Universität Augsburg
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Master's thesis

Classification of meromorphic connections using Stokes structures and Stokes groups

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Contents

1 Stokes Structures

Let $(\mathcal{M}^{nf}, \nabla^{nf})$ be a fixed model with the corresponding normal form A^0 and let us also fix a normal solution $\mathcal{Y}_0$ of A^0. Here we want to introduce the Stokes structures, which will characterize the isomorphism classes of meromorphic connections uniquely, i.e. an space which is isomorphic to the set $\mathcal{H}(A^0)$ of isomorphism classes of marked pairs. A great overview of this topic is given by Varadarajan in [Var96].

One of the most important theorems, which is fundamental for the whole chapter, is the Malgrange-Sibuya Theorem. It states that the classifying set $\mathcal{H}(A^0)$ is via an map exp isomorphic to the first non abelian cohomology space $H^1(S^1; \Lambda(A^0)) =: \mathcal{S}t(A^0)$ of the Stokes sheaf $\Lambda(A^0)$ and will be proven in the first section. In Section **??** we will improve the Malgrange-Sibuya Theorem by showing that each 1-cohomology class in $\mathcal{S}t(A^0)$ contains a unique 1-cocycle of a special form called *the Stokes cocycle*. We will further show that such cocycles can be identified with their germs at some special directions, i.e. anti-Stokes directions. These germs are called Stokes germs and for an anti-Stokes direction α do these germs form the Stokes groups $\mathrm{Sto}_\alpha(A^0)$ (cf. Section **??**). The morphism, which maps each product of Stokes germs to its corresponding 1-cocycle will be denoted by h. This will be further improved in Section **??**, where we will collect multiple Stokes germs to their product to obtain a more robust version of the Stokes space.

If one introduces the map g, which arises from the theory of summation[1] and takes an equivalence class[2] and returns the corresponding Stokes cocycle in an canonically way one obtains the following commutative diagram.

$$\begin{tikzcd}
\mathcal{H}(A^0) \arrow[r, "\exp"] \arrow[d, "\mathrm{g}"'] & \mathcal{S}t(A^0) \\
\prod_{\theta\in\mathbb{A}} \mathrm{Sto}_\theta(A^0) \arrow[ur, "h"'] &
\end{tikzcd}$$

This diagram will be enhanced in Section **??** by adding a couple of isomorphisms.

In the first section of this chapter we will use Sabbah's book [Sab07, section II] as its main resource. In the Sections **??** and **??** will Loday-Richaud's paper [Lod94] and her book [Lod14, Sec.4] be useful. Stokes groups are also discussed Boalch's paper [Boa01] (resp. his thesis [Boa99]) which looks only at the single leveled case or the paper [MR91] from Martinet and Ramis. Another very important paper is [BJL79] from Balser, Jurkat and Lutz.

[1] We will not use the theory of summation and only think of it as a black box, although we will roughly discuss the theory of summation in Appendix **??**.

[2] resp. an ambassador of such a class.

1.1 Stokes structures: Malgrange-Sibuya isomorphism

Here we will look at the set $\mathcal{H}(A^0)$ of isomorphism classes of marked pairs and we will proof that it is isomorphic to the first non abelian cohomology set $H^1(S^1; \Lambda(A^0))$ which will be denoted as $\mathcal{S}t(A^0)$. If we talk about cocycles or cochains, we will in the following always mean 1-cocycles or 1-cochains.

Let us first define the Stokes sheaf $\Lambda(A^0)$ on S^1 as the sheaf of flat isotropies of $[A^0]$.

Definition 1.1

The Stokes sheaf $\Lambda(A^0)$ of A^0 is defined as the subsheaf of $\mathrm{GL}_n(\mathcal{A})$ in the following way. For some $\theta \in S^1$ is the stalk at θ the subgroup of $\mathrm{GL}_n(\mathcal{A})_\theta$ of elements f which satisfy

1. Multiplicatively flatness: f is asymptotic to the identity, i.e. $f \sim_{\mathfrak{s}} 1$;
2. Isotropy of A^0: ${}^fA^0 = A^0$.

