Contents

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1 Stokes Structures

Stokes structures contain exactly necessary information to classify meromorphic classification, i.e. with the Stokes structures we are able to construct a space PROBLEM: Why space, which is isomorphic to the classifying set.

A great overview of this topic is given by Varadarajan in [Var96]. Other resources we will use are for example Sabbah's book [Sab07, section II] for Section ??. In the Sections 1.1 and 1.2 will Loday-Richaud's paper [Lod94] and her book [Lod14, Sec.4] be useful. Stokes groups are also discussed Boalch's paper [Boa01] (resp. his thesis [Boa99]) which looks only at the single leveled case or the paper [MR91] from Martinet and Ramis.

Let $(\mathcal{M}^{nf}, \nabla^{nf})$ be a fixed model with the corresponding normal form A^0 and let us also fix a normal solution $\mathcal{Y}_0$ of A^0. The purpose of the next section (Section ??) is, to proof the Malgrange-Sibuya Theorem. It states that the classifying set $\mathcal{H}(A^0)$ is via an map exp isomorphic to the first non abelian cohomology $H^1(S^1; \Lambda(A^0)) =: \mathcal{S}t(A^0)$ of the Stokes sheaf $\Lambda(A^0)$. In Section 1.2 we will improve the Malgrange-Sibuya Theorem by showing that each 1-cohomology class in $\mathcal{S}t(A^0)$ contains a unique 1-cocycle of a special form called *the Stokes cocycle* (cf. Section 1.1). The morphism, which maps each Stokes cocycle to its corresponding 1-cocycle will be denoted by h. This will be further improved in Section ??.

If one introduces the map g, which arises from the theory of summation and takes an equivalence class (resp. an ambassador of such a class) and returns a corresponding Stokes cocycle in an canonically way (cf. Appendix ?? where the theory of summation will be roughly discussed), as a black-box one can write the following commutative diagram.

$$\begin{tikzcd}
\mathcal{H}(A^0) \arrow[r, "\exp"] \arrow[d, "g"'] & \mathcal{S}t(A^0) \\
\prod_{\theta \in \mathbb{A}} \mathrm{Sto}_\theta(A^0) \arrow[ur, "h"'] &
\end{tikzcd}$$

This diagram will be enhanced in Section ?? by adding a couple of isomorphisms.

1.1 The Stokes groups

Here we want to introduce the notion of Stokes groups. They are for example also introduced by Loday-Richaud in [Lod94; Lod14] or section 4 of [MR91] by Martinet and Ramis.

Let us recall, that the normal form A^0 can be written as $A^0 = Q'(t^{-1}) + L\frac{1}{t}$ and a normal solution is given by $\mathcal{Y}_0(t) = t^L e^{Q(t^{-1})}$ (cf. Proposition **??**), where

- $Q(t^{-1}) = \bigoplus_{j\in\{1,\dots,s\}} q_j(t^{-1}) \cdot \mathrm{id}_{n_j}$ and
- the block structure of L is finer then the structure of Q (cf. Definition **??**).

Let $\{q_1(t^{-1}), \dots, q_s(t^{-1})\}$ be the *set of all determining polynomials of* $[A^0]$ and denote by

$$\mathcal{Q}(A^0) := \left\{ q_j - q_l \mid q_j \text{ and } q_l \text{ determining polynomials of } [A^0],\ q_j \neq q_l \right\}$$

the *set of all determining polynomials of* $[\operatorname{End} A^0]$. Instead of $q_j - q_l \in \mathcal{Q}(A^0)$ we will sometimes talk of (ordered) pairs $(q_j, q_l) \in \mathcal{Q}(A^0)$.

Definition 1.1

We call

- $a_{jl} \in \mathbb{C}\backslash\{0\}$ the *leading factor*,
- $\frac{a_{jl}}{t^{k_{jl}}}$ the *leading coefficient* and
- $k_{jl} \in \mathbb{Q}$ the *degree*

of $q_j - q_l \in \mathcal{Q}(A^0)$ if

$$q_j - q_l \in \left\{ \frac{a_{jl}}{t^{k_{jl}}} + h \mid h \in o(t^{-k_{jl}}), a_{jl} \neq 0 \right\}.$$

Remark 1.1.1

1. It is obvious that $k_{jl} = k_{lj}$ and $\frac{a_{jl}}{t^{k_{jl}}} = \frac{-a_{lj}}{t^{k_{lj}}}$.
2. In Boalch's paper [Boa01] (and also in [Boa99]) are the degrees of the pairs always incremented by one. We will prefer the other notion, which is also used in Loday-Richaud's paper [Lod94].
3. In Loday-Richaud's book [Lod14, Def.4.3.6] a_{jl} is negated to be consistent with calculations at ∞. Here this is not necessary, since we use the clockwise orientation on S^1 (cf. Definition 1.5).

The degrees of the elements in $\mathcal{Q}(A^0)$ are defined to be the *levels* of A^0. The set of all levels of A^0 will be denoted by

$$\mathcal{K} = \{k_1 < \cdots < k_r\} \subset \mathbb{Q}.$$

Remark 1.1.2

The system $[A^0]$ is unramified if and only if $\mathcal{K} \subset \mathbb{Z}$. Since we only want to consider the unramified case, this will be always the case.

1.1.1 Anti-Stokes directions and the Stokes group

Definition 1.2

[Lod94, p. I.4]

Let $k \in \mathbb{N}$ and $a \in \mathbb{C}$. We say that an exponential $e^{q(t^{-1})}$, where $q(t^{-1}) \in \frac{a}{t^k} + o(t^{-k})$, has *maximal decay in a direction* $\theta \in S^1$ if and only if $ae^{-ik\widetilde{\theta}}$ is real negative. We say that an matrix has maximal decay, if every entry has maximal decay.

[HTT08, p. 130], [Lod14, p. 79]

$be^0 \in \mathbb{C}$ corresponding to has maximal decay if and only if PROBLEM: 1 has maximal decay?

On the determining polynomials of $[A^0]$ we define the following (partial) order relations:

Definition 1.3

Let $\widetilde{\theta}$ be a determination of $\theta \in S^1$.

- We define the relation $\boldsymbol{q_j} \underset{\widetilde{\boldsymbol{\theta}}}{\prec} \boldsymbol{q_l}$ to be equivalent to the condition

  $e^{(q_j - g_l)(t^{-1})}$ is flat at 0 in a neighbourhood of the direction $\widetilde{\theta}$, i.e. if and only if $\Re(a_{jl}e^{-ik_{jl}\widetilde{\theta}}) < 0$.

- Let us define another relation $\boldsymbol{q_j} \underset{\widetilde{\boldsymbol{\theta}}}{\prec\!\!\!\cdot} \boldsymbol{q_l}$ equivalent to

  $e^{(q_j - g_l)(t^{-1})}$ is of maximal decay in the direction $\widetilde{\theta}$,

  which by itself is equivalent to

  $a_{jl}e^{-ik_{jl}\widetilde{\theta}}$ is a real negative number, i.e. $q_j \underset{\widetilde{\theta}}{\prec} q_l$ and $\Im(a_{jl}e^{-ik_{jl}\widetilde{\theta}}) = 0$.

Remark 1.3.1

In the unramified case do these relations not depend on the determination $\widetilde{\theta}$ of θ. As a consequence we will only write $\underset{\theta}{\prec}$ and $\underset{\theta}{\prec\!\!\!\cdot}$.

To understand the previous definition better, it is convenient to look closer at functions of the form $f : \theta \mapsto ae^{-ik\theta}$, $k \in \mathbb{Z}$, corresponding to some pair (q_j, q_l). Write a as $a = |a|e^{i\arg(a)}$, thus the function f writes as

$$\begin{aligned} f(\theta) &= |a|e^{i(\arg(a) - k\theta)} \\ &= |a|(\cos(\arg(a) - k\theta) + i\sin(\arg(a) - k\theta)) . \end{aligned}$$

In the Figure 1.1, we illustrate the real and the imaginary part of f.

The graphs, corresponding to the flipped pair (q_l, q_j) are then obtained by the transformation $\arg(a) \to \arg(-a) = \arg(a) + \pi$, i.e. the shift by $\frac{\pi}{k}$ to the right. This $\frac{\pi}{k}$ is exactly a half period, thus the new graphs are obtained by mirroring at the line $t = 0$.

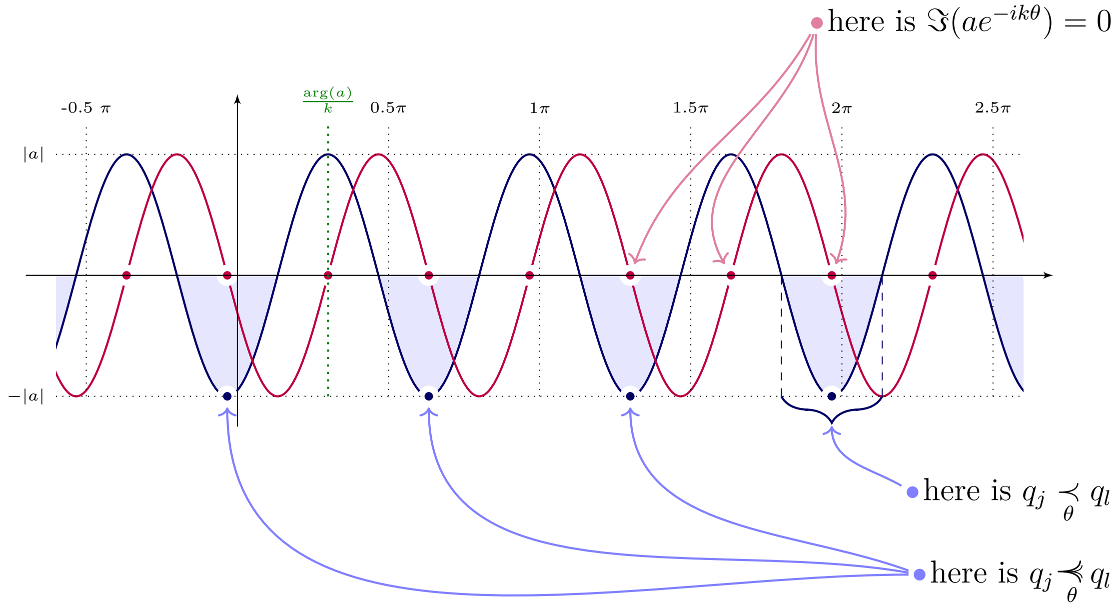


Figure 1.1: In this plot is the real part of $f(\theta) = ae^{-ik\theta}$, corresponding to some pair (q_j, q_l), in blue and the imaginary part in purple sketched.

Remark 1.4

Let k_{jl} be the degree of $q_j - q_l$. It is easy to see (cf. Figure 1.1), that the condition $q_j \underset{\theta}{\prec} q_l$ is equivalent to

there is a $\theta' \in U(\theta, \frac{\pi}{k_{jl}})$ such that $q_j \underset{\theta}{\prec\!\!\prec} q_l$.

Let us now use the defined relations to say, which are the interesting directions of S^1.

Definition 1.5

Let $\theta \in S^1$ be an direction.

1. θ is an *anti-Stokes direction* if there is at least one pair (q_j, q_l) in $\mathcal{Q}(A^0)$, which satisfies $q_j \underset{\theta}{\prec\!\!\prec} q_l$.

   Let $\mathbb{A} = \{\alpha_1, \ldots, \alpha_\nu\}$ denote the set of all anti-Stokes directions in a clockwise ordering. For a uniform notation later, define $\mathbb{A}$ to contain a single, arbitrary direction if $\mathcal{K} = \{0\}$.

Remark 1.5.1

The clockwise ordering is chosen, similar to Loday-Richaud's paper [Lod94], since the calculations are then compatible with the calculations, which look at ∞ and take a counterclockwise ordering. Boalch uses in [Boa01] and [Boa99] the inverse ordering, but looks also at 0, thus there might be some incompatibilities. In Loday-Richaud's book [Lod14] this problem is solved by an additional minus sign for some coefficient.

2. The direction θ is a *Stokes direction* if there is at least one pair (q_j, q_l) in $\mathcal{Q}(A^0)$, which satisfies neither $q_j \underset{\theta}{\prec} q_l$ nor $q_l \underset{\theta}{\prec} q_j$.

Let $\mathbb{S} = \{\sigma_1 < \cdots < \sigma_\mu\}$ be the set of Stokes directions.

We will use the Greek letter α whenever we want to emphasize that a direction is an anti-Stokes direction. For generic directions, we will use θ. In fact will most of the following definitions and constructions work for every $\theta \in S^1$, but the Stokes group (cf. Definition 1.7) for example will be trivial for every $\theta \notin \mathbb{A}$. Thus the interesting directions are only the anti-Stokes directions $\alpha \in \mathbb{A}$.

