Contents

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1 Stokes Structures

Stokes structures contain exactly necessary information to classify meromorphic classification, i.e. with the Stokes structures we are able to construct a space PROBLEM: Why space, which is isomorphic to the classifying set.

A great overview of this topic is given by Varadarajan in [Var96]. Other resources we will use are for example Sabbah's book [Sab07, section II] for Section **??**. In the Sections **??** and 1.1 will Loday-Richaud's paper [Lod94] and her book [Lod14, Sec.4] be useful. Stokes groups are also discussed Boalch's paper [Boa01] (resp. his thesis [Boa99]) which looks only at the single leveled case or the paper [MR91] from Martinet and Ramis.

Let $(\mathcal{M}^{nf}, \nabla^{nf})$ be a fixed model with the corresponding normal form A^0 and let us also fix a normal solution $\mathcal{Y}_0$ of A^0. The purpose of the next section (Section **??**) is, to proof the Malgrange-Sibuya Theorem. It states that the classifying set $\mathcal{H}(A^0)$ is via an map exp isomorphic to the first non abelian cohomology $H^1(S^1; \Lambda(A^0)) =: \mathcal{S}t(A^0)$ of the Stokes sheaf $\Lambda(A^0)$. In Section 1.1 we will improve the Malgrange-Sibuya Theorem by showing that each 1-cohomology class in $\mathcal{S}t(A^0)$ contains a unique 1-cocycle of a special form called *the Stokes cocycle* (cf. Section **??**). The morphism, which maps each Stokes cocycle to its corresponding 1-cocycle will be denoted by h. This will be further improved in Section **??**.

If one introduces the map g, which arises from the theory of summation and takes an equivalence class (resp. an ambassador of such a class) and returns a corresponding Stokes cocycle in an canonically way (cf. Appendix **??** where the theory of summation will be roughly discussed), as a black-box one can write the following commutative diagram.

$$\begin{tikzcd}
\mathcal{H}(A^0) \arrow[r, "\exp"] \arrow[d, "g"'] & \mathcal{S}t(A^0) \\
\prod_{\theta \in \mathbb{A}} \mathrm{Sto}_\theta(A^0) \arrow[ur, "h"'] &
\end{tikzcd}$$

This diagram will be enhanced in Section **??** by adding a couple of isomorphisms.

1.1 Stokes structures: using Stokes groups

The goal in this section is to prove that there is a bijective and natural map [Lod94], [Lod14, Thm.4.3.11] and [Boa01; Boa99] and [BV89] and [BJL79] and [MR91, Chapter 4]

$$h : \prod_{\alpha \in \mathbb{A}} \mathrm{Sto}_\alpha(A^0) \longrightarrow \mathcal{S}t(A^0)$$

which endows $\mathcal{S}t(A^0)$ with the structure of a unipotent Lie group. And since $\mathrm{Sto}_\alpha(A^0)$ has $\mathbb{S}\mathrm{to}_\alpha(A^0)$ as a faithful representation, we also get the isomorphism $\prod_{\alpha\in\mathbb{A}} \mathbb{S}\mathrm{to}_\alpha(A^0) \cong \mathcal{S}t(A^0)$ as a corollary. TODO: This goes back to [BJL79]?

Let us recall, that $\mathcal{S}t(A^0)$ is defined to be $H^1(S^1; \Lambda(A^0))$ (cf. Section **??**). The elements of $\prod_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha(A^0)$ define in a canonical way cocycles of the sheaf $\Lambda(A^0)$ (cf. Equation (1.1)), called Stokes cocycles (cf. Definition 1.5). In fact, will h map such cocycles to the cohomology class, to which they correspond. Thus the statement, that h is a bijection, is equivalent to the statement that

in each cohomology class of $\mathcal{S}t(A^0)$ is a unique 1-cocycle, which is a Stokes cocycle.

Cyclic coverings

To formulate the theorem in the next section, we use the notion of cyclic coverings and nerves of such coverings, which are defined as follows.

Definition 1.1

[Lod94, Sec.II.1] and [Lod94, Sec.II.3.1]

Let J be a finite set, identified to $\{1, \dots, p\} \subset \mathbb{Z}$.

1. A *cyclic covering* of S^1 is a finite covering $\mathcal{U} = \big(U_j := U(\theta_j, \varepsilon_j)\big)_{j\in J}$ consisting of arcs, which satisfies that
   a) $\widetilde{\theta}_j \geq \widetilde{\theta}_{j+1}$ for $j \in \{1, \dots, p-1\}$, i.e. the center points are ordered in ascending order with respect to the clockwise orientation of S^1 and
   b) $\widetilde{\theta}_j + \frac{\varepsilon_j}{2} \geq \widetilde{\theta}_{j+1} + \frac{\varepsilon_{j+1}}{2}$ for $j \in \{1, \dots, p-1\}$ and $\widetilde{\theta}_p + \frac{\varepsilon_p}{2} \geq \widetilde{\theta}_1 - 2\pi + \frac{\varepsilon_1}{2}$, i.e. the arcs are not encased by another arc,

   where the $\widetilde{\theta}_j \in [0, 2\pi[$ are determinations of the $\theta_j \in S^1$.

   a) the θ_j are in ascending order with respect to the clockwise orientation of S^1;
   b) the $U_j \cap U_{j+1}$ have only one connected component when $\#J > 2$;
   c) the U_j are not encased by another arc, this means that the open sets $U_j \backslash U_l$ are connected for all $j, l \in J$.

2. The *nerve* of a cyclic covering $\mathcal{U} = \{U_j; j \in J\}$ is the family $\dot{\mathcal{U}} = \{\dot{U}_j; j \in J\}$ defined by:
   - $\dot{U}_j = U_j \cap U_{j+1}$ when $\#J > 2$,
   - $\dot{U}_1$ and $\dot{U}_2$ the connected components of $U_1 \cap U_2$ when $\#J = 2$.

Remark 1.1.1

The nerve of the cyclic covering $\mathcal{U} = (U(\theta_j, \varepsilon_j))_{j\in J}$ is explicitly given by

$$\dot{\mathcal{U}} = \left(\left(\theta_j - \frac{\varepsilon_j}{2}, \theta_{j+1} + \frac{\varepsilon_{j+1}}{2}\right)\right)_{j\in J}.$$

The cyclic coverings correspond one-to-one to nerves of cyclic coverings. If one starts with a nerve $\{\dot{U}_j \mid j \in J\}$, one obtains a cyclic covering as $\mathcal{U} = \{U_j \mid j \in J\}$ where the arc U_j are the connected clockwise hulls from $\dot{U}_{j-1}$ to $\dot{U}_j$.

Definition 1.2

A covering $\mathcal{V}$ is said to *refine* a covering $\mathcal{U}$ if, to each open set $V \in \mathcal{V}$ there is at least one $U \in \mathcal{U}$ with $V \subset U$.

Each refined covering of $\mathcal{U}$ is obtained by successively

1. narrowing an arc $U \in \mathcal{U}$ to a smaller arc $\widetilde{U} \subset U$ or
2. splitting an arc $U \in \mathcal{U}$ into two smaller arcs U' and U'' satisfying $U = U' \cup U''$.

This can be used to see the following proposition (cf. [Lod94, Prop.II.1.3]).

Proposition 1.3

The covering $\mathcal{V}$ refines $\mathcal{U}$ if and only if the corresponding nerves $\dot{\mathcal{U}} = \{\dot{U}_j\}$ and $\dot{\mathcal{V}} = \{\dot{V}_l\}$ satisfy

each $\dot{U}_j$ contains at least one $\dot{V}_l$.

The cyclic coverings and especially the nerves of such coverings will be useful, since we have the following proposition (cf. [Lod04, Prop.2.6]).

Proposition 1.4

The set of 1-cocycles of $\mathcal{U}$ is canonically isomorphic to the set of 1-cochains restricted to $\dot{\mathcal{U}} = \left(\dot{U}_j\right)_{j \in J}$ without any cocycle condition, i.e. the product $\prod_{j \in J} \Gamma(\dot{U}_j, \Lambda(A^0))$.

