



Universität Augsburg  
Institut für Mathematik

Master's thesis

---

# Classification of meromorphic connections, using Stokes structures and the representations of the corresponding groups

---

by Maximilian Huber

submitted to the Institut für Mathematik  
of Universität Augsburg

approved by Prof. Dr. Marco Hien  
and Prof. Dr. Marc Nieper-Wißkirchen

submitted on May 17, 2015



# Contents

<b>Introduction/motivation</b>	<b>v</b>
<b>1 Asymptotic analysis</b>	<b>1</b>
1.1 Borel-Ritt Lemma . . . . .	5
<b>2 Systems and meromorphic connections</b>	<b>8</b>
2.1 (Global) meromorphic connections . . . . .	8
2.2 Local expression of meromorphic connections and systems . . . . .	10
2.2.1 Transformation of systems . . . . .	13
2.2.2 Fundamental solutions and the monodromy of a system . . . . .	17
2.3 Formal classification . . . . .	18
2.3.1 In the language of meromorphic connections: models . . . . .	19
2.3.2 In the language of systems: normal forms . . . . .	21
2.4 The main asymptotic existence theorem (M.A.E.T) . . . . .	24
2.5 The classifying set . . . . .	25
<b>3 Stokes Structures</b>	<b>29</b>
3.1 Stokes structures: Malgrange-Sibuya isomorphism . . . . .	30
3.1.1 The theorem . . . . .	30
3.1.2 Proof of Theorem 3.4 . . . . .	33
3.2 The Stokes group . . . . .	36
3.2.1 Anti-Stokes directions and the Stokes group . . . . .	36
3.2.2 Stokes matrices . . . . .	41
3.2.3 Filtration of the Stokes group by levels . . . . .	44
3.3 Stokes structures: using Stokes groups . . . . .	48
3.3.1 The theorem . . . . .	50
3.3.2 Proof of Theorem 3.30 . . . . .	52
3.4 Further improvements . . . . .	61
3.5 The complete Diagram . . . . .	66
<b>4 Structure of Stokes germs and Stokes matrices</b>	<b>68</b>
4.1 Simplest, multileveled case . . . . .	68
4.1.1 What do the Stokes germs $\varphi_\alpha$ look like? . . . . .	68
4.1.2 What do Stokes cocycles look like? . . . . .	72
4.2 In the more general multileveled context . . . . .	73
<b>Appendix</b>	<b>74</b>

<b>A</b>	<b>More basics</b>	<b>75</b>
A.1	Semidirect products . . . . .	75
A.2	Faithful representations . . . . .	76
A.3	Multisummability . . . . .	77
A.3.1	Factorization . . . . .	78
A.3.2	The map $g : G \backslash \widehat{G}(A^0) \rightarrow \prod_{\alpha \in \mathcal{A}} \text{Sto}_{\alpha}(A^0)$ . . . . .	79
	<b>Bibliography</b>	<b>80</b>

# Introduction/motivation

The basic problem, from which this masters-thesis arises, is that there are differential equations  $(\frac{d}{dt} - A)\hat{v} = w$  with coefficients in convergent powers series, which have no solution with converging entries. One can use the theory of meromorphic connections, to look at this problem and the interesting object is then the classifying set.

We will only be interested in local information and thus only in local classification of meromorphic connections. This classification can be splitted in the coarse formal classification and the fine meromorphic classification.

The formal classification problem was solved by the Levelt-Turittin Theorem (cf. Theorem 2.27). It states that in every formal equivalence class are some meromorphic connections of special form, which will be called models. These models are meromorphic connections, which are defined to be isomorphic to some direct sum of elementary meromorphic connections and elementary meromorphic connections are well understood.

Starting with a formal equivalence class corresponding to some model  $A^0$ , we can apply the meromorphic classification. The obtained set of meromorphic classes will be called the classifying set  $\mathcal{H}(A^0)$  (cf. Section 2.5). The tool to describe the classifying set will be the Stokes structures which turn out to deliver exactly the needed information. This idea is formulated in the Malgrange-Sibuya Theorem (cf. Theorem 3.4), where the Stokes structures appear as the first cohomology  $H^1(S^1; \Lambda(A^0))$  of the Stokes sheaf  $\Lambda(A^0)$  on  $S^1$  (cf. Definition 3.1).

The Malgrange-Sibuya theorem can be improved by showing that in each element in the  $H^1(S^1; \Lambda(A^0))$  contains a unique cocycle called the Stokes cocycle (cf. Definition 3.29) of special form. These Stokes cocycles are given by the elements in the product over some special directions  $\theta \in \mathbb{A} \subset S^1$  determined by  $A^0$  (cf. Definition 3.13) of groups  $\text{Sto}_\theta(A^0) \subset \Lambda_\theta(A^0)$  (cf. Definition 3.15). We will see that  $\text{Sto}_\theta(A^0)$  has faithful representation  $\text{Sto}_\theta(A^0)$  (cf. Proposition 3.17). The elements of  $\text{Sto}_\theta(A^0)$  are the so-called Stokes matrices and it is easy to see that they are nilpotent. This characteristic can be used to define the structure of a nilpotent Lie group on the classifying set  $\mathcal{H}(A^0)$ .

Stokes matrices provide exactly the required information in order to describe a meromorphic class of a meromorphic connection, and since we know which restrictions hold for these matrices, we can explicitly give the isomorphism  $\mathbb{C}^n \rightarrow H^1(S^1; \Lambda(A^0))$  for a convenient  $n$  (cf. Chapter 4).

# List of used symbols

## Asymptotic analysis

$\mathfrak{s}_{a,b}(r)$ .....	the sector $\{t \in \mathbb{C} \mid a < \arg(t) < b, 0 <  t  < r\}$
$\mathfrak{s}_I(r)$ .....	the sector $\mathfrak{s}_{a,b}(r)$ for $I = (a, b)$ or the arc $I$
$\bar{\mathfrak{s}}_{a,b}(r)$ .....	the closure of $\mathfrak{s}_{a,b}(r)$ in $\mathbb{C}^*$ .
$(\theta, \theta')$ .....	the arc from $\theta \in S^1$ to $\theta'$
$U(\theta, \varepsilon)$ .....	the arc $(\theta - \frac{\varepsilon}{2}, \theta + \frac{\varepsilon}{2}) \subset S^1$ resp. the corresponding sector
$\mathfrak{s} \Subset \mathfrak{s}'$ .....	$\mathfrak{s}$ is proper sub-sector of $\mathfrak{s}'$
$\hat{f}$ .....	the asymptotic expansion of $f$ (if not used in another way)
$f \sim_{\mathfrak{s}} \hat{f}$ .....	$f$ has $\hat{f}$ as asymptotic expansion on the sector $\mathfrak{s}$
$f \sim_I \hat{f}$ .....	$f$ has $\hat{f}$ as asymptotic expansion on the arc $I$
$f \sim_{\theta} \hat{f}$ .....	$f$ has $\hat{f}$ as asymptotic expansion in the direction $\theta$
$\mathcal{A}$ .....	the sheaf of asymptotic expansions
$T_I$ .....	the Taylor map on the arc $I$
$T_{\mathfrak{s}}$ .....	the Taylor map on the sector $\mathfrak{s}$
$\mathcal{A}^{<0}(I)$ .....	$\ker(T_I)$ ; the functions asymptotic to 0 on $I$
$\mathcal{A}^{<0}(\mathfrak{s})$ .....	$\ker(T_{\mathfrak{s}})$ ; the functions asymptotic to 0 on $\mathfrak{s}$

## Systems and meromorphic connections

$M$ .....	a riemanian surface
$Z$ .....	a effective divisor on $M$
$\mathcal{M}$ .....	a holomorphic bundle over $M$
$\mathcal{M}; (\mathcal{M}, \nabla)$ .....	a meromorphic connection on $\mathcal{M}$ with poles on $Z$
$\mathcal{M}; (\mathcal{M}, \nabla)$ .....	germ of a meromorphic connection $(\mathcal{M}, \nabla)$ at $0 \in M$
$\Delta$ .....	a differential operator
$[A]$ .....	the system corresponding to the connection matrix $A$
$G[t]$ .....	$\mathrm{GL}_n(\mathbb{C}[t])$
$G\{t\}$ .....	$\mathrm{GL}_n(\mathbb{C}\{t\})$ ; the analytic transformations
$G(\{t\})$ .....	$\mathrm{GL}_n(\mathbb{C}\{t\}[t^{-1}])$ ; the meromorphic transformations
$G[[t]]$ .....	$\mathrm{GL}_n(\mathbb{C}[[t]])$ ; the maybe not applicable formal transformations

$G((t))$ .....	$\mathrm{GL}_n(\mathbb{C}[[t]][t^{-1}])$ ; the maybe not applicable formal meromorphic transformations
${}^F A$ .....	$(dF)F^{-1} + FAF^{-1}$ ; the transforamtion of $A$ by $F$
$\widehat{G}(A)$ .....	the set of all (applicable) formal transformations
$[A, B]$ .....	the linear differential system $\frac{dF}{dt} = BF - FA$
$[\mathrm{End} A]$ .....	$[A^0, A^0]$
$G_0(A)$ .....	the set of all isotropies of $A$
$\mathcal{Y}$ .....	a fundamental solution
$\mathcal{E}^\varphi$ .....	the germ $(\mathbb{C}(\{t\}), d - \varphi')$
$\mathcal{N}_{\alpha,0}$ .....	the germ $(\mathbb{C}(\{t\}), d + \frac{\alpha}{t})$
$\mathcal{N}_{\alpha,d}$ .....	an elementary regular model
$(\mathcal{E}^\varphi, \nabla) \otimes (\mathcal{R}, \nabla)$ ....	a elementary meromorphic connection
$(\mathcal{M}^{nf}, \nabla^{nf})$ .....	a model
$\lambda$ .....	the isomorphism from definition 2.26
$Q(t^{-1})$ .....	the irregular part of a system $[A]$
$L \in \mathrm{GL}_n(\mathbb{C})$ .....	the matrix of formal monodromy of $[A]$
${}^0C(\mathcal{M}^{nf}, \nabla^{nf})$ .....	all isomorphism classes of meromorphic connections, which are formally isomorphic to $(\mathcal{M}^{nf}, \nabla^{nf})$
${}^0C(A^0)$ .....	the system variant of ${}^0C(\mathcal{M}^{nf}, \nabla^{nf})$
$\mathcal{H}; \mathcal{H}(\mathcal{M}^{nf}, \nabla^{nf})$ ....	the space of marked (meromorphic) pairs
$\mathrm{Syst}_m(A^0)$ .....	the systems formally meromorphic equivalent to $A^0$
$\mathcal{H}(A^0)$ .....	the system version of $\mathcal{H}(\mathcal{M}^{nf}, \nabla^{nf})$
$\widehat{\mathrm{Syst}}_m(A^0)$ .....	the marked pairs corresponding to $A^0$

## Stokes Structures

$\Lambda(A^0)$ .....	the Stokes sheaf (on $S^1$ )
$St(A^0)$ .....	$H^1(S^1; \Lambda(A^0))$ ; the first non abelian cohomology of the Stokes sheaf
$\exp$ .....	the Malgrange-Sibuya isomorphism
$\exp_A$ .....	the Malgrange-Sibuya isomorphism corresponding to the system $[A]$
$\mathcal{G}$ .....	the sheaf of flat functions
$\Theta$ .....	$G[[t]]/G\{t\} \rightarrow H^1(S^1; \mathcal{G})$ ; the isomorphism from the other Malgrange-Sibuya theorem
$\mathcal{Q}(A^0)$ .....	the set of all determining polynomials of $[\mathrm{End} A^0]$
$a_{jl} \in \mathbb{C} \setminus \{0\}$ .....	the leading factor of $q_j - q_l \in \mathcal{Q}(A^0)$
$q_{jl}(t^{-1})$ .....	the leading coefficient of $q_j - q_l \in \mathcal{Q}(A^0)$
$k_{jl}$ .....	the degree of $q_j - q_l \in \mathcal{Q}(A^0)$
$\mathcal{K}$ .....	$\{k_1 < \dots < k_r\}$ ; the set of all levels of a system.
$q_j \prec_\theta q_l$ .....	the first relation from definition ??

$q_j \prec_{\max, \theta} q_l$ .....	the second relation from definition ??
$\Re(z)$ .....	the real part of a complex number $z$
$\Im(z)$ .....	the imaginary part of a complex number $z$
$\arg(z)$ .....	the argument of a complex number $z$
$\mathbb{A}$ .....	$\{\alpha_1, \dots, \alpha_\nu\}$ ; the set of all anti-Stokes directions
$\mathbb{S}$ .....	the set of all Stokes directions
$\text{Sto}_\theta(A^0)$ .....	the Stokes group of $A^0$ in direction $\theta$ whose elements are Stokes germs
$\delta_{jl}$ .....	a block matrix version of Kronecker's delta, corresponding to the structure of $Q$
$\text{Sto}_\theta(A^0)$ .....	the group of all Stokes matrices of $A^0$ in direction $\theta$
$\rho_\theta$ .....	$\text{Sto}_\theta(A^0) \rightarrow \text{Sto}_\theta(A^0)$ ; the map from proposition 3.17
$C_{\varphi_\theta}$ .....	$\rho_\theta(\varphi_\theta)$ ; the Stokes matrix corresponding to $\varphi_\theta$
$\Lambda^k(A^0)$ .....	the subsheaf of $\Lambda(A^0)$ of all germs, which are generated by $k$ -germs
$\Lambda^{\leq k}(A^0)$ .....	the subsheaf of $\Lambda(A^0)$ generated by $k'$ -germs for all $k' \leq k$
$\Lambda^{< k}(A^0)$ .....	the subsheaf of $\Lambda(A^0)$ generated by $k'$ -germs for all $k' < k$
$\Lambda^{\geq k}(A^0)$ .....	the subsheaf of $\Lambda(A^0)$ generated by $k'$ -germs for all $k' \geq k$
$\text{Sto}_\theta^*(A^0)$ .....	$\text{Sto}_\theta(A^0) \cap \Lambda_\theta^*(A^0)$ ; the restriction of the Stokes sheaf for $\star \in \{k, < k, \leq k, \dots\}$
$\text{Sto}_\theta^*(A^0)$ .....	the groups of representations, which correspond to elements of $\text{Sto}_\theta^*(A^0)$
$\mathbb{A}^k$ .....	the set of anti-Stokes directions bearing the level $k$
$\mathbb{A}^{\leq k}$ .....	$\bigcup_{k' \leq k} \mathbb{A}^{k'}$
$\mathbb{A}^{< k}$ .....	$\bigcup_{k' < k} \mathbb{A}^{k'}$
$\mathbb{A}^{\geq k}$ .....	$\bigcup_{k' \geq k} \mathbb{A}^{k'}$
$\mathcal{K}_\alpha$ .....	the set of levels beared by $\alpha \in \mathbb{A}$
$K_\alpha$ .....	the maximal level beared by $\alpha$
$i_\alpha$ .....	$\text{Sto}_\alpha(A^0) \rightarrow \prod_{k \in \mathcal{K}_\alpha} \text{Sto}_\alpha^k(A^0)$ ; the isomorphism corresponding to a factorization by a given order in a semidirect product
$\rho_\alpha^k$ .....	$\text{Sto}_\alpha^k(A^0) \rightarrow \text{Sto}_\alpha^k(A^0)$ ; the restriction of the map $\rho_\alpha$ to the level $k$ .
$i_\alpha$ .....	$\text{Sto}_\alpha(A^0) \rightarrow \prod_{k \in \mathcal{K}_\alpha} \text{Sto}_\alpha^k(A^0)$ ; the Stokes matrix version of the $i_\alpha$ above
$\mathcal{U}$ .....	the nerve of the covering $\mathcal{U}$
$h$ .....	$\prod_{\alpha \in \mathbb{A}} \text{Sto}_\alpha(A^0) \rightarrow \mathcal{St}(A^0)$ ; the isomorphism from theorem 3.30
$\mathcal{U}^\star$ .....	for $\star \in \{k, < k, \leq k\}$ the adequate coverings defined in section 3.3.2



$\Gamma(\dot{U}^\star; \Lambda^\star(A^0))$	.....	$\prod_{\alpha \in \mathbb{A}^\star} \Gamma(\dot{U}_\alpha^\star; \Lambda^\star(A^0))$ for every $\star \in \{k, < k, \leq k, \dots\}$
$s^k$	.....	$\Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A^0)) \rightarrow H^1(S^1; \Lambda^k(A^0))$ ; the quotient map
$\sigma^k$	.....	$\Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A^0)) \rightarrow \Gamma(\dot{\mathcal{U}}^{\leq k}; \Lambda^{\leq k}(A^0))$ ; an injective map defined in definition 3.40
$\sigma^{<k}$	.....	$\Gamma(\dot{\mathcal{U}}^{<k}; \Lambda^{<k}(A^0)) \rightarrow \Gamma(\dot{\mathcal{U}}^{\leq k}; \Lambda^{\leq k}(A^0))$ ; an injective map defined in definition 3.40
$\mathfrak{S}^{\leq k}$	.....	$\Gamma(\dot{\mathcal{U}}^{<k}; \Lambda^{<k}(A^0)) \times \Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A^0)) \rightarrow \Gamma(\dot{\mathcal{U}}^{\leq k}; \Lambda^{\leq k}(A^0))$ ; the product map of cocycles
$\alpha^+$	.....	the nearest anti-Stokes direction on the right of $\alpha$
$\tau$	.....	$\prod_{k \in \mathcal{K}} \Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A^0)) \rightarrow \Gamma(\dot{\mathcal{U}}; \Lambda(A^0))$ ; the product map of single-leveled cocycles
$\mathcal{T}$	.....	$\prod_{k \in \mathcal{K}} \Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A^0)) \rightarrow H^1(\mathcal{U}; \Lambda(A^0))$ ; the isomorphism induced by $\tau$ on the cohomology
$\mathfrak{T}$	.....	a map introduced in the proof of theorem 3.30



# 1 Asymptotic analysis

In this chapter, we want to look at Poincaré asymptotic expansions at  $t_0$ . There are various literature references for Poincaré asymptotic expansions, or for short, asymptotic expansions. We will mostly refer to Loday-Richaud's book [Lod14, chapter 2] and the book [PS03, chapter 7] written by van der Put and Singer although this topic is scratched in many publications. We will assume that  $t_0 = 0$  and this is without loss of generality, since asymptotic expansions at  $t_0 \in \mathbb{C}$  reduce to asymptotic expansions at 0 after the change of variable  $t \mapsto s = t - t_0$  and asymptotic expansions at  $t_0 = \infty$  are obtained by  $t \mapsto s = \frac{1}{t}$ .