Lemma 1.6

Let $\alpha \in \mathbb{A}$ together with a pair $(q_j - q_l)(t^{-1}) \in \mathcal{Q}(A^0)$ of degree k_{jl}, such that $q_j \underset{\alpha}{\prec\!\!\!\cdot} q_l$ be given. We then know for every $m \in \mathbb{N}$ that

$$\underbrace{\alpha + m\frac{\pi}{k_{jl}}}_{\stackrel{!!}{\alpha'}} \in \mathbb{A}\,.$$

Especially is either $q_j \underset{\alpha'}{\prec\!\!\!\cdot} q_l$ (in the case, when m is even) or $q_l \underset{\alpha'}{\prec\!\!\!\cdot} q_j$ (when m is uneven) satisfied (see Figure 1.1).

Corollary 1.6.1

It follows that in the case $\mathcal{K} = \{k\}$, the set $\mathbb{A}$ has $\frac{\pi}{k}$-rotational symmetry.

Proof. Let (j, l) be a pair such that $q_j \underset{\alpha}{\prec\!\!\!\cdot} q_l$, i.e. such that $a_{jl}e^{-ik_{jl}\alpha} \in \mathbb{R}_{<0}$. Hence for $m \in \mathbb{N}$ [Boa99, p. 8]

$$a_{jl}e^{-ik_{jl}\left(\alpha + m\frac{\pi}{k_{jl}}\right)} = a_{jl}e^{-ik_{jl}\alpha}e^{-im\pi} = \begin{cases} a_{jl}e^{-ik_{jl}\alpha} \in \mathbb{R}_{<0} & \text{, if } m \text{ is even} \\ -a_{jl}e^{-ik_{jl}\alpha} \in \mathbb{R}_{>0} & \text{, if } m \text{ is uneven} \end{cases}$$

is in the case when m is even, also real and negative. In the other case, when n is uneven, we use that $a_{jl} = -a_{lj}$ and $k_{jl} = k_{lj}$ to obtain $a_{lj}e^{-ik_{lj}\left(\alpha + m\frac{\pi}{k_{lj}}\right)} \in \mathbb{R}_{<0}$.

Thus, for $\alpha' := \alpha + m\frac{\pi}{k_{jl}}$, we have $\alpha' \in \mathbb{A}$ since

- $q_j \underset{\alpha'}{\prec} q_l$ when m is even or
- $q_l \underset{\alpha'}{\prec} q_j$ when m is uneven.

□

As a subgroup of the stalk at θ of the in Definition **??** defined Stokes sheaf $\Lambda(A^0)$ we define the Stokes group as follows.

Definition 1.7

Define the *Stokes group*

$$\mathrm{Sto}_\theta(A^0) := \left\{ \varphi_\theta \in \Lambda_\theta(A^0) \mid \varphi_\theta \text{ has maximal decay at } \theta \right\}$$

whose elements are called *Stokes germs.* TODO: This is in fact a group, since...

Remark 1.7.1

For $\theta \notin \mathbb{A}$ the group $\mathrm{Sto}_\theta(A^0)$ is trivial, since at θ no flat isotropy has maximal decay, but the identity.

1.1.2 Stokes matrices

[Boa99, 9f], [Lod94, ??] Stokes matrices, which Wasow calls in his book [Was02] Stokes multipliers and Boalch calls Stokes factors in [Boa01; Boa99], arise either

> as faithful representations of Stokes germs

or, if one starts by comparing the actual fundamental solutions on arcs, as

> the matrices describing the blending between two adjacent fundamental solutions, with some additional assumptions (cf. Definition [Lod14, p. 80]).

Definition 1.8

Let us use

$$\boldsymbol{\delta}_{jl} := \begin{cases} 0 \in \mathbb{C}^{n_j \times n_l} & \text{, if } j \neq l \\ \mathrm{id} \in \mathbb{C}^{n_j \times n_l} & \text{, if } j = l \end{cases}$$

as a block version of Kronecker's delta corresponding to the structure of the normal solution $\mathcal{Y}_0$, which was fixed. Define the group

$$\mathbb{S}\mathrm{to}_\theta(A^0) = \left\{ K = (K_{jl})_{j,l \in \{1,\dots,s\}} \in \mathrm{GL}_n(\mathbb{C}) \mid K_{jl} = \boldsymbol{\delta}_{jl} \text{ unless } q_j \underset{\theta}{\prec} q_l \right\}$$

of all *Stokes matrices* of A^0 in the direction θ. They will arise as a faithful representation (cf. Section **??**) of $\mathrm{Sto}_\theta(A^0)$.

Remark 1.8.1

There is obviously a bijection $\vartheta_\theta : \prod_{q_j \not\prec_\theta q_l} \mathbb{C}^{n_j \cdot n_l} \xrightarrow{\cong} \mathbb{S}\mathrm{to}_\theta(A^0)$.

Proposition 1.9

In this situation is [Lod94, Def.I.4.7] [Lod14, 78f]

$$\begin{aligned} \rho_\theta : \mathrm{Sto}_\theta(A^0) &\longrightarrow \mathbb{S}\mathrm{to}_\theta(A^0) \\ \varphi_\theta &\longmapsto C_{\varphi_\theta} := \mathcal{Y}_0 \varphi_\theta \mathcal{Y}_0^{-1} \end{aligned}$$

an isomorphism which maps a germ of $\mathrm{Sto}_\theta(A^0)$ to the corresponding Stokes matrix C_{φ_θ} such that Boalch uses $C_{\varphi_\theta} := \mathcal{Y}_0^{-1}\varphi_\theta\mathcal{Y}_0$

$$\varphi_\theta(t)\mathcal{Y}_0(t) = \mathcal{Y}_0(t) C_{\varphi_\theta} \tag{1.1}$$

near θ. The matrix C_{φ_θ} is then called a *representation of* φ_θ.

Remark 1.9.1

1. In the ramified case does this morphism depend on the choice of the determination $\widetilde{\theta}$ of θ and the corresponding choice of a realization of the fundamental solution with that determination of the argument near the direction θ (cf. [Lod94] or [Lod14, 78f]).
2. This construction defines also a morphism, which takes a germ $\varphi_\theta \in \Lambda_\theta(A^0) \supset \mathrm{Sto}_\theta(A^0)$ into its unique representation matrix [Lod94, Defn.I.4.7]

$$C_{\varphi_\theta} \in \underset{\substack{\cup \\ \mathbb{S}\mathrm{to}_\theta(A^0)}}{\widehat{\mathbb{S}\mathrm{to}}_\theta(A^0)} := \left\{ (K_{jl})_{j,l\in\{1,\dots,s\}} \in \mathrm{GL}_n(\mathbb{C}) \mid K_{jl} = \boldsymbol{\delta}_{jl} \text{ unless } q_j \underset{\theta}{\prec} q_l \right\}$$

and there is a bijection $\widehat{\vartheta}_\theta : \prod_{q_j \prec_\theta q_l} \mathbb{C}^{n_j \cdot n_l} \xrightarrow{\cong} \widehat{\mathbb{S}\mathrm{to}}_\theta(A^0)$.

Does this define a local-constant sheaf

$$I \mapsto \widehat{\mathbb{S}\mathrm{to}}_I(A^0) := \left\{ (K_{jl})_{j,l\in\{1,\dots,s\}} \in \mathrm{GL}_n(\mathbb{C}) \mid K_{jl} = \boldsymbol{\delta}_{jl} \text{ unless } q_j \underset{\theta}{\prec} q_l \text{ for some } \theta \in I \right\}$$

and a skyscraper sheaf

$$I \mapsto \mathbb{S}\mathrm{to}_I(A^0)\,.$$

PROBLEM

Proof. It is well known (cf. [Boa99, p. 10]), that the morphism ρ_θ, i.e. conjugation by the fundamental solution, relates solutions φ_θ of $[\mathrm{End}(A^0)] = [A^0, A^0]$ to solutions of $[0,0]$ which are the constant matrices $\mathrm{GL}_n(\mathbb{C})$. Thus we have to show, that the image of $\mathrm{Sto}_\theta(A^0)$ under ρ_θ is $\mathbb{S}\mathrm{to}_\theta(A^0)$.

To see that the obtained matrix has the necessary zeros, to lie in $\mathbb{S}\mathrm{to}_\theta(A^0)$ we look at Equation (1.1) and deduce

$$\varphi_\theta(t) = t^L e^{Q(t^{-1})} C_{\varphi_\theta} e^{-Q(t^{-1})} t^{-L} \tag{1.2}$$

with the given choice of the argument near θ. After decomposing C_{φ_θ} into

$$\begin{aligned} C_{\varphi_\theta} &= 1_n + \begin{pmatrix} c_{(1,1)} & c_{(1,2)} & \cdots & \\ c_{(2,1} & \ddots & & \\ \vdots & & & \\ & & & c_{(s,s)} \end{pmatrix} \\ &= 1_n + \underbrace{\begin{pmatrix} c_{(1,1)} & 0 & \cdots \\ 0 & & \\ \vdots & & \end{pmatrix}}_{C^{(1,1)}_{\varphi_\theta}} + \underbrace{\begin{pmatrix} 0 & c_{(1,2)} & 0 & \cdots \\ & 0 & & \\ & \vdots & & \end{pmatrix}}_{C^{(1,2)}_{\varphi_\theta}} + \cdots + \underbrace{\begin{pmatrix} & & \vdots \\ & & 0 \\ \cdots & 0 & c_{(s,s)} \end{pmatrix}}_{C^{(s,s)}_{\varphi_\theta}} \\ &= 1_n + \sum_{(l,j)} C^{(l,j)}_{\varphi_\theta} \end{aligned}$$

One can ignore the block structure by using 1×1 sized blocks. But one looses the uniqueness of the q_j's.

where the $c_{(j,l)}$ are blocks of size $n_j \times n_l$ which correspond to the structure of Q. After rewriting the Equation (1.2) we get

$$\varphi_\theta = t^L \left(1_n + \sum_{(l,j)} C^{(l,j)}_{\varphi_\theta} e^{(q_l - q_j)(t^{-1})} \right) t^{-L} .$$

$$\begin{aligned} \varphi_\theta(t) &= t^L e^{Q(t^{-1})} \left(1_n + C_{\varphi_\theta}\right) e^{-Q(t^{-1})} t^{-L} \\ &= t^L e^{Q(t^{-1})} \left(1_n + \sum_{(l,j)} C^{(l,j)}_{\varphi_\theta} \right) e^{-Q(t^{-1})} t^{-L} \\ &= t^L \left(1_n + \sum_{(l,j)} e^{Q(t^{-1})} C^{(l,j)}_{\varphi_\theta} e^{-Q(t^{-1})} \right) t^{-L} \\ &= t^L \left(1_n + \sum_{(l,j)} C^{(l,j)}_{\varphi_\theta} e^{(q_l - q_j)(t^{-1})} \right) t^{-L} . \end{aligned}$$

Thus, for φ_θ to be flat in direction θ, it is necessary and sufficient that if $e^{(q_l - q_j)(t^{-1})}$ does not have maximal decay in direction θ the corresponding block $C^{(l,j)}_{\varphi_\theta}$ vanishes. Thus we have seen, that C_{φ_θ} is an element of $\mathbb{S}\mathrm{to}_\theta(A^0)$.

The **surjectivity** can now be seen easily since every constant matrix with zeros at the necessary positions characterizes a unique element of $\mathrm{Sto}_\theta(A^0)$:

Let $C = 1_n + \sum_{(l,j)|q_j \underset{\theta}{\not\prec} q_l} C^{(l,j)}$ be an element of $\mathbb{S}\mathrm{to}_\theta(A^0)$. Then is a pre-image of C given by $t^L e^{Q(t^{-1})} C e^{-Q(t^{-1})} t^{-L}$ which lies in $\mathrm{Sto}_\theta(A^0)$, since it satisfies the condition discussed above.

The map $\rho_{\widetilde{\theta}}$ is also **injective**, since it is the conjugation by an invertible matrix. $\square$

From the calculations in the proof it is clear that

1. for $j = l$ the (diagonal) blocks $C^{(l,j)}_{\varphi_\theta}$ vanish since $q_l - q_j = 0$ does not have maximal decay and
2. if $e^{q_j - q_l}$ has has maximal decay, then $e^{q_l - q_j}$ has not. Thus if $C^{(l,j)}_{\varphi_\theta}$ is not equal to zero, the block $C^{(j,l)}_{\varphi_\theta}$ is necessarily zero.