1.1.1 The theorem

Let $\{\theta_j \mid j \in J\} \subset S^1$ be a finite set and $\dot{\varphi} = (\dot{\varphi}_{\theta_j})_{j \in J} \in \prod_{j \in J} \Lambda_{\theta_j}(A^0)$ be a finite family of germs. Let $\dot{\varphi}_j$ be the function representing the germ $\dot{\varphi}_{\theta_j}$ on its (maximal) arc of definition Ω_j around θ_j. In the following way, one can associate a cohomology class in $\mathcal{S}t(A^0)$ to $\dot{\varphi}$: [Lod94, p. 868]

for every cyclic covering $\mathcal{U} = (U_j)_{j \in J}$, which satisfies $\dot{U}_j \subset \Omega_j$ for all $j \in J$, one can define the 1-cocycle $(\dot{\varphi}_{j|\dot{U}_j})_{j \in J} \in \Gamma(\dot{\mathcal{U}}; \Lambda(A^0))$.

To a different cyclic covering, satisfying the condition above, this construction yields a cohomologous 1-cocycle, thus the induced map

$$\prod_{j \in J} \Lambda_{\theta_j}(A^0) \longrightarrow H^1(S^1; \Lambda(A^0)) = \mathcal{S}t(A^0) \tag{1.1}$$

is welldefined (cf. [Lod94, p. 868]).

Recall that $\nu = \#\mathbb{A}$ is the number of all anti-Stokes directions and the set of all anti-Stokes directions is denoted by $\mathbb{A} = \{\alpha_1, \ldots, \alpha_\nu\}$.

Definition 1.5

[Lod94, Def.II.1.8]
, [Lod14, p. 4.3.10]
, [MR91, Defn 6 on p 374]

A *Stokes cocycle* is a 1-cocycle $(\varphi_j)_{j\in\{1,\dots,\nu\}} \in \prod_{j\in\{1,\dots,\nu\}} \Gamma(U_j;\Lambda(A^0))$ corresponding to some cyclic covering with nerve $\dot{\mathcal{U}} = (\dot{U}_j)_{j\in\{1,\dots,\nu\}}$, which satisfies for every $j \in \{1,\dots,\nu\}$

- $\alpha_j \in \dot{U}_j$ and
- the germ $\varphi_{\alpha_j} := \varphi_{j,\alpha_j}$ of φ_j at α_j is an element of $\mathrm{Sto}_{\alpha_j}(A^0)$.

Remark 1.5.1

PROBLEM: refactor!remove? The sections $\Gamma(\dot{U}_j;\Lambda(A^0))$ are uniquely determined as the extension of the germ at α_j, since the sheaf $\Lambda(A^0)$ defined via the system $[A^0, A^0]$ (cf. Definition **??**). We thus have an injective map

$$\prod_{j\in\{1,\dots,\nu\}} \Gamma(\dot{U}_j;\Lambda(A^0)) \hookrightarrow \prod_{j\in\{1,\dots,\nu\}} \mathrm{Sto}_{\alpha_j}(A^0)\,,$$

which takes an Stokes cocycle and yields the corresponding Stokes germs. For a fine enough covering $\mathcal{U}$, i.e. a covering $\mathcal{U}$ with a nerve $\dot{\mathcal{U}}$ which consists of small enough arcs satisfying the conditions above, is this map a bijection.

We will use this fact implicitly and assume that the covering is always fine enough to call elements of $\prod_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha(A^0)$ Stokes cocycles.

We can use the mapping (1.1), corresponding to the construction at the beginning of this section, to obtain a mapping

$$h : \prod_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha(A^0) \longrightarrow \prod_{\alpha\in\mathbb{A}} \Lambda_\alpha(A^0) \xrightarrow{(1.1)} \mathcal{S}t(A^0),$$

which takes a complete set of Stokes germs to its corresponding cohomology class, given by a Stokes cocycle.

Theorem 1.6

[Lod94, p. 869],[Lod94, Sec.III.3.3]

The map

$$h : \prod_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha(A^0) \longrightarrow \mathcal{S}t(A^0)$$

is a bijection and natural.

Remark 1.6.1

Natural means that h commutes to isomorphisms and constructions over systems or connections they represent.

To define the inverse map of h, one has to find in each cocycle in $\mathcal{S}t(A^0)$ the Stokes cocycle and take the germs. Loday-Richaud gives an algorithm in Section II.3.4 of her paper [Lod94], which takes a cocycle over an arbitrary cyclic covering and outputs cohomologous Stokes cocycle and thus solves this problem. This means, that the inverse of h is constructible.

Corollary 1.7

PROBLEM: mentioned twice PROBLEM: remove? Using the isomorphisms $\mathrm{Sto}_\theta(A^0) \cong \mathbb{S}\mathrm{to}_\theta(A^0)$ from Proposition **??** we obtain

$$\mathcal{S}t(A^0) \cong \prod_{\alpha \in \mathbb{A}} \mathbb{S}\mathrm{to}_\alpha(A^0)$$

Since the product of Lie groups is in an obvious way again a Lie group, this endows $\mathcal{S}t(A^0)$ with the structure of an unipotent Lie group with the finite complex dimension $N := \dim_{\mathbb{C}} \mathcal{S}t(A^0)$ (cf. [Lod94, Sec.III.1]).

Remark 1.8

This number N is known to be the *irregularity* of $[\operatorname{End} A^0]$ and can be rewritten in the following way:

$$N = \sum_{\alpha \in \mathbb{A}} \dim_{\mathbb{C}} \mathbb{S}\mathrm{to}_\alpha(A^0) = \sum_{\alpha \in \mathbb{A}} \sum_{q_j \underset{\alpha}{\prec\!\!<} q_l} n_j \cdot n_l = \sum_{\substack{1 \le j,l \le n \\ j<l}} 2 \cdot \deg(q_j - q_l) \cdot n_j \cdot n_l \,.$$

One can also define the structure of a linear affine variety on the set $\mathcal{S}t(A^0)$. This was for example done in Section [BV89, Sec.II.3] or in [Lod94, Sec.III.1], where actually multiple structures of linear affine varieties on $\mathcal{S}t(A^0)$ are defined. In [Var96, 35ff] is mentioned that one can also define a scheme structure on $\mathcal{S}t(A^0)$.

1.1.2 Proof of Theorem 1.6

We will only look at the unramified case, for which we refer to [Lod94, Sec.II.3]. The proof in the ramified case can be found in Section II.4 of Loday-Richaud's Paper [Lod94] and a sketch of the complete proof is also in [Lod04]. We first have to introduce the notion of adequate coverings, which will be used in the proof.

Adequate coverings

TODO: Adequate is acyclic in Loday2004?

Definition 1.9

1. [MR91, p. 371] defines adapted coverings
2. [Lod04, p. 5269] defines acyclic coverings
   - uses the **theorem of Leray** `https://de.wikipedia.org/wiki/Satz_von_Leray`

Let $\star \in \{k, < k, \leq k, \dots\}$. A covering $\mathcal{U}$ beyond which the inductive limit $\varinjlim_{\mathcal{U}} H^1(\mathcal{U}; \Lambda^\star(A^0))$ is stationary is said to be *adequate* to describe $H^1(S^1; \Lambda^\star(A^0))$ or *adequate* to $\Lambda^\star(A^0)$.

A covering $\mathcal{U}$ is said to be *adequate* to describe $H^1(S^1; \Lambda^\star(A^0))$ or *adequate* to $\Lambda^\star(A^0)$ if for every element in $\varinjlim_{\mathcal{U}} H^1(\mathcal{U}; \Lambda^\star(A^0))$ given by some covering $\mathcal{U}'$ and an element of $\Gamma(\mathcal{U}'; \Lambda^\star(A^0))$ there exists

- an element in $\Gamma(\mathcal{U}; \Lambda^\star(A^0))$ and
- an common refinement of $\mathcal{U}$ and $\mathcal{U}'$

such that PROBLEM:the elements are ?? on the refined covering.

In other words is a covering $\mathcal{U}$ adequate to $\Lambda^\star(A^0)$, if and only if the quotient map

$$\Gamma(\mathcal{U}; \Lambda^\star(A^0)) \longrightarrow H^1(S^1; \Lambda^\star(A^0))$$

is surjective. TODO: Proof? check??