We introduce the following notations:

## Notations 1.1

We denote by

- $\mathfrak{s}_{a,b}(r)$ ,  $a, b \in \mathbb{S}^1$  and  $r \in \mathbb{R}_{>0}$  the open sector

$$\mathfrak{s}_{a,b}(r) := \{t \in \mathbb{C} \mid a < \arg(t) < b, 0 < |t| < r\}$$

of all points  $t \in \mathbb{C}$  satisfying  $a < \arg(t) < b$  and  $0 < |t| < r$ ;

- $\mathfrak{s}_I(r) = \mathfrak{s}_{a,b}(r)$  for an open arc<sup>a</sup>  $I = (a, b)$ ;
- $\bar{\mathfrak{s}}_{a,b}(r)$  the closure of  $\mathfrak{s}_{a,b}(r)$  in  $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ .

The notion of sectors can easily be extended to sectors on a  $q$ -sheet cover or sectors on the Universal cover of  $\mathbb{C}^*$ , but we will not use that.

---

<sup>a</sup>i.e. an open interval of  $S^1$

## Definition 1.2

We will say that a sector  $\mathfrak{s}_I(r)$  contains  $\theta \in S^1$  if  $\theta \in I$ . In the same way we will say that  $\mathfrak{s}_I(r)$  contains  $U \subset S^1$  if  $U \subset I$  or that  $U \subset S^1$  contains a sector  $\mathfrak{s}_I(r)$  if  $I \subset U$ .

## Remark 1.3

Since the explicit value  $r$  of the radius does not matter as long as it is small enough, we shall simply speak of an *arc* or *interval*  $I$  as  $\mathfrak{s}_I(r)$  for small enough  $r$ .

It will be sometimes convenient to talk of arcs

$$U(\theta, \varepsilon) := \left(\theta - \frac{\varepsilon}{2}, \theta + \frac{\varepsilon}{2}\right) \subset S^1$$

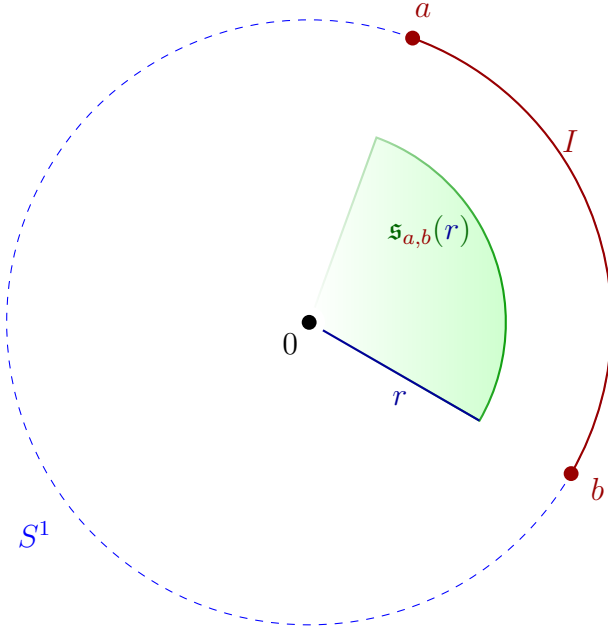


Figure 1.1: An exemplary arc  $\mathfrak{s}_{a,b}(r)$  spanning from  $a$  to  $b$  with radius  $r$ .

defined by the midpoint together with the radius.

#### Definition 1.4

A sector  $\mathfrak{s}_{a',b'}(r')$  is said to be a proper sub-sector of the sector  $\mathfrak{s}_{a,b}(r)$ ,  $\mathfrak{s}_{a',b'}(r') \subseteq \mathfrak{s}_{a,b}(r)$ , if its closure  $\bar{\mathfrak{s}}_{a',b'}(r')$  is included in  $\mathfrak{s}_{a,b}(r)$ .

##### Remark 1.4.1

$\mathfrak{s}_{a',b'}(r') \subseteq \mathfrak{s}_{a,b}(r)$  if and only if  $a < a' < b' < b$  and  $r' < r$ .

#### Definition 1.5

A function  $f \in \mathcal{O}(\mathfrak{s})$  is said to have the formal<sup>a</sup> Laurent series  $\hat{g} = \sum_{n \geq n_0} a_n t^n$  as asymptotic expansion (or to be asymptotic to the series) on the sector  $\mathfrak{s}$  if

- for all proper sub-sectors  $\mathfrak{s}' \subseteq \mathfrak{s}$  and
- all  $\mathbf{N} \in \mathbb{N}$ ,

there exists a constant  $C(\mathbf{N}, \mathfrak{s}')$  such that the following estimate<sup>b</sup> holds

$$\left| f(x) - \sum_{n_0 \leq n \leq \mathbf{N}-1} c_n t^n \right| \leq C(\mathbf{N}, \mathfrak{s}') |t|^{\mathbf{N}} \quad \text{for all } t \in \mathfrak{s}'.$$

We will use the following notations:

1. We will say that  $f$  has the formal Laurent series  $\hat{g}$  as asymptotic expansion on the interval  $I \subset S^1$  if there exists a  $r \in \mathbb{R}_{>0}$  such that  $f$  has the formal Laurent series  $\hat{g}$  as asymptotic expansion on  $\mathfrak{s}_I(r)$ .
2. We will say that  $f$  has the formal Laurent series  $\hat{g}$  as asymptotic expansion in the direction  $\theta$  if there exists an interval  $\theta \in I \subset S^1$  such that  $f$  has formal Laurent series  $\hat{g}$  as asymptotic expansion on the interval  $I$ .

<sup>a</sup>The term “formal” emphasizes that we do not restrict the coefficients  $a_n \in \mathbb{C}$  in any way.

<sup>b</sup>Sometimes, for example in [Sab90], this condition is written as

$$\lim_{z \rightarrow 0, z \in \mathfrak{s}'} |t|^{-(\mathbf{N}-1)} \left| f(x) - \sum_{n_0 \leq n \leq \mathbf{N}-1} c_n t^n \right| = 0 \quad \text{for all } t \in \mathfrak{s}'.$$

If  $f$  has  $\hat{f}$  as asymptotic expansion on the sector  $\mathfrak{s}$  (or on the interval  $I \subset S^1$ , or in the direction  $\theta$ ), we denote that by  $f \sim_{\mathfrak{s}} \hat{f}$  (or  $f \sim_I \hat{f}$ , or  $f \sim_{\theta} \hat{f}$ ).

### Definition 1.6

1. Define  $\mathcal{A}(\mathfrak{s})$  as the set of holomorphic functions  $f \in \mathcal{O}(\mathfrak{s})$  which admit an asymptotic expansion at 0 on  $\mathfrak{s}$ .

#### Remark 1.6.1

- a)  $\mathcal{A}(\mathfrak{s})$  is a subring of  $\mathcal{O}(\mathfrak{s})$ .
- b)  $\mathcal{A}(\mathfrak{s})$  contains  $\mathbb{C}(\{t\})$  as a subfield.

2. For  $(a, b) = I$  define  $\mathcal{A}(a, b) = \mathcal{A}(I)$  as the limit  $\varinjlim_{r \rightarrow 0} \mathcal{A}(\mathfrak{s}_{a,b}(r))^a$ .

<sup>a</sup>In more detail: the elements of  $\mathcal{A}(a, b)$  are pairs  $(f, \mathfrak{s}_{a,b}(r))$  mit  $f \in \mathcal{A}(\mathfrak{s}_{a,b}(r))$ . The equivalence relation is given by  $(f_1, \mathfrak{s}_{a,b}(r_1)) \sim (f_2, \mathfrak{s}_{a,b}(r_2))$  if there is a pair  $(f_3, \mathfrak{s}_{a,b}(r_3))$  such that  $r_3 < \min(r_1, r_2)$  and  $f_3 = f_1 = f_2$  on  $\mathfrak{s}_{a,b}(r_3)$ .

It can easily be seen that  $\mathcal{A} : U \rightarrow \mathcal{A}(U)$  defines a sheaf on  $S^1$ .

### Definition 1.7

- One writes

$$\begin{array}{ccc} T_I : \mathcal{A}(I) \rightarrow \mathbb{C}((t)) & (\text{resp.} & T_{\mathfrak{s}} : \mathcal{A}(\mathfrak{s}) \rightarrow \mathbb{C}((t)) \\ f \mapsto \hat{f} & & ) \end{array}$$

for the mapping, called *Taylor map*, which associates to each function  $f \in \mathcal{A}(I)$  (resp.  $f \in \mathcal{A}(\mathfrak{s})$ ) its asymptotic expansion.

- A function  $f$  which is asymptotic to the identity, i.e. with  $T_{\mathfrak{s}}(f) = \hat{f} = \text{id}_{\mathfrak{s}}$ , is called *flat* (on  $\mathfrak{s}$ ).
- Denote by  $\mathcal{A}^{<0}(I)$  (resp.  $\mathcal{A}^{<0}(\mathfrak{s})$ ) the set  $\ker(T_I) \subset \mathcal{A}(I)$  (resp.  $\ker(T_{\mathfrak{s}}) \subset \mathcal{A}(\mathfrak{s})$ ) of functions asymptotic to zero at 0.

**Remark 1.7.1**

The kernel  $\mathcal{A}^{<0}(I)$  is not zero in general. Let  $I = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ , then  $\mathcal{A}^{<0}(I)$  contains the element  $e^{-\frac{1}{\sqrt{t}}}$  where  $\sqrt{t}$  stands for the principal determination of  $x^{\frac{1}{2}}$ .

**Example 1.8: Trivial example**

If  $f$  is an analytic function on  $D = \{t \mid |t| < r\}$  then  $f$  is asymptotic to its Taylor series at 0 on  $D^* = D \setminus \{0\}$ . Reciprocally, if  $f$  is analytic on  $D^*$  and has an asymptotic expansion at 0 on  $D^*$  then,  $f$  is bounded near 0 and according to the removable singularity theorem,  $f$  is analytic on  $D$ .

**Remark 1.8.1**

This implies that  $\mathcal{A}(S^1) \cong \mathbb{C}(\{t\})$ .

For many more extensively discussed examples, like the Euler function or the exponential integral, see Loday-Richaud's book [Lod14, Sec.2.2].

**Proposition 1.9**

For every arc  $I \subset S^1$  is  $\mathcal{A}(I)$  is stable under derivation, i.e.  $f'(t)$  is on  $I$  asymptotic to  $\hat{f}'(t)$  for every function  $f$  asymptotic to  $\hat{f}$  on  $I$ .

**Remark 1.9.1**

Let  $f_1$  and  $f_2$  be functions which satisfying  $f_1 \sim_I \hat{f}_1$  and  $f_2 \sim_I \hat{f}_2$ . We have also the rules:

1.  $f_1(t) + f_2(t)$  is on  $I$  asymptotic to  $\hat{f}_1(t) + \hat{f}_2(t)$  (cf. Theorem [Bal00, 4.5.Thm.13]).
2.  $f_1(t)f_2(t)$  is on  $I$  asymptotic to  $\hat{f}_1(t)\hat{f}_2(t)$  (cf. Theorem [Bal00, 4.5.Thm.14]).
3.  $\int_0^t f(s) ds$  is on  $I$  asymptotic to  $\int_0^t \hat{f}(s) ds$  (cf. Theorem [Bal00, 4.5.Thm.20]).
4. If  $f \in \mathcal{A}(I) \setminus \mathcal{A}^{<0}(I)$  then  $f^{-1}(t)$  is on  $I$  asymptotic to  $\hat{f}^{-1}(t)$  (cf. Theorem [Bal00, 4.5.Thm.21]).

*Proof of Proposition 1.9.* Let  $f \in \mathcal{A}(I)$  with asymptotic expansion  $\hat{f} = \sum_{-n_0 \leq n} a_n x^n$  its asymptotic expansion where we assume without restriction that  $n_0 = 0$ . One has for each  $m \geq 0$

$$f(t) = \sum_{n=0}^m a_n t^n + R_m(t) t^m$$

with  $\lim_{\substack{t \rightarrow 0 \\ t \in \mathfrak{s}_I(r)}} R_m(t) = 0$ . This implies that  $R_m$  is holomorphic in  $\mathfrak{s}_I(r)$ . Thus one has

$$f'(t) = \sum_{n=0}^m n a_n t^{n-1} + m R_m(t) t^{m-1} + R'_m(t) t^m$$

Let  $C_{x,\rho}$  be the circle at  $x$  with radius  $\rho$  contained in  $\mathfrak{s}_I(r)$ . The Cauchy theorem implies then that

$$|R'_m(t)| \leq \frac{1}{\rho} \max_{s \in \mathbb{C}_{x,\rho}} |R(s)|.$$

Let  $J$  be a relatively compact open set in  $I$ , then there exists a positive number  $\alpha$  such that for all  $x \in \mathfrak{s}_I(r)$  one has  $C_{x,\alpha|x|} \subset \mathfrak{s}_I(r)$ . This proves that  $\hat{f}'$  is an asymptotic expansion for  $f'$  in  $\mathfrak{s}_I(r)$ .  $\square$

## 1.1 Borel-Ritt Lemma

The most important theorem here is the Borel-Ritt Lemma.

**Theorem 1.10: Borel-Ritt**

Let  $I \subsetneq S^1$  be an open sub-arc, i.e. with a opening less than  $2\pi$ . Then the Taylor map

$$T_I : \mathcal{A}(I) \rightarrow \mathbb{C}((t))$$

is onto.

**Remark 1.10.1**

This means that to each  $\hat{f} \in \mathbb{C}((t))$  and every arc  $I \subsetneq S^1$  there are functions  $f$  asymptotic to  $\hat{f}$  on  $I$ . However, when  $\hat{f}$  satisfies an equation, these asymptotic functions do not necessarily satisfy the same equation in general. This gap will be filled in section 2.4 for small enough sectors.

There are multiple approaches to obtain a function  $f \in \mathcal{A}(I)$  asymptotic to a given function  $\hat{f} \in \mathbb{C}((t))$  in a canonical way. For example (multi-)multisummability which is in depth described in Loday-Richaud's book [Lod14]. This topic is also discussed in the paper [MR91] from Martinet and Ramis.

*Proof of theorem 1.10.* Let  $I := (-\pi, \pi)$  and  $R \in \mathbb{R}_{>0}$ . We will prove this for the sector

$$\mathfrak{s} := \mathfrak{s}_I(R) = \{t \in \mathbb{C} \mid |\arg(t)| < \pi, 0 < |t| < R\}$$

in which, after rotation and for large enough  $R$ , every sector  $\mathfrak{s}'' \subsetneq \mathbb{C}^*$  lies. Let  $\sum a_n z^n$  be a formal Laurent series. We look for a function  $f \in \mathcal{A}(I)$  with Taylor series  $T_I f = \sum a_n z^n$ . By subtracting the principal part we may assume that this series has no terms with negative degree. Let  $b_n$  be a sequence, which satisfies

the series  $\sum |a_n| b_n R^{n-\frac{1}{2}}$  is convergent.

For example, one may set

$$b_n = \begin{cases} 0 & \text{when } n = 0 \\ \frac{1}{n!} |a_n| & \text{when } n > 0 \end{cases}$$

Let  $\sqrt{t}$  be the branch of the square root function that satisfies  $|\arg(\sqrt{t})| < \frac{\pi}{2}$  for all  $t \in \mathfrak{s}$ . For any real number  $b_n$ , the function  $\beta_n(t) := 1 - e^{-\frac{b_n}{\sqrt{t}}}$  satisfies

- (a)  $|\beta_n(t)| \leq \frac{b_n}{\sqrt{|t|}}$  since  $1 - e^t = -\int_0^t e^s ds$  implies that  $|1 - e^t| < |t|$  for  $\Re(t) < 0$   
and



- (b)  $\beta_n$  has asymptotic expansion 1 on  $\mathfrak{s}$  (thus  $\beta_n - 1$  has asymptotic expansion 0 on  $\mathfrak{s}$ ).

Define  $f(t) := \sum a_n \beta_n(t) t^n$ . Since

$$|a_n \beta_n(t) t^n| \leq |a_n| |b_n| |z|^{n-\frac{1}{2}} \leq |a_n| |b_n| R^{n-\frac{1}{2}},$$

the series  $\sum a_n \beta_n(t) t^n$  converges and its sum  $f(t)$  is in  $\mathcal{O}(\mathfrak{s})$ .