This implies that the matrix C_{φ_θ} is unipotent, and hence is $\mathrm{Sto}_\theta(A^0)$ is a unipotent Lie group.

One can use the Stokes matrices to give an alternative characterization of Stokes germs:

> a germ $\varphi_\theta \in \Lambda_\theta(A^0)$ is in $\mathrm{Sto}_\theta(A^0)$ if and only if there exists a $C \in \mathbb{S}\mathrm{to}_\theta(A^0)$ such that $\varphi_\theta = \mathcal{Y}_0 C \mathcal{Y}_0^{-1}$.

Formulated is this in the following corollary.

Corollary 1.10

A germ $\varphi_\theta \in \Lambda_\theta(A^0)$ is a Stokes germ, i.e. an element in $\mathrm{Sto}_\theta(A^0)$, if and only if it has a representation C_{φ_θ} where [Lod94, Def.I.4.12]

$$C_{\varphi_\theta} = 1_n + \sum_{(l,j)|q_j \underset{\theta}{\not\prec} q_l} C^{(l,j)}_{\varphi_\theta}$$

and the $C^{(l,j)}_{\varphi_\theta}$ have the necessary block structure, i.e. it is in $\mathbb{S}\mathrm{to}_\theta(A^0)$.

Remark 1.10.1

In Loday-Richaud's book [Lod14, p. 78] are the elements of $\mathrm{Sto}_\theta(A^0)$ actually characterized as the flat transformations, such that Equation (1.1) is satisfied for some unique constant invertible matrix $C \in \mathbb{S}\mathrm{to}_\theta(A^0)$.

Definition 1.11

We denote the set of *levels of the germ* $\varphi_\theta \in \Lambda_\theta(A^0)$ by

$$\mathcal{K}(\varphi_\theta) := \left\{ \deg(q_j - q_l) \mid C^{(l,j)}_{\varphi_\theta} \neq 0 \text{ in some representation of } \varphi_\theta \right\} \subset \mathcal{K}\,.$$

A germ φ_θ is called a *k-germ* when $\mathcal{K}(\varphi_\theta) \subset \{k\}$, i.e. it has at most the level k.

PROBLEM: This would be good

Lemma 1.13

Every k-germ in direction α can be extended to the Stokes arc $U(\frac{\pi}{k}, \alpha)$.

Corollary 1.13

Every germ $\varphi_\alpha \in \mathrm{Sto}_\alpha(A^0)$ can be extended to the arc $U(\frac{\pi}{\max \mathcal{K}(\varphi_\alpha)}, \alpha)$, i.e. there is a section $\varphi \in \Gamma\left(U(\frac{\pi}{\max \mathcal{K}(\varphi_\alpha)}, \alpha), \Lambda(A^0)\right)$ which has φ_α as its germ at α.

Let φ_α be a **simple** k-germ in the sense that it is build from a single block, i.e.

$$\varphi_\theta = t^L \left(1_n + C^{(l,j)}_{\varphi_\theta} e^{(q_l - q_j)(t^{-1})}\right) t^{-L} .$$

for some pair (j, l). Assume also that the block has size 1×1.

Let φ be the extension of φ_α around alpha, i.e. the matrix which has germ φ_α at α and which solves [End A^0] and is multiplicatively flat.

The system [End A^0] is

$$\frac{dF}{dt} = A^0 F - F A^0 .$$

Question: Which form has the extension φ around α of the germ φ_α, does it retain the structure?

1. Look at a diagonal element $\varphi^{j,j}$ an (j, j):
   - it satisfies $\varphi^{j,j}_\alpha = 1$ and
   - is satisfies some complicated equation

     Question: Is it constantly 1

     hopefully yes
2. Look at an off-diagonal position at (j, l):

case a: $\varphi^{j,l}_\alpha \neq 0$

case b: $\varphi^{j,l}_\alpha = 0$

1.1.3 Decomposition of the Stokes group by levels

[Lod94], [MR91, 362ff]

The goal of this section is, to introduce a filtration of $\Lambda(A^0)$, which will be restricted to $\mathrm{Sto}_\theta(A^0)$ and defines there a filtration. This leads to a decomposition of $\mathrm{Sto}_\theta(A^0)$ into a semidirect product (cf. Proposition 1.20).

Let us introduce a couple of notations and definitions, which coincide with the notations used in Loday-Richaud's paper [Lod94]. Another good resource, which uses similar notations, is for example the paper [MR91, 362f] from Martinet and Ramis.

Notations 1.14

For every level $k \in \mathcal{K}$ and direction $\theta \in S^1$ we set [Lod94, Not.I.4.15], [MR91, p. 362]

- $\Lambda^k(A^0)$ as the subsheaf of $\Lambda(A^0)$ of all germs, which are generated by k-germs;
- $\Lambda^{\leq k}(A^0)$ (resp. $\Lambda^{<k}(A^0)$ or $\Lambda^{\geq k}(A^0)$) as the subsheaf of $\Lambda(A^0)$ generated by k'-germs for all $k' \leq k$ (resp. $k' < k$ or $k' \geq k$).

Let $\star \in \{k, <k, \leq k, \dots\}$. The restrictions to Sto_θ yield the groups

$$\mathrm{Sto}_\theta^\star(A^0) := \mathrm{Sto}_\theta(A^0) \cap \Lambda_\theta^\star(A^0)$$

and let us also define $\mathbb{S}\mathrm{to}_\theta^\star(A^0)$ as the groups of representations, which correspond to elements of $\mathrm{Sto}_\theta^\star(A^0)$.

Corresponding to the definitions above, one can define $\mathbb{A}^\star := \{\alpha \in \mathbb{A} \mid \mathrm{Sto}_\alpha^\star(A^0) \neq \{\mathrm{id}\}\}$ for $\star \in \{k, <k, \leq k, \dots\}$ and we say that α *is bearing the level* k if $\alpha \in \mathbb{A}^k$.

Remark 1.15

It is clear that for every $k \in \mathcal{K}$ we have the canonical inclusions $\mathbb{A}^k \hookrightarrow \mathbb{A}^{\leq k}$ and $\mathbb{A}^{<k} \hookrightarrow \mathbb{A}^{\leq k}$.

Sometimes it is also useful to talk about the *set of levels beared by an direction* $\alpha \in \mathbb{A}$:

$$\mathcal{K}_\alpha := \left\{k \in \mathcal{K} \mid \mathrm{Sto}_\alpha^k(A^0) \neq \{\mathrm{id}\}\right\} .$$

Corollary 1.16

The Lemma 1.6 implies that from $k \in \mathcal{K}_\alpha$ follows that $k \in \mathcal{K}_{\alpha + m\frac{\pi}{k}}$ for $m \in \mathbb{N}$.

Let us now study the sheaves $\Lambda^\star(A^0)$ and discuss how they correlate and how they can be composed from the others.

The following proposition can be found as [Lod94, Prop.I.5.1] and the key-statement is also given in [MR91, Prop.4.10].

Proposition 1.17

For any level $k \in \mathcal{K}$ one has that $\Lambda^k(A^0)$, $\Lambda^{\leq k}(A^0)$ and $\Lambda^{<k}(A^0)$ are sheaves of subgroups of $\Lambda(A^0)$ and the sheaf $\Lambda^k(A^0)$ is normal in $\Lambda^{\leq k}(A^0)$. [Lod94, Prop.I.5.1]

A subgroup N is normal in G ($N \lhd G$) if it is stable under conjugation, i.e.

$$N \lhd G \Leftrightarrow \forall n \in N \forall g \in G, gng^{-1} \in N, .$$

[MR91, Proposition 10]

We even know more, let

- $i : \Lambda^k(A^0) \hookrightarrow \Lambda^{\leq k}(A^0)$ be the canonical inclusion and
- $p : \Lambda^{\leq k}(A^0) \twoheadrightarrow \Lambda^{<k}(A^0)$ be the truncation to terms of levels $< k$.

Then does the exact sequence of sheaves

$$1 \longrightarrow \Lambda^k(A^0) \xrightarrow{i} \Lambda^{\leq k}(A^0) \xrightarrow{p} \Lambda^{<k}(A^0) \longrightarrow 1\,,$$

split.

From the splitting of the sequence, we obtain immediately the following decomposition into a semidirect product.

Corollary 1.18

[Lod94, Cor.I.5.2]

For any $k \in \mathcal{K}$, there are the two following ways of factoring $\Lambda^{\leq k}(A^0)$ in a semidirect product:

$$\begin{aligned}\Lambda^{\leq k}(A^0) &\cong \Lambda^{<k}(A^0) \ltimes \Lambda^k(A^0) \\ &\cong \Lambda^k(A^0) \ltimes \Lambda^{<k}(A^0)\,.\end{aligned}$$

This means that any germ $f^{\leq k} \in \Lambda^{\leq k}(A^0)$ can be uniquely written as

- $f^{\leq k} = f^{<k} g^k$, where $f^{<k} \in \Lambda^{<k}$ and $g^k \in \Lambda^k$, or
- $f^{\leq k} = f^k f^{<k}$, where $f^k \in \Lambda^k$ and $f^{<k} \in \Lambda^{<k}$.

Remark 1.18.1

[Lod94, Cor.I.5.2(ii)]

We can get the factor $f^{<k}$ common to both factorizations by truncation of $f^{\leq k}$ to terms of level $< k$, i.e. by applying the map p from Proposition 1.17. This truncation can explicitly be achieved, in terms of Stokes matrices, by keeping in representations $1 + \sum C^{(j,l)}$ of $f^{\leq k}$ only the blocks $C^{(j,l)}$ such that $\deg(q_j - q_l) < k$.

A factorization algorithm could then be:

> get the factor $f^{<k}$ common to both factorizations by truncation of $f^{\leq k}$ to terms of level $< k$ and set $g^k := (f^{<k})^{-1} f^{\leq k}$ and $f^k := f^{\leq k} (f^{<k})^{-1}$.

This decomposition in a semidirect product can be extended to all levels, since $\Lambda^{<k}(A^0) = \Lambda^{\leq \max\{k' \in \mathcal{K} | k' < k\}}$. Thus

$$\Lambda(A^0) \cong \bigltimes_{k \in \mathcal{K}} \Lambda^k(A^0)\,,$$

where the semidirect product is taken in an ascending or descending order of levels.

Remark 1.19

Loday-Richaud states in her paper [Lod94, Prop.I.5.3] the following proposition, which is a more general version of Proposition 1.17.

Proposition 1.19.1

For any levels k,$k' \in \mathcal{K}$ with $k' < k$ one has: [Lod94, Prop.I.5.3]

1. the sheaf $\Lambda^{\geq k'}(A^0) \cap \Lambda^{\leq k'}(A^0)$ is normal in $\Lambda^{\leq k}(A^0)$;
2. the exact sequence of sheaves

$$1 \longrightarrow \Lambda^{\geq k'}(A^0) \cap \Lambda^{\leq k}(A^0) \xrightarrow{i} \Lambda^{\leq k}(A^0) \xrightarrow{p} \Lambda^{<k'}(A^0) \longrightarrow 1\,,$$

   where
   - i is the canonical inclusion and
   - p is the truncation to terms of levels $< k'$,

   splits.

TODO: is $\Lambda^{\geq k'}(A^0) \cap \Lambda^{\leq k}(A^0) = \Lambda^{k}(A^0)$ and thus the first proposition a corollary of this?

We can use this proposition to follow (cf. [Lod94, Cor.I.5.4]) that

1. the filtration

$$\Lambda^{k_r}(A^0) = \Lambda^{\geq k_r}(A^0) \subset \Lambda^{\geq k_{r-1}}(A^0) \subset \cdots \subset \Lambda^{\geq k_1}(A^0) = \Lambda(A^0)$$

   is normal and
2. we can use this to achieve the decomposition

$$\Lambda(A^0) \cong \rtimes_{k \in \mathcal{K}} \Lambda^{k}(A^0)$$

   taken in an arbitrary order. In fact, one can also extend the algorithm from Remark 1.18.1 to an arbitrary order of levels.

The important statement, which we will use later, is then the following Proposition. It is stated by Loday-Richaud in her Paper [Lod94] as Proposition I.5.5 or in the Paper [MR91, Thm.4.8] by Martinet and Ramis.

Proposition 1.20

The results from above can be restricted to the Stokes groups. Thus, for $\alpha \in \mathbb{A}$, one has

$$\mathrm{Sto}_\alpha(A^0) \cong \ltimes_{k \in \mathcal{K}_\alpha} \mathrm{Sto}_\alpha^k(A^0)$$

the semidirect product being taken in an arbitrary order (we will only be interested in the ascending order).