[MR91, p. 371] introduces the following definition

Definition 1.9.1

A covering $\mathcal{U}$ is *adapted* if every anti-Stokes direction is contained in exactly one element of the nerve $\dot{\mathcal{U}}$.

The following proposition is in Loday-Richaud's paper [Lod94] given as Proposition II.1.7. It contains a simple characterization, which will be used to see, that our defined coverings are adequate.

Proposition 1.10

[Lod94, Prop.II.1.7]

Let $k \in \mathcal{K}_\alpha$.

Definition 1.10.1

Let $\alpha \in \mathbb{A}^k$. An arc $U\left(\alpha, \frac{\pi}{k}\right)$ is called a *Stokes arc of level k at α*. If $(q_j, q_l) \in \mathcal{Q}(A^0)$ is a pair, such that $q_j \underset{\alpha}{\prec} q_l$, is the arc $U\left(\alpha, \frac{\pi}{k}\right)$ exactly the arc of decay of the exponential $e^{q_j - q_l}$ (cf. [Lod04, p. 5269] and Remark **??**).

A cyclic covering $\mathcal{U} = (U_j)_{j \in J}$ which satisfies

> for every $\alpha \in \mathbb{A}^k$ contains the Stokes arc $U\left(\alpha, \frac{\pi}{k}\right)$ at least one arc $\dot{U}_j$ from the nerve $\dot{\mathcal{U}}$ of $\mathcal{U}$

is adequate to $\Lambda^k(A^0)$.

The covering $\mathcal{U}$ is adequate to $\Lambda^{\leq k}(A^0)$ (resp. $\Lambda^{<k}(A^0)$) if it is adequate to $\Lambda^{k'}(A^0)$ for every $k' \leq k$ (resp. $k' < k$).

Proof. Show that for every $U_j, U_l \in \mathcal{U}$ is

$$H^1(U_j \cap U_l; \Lambda^k(A^0)) = 0\,.$$

Then the theorem of Leray implies that $H^1(S^0; \Lambda^k(A^0)) = H^1(\mathcal{U}; \Lambda^k(A^0))$ □

Let $k \in \mathcal{K}$. We want to define the three cyclic coverings $\mathcal{U}^k$, $\mathcal{U}^{\leq k}$ and $\mathcal{U}^{<k}$ which will be adequate to $\Lambda^k(A^0)$, $\Lambda^{\leq k}(A^0)$ and $\Lambda^{<k}(A^0)$. Furthermore will the coverings be comparable at the different levels.

1. The first covering $\mathcal{U}^k = \{\dot{U}^k_\alpha \mid \alpha \in \mathbb{A}^k\}$ is the cyclic covering determined by the nerve

$$\dot{\mathcal{U}}^k := \left\{\dot{U}^k_\alpha = U\left(\alpha, \frac{\pi}{k}\right) \mid \alpha \in \mathbb{A}^k\right\}$$

consisting of all Stokes arcs of level k for anti-Stokes directions bearing the level k.

Since $\dot{\mathcal{U}}^k$ consists only of arcs with equal opening is this canonically a nerve.

Remark 1.11

Boalch, who looks only at the single-leveled case, introduces in his publications [Boa01, p. 19] and [Boa99, Def.1.23] the notion of supersectors which are defined as follows:

> write the anti-Stokes directions as $\mathbb{A} = \{\alpha_1, \dots, \alpha_\nu\}$ arranged according to the clockwise ordering, then is the i-th *supersector* defined as the arc
>
> $$\widehat{\mathrm{Sect}}^k_i := \left(\alpha_{i+1} - \frac{\pi}{2k}, \alpha_i + \frac{\pi}{2k}\right)\,.$$

This yields a cyclic covering $(\widehat{\mathrm{Sect}}^k_i)_{i\in\{1,\dots,\nu\}}$ whose nerve is exactly $\dot{\mathcal{U}}^k$, which was defined above.

If we extend to a subset J of $\mathcal{K}$ containing more then one level level, $\#J > 1$, the set

$$\bigcup_{k\in J}\left\{U\left(\alpha, \frac{\pi}{k}\right) \mid \alpha \in \mathbb{A}^k\right\} = \bigcup_{k\in J} \dot{\mathcal{U}}^k$$

is no longer guaranteed to be a nerve. Hence we have to define the coverings $\mathcal{U}^{\leq k}$ and $\mathcal{U}^{<k}$ in a different way. We will start by adding the arcs corresponding to the

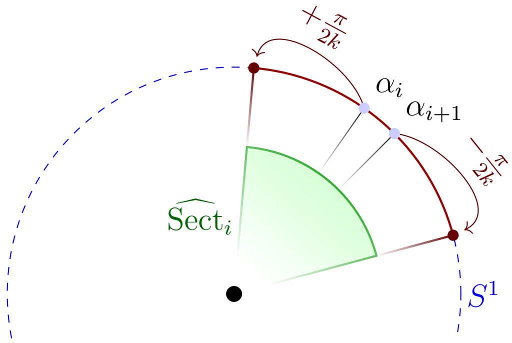

Figure 1.1: An exemplary supersector $\widehat{\mathrm{Sect}}_i$ corresponding to the anti-Stokes directions α_i and α_{i+1}.

largest degree first and continue by adding the arcs corresponding to smaller degrees successively as described below.

Denote by

$$\{K_1 < \cdots < K_s = k\} = \left\{\max\left(\mathcal{K}_\alpha \cap [0,k]\right) \mid \alpha \in \mathbb{A}^{\leq k}\right\}$$

the set of all k-*maximum levels* for all $\alpha \in \mathbb{A}^{\leq k}$.

2. The cyclic covering $\mathcal{U}^{\leq k} = \left\{U_\alpha^{\leq k} \mid \alpha \in \mathbb{A}^{\leq k}\right\}$ will be defined by decreasing induction on the levels. Let us assume that

> the $\dot{U}_\alpha^{\leq k}$ are defined for all $\alpha \in \mathbb{A}^{\leq k}$ with k-maximum level greater than K_i such that their complete family is a nerve.

For every anti-Stokes direction α with k-maximum level K_i let α^- (resp. α^+) be the next anti-Stokes direction with k-maximum level greater then K_i on the left (resp. on the right).

Define $\dot{U}_{\alpha^-,\alpha^+}$ as the clockwise hull of the arcs $\dot{U}_{\alpha^-}^{\leq k}$ and $\dot{U}_{\alpha^+}^{\leq k}$ already defined by induction. If there are no anti-Stokes directions with k-maximum level greater then K_i we set $\dot{U}_{\alpha^-,\alpha^+} = S^1$.

We then add for every anti-Stokes direction α with k-maximum level K_i the arc

$$\dot{U}_\alpha^{\leq k} := U\Big(\alpha, \frac{\pi}{K_i}\Big) \cap \dot{U}_{\alpha^-,\alpha^+}$$

to $\dot{\mathcal{U}}^{\leq k}$ and the received family is still a nerve.

> **Remark 1.12**
>
> If α has a k-maximum level equal to k then is $\dot{U}_\alpha^{\leq k}$ equal to the Stokes arc $U\left(\alpha, \frac{\pi}{k}\right) = \dot{U}_\alpha^k$. ...and then no 0-cochain with level k or $\geq k$ can exists on the covering $\mathcal{U}^{\leq k}$.

3. The last cyclic covering, $\mathcal{U}^{<k} = \left\{U_\alpha^{<k} \mid \alpha \in \mathbb{A}^{<k}\right\}$, is defined as $\mathcal{U}^{<k} := \mathcal{U}^{\leq k'}$ where $k' := \max\{k'' \in \mathcal{K} \mid k'' < k\}$.