Remark 1.1.1

This definition makes also sense as $\Lambda(A)$ where A stands for a systems wich is not in normal form. The elements of $\Lambda(A)$ then have to be isotropies of a normal form A^0 of A.

Since the sheaf by definition only contains solutions of the differential equation [End A^0] we obtain the following proposition.

Proposition 1.2

The sheaf $\Lambda(A^0)$ is piecewise-constant (cf. [Lod04, Prop.2.1]).

1.1.1 The theorem

Let us now state the Malgrange-Sibuya Theorem. We will first give it in the language of meromorphic connections and after that we will give the same theorem in the language of systems. The second formulation of this theorem will be proven in the next section.

In the language of meromorphic connections is the map, of the Malgrange-Sibuya Theorem below, described as follows.

Let $(\mathcal{M}, \nabla, \widehat{f})$ be a marked germ of a meromorphic connection. By Theorem **??** there exists an open covering $\mathcal{U} = (U_j)_{j\in J}$ and for every open set, an isomorphism

$$f_j : (\widetilde{\mathcal{M}}, \widetilde{\nabla})_{|U_j} \longrightarrow (\widetilde{\mathcal{M}}^{nf}, \widetilde{\nabla}^{nf})_{|U_j}$$

such that $f_j \sim_{U_j} \widehat{f}$. By $(f_k f_j^{-1})_{jk}$ is then a cocycle of the sheaf $\mathcal{S}t(A^0)$, relative to the covering $\mathcal{U}$, defined. This defines a mapping of pointed sets

$$\exp : \mathcal{H}(\mathcal{M}^{nf}, \nabla^{nf}) \longrightarrow H^1(S^1; \Lambda(A^0))$$

to the first non abelian cohomology of $\Lambda(A^0)$, which sends the class of $(\mathcal{M}^{nf}, \nabla^{nf}, \widehat{\mathrm{id}})$ to that of id, i.e. the trivial cohomology class.

Theorem 1.3: Malgrange-Sibuya

The homomorphism

$$\exp : \mathcal{H}(\mathcal{M}^{nf}, \nabla^{nf}) \longrightarrow \mathcal{S}t(A^0) := H^1(S^1; \Lambda(A^0))$$

is an isomorphism of pointed sets.

The theorem (system version)

Since the language of meromorphic connections is equivalent to the one of systems, there is also the translated version of the Malgrange-Sibuya isomorphism to the language of systems. The corresponding map is then build as follows.

Let $(A, \widehat{F})$ be a marked pair, thus $\widehat{F}$ solves $[A^0, A]$. By the M.A.E.T (Theorem **??**) there exists an open covering $\mathcal{U} = (U_j)_{j \in J}$ together with for every open set U_j a lift $F_j \in \mathrm{GL}_n(\mathcal{A}(U_j))$ of $\widehat{F}$ (cf. Definition **??**), which solves $[A, A^0]$. By the cocycle $(F_l^{-1} F_j)_{jl} \in \Gamma(\dot{\mathcal{U}}; \Lambda(A^0))$ is then a cohomology class in $\mathcal{S}t(A^0)$ relative to the covering $\mathcal{U}$ determined. For other lifts F'_j of $\widehat{F}$ on U_j is $(G_j = F_j^{-1} F'_j)$ a 0-cochain of $\Lambda(A^0)$ relative to $\mathcal{U}$, which satisfies

$$F_k^{-1} F_j = G_k F_k'^{-1} F'_j G_j^{-1} .$$

Thus the cochians associated to (F_j) and (F'_j) determine the same cohomology class in $\mathcal{S}t(A^0)$. One can also check that, if $(A, \widehat{F})$ and $(A', \widehat{F}')$ are equivalent, the corresponding cocycles define the same cohomology class. This defines a welldefined mapping of pointed sets

$$\mathcal{H}(A^0) \to H^1(S^1; \Lambda(A^0))$$

to the first non abelian cohomology of $\Lambda(A^0)$, which we call exp. It maps the class of $([A^0], \widehat{\mathrm{id}})$ to that of id, i.e. the trivial cohomology class.