Definition 1.20.1

We will denote the map which gives the factors of this factorization by

$$i_\alpha : \mathrm{Sto}_\alpha(A^0) \xrightarrow{\cong} \prod_{k \in \mathcal{K}_\alpha} \mathrm{Sto}_\alpha^k(A^0)\,,$$

where the factorization is taken in ascending order.

Remark 1.20.1

Write $\rho_\alpha^k : \mathrm{Sto}_\alpha^k(A^0) \to \mathbb{S}\mathrm{to}_\alpha^k(A^0)$ for the restriction of the map ρ_α (cf. Proposition 1.9) to the level k. Then, one can denote, by abuse of notation, the induced decomposition also by

$$i_\alpha : \mathbb{S}\mathrm{to}_\alpha(A^0) \xrightarrow{\cong} \prod_{k \in \mathcal{K}_\alpha} \mathbb{S}\mathrm{to}_\alpha^k(A^0)$$

and the corresponding diagram

$$\begin{tikzcd}
\mathrm{Sto}_\alpha(A^0) \arrow[r, "i_\alpha"] \arrow[d, "\rho_\alpha"] & \prod_{k \in \mathcal{K}_\alpha} \mathrm{Sto}_\alpha^k(A^0)\,, \arrow[d, "\prod_{k \in \mathcal{K}} \rho_\alpha^k"] \\
\mathbb{S}\mathrm{to}_\alpha(A^0) \arrow[r, "i_\alpha"] & \prod_{k \in \mathcal{K}_\alpha} \mathbb{S}\mathrm{to}_\alpha^k(A^0)\,,
\end{tikzcd}$$

commutes.

1.2 Stokes structures: using Stokes groups

[Lod94], [Lod14, Thm.4.3.11] and [Boa01; Boa99] and [BV89] and [BJL79] and [MR91, Chapter 4]

The goal in this section is to prove that there is a bijective and natural map

$$h : \prod_{\alpha \in \mathbb{A}} \mathrm{Sto}_\alpha(A^0) \longrightarrow \mathcal{S}t(A^0)$$

which endows $\mathcal{S}t(A^0)$ with the structure of a unipotent Lie group. And since $\mathrm{Sto}_\alpha(A^0)$ has $\mathbb{S}\mathrm{to}_\alpha(A^0)$ as a faithful representation, we also get the isomorphism $\prod_{\alpha\in\mathbb{A}} \mathbb{S}\mathrm{to}_\alpha(A^0) \cong \mathcal{S}t(A^0)$ as a corollary. TODO: This goes back to [BJL79]?

Let us recall, that $\mathcal{S}t(A^0)$ is defined to be $H^1(S^1; \Lambda(A^0))$ (cf. Section **??**). The elements of $\prod_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha(A^0)$ define in a canonical way cocycles of the sheaf $\Lambda(A^0)$ (cf. Equation (1.3)), called Stokes cocycles (cf. Definition 1.24). In fact, will h map such cocycles to the cohomology class, to which they correspond. Thus the statement, that h is a bijection, is equivalent to the statement that

in each cohomology class of $\mathcal{S}t(A^0)$ is a unique 1-cocycle, which is a Stokes cocycle.

Cyclic coverings

To formulate the theorem in the next section, we use the notion of cyclic coverings and nerves of such coverings, which are defined as follows.

Definition 1.21

Let J be a finite set, identified to $\{1,\dots,p\} \subset \mathbb{Z}$. [Lod94, Sec.II.1] and [Lod94, Sec.II.3.1]

1. A *cyclic covering* of S^1 is a finite covering $\mathcal{U} = \Big(U_j = U(\theta_j,\varepsilon_j)\Big)_{j\in J}$ consisting of arcs, which satisfies that
   a) $\widetilde{\theta}_j \geq \widetilde{\theta}_{j+1}$ for $j \in \{1,\dots,p-1\}$, i.e. the center points are ordered and
   b) $\widetilde{\theta}_j + \frac{\varepsilon_j}{2} \geq \widetilde{\theta}_{j+1} + \frac{\varepsilon_{j+1}}{2}$ for $j \in \{1,\dots,p-1\}$ and $\widetilde{\theta}_p + \frac{\varepsilon_p}{2} \geq \widetilde{\theta}_1 - 2\pi + \frac{\varepsilon_1}{2}$, i.e. the arcs are not encased by another arc,

   where the $\widetilde{\theta}_j \in [0, 2\pi[$ are determinations of the $\theta_j \in S^1$.

   a) the θ_j are in ascending order with respect to the clockwise orientation of S^1;
   b) the $U_j \cap U_{j+1}$ have only one connected component when $\#J > 2$;
   c) the U_j are not encased by another arc, this means that the open sets $U_j \backslash U_l$ are connected for all $j, l \in J$.

2. The *nerve* of a cyclic covering $\mathcal{U} = \{U_j; j \in J\}$ is the family $\dot{\mathcal{U}} = \{\dot{U}_j; j \in J\}$ defined by:
   - $\dot{U}_j = U_j \cap U_{j+1}$ when $\#J > 2$,
   - $\dot{U}_1$ and $\dot{U}_2$ the connected components of $U_1 \cap U_2$ when $\#J = 2$.

   Remark 1.21.1

   The nerve of the cyclic covering $\mathcal{U} = (U(\theta_j,\varepsilon_j))_{j\in J}$ is explicitly given by

   $$\dot{\mathcal{U}} = \left(\left(\theta_{j+1} - \frac{\varepsilon_{j+1}}{2}, \theta_j + \frac{\varepsilon_j}{2}\right)\right)_{j\in J} .$$

The cyclic coverings correspond one-to-one to nerves of cyclic coverings. If one starts with a nerve $\{\dot{U}_j \mid j \in J\}$, one obtains a cyclic covering as $\mathcal{U} = \{U_j \mid j \in J\}$ where the arc U_j is the connected clockwise hull from $\dot{U}_{j-1}$ to $\dot{U}_j$.

Definition 1.22

A covering $\mathcal{V}$ is said to *refine* a covering $\mathcal{U}$ if, to each open set $V \in \mathcal{V}$ there is at least one $U \in \mathcal{U}$ with $V \subset U$.

Proposition 1.23

[Lod94, Prop.II.1.3]

The covering $\mathcal{V}$ refines $\mathcal{U}$ if and only if the corresponding nerves $\dot{\mathcal{U}} = \{\dot{U}_j\}$ and $\dot{\mathcal{V}} = \{\dot{V}_l\}$ satisfy

each $\dot{U}_j$ contains at least one $\dot{V}_l$.

1.2.1 The theorem

[Lod94, p. 868]

Let $\{\theta_j \mid j \in J\} \subset S^1$ be a finite set and $\dot{\varphi} = (\dot{\varphi}_{\theta_j})_{j \in J} \in \prod_{j \in J} \Lambda_{\theta_j}(A^0)$ be a finite family of germs. Let $\dot{\varphi}_j$ be the function representing the germ $\dot{\varphi}_{\theta_j}$ on its (maximal) arc of definition Ω_j around θ_j. In the following way, one can associate a cohomology class in $\mathcal{S}t(A^0)$ to $\dot{\varphi}$:

for every cyclic covering $\mathcal{U} = (U_j)_{j \in J}$ which satisfies $\dot{U}_j \subset \Omega_j$ for all $j \in J$, one can define the 1-cocycle $(\dot{\varphi}_{j|\dot{U}_j})_{j \in J} \in \Gamma(\dot{\mathcal{U}}; \Lambda(A^0))$.

To a different cyclic covering, satisfying the condition above, this construction yields a cohomologous 1-cocycle, thus the induced map

$$\prod_{j \in J} \Lambda_{\theta_j}(A^0) \longrightarrow H^1(S^1; \Lambda(A^0)) = \mathcal{S}t(A^0) \tag{1.3}$$

is welldefined (cf. [Lod94, p. 868]).

Definition 1.24

[Lod94, Def.II.1.8], [Lod14, p. 4.3.10], [MR91, Defn 6 on p 374]

Let $\nu = \#\mathbb{A}$ the number of all anti-Stokes directions and write $\mathbb{A} = \{\alpha_1, \dots, \alpha_\nu\}$.

A *Stokes cocycle* is a 1-cocycle $(\varphi_j)_{j \in \{1,\dots,\nu\}} \in \prod_{j \in \{1,\dots,\nu\}} \Gamma(U_j; \Lambda(A^0))$ corresponding to some cyclic covering with nerve $\dot{\mathcal{U}} = (\dot{U}_j)_{j \in \{1,\dots,\nu\}}$, which satisfies for every $j \in \{1, \dots, \nu\}$

- $\alpha_j \in \dot{U}_j$ and
- the germ $\varphi_{\alpha_j} := \varphi_{j,\alpha_j}$ of φ_j at α_j is an element of $\mathrm{Sto}_{\alpha_j}(A^0)$.

Remark 1.24.1

PROBLEM:refactor! The sections in $\Gamma(\dot{U}_j; \Lambda(A^0))$ are uniquely determined as the extension of the germ at α_j, since the sheaf $\Lambda(A^0)$ defined via the system $[A^0, A^0]$ (cf. Definition **??**). We thus have an injective map

$$\prod_{j\in\{1,\dots,\nu\}} \Gamma(\dot{U}_j; \Lambda(A^0)) \hookrightarrow \prod_{j\in\{1,\dots,\nu\}} \mathrm{Sto}_{\alpha_j}(A^0)\,,$$

which takes an Stokes cocycle and yields the corresponding Stokes germs. For a fine enough covering $\mathcal{U}$, i.e. a covering $\mathcal{U}$ with a nerve $\dot{\mathcal{U}}$ which consists of small enough arcs satisfying the conditions above, is this map a bijection.

We will use this fact implicitly and assume that the covering is always fine enough to call elements of $\prod_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha(A^0)$ Stokes cocycles.

We can use the Equation (1.3) to obtain for Stokes cocycles a mapping

$$h : \textstyle\prod_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha(A^0) \longrightarrow \prod_{\alpha\in\mathbb{A}} \Lambda_\alpha(A^0) \xrightarrow{(1.3)} \mathcal{S}t(A^0),$$

which takes a Stokes cocycle to its corresponding cohomology class.

Theorem 1.25

The map

$$h : \textstyle\prod_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha(A^0) \longrightarrow \mathcal{S}t(A^0)$$

is a bijection and natural.

Remark 1.25.1

Natural means that h commutes to isomorphisms and constructions over systems or connections they represent.

[Lod94, p. 869],[Lod94, Sec.III.3.3]

To define the inverse map of h, one has to find in each cocycle in $\mathcal{S}t(A^0)$ the Stokes cocycle. Loday-Richaud gives an algorithm in section II.3.4 of her paper [Lod94], which takes a cocycle over an arbitrary cyclic covering and outputs cohomologous Stokes cocycle and thus solves this problem.

From Theorem 1.25 and Proposition 1.9 we get the following corollary.

Corollary 1.26

PROBLEM:mentioned twice Using the isomorphisms $\mathrm{Sto}_\theta(A^0) \cong \mathbb{S}\mathrm{to}_\theta(A^0)$ from Proposition 1.9 we obtain

$$\mathcal{S}t(A^0) \cong \prod_{\alpha\in\mathbb{A}} \mathbb{S}\mathrm{to}_\alpha(A^0)$$

which endows $\mathcal{S}t(A^0)$ with the structure of an unipotent Lie group with the finite complex dimension $N := \dim_{\mathbb{C}} \mathcal{S}t(A^0)$ (cf. [Lod94, Sec.III.1]). This can be rewritten in the following way:

$$N = \sum_{\alpha\in\mathbb{A}} \dim_{\mathbb{C}} \mathbb{S}\mathrm{to}_\alpha(A^0) = \sum_{\alpha\in\mathbb{A}} \sum_{q_j \underset{\alpha}{\lessdot} q_l} n_j \cdot n_l = \sum_{\substack{1\le j,l\le n \\ j<l}} 2\cdot \deg(q_j - q_l)\cdot n_j \cdot n_l \,.$$

Remark 1.26.1

This number N is known to be the *irregularity* of $[\mathrm{End}\, A^0]$.

One can also define the structure of a linear affine variety on the set $\mathcal{S}t(A^0)$. This was for example done in Section [BV89, Sec.III.3] or in [Lod94, 880f], where actually multiple structures of linear affine varieties on $\mathcal{S}t(A^0)$ are defined. In [Var96, 35ff] is mentioned that one can also define a scheme structure of $\mathcal{S}t(A^0)$.

1.2.2 Proof of Theorem 1.25

We will only look at the unramified case, for which we refer to [Lod94, Sec.II.3]. The proof in the ramified case can be found in [Lod94, Sec.II.4]. We first have to introduce adequate coverings, which will be used in the proof.