Consider a proper sub-sector  $\mathfrak{s}' \Subset \mathfrak{s}$  and  $t \in \mathfrak{s}'$ . Then, for every  $N > 0$

$$\begin{aligned} \left| f(t) - \sum_{n=0}^{N-1} a_n t^n \right| &\leq \left| \sum_{n=0}^{N-1} a_n (\beta_n(t) - 1) t^n \right| + |t|^N \sum_{n \geq N} |a_n \beta_n(t) t^{n-N}| \\ &= \left| \sum_{n=0}^{N-1} a_n e^{-\frac{b_n}{\sqrt{i}} t^n} \right| + |t|^N \sum_{n \geq N} |a_n \beta_n(t) t^{n-N}| \end{aligned}$$

The first summand is a finite sum of terms, which are all asymptotic to 0 and then, is majorized by  $C'|t|^N$  for a convenient constant  $C'$ . The second summand is majorized by

$$|t|^N \left( 2|a_n| + \sum_{n \geq N+1} |a_n| |b_n| R^{n-\frac{1}{2}-N} \right).$$

By setting  $C = C' + 2|a_n| + \sum_{n \geq N+1} |a_n| |b_n| R^{n-\frac{1}{2}-N}$  we obtain a positive constant  $C$ , which depends on  $N$  and the radius of  $\mathfrak{s}' < R$ , such that

$$\left| f(t) - \sum_{n=0}^{N-1} a_n t^n \right| \leq C |t|^N \quad \text{for all } t \in \mathfrak{s}'.$$

□

---

<sup>a</sup>For  $t = re^{i\varphi}$  is  $\sqrt{t} = \sqrt{r}e^{i\frac{\varphi}{2}}$  with  $-\pi < \varphi < \pi$ .

## 2 Systems and meromorphic connections

There are multiple languages, which can be used for talking about meromorphic connections. Firstly the languages of meromorphic connections which can be used to talk about global information. For local description one can use **germs of meromorphic connections** or the coordinate dependent **systems** and **connection matrices**. Other coordinate independent approaches arise for example from the theory of (localized holonomic)  $\mathcal{D}$ -modules.

Meromorphic connections are introduced and discussed in many resources. A good starting point are Sabbah's lecture notes [Sab90]. More advanced resources are for example Sabbah's book [Sab07], Varadarajan's book [Var96] or the book [HTT08] from Hotta et al. The necessary facts about meromorphic connections are also stated in Boalch's paper [Boa01] (resp. his thesis [Boa99]), and Loday-Richaud's paper [Lod94].

Although the language of meromorphic connections is often preferred, we will use most of the time the language of systems. Systems are, for example, discussed in the book [HTT08] from Hotta et al, Loday-Richaud's paper [Lod94] and his book [Lod14] and Boalch's publications [Boa01; Boa99]. Another resource might Remy's paper [Rem14] be.

We will use all of the mentioned resources.

### 2.1 (Global) meromorphic connections

Let  $M$  be a riemanian surface and let  $Z = k_1(a_1) + \cdots + k_m(a_m) > 0$  be an effective divisor<sup>[1]</sup> on  $M$ . It is sufficient to think  $M = \mathbb{P}^1$  and  $0 \in |Z|^{[2]}$ , since we will only be interested in local information (at 0).

Let  $\mathcal{M}$  be a holomorphic Bundle over  $M$  i.e. a locally free  $\mathcal{O}_M$ -module of rank  $n$ .

A (global) meromorphic connection is then defined as follows.

**Definition 2.1**

---

<sup>[1]</sup>The  $a_i \in M$  are distinct points and the  $k_i$  are positive integers.

<sup>[2]</sup>If  $Z = k_1(a_1) + \cdots + k_m(a_m)$  then  $|Z| := \{a_1, \dots, a_m\}$ .

A meromorphic connection  $(\mathcal{M}, \nabla)$  on  $\mathcal{M}$  with poles on  $Z$  is defined by a  $\mathbb{C}$ -linear morphism of sheaves

$$\nabla : \mathcal{M} \rightarrow \Omega_M^1(*Z) \otimes \mathcal{M}$$

satisfying, for each  $U \subseteq M$ , the *Leibniz rule*

$$\nabla(fs) = f\nabla s + (df) \otimes s$$

for  $s \in \Gamma(U, \mathcal{M})$  and  $f \in \mathcal{O}_M(U)$ . The *rank* of the meromorphic connection  $(\mathcal{M}, \nabla)$  is defined to be the rank of the Bundle  $\mathcal{M}$ .

### Remark 2.1.1

Some authors use the factors  $k_i$  of the divisor  $Z$  to limit the pole orders at the points  $a_i$ . Since we do not need this restriction, we allow arbitrary pole orders. Denoted is this by the  $*$  in  $\Omega_M^1(*Z)$ . The sheaf  $\mathcal{O}_M(*Z)$  of functions, which are meromorphic along  $Z$ , is defined in Sabbah's book [Sab07, Sec.0.8] and  $\Omega_M^1(*Z)$  is then defined as

$$\Omega_M^1(*Z) := \mathcal{O}_M(*Z) \otimes_{\mathcal{O}_M} \Omega_M^1$$

the *sheaf of meromorphic differential 1-forms* (cf. [Sab07, Sec.0.9.b]).

The word “global” used to emphasize that the connection is “on  $M$ ”, i.e. not only a germ at some point of  $M$ , like it will be used later (cf. Remark 2.5).

### Remark 2.2

1. It will occasionally be convenient to omit the  $\nabla$  and simply call  $\mathcal{M}$  the meromorphic connection.
2. Here, the variant ‘holomorphic bundle with meromorphic connection’ is chosen, like in Boalch's paper [Boa01]. There is also the transposed description ‘meromorphic bundle with holomorphic connection’ which is for example used in Sabbah's book [Sab07].

By choosing a lattice of a meromorphic bundle (cf. [Sab07, Def.0.8.3]), one gets a holomorphic bundle but if the meromorphic bundle had a holomorphic connection, the induced connection on the lattice is no longer guaranteed to be holomorphic. Thus we obtain a meromorphic connection on a holomorphic bundle in our sense.

### Definition 2.3

The connection  $\nabla : \mathcal{M} \rightarrow \Omega_M^1 \otimes_{\mathcal{O}_M} \mathcal{M}$  is said to be *integrable* or *flat*, if

its curvature vanishes, i.e.  $R_{\nabla} \equiv 0$  where

$$- R_{\nabla} := \nabla \circ \nabla : \mathcal{E} \rightarrow \Omega_M^2 \otimes_{\mathcal{O}_M} \mathcal{E} \text{ is a } \mathcal{O}_M\text{-linear morphism.}$$

**Remark 2.3.1**

Here are all connections flat, since only look at Dimension one.

## 2.2 Local expression of meromorphic connections and systems

We will usually be interested in local information of meromorphic connections. This means that we look at a connection in a neighbourhood of 0 and allow only one singularity at 0. There are multiple ways of expressing the local information, we will either talk about germs of meromorphic connections or systems, which are coordinate- and trivialization-dependent.

**Proposition 2.4**

A germ of a meromorphic connection  $(\mathcal{M}, \nabla)$  is the sheaf-theoretic germ (at  $t = 0$ ), thus is given by a tuple  $(\mathcal{M}, \nabla)$  where

- $\mathcal{M}$  is the germ at 0 of the holomorphic bundle  $\mathcal{M}$  and thus a  $\mathbb{C}(\{t\})$ -vectorspace of dimension  $n$ , since the ring of germs of meromorphic functions with poles at 0 is the ring  $\mathbb{C}(\{t\})$ , and
- $\nabla : \mathcal{M} \rightarrow \mathcal{M}$  is a additive map, which satisfies the *Leibniz rule*

$$\nabla(fm) = \frac{d}{dt}f \cdot m + f\nabla(m)$$

for all  $f \in \mathbb{C}(\{t\})$  and  $m \in \mathcal{M}$ .

**Remark 2.4.1**

Loday-Richaud calls the meromorphic connections in her book [Lod14, Def.4.2.1] *differential modules*.

**Remark 2.5**

From now on we will mostly talk about **germs of** meromorphic connections  $(\mathcal{M}, \nabla)$  and we will call them meromorphic connection. If we want to talk about meromorphic connections in the sense of Definition 2.1 we will emphasize this by the word ‘global’ or by talking about a meromorphic connection **on**  $M$ .

**Definition 2.6**

A (iso-) morphism of meromorphic connections  $\Phi : (\mathcal{M}, \nabla) \xrightarrow{\sim} (\mathcal{M}', \nabla')$  is a (iso-) morphism of  $\mathbb{C}(\{t\})$ -vectorspaces  $\Phi : \mathcal{M} \xrightarrow{\sim} \mathcal{M}'$  which commutes with the connections, i.e. which satisfies  $\nabla \circ \varphi = (\text{id} \otimes \varphi) \circ \nabla$ .

Choose a  $\mathbb{C}(\{t\})$ -basis  $\underline{e} = (e_1, e_2, \dots, e_n)$  of  $\mathcal{M}$ . Let  $A = (a_{jk})_{j,k \in \{1, \dots, n\}}$  be a  $n \times n$  matrix with entries in  $\mathbb{C}(\{t\})$  such that it describes the action of  $\nabla$ , i.e. it satisfies

$$\nabla e_k = - \sum_{1 \leq j \leq n} a_{jk}(t) e_j, \text{ for every } k \in \{1, \dots, n\}.$$

**Definition 2.7**

The matrix  $A$  is called a *connection matrix* of  $(\mathcal{M}, \nabla)$ .

The connection  $\nabla$  is fully determined by  $A$ . Indeed, let  $x = \sum_{0 \leq j \leq n} x_j e_j$  be an arbitrary element of  $\mathcal{M}$  which is in matrix notation described as  $x = \underline{e} \cdot X$  with the column matrix  $X = {}^t(x_1, x_2, \dots, x_n)$ . Then, applying  $\nabla$  using the Leibniz rule, yields

$$\begin{aligned} \nabla x &= \nabla (\underline{e} \cdot X) \\ &= \underline{e} \cdot dX + \nabla \underline{e} \cdot X \\ &= \underline{e} (dX - AX). \end{aligned}$$

Such that horizontal sections of  $(\mathcal{M}, \nabla)$ , i.e. sections which satisfy  $\nabla x = 0$ , correspond to solutions of

$$\frac{d}{dt} x = Ax. \tag{2.1}$$

Thus, with the connection  $\nabla$  and the  $\mathbb{C}(\{t\})$ -basis  $\underline{e}$  is naturally associated the differential operator  $\Delta = d - A$ , which has order one and dimension  $n$ .

**Definition 2.8**

We call (2.1), determined by the differential operator  $\Delta = d - A$ , a *germ of a meromorphic linear differential system*<sup>a</sup> of rank  $n$ , or just a *system*.

**Proposition 2.8.1**

Thus, the set of systems is isomorphic to the set

$$\text{End}(E) \otimes \mathbb{C}(\{t\}) = \mathfrak{gl}_n(\mathbb{C}(\{t\}))$$

of all connection matrices.

<sup>a</sup>Martinet and Ramis call them in [MR91] *germs of meromorphic differential operators*.

Such a system will be denoted by  $[A] = d - A$  and we will call  $A$  the connection matrix of the system  $[A]$ .

If we start with a system  $[A]$  and we want a meromorphic connection  $(\mathcal{M}, \nabla)$ , which has  $A$  as connection matrix, we can do this in the following way.

**Proposition 2.9**

If we start with either a system  $[A]$  of rank  $n$ , or a connection matrix  $A \in \mathfrak{gl}_n(\mathbb{C}(\{t\}))$ , we get a germ of a meromorphic connection via

$$(\mathcal{M}_A, \nabla_A) = (\mathbb{C}(\{t\})^n, d - A)$$

which has  $A$  as its connection matrix.

**Proposition 2.10**

Let  $(\mathcal{M}_1, \nabla_1)$  and  $(\mathcal{M}_2, \nabla_2)$  be two meromorphic connections with the connection matrices  $A_1$  and  $A_2$ . A connection matrix of  $(\mathcal{M}_1, \nabla_1) \oplus (\mathcal{M}_2, \nabla_2)$  is then given by the block-diagonal matrix  $\text{diag}(A_1, A_2)$ .

*Proof.* Use Proposition 2.9 to write the connection as

$$(\mathbb{C}(\{t\})^{n_1}, d - A_1) \oplus (\mathbb{C}(\{t\})^{n_2}, d - A_2)$$

and we want to show that it is isomorphic to

$$\left( \mathbb{C}(\{t\})^{n_1+n_2}, d - \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \right).$$

Denote by  $A := \text{diag}(A_1, A_2)$  the block-diagonal matrix build from  $A_1$  and  $A_2$ . For every  $j \in \{1, 2\}$  we have the corresponding inclusion  $i_j : \mathbb{C}(\{t\})^{n_j} \hookrightarrow \mathbb{C}(\{t\})^{n_1+n_2}$  and the diagram

$$\begin{array}{ccc} \mathbb{C}(\{t\})^{n_j} & \xrightarrow{d-A_j} & \mathbb{C}(\{t\})^{n_j} \\ \downarrow i_j & & \downarrow i_j \\ \mathbb{C}(\{t\})^{n_1+n_2} & \xrightarrow{d-A} & \mathbb{C}(\{t\})^{n_1+n_2} \end{array}$$

which commutes, since the derivation commutes with the inclusion and the matrix  $A$  is build in the correct way, to satisfy  $i_j(A_j x) = A_j(i_j(x))$ :

$$\begin{aligned} i_j(dx - A_j x) &= i_j(dx) - i_j(A_j x) \\ &= d(i_j(x)) - A_j(i_j(x)) \\ &= ((d - A) \circ i_j)(x). \end{aligned}$$

□

### Remark 2.11

Let  $(\mathcal{M}_1, \nabla_1)$  and  $(\mathcal{M}_2, \nabla_2)$  be meromorphic connections. Then

1.  $\mathcal{M}_1 \otimes \mathcal{M}_2$  is endowed with the structure of a meromorphic connection by

$$\nabla(u_1 \otimes u_2) = \nabla_1 u_1 \otimes u_2 + u_1 \otimes \nabla_2 u_2$$

where  $u_i \in \mathcal{M}_i$  and

2.  $\text{Hom}_{\mathbb{C}(\{t\})}(\mathcal{M}_1, \mathcal{M}_2)$  is endowed with the structure of a meromorphic connection by

$$(\nabla \varphi)(u_1) = \nabla_2(\varphi(u_1)) - \varphi(\nabla_1 u_1)$$

where  $\varphi \in \text{Hom}_{\mathbb{C}(\{t\})}(\mathcal{M}_1, \mathcal{M}_2)$  and  $u_i \in \mathcal{M}_i$ .

## 2.2.1 Transformation of systems

### Notations 2.12

We will use the following notations

- $G[t] = \text{GL}_n(\mathbb{C}[t])$ ;
- $G\{t\} = \text{GL}_n(\mathbb{C}\{t\})$  the analytic transformations;

- $G(\{t\}) = \mathrm{GL}_n(\mathbb{C}\{t\}[t^{-1}])$  the meromorphic<sup>a</sup> transformations;
- $G[[t]] = \mathrm{GL}_n(\mathbb{C}[[t]])$  the maybe not applicable formal transformations;
- $G((t)) = \mathrm{GL}_n(\mathbb{C}((t)))$  the maybe not applicable formal meromorphic transformations.

<sup>a</sup>We use the term meromorphic in the sens of convergent meromorphic. Otherwise we say formal meromorphic.

By *meromorphic transformation*, or just *transformation*, of a system we mean a  $\mathbb{C}\{t\}$ -linear change of the trivialization. Such a change is given by a matrix  $F \in G(\{t\})$  and the transformed connection matrix  ${}^F A$  is obtained through the Gauge transformation

$${}^F A = (dF)F^{-1} + FAF^{-1}.$$

If  $F$  is formal i.e.  $F \in G((t))$ , it will usually be denoted by  $\hat{F}$ . The transformation of  $A$  by  $\hat{F}$  is not guaranteed to have convergent entries. We denote by  $\hat{G}(A)$  the set of all (*applicable*) *formal transformations*

$$\hat{G}(A) := \left\{ \hat{F} \in G((t)) \mid \hat{F}A \text{ has convergent entries, i.e. } \hat{F}A \in G(\{t\}) \right\}.$$

Let  $\hat{F}' \in \hat{G}(A^0)$  and  $A' := \hat{F}'A$ , then are the sets  $\hat{G}(A)$  and  $\hat{G}(A')$  related by

$$\hat{G}(A') = \hat{G}(A)\hat{F}'^{-1} = \left\{ \hat{F} \in G(\{t\}) \mid \hat{F}\hat{F}' \in \hat{G}(A) \right\}.$$

### Remark 2.13

The condition

$B$  is obtained from  $A$  by transformation  $F$

is clearly equivalent to

$F$  solves the linear differential system

$$\frac{dF}{dt} = BF - FA$$

which is denoted by  $[A, B]$ .