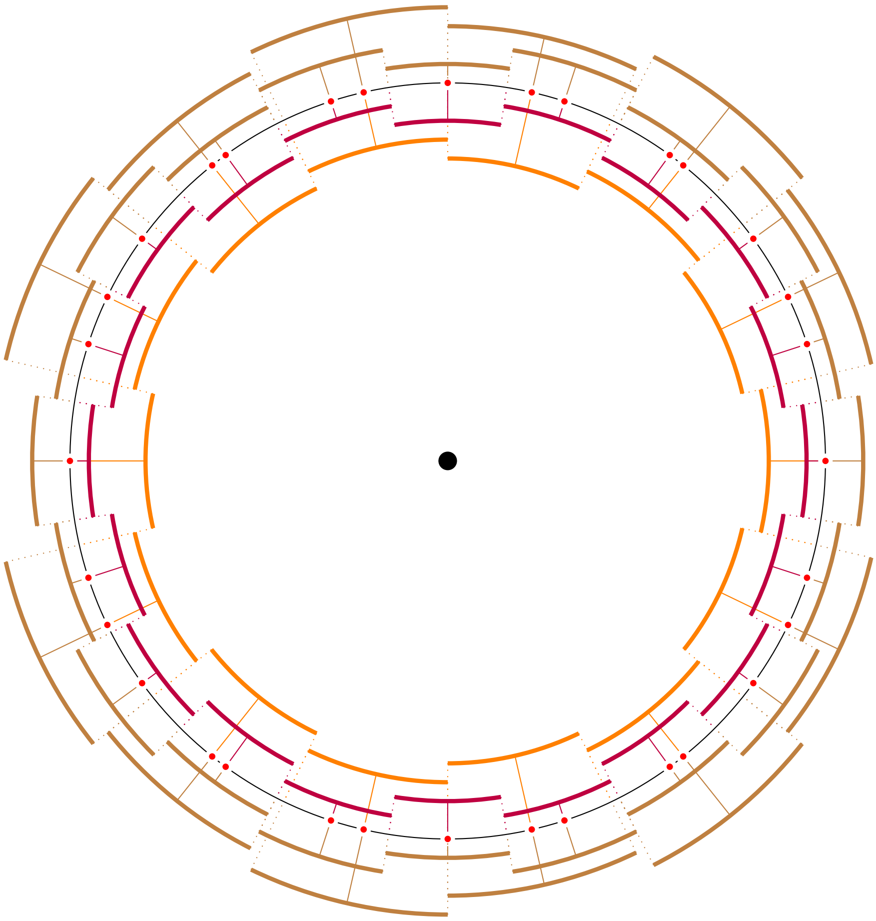

Figure 1.2: The adequate coverings for an example with $\mathcal{K} = \{7, 10\}$ and $\mathbb{A} = \left\{\frac{j\cdot\pi}{k} \mid k \in \mathcal{K},\ j \in \mathbb{N}\right\}$. The anti-Stokes directions are marked by the red dots. The arcs of $\dot{\mathcal{U}}^7 = \dot{\mathcal{U}}^{\leq 7}$ are orange, the arcs of $\dot{\mathcal{U}}^{10}$ are purple and the arcs of $\dot{\mathcal{U}}^{\leq 10} = \dot{\mathcal{U}}$ are brown.

Remark 1.13

The coverings $\mathcal{U}^k$, $\mathcal{U}^{\leq k}$ and $\mathcal{U}^{<k}$ depend only on $\mathcal{Q}(A^0)$. Hence they depend only on the determining polynomials.

For every $k \in \mathcal{K}$ and every $\alpha \in \mathcal{K}^k$ (resp. $\alpha \in \mathcal{K}^{<k}$) is $\dot{U}_\alpha^{\leq k} \subset \dot{U}_\alpha^k$ (resp. $\dot{U}_\alpha^{\leq k} \subset \dot{U}_\alpha^{<k}$), thus the covering $\mathcal{U}^{\leq k}$ refines $\mathcal{U}^k$ and $\mathcal{U}^{<k}$. Furthermore are the coverings defined, such that they satisfy the condition in Proposition 1.10, such that the first property in the following proposition is satisfied. The other two can be found at [Lod94, Prop.II.3.1 (iv)] or on page 5269 of [Lod04].

Proposition 1.14

[Lod94, Prop.II.3.1] PROBLEM: Was bedeutet das? Let $k \in \mathcal{K}$, then

1. the coverings $\mathcal{U}^k$, $\mathcal{U}^{\leq k}$ and $\mathcal{U}^{<k}$ are adequate to $\Lambda^k(A^0)$, $\Lambda^{\leq k}(A^0)$ and $\Lambda^{<k}(A^0)$, respectively;
2. there exists no 0-cochain in $\Lambda^k(A^0)$ on $\mathcal{U}^k$;
3. on $\mathcal{U}^{\leq k}$ there is no 0-cochain in $\Lambda^{\leq k}(A^0)$ of level k, i.e. all 0-cochains of $\Lambda^{\leq k}(A^0)$ belong to $\Lambda^{<k}(A^0)$.

To have a shorter notation, we denote the product $\prod_{\alpha \in \mathbb{A}^\star} \Gamma(\dot{U}_\alpha^\star; \Lambda^\star(A^0))$ by $\Gamma(\dot{U}^\star; \Lambda^\star(A^0))$ for every $\star \in \{k, <k, \leq k, \dots\}$.

The case of a unique level

[Lod94, p. II.3.2] First we will proof Theorem 1.6 in the case of a unique level. This means that

- either $\Lambda(A^0)$ has only one level k, thus
  - $\Lambda(A^0) = \Lambda^k(A^0)$ and
  - $\mathrm{Sto}_\theta(A^0) = \mathrm{Sto}_\theta^k(A^0)$ for every θ,
- or we restrict to a given level $k \in \mathcal{K}$.

The following lemma, which will also be required for the multileveled case, solves the case of a unique level.

Lemma 1.15

Let $k \in \mathcal{K}$. The morphism h from Theorem 1.6 is in the case of a unique level build as

$$\begin{array}{ccccc}
\prod_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha^k(A^0) & \xrightarrow{i^k} & \Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A^0)) & \xrightarrow{s^k} & H^1(S^1; \Lambda^k(A^0)) \\
\| & & & & \| \\
\prod_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha(A^0) & & \xrightarrow{\qquad h \qquad} & & H^1(S^1; \Lambda(A^0))
\end{array}$$

only in the single leveled case

from

- the canonical injective map

$$i^k : \prod_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha^k(A^0) \longrightarrow \Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A^0))\,,$$

  i.e. the map which is the canonical extension of germs to their natural arc of definition, and

- the quotient map

$$s^k : \Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A^0)) \longrightarrow H^1(S^1; \Lambda^k(A^0))$$

which are both isomorphisms.

Proof. 1. The map

$$i^k : \prod_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha^k(A^0) \longrightarrow \underbrace{\prod_{\alpha\in\mathbb{A}^k} \Gamma(\dot{\mathcal{U}}_\alpha^k; \Lambda^k(A^0))}_{\substack{\parallel \\ \Gamma(\dot{\mathcal{U}}^k;\Lambda^k(A^0))}}$$

is welldefined, i.e. the germs at α can be uniquely extended to the arc $\dot{\mathcal{U}}_\alpha^k$, since

- the sheaf $\Lambda(A^0)$, and thus $\Lambda^k(A^0)$, is a piecewise-constant sheaf, we know that the sections on arcs are uniquely determined by their germs and
- arcs of $\dot{\mathcal{U}}_\alpha^k$ are the natural domains of existence of the corresponding Stokes germs.

The second fact can be found in Loday-Richaud's paper [Lod04] on page 5269.

In [Lod04, p. 5269] is also stated, that only the Stokes germs at α are sections of $\Lambda(A^0)$ on $\dot{\mathcal{U}}_\alpha^k$ and this implies surjectivity.

2. The second map

$$s^k : \overbrace{\prod_{\alpha\in\mathbb{A}^k} \Gamma(\dot{\mathcal{U}}_\alpha^k; \Lambda^k(A^0))}^{\substack{\Gamma(\dot{\mathcal{U}}^k;\Lambda^k(A^0)) \\ \parallel}} \longrightarrow H^1(S^1; \Lambda^k(A^0))$$

is a bijection, since from Proposition 1.14 we know that it is

- **surjective**, since $\mathcal{U}^k$ is adequate to $\Lambda^k(A^0)$ and
- **injective**, since on $\mathcal{U}^k$ there is no 0-cochain in $\Lambda^k(A^0)$.