Adequate coverings

Definition 1.27

[MR91, p. 371] defines adapted coverings

Let $\star \in \{k, < k, \le k, \dots\}$. A covering $\mathcal{U}$ beyond which the inductive limit $\varinjlim_{\mathcal{U}} H^1(\mathcal{U}; \Lambda^\star(A^0))$ is stationary is said to be *adequate* to describe $H^1(S^1; \Lambda^\star(A^0))$ or *adequate* to $\Lambda^\star(A^0)$.

A covering $\mathcal{U}$ is said to be *adequate* to describe $H^1(S^1; \Lambda^\star(A^0))$ or *adequate* to $\Lambda^\star(A^0)$ if for every element in $\varinjlim_{\mathcal{U}} H^1(\mathcal{U}; \Lambda^\star(A^0))$ given by some covering $\mathcal{U}'$ and an element of $\Gamma(\mathcal{U}'; \Lambda^\star(A^0))$ there exists

- an element in $\Gamma(\mathcal{U}; \Lambda^\star(A^0))$ and
- an common refinement of $\mathcal{U}$ and $\mathcal{U}'$

such that PROBLEM: the elements are ?? on the refined covering.

In other words is a covering $\mathcal{U}$ adequate, if and only if the quotient map

$$\Gamma(\mathcal{U}; \Lambda^\star(A^0)) \longrightarrow H^1(S^1; \Lambda^\star(A^0))$$

is surjective. TODO: Proof? check??

[MR91, p. 371] introduces the following definition

Definition 1.27.1

A covering $\mathcal{U}$ is *adapted* if every anti-Stokes direction is contained in exactly one element of the nerve $\dot{\mathcal{U}}$.

The following proposition is in Loday-Richaud's paper [Lod94] given as Proposition II.1.7. It contains a simple characterization, which will be used to see, that our defined coverings are adequate.

Proposition 1.28

Let $k \in \mathcal{K}_\alpha$. [Lod94, Prop.II.1.7]

Definition 1.28.1

Let $\alpha \in \mathbb{A}^k$. An arc $U(\alpha, \frac{\pi}{k})$ is called a *Stokes arc of level k at α*.

A cyclic covering $\mathcal{U} = (U_j)_{j \in J}$, which satisfies

> for every $\alpha \in \mathbb{A}^k$ contains the Stokes arc $U(\alpha, \frac{\pi}{k})$ at least one arc $\dot{U}_j$ from the nerve $\dot{\mathcal{U}}$ of $\mathcal{U}$

is adequate to $\Lambda^k(A^0)$.

The covering $\mathcal{U}$ is adequate to $\Lambda^{\leq k}(A^0)$ (resp. $\Lambda^{<k}(A^0)$) if it is adequate to $\Lambda^{k'}(A^0)$ for every $k' \leq k$ (resp. $k' < k$).

Let $k \in \mathcal{K}$. We want to define the three cyclic coverings $\mathcal{U}^k$, $\mathcal{U}^{\leq k}$ and $\mathcal{U}^{<k}$ which will be adequate to $\Lambda^k(A^0)$, $\Lambda^{\leq k}(A^0)$ and $\Lambda^{<k}(A^0)$. Furthermore will the coverings be comparable at the different levels.

1. The first covering $\mathcal{U}^k = \{\dot{U}^k_\alpha \mid \alpha \in \mathbb{A}^k\}$ is the cyclic covering with nerve

$$\dot{\mathcal{U}}^k := \left\{ \dot{U}^k_\alpha = U(\alpha, \frac{\pi}{k}) \mid \alpha \in \mathbb{A}^k \right\}$$

consisting of all Stokes arcs of level k for anti-Stokes directions bearing the level k.

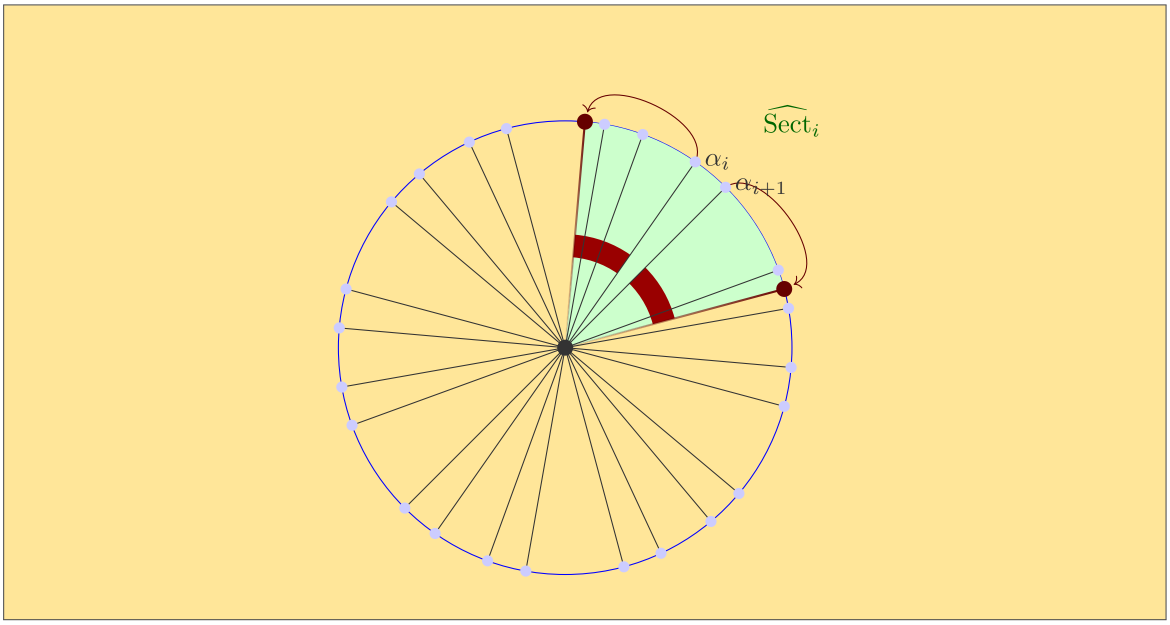
Remark 1.29

Boalch introduces in his publications [Boa01, p. 19] and [Boa99, Def.1.23] the notion of *supersectors*, they are in the case of a single level k, defined as follows:

> write the anti-Stokes directions as $\mathbb{A} = \{\alpha_1, \dots, \alpha_\nu\}$ arranged according to the clockwise ordering, then is the i-th supersector defined as the arc

$$\widehat{\mathrm{Sect}}^k_i := \left(\alpha_i - \frac{\pi}{2k}, \alpha_{i+1} + \frac{\pi}{2k} \right) .$$

This yields a cyclic covering $(\widehat{\mathrm{Sect}}_i^k)_{i\in\{1,\ldots,\nu\}}$ whose nerve is exactly $\dot{\mathcal{U}}^k$ defined above.



If we extend to more then one level level, $\#\mathcal{K} > 1$, the set $\bigcup_{k\in\mathcal{K}}\left\{U(\alpha,\frac{\pi}{k}) \mid \alpha\in\mathbb{A}^k\right\}$ is no longer a nerve. Hence we have to define the coverings $\mathcal{U}^{\leq k}$ and $\mathcal{U}^{<k}$ in a different way. Denote by

$$\{K_1 < \cdots < K_s = k\} = \left\{\max\left(\mathcal{K}_\alpha \cap [0,k]\right) \mid \alpha \in \mathbb{A}^{\leq k}\right\}$$

the set of all *k-maximum levels* for $\alpha \in \mathbb{A}^{\leq k}$.

2. The cyclic covering $\mathcal{U}^{\leq k} = \left\{U_\alpha^{\leq k} \mid \alpha\in\mathbb{A}^{\leq k}\right\}$ will be defined by induction. Let us assume that

   the $\dot{U}_\alpha^{\leq k}$ are defined for all $\alpha\in\mathbb{A}^{\leq k}$ with k-maximum level greater than K_i such that their complete family is a nerve.

   Let

   - α be a anti-Stokes direction with k-maximum level K_i and
   - α^- (resp. α^+) be the next anti-Stokes direction with k-maximum level greater then K_i on the left (resp. on the right) and define $\dot{U}_{\alpha^-,\alpha^+}$ as the clockwise hull of the arcs $\dot{U}_{\alpha^-}^{\leq k}$ and $\dot{U}_{\alpha^+}^{\leq k}$ already defined by induction. If there are no anti-Stokes directions with k-maximum level greater then K_i we set $\dot{U}_{\alpha^-,\alpha^+} = S^1$.

   We then set

   $$\dot{U}_\alpha^{\leq k} := U\left(\alpha, \frac{\pi}{K_i}\right) \cap \dot{U}_{\alpha^-,\alpha^+}$$

   and the family of all $\dot{U}_\alpha^{\leq k}$ is a nerve.

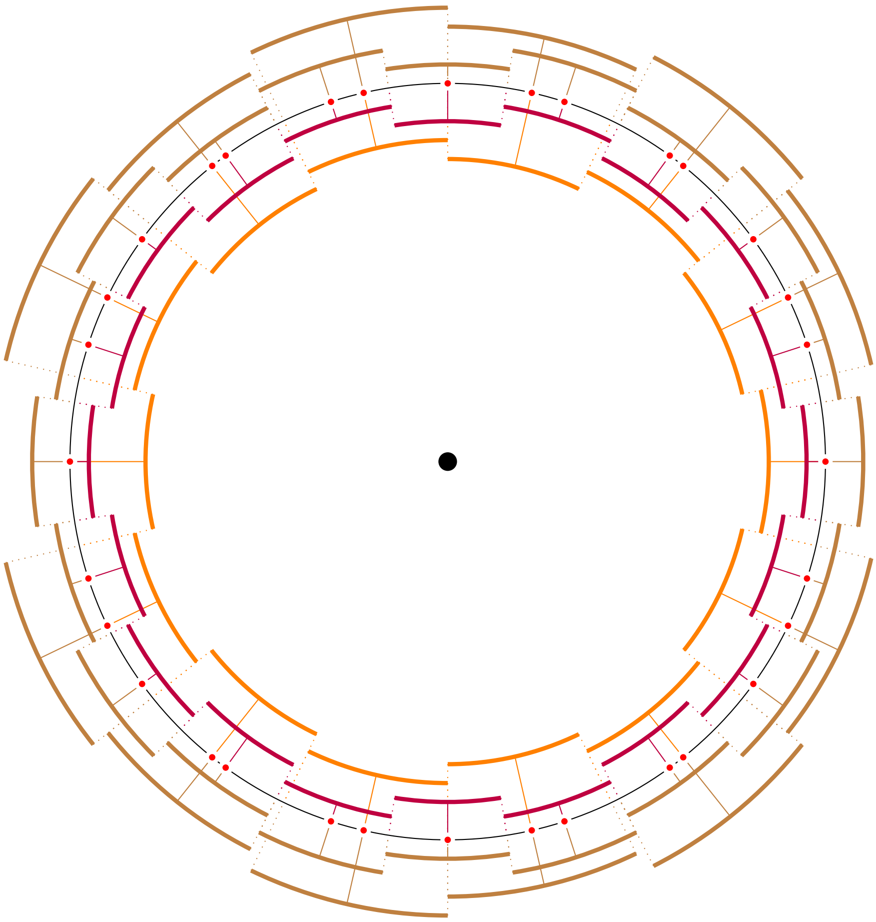


Figure 1.2: The adequate coverings for an example with $\mathcal{K} = \{7, 10\}$ and $\mathbb{A} = \left\{\frac{j \cdot \pi}{k} \mid k \in \mathcal{K},\, j \in \mathbb{N}\right\}$. The anti-Stokes directions are marked by the red dots. The arcs of $\dot{\mathcal{U}}^7 = \dot{\mathcal{U}}^{\leq 7}$ are orange, the arcs of $\dot{\mathcal{U}}^{10}$ are purple and the arcs of $\dot{\mathcal{U}}^{\leq 10} = \dot{\mathcal{U}}$ are brown.

Remark 1.30

If α has a k-maximum level equal to k then is $\dot{U}_\alpha^{\leq k}$ equal to the Stokes arc $U\left(\alpha, \frac{\pi}{k}\right) = \dot{U}_\alpha^k$. ...and then no 0-cochain with level k or $\geq k$ can exists on the covering $\mathcal{U}^{\leq k}$.

3. The last cyclic covering, $\mathcal{U}^{<k} = \left\{U_\alpha^{<k} \mid \alpha \in \mathbb{A}^{<k}\right\}$, is defined as $\mathcal{U}^{<k} := \mathcal{U}^{\leq k'}$ where $k' := \max\{k'' \in \mathcal{K} \mid k'' < k\}$.