**Remark 2.13.1**

If we start with

- two base choices  $\mathcal{M} \xrightarrow{\sim} \mathbb{C}(\{t\})^n$  and  $\mathcal{M}' \xrightarrow{\sim} \mathbb{C}(\{t\})^n$  and
- an isomorphism  $\Phi : (\mathcal{M}, \nabla) \xrightarrow{\sim} (\mathcal{M}', \nabla')$  together with the corresponding base change  $F \in G(\{t\})$

we have the following commutative diagram:

$$\begin{array}{ccccc}
 \mathbb{C}(\{t\})^n & \xrightarrow{\quad F \quad} & \mathbb{C}(\{t\})^n & & \\
 & \nwarrow & \nearrow & & \\
 & \mathcal{M} & \xrightarrow{\quad \Phi \quad} & \mathcal{M}' & \\
 & \downarrow \nabla & & \downarrow \nabla' & \\
 & \mathcal{M} & \xrightarrow{\quad \Phi \quad} & \mathcal{M}' & \\
 & \nwarrow & \nearrow & & \\
 \mathbb{C}(\{t\})^n & \xrightarrow{\quad F \quad} & \mathbb{C}(\{t\})^n & & 
 \end{array}$$

$d-B$  (left vertical arrow),  $d-A$  (right vertical arrow),  $F$  (top and bottom horizontal arrows)

Thus the commutation property for the outer rectangle reads

$$(d - B) \circ F = F \circ (d - A)$$

which is equivalent to

$$\frac{dF}{dt} = BF - FA.$$

**Remark 2.14**

The simple but useful rule

$${}^{(F_2 F_1)}A = {}^{F_2}({}^{F_1}A)$$

can be seen by calculation:

$$\begin{aligned}
 {}^{(F_2 F_1)}A &= d(F_2 F_1)(F_2 F_1)^{-1} + F_2 F_1 A (F_2 F_1)^{-1} \\
 &= ((dF_2) F_1 + F_2 (dF_1)) F_1^{-1} F_2^{-1} + F_2 F_1 A F_1^{-1} F_2^{-1} \\
 &= (dF_2) F_2^{-1} + F_2 (dF_1) F_1^{-1} F_2^{-1} + F_2 ({}^{F_1}A - (dF_1) F^{-1}) F_2^{-1} \\
 &= (dF_2) F_2^{-1} + F_2 {}^{F_1}A F_2^{-1} \\
 &= {}^{F_2}({}^{F_1}A) .
 \end{aligned}$$

**Definition 2.15**

We define the *(formal) equivalence relation on the connection matrices (resp. on the corresponding systems)* as

**$A$  is (formally) equivalent to  $B$**

if and only if

**$B$  is obtained from  $A$  by (formal) transformation.**

The *class of a connection matrix* is the orbit under the Gauge transformation by  $G(\{t\})$ . The *formal class* is the orbit by  $\hat{G}(A)$ .

**Remark 2.15.1**

Thus  $A$  is (formally) equivalent to  $B$  if and only if there is a (formal) solution of  $[A, B]$ .

The defined equivalence relations are compatible with the isomorphisms relations on meromorphic connections (cf. [HTT08, Lem.5.1.3]), i.e. the following proposition holds.

**Proposition 2.16**

Two germs of meromorphic connections are (formally) isomorphic if and only if their corresponding connection matrices are (formally) equivalent.

**Definition 2.17**

An *isotropy* of  $A^0$  or of  $[A^0]$  is a transformation  $\hat{F}$  which satisfies  $\hat{F}A^0 = A^0$ . Thus, the isotropies are the solutions of the system  $[\text{End } A^0] := [A^0, A^0]$ .

Let  $G_0(A^0)$  denote the set of all isotropies of  $A^0$ .

**Remark 2.17.1**

The isotropies are, a priori, formal transformations. Loday-Richaud mentions in her paper [Lod94, p. 853], that actually  $G_0(A^0)$  is a subgroup of  $\text{GL}_n(\mathbb{C}[1/t, t])$ .

**Lemma 2.18**

Two formal transformations  $\hat{F}_1$  and  $\hat{F}_2$  take  $A^0$  into equivalent matrices  $\hat{F}_1 A^0$  and  $\hat{F}_2 A^0$  if and only if there exists  $f_0 \in G_0(A^0)$  such that  $\hat{F}_1 = \hat{F}_2 f_0$ .

## Regular / irregular singularities

The meromorphic connections are distinguished into regular and irregular meromorphic connections.

### Definition 2.19

A connection with connection matrix  $A$  has *regular singularity* at 0 if there exists a convergent transformation, by which  $A$  is obtained from a matrix with at most a simple pole at  $t = 0$ . Otherwise, the singularity is called *irregular*.

### Remark 2.19.1

This implies that, if  $A$  has irregular (resp. regular) singularity, then also all meromorphic equivalent matrices  ${}^F A$  have irregular (resp. regular) singularity.

### Theorem 2.20

Let  $(\mathcal{M}, \nabla)$  be a regular singular meromorphic connection and  $A$  its connection matrix. Then there exists a matrix  $F \in G(\{t\})$  such that after transformation by  $F$  the matrix  $B = {}^F A$  is constant i.e.  ${}^F A \in \mathrm{GL}_n(\mathbb{C})$  (cf. [Sab07, Thm.II.2.8] or [HTT08, Sec.5.1.2]).

## 2.2.2 Fundamental solutions and the monodromy of a system

It is well known in the theory of linear ODEs that the set of solutions to a system like (2.1) forms a vector space of dimension  $n$  over  $\mathbb{C}$ , i.e. if  $x_0(t)$  and  $x_{00}(t)$  are two solutions of (2.1) and  $C_1, C_2 \in \mathbb{C}$  are two constants, then is also  $C_1 x_0(t) + C_2 x_{00}(t)$  a solution of the same system.

### Definition 2.21

A *fundamental matrix of (formal) solutions* or *(formal) fundamental solution*  $\mathcal{Y}$  on a sector  $\mathfrak{s}$  of the system  $[A]$  is an invertible  $n \times n$  matrix (with formal entries), which solves  $[0, A]$  on  $\mathfrak{s}$ .

**Remark 2.21.1**

- This means that the columns of  $\mathcal{Y}$  are  $n$   $\mathbb{C}$ -linearly independent solutions of the system  $[A]$  on  $\mathfrak{s}$ .
- Some authors introduce multi-valued solutions, to avoid the restriction to sectors.

*Multi-valued* are functions, which are not single-valued and *single-valued* is a function  $f$ , which satisfies

$$f(t) = f(t \exp(2\pi i)) \quad \text{whenever both sides are defined.}$$

The function  $t \rightarrow t^\alpha$ , for example, is single-valued whenever  $\alpha \in \mathbb{Z}$ .

The Stokes phenomenon, which will be discussed in the chapter 3, results from the fact that there is always a formal fundamental solution, which solves a system on the full arc  $S^1$ . But holomorphic solutions, which are asymptotic to the formal solution, may exist only on small sub-sectors of  $S^1$ .

**Remark 2.22**

If the trivialization is changed by  $F$  (resp.  $\hat{F}$ ) the fundamental solution  $\mathcal{Y} \in G(\{t\})$  changes to  $F\mathcal{Y}$  (resp.  $\hat{F}\mathcal{Y}$ ). (cf. [Lod14, Thm.4.3.1] or [FJ09, p. 2.1.3])

Choose a fundamental solution  $\mathcal{Y}$ . Then analytic continuation along a closed path  $\gamma$  in  $M \setminus Z$  provides another fundamental solution  $\mathcal{Y}' = \rho^{-1}(\gamma)\mathcal{Y}$  where  $\rho(\gamma)$  is called the *monodromy along the path  $\gamma$* .

**Definition 2.23**

Let  $[A]$  be a system with fundamental solution  $\mathcal{Y}$ . The analytic continuation of  $\mathcal{Y}$  along a small circle around  $t = 0$  yields the fundamental solution

$$\lim_{s \rightarrow 2\pi} \mathcal{Y}(e^{\sqrt{-1}s}t) = \mathcal{Y}(t)L$$

where  $L \in \mathrm{GL}_n(\mathbb{C})$  is called the *monodromy matrix* of  $[A]$ .

## 2.3 Formal classification

In every formal equivalence class of meromorphic connections, there are some meromorphic connections of special form, which we will call models. They are not unique but all of them, which are formally isomorphic to a given meromorphic connection, lie

in the same convergent equivalence class. In fact, every element of this convergent equivalence class will be a model in our definition.

The first part is given by the Levelt-Turittin theorem, which says that each meromorphic connection is, after potentially needed ramification, formally isomorphic to such a model. Thus the Levelt-Turittin theorem solves the *formal classification problem*.

### 2.3.1 In the language of meromorphic connections: models

This approach is discussed in [Sab90] and in a more general context in [Sab07, Sec.II.5].

#### Definition 2.24

- For a  $\varphi \in \mathbb{C}(\{t\})$  we use  $\mathcal{E}^\varphi$  to denote the germ

$$(\mathcal{E}^\varphi, \nabla) = (\mathbb{C}(\{t\}), d - \varphi').$$

This corresponds to the system satisfied by the function  $e^\varphi$ , since  $(d - \varphi')e^\varphi = 0$ .

#### Corollary 2.24.1

$\mathcal{E}^\varphi$  is determined by the class of  $\varphi$  in  $\mathbb{C}(\{t\})/\mathbb{C}\{t\} \cong t^{-1}\mathbb{C}[t^{-1}]$ . In the following, we will only consider the unique ambassador  $\varphi$  in each class which has no holomorphic part.

- For  $\alpha \in \mathbb{C}$ , define the *elementary regular meromorphic connection of rank one*  $\mathcal{N}_{\alpha,0}$  as the germ

$$(\mathcal{N}_{\alpha,0}, \nabla) = \left( \mathbb{C}(\{t\}), d + \frac{\alpha}{t} \right).$$

This corresponds to the system satisfied by  $t^{-\alpha}$ .

An *elementary regular model of arbitrary rank* is a meromorphic connection which has a basis, in which the connection Matrix can be written as

$$\frac{1}{t}(\alpha \text{id} + N)$$

where  $N \in \text{gl}_{d+1}(\mathbb{C})$  is a nilpotent matrix. If  $\alpha \text{id} + N$  is a single Jordan Block, we denote the corresponding connection by  $\mathcal{N}_{\alpha,d}$ .

**Remark 2.24.1**

The Corollary [Sab07, Cor.II.2.9] states, that every germ of a regular meromorphic connection  $(\mathcal{R}, \nabla)$  is isomorphic to some direct sum

$$(\mathcal{R}, \nabla) = \bigoplus_{\alpha, d} (\mathcal{N}_{\alpha, d}, \nabla).$$

of elementary regular meromorphic connections. This solves the classification problem, for regular meromorphic connections. For a detailed analysis of regular meromorphic connections see Sabbah's book [Sab07, Sec.II.2] or the book [HTT08, Sec.5.2] from Hotta et al.

**Definition 2.25**

A germ  $(\mathcal{M}, \nabla)$  is called *elementary* if it is isomorphic to some germ  $(\mathcal{E}^\varphi, \nabla) \otimes (\mathcal{R}, \nabla)$  where

- $(\mathcal{E}^\varphi, \nabla)$  is defined as in definition 2.24 and
- $(\mathcal{R}, \nabla)$  has regular singularity at  $\{0\}$ , i.e. is isomorphic to a direct sum of regular elementary meromorphic connections.

**Definition 2.26**

A germ  $(\mathcal{M}', \nabla')$  is a *model* if there exists, after ramification  $\mathcal{M} = \pi^* \mathcal{M}'$  by  $\pi$ , a isomorphism to a direct sum of elementary meromorphic connections:

$$\lambda : (\mathcal{M}, \nabla) \xrightarrow{\cong} \bigoplus_{\varphi} \mathcal{E}^\varphi \otimes \mathcal{R}_\varphi.$$

• is irregular singular

• has regular singularity at  $\{0\}$

•  $\varphi \in t^{-1}\mathbb{C}[t^{-1}]$  pairwise distinct

**Remark 2.26.1**

Here it is not necessary to understand ramification, since we restrict to the unramified case.

The important theorem here is the Levelt-Turittin theorem, which solves the formal classification problem.

**Theorem 2.28: Levelt-Turittin**

To each germ  $(\mathcal{M}', \nabla')$  of a meromorphic connection there exists, after potentially needed pullback  $\pi^*\mathcal{M}' =: \mathcal{M}$  by some suitable ramification  $t = z^q$  of order  $q \geq 1$ , a **formal** isomorphism

$$\widehat{\lambda} : \widehat{\mathcal{M}} \xrightarrow{\cong} \widehat{\mathcal{M}}^{nf} := \widehat{\mathcal{O}}_M \otimes \mathcal{M}^{nf}$$

to a model  $\mathcal{M}^{nf}$ . We then call  $\mathcal{M}^{nf}$  a *formal decomposition* or *formal model* of  $\mathcal{M}$  or  $\mathcal{M}'$ .

**Remark 2.28**

But there is in general **no** lift of the isomorphism  $\widehat{\lambda}$ , i.e. there is no isomorphism making the diagram

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{?} & \mathcal{M}^{nf} \\ \downarrow & & \downarrow \\ \widehat{\mathcal{M}} & \xrightarrow{\widehat{\lambda}} & \widehat{\mathcal{M}}^{nf} \end{array}$$

kommutative. But sectorwise, there are lifts given by the main asymptotic existence theorem (cf. Theorem 2.35).

Proofs of this theorem can be found in multiple places, for example in [Sab90, Thm.5.4.7].

**2.3.2 In the language of systems: normal forms**

Here we will see, that the fundamental solution of a model can be written in a special form and that this property characterizes models.

**Definition 2.29**

The fundamental solution corresponding to a model (resp. a normal form, see Definition 2.32) will be called a *normal solution*.

Let  $(\mathcal{M}, \nabla)$  be an unramified meromorphic connection which is via some  $\widehat{\lambda}$  formally isomorphic to some model  $\mathcal{M}^{nf} = \bigoplus_{j=1}^s \mathcal{E}^{\varphi_j} \otimes \mathcal{R}_j$ . For every  $j$  is a connection matrix, corresponding to  $\mathcal{R}_j$  given by  $\frac{1}{t}L_j$ , where  $L$  is in Jordan normal form. A connection matrix to  $\mathcal{E}^{\varphi_j} \otimes \mathcal{R}_j$  is then by  $q'_j(t^{-1}) \cdot \text{id}_{n_j} + \frac{1}{t}L_j$  given, where

$q_j(t^{-1}) = \varphi_j(t)$  and  $q'_j(t^{-1}) = \frac{d}{dt}\varphi_j(t)$ <sup>[3]</sup>. A connection matrix for  $\mathcal{M}^{nf}$  is then obtained by

$$\begin{aligned} A^0 &= \bigoplus_{j=1}^s \left( q'_j(t^{-1}) \cdot \text{id}_{n_j} + \frac{1}{t} L_j \right) \\ &= \bigoplus_{j=1}^s q'_j(t^{-1}) \cdot \text{id}_{n_j} + \frac{1}{t} \bigoplus_{j=1}^s L_j \\ &= Q'(t^{-1}) + \frac{1}{t} L, \end{aligned}$$

where  $Q(t^{-1}) := \bigoplus_{j=1}^s q_j(t^{-1}) \cdot \text{id}_{n_j}$  and  $L := \bigoplus_{j=1}^s L_j$ .

### Proposition 2.30

Let  $L \in \text{GL}_n(\mathbb{C})$  be constant and  $Q(t^{-1}) = \text{diag}(q_1(t^{-1}), \dots, q_n(t^{-1}))$  a diagonal matrix of polynomials in  $t^{-1}$ . The matrix  $\mathcal{Y}_0 := t^L e^{Q(t^{-1})}$  is then a fundamental solution of the system determined by the matrix

$$A^0 = t^L Q'(t^{-1}) t^{-L} + L \frac{1}{t}.$$

*Proof.* It is easy to see that  $\frac{d}{dt} e^{Q(t^{-1})} = Q'(t^{-1}) e^{Q(t^{-1})}$  and  $\frac{d}{dt} t^L = L \frac{1}{t} t^L$ , thus we can prove that  $\mathcal{Y}_0$  is a matrix consisting of solutions:

$$\begin{aligned} \frac{d}{dt} \mathcal{Y}_0 &= \frac{d}{dt} \left( t^L e^{Q(t^{-1})} \right) \\ &= t^L \frac{d}{dt} e^{Q(t^{-1})} + \frac{d}{dt} t^L e^{Q(t^{-1})} \\ &= t^L Q'(t^{-1}) e^{Q(t^{-1})} + L \frac{1}{t} t^L e^{Q(t^{-1})} \\ &= \left( t^L Q'(t^{-1}) t^{-L} + L \frac{1}{t} \right) t^L e^{Q(t^{-1})} \\ &= A^0 \mathcal{Y}_0. \end{aligned}$$

The invertability condition is clear. □

The matrices  $L$  and  $Q$  defined above satisfy  $t^L Q'(t^{-1}) t^{-L} = Q'(t^{-1})$  since the block structure of  $L$  is finer than the structure of  $Q(t^{-1})$ , and thus the structure of  $Q'(t^{-1})$ . Thus one can look at the problem  $t^L Q'(t^{-1}) t^{-L} = Q'(t^{-1})$  blockwise, where  $Q'(t^{-1})$  can be written as  $q'(t^{-1}) \cdot \text{id}$  for some  $q'(t^{-1})$ . The equation is then clearly satisfied. Using this fact and the Proposition 2.30 we have, that  $\mathcal{Y}_0 := t^L e^{Q(t^{-1})}$  is a fundamental for the system corresponding to the matrix  $A^0$  defined above.

<sup>[3]</sup>By abuse of notation we denote by  $q'_j(t^{-1})$  the derivation  $\frac{d}{dt}(q_j(t^{-1})) = \left( \left( \frac{d}{dt} q_j \right) \circ t^{-1} \right) \cdot \frac{d}{dt} t^{-1}$ . The same does apply for  $Q'(t^{-1})$ .



To  $\hat{\lambda}$  and the (implicitly) used choices of bases corresponds a formal transformation  $\hat{F}$  and thus we have

- a fundamental solution  $\hat{F}t^Le^{Q(t^{-1})}$  and
- a connection matrix  $\hat{F}A^0$

of  $(\mathcal{M}, \nabla)$  defined at the beginning of this subsection.