PROBLEM:Show that this is the correct h PROBLEM:Naturality? $\square$

The case of several levels

In the proof of the case of several levels, we will still use Loday-Richaud's paper [Lod94] as reference.

Definition 1.16

Here we want to define a *product map of cocycles* $\mathfrak{S}^{\leq k}$. This map will be composed from the following injective maps:

1. The first map is defined as

$$\begin{aligned} \sigma^k : \Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A^0)) &\longrightarrow \Gamma(\dot{\mathcal{U}}^{\leq k}; \Lambda^{\leq k}(A^0)) \\ \dot{f} = (\dot{f}_\alpha)_{\alpha \in \mathbb{A}^k} &\longmapsto (\dot{G}_\alpha)_{\alpha \in \mathbb{A}^{\leq k}} \end{aligned}$$

   where

$$\dot{G}_\alpha = \begin{cases} \dot{f}_\alpha \text{ restricted to } \dot{U}_\alpha^{\leq k} \text{ and seen as being in } \Lambda^{\leq k}(A^0) & \text{when } \alpha \in \mathbb{A}^k \\ \mathrm{id} \ \text{ (the identity)} & \text{when } \alpha \notin \mathbb{A}^k \end{cases}$$

2. and the second map

$$\begin{aligned} \sigma^{<k} : \Gamma(\dot{\mathcal{U}}^{<k}; \Lambda^{<k}(A^0)) &\longrightarrow \Gamma(\dot{\mathcal{U}}^{\leq k}; \Lambda^{\leq k}(A^0)) \\ \dot{f} = (\dot{f}_\alpha)_{\alpha \in \mathbb{A}^{<k}} &\longmapsto (\dot{F}_\alpha)_{\alpha \in \mathbb{A}^{\leq k}} \end{aligned}$$

   is defined, in a similar way, as

$$\dot{F}_\alpha = \begin{cases} \dot{f}_\alpha \text{ restricted to } \dot{U}_\alpha^{\leq k} \text{ and seen as being in } \Lambda^{\leq k}(A^0) & \text{when } \alpha \in \mathbb{A}^{<k} \\ \mathrm{id} \ \text{ (the identity)} & \text{when } \alpha \notin \mathbb{A}^{<k} \end{cases}$$

Thus we can define

$$\begin{aligned} \mathfrak{S}^{\leq k} : \Gamma(\dot{\mathcal{U}}^{<k}; \Lambda^{<k}(A^0)) \times \Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A^0)) &\longrightarrow \Gamma(\dot{\mathcal{U}}^{\leq k}; \Lambda^{\leq k}(A^0)) \\ (\dot{f}, \dot{g}) &\longmapsto (\dot{F}_\alpha \dot{G}_\alpha)_{\alpha \in \mathbb{A}^{\leq k}} \end{aligned}$$

where $(\dot{F}_\alpha)_{\alpha \in \mathbb{A}^{\leq k}} = \sigma^{<k}(\dot{f})$ and $(\dot{G}_\alpha)_{\alpha \in \mathbb{A}^{\leq k}} = \sigma^k(\dot{g})$ are defined as above.

Remark 1.16.1

This map $\mathfrak{S}^{\leq k}$ is injective, since injectivity for germs implies injectivity for sections.

Lemma 1.17

[Lod94, Lem.II.3.3] Let $k \in \mathcal{K}$.

1. If the cocycles $\mathfrak{S}^{\leq k}(\dot{f}, \dot{g})$ and $\mathfrak{S}^{\leq k}(\dot{f}', \dot{g}')$ are cohomologous in $\Gamma(\dot{\mathcal{U}}^{\leq k}; \Lambda^{\leq k}(A^0))$ then $\dot{f}$ and $\dot{f}'$ are cohomologous in $\Gamma(\dot{\mathcal{U}}^{<k}; \Lambda^{<k}(A^0))$.
2. Any cocycle in $\Gamma(\dot{\mathcal{U}}^{\leq k}, \Lambda^{\leq k}(A^0))$ is cohomologous to a cocycle in the range of $\mathfrak{S}^{\leq k}$.

Proof. 1. Denote by α^+ the nearest anti-Stokes direction in $\mathbb{A}^{\leq k}$ on the right[a] of α. The cocycles $\mathfrak{S}^{\leq k}(\dot{f}, \dot{g})$ and $\mathfrak{S}^{\leq k}(\dot{f}', \dot{g}')$ are cohomologous if and only if there is a 0-cochain $c = (c_\alpha)_{\alpha \in \mathbb{A}^{\leq k}} \in \Gamma(\mathcal{U}^{\leq k}, \Lambda^{\leq k}(A^0)$ such that

$$\dot{F}_\alpha \dot{G}_\alpha = c_\alpha^{-1} \dot{F}'_\alpha \dot{G}'_\alpha c_{\alpha^+} \tag{1.2}$$

for every $\alpha \in \mathbb{A}$. From Proposition 1.14 follows, that c is with values in $\Lambda^{<k}(A^0)$. The fact that $\Lambda^k(A^0)$ is normal in $\Lambda^{\leq k}(A^0)$ in Proposition **??**, can be used to see that $c_{\alpha^+}^{-1} G'_\alpha c_{\alpha^+} \in \Gamma(\mathcal{U}^{\leq k}; \Lambda^k(A^0))$. Thus, we rewrite the relation (1.2) to

$$\dot{F}_\alpha \dot{G}_\alpha = (c_\alpha^{-1} \dot{F}'_\alpha c_{\alpha^+})(c_{\alpha^+}^{-1} G'_\alpha c_{\alpha^+}), \qquad \text{for } \alpha \in \mathbb{A}^{\leq k}.$$

Since Corollary **??** tells us, that the factorization into the factors of the semidirect product are unique, we get for all $\alpha \in \mathbb{A}^k$

$$\dot{F}_\alpha = c_\alpha^{-1} \dot{F}'_\alpha c_{\alpha^+} \qquad \text{and} \qquad \dot{G}_\alpha = c_{\alpha^+}^{-1} \dot{G}'_\alpha c_{\alpha^+}.$$

The former relation implies that $(\dot{F}_\alpha)$ and $(\dot{F}'_\alpha)$ are cohomologous with values in $\Lambda^{<k}(A^0)$ on $\mathcal{U}^{\leq k}$. Since $\mathcal{U}^{<k}$ is already adequate to $\Lambda^{<k}(A^0)$, are $(\dot{F}_\alpha)$ and $(\dot{F}'_\alpha)$ already on $\mathcal{U}^{<k}$, i.e. in $\Gamma(\dot{\mathcal{U}}^{<k}; \Lambda^{<k}(A^0))$, cohomologous.

2. The proof of part 2. (together with a proof of part 1.) can be found in Loday-Richaud's paper [Lod94, Proof of Lem.II.3.3].

□

[a] In clockwise direction.

Let $k \in \mathcal{K}$ and $k' = \max\{k' \in \mathcal{K} \mid k' < k\}$. We then know by definition that $\mathcal{U}^{<k} = \mathcal{U}^{\leq k'}$ as well as $\Lambda^{<k}(A^0) = \Lambda^{\leq k'}(A^0)$ and thus $\Gamma(\dot{\mathcal{U}}^{<k}; \Lambda^{<k}(A^0)) = \Gamma(\dot{\mathcal{U}}^{\leq k'}; \Lambda^{\leq k'}(A^0))$ and obtain the following proposition.