Remark 1.31

The coverings $\mathcal{U}^k$, $\mathcal{U}^{\leq k}$ and $\mathcal{U}^{<k}$ depend only on $\mathcal{Q}(A^0)$. Hence they depend only on the determining polynomials.

It is obvious, that for every $k \in \mathcal{K}$ the covering $\mathcal{U}^{\leq k}$ refines $\mathcal{U}^k$ and $\mathcal{U}^{<k}$. Furthermore are the coverings defined, such that they satisfy the condition in Proposition 1.28. Thus

the first property in the following proposition is satisfied. The other two can be found at [Lod94, Prop.II.3.1 (iv)].

Proposition 1.32

[Lod94, Prop.II.3.1] PROBLEM: Was bedeutet das? Let $k \in \mathcal{K}$, then

1. the coverings $\mathcal{U}^k$, $\mathcal{U}^{\leq k}$ and $\mathcal{U}^{<k}$ are adequate to $\Lambda^k(A^0)$, $\Lambda^{\leq k}(A^0)$ and $\Lambda^{<k}(A^0)$, respectively;
2. there exists no 0-cochain in $\Lambda^k(A^0)$ on $\mathcal{U}^k$;
3. on $\mathcal{U}^{\leq k}$ there is no 0-cochain in $\Lambda^{\leq k}(A^0)$ of level k, i.e. all 0-cochains of $\Lambda^{\leq k}(A^0)$ belong to $\Lambda^{<k}(A^0)$.

To have a shorter notation, we denote the product $\prod_{\alpha\in\mathbb{A}^\star}\Gamma(\dot{U}^\star_\alpha;\Lambda^\star(A^0))$ by $\Gamma(\dot{U}^\star;\Lambda^\star(A^0))$ for every $\star \in \{k, <k, \leq k, \dots\}$.

The case of a unique level

[Lod94, p. II.3.2] First we will proof Theorem 1.25 in the case of a unique level. This means that

- either $\Lambda(A^0)$ has only one level k, thus
  - $\Lambda(A^0) = \Lambda^k(A^0)$ and
  - $\mathrm{Sto}_\theta(A^0) = \mathrm{Sto}^k_\theta(A^0)$ for every θ,
- or we restrict to a given level $k \in \mathcal{K}$.

Lemma 1.33

Let $k \in \mathcal{K}$. The morphism h from Theorem 1.25 is in the case of a unique level build as

$$\begin{array}{ccccc} \prod\limits_{\alpha\in\mathbb{A}} \mathrm{Sto}^k_\alpha(A^0) & \xrightarrow{i^k} & \Gamma(\dot{\mathcal{U}}^k;\Lambda^k(A^0)) & \xrightarrow{s^k} & H^1(S^1;\Lambda^k(A^0)) \\ \| & & & & \| \\ \prod\limits_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha(A^0) & & \xrightarrow{\qquad h \qquad} & & H^1(S^1;\Lambda(A^0)) \end{array}$$

only in the single leveled case

from

- the canonical injective map

$$i^k : \prod_{\alpha\in\mathbb{A}} \mathrm{Sto}^k_\alpha(A^0) \longrightarrow \Gamma(\dot{\mathcal{U}}^k;\Lambda^k(A^0)),$$

i.e. the map which is the canonical extension of germs to their natural arc of definition, and

- the quotient map

$$s^k : \Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A^0)) \longrightarrow H^1(S^1; \Lambda^k(A^0))$$

which are both isomorphisms.

Proof. 1. The map

$$i^k : \prod_{\alpha \in \mathbb{A}} \mathrm{Sto}^k_\alpha(A^0) \longrightarrow \underbrace{\prod_{\alpha \in \mathbb{A}^k} \Gamma(\dot{\mathcal{U}}^k_\alpha; \Lambda^k(A^0))}_{\substack{\| \\ \Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A^0))}}$$

is welldefined, since the sections of $\Lambda^k(A^0)$ are solution of the system $[A^0, A^0]$ and it is very well known from the theory of differential equations TODO: source? that an element $f_\alpha \in \Gamma(\dot{\mathcal{U}}^k_\alpha; \Lambda^k(A^0))$ is uniquely determined as the extension of its germ at some point α.

It is also a isomorphism groups (cf. [Lod94]).

Problems:

- Show that a element of $\Lambda^k_\alpha(A^0)$ is extensionable to the arc $U_\alpha \in \dot{\mathcal{U}}^k_\alpha$ if and only if it has maximal decay in direction α.
- PROBLEM: $\mathrm{Sto}^k_\alpha(A^0) \subsetneq \Lambda^k_\alpha(A^0)$ and thus

$$i^k : \prod_{\alpha \in \mathbb{A}} \mathrm{Sto}^k_\alpha(A^0) \subsetneq \prod_{\alpha \in \mathbb{A}} \Lambda^k_\alpha(A^0) \to \prod_{\alpha \in \mathbb{A}^k} \Gamma(\dot{\mathcal{U}}^k_\alpha; \Lambda^k(A^0))$$

Facts:

- Every arc $U \in \dot{\mathcal{U}}^k_\alpha$ has the width $\frac{\pi}{k}$ and is delimited by Stokes directions, i.e. is U^k_α the largest arc, to contain no **corresponding** Stokes ray

    Then might [BJL79, Lemma 1] on page 72-73 help? **NO!**

IDEAS:

- See [MR91, p. 375] and **[MR91, Def.5 on 372]**
- **See [BV89]** and **[BV89, p. 72]**
- or maybe [Was02]

2. The second map

$$s^k : \overbrace{\prod_{\alpha \in \mathbb{A}^k} \Gamma(\dot{\mathcal{U}}^k_\alpha; \Lambda^k(A^0))}^{\substack{\Gamma(\dot{\mathcal{U}}^k;\Lambda^k(A^0)) \\ \|}} \longrightarrow H^1(S^1; \Lambda^k(A^0))$$

is a bijection, since from Proposition 1.32 we know that it is

- **surjective**, since $\mathcal{U}^k$ is adequate to $\Lambda^k(A^0)$ and
- **injective**, since on $\mathcal{U}^k$ there is no 0-cochain in $\Lambda^k(A^0)$.

PROBLEM:Show that this is the correct h PROBLEM:Naturality? □

The case of several levels

In the proof of the case of several levels, we will still use Loday-Richaud's paper [Lod94] as reference.

Definition 1.34

Here we want to define a *product map of cocycles* $\mathfrak{S}^{\leq k}$. This map will be composed from the following injective maps:

1. The first map is defined as

$$\begin{aligned} \sigma^k : \Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A^0)) &\longrightarrow \Gamma(\dot{\mathcal{U}}^{\leq k}; \Lambda^{\leq k}(A^0)) \\ \dot{f} = (\dot{f}_\alpha)_{\alpha \in \mathbb{A}^k} &\longmapsto (\dot{G}_\alpha)_{\alpha \in \mathbb{A}^{\leq k}} \end{aligned}$$

where

$$\dot{G}_\alpha = \begin{cases} \dot{f}_\alpha \text{ restricted to } \dot{U}^{\leq k}_\alpha \text{ and seen as being in } \Lambda^{\leq k}(A^0) & \text{when } \alpha \in \mathbb{A}^k \\ \text{id (the identity)} & \text{when } \alpha \notin \mathbb{A}^k \end{cases}$$

2. and the second map

$$\begin{aligned} \sigma^{<k} : \Gamma(\dot{\mathcal{U}}^{<k}; \Lambda^{<k}(A^0)) &\longrightarrow \Gamma(\dot{\mathcal{U}}^{\leq k}; \Lambda^{\leq k}(A^0)) \\ \dot{f} = (\dot{f}_\alpha)_{\alpha \in \mathbb{A}^{<k}} &\longmapsto (\dot{F}_\alpha)_{\alpha \in \mathbb{A}^{\leq k}} \end{aligned}$$

is defined, in a similar way, as

$$\dot{F}_\alpha = \begin{cases} \dot{f}_\alpha \text{ restricted to } \dot{U}^{\leq k}_\alpha \text{ and seen as being in } \Lambda^{\leq k}(A^0) & \text{when } \alpha \in \mathbb{A}^{<k} \\ \text{id (the identity)} & \text{when } \alpha \notin \mathbb{A}^{<k} \end{cases}$$

Thus we can define

$$\mathfrak{S}^{\leq k} : \Gamma(\dot{\mathcal{U}}^{<k}; \Lambda^{<k}(A^0)) \times \Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A^0)) \longrightarrow \Gamma(\dot{\mathcal{U}}^{\leq k}; \Lambda^{\leq k}(A^0))$$

$$(\dot{f}, \dot{g}) \longmapsto (\dot{F}_\alpha \dot{G}_\alpha)_{\alpha \in \mathbb{A}^{\leq k}}$$

where $(\dot{F}_\alpha)_{\alpha\in\mathbb{A}^{\leq k}} = \sigma^{<k}(\dot{f})$ and $(\dot{G}_\alpha)_{\alpha\in\mathbb{A}^{\leq k}} = \sigma^k(\dot{g})$ are defined as above.

Remark 1.34.1

This map $\mathfrak{S}^{\leq k}$ is injective, since injectivity for germs implies injectivity for sections.

Lemma 1.35

Let $k \in \mathcal{K}$. [Lod94, Lem.II.3.3]

1. If the cocycles $\mathfrak{S}^{\leq k}(\dot{f}, \dot{g})$ and $\mathfrak{S}^{\leq k}(\dot{f}', \dot{g}')$ are cohomologous in $\Gamma(\dot{\mathcal{U}}^{\leq k}; \Lambda^{\leq k}(A^0))$ then $\dot{f}$ and $\dot{f}'$ are cohomologous in $\Gamma(\dot{\mathcal{U}}^{<k}; \Lambda^{<k}(A^0))$.
2. Any cocycle in $\Gamma(\dot{\mathcal{U}}^{\leq k}, \Lambda^{\leq k}(A^0))$ is cohomologous to a cocycle in the range of $\mathfrak{S}^{\leq k}$.

Proof.

1. Denote by α^+ the nearest anti-Stokes direction in $\mathbb{A}^{\leq k}$ on the right[a] of α. The cocycles $\mathfrak{S}^{\leq k}(\dot{f}, \dot{g})$ and $\mathfrak{S}^{\leq k}(\dot{f}', \dot{g}')$ are cohomologous if and only if there is a 0-cochain $c = (c_\alpha)_{\alpha\in\mathbb{A}^{\leq k}} \in \Gamma(\mathcal{U}^{\leq k}, \Lambda^{\leq k}(A^0)$ such that

$$\dot{F}_\alpha \dot{G}_\alpha = c_\alpha^{-1} \dot{F}'_\alpha \dot{G}'_\alpha c_{\alpha^+} \tag{1.4}$$

   for every $\alpha \in \mathbb{A}$. From Proposition 1.32 follows, that c is with values in $\Lambda^{<k}(A^0)$. The fact that $\Lambda^k(A^0)$ is normal in $\Lambda^{\leq k}(A^0)$ in Proposition 1.17, can be used to see that $c_{\alpha^+}^{-1} G'_\alpha c_{\alpha^+} \in \Gamma(\mathcal{U}^{\leq k}; \Lambda^k(A^0))$. Thus, we rewrite the relation (1.4) to

$$\dot{F}_\alpha \dot{G}_\alpha = (c_\alpha^{-1} \dot{F}'_\alpha c_{\alpha^+})(c_{\alpha^+}^{-1} G'_\alpha c_{\alpha^+}), \qquad \text{for } \alpha \in \mathbb{A}^{\leq k}.$$

   Since Corollary 1.18 tells us, that the factorization into the factors of the semidirect product are unique, we get for all $\alpha \in \mathbb{A}^k$

$$\dot{F}_\alpha = c_\alpha^{-1} \dot{F}'_\alpha c_{\alpha^+} \qquad \text{and} \qquad \dot{G}_\alpha = c_{\alpha^+}^{-1} \dot{G}'_\alpha c_{\alpha^+}.$$

   The former relation implies that $(\dot{F}_\alpha)$ and $(\dot{F}'_\alpha)$ are cohomologous with values in $\Lambda^{<k}(A^0)$ on $\mathcal{U}^{\leq k}$. Since $\mathcal{U}^{<k}$ is already adequate to $\Lambda^{<k}(A^0)$, are $(\dot{F}_\alpha)$ and $(\dot{F}'_\alpha)$ already on $\mathcal{U}^{<k}$, i.e. in $\Gamma(\dot{\mathcal{U}}^{<k}; \Lambda^{<k}(A^0))$, cohomologous.
2. The proof of part 2. (together with a proof of part 1.) can be found in Loday-Richaud's paper [Lod94, Proof of Lem.II.3.3].