Theorem 1.4: Malgrange-Sibuya (system version)

The homomorphism

$$\exp : \mathcal{H}(A^0) \longrightarrow \mathcal{S}t(A^0) := H^1(S^1; \Lambda(A^0))$$

is an isomorphism of pointed sets.

Remark 1.5

The Theorem [BV89, Thm.III.1.1.2] in the book from Babbitt and Varadarajan, states that $\mathcal{S}t(A^0)$ is actually a local moduli space for marked pairs, which are formally isomorphic to a given system $[A^0]$. In fact is the whole third part of [BV89] dedicated to this topic.

Since the morphism exp depends on the choice of the normal form, we will denote that, if it is not clear, by $\exp_{A^0} = \exp$.

Remark 1.6

To another normal form $A^1 = {}^{\Phi}A^0$ there correspond cochains, which are conjugated via $\Phi \in G(\{t\})$. This is formulated in the following commutative diagram:

$$\begin{tikzcd}
G\backslash\widehat{G}(A^1) \arrow[rr, "\cdot\Phi"] \arrow[dd, "\exp_{A^1}"] \arrow[dr, phantom, sloped, "\ni"] & & G\backslash\widehat{G}(A^0) \arrow[dd, "\exp_{A^0}"] \arrow[dr, phantom, sloped, "\ni"] & \\
& \widehat{F} \arrow[rr, mapsto] \arrow[dd, mapsto] & & \widehat{F}\Phi \arrow[dd, mapsto] \\
H^1(S^1; \Lambda(A^1)) \arrow[rr] \arrow[dr, phantom, sloped, "\ni"] & & H^1(S^1; \Lambda(A^0)) \arrow[dr, phantom, sloped, "\ni"] & \\
& \exp_{A^1}(\widehat{F}) \arrow[rr, mapsto] & & \exp_{A^0}(\widehat{F}\Phi)
\end{tikzcd}$$

where $\exp_{A^0}(\widehat{F}\Phi) = \Phi^{-1} \exp_{A^0}(\widehat{F})\Phi$.

1.1.2 Proof of Theorem 1.4

We will mainly refer to [BV89, Proof of Theorem 4.5.1] and [Sab07, Section 6.d], where a slightly more complicated case with deformation space is proven. These both resources proof the theorem using the languages of meromorphic connections whereas we will use systems.

We will start by proofing the injectivity of the morphism exp.

Proof of the injectivity. Consider the two marked pairs $(A, \widehat{F})$ and $(A', \widehat{F}')$ in $\widehat{\mathrm{Syst}}_m(A^0)$, whose classes in $\mathcal{H}(A^0)$ get mapped to same element

$$\exp([(A, \widehat{F})]) = \lambda = \exp([(A', \widehat{F}')]) \in H^1(S^1; \Lambda(A^0)) .$$

By using refined coverings, it is possible to find a common finite covering $\mathcal{U} = \{U_j; j \in J\}$ of S^1 such that λ is the class of the cocycles $(F_l^{-1}F_j)$ and $(F_l'^{-1}F_j')$, where F_j (resp. F_j') are lifts of $\widehat{F}$ (resp. $\widehat{F}'$) on $U_j \in \mathcal{U}$. From $[(F_l^{-1}F_j)] = [(F_l'^{-1}F_j')]$ follows that there exists a 0-cochain $(G_j)_{j\in J}$ of the sheaf $\Lambda(A^0)$ relative to the covering $\mathcal{U}$, such that

$$F_l'^{-1}F_j' = G_l F_l^{-1} F_j G_j^{-1} \text{ on the arc } U_j \cap U_l \,,$$

which can be rewritten to

$$F_j' G_j F_j^{-1} = F_l' G_l F_l^{-1} \text{ on the arc } U_j \cap U_l \,. \tag{1.1}$$