Proposition 1.18

By applying $\mathfrak{S}^{\leq k}$ successively for different k's in decending order, one obtains the *product map of single leveled cocycles* τ in the following way

$$\begin{tikzcd}[column sep=large]
\underbrace{\Gamma(\dot{\mathcal{U}}^{<k_r};\Lambda^{<k_r}(A^0))} \times \Gamma(\dot{\mathcal{U}}^{k_r};\Lambda^{k_r}(A^0)) \arrow[r, "\mathfrak{S}^{\leq k_r}"] \arrow[dr, bend right, "="] & \overset{\Gamma(\dot{\mathcal{U}};\Lambda(A^0))}{\overset{\parallel}{\Gamma(\dot{\mathcal{U}}^{\leq k_r};\Lambda^{\leq k_r}(A^0))}} \\
\underbrace{\Gamma(\dot{\mathcal{U}}^{<k_{r-1}};\Lambda^{<k_{r-1}}(A^0))} \times \Gamma(\dot{\mathcal{U}}^{k_{r-1}};\Lambda^{k_{r-1}}(A^0)) \arrow[r, "\mathfrak{S}^{\leq k_{r-1}}"] \arrow[dr, bend right, "="] & \Gamma(\dot{\mathcal{U}}^{\leq k_{r-1}};\Lambda^{\leq k_{r-1}}(A^0)) \\
\cdots \arrow[r, "\mathfrak{S}^{\leq k_{r-2}}"] & \Gamma(\dot{\mathcal{U}}^{\leq k_{r-2}};\Lambda^{\leq k_{r-2}}(A^0))
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=large]
\underbrace{\Gamma(\dot{\mathcal{U}}^{<k_3};\Lambda^{<k_3}(A^0))} \times \Gamma(\dot{\mathcal{U}}^{k_3};\Lambda^{k_3}(A^0)) \arrow[r, "\mathfrak{S}^{\leq k_3}"] \arrow[dr, bend right, "="] & \cdots \\
\Gamma(\dot{\mathcal{U}}^{k_1};\Lambda^{k_1}(A^0)) \times \Gamma(\dot{\mathcal{U}}^{k_2};\Lambda^{k_2}(A^0)) \arrow[r, "\mathfrak{S}^{\leq k_2}"] & \Gamma(\dot{\mathcal{U}}^{\leq k_2};\Lambda^{\leq k_2}(A^0))
\end{tikzcd}$$

which can be written in the following compact form

$$\begin{aligned}
\tau : \prod_{k\in\mathcal{K}} \Gamma(\dot{\mathcal{U}}^k;\Lambda^k(A^0)) &\longrightarrow \Gamma(\dot{\mathcal{U}};\Lambda(A^0)) \\
(\dot{f}^k)_{k\in\mathcal{K}} &\longmapsto \prod_{k\in\mathcal{K}} \tau^k(\dot{f}^k)
\end{aligned}$$

where the product is following an ascending order of levels and the maps τ_k are defined as

$$\begin{aligned}
\tau^k : \Gamma(\dot{\mathcal{U}}^k;\Lambda^k(A^0)) \xrightarrow{\sigma^k} \Gamma(\dot{\mathcal{U}}^{\leq k};\Lambda^{\leq k}(A^0)) &\longrightarrow \Gamma(\dot{\mathcal{U}};\Lambda(A^0)) \\
(\dot{f}_\alpha)_{\alpha\in\mathbb{A}^k} \longmapsto\quad &(\dot{G}_\alpha)_{\alpha\in\mathbb{A}}
\end{aligned}$$

with

$$\dot{G}_\alpha = \begin{cases} \dot{f}_\alpha \text{ restricted to } \dot{U}_\alpha \text{ and seen as being in } \Lambda(A^0) & \text{when } \alpha\in\mathbb{A}^k \\ \mathrm{id} \ \text{ (the identity on } \dot{U}_\alpha) & \text{when } \alpha\notin\mathbb{A}^k \end{cases}$$

The defined map τ is clearly injective and it can be extended to an arbitrary order of levels (cf. Remark [Lod94, Rem.II.3.5]).

Definition 1.19

Define the injective map

$$\begin{aligned}\tau^k : \Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A^0)) &\longrightarrow \Gamma(\dot{\mathcal{U}}; \Lambda(A^0))\\ \dot{f} = (\dot{f}_\alpha)_{\alpha \in \mathbb{A}^k} &\longmapsto (\dot{F}_\alpha)_{\alpha \in \mathbb{A}}\end{aligned}$$

where

$$\dot{F}_\alpha = \begin{cases} \dot{f}_\alpha \text{ restricted to } \dot{U}_\alpha \text{ and seen as being in } \Lambda(A^0) & \text{when } \alpha \in \mathbb{A}^k \\ \text{id } \text{ (the identity on } \dot{U}_\alpha) & \text{when } \alpha \notin \mathbb{A}^k \end{cases}$$

The *product map of single-leveled cocycles* is then defined as [Lod94, Prop.II.3.4]

$$\begin{aligned}\tau : \prod_{k \in \mathcal{K}} \Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A^0)) &\longrightarrow \Gamma(\dot{\mathcal{U}}; \Lambda(A^0))\\ (\dot{f}^k)_{k \in \mathcal{K}} &\longmapsto \prod_{k \in \mathcal{K}} \tau^k(\dot{f}^k)\end{aligned}$$

following an ascending order of levels.

Remark 1.19.1

The map τ

1. is injective since it is composed from σ^k and the clearly injetive mapping

$$\begin{aligned}\Gamma(\dot{\mathcal{U}}^{\leq k}; \Lambda^{\leq k}(A^0)) &\longrightarrow \Gamma(\dot{\mathcal{U}}; \Lambda(A^0))\\ (\dot{F}_\alpha)_{\alpha \in \mathbb{A}^{\leq k}} &\longmapsto (\dot{F}'_\alpha)_{\alpha \in \mathbb{A}}\end{aligned}$$

   where

$$\dot{F}'_\alpha = \begin{cases} \dot{F}_\alpha \text{ restricted to } \dot{U}_\alpha \text{ and seen as being in } \Lambda(A^0) & \text{when } \alpha \in \mathbb{A}^{\leq k} \\ \text{id } \text{ (the identity on } \dot{U}_\alpha) & \text{when } \alpha \notin \mathbb{A}^{\leq k} \end{cases}$$

   and

2. it can be extended to an arbitrary order of levels (cf. Remark [Lod94, Rem.II.3.5]).

Corollary 1.20

The product map of single-leveled cocycles τ induces on the cohomology a bijective and natural map

$$\begin{array}{ccc}\mathcal{T} : \prod_{k \in \mathcal{K}} \Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A^0)) & \longrightarrow & H^1(\mathcal{U}; \Lambda(A^0))\,. \\ \mathrel{\text{\rotatebox{90}{$\cong$}}} & & \mathrel{\text{\rotatebox{90}{$\cong$}}} \\ \prod_{k \in \mathcal{K}} H^1(S^1; \Lambda^k(A^0)) & & H^1(S^1; \Lambda(A^0))\end{array}$$

Composing functions to obtain h We have the ingredients to define the function h from Theorem 1.6 by composition of already bijective maps.

Proof of Theorem 1.6. Let $i_\alpha : \mathrm{Sto}_\alpha(A^0) \to \prod_{k\in\mathcal{K}} \mathrm{Sto}^k_\alpha(A^0)$ be the map which corresponds to the filtration from Proposition **??** and denote the composition

$$\begin{tikzcd}
\prod\limits_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha(A^0) \arrow[r, "\prod_{\alpha\in\mathbb{A}} i_\alpha"] \arrow[drr, bend right, "\mathfrak{T}"'] & \prod\limits_{\alpha\in\mathbb{A}} \prod\limits_{k\in\mathcal{K}} \mathrm{Sto}^k_\alpha(A^0) \arrow[d, phantom, "\cong"] & \\
& \prod\limits_{k\in\mathcal{K}} \prod\limits_{\alpha\in\mathbb{A}} \mathrm{Sto}^k_\alpha(A^0) \arrow[r, "\prod_{k\in\mathcal{K}} i^k"] & \prod\limits_{k\in\mathcal{K}} \Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A^0))
\end{tikzcd}$$

by $\mathfrak{T}$. The bijection h is then obtained as

$$\mathcal{T} \circ \mathfrak{T} : \prod_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha(A^0) \longrightarrow H^1(\mathcal{U}; \Lambda(A^0))\,.$$

PROBLEM:naturality (is obvious?) PROBLEM:Show that this is the correct h □

1.1.3 Some exemplary calculations

Here we want to discuss which information is required to describe the Stokes cocycle corresponding to a multileveled system in more depth. We will look at a sigle-leveled system corresponding to a normal form $A^0 \in \mathrm{GL}_3(\mathbb{C}(\{t\}))$ with exactly two levels and will apply the techniques developed in the previous sections in an rather explicit way.