□

[a] In clockwise direction.

Let $k \in \mathcal{K}$ and $k' = \max\{k' \in \mathcal{K} \mid k' < k\}$. We then know by definition that $\mathcal{U}^{<k} = \mathcal{U}^{\leq k'}$ as well as $\Lambda^{<k}(A^0) = \Lambda^{\leq k'}(A^0)$ and thus $\Gamma(\dot{\mathcal{U}}^{<k}; \Lambda^{<k}(A^0)) = \Gamma(\dot{\mathcal{U}}^{\leq k'}; \Lambda^{\leq k'}(A^0))$ and obtain the following proposition.

Proposition 1.36

By applying $\mathfrak{S}^{\leq k}$ successively for different k's in decending order, one obtains the *product map of single leveled cocycles* τ in the following way

$$\underbrace{\Gamma(\dot{\mathcal{U}}^{<k_r};\Lambda^{<k_r}(A^0))}_{=} \times \Gamma(\dot{\mathcal{U}}^{k_r};\Lambda^{k_r}(A^0)) \xrightarrow{\quad\mathfrak{S}^{\leq k_r}\quad} \overset{\Gamma(\dot{\mathcal{U}};\Lambda(A^0))}{\overset{\|}{\Gamma(\dot{\mathcal{U}}^{\leq k_r};\Lambda^{\leq k_r}(A^0))}}$$

$$\underbrace{\Gamma(\dot{\mathcal{U}}^{<k_{r-1}};\Lambda^{<k_{r-1}}(A^0))}_{=} \times \Gamma(\dot{\mathcal{U}}^{k_{r-1}};\Lambda^{k_{r-1}}(A^0)) \xrightarrow{\mathfrak{S}^{\leq k_{r-1}}} \Gamma(\dot{\mathcal{U}}^{\leq k_{r-1}};\Lambda^{\leq k_{r-1}}(A^0))$$

$$\cdots \xrightarrow{\quad\mathfrak{S}^{\leq k_{r-2}}\quad} \Gamma(\dot{\mathcal{U}}^{\leq k_{r-2}};\Lambda^{\leq k_{r-2}}(A^0))$$

$$\underbrace{\Gamma(\dot{\mathcal{U}}^{<k_3};\Lambda^{<k_3}(A^0))}_{=} \times \Gamma(\dot{\mathcal{U}}^{k_3};\Lambda^{k_3}(A^0)) \xrightarrow{\quad\mathfrak{S}^{\leq k_3}\quad} \cdots$$

$$\Gamma(\dot{\mathcal{U}}^{k_1};\Lambda^{k_1}(A^0)) \times \Gamma(\dot{\mathcal{U}}^{k_2};\Lambda^{k_2}(A^0)) \xrightarrow{\quad\mathfrak{S}^{\leq k_2}\quad} \Gamma(\dot{\mathcal{U}}^{\leq k_2};\Lambda^{\leq k_2}(A^0))$$

which can be written in the following compact form

$$\begin{aligned} \tau : \prod_{k\in\mathcal{K}} \Gamma(\dot{\mathcal{U}}^k;\Lambda^k(A^0)) &\longrightarrow \Gamma(\dot{\mathcal{U}};\Lambda(A^0)) \\ (\dot{f}^k)_{k\in\mathcal{K}} &\longmapsto \prod_{k\in\mathcal{K}} \tau^k(\dot{f}^k) \end{aligned}$$

where the product is following an ascending order of levels and the maps τ_k are defined as

$$\begin{aligned} \tau^k : \Gamma(\dot{\mathcal{U}}^k;\Lambda^k(A^0)) \xrightarrow{\ \sigma^k\ } \Gamma(\dot{\mathcal{U}}^{\leq k};\Lambda^{\leq k}(A^0)) &\longrightarrow \Gamma(\dot{\mathcal{U}};\Lambda(A^0)) \\ (\dot{f}_\alpha)_{\alpha\in\mathbb{A}^k} &\longmapsto (\dot{G}_\alpha)_{\alpha\in\mathbb{A}} \end{aligned}$$

with

$$\dot{G}_\alpha = \begin{cases} \dot{f}_\alpha \text{ restricted to } \dot{U}_\alpha \text{ and seen as being in } \Lambda(A^0) & \text{when } \alpha\in\mathbb{A}^k \\ \mathrm{id} \ \text{ (the identity on } \dot{U}_\alpha) & \text{when } \alpha\notin\mathbb{A}^k \end{cases}$$

The defined map τ is clearly injective and it can be extended to an arbitrary order of levels (cf. Remark [Lod94, Rem.II.3.5]).

Definition 1.37

Define the injective map

$$\begin{aligned}\tau^k : \Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A^0)) &\longrightarrow \Gamma(\dot{\mathcal{U}}; \Lambda(A^0))\\ \dot{f} = (\dot{f}_\alpha)_{\alpha\in\mathbb{A}^k} &\longmapsto (\dot{F}_\alpha)_{\alpha\in\mathbb{A}}\end{aligned}$$

where

$$\dot{F}_\alpha = \begin{cases}\dot{f}_\alpha \text{ restricted to } \dot{U}_\alpha \text{ and seen as being in } \Lambda(A^0) & \text{when } \alpha \in \mathbb{A}^k\\ \text{id} \ \ \text{(the identity on } \dot{U}_\alpha) & \text{when } \alpha \notin \mathbb{A}^k\end{cases}$$

The *product map of single-leveled cocycles* is then defined as [Lod94, Prop.II.3.4]

$$\begin{aligned}\tau : \prod_{k\in\mathcal{K}} \Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A^0)) &\longrightarrow \Gamma(\dot{\mathcal{U}}; \Lambda(A^0))\\ (\dot{f}^k)_{k\in\mathcal{K}} &\longmapsto \prod_{k\in\mathcal{K}} \tau^k(\dot{f}^k)\end{aligned}$$

following an ascending order of levels.

Remark 1.37.1

The map τ

1. is injective since it is composed from σ^k and the clearly injetive mapping

$$\begin{aligned}\Gamma(\dot{\mathcal{U}}^{\leq k}; \Lambda^{\leq k}(A^0)) &\longrightarrow \Gamma(\dot{\mathcal{U}}; \Lambda(A^0))\\ (\dot{F}_\alpha)_{\alpha\in\mathbb{A}^{\leq k}} &\longmapsto (\dot{F}'_\alpha)_{\alpha\in\mathbb{A}}\end{aligned}$$

   where

$$\dot{F}'_\alpha = \begin{cases}\dot{F}_\alpha \text{ restricted to } \dot{U}_\alpha \text{ and seen as being in } \Lambda(A^0) & \text{when } \alpha \in \mathbb{A}^{\leq k}\\ \text{id} \ \ \text{(the identity on } \dot{U}_\alpha) & \text{when } \alpha \notin \mathbb{A}^{\leq k}\end{cases}$$

   and

2. it can be extended to an arbitrary order of levels (cf. Remark [Lod94, Rem.II.3.5]).

Corollary 1.38

The product map of single-leveled cocycles τ induces on the cohomology a bijective and natural map

$$\begin{array}{ccc}\mathcal{T} : \prod_{k\in\mathcal{K}} \Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A^0)) & \longrightarrow & H^1(\mathcal{U}; \Lambda(A^0))\,.\\ \wr\Vert & & \wr\Vert\\ \prod_{k\in\mathcal{K}} H^1(S^1;\Lambda^k(A^0)) & & H^1(S^1;\Lambda(A^0))\end{array}$$

Composing functions to obtain h We have the ingredients to define the function h from Theorem 1.25 by composition of already bijective maps.

Proof of Theorem 1.25. Let $i_\alpha : \mathrm{Sto}_\alpha(A^0) \to \prod_{k\in\mathcal{K}} \mathrm{Sto}^k_\alpha(A^0)$ be the map which corresponds to the filtration from Proposition 1.20 and denote the composition

$$
\begin{tikzcd}
\prod\limits_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha(A^0) \arrow[r, "\prod_{\alpha\in\mathbb{A}} i_\alpha"] \arrow[drr, bend right, "\mathfrak{T}"'] & \prod\limits_{\alpha\in\mathbb{A}}\prod\limits_{k\in\mathcal{K}} \mathrm{Sto}^k_\alpha(A^0) \arrow[d, phantom, "\rotatebox{90}{$\equiv$}"] & \\
& \prod\limits_{k\in\mathcal{K}}\prod\limits_{\alpha\in\mathbb{A}} \mathrm{Sto}^k_\alpha(A^0) \arrow[r, "\prod_{k\in\mathcal{K}} i^k"] & \prod\limits_{k\in\mathcal{K}} \Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A^0))
\end{tikzcd}
$$

by $\mathfrak{T}$. The bijection h is then obtained as

$$\mathcal{T}\circ\mathfrak{T} : \prod_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha(A^0) \longrightarrow H^1(\mathcal{U};\Lambda(A^0))\,.$$

PROBLEM:naturality (is obvious?) PROBLEM:Show that this is the correct h ∎

1.2.3 Some exemplary calculations

Here we want to discuss which information is required to describe the Stokes cocycle corresponding to a multileveled system in more depth. We will look at a sigle-leveled system corresponding to a normal form $A^0 \in \mathrm{GL}_3(\mathbb{C}(\{t\}))$ with exactly two levels and will apply the techniques developed in the previous sections in an rather explicit way.

Let A^0 be a normal form with dimension $n = 3$ and two levels $\mathcal{K} = \{k_1 < k_2\}$ which satisfies that there is at least one anti-Stokes direction θ which is beared by both levels. Let $q_j(t^{-1})$ be the determining polynomials and let k_{jl} be the degrees of $(q_j - q_l)(t^{-1})$. Up to permutation, we know that in our case are the leading terms of $(q_1 - q_2)(t^{-1})$ and $(q_1 - q_3)(t^{-1})$ equal and thus

- up to permutation is $k_2 = k_{1,2} = k_{1,3}$ and $k_1 = k_{2,3}$, i.e. the larger degree appears twice, and

let q_1, q_2, q_3 be polynomials, such that

$$\deg(q_1 - q_2) =: k_2 > k_1 := \deg(q_2 - q_3)$$

then is the degree of $q_1 - q_3$ given by

case 1 $\deg(q_1) < k_2$: then is $\deg(q_2) = k_2$ and thus $\deg(q_3) = k_2$.

case 2 $\deg(q_2) < k_2$: then is $\deg(q_1) = k_2$ and $\deg(q_3) \leq \deg(q_2)$.

case 3 $\deg(q_1) = \deg(q_2) = k_2$: thus follows that the leading term of q_1 and q_2 are different.

subcase 3.a $\deg(q_3) < k_2$: everything is clear

subcase 3.b $\deg(q_3) = k_2$: here has the leading term of q_3 be equal to them from q_1 to satisfy that $k_2 > \deg(q_2 - q_3)$.

From this follows that $\deg(q_1 - q_3) = k_2$.

- $q_0 \underset{\alpha}{\prec} q_2$ (resp. $q_2 \underset{\alpha}{\prec} q_1$) if and only if $q_1 \underset{\alpha}{\prec} q_3$ (resp. $q_3 \underset{\alpha}{\prec} q_1$) and thus do they determine the same anti-Stokes directions.

The set of all anti-Stokes directions is then given as

$$\mathbb{A} = \left\{ \theta + \frac{\pi}{k} \cdot j \mid k \in \mathcal{K},\ j \in \mathbb{N} \right\} .$$

Denote by $\mathcal{Y}_0(t)$ a normal solution of $[A^0]$.

Let us start by looking at a single germ in depth. The Proposition 1.9 states that every Stokes germ φ_α can be written as its matrix representation conjugated by the normal solution, i.e. as $\varphi_\alpha = \mathcal{Y}_0 C_{\varphi_\alpha} \mathcal{Y}_0^{-1} = \rho_\alpha^{-1}(C_{\varphi_\alpha})$.

Look at an example in which we will demonstrate, from which relations on the determining polynomials which restriction on the form of the Stokes matrices arise.

Example 1.39

Let $\alpha \in \mathbb{A}$ be an anti-Stokes direction. From the definition of $\mathbb{S}\mathrm{to}_\alpha(A^0)$ (cf. Definition 1.8) we know that, if one has $q_1 \underset{\alpha}{\prec} q_2$, the Stokes matrix has the form

$$\begin{pmatrix} 1 & \boldsymbol{c_1} & \star \\ \mathbf{0} & 1 & \star \\ \star & \star & 1 \end{pmatrix}$$

where $c_j \in \mathbb{C}$ and $\star \in \mathbb{C}$.