### Corollary 2.31

From the Levelt-Turittin theorem we deduce that every meromorphic connection  $(\mathcal{M}, \nabla)$  and thus every system  $[A]$  has a fundamental solution of the form

$$\mathcal{Y} = \hat{F}t^Le^{Q(t^{-1})}$$

where  $\hat{F}$  is a formal transformation, solving the equation corresponding to the formal isomorphism  $(\mathcal{M}^{nf}, \nabla^{nf}) \rightarrow (\mathcal{M}, \nabla)$ . In the other way is a system, which has  $\mathcal{Y}$  as fundamental solution is given by

$$\bar{A} = \hat{F} \left( Q'(t^{-1}) + L \frac{1}{t} \right).$$

In the special case when  $\hat{\lambda}$  is a convergent isomorphism, i.e. in the case, when  $A$  is already a model, we see that it has a fundamental solution of the form  $Ft^Le^{Q(t^{-1})}$ , where  $F$  is a convergent transformation corresponding to  $\hat{\lambda}$ . In fact are the models uniquely characterized as the meromorphic connections, with a fundamental solution which can be written as  $\mathcal{Y}_0 = Ft^Le^{Q(t^{-1})}$  and we will use this fact, to say when a system is something like a model.

### Definition 2.32

Let  $[A]$  be a system. We call  $[A]$  (or  $A$ ) a *normal form* if a fundamental solution of  $[A]$  can be written as

$$\mathcal{Y}_0(t) = Ft^Le^{Q(t^{-1})}$$

with

- an *irregular part*  $e^{Q(t^{-1})}$  of  $\mathcal{Y}_0$  determined by

$$Q(t^{-1}) = \bigoplus_{j=1}^s q_j(t^{-1}) \text{id}_{n_j} = \text{diag}(\underbrace{q_1, \dots, q_1}_{n_1\text{-times}}, q_2, \dots, q_s)$$

where the  $q_i(t^{-1})$  are polynomials in  $\frac{1}{t}$  (or in a fractional power  $\frac{1}{s} = \frac{1}{t^{1/p}}$  of  $\frac{1}{t}$  for the ramified case) such that  $q_j(0) = 0$ , i.e. without constant term,

- a constant matrix  $L \in \mathfrak{gl}_n(\mathbb{C})$  called the *matrix of formal monodromy*, where  $t^L$  means  $e^{L \ln t}$

**Remark 2.32.1**

Since we only look at unramified connections, we are able to assume that  $L$  is in Jordan normal form, since in this case  $Q$  and  $L$  commute. It then has a block structure, which is finer than the structure of  $Q = \bigoplus_{j=1}^s q_j(t^{-1}) \text{id}_{n_j}$  (cf. [Rem14, Sec.1] or [MR91, Sec.4]).

and

- a (convergent) transformation  $F \in G(\{t\})$ .

**Remark 2.32.2**

By changing the basis via  $F^{-1}$  we obtain from  $[A^0]$  the equivalent system  $[F^{-1}A^0]$  with the normal solution

$$F^{-1}\mathcal{Y}_0(t) = t^L e^{Q(t^{-1})}.$$

Thus it is always possible to assume that the transformation matrix  $F$  is trivial.

The normal forms will often be denoted  $A^0$ . If  $A$  is formally equivalent to a normal form  $A^0$  we say that  $A^0$  is a *normal form for  $A$*  and for  $[A]$ .

**Corollary 2.33**

A meromorphic connection  $(\mathcal{M}, \nabla)$  is a model if and only if its connection matrix  $A$  is a normal form.

## 2.4 The main asymptotic existence theorem (M.A.E.T)

Here we want to state the main asymptotic expansion theorem (or often M.A.E.T.) which is essentially a deduction from the Borel-Ritt lemma. It states that to every formal solution of a system of meromorphic differential equations and every sector with sufficiently small opening, one can find a holomorphic solution of the system having the formal one as its asymptotic expansion.

**Definition 2.34**

Let  $A$  be via  $\widehat{F}$  formally equivalent to  $A^0$ . We call  $F$  a *lift of  $\widehat{F}$  on  $I \subset S^1$*  if

- $F \sim_I \widehat{F}$  (cf. Page 3) and
- $F$  satisfies the same system  $[A^0, A]$  as  $\widehat{F}$ .

The following theorem, often called the main asymptotic existence theorem, can be for example found in as Theorem A in the paper [BJL79] from Balser, Jurkat and Lutz, Theorem 3.1 in Boalch's paper [Boa01, Thm.3.1] or Theorem 4.4.1 in Loday-Richaud's Book [Lod14].

**Theorem 2.35: M.A.E.T**

To every  $\widehat{F} \in G((t))$  and to every small enough arc  $I \subsetneq S^1$  there exists a lift  $F$  on  $I$ .

**Remark 2.35.1**

If we write the system  $[A^0, A]$  as a differential operator  $D$ . The theorem then states that  $D$  acts linearly and surjectively on the sheaf  $\mathcal{A}^{<0}$ , i.e. the sequence

$$\mathcal{A}^{<0} \xrightarrow{D} \mathcal{A}^{<0} \longrightarrow 0$$

are exact sequences of sheaves of  $\mathbb{C}$ -vector spaces.

## 2.5 The classifying set

We want to understand the Set  $\left\{[(\mathcal{M}, \nabla)]\right\}$  of the (convergent) isomorphism classes of all meromorphic connections. Since we can use the formal classification (cf. Section 2.3) and we know that all elements in a convergent isomorphism class lie in the same formal isomorphism class (In other words: the convergent classification is finer than the formal classification) we can reduce the problem by fixing a model  $(\mathcal{M}^{nf}, \nabla^{nf})$  with the corresponding normal form  $A^0$ . Thus we can restrict ourself to the subset

$${}^0C(\mathcal{M}^{nf}, \nabla^{nf}) = \left\{[(\mathcal{M}, \nabla)] \mid \text{there exists a formal isomorphism} \right. \\ \left. \widehat{f} : (\widehat{\mathcal{M}}, \widehat{\nabla}) \xrightarrow{\sim} (\widehat{\mathcal{M}}^{nf}, \widehat{\nabla}^{nf}) \right\}$$

of all isomorphism classes of meromorphic connections, which are formally isomorphic to  $(\mathcal{M}^{nf}, \nabla^{nf})$ . This is the set that we will be calling the *classifying set* (to  $(\mathcal{M}^{nf}, \nabla^{nf})$ )

and we will also denote it also by  ${}^0C(A^0)$ , if we are using the language of systems.

**Remark 2.36**

Note that we classify

meromorphic connections within fixed **formal meromorphic classes, modulo meromorphic equivalence**.

Whereas for example Boalch in [Boa01] and [Boa99] classifies

meromorphic connections within fixed **formal analytic classes, modulo analytic equivalence**

as it was done in the older literature. This makes no difference, since the resulting classifying sets are isomorphic (cf. [Boa99] or [BV89]).

This distinction relates to the difference between ‘**regular singular**’ connections and ‘**logarithmic**’ connections.

It is convenient to look at the slightly larger space of *marked (meromorphic) pairs*

$$\mathcal{H} = \mathcal{H}(\mathcal{M}^{nf}, \nabla^{nf}) = \left\{ [(\mathcal{M}, \nabla, \hat{f})] \mid \hat{f} : (\widehat{\mathcal{M}}, \widehat{\nabla}) \xrightarrow{\sim} (\widehat{\mathcal{M}}^{nf}, \widehat{\nabla}^{nf}) \right\}$$

in which we also handle the additional information, of the formal isomorphism, by which the meromorphic connection is formally isomorphic to the model. Where the isomorphisms of marked pairs are defined as follows:

**Definition 2.37**

Two germs  $(\mathcal{M}, \nabla, \hat{f})$  and  $(\mathcal{M}', \nabla', \hat{f}')$  are isomorphic if there exists an isomorphism  $g : (\mathcal{M}, \nabla) \xrightarrow{\sim} (\mathcal{M}', \nabla')$  such that  $\hat{f} = \hat{f}' \circ \hat{g}$ .

**Remark 2.37.1**

Sabbah states in [Sab07, p. 111] that such an isomorphism is then unique.

Equivalently, one can talk in terms of systems. We then denote by

$$\text{Syst}_m(A^0) := \{[A] \mid A = \hat{F}A^0 \text{ for some } \hat{F} \in G((t))\}$$

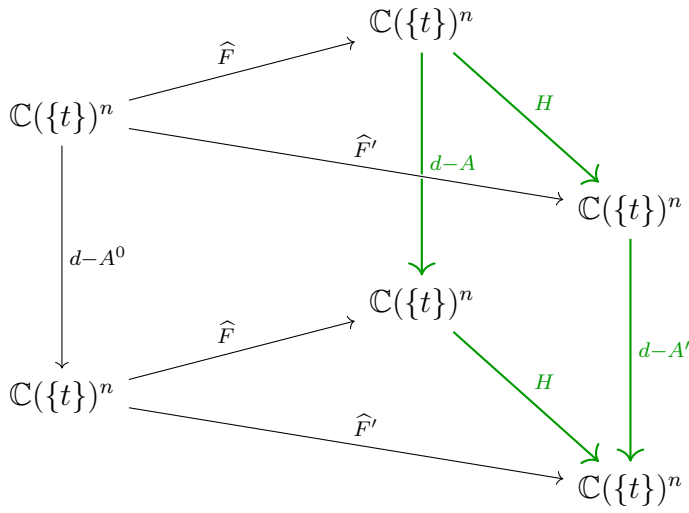
the set of systems formally meromorphic equivalent to  $A^0$ . Since we use meromorphic equivalences, in contrast to [Boa01; Boa99], we denote that in  $\text{Syst}_m$  by the subscript  $m$ . Thus  ${}^0C(\mathcal{M}^{nf}, \nabla^{nf})$  corresponds to the set  ${}^0C(A^0) := \text{Syst}_m(A^0)/G(\{t\})$  of meromorphic classes which are formally equivalent to  $A^0$ . Analogous,  $\mathcal{H}(\mathcal{M}^{nf}, \nabla^{nf})$  corresponds to the set  $\mathcal{H}(A^0)$  of equivalence classes, i.e. orbits of  $G(\{t\})$ , in

$$\widehat{\text{Syst}}_m(A^0) := \left\{ (A, \hat{F}) \mid A = \hat{F}A^0 \text{ for some } \hat{F} \in G((t)) \right\}.$$

Especially follows, that two marked pairs  $(A, \widehat{F})$  and  $(A', \widehat{F}')$  are equivalent, if and only if there is a base change  $H \in G(\{t\})$  such that

- $A' = {}^H A$ , i.e.  $H$  is a solution of  $[A, A']$ , and
- $\widehat{F}' = \widehat{H}\widehat{F}$ .

In the following diagram is the first condition equivalent to the commutation property of the green square. The second property corresponds to the commutation property of the top (resp. bottom) triangle after formalization.



### Lemma 2.38

Since  $G_0(A^0)$  is by definition the stabilizer of  $A^0$  (cf. definition 2.17) and  $\text{Syst}_m(A^0)$  is the corresponding orbit we can use the Proposition [Wie64, Prop.3.1] in the book from Wielandt we deduce

$$\text{Syst}_m(A^0) \cong \widehat{G}(A^0)/G_0(A^0).$$

#### Corollary 2.38.1

Thus the *set of meromorphic classes of systems formally equivalent to  $A^0$*  are just the orbits of  $G(\{t\})$  in  $\text{Syst}_m(A^0)$  that is

$${}^0C(A^0) \cong G(\{t\}) \backslash \widehat{G}(A^0) / G_0(A^0)$$

whereas the *set of meromorphic classes of transformations  $\mathcal{H}(A^0)$  of  $[A^0]$*  is canonically isomorphic to the left quotient  $G(\{t\}) \backslash \widehat{G}(A^0)$  (cf. [Boa99, Lem.1.17]).

The group  $G_0(A^0)$  is easy to compute and is often trivial. In fact, the elements are block-diagonal, see [Lod14, p. 77]. Thus the structure of  ${}^0C(A^0)$  is easily deduced from the structure of  $G(\{t\})\backslash\widehat{G}(A^0)$ .

### 3 Stokes Structures

Stokes structures contain exactly necessary information to classify meromorphic classification, i.e. with the Stokes structures we are able to construct a space, which is isomorphic to the classifying set.

A great overview of this topic is given by Varadarajan in [Var96]. Other resources we will use are for example Sabbah's book [Sab07, section II] for section 3.1. For the sections 3.2 and 3.3 will Loday-Richaud's paper [Lod94] and the book [Lod14] be useful. Also useful was Boalch's paper [Boa01] (resp. his thesis [Boa99]) which looks only at the single leveled case.

Let  $(\mathcal{M}^{nf}, \nabla^{nf})$  be a fixed model with the corresponding normal form  $A^0$ . The first objective is to proof the Malgrange-Sibuya theorem in section 3.1. It states that the classifying set  $\mathcal{H}(A^0)$  is via an map  $\exp$  isomorphic to the first cohomology  $H^1(S^1; \Lambda(A^0)) =: \mathcal{St}(A^0)$ . In section 3.3 we will improve the Malgrange-Sibuya Theorem by showing that each 1-cohomology class in  $\mathcal{St}(A^0)$  contains a unique 1-cocycle of a special form called *the Stokes cocycle* (cf. Section 3.2). The morphism, which maps each Stokes cocycle to its corresponding 1-cocycle will be denoted by  $h$ .

We have the following commutative diagram of isomorphisms.

$$\begin{array}{ccc}
 \mathcal{H} & \xrightarrow{\text{exp}} & \mathcal{St}(\mathcal{M}^{nf}) \\
 \parallel & & \nearrow \\
 \mathcal{H}(A^0) & \xrightarrow{\text{exp}_{A^0}} & \mathcal{St}(A^0) \\
 \downarrow g & \nearrow h & \\
 \prod_{\theta \in \mathbb{A}} \text{Sto}_{\theta}(A^0) & & 
 \end{array}$$

The map  $g$ , which makes the diagram commute, is defined using the theory of summation, which will be roughly discussed the appendix in section A.3.

### 3.1 Stokes structures: Malgrange-Sibuya isomorphism

Here we will look at the classifying set and we will proof that it is isomorphic to the first non abelian cohomology set  $H^1(S^1; \Lambda(A^0)) =: \mathcal{St}(A^0)$ . If we talk about cocycles or cochains, we will in the following always mean 1-cocycles or 1-cochains.

Let us first define the Stokes sheaf  $\Lambda(A^0)$  on  $S^1$ , as the sheaf of flat isotropies of  $[A^0]$ .

#### Definition 3.1

The Stokes sheaf  $\Lambda(A^0)$  of  $A^0$ , is defined as the subsheaf of  $\mathrm{GL}_n(\mathcal{A})$ , in the following way. For some  $\theta \in S^1$  is the stalk at  $\theta$  the subgroup of  $\mathrm{GL}_n(\mathcal{A})_\theta$  of elements  $f$  which satisfy

1. Multiplicatively flatness:  $f$  is asymptotic to the identity, i.e.  $f \sim_{\mathfrak{s}} 1$ ;
2. Isotropy of  $A^0$ :  ${}^f A^0 = A^0$ .

#### Remark 3.1.1

This definition makes also sense for systems wich are not in normal form.

#### Remark 3.1.2

Sabbah [Sab07, p. 110] talks about (global) meromorphic connections  $\mathcal{M}$  on a small disk  $D$  around 0 instead of germs of meromorphic connections.

Define on  $S^1$  the sheaf  $\mathrm{Aut}^{<0}(\widetilde{\mathcal{M}}^{nf})$  of automorphisms of  $\widetilde{\mathcal{M}}^{nf} = \mathcal{A}_D \otimes_{\mathcal{O}_D} \mathcal{M}^{nf}$  which

- are compatible with the connection and
- are formally equal to the identity, i.e. induce the identity on  $\widehat{\mathcal{M}}^{nf} = \widehat{\mathcal{O}}_D \otimes_{\mathcal{O}_D} \mathcal{M}^{nf}$

The sheaf  $\mathrm{Aut}^{<0}(\widetilde{\mathcal{M}}^{nf})$  corresponds to our  $\Lambda(A^0)$ .

#### 3.1.1 The theorem

In the language of meromorphic connections is the map, in the Malgrange-Sibuya Theorem bellow, described as follows.



Let  $(\mathcal{M}, \nabla, \hat{f})$  be a marked germ of a meromorphic connection. By Theorem 2.35 there exists an open covering  $\mathcal{U} = (U_j)_{j \in J}$  and for every open set, an isomorphism

$$f_j : (\widetilde{\mathcal{M}}, \widetilde{\nabla})|_{U_j} \rightarrow (\widetilde{\mathcal{M}}^{nf}, \widetilde{\nabla}^{nf})|_{U_j}$$

such that  $f_j \sim_{U_j} \hat{f}$ . By  $(f_k f_j^{-1})_{jk}$  is then a cocycle of the sheaf  $\mathcal{S}t(A^0)$ , relative to the covering  $\mathcal{U}$ , defined. This defines a mapping of pointed sets

$$\mathcal{H} \rightarrow H^1(S^1; \Lambda(A^0))$$

to the first non abelian cohomology of  $\Lambda(A^0)$ , which sends the class of  $(\mathcal{M}^{nf}, \nabla^{nf}, \hat{\text{id}})$  to that of  $\text{id}$ , i.e. the trivial cohomology class.

### Theorem 3.2: Malgrange-Sibuya

The homomorphism

$$\exp : \mathcal{H} \longrightarrow \mathcal{S}t(A^0) := H^1(S^1; \Lambda(A^0))$$

is an isomorphism of pointed sets.

### Remark 3.3

The theorem [BV89, Thm.III.1.1.2], in the book from Babbitt and Varadarajan, states that  $\mathcal{S}t(A^0)$  is actually a local moduli space for marked pairs, which are formally isomorphic to a given connection  $(\mathcal{M}^{nf}, \nabla^{nf})$ . This means that

- the morphism property,
- the criterion of equivalence and
- the existence of universal families

are satisfied (cf. [BV89, p. 169]). In fact is the whole third part of [BV89] dedicated to this topic.