Let A^0 be a normal form with dimension $n = 3$ and two levels $\mathcal{K} = \{k_1 < k_2\}$ which satisfies that there is at least one anti-Stokes direction θ which is beared by both levels. Let $q_j(t^{-1})$ be the determining polynomials and let k_{jl} be the degrees of $(q_j - q_l)(t^{-1})$. Up to permutation, we know that in our case are the leading terms of $(q_1 - q_2)(t^{-1})$ and $(q_1 - q_3)(t^{-1})$ equal and thus

- up to permutation is $k_2 = k_{1,2} = k_{1,3}$ and $k_1 = k_{2,3}$, i.e. the larger degree appears twice, and

let q_1, q_2, q_3 be polynomials, such that

$$\deg(q_1 - q_2) =: k_2 > k_1 := \deg(q_2 - q_3)$$

then is the degree of $q_1 - q_3$ given by

case 1 $\deg(q_1) < k_2$: then is $\deg(q_2) = k_2$ and thus $\deg(q_3) = k_2$.

case 2 $\deg(q_2) < k_2$: then is $\deg(q_1) = k_2$ and $\deg(q_3) \le \deg(q_2)$.

case 3 $\deg(q_1) = \deg(q_2) = k_2$: thus follows that the leading term of q_1 and q_2 are different.

subcase 3.a $\deg(q_3) < k_2$: everything is clear

subcase 3.b $\deg(q_3) = k_2$: here has the leading term of q_3 be equal to them from q_1 to satisfy that $k_2 > \deg(q_2 - q_3)$.

From this follows that $\deg(q_1 - q_3) = k_2$.

- $q_0 \underset{\alpha}{\prec} q_2$ (resp. $q_2 \underset{\alpha}{\prec} q_1$) if and only if $q_1 \underset{\alpha}{\prec} q_3$ (resp. $q_3 \underset{\alpha}{\prec} q_1$) and thus do they determine the same anti-Stokes directions.

The set of all anti-Stokes directions is then given as

$$\mathbb{A} = \left\{ \theta + \frac{\pi}{k} \cdot j \mid k \in \mathcal{K},\ j \in \mathbb{N} \right\} .$$

Denote by $\mathcal{Y}_0(t)$ a normal solution of $[A^0]$.

Let us start by looking at a single germ in depth. The Proposition **??** states that every Stokes germ φ_α can be written as its matrix representation conjugated by the normal solution, i.e. as $\varphi_\alpha = \mathcal{Y}_0 C_{\varphi_\alpha} \mathcal{Y}_0^{-1} = \rho_\alpha^{-1}(C_{\varphi_\alpha})$.

Look at an example in which we will demonstrate, from which relations on the determining polynomials which restriction on the form of the Stokes matrices arise.

Example 1.21

Let $\alpha \in \mathbb{A}$ be an anti-Stokes direction. From the definition of $\mathbb{S}\mathrm{to}_\alpha(A^0)$ (cf. Definition **??**) we know that, if one has $q_1 \underset{\alpha}{\prec} q_2$, the Stokes matrix has the form

$$\begin{pmatrix} 1 & \boldsymbol{c_1} & \star \\ \mathbf{0} & 1 & \star \\ \star & \star & 1 \end{pmatrix}$$

where $c_j \in \mathbb{C}$ and $\star \in \mathbb{C}$.

We have seen that $q_1 \underset{\alpha}{\prec} q_2 \Rightarrow q_1 \underset{\alpha}{\prec} q_3$ thus the representation has the form

$$\begin{pmatrix} 1 & c_1 & \boldsymbol{c_2} \\ 0 & 1 & \star \\ \mathbf{0} & \star & 1 \end{pmatrix}$$

and if we also know that neither $q_2 \underset{\alpha}{\prec} q_3$ nor $q_3 \underset{\alpha}{\prec} q_2$ it has the form

$$\begin{pmatrix} 1 & c_1 & c_2 \\ 0 & 1 & \mathbf{0} \\ 0 & \mathbf{0} & 1 \end{pmatrix} .$$

We also know that every matrix of this form is a representation to some Stokes germ. Thus we have an isomorphism

$$\begin{aligned} \vartheta_\alpha : \mathbb{C}^2 &\longrightarrow \mathbb{S}\mathrm{to}_\alpha(A^0) \\ (c_1, c_2) &\longmapsto \begin{pmatrix} 1 & c_1 & c_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

In fact, the following 9 cases of Stokes matrices can arise:

	$q_2 \underset{\alpha}{\prec} q_3$	$q_3 \underset{\alpha}{\prec} q_2$	else
$q_1 \underset{\alpha}{\prec} q_2$ and $q_1 \underset{\alpha}{\prec} q_3$	$\begin{pmatrix} 1 & c_2 & c_3 \\ 0 & 1 & c_1 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & c_2 & c_3 \\ 0 & 1 & 0 \\ 0 & c_1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & c_2 & c_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
$q_2 \underset{\alpha}{\prec} q_1$ and $q_3 \underset{\alpha}{\prec} q_1$	$\begin{pmatrix} 1 & 0 & 0 \\ c_2' & 1 & c_1 \\ c_3 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ c_2 & 1 & 0 \\ c_3' & c_1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ c_2 & 1 & 0 \\ c_3 & 0 & 1 \end{pmatrix}$
else	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & c_1 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c_1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

In the blue cases we have $\mathcal{K}_\alpha = \mathcal{K}$ and $\mathbb{C}^3 \underset{\cong}{\xrightarrow{\vartheta_\alpha}} \mathbb{S}\mathrm{to}_\alpha(A^0)$. In the green cases $\mathcal{K}_\alpha = \{k_2\}$ and $\mathbb{C}^2 \underset{\cong}{\xrightarrow{\vartheta_\alpha}} \mathbb{S}\mathrm{to}_\alpha(A^0)$ as well as in the purple cases $\mathcal{K}_\alpha = \{k_1\}$ and $\mathbb{C}^1 \underset{\cong}{\xrightarrow{\vartheta_\alpha}} \mathbb{S}\mathrm{to}_\alpha(A^0)$. Thus, for every $\alpha \in \mathbb{A}$, we have an isomorphism $\rho_\alpha^{-1} \circ \vartheta_\alpha$. We will replace c_2' by $c_2 + c_1 c_3$ and c_3' by $c_1 c_2 + c_3$ to be consistent with the decomposition in the next part (cf. Example 1.23).

Corollary 1.22

The morphism $\prod_{\alpha \in \mathbb{A}} \vartheta_\alpha$ is an isomorphism of pointed sets, which maps the element only containing zeros to

$$(\mathrm{id}, \mathrm{id}, \dots, \mathrm{id}) \in \prod_{\alpha \in \mathbb{A}} \mathbb{S}\mathrm{to}_\alpha(A^0),$$

which gets by $(\prod_{\alpha \in \mathbb{A}})^{-1} \circ h$ mapped to the trivial cohomology class in $\mathcal{S}t(A^0)$.

In proposition **??** and especially Remark **??** we have defined a decomposition of the Stokes group $\mathrm{Sto}_\alpha(A^0)$ in subgroups generated by k-germs for $k \in \mathcal{K}$. In our case, we have at most two nontrivial factors. Especially is this decomposition given by

$$\varphi_\alpha = \varphi_\alpha^{k_1} \varphi_\alpha^{k_2} \xmapsto{i_\alpha} \left(\varphi_\alpha^{k_1}, \varphi_\alpha^{k_2} \right) \in \mathrm{Sto}_\alpha^{k_1}(A^0) \times \mathrm{Sto}_\alpha^{k_2}(A^0),$$

and i_α is the map, wich gives the factors of this factorization in ascending order. This decomposition, of a germ φ_α, is trivial if $\#\mathcal{K}(\varphi_\alpha) \leq 1$, thus the interesting cases are the blue cases.

