $f(2) = -(H_t(1) - H_t(2))$, so $\pi_t(2)$ can be re-expressed as,

$$\pi_t(2) = \frac{1}{1 + e^{-f(2)}} = L(2).$$

The above equality demonstrates the equivalence of logistic classifiers and the softmax distributions with two actions, or classes.

2.10:

We are presented with a 2-armed bandit problem, and our two actions vary between two sets of reward values for scenario A and scenario B, $Q_A(1) = .1$, $Q_B(2) = .2$, and $Q_B(1) = .9$, $Q_B(2) = .8$, where both scenarios occur with a probability of .5.

In the case that we are unable to determine what scenario we are in, the expected reward of each action is,

$$Q(a) = \frac{Q_A(a) + Q_B(a)}{2}$$

In this case $Q(1) = Q(2) = .5$. The expected rewards are the same for both action, meaning we can achieve the best possible result of .5 by picking actions arbitrarily, possible at random.

In the case that we are told at each time step whether we are facing scenario A or scenario B, we should select action $a = 2$ in scenario A, and action $a = 1$ in scenario B. So the average reward at each time step will be,

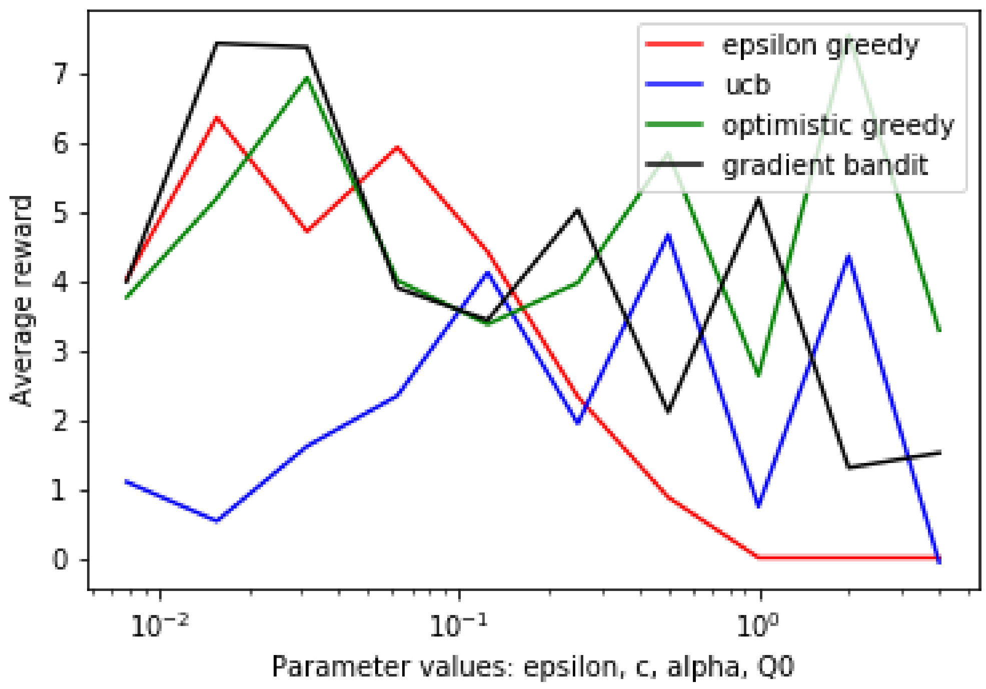
$$Q = \frac{Q_A(2) + Q_B(1)}{2} = .55$$

```
In [111]: # EXERCISE 2.11

          def compare_bandits_parameters(steps = 200000):
              # epsilon greedy varied over epsilon
              # ucb varied over c
              # gradient bandit varied over alpha
              # optimistic epsilon greedy varied over Q0
              average_range = int((steps / 5) - 1)
              epsilon_greedy = []
              optimistic_greedy = []
              ucb = []
              gradient_bandit = []

              for i in range(10):
                  parameter_value = 2 ** (i - 7)

                  if parameter_value < 1:
                      epsilon_greedy.append(np.average(ten_armed_bandit(steps, epsilon =
          parameter_value)["rewards"][average_range:]))
                  else:
                      epsilon_greedy.append(0)

                  optimistic_greedy.append(np.average(ten_armed_bandit(steps, Q0 =
          parameter_value, method = methods.OPTIMISTIC_GREEDY)["rewards"][average_range:]))

                  ucb.append(np.average(ten_armed_bandit(steps, c = parameter_value, method =
          methods.UCB)["rewards"][average_range:]))

                  gradient_bandit.append(np.average(ten_armed_bandit(steps, constant_alpha =
          parameter_value, method = methods.GRADIENT_BANDIT)["rewards"][average_range:]))

              return {
```

```
            "epsilon_greedy": np.array(epsilon_greedy),
            "optimistic_greedy": np.array(optimistic_greedy),
            "ucb": np.array(ucb),
            "gradient_bandit": np.array(gradient_bandit),
        }

    def plot_bandits_parameter_comparison(comparison):
        parameter_values = list(map(lambda i: 2 ** (i - 7), range(10)))
        plt.plot(parameter_values, comparison["epsilon_greedy"], "r", label = "epsilon
    greedy")
        plt.plot(parameter_values, comparison["ucb"], "b", label = "ucb")
        plt.plot(parameter_values, comparison["optimistic_greedy"], "g", label = "optimistic
    greedy")
        plt.plot(parameter_values, comparison["gradient_bandit"], "k", label = "gradient
    bandit")
        plt.xscale('log')
        plt.xlabel("Parameter values: epsilon, c, alpha, Q0")
        plt.ylabel("Average reward")
        plt.legend()
        plt.show()
```

```
In [116]: bandits_parameter_comparison = compare_bandits_parameters(200000)
```

```
In [117]: plot_bandits_parameter_comparison(bandits_parameter_comparison)
```



As expected the UCB method performs poorly for non-stationary reward distributions, and epsilon greedy and gradient bandit perform the best.