### The theorem (system version)

Since the language of meromorphic connections is equivalent to the one of systems, there is also the translated version of the Malgrange-Sibuya isomorphism to the language of systems. The corresponding map is then build as follows.

Let  $(A, \widehat{F})$  be a marked pair, thus  $\widehat{F}$  solves  $[A^0, A]$ . By the M.A.E.T there exists an open covering  $\mathcal{U} = (U_j)_{j \in J}$  together with, for every open set  $U_j$  a lift  $F_j \in \mathrm{GL}_n(\mathcal{A}(U_j))$  of  $\widehat{F}$  (cf. Definition 2.34), which solves  $[A, A^0]$ . By the cocycle  $(F_l^{-1}F_j)_{jl} \in \Gamma(\mathcal{U}; \Lambda(A^0))$  is then a cohomology class in  $\mathcal{S}t(A^0)$ , relative to the covering  $\mathcal{U}$ , determined. For other lifts  $F'_j$  of  $\widehat{F}$  on  $U_j$  is  $(G_j = F_j^{-1}F'_j)$  a 0-cochain of  $\Lambda(A^0)$  relative to  $\mathcal{U}$ , which satisfies

$$F_k^{-1}F_j = G_k F'_k{}^{-1} F'_j G_j^{-1}.$$

Thus the cochians associated to  $(F_j)$  and  $(F'_j)$  determine the same cohomology class in  $\mathcal{S}t(A^0)$ . One can also check that, if  $(A, \widehat{F})$  and  $(A', \widehat{F}')$  are equivalent, the corresponding cocycles define the same cohomology class. This defines a welldefined mapping of pointed sets

$$\mathcal{H}(A^0) \rightarrow H^1(S^1; \Lambda(A^0))$$

to the first non abelian cohomology of  $\Lambda(A^0)$ , which we call  $\exp$ .

### Theorem 3.4: Malgrange-Sibuya (system version)

The homomorphism

$$\exp : \mathcal{H}(A^0) \longrightarrow \mathcal{S}t(A^0) := H^1(S^1; \Lambda(A^0))$$

is an isomorphism of pointed<sup>a</sup> sets.

<sup>a</sup>Which maps the class of  $(A^0, \widehat{\mathrm{id}})$  to that of  $\mathrm{id}$ , i.e. the trivial cohomology class.

### Remark 3.5

The theorem [BV89, Thm.III.1.1.2], in the book from Babbitt and Varadarajan, states that  $\mathcal{S}t(A^0)$  is actually a local moduli space for marked pairs, which are formally isomorphic to a given system  $[A^0]$ . This means that

- the morphism property,
- the criterion of equivalence and
- the existence of universal families

are satisfied (cf. [BV89, p. 169]). In fact is the whole third part of [BV89] dedicated to this topic.

Since the morphism  $\exp$  depends on the choice of the normal form, we will denote that, if it is not clear, by  $\exp_{A^0} = \exp$ .

**Remark 3.6**

To another normal form  $A^1 = \Phi A^0$  there correspond cochains, which are conjugated via  $\Phi$ . We get the following commutative diagram:

$$\begin{array}{ccc}
 G \backslash \widehat{G}(A^1) & \xrightarrow{\cdot \Phi} & G \backslash \widehat{G}(A^0) \\
 \downarrow \wr & & \downarrow \wr \\
 \widehat{F} & \xrightarrow{\quad} & \widehat{F} \Phi \\
 \exp_{A^1} \downarrow & & \exp_{A^0} \downarrow \\
 H^1(S^1; \lambda(A^1)) & \xrightarrow{\quad} & H^1(S^1; \lambda(A^0)) \\
 \downarrow \wr & & \downarrow \wr \\
 \exp_{A^1}(\widehat{F}) & \xrightarrow{\quad} & \exp_{A^0}(\widehat{F})
 \end{array}$$

where  $\exp_{A^0}(\widehat{F} \Phi) = \Phi^{-1} \exp_{A^0}(\widehat{F}) \Phi$ .

**3.1.2 Proof of Theorem 3.4**

We will mainly refer to [BV89, Proof of Theorem 4.5.1] and [Sab07, Section 6.d], where a slightly more complicated case with deformation space is proven. These both resources proof the theorem using the languages of meromorphic connections whereas we will use systems.

We will start by proving the injectivity of the morphism  $\exp$ .

*Proof of the injectivity.* Consider the two marked pairs  $(A, \widehat{F})$  and  $(A', \widehat{F}')$  in  $\widehat{\text{Syst}}_m(A^0)$ , whose classes in  $\mathcal{H}(A^0)$  get mapped to same element

$$\exp((A, \widehat{F})) = \lambda = \exp((A', \widehat{F}')) \in H^1(S^1; \Lambda(A^0)).$$

By using refined coverings, it is possible to find a finite covering  $\mathcal{U} = \{U_j; j \in J\}$  of  $S^1$  such that  $\lambda$  is the class of the cocycles  $(F_l^{-1} F_j)$  and  $(F_l'^{-1} F_j')$ , where  $F_j$  (resp.  $F_j'$ ) are lifts of  $\widehat{F}$  (resp.  $\widehat{F}'$ ) on  $U_j \in \mathcal{U}$ . Since  $[(F_l^{-1} F_j)] = [(F_l'^{-1} F_j')]$  by assumption there exists a 0-cochain  $(G_j)_{j \in J}$  of the sheaf  $\Lambda(A^0)$  relative to the covering  $\mathcal{U}$ , such that

$$F_l'^{-1} F_j' = G_l F_l^{-1} F_j G_j^{-1} \text{ on the arc } U_j \cap U_l,$$

which can be rewritten to

$$F'_j G_j F_j^{-1} = F'_l G_l F_l^{-1} \text{ on the arc } U_j \cap U_l.$$

If we set  $H_j := F'_j G_j F_j^{-1}$  on  $U_j$ , we get

- a solution of  $[A, A']$  on every  $U_j$ , i.e. it satisfies there  $^{H_j}A = A'$ , since

$$\begin{aligned} ^{H_j}A &= F'_j G_j F_j^{-1} A \\ &= F'_j G_j A^0 && (\text{since } F'_j \text{ is a lift of } \widehat{F}' \text{ on } U_j) \\ &= F'_j A^0 && (\text{since } G_j \text{ is an isotropy of } A^0) \\ &= A' && (\text{since } F_j \text{ is a lift of } \widehat{F} \text{ on } U_j) \end{aligned}$$

and

- which satisfies  $\widehat{F}' = \widehat{H}_j \widehat{F}$  on every  $U_j$ , since

$$\begin{aligned} \widehat{H}_j \widehat{F} &= \widehat{F'_j G_j F_j^{-1} \widehat{F}} \\ &= \widehat{F'_j} \underbrace{\widehat{G_j}}_{\parallel \text{id}} \widehat{F_j^{-1} \widehat{F}} && (\text{since } G_j \text{ is flat, i.e. } \widehat{G_j} = \text{id}) \\ &= \widehat{F'} \underbrace{\widehat{F_j^{-1} \widehat{F}}}_{\parallel \text{id}} \\ &= \widehat{F'} \end{aligned}$$

Therefore are  $(A, \widehat{F})$  and  $(A, \widehat{F}')$  equivalent (cf. page 27) and injectivity is proven.  $\square$

For the proof of the surjectivity we will use another result from Malgrange and Sibuya, which is also called the Malgrange-Sibuya Theorem. It can for example be found in Babbitt and Varadarajans's book [BV89, 65ff] as Theorem 4.2.1.

Let  $\widehat{F} \in G((t))$  be a matrix with formally meromorphic entries. By the Borel-Ritt Lemma (cf. Theorem 1.10) we then know, that there exists for every sector  $\mathfrak{s} \subsetneq S^1$  a holomorphic function  $G : \mathfrak{s} \rightarrow \text{gl}_n(\mathbb{C})$  which is asymptotic to  $\widehat{F}$ . We will denote the set of all such holomorphic functions, which are on the arc  $I$  asymptotic to  $\text{id} \in G((t))$  by

$$\mathcal{G}(I) = \{G \in \text{GL}_n(\mathcal{A}(I)) \mid g \sim_I \text{id}\},$$

and this defines a sheaf  $\mathcal{G}$  on  $S^1$ . The statement of the (second) Malgrange-Sibuya Theorem is then, that the difference between formal and konvergent invertible matrices is described by the first sheaf cohomology  $H^1(S^1; \mathcal{G})$  of  $\mathcal{G}$  via the map

$$\Theta : G[[t]]/G\{t\} \rightarrow H^1(S^1; \mathcal{G}),$$

which will turn out to be an isomorphism. It is set up as follows:

Let  $[\hat{F}] \in G[[t]]/G\{t\}$  with ambassador  $\hat{F}$  and let  $\mathcal{U} = \{U_j \mid j \in J\}$  be a finite covering of  $S^1$  by open arcs. The Borel-Ritt Lemma yields for every arc  $j \in J \subsetneq S^1$  a holomorphic function  $F_j$  which satisfies  $F_j \sim_{U_j} \hat{F}$ . By  $(F_l F_j^{-1})_{j,l \in J}$  is then a cocycle for  $\mathcal{G}$  defined, and write  $\Theta([\hat{F}])$  for the corresponding cohomology class.

This construction is similar to the definition of the map of Theorem 3.4. The difference is, that we instead of M.A.E.D, to obtain lifts in the sense of Definition 2.34, we use only the Borel-Ritt Lemma to obtain only asymptotic lifts.

It can be verified, that the class  $\Theta([\hat{F}])$  does not depend on

- the choice of an ambassador  $\hat{F}$  in  $[\hat{F}] \in G[[t]]/G\{t\}$ ,
- the choice of the covering  $\mathcal{U}$  nor
- the choice the  $F_j$ .

### Lemma 3.7

The mapping  $\Theta$  is injective.

*Proof.* Let  $\hat{F}$  and  $\hat{F}' \in G[[t]]$  such that  $\Theta([\hat{F}]) = \Theta([\hat{F}'])$ . We then can find a covering  $\mathcal{U} = \{U_j \mid j \in J\}$  together with holomorphic functions  $F_j$  and  $F'_j$ , which satisfy  $F_j \sim_{U_j} \hat{F}$  and  $F'_j \sim_{U_j} \hat{F}'$ , such that  $(F_l^{-1} F_j)_{j,l \in J}$  and  $(F_l'^{-1} F'_j)_{j,l \in J}$  determine the classes  $\Theta([\hat{F}])$  and  $\Theta([\hat{F}'])$ . This implies that there are maps  $G_j$ , which are on  $U_j$  holomorphic and satisfy  $G_j \sim_{U_j} \text{id}$  such that

$$F_l'^{-1} F'_j = G_l F_l^{-1} F_j G_j^{-1} \text{ on the arc } U_j \cap U_l$$

This equation can be rewritten to

$$F'_j G_j F_j^{-1} = F'_l G_l F_l^{-1} \text{ on the arc } U_j \cap U_l.$$

Since this tells us, that the functions  $F'_j G_j F_j^{-1}$  coincide on the overlapping and define a holomorphic map from the arc  $S^1$  (i.e. a punctured disc with a small radius) into  $\text{gl}_n(\mathbb{C})$ , which will be called  $G$ . Since  $F'_j G_j F_j^{-1} \sim_{U_j} \text{id}$  for all  $j \in J$ , we have  $G \sim \text{id}$ . Thus the defined  $G$  satisfies  $G = F'^{-1} F$ , so that  $[F] = [F']$ .  $\square$

### Theorem 3.8: Malgrange-Sibuya

The map  $\Theta$  is an isomorphism

$$G[[t]]/G\{t\} \cong H^1(S^1; \mathcal{G}).$$

**Remark 3.8.1**

Since we know that  $G[[t]]/G\{t\} \cong G((t))/G(\{t\})$ , we get also

$$G((t))/G(\{t\}) \cong H^1(S^1; \mathcal{G}).$$

This theorem is proven in Section 4.4 of Babbitt and Varadarajan's book [BV89] or on page 371 of [MR91].

We are now able to proof the surjectivity of the map from Theorem 3.4.

*Proof of surjectivity.* Let the cohomology class  $\lambda \in H^1(S^1; \Lambda(A^0))$  be represented by a cocycle  $(F_{jl})_{j,l \in J}$  associated with some finite covering  $\mathcal{U} = \{U_j; j \in J\}$  of  $S^1$ . We especially know, that

- $F_{jl}$  is on  $U_j \cap U_l$  asymptotic to id and
- it is a isotropy, i.e.  $F_{jl} A^0 = A^0$ .

The cocycle  $(F_{jl})_{j,l \in J}$  also determines an element in  $\sigma \in H^1(S^1; \mathcal{G})$ . From the Theorem 3.8 we know, that there is a  $\hat{F} \in G[[t]] \subset G((t))$  whose class  $[\hat{F}]$  gets via  $\Theta$  mapped to  $\sigma$ . Thus there exists holomorphic functions  $F_j : \mathfrak{s}_{U_j} \rightarrow \mathfrak{gl}_n(\mathbb{C})$  with  $F_j \sim_{U_j} \hat{F}$  and  $F_l^{-1} F_j = F_{jl}$  on  $\mathfrak{s}_{U_j \cap U_l}$  for all  $j, l \in J$ .

Define on every arc  $U_j$  the matrix  $A_j := F_j A^0$ . On the intersections  $U_j \cap U_l$  we know that  $A_j = A_l$ , since from  $F_l^{-1} F_j \in \Lambda(A^0)$  follows on  $U_j \cap U_l$  that

$$A^0 = F_l^{-1} (F_j A^0) \quad \implies \quad \underbrace{F_l A^0}_{=A_l} = \underbrace{F_j A^0}_{=A_j}.$$

Thus the  $A_j$  glue to a section  $A$ , which satisfies  $\hat{F} A = A_0$  by construction. We have found an element  $(A, \hat{F}) \in \mathcal{H}(A^0)$  whose image under  $\exp$  is  $\sigma$ .  $\square$

## 3.2 The Stokes group

Here we want to introduce the notion of Stokes groups. They are for example also introduced in [Lod94], [Lod14] or section 4 of [MR91].

### 3.2.1 Anti-Stokes directions and the Stokes group

Let us recall, that the normal form  $A^0$  can be written as  $A^0 = Q'(t^{-1}) + L_t^1$  and a normal solution is given by  $\mathcal{Y}_0(t) = t^L e^{Q(t^{-1})}$  (cf. Proposition 2.30), where

$Q(t^{-1}) = \bigoplus_{j \in \{1, \dots, s\}} q_j(t^{-1}) \text{id}_{n_j}$ . Let  $\{q_1(t^{-1}), \dots, q_s(t^{-1})\}$  be the set of all determining polynomials of  $[A^0]$  and denote by

$$\mathcal{Q}(A^0) := \{(q_j - q_l)(t^{-1}) \mid q_j, q_l \in \{q_1(t^{-1}), \dots, q_s(t^{-1})\}, q_j \neq q_l\}$$

the set of all determining polynomials of  $[\text{End } A^0]$ . Instead of  $q_j - q_l \in \mathcal{Q}(A^0)$  we will sometimes talk of (ordered) pairs  $(q_j, q_l) \in \mathcal{Q}(A^0)$ .

### Definition 3.9

We call

- $a_{jl} \in \mathbb{C} \setminus \{0\}$  the *leading factor*,
- $\frac{a_{jl}}{t^k} =: q_{jl}(t^{-1})$  the *leading coefficient* and
- $k = k_{jl}$  the *degree*

of  $q_j - q_l \in \mathcal{Q}(A^0)$  if

$$q_j - q_l \in \left\{ \frac{a_{jl}}{t^k} + h \mid h \in o(t^{-k}), a_{jl} \neq 0 \right\}.$$

### Remark 3.9.1

1. It is obvious that  $k_{jl} = k_{lj}$  and  $q_{jl}(t^{-1}) = -q_{lj}(t^{-1})$ .
2. In Boalch's paper [Boa01] (and also in [Boa99]) is the  $k$  always incremented by one. We will prefer the other notion, which is also used in Loday-Richaud's paper [Lod94].
3. In Loday-Richaud's book [Lod14, Def.4.3.6]  $a_{jl}$  is negated to be consistent with calculations at  $\infty$ . Here this is not necessary, since we use the clockwise orientation on  $S^1$  (cf. Definition 3.13).

The degrees of the elements in  $\mathcal{Q}(A^0)$  are defined to be the *levels* of  $A^0$ . The set of all levels of  $A^0$  will be denoted by

$$\mathcal{K} = \{k_1 < \dots < k_r\} \subset \mathbb{Q}.$$

### Remark 3.9.2

$[A^0]$  is unramified if and only if  $\mathcal{K} \subset \mathbb{Z}$ . Since we only want to consider the unramified case, this will be always the case.

**Definition 3.10**

Let  $k \in \mathbb{N}$ . An exponential  $e^{q(1/t)} = e^{-\frac{a}{t^k}(x+o(t^{-k}))}$  has *maximal decay in a direction*  $\theta \in S^1$  if and only if  $-ae^{-ik\tilde{\theta}}$  is real negative.

On the elements of  $\mathcal{Q}(A^0)$  we define the following (partial) order relations:

**Definition 3.11**

Let  $\tilde{\theta}$  be a determination of  $\theta$ .

- We define the relation  $q_j \prec_{\tilde{\theta}} q_l$  to be equivalent to the condition

$e^{(q_j - q_l)(t^{-1})}$  is flat at 0 in a neighbourhood of the direction  $\tilde{\theta}$ .

- Let us define another relation  $q_j \prec_{\tilde{\theta}, \max} q_l$  equivalent to

$e^{(q_j - q_l)(t^{-1})}$  is of maximal decay in the direction  $\tilde{\theta}$ .