If we set $H_j := F_j' G_j F_j^{-1}$ on U_j, we get

- that from equation (1.1) that the H_j glue together,
- that H_j is a solution of $[A, A']$ on every U_j, i.e. it satisfies there ${}^{H_j}A = A'$, since

$$\begin{aligned}
{}^{H_j}A &= {}^{F_j' G_j F_j^{-1}}A \\
&= {}^{F_j' G_j}A^0 && \text{(since } F_j' \text{ is a lift of } \widehat{F}' \text{ on } U_j) \\
&= {}^{F_j'}A^0 && \text{(since } G_j \text{ is a is an isotropy of } A^0) \\
&= A' && \text{(since } F_j \text{ is a lift of } \widehat{F} \text{ on } U_j)
\end{aligned}$$

  and

- which satisfies $\widehat{F}' = \widehat{H}_j \widehat{F}$ on every U_j, since

$$\begin{aligned}
\widehat{H}_j \widehat{F} &= \widehat{F_j' G_j F_j^{-1}} \widehat{F} \\
&= \widehat{F}_j' \underbrace{\widehat{G}_j}_{\substack{\parallel \\ \mathrm{id}}} \widehat{F_j^{-1}} \widehat{F} && \text{(since } G_j \text{ is flat, i.e. } \widehat{G}_j = \mathrm{id}) \\
&= \widehat{F}' \underbrace{\widehat{F^{-1}} \widehat{F}}_{\substack{\parallel \\ \mathrm{id}}} \\
&= \widehat{F}'
\end{aligned}$$

Therefore are $(A, \widehat{F})$ and $(A', \widehat{F}')$ equivalent (cf. page **??**) and injectivity is proven. □

For the proof of the surjectivity we will use another result from Malgrange and Sibuya, which is also called the Malgrange-Sibuya Theorem (Theorem 1.8). It will be stated bellow but one can also be found in Babbitt and Varadarajans's book [BV89, 65ff] as Theorem 4.2.1.

Let $\widehat{F} \in G(\!(t)\!)$ be a matrix with formally meromorphic entries. By the Borel-Ritt Lemma (cf. Theorem **??**) we then know, that there exists for every arc $I \subsetneq S^1$ a holomorphic function $G : \mathfrak{s}_I \to \mathrm{GL}_n(\mathbb{C})$ which is asymptotic to $\widehat{F}$. We will denote the set of all such holomorphic functions, which are on the arc I asymptotic to $\mathrm{id} \in G(\!(t)\!)$ by

$$\mathcal{G}(I) = \{G \in \mathrm{GL}_n(\mathcal{A}(I)) \mid G \sim_I \mathrm{id}\} \;,$$

and this defines a sheaf $\mathcal{G}$ on S^1. The statement of the (second) Malgrange-Sibuya Theorem (Theorem 1.8) is then, that the difference between formal and convergent invertible matrices is described by the first sheaf cohomology $H^1(S^1; \mathcal{G})$ of $\mathcal{G}$ via the map

$$\Theta : G(\!(t)\!)/G(\{t\}) \longrightarrow H^1(S^1; \mathcal{G}) \,,$$

which will turn out to be an isomorphism. It is set up as follows:

> Let $[\widehat{F}] \in G(\!(t)\!)/G(\{t\})$ with ambassador $\widehat{F}$ and let $\mathcal{U} = \{U_j \mid j \in J\}$ be a finite covering of S^1 by open arcs. The Borel-Ritt Lemma yields for every arc $j \in J \subsetneq S^1$ a holomorphic function F_j which satisfies $F_j \sim_{U_j} \widehat{F}$. By $(F_l F_j^{-1})_{j,l \in J}$ is then a cocycle for $\mathcal{G}$ defined, and write $\Theta([\widehat{F}])$ for the corresponding cohomology class.

This construction is similar to the definition of the map of Theorem 1.4. The difference is, that we instead of M.A.E.D, to obtain lifts in the sense of Definition **??**, we use only the Borel-Ritt Lemma to obtain only asymptotic lifts.

It can be verified, that the class $\Theta([\widehat{F}])$ does not depend on

- the choise of an ambassador $\widehat{F}$ in $[\widehat{F}] \in G(\!(t)\!)/G(\{t\})$,
- the choice of the covering $\mathcal{U}$ nor
- the choice the lifts F_j.