Example 1.23

Look at the example

$$\vartheta_\alpha(c_1, c_2, c_3) = \mathcal{Y}_0 \begin{pmatrix} 1 & 0 & 0 \\ c_2 & 1 & 0 \\ c_1c_2 + c_3 & c_1 & 1 \end{pmatrix} \mathcal{Y}_0^{-1} = \varphi_\alpha \,.$$

According to Remark **??** the factor $\varphi_\alpha^{k_1} \in \mathrm{Sto}_\alpha^{k_1}(A^0)$, is given by

$$\varphi_\alpha^{k_1} = \mathcal{Y}_0 \begin{pmatrix} 1 & 0 & 0 \\ \mathbf{0} & 1 & 0 \\ \mathbf{0} & c_1 & 1 \end{pmatrix} \mathcal{Y}_0^{-1} \,.$$

The other factor $\varphi_\alpha^{k_2}$ is then obtained as

$$\begin{aligned} \varphi_\alpha^{k_2} &= \left(\varphi_\alpha^{k_1}\right)^{-1} \varphi_\alpha^{k_2} \\ &= \mathcal{Y}_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -c_1 & 1 \end{pmatrix} \underbrace{\mathcal{Y}_0^{-1}\mathcal{Y}_0}_{=\mathrm{id}} \begin{pmatrix} 1 & 0 & 0 \\ c_2 & 1 & 0 \\ c_1c_2 + c_3 & c_1 & 1 \end{pmatrix} \mathcal{Y}_0^{-1} \\ &= \mathcal{Y}_0 \begin{pmatrix} 1 & 0 & 0 \\ c_2 & 1 & 0 \\ c_3 & 0 & 1 \end{pmatrix} \mathcal{Y}_0^{-1} \,. \end{aligned}$$

The four nontrivial decomposition in our situation, are given by:

1. $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & c_1 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & c_2 & c_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & c_2 & c_3 \\ 0 & 1 & c_1 \\ 0 & 0 & 1 \end{pmatrix}$$

2. $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c_1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & c_2 & c_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & c_2 & c_3 \\ 0 & 1 & 0 \\ 0 & c_1 & 1 \end{pmatrix}$$

3. $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & c_1 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ c_2 & 1 & 0 \\ c_3 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ c_2 + c_1c_3 & 1 & c_1 \\ c_3 & 0 & 1 \end{pmatrix}$$

4. $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c_1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ c_2 & 1 & 0 \\ c_3 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ c_2 & 1 & 0 \\ c_1c_2 + c_3 & c_1 & 1 \end{pmatrix}$$

Explicit example

Even more explicit, we can fix the levels $k_1 = 1$ and $k_2 = 3$ together with $\theta = 0$. Assume without any restriction that $q_1 \underset{\theta}{\prec} q_2$ and $q_1 \underset{\theta}{\prec} q_3$ as well as $q_2 \underset{\theta}{\prec} q_3$. Other choices would result in reordering of the tuples below. Let the matrix L be given as $L = \operatorname{diag}(l_1, l_2, l_3) \in \mathrm{GL}_n(\mathbb{C})$.

The classification space is in this case isomorphic to $\mathbb{C}^{2\cdot(1+2\cdot 3)} = \mathbb{C}^{14}$. The element

$$({}^1c_1, {}^2c_1, {}^1c_2, {}^1c_3, {}^2c_2, {}^2c_3, \dots, {}^6c_2, {}^6c_3) \in \mathbb{C}^{14}$$

gets, via the isomorphism $\prod_{\alpha\in\mathbb{A}} j_\alpha$, mapped to

$$\left(\left(\begin{pmatrix}1&0&0\\0&1&{}^1c_1\\0&0&1\end{pmatrix},\begin{pmatrix}1&0&0\\0&1&0\\0&{}^2c_1&1\end{pmatrix}\right),\right.$$
$$\left.\left(\begin{pmatrix}1&{}^1c_2&{}^1c_3\\0&1&0\\0&0&1\end{pmatrix},\begin{pmatrix}1&0&0\\{}^2c_2&1&0\\{}^2c_3&0&1\end{pmatrix},\dots,\begin{pmatrix}1&0&0\\{}^6c_2&1&0\\{}^6c_3&0&1\end{pmatrix}\right)\right)$$

in $\prod_{\alpha\in\mathbb{A}^1} \mathbb{S}\mathrm{to}^1_\alpha(A^0) \times \prod_{\alpha\in\mathbb{A}^3} \mathbb{S}\mathrm{to}^3_\alpha(A^0)$ and thus the element

$$\left(\left(\begin{pmatrix}1&0&0\\0&1&{}^1c_1\\0&0&1\end{pmatrix},\mathrm{id},\mathrm{id},\begin{pmatrix}1&0&0\\0&1&0\\0&{}^2c_1&1\end{pmatrix},\mathrm{id},\mathrm{id}\right),\right.$$
$$\left.\left(\begin{pmatrix}1&{}^1c_2&{}^1c_3\\0&1&0\\0&0&1\end{pmatrix},\begin{pmatrix}1&0&0\\{}^2c_2&1&0\\{}^2c_3&0&1\end{pmatrix},\dots,\begin{pmatrix}1&0&0\\{}^6c_2&1&0\\{}^6c_3&0&1\end{pmatrix}\right)\right)$$

in $\prod_{\alpha\in\mathbb{A}} \mathbb{S}\mathrm{to}^1_\alpha(A^0) \times \prod_{\alpha\in\mathbb{A}} \mathbb{S}\mathrm{to}^3_\alpha(A^0)$. Using the morphism $\prod_{\alpha\in\mathbb{A}} i_\alpha^{-1}$ we get a complete set of Stokes matrices as

$$\left(\begin{pmatrix}1&{}^1c_2&{}^1c_3\\0&1&{}^1c_1\\0&0&1\end{pmatrix},\begin{pmatrix}1&0&0\\{}^2c_2&1&0\\{}^2c_3&0&1\end{pmatrix},\begin{pmatrix}1&{}^3c_2&{}^3c_3\\0&1&0\\0&0&1\end{pmatrix},\right.$$
$$\left.\begin{pmatrix}1&0&0\\{}^4c_2&1&0\\{}^2c_1{}^4c_2+{}^4c_3&{}^2c_1&1\end{pmatrix},\begin{pmatrix}1&{}^5c_2&{}^5c_3\\0&1&0\\0&0&1\end{pmatrix},\begin{pmatrix}1&0&0\\{}^6c_2&1&0\\{}^6c_3&0&1\end{pmatrix}\right) \in \prod_{\alpha\in\mathbb{A}} \mathbb{S}\mathrm{to}_\alpha(A^0)\,.$$

Applying the isomorphism $\prod_{\alpha\in\mathbb{A}} \rho_\alpha^{-1}$, i.e. conjugation by the fundamental solution $\mathcal{Y}_0(t) = t^L e^{Q(t^{-1})}$ (cf. Proposition ??), yields then the corresponding Stokes cocycle in $\prod_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha(A^0)$ and thus an element in $\mathcal{S}t(A^0)$.

This element is explicitly given as

$$\left(\begin{pmatrix}1&{}^1c_2 t^{l_2-l_1} e^{(q_2-q_1)(t^{-1})}&{}^1c_3 t^{l_3-l_1} e^{(q_3-q_1)(t^{-1})}\\0&1&{}^1c_1 t^{l_3-l_2} e^{(q_3-q_2)(t^{-1})}\\0&0&1\end{pmatrix},\mathcal{Y}_0^{-1}\begin{pmatrix}1&0&0\\{}^2c_2&1&0\\{}^2c_3&0&1\end{pmatrix}\mathcal{Y}_0,\right.$$

$$\mathcal{Y}_0^{-1}\begin{pmatrix}1 & {}^3c_2 & {}^3c_3\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\mathcal{Y}_0, \mathcal{Y}_0^{-1}\begin{pmatrix}1 & 0 & 0\\ {}^4c_2 & 1 & 0\\ {}^2c_1{}^4c_2+{}^4c_3 & {}^2c_1 & 1\end{pmatrix}\mathcal{Y}_0,$$
$$\left.\mathcal{Y}_0^{-1}\begin{pmatrix}1 & {}^5c_2 & {}^5c_3\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\mathcal{Y}_0, \mathcal{Y}_0^{-1}\begin{pmatrix}1 & 0 & 0\\ {}^6c_2 & 1 & 0\\ {}^6c_3 & 0 & 1\end{pmatrix}\mathcal{Y}_0\right) \in \prod_{\alpha\in\mathbb{A}}\mathrm{Sto}_\alpha(A^0)\,.$$