**Remark 3.11.1**

In the unramified case do these relations not depend on the determination  $\tilde{\theta}$  of  $\theta$ . As a consequence we will only write  $\prec_{\theta}$  and  $\prec_{\theta, \max}$ .

We know that

1. the condition  $q_j \prec_{\theta} q_l$  is satisfied if and only if  $\Re(a_{jl}e^{-ik_{jl}\theta}) < 0$  and
2. the condition  $q_j \prec_{\theta, \max} q_l$  is equivalent to

$a_{jl}e^{-ik_{jl}\theta}$  is a real negative number, i.e.  $q_j \prec_{\theta} q_l$  and  $\Im(a_{jl}e^{-ik_{jl}\theta}) = 0$ .

Thus it is convenient to look closer at functions of the form  $f : \theta \mapsto ae^{-ik\theta}$ , corresponding to some pair  $(q_j, q_l)$ . Write  $a$  as  $a = |a|e^{i \arg(a)}$ , thus the function writes as

$$\begin{aligned} f(\theta) &= |a|e^{i(\arg(a) - k\theta)} \\ &= |a|(\cos(\arg(a) - k\theta) + i \sin(\arg(a) - k\theta)). \end{aligned}$$

In the figure 3.1, we illustrate the real and the imaginary part of  $f$ .

The graphs, corresponding to the flipped pair  $(q_l, q_j)$  are then obtained by the transformation  $\arg(a) \rightarrow \arg(-a) = \arg(a) + \pi$ , i.e. the shift by  $\frac{\pi}{k}$  to the right. This  $\frac{\pi}{k}$  is exactly a halve period, thus the new graphs are obtained by mirroring at the line  $t = 0$ .



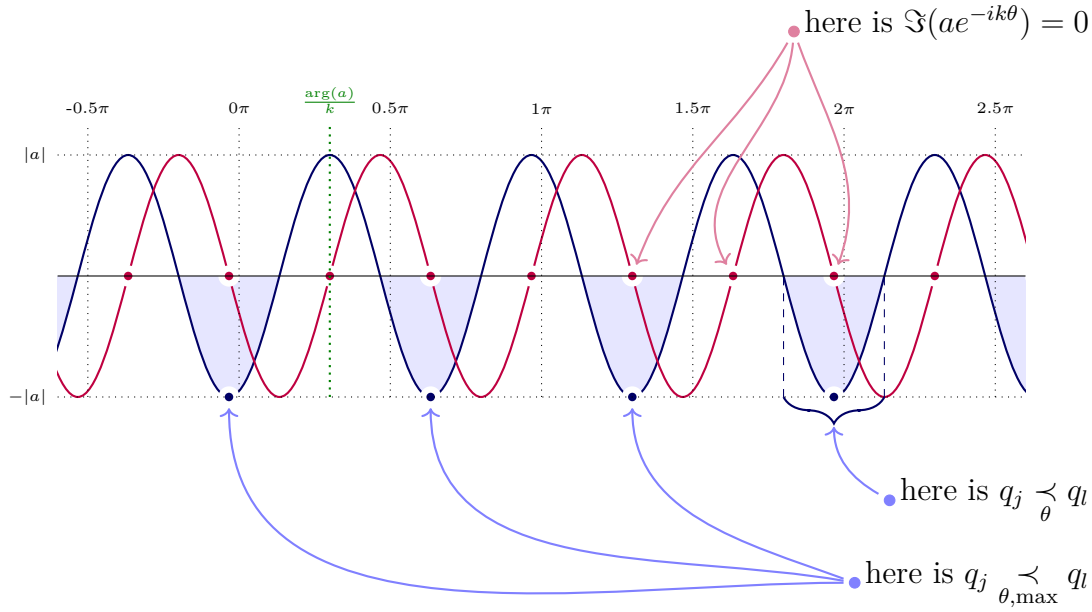


Figure 3.1: In this plot is the real part of  $f(\theta) = ae^{-ik\theta}$ , corresponding to some pair  $(q_j, q_l)$ , in blue and the imaginary part in purple sketched.

### Remark 3.12

Let  $k = k_{jl}$  be the degree of  $q_j - q_l$ . It is easy to see (cf. Figure 3.1), that the condition

$$q_j \prec_{\theta} q_l$$

is equivalent to

$$\text{there is a } \theta' \in U(\theta, \frac{\pi}{k}) \text{ such that } q_j \prec_{\theta, \max} q_l.$$

### Definition 3.13

1.  $\alpha$  is an *anti-Stokes direction* if there is at least one pair  $(q_j, q_l)$  in  $\mathcal{Q}(A^0)$ , which satisfies  $q_j \prec_{\alpha, \max} q_l$ .
  - Let  $\mathbb{A} = \{\alpha_1, \dots, \alpha_\nu\}$  denote the set of all anti-Stokes directions in a clockwise ordering. For a uniform notation later, define  $\mathbb{A}$  to contain a single, arbitrary direction if  $\mathcal{K} = \{0\}$ .

**Remark 3.13.1**

The clockwise ordering is chosen, similar to Loday-Richaud's paper [Lod94], since the calculations are then compatible with the calculations, which look at  $\infty$  and take a counterclockwise ordering. Boalch uses in [Boa01] and [Boa99] the inverse ordering, but looks also at 0, thus there might be some incompatibilities. In Loday-Richaud's book [Lod14] this problem is solved by an additional minus sign for some coefficients.

**Remark 3.13.2**

We will use the greek letter  $\alpha$  when we want to emphasize that a direction is an anti-Stokes direction. For generic directions, we will use  $\theta$ . In fact most of the following definitions and constructions work for every  $\theta \in S^1$ , the Stokes group (cf. Definition 3.15 for example) will be trivial for every  $\theta \notin \mathbb{A}$ . Thus the interesting directions are still the anti-Stokes directions  $\alpha \in \mathbb{A}$ .

2.  $\theta$  is a *Stokes direction* if there is at least one pair  $(q_j, q_l)$  in  $\mathcal{Q}(A^0)$ , which satisfies neither  $q_j \prec_{\theta} q_l$  nor  $q_l \prec_{\theta} q_j$ .
  - Let  $\mathbb{S} = \{\sigma_1 < \dots < \sigma_{\mu}\}$  be the set of Stokes directions.

**Lemma 3.14**

If  $\alpha \in \mathbb{A}$  together with a determining pair  $(q_j - q_l)(t^{-1}) \in \mathcal{Q}(A^0)$  of degree  $k$  is given, we have for every  $m \in \mathbb{N}$  that

$$\underbrace{\alpha + m \frac{\pi}{k}}_{=: \alpha'} \in \mathbb{A}.$$

Especially is either  $q_j \prec_{\alpha', \max} q_l$  (in the case, when  $m$  is even) or  $q_l \prec_{\alpha', \max} q_j$  (when  $m$  is uneven) satisfied (see figure 3.1).

**Corollary 3.14.1**

It follows that in the case  $\mathcal{K} = \{k\}$ , the set  $\mathbb{A}$  has  $\frac{\pi}{k}$ -rotational symmetry.

*Proof.* Let  $(j, l)$  be a pair such that  $q_j \prec_{\alpha, \max} q_l$ , i.e. such that  $a_{jl}e^{-ik_{jl}\alpha} \in \mathbb{R}_{<0}$ . Hence, for  $m \in \mathbb{N}$ ,

$$a_{jl}e^{-ik_{jl}\left(\alpha + m \frac{\pi}{k_{jl}}\right)} = a_{jl}e^{-ik_{jl}\alpha}e^{-im\pi} = \begin{cases} a_{jl}e^{-ik_{jl}\alpha} \in \mathbb{R}_{<0} & , \text{ if } m \text{ is even} \\ -a_{jl}e^{-ik_{jl}\alpha} \in \mathbb{R}_{>0} & , \text{ if } m \text{ is uneven} \end{cases}$$

is, in the case when  $m$  is even, also real and negative. In the other case, when  $n$  is uneven, we use that  $a_{jl} = -a_{lj}$  and  $k_{jl} = k_{lj}$  to obtain  $a_{lj} e^{-ik_{lj} \left( \alpha + m \frac{\pi}{k_{lj}} \right)} \in \mathbb{R}_{<0}$ .

Thus, for  $\alpha' := \alpha + m \frac{\pi}{k_{jl}}$ , we have  $\alpha' \in \mathbb{A}$  since

- $q_j \prec_{\alpha', \max} q_l$  when  $m$  is even or
- $q_l \prec_{\alpha', \max} q_j$  when  $m$  is uneven.

□

As a subgroup of the stalk at  $\theta$  of the in Definition 3.1 defined Stokes sheaf  $\Lambda(A^0)$  we define the Stokes group as follows.

### Definition 3.15

Define the *Stokes group*

$$\text{Sto}_\theta(A^0) := \left\{ \varphi_\theta \in \Lambda_\theta(A^0) \mid \varphi_\theta \text{ has maximal decay at } \theta \right\}$$

whose elements are called *Stokes germs*.

#### Remark 3.15.1

For  $\theta \notin \mathbb{A}$  the group  $\text{Sto}_\theta(A^0)$  is trivial, since no flat isotropy has maximal decay, but the identity.

## 3.2.2 Stokes matrices

### Definition 3.16

Let us use

$$\delta_{jl} := \begin{cases} 0 \in \mathbb{C}^{n_j \times n_l} & , \text{ if } j \neq l \\ \text{id} \in \mathbb{C}^{n_j \times n_l} & , \text{ if } j = l \end{cases}$$

as a block version of Kronecker's delta. Define the group

$$\text{Sto}_\theta(A^0) = \left\{ K = (K_{jl})_{j,l \in \{1, \dots, s\}} \in \text{GL}_n(\mathbb{C}) \mid K_{jl} = \delta_{jl} \text{ unless } q_j \prec_{\theta, \max} q_l \right\}$$

of all *Stokes matrices* of  $A^0$  in the direction  $\theta$ . They will arise as a faithful representation (cf. Section A.2) of  $\text{Sto}_\theta(A^0)$ .

**Remark 3.16.1**

In Boalch's publications [Boa01; Boa99] are our Stokes matrices called Stokes factors, since he introduces other objects, he wants to call Stokes matrices.

Let  $\mathcal{Y}_0 = t^L e^{Q(t^{-1})}$  be a normal solution corresponding to  $A^0$  (cf. Definition 2.32).

**Proposition 3.17**

In this situation is the morphism

$$\begin{aligned} \rho_\theta : \text{Sto}_\theta(A^0) &\longrightarrow \text{Sto}_\theta(A^0) \\ \varphi_\theta &\longmapsto C_{\varphi_\theta} := \mathcal{Y}_0^{-1} \varphi_\theta \mathcal{Y}_0 \end{aligned}$$

an isomorphism which maps a germ of  $\text{Sto}_\theta(A^0)$  to the corresponding Stokes matrix  $C_{\varphi_\theta}$  such that

$$\varphi_\theta(t) \mathcal{Y}_0(t) = \mathcal{Y}_0(t) C_{\varphi_\theta} \quad (3.1)$$

near  $\theta$ . The matrix  $C_{\varphi_\theta}$  is then called a *representation* of  $\varphi_\theta$ .

**Remark 3.17.1**

1. In the unramified case does this morphism depend on the choice of the determination  $\tilde{\theta}$  of  $\theta$  and the corresponding choice of a realization of the fundamental solution with that determination of the argument near the direction  $\theta$  (cf. [Lod94] or [Lod14, 78f]).
2. This construction defines also a morphism, which takes a germ  $\varphi_\theta \in \Lambda_\theta(A^0) \supset \text{Sto}_\theta(A^0)$  into its unique representation matrix

$$C_{\varphi_\theta} \in \left\{ K \in G \mid K_{jl} = \delta_{jl} \text{ unless } q_j \prec_\theta q_l \right\} \supset \text{Sto}_\theta(A^0).$$

*Proof.* It is well known (cf. [Boa99, p. 10]), that the morphism  $\rho_\theta$ , i.e. conjugation by the fundamental solution, relates solutions  $\varphi_\theta$  of  $[\text{End}(A^0)] = [A^0, A^0]$  to solutions of  $[0, 0]$  which are the constant matrices  $\text{GL}_n(\mathbb{C})$ . Thus we have to show, that the image of  $\text{Sto}_\theta(A^0)$  under  $\rho_\theta$  is  $\text{Sto}_\theta(A^0)$ .

To see that the obtained matrix has the necessary zeros, to lie in  $\text{Sto}_\theta(A^0)$  we look at equation (3.1) and deduce

$$\varphi_\theta(t) = t^L e^{Q(t^{-1})} C_{\varphi_\theta} e^{-Q(t^{-1})} t^{-L} \quad (3.2)$$

with the given choice of the argument near  $\theta$ . After decomposing  $C_{\varphi_\theta}$  into

$$\begin{aligned}
 C_{\varphi_\theta} &= 1_n + \begin{pmatrix} c_{(1,1)} & c_{(1,2)} & \cdots \\ c_{(2,1)} & \ddots & \\ \vdots & & c_{(s,s)} \end{pmatrix} \\
 &= 1_n + \underbrace{\begin{pmatrix} c_{(1,1)} & 0 & \cdots \\ 0 & & \\ \vdots & & \end{pmatrix}}_{C_{\varphi_\theta}^{(1,1)}} + \underbrace{\begin{pmatrix} 0 & c_{(1,2)} & 0 & \cdots \\ & 0 & & \\ \vdots & & & \end{pmatrix}}_{C_{\varphi_\theta}^{(1,2)}} + \cdots + \underbrace{\begin{pmatrix} & & \vdots \\ & & 0 \\ \cdots & 0 & c_{(s,s)} \end{pmatrix}}_{C_{\varphi_\theta}^{(s,s)}} \\
 &= 1_n + \sum_{(l,j)} C_{\varphi_\theta}^{(l,j)}
 \end{aligned}$$

where the  $c_{(j,l)}$  are blocks of size  $n_j \times n_l$  which correspond to the structure of  $Q$ . After rewriting the Equation (3.2) we get

$$\varphi_\theta = t^L \left( 1_n + \sum_{(l,j)} C_{\varphi_\theta}^{(l,j)} e^{(q_l - q_j)(t^{-1})} \right) t^{-L}.$$

Thus, for  $\varphi_\theta$  to be flat in direction  $\theta$ , it is necessary and sufficient that if  $e^{(q_l - q_j)(t^{-1})}$  does not have maximal decay in direction  $\theta$  the corresponding block  $C_{\varphi_\theta}^{(l,j)}$  vanishes. Thus we have seen, that  $C_{\varphi_\theta}$  is an element of  $\text{Sto}_\theta(A^0)$ .

The map  $\rho_\theta$  is **injective**, and thus a faithful representation, since conjugation by an invertible matrix defines an isomorphism to its image.

The **surjectivity** can also be seen easily, since every constant (unipotent) matrix, with zeros at the necessary positions, characterizes a unique element of  $\text{Sto}_\theta(A^0)$ .  $\square$

In particular, from the proof, we get that

1. for  $j = l$  the (diagonal) blocks  $C_{\varphi_\theta}^{(l,j)}$  vanish since  $q_l - q_j = 0$  does not have maximal decay and
2. if  $e^{q_j - q_l}$  has maximal decay, then  $e^{q_l - q_j}$  has not. Thus if  $C_{\varphi_\theta}^{(l,j)}$  is not equal to zero, the block  $C_{\varphi_\theta}^{(j,l)}$  is necessarily zero.

### Corollary 3.18

A germ  $\varphi_\theta \in \Lambda_\theta(A^0)$  is a Stokes germ, i.e. an element in  $\text{Sto}_\theta(A^0)$ , if and only if it has a representation  $C_{\varphi_\theta}$  where

$$C_{\varphi_\theta} = 1_n + \sum_{(l,j) | q_j \prec_{\theta, \max} q_l} C_{\varphi_\theta}^{(l,j)}$$

and the  $C_{\varphi_\theta}^{(l,j)}$  have the necessary block structure.

In other words:

a germ  $\varphi_\theta \in \Lambda_\theta(A^0)$  is in  $\text{Sto}_\theta(A^0)$  if and only if there exists a  $C \in \text{Sto}_\theta(A^0)$  such that  $\varphi_\theta = \mathcal{Y}_0 C \mathcal{Y}_0^{-1}$ .

**Remark 3.18.1**

In Loday-Richaud's book [Lod14, p. 78] are the elements of  $\text{Sto}_\theta(A^0)$  actually characterized as the flat transformations, such that equation (3.1) is satisfied for some unique constant invertible matrix  $C \in \text{Sto}_\theta(A^0)$ .

### 3.2.3 Filtration of the Stokes group by levels

Let us introduce a couple of notations and definitions, which coincide with the notations used in Loday-Richaud's paper [Lod94]. Another resource is for example the paper [MR91, 362f] from Martinet and Ramis.

**Definition 3.19**

We denote the set of *levels of the germ*  $\varphi_\theta \in \Lambda_\theta(A^0)$  by

$$\mathcal{K}(\varphi_\theta) := \{ \deg(q_j - q_l) \mid C_{\varphi_\theta}^{(l,j)} \neq 0 \text{ in some representation of } \varphi_\theta \} \subset \mathcal{K}.$$

A germ  $\varphi_\theta$  is called a *k-germ* when  $\mathcal{K}(\varphi_\theta) \subset \{k\}$ , i.e. its only level, if present, is  $k$ .