Lemma 1.7

The mapping Θ is injective.

Proof. Let $\widehat{F}$ and $\widehat{F}' \in G(\!(t)\!)$ such that $\Theta([\widehat{F}]) = \Theta([\widehat{F}'])$. We then can find a covering $\mathcal{U} = \{U_j \mid j \in J\}$ together with holomorphic functions F_j and F_j', which satisfy $F_j \sim_{U_j} \widehat{F}$ and $F_j' \sim_{U_j} \widehat{F}'$, such that $(F_l^{-1} F_j)_{j,l \in J}$ and $(F_l'^{-1} F_j')_{j,l \in J}$ determine the classes $\Theta([\widehat{F}])$ and $\Theta([\widehat{F}'])$. This implies that there are maps G_j, which are on U_j holomorphic and satisfy $G_j \sim_{U_j} \mathrm{id}$ such that

$$F_l'^{-1} F_j' = G_l F_l^{-1} F_j G_j^{-1} \text{ on the arc } U_j \cap U_l$$

This equation can be rewritten to

$$F'_j G_j F_j^{-1} = F'_l G_l F_l^{-1} \text{ on the arc } U_j \cap U_l\,.$$

Since this tells us, that the functions $F'_j G_j F_j^{-1}$ coincide on the overlapping and define a holomorphic map from the arc S^1 (i.e. a punctured disc with a small radius) into $\mathrm{gl}_n(\mathbb{C})$, which will be called H. Since $F'_j G_j F_j^{-1} \sim_{U_j}$ id for all $j \in J$, we have $H \sim_{S^1}$ id. Thus the defined H meromorphic at 0 and satisfies $H = F'^{-1}F$, so that $[F] = [F']$. □

Theorem 1.8: Malgrange-Sibuya

The map $\Theta : G(\!(t)\!)/G(\{t\}) \to H^1(S^1; \mathcal{G})$ is an isomorphism.

This Theorem is proven in Section 4.4 of Babbitt and Varadarajan's book [BV89] or on page 371 of [MR91] from Martinet and Ramis. It is also mentioned on page 30 of [Var96].

We are now able to proof the surjectivity of the map from Theorem 1.4.

Proof of surjectivity. Let the cohomology class $\lambda \in H^1(S^1; \Lambda(A^0))$ be represented by a cocycle $(F_{jl})_{j,l\in J}$ associated with some finite covering $\mathcal{U} = \{U_j; j \in J\}$ of S^1. The sections of $\Lambda(A^0)$ are asymptotic to id and thus the cocycle $(F_{jl})_{j,l\in J}$ also determines an element $\sigma \in H^1(S^1; \mathcal{G})$. From the Theorem 1.8 we know, that there is a $\widehat{F} \in G(\!(t)\!)$ whose class $[\widehat{F}]$ gets via Θ mapped to σ. Thus there exists holomorphic functions $F_j : \mathfrak{s}_{U_j} \to \mathrm{gl}_n(\mathbb{C})$ with $F_j \sim_{U_j} \widehat{F}$ and $F_l^{-1}F_j = F_{jl}$ on $\mathfrak{s}_{U_j \cap U_l}$ for all $j, l \in J$.

Define on every arc U_j the matrix $A_j := {}^{F_j}A^0$. On the intersections $U_j \cap U_l$ we know that $A_j = A_l$, since from $F_l^{-1}F_j \in \Lambda(A^0)$ follows on $U_j \cap U_l$ that

$$A^0 = {}^{F_l^{-1}}({}^{F_j}A^0) \qquad \Longrightarrow \qquad \underbrace{{}^{F_l}A^0}_{=A_l} = \underbrace{{}^{F_j}A^0}_{=A_j}.$$

Thus the A_j glue to a section A, which satisfies ${}^{\widehat{F}}A = A_0$ by construction. We have found an element $(A, \widehat{F}) \in \mathcal{H}(A^0)$ whose image under exp is σ. □