We have seen that $q_1 \underset{\alpha}{\prec} q_2 \Rightarrow q_1 \underset{\alpha}{\prec} q_3$ thus the representation has the form

$$\begin{pmatrix} 1 & c_1 & \boldsymbol{c_2} \\ 0 & 1 & \star \\ \mathbf{0} & \star & 1 \end{pmatrix}$$

and if we also know that neither $q_2 \underset{\alpha}{\prec} q_3$ nor $q_3 \underset{\alpha}{\prec} q_2$ it has the form

$$\begin{pmatrix} 1 & c_1 & c_2 \\ 0 & 1 & \mathbf{0} \\ 0 & \mathbf{0} & 1 \end{pmatrix} .$$

We also know that every matrix of this form is a representation to some Stokes germ. Thus we have an isomorphism

$$\begin{aligned}\vartheta_\alpha : \mathbb{C}^2 &\longrightarrow \mathbb{S}\mathrm{to}_\alpha(A^0)\\ (c_1, c_2) &\longmapsto \begin{pmatrix} 1 & c_1 & c_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\end{aligned}$$

In fact, the following 9 cases of Stokes matrices can arise:

	$q_2 \underset{\alpha}{\prec} q_3$	$q_3 \underset{\alpha}{\prec} q_2$	else
$q_1 \underset{\alpha}{\prec} q_2$ and $q_1 \underset{\alpha}{\prec} q_3$	$\begin{pmatrix} 1 & c_2 & c_3 \\ 0 & 1 & c_1 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & c_2 & c_3 \\ 0 & 1 & 0 \\ 0 & c_1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & c_2 & c_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
$q_2 \underset{\alpha}{\prec} q_1$ and $q_3 \underset{\alpha}{\prec} q_1$	$\begin{pmatrix} 1 & 0 & 0 \\ c_2' & 1 & c_1 \\ c_3 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ c_2 & 1 & 0 \\ c_3' & c_1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ c_2 & 1 & 0 \\ c_3 & 0 & 1 \end{pmatrix}$
else	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & c_1 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c_1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

In the blue cases we have $\mathcal{K}_\alpha = \mathcal{K}$ and $\mathbb{C}^3 \xrightarrow[\cong]{\vartheta_\alpha} \mathbb{S}\mathrm{to}_\alpha(A^0)$. In the green cases $\mathcal{K}_\alpha = \{k_2\}$ and $\mathbb{C}^2 \xrightarrow[\cong]{\vartheta_\alpha} \mathbb{S}\mathrm{to}_\alpha(A^0)$ as well as in the purple cases $\mathcal{K}_\alpha = \{k_1\}$ and $\mathbb{C}^1 \xrightarrow[\cong]{\vartheta_\alpha} \mathbb{S}\mathrm{to}_\alpha(A^0)$. Thus, for every $\alpha \in \mathbb{A}$, we have an isomorphism $\rho_\alpha^{-1} \circ \vartheta_\alpha$. We will replace c_2' by $c_2 + c_1 c_3$ and c_3' by $c_1 c_2 + c_3$ to be consistent with the decomposition in the next part (cf. Example 1.41).

Corollary 1.40

The morphism $\prod_{\alpha\in\mathbb{A}} \vartheta_\alpha$ is an isomorphism of pointed sets, which maps the element only containing zeros to

$$(\mathrm{id}, \mathrm{id}, \dots, \mathrm{id}) \in \prod_{\alpha\in\mathbb{A}} \mathbb{S}\mathrm{to}_\alpha(A^0),$$

which gets by $(\prod_{\alpha\in\mathbb{A}})^{-1} \circ h$ mapped to the trivial cohomology class in $\mathcal{S}t(A^0)$.

In proposition 1.20 and especially Remark 1.20.1 we have defined a decomposition of the Stokes group $\mathrm{Sto}_\alpha(A^0)$ in subgroups generated by k-germs for $k \in \mathcal{K}$. In our case, we have at most two nontrivial factors. Especially is this decomposition given by

$$\varphi_\alpha = \varphi_\alpha^{k_1}\varphi_\alpha^{k_2} \overset{i_\alpha}{\longmapsto} \left(\varphi_\alpha^{k_1}, \varphi_\alpha^{k_2}\right) \in \mathrm{Sto}_\alpha^{k_1}(A^0) \times \mathrm{Sto}_\alpha^{k_2}(A^0)\,,$$

and i_α is the map, wich gives the factors of this factorization in ascending order. This decomposition, of a germ φ_α, is trivial if $\#\mathcal{K}(\varphi_\alpha) \leq 1$, thus the interesting cases are the blue cases.

Example 1.41

Look at the example

$$\vartheta_\alpha(c_1, c_2, c_3) = \mathcal{Y}_0 \begin{pmatrix} 1 & 0 & 0 \\ c_2 & 1 & 0 \\ c_1c_2 + c_3 & c_1 & 1 \end{pmatrix} \mathcal{Y}_0^{-1} = \varphi_\alpha \, .$$

According to Remark 1.18.1 the factor $\varphi_\alpha^{k_1} \in \mathrm{Sto}_\alpha^{k_1}(A^0)$, is given by

$$\varphi_\alpha^{k_1} = \mathcal{Y}_0 \begin{pmatrix} 1 & 0 & 0 \\ \mathbf{0} & 1 & 0 \\ \mathbf{0} & c_1 & 1 \end{pmatrix} \mathcal{Y}_0^{-1} \, .$$

The other factor $\varphi_\alpha^{k_2}$ is then obtained as

$$\begin{aligned} \varphi_\alpha^{k_2} &= \left(\varphi_\alpha^{k_1}\right)^{-1} \varphi_\alpha^{k_2} \\ &= \mathcal{Y}_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -c_1 & 1 \end{pmatrix} \underbrace{\mathcal{Y}_0^{-1}\mathcal{Y}_0}_{=\mathrm{id}} \begin{pmatrix} 1 & 0 & 0 \\ c_2 & 1 & 0 \\ c_1c_2 + c_3 & c_1 & 1 \end{pmatrix} \mathcal{Y}_0^{-1} \\ &= \mathcal{Y}_0 \begin{pmatrix} 1 & 0 & 0 \\ c_2 & 1 & 0 \\ c_3 & 0 & 1 \end{pmatrix} \mathcal{Y}_0^{-1} \, . \end{aligned}$$

The four nontrivial decomposition in our situation, are given by:

1. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & c_1 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & c_2 & c_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & c_2 & c_3 \\ 0 & 1 & c_1 \\ 0 & 0 & 1 \end{pmatrix}$

2. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c_1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & c_2 & c_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & c_2 & c_3 \\ 0 & 1 & 0 \\ 0 & c_1 & 1 \end{pmatrix}$

3. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & c_1 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ c_2 & 1 & 0 \\ c_3 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ c_2 + c_1c_3 & 1 & c_1 \\ c_3 & 0 & 1 \end{pmatrix}$

4. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c_1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ c_2 & 1 & 0 \\ c_3 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ c_2 & 1 & 0 \\ c_1c_2 + c_3 & c_1 & 1 \end{pmatrix}$

Explicit example

Even more explicit, we can fix the levels $k_1 = 1$ and $k_2 = 3$ together with $\theta = 0$. Assume without any restriction that $q_1 \underset{\theta}{\prec} q_2$ and $q_1 \underset{\theta}{\prec} q_3$ as well as $q_2 \underset{\theta}{\prec} q_3$. Other choices would result in reordering of the tuples below. Let the matrix L be given as $L = \mathrm{diag}(l_1, l_2, l_3) \in \mathrm{GL}_n(\mathbb{C})$.

The classification space is in this case isomorphic to $\mathbb{C}^{2\cdot(1+2\cdot 3)} = \mathbb{C}^{14}$. The element

$$({}^1c_1, {}^2c_1, {}^1c_2, {}^1c_3, {}^2c_2, {}^2c_3, \ldots, {}^6c_2, {}^6c_3) \in \mathbb{C}^{14}$$

gets, via the isomorphism $\prod_{\alpha\in\mathbb{A}} j_\alpha$, mapped to

$$\left(\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & {}^1c_1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & {}^2c_1 & 1 \end{pmatrix} \right), \left(\begin{pmatrix} 1 & {}^1c_2 & {}^1c_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ {}^2c_2 & 1 & 0 \\ {}^2c_3 & 0 & 1 \end{pmatrix}, \ldots, \begin{pmatrix} 1 & 0 & 0 \\ {}^6c_2 & 1 & 0 \\ {}^6c_3 & 0 & 1 \end{pmatrix} \right) \right)$$

in $\prod_{\alpha\in\mathbb{A}^1} \mathbb{S}\mathrm{to}^1_\alpha(A^0) \times \prod_{\alpha\in\mathbb{A}^3} \mathbb{S}\mathrm{to}^3_\alpha(A^0)$ and thus the element

$$\left(\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & {}^1c_1 \\ 0 & 0 & 1 \end{pmatrix}, \mathrm{id}, \mathrm{id}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & {}^2c_1 & 1 \end{pmatrix}, \mathrm{id}, \mathrm{id} \right), \left(\begin{pmatrix} 1 & {}^1c_2 & {}^1c_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ {}^2c_2 & 1 & 0 \\ {}^2c_3 & 0 & 1 \end{pmatrix}, \ldots, \begin{pmatrix} 1 & 0 & 0 \\ {}^6c_2 & 1 & 0 \\ {}^6c_3 & 0 & 1 \end{pmatrix} \right) \right)$$

in $\prod_{\alpha\in\mathbb{A}} \mathbb{S}\mathrm{to}^1_\alpha(A^0) \times \prod_{\alpha\in\mathbb{A}} \mathbb{S}\mathrm{to}^3_\alpha(A^0)$. Using the morphism $\prod_{\alpha\in\mathbb{A}} i_\alpha^{-1}$ we get a complete set of Stokes matrices as

$$\left(\begin{pmatrix} 1 & {}^1c_2 & {}^1c_3 \\ 0 & 1 & {}^1c_1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ {}^2c_2 & 1 & 0 \\ {}^2c_3 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & {}^3c_2 & {}^3c_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ {}^4c_2 & 1 & 0 \\ {}^2c_1{}^4c_2 + {}^4c_3 & {}^2c_1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & {}^5c_2 & {}^5c_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ {}^6c_2 & 1 & 0 \\ {}^6c_3 & 0 & 1 \end{pmatrix} \right) \in \prod_{\alpha\in\mathbb{A}} \mathbb{S}\mathrm{to}_\alpha(A^0)\,.$$

Applying the isomorphism $\prod_{\alpha\in\mathbb{A}} \rho_\alpha^{-1}$, i.e. conjugation by the fundamental solution $\mathcal{Y}_0(t) = t^L e^{Q(t^{-1})}$ (cf. Proposition 1.9), yields then the corresponding Stokes cocycle in $\prod_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha(A^0)$ and thus an element in $\mathcal{S}t(A^0)$.

This element is explicitly given as

$$\left(\begin{pmatrix} 1 & {}^1c_2 t^{l_2-l_1} e^{(q_2-q_1)(t^{-1})} & {}^1c_3 t^{l_3-l_1} e^{(q_3-q_1)(t^{-1})} \\ 0 & 1 & {}^1c_1 t^{l_3-l_2} e^{(q_3-q_2)(t^{-1})} \\ 0 & 0 & 1 \end{pmatrix}, \mathcal{Y}_0^{-1} \begin{pmatrix} 1 & 0 & 0 \\ {}^2c_2 & 1 & 0 \\ {}^2c_3 & 0 & 1 \end{pmatrix} \mathcal{Y}_0, \right.$$

$$\mathcal{Y}_0^{-1}\begin{pmatrix}1 & {}^3c_2 & {}^3c_3\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\mathcal{Y}_0, \mathcal{Y}_0^{-1}\begin{pmatrix}1 & 0 & 0\\ {}^4c_2 & 1 & 0\\ {}^2c_1{}^4c_2 + {}^4c_3 & {}^2c_1 & 1\end{pmatrix}\mathcal{Y}_0,$$
$$\left.\mathcal{Y}_0^{-1}\begin{pmatrix}1 & {}^5c_2 & {}^5c_3\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\mathcal{Y}_0, \mathcal{Y}_0^{-1}\begin{pmatrix}1 & 0 & 0\\ {}^6c_2 & 1 & 0\\ {}^6c_3 & 0 & 1\end{pmatrix}\mathcal{Y}_0\right) \in \prod_{\alpha\in\mathbb{A}}\mathrm{Sto}_\alpha(A^0)\,.$$