**Notations 3.20**

For  $k \in \mathcal{K}$  and  $\theta \in S^1$  we set:

- $\Lambda^k(A^0) :=$  the subsheaf of  $\Lambda(A^0)$  of all germs, which are generated by  $k$ -germs;
- $\Lambda^{\leq k}(A^0)$  (resp.  $\Lambda^{< k}(A^0)$  or  $\Lambda^{\geq k}(A^0)$ ) as the subsheaf of  $\Lambda(A^0)$  generated by  $k'$ -germs for all  $k' \leq k$  (resp.  $k' < k$  or  $k' \geq k$ );

The restriction to  $\text{Sto}_\theta$  yields the groups

$$\text{Sto}_\theta^*(A^0) := \text{Sto}_\theta(A^0) \cap \Lambda_\theta^*(A^0)$$

for  $\star \in \{k, < k, \leq k, \dots\}$ . We define also

- $\text{Sto}_\theta^\star(A^0)$  as the groups of representations, which correspond to elements of  $\text{Sto}_\theta^\star(A^0)$ ;
- $\mathbb{A}^k := \{\alpha \in \mathbb{A} \mid \text{Sto}_\alpha^k(A^0) \neq \{\text{id}\}\}$  as the set of anti-Stokes directions *bearing the level  $k$* ;
- $\mathbb{A}^{\leq k} := \bigcup_{k' \leq k} \mathbb{A}^{k'}$  (resp.  $\mathbb{A}^{< k} := \bigcup_{k' < k} \mathbb{A}^{k'}$  and  $\mathbb{A}^{\geq k} := \bigcup_{k' \geq k} \mathbb{A}^{k'}$ );

### Remark 3.20.1

It is clear that

- $\mathbb{A}^\star = \{\alpha \in \mathbb{A} \mid \text{Sto}_\alpha^\star(A^0) \neq \{\text{id}\}\}$  and
- we have the canonical inclusions  $\mathbb{A}^k \hookrightarrow \mathbb{A}^{\leq k}$  and  $\mathbb{A}^{< k} \hookrightarrow \mathbb{A}^{\leq k}$ .

- $\mathcal{K}_\alpha := \{k \in \mathcal{K} \mid \text{Sto}_\alpha^k(A^0) \neq \{\text{id}\}\}$  the set of levels *beared by  $\alpha \in \mathbb{A}$* , and  $K_\alpha = \max \mathcal{K}_\alpha$  the *maximal level beared by  $\alpha$* .

### Remark 3.20.2

The Lemma 3.14 implies that from  $k \in \mathcal{K}_\alpha$  follows that  $k \in \mathcal{K}_{\alpha+m\frac{\pi}{k}}$  for  $m \in \mathbb{N}$ .

We will now introduce a filtration of  $\Lambda(A^0)$ , wich will be restricted to  $\text{Sto}_\theta(A^0)$  and defines there a filtration. This leads to a decomposition of  $\text{Sto}_\theta(A^0)$  into a semidirect product. Let us start by studying the sheaves  $\Lambda^\star(A^0)$ , and discuss how they correlate.

The following proposition can be found as [Lod94, Prop.I.5.1] and the key-statement is also given in [MR91, Prop.4.10].

### Proposition 3.21

For any level  $k \in \mathcal{K}$  one has

1.  $\Lambda^k(A^0)$ ,  $\Lambda^{\leq k}(A^0)$  and  $\Lambda^{< k}(A^0)$  are sheaves of subgroups of  $\Lambda(A^0)$ ;
2. the sheaf  $\Lambda^k(A^0)$  is normal in  $\Lambda^{\leq k}(A^0)$ ;
3. the exact sequence of sheaves

$$1 \longrightarrow \Lambda^k(A^0) \xrightarrow{i} \Lambda^{\leq k}(A^0) \xrightarrow{p} \Lambda^{< k}(A^0) \longrightarrow 1,$$

where

- $i : \Lambda^k(A^0) \hookrightarrow \Lambda^{\leq k}(A^0)$  is the canonical inclusion and

- $p$  is the truncation to terms of levels  $< k'$ , splits.

From the splitting of the sequence, we obtain immediately the following decomposition into a semidirect product.

### Corollary 3.22

1. For any  $k \in \mathcal{K}$ , there are the two following ways of factoring  $\Lambda^{\leq k}(A^0)$  in a semidirect product:

$$\begin{aligned}\Lambda^{\leq k}(A^0) &\cong \Lambda^{< k}(A^0) \ltimes \Lambda^k(A^0) \\ &\cong \Lambda^k(A^0) \ltimes \Lambda^{< k}(A^0).\end{aligned}$$

Thus any germ  $f^{\leq k} \in \Lambda^{\leq k}(A^0)$  can be uniquely written as

- $f^{\leq k} = f^{< k} g^k$ , where  $f^{< k} \in \Lambda^{< k}$  and  $g^k \in \Lambda^k$ , or
- $f^{\leq k} = f^k f^{< k}$ , where  $f^k \in \Lambda^k$  and  $f^{< k} \in \Lambda^{< k}$ .

#### Remark 3.22.1

We can get the factor  $f^{< k}$  common to both factorizations by truncation of  $f^{\leq k}$  to terms of level  $k$ . This truncation can explicitly be achieved, in terms of Stokes matrices, by keeping in any representation  $1 + \sum C^{(j,l)}$  of  $f^{\leq k}$  only the blocks  $C^{(j,l)}$  such that  $\deg(q_j - q_l) < k$ .

A factorization algorithm could then be:

get the factor  $f^{< k}$  common to both factorizations by truncation of  $f^{\leq k}$  to terms of level  $k$  and set  $g^k := (f^{< k})^{-1} f^{\leq k}$  and  $f^k := f^{\leq k} (f^{< k})^{-1}$ .

2. This decomposition in semidirect product can be extended to all levels (cf. [Lod94, Cor.I.5.2]). Thus

$$\Lambda(A^0) \cong \bigtimes_{k \in \mathcal{K}} \Lambda^k(A^0),$$

where the semidirect product is taken in an ascending or descending order of levels  $k$ .

### Remark 3.23

Loday-Richaud states in her paper [Lod94, Prop.I.5.3] the following proposition.



**Proposition 3.23.1**

For any levels  $k, k' \in \mathcal{K}$  with  $k' < k$  one has:

1. the sheaf  $\Lambda^{\geq k'}(A^0) \cap \Lambda^{\leq k'}(A^0)$  is normal in  $\Lambda^{\leq k}(A^0)$ ;
2. the exact sequence of sheaves

$$1 \longrightarrow \Lambda^{\geq k'}(A^0) \cap \Lambda^{\leq k'}(A^0) \xrightarrow{i} \Lambda^{\leq k}(A^0) \xrightarrow{p} \Lambda^{< k'}(A^0) \longrightarrow 1,$$

where

- $i$  is the canonical inclusion and
- $p$  is the truncation to terms of levels  $< k'$ ,

splits.

We can use this proposition to follow (cf. [Lod94, Cor.I.5.4]) that

1. the filtration

$$\Lambda^{k_r}(A^0) = \Lambda^{\geq k_r}(A^0) \subset \Lambda^{\geq k_{r-1}}(A^0) \subset \cdots \subset \Lambda^{\geq k_1}(A^0) = \Lambda(A^0)$$

is normal and

2. we can use this to achieve the decomposition

$$\Lambda(A^0) \cong \bigtimes_{k \in \mathcal{K}} \Lambda^k(A^0)$$

taken in an arbitrary order. In fact, one can also extend the algorithm from Remark 3.22.1 to an arbitrary order of levels.

**Proposition 3.24**

The results can be restricted to the Stokes groups (cf. [Lod94, Prop.I.5.5]). Thus, for  $\alpha \in \mathbb{A}$ , one has

$$\text{Sto}_\alpha(A^0) \cong \bigtimes_{k \in \mathcal{K}_\alpha} \text{Sto}_\alpha^k(A^0)$$

the semidirect product being taken in an arbitrary order.

**Definition 3.24.1**

We will denote the map, which gives the factors of this factorization by

$$i_\alpha : \text{Sto}_\alpha(A^0) \xrightarrow{\cong} \prod_{k \in \mathcal{K}_\alpha} \text{Sto}_\alpha^k(A^0),$$

where the factorization is taken in ascending order.

**Remark 3.24.1**

Write  $\rho_\alpha^k : \text{Sto}_\alpha^k(A^0) \rightarrow \text{Sto}_\alpha^k(A^0)$  for the restriction of the map  $\rho_\alpha$  (cf. Proposition 3.17) to the level  $k$ . Then, one can denote the induced decomposition also by

$$i_\alpha : \text{Sto}_\alpha(A^0) \xrightarrow{\cong} \prod_{k \in \mathcal{K}_\alpha} \text{Sto}_\alpha^k(A^0)$$

and the corresponding diagram

$$\begin{array}{ccc} \text{Sto}_\alpha(A^0) & \xrightarrow{i_\alpha} & \prod_{k \in \mathcal{K}_\alpha} \text{Sto}_\alpha^k(A^0), \\ \downarrow \rho_\alpha & & \downarrow \prod_{k \in \mathcal{K}} \rho_\alpha^k \\ \text{Sto}_\alpha(A^0) & \xrightarrow{i_\alpha} & \prod_{k \in \mathcal{K}_\alpha} \text{Sto}_\alpha^k(A^0), \end{array}$$

commutes.

### 3.3 Stokes structures: using Stokes groups

The goal in this section is to prove that there is a bijective and natural map

$$h : \prod_{\alpha \in \mathbb{A}} \text{Sto}_\alpha(A^0) \longrightarrow \mathcal{St}(A^0).$$

And since  $\text{Sto}_\alpha(A^0)$  has  $\text{Sto}_\alpha(A^0)$  as a faithful representation, we also get the isomorphism  $\prod_{\alpha \in \mathbb{A}} \text{Sto}_\alpha(A^0) \cong \mathcal{St}(A^0)$  as a corollary.

Let us recall, that  $\mathcal{St}(A^0)$  is defined to be  $H^1(S^1; \Lambda(A^0))$  (cf. Section 3.1). The elements of  $\prod_{\alpha \in \mathbb{A}} \text{Sto}_\alpha(A^0)$  define in a canonical way cocycles of the sheaf  $\Lambda(A^0)$  (cf. Remark 3.28), called Stokes cocycles (cf. Definition 3.29). In fact, will  $h$  map such cocycles to the cohomology class, to which they correspond. Thus the statement, that  $h$  is a bijection, is equivalent to the statement that

in each cohomology class of  $\mathcal{S}t(A^0)$  is an unique 1-cocycle, which is a Stokes cocycle.

## Coverings

To formulate the following theorem, we use the notion of cyclic coverings and nerves of such coverings, which are defined as follows.

### Definition 3.25

Let  $J = \mathbb{Z}/p\mathbb{Z}$  be a finite set with a cyclic additive structure.

1. A *cyclic covering* of  $S^1$  is a finite covering  $\mathcal{U} = (U_j = U(\theta_j, \varepsilon_j))_{j \in J}$  consisting of arcs, which satisfies that
  - a) the  $U_j \cap U_{j+1}$  have only one connected component when  $\#J > 2$ ;
  - b) the  $\theta_j$  are in ascending order with respect to the clockwise orientation of  $S^1$ ;
  - c) the  $U_j$  are not encased, this means that the arcs  $U_j \setminus U_l$  are connected arcs for all  $j, l \in J$ .
2. The *nerve* of a cyclic covering  $\mathcal{U} = \{U_j; j \in J\}$  is the family  $\dot{\mathcal{U}} = \{\dot{U}_j; j \in J\}$  defined by:
  - $\dot{U}_j = U_j \cap U_{j+1}$  when  $\#J > 2$ ,
  - $\dot{U}_1$  and  $\dot{U}_2$  the connected components of  $U_1 \cap U_2$  when  $\#J = 2$ .

The cyclic coverings correspond one-to-one to nerves of cyclic coverings. If one starts with a nerve  $\{\dot{U}_j \mid j \in J\}$ , one obtains a cyclic covering as  $\mathcal{U} = \{U_j \mid j \in J\}$  where the arc  $U_j$  is the connected clockwise hull from  $\dot{U}_{j-1}$  to  $\dot{U}_j$ .

### Definition 3.26

A covering  $\mathcal{V}$  is said to *refine* a covering  $\mathcal{U}$  if, to each open set  $V \in \mathcal{V}$  there is at least one  $U \in \mathcal{U}$  with  $V \subset U$ .

### Proposition 3.27

The covering  $\mathcal{V}$  refines  $\mathcal{U}$  if and only if the corresponding nerves  $\dot{\mathcal{U}} = \{\dot{U}_j\}$  and  $\dot{\mathcal{V}} = \{\dot{V}_l\}$  satisfy

each  $\dot{U}_j$  contains at least one  $\dot{V}_l$ .

### 3.3.1 The theorem

#### Remark 3.28

Let  $\{\theta_j \mid j \in J\} \subset S^1$  be a finite set and  $\dot{\varphi} = (\dot{\varphi}_{\theta_j})_{j \in J} \in \prod_{j \in J} \Lambda_{\theta_j}(A^0)$  be a finite family of germs. In the following way, one can associate a cohomology class in  $\mathcal{S}t(A^0)$  to any  $\dot{\varphi}$ :

let  $\dot{\varphi}_j$  be the function representing the germ  $\dot{\varphi}_{\theta_j}$  on its maximal arc of definition  $\Omega_j$  around  $\theta_j$ . Then, for every cyclic covering  $\mathcal{U} = (U_j)_{j \in J}$  which satisfies  $\dot{U}_j \subset \Omega_j$  for all  $j \in J$ , one can define the 1-cocycle  $(\dot{\varphi}_j|_{\dot{U}_j})_{j \in J} \in \Gamma(\dot{\mathcal{U}}; \Lambda(A^0))$ .

To a different cyclic covering, satisfying the condition above, this construction yields a cohomologous 1-cocycle, thus the induced map

$$\prod_{j \in J} \Lambda_{\theta_j}(A^0) \longrightarrow H^1(S^1; \Lambda(A^0)) = \mathcal{S}t(A^0)$$

is welldefined.

#### Definition 3.29

Let  $\nu = \#\mathbb{A}$  the number of all anti-Stokes directions and write  $\mathbb{A} = \{\alpha_1, \dots, \alpha_\nu\}$ .

A *Stokes cocycle* is a 1-cocycle  $(\varphi_j)_{j \in \{1, \dots, \nu\}} \in \prod_{j \in \{1, \dots, \nu\}} \Gamma(U_j; \Lambda(A^0))$  corresponding to some cyclic covering with nerve  $\dot{\mathcal{U}} = (\dot{U}_j)_{j \in \{1, \dots, \nu\}}$ , which satisfies for every  $j \in \{1, \dots, \nu\}$

- $\alpha_j \in \dot{U}_j$  and
- the germ  $\varphi_{\alpha_j} := \varphi_{j, \alpha_j}$  of  $\varphi_j$  at  $\alpha_j$  is an element of  $\text{Sto}_{\alpha_j}(A^0)$ .

#### Remark 3.29.1

Since the sections in  $\Gamma(\dot{U}_j; \Lambda(A^0))$  are uniquely determined as the extension of the germ at  $\alpha_j$  have that the set of all Stokes cocycles can be identified with  $\prod_{\alpha \in \mathbb{A}} \text{Sto}_{\alpha}(A^0)$ .

We can use Remark 3.28 to obtain, in the case  $J = \mathbb{A}$ , a mapping

$$h : \prod_{\alpha \in \mathbb{A}} \text{Sto}_{\alpha}(A^0) \hookrightarrow \prod_{\alpha \in \mathbb{A}} \Lambda_{\alpha}(A^0) \longrightarrow \mathcal{S}t(A^0),$$

which takes a Stokes cocycle to its corresponding cohomology class.

**Theorem 3.30**

The map

$$h : \prod_{\alpha \in \mathbb{A}} \text{Sto}_{\alpha}(A^0) \longrightarrow \mathcal{St}(A^0)$$

is a bijection and natural.

**Remark 3.30.1**

Natural means that  $h$  commutes to isomorphisms and constructions over systems or connections they represent.

From theorem 3.30 and Proposition 3.17 we get the following corollary.

**Corollary 3.31**

Using the isomorphisms from the Stokes groups to the group of all Stokes matrices (cf. Proposition 3.17) we obtain

$$\mathcal{St}(A^0) \cong \prod_{\alpha \in \mathbb{A}} \text{Sto}_{\alpha}(A^0)$$

which endows  $\mathcal{St}(A^0)$  with the structure of an unipotent Lie group with the finite complex dimension  $N := \dim_{\mathbb{C}} \mathcal{St}(A^0) = \sum_{\alpha \in \mathbb{A}} \dim_{\mathbb{C}} \text{Sto}_{\alpha}(A^0)$  where  $\dim_{\mathbb{C}} \text{Sto}_{\alpha}(A^0) = \sum_{q_j \prec_{\alpha, \max} q_l} n_j \cdot n_l$  (cf. [Lod94, Sec.III.1]). This can be rewritten in the following way

$$\begin{aligned} N &= \sum_{\alpha \in \mathbb{A}} \sum_{q_j \prec_{\alpha, \max} q_l} n_j \cdot n_l \\ &= \sum_{1 \leq j, l \leq n} \deg(q_j - q_l) \cdot n_j \cdot n_l \\ &= \sum_{\substack{1 \leq j, l \leq n \\ j < l}} 2 \cdot \deg(q_j - q_l) \cdot n_j \cdot n_l \end{aligned}$$

**Remark 3.31.1**

This number  $N$  is known to be the *irregularity* of  $[\text{End } A^0]$ .

**Remark 3.32**

To define the inverse map of  $h$ , one has to find in each cocycle in  $\mathcal{St}(A^0)$  the Stokes cocycle. Loday-Richaud gives an algorithm in section II.3.4 of her paper [Lod94],

which takes a cocycle over an arbitrary cyclic covering and outputs cohomologous Stokes cocycle and thus solves this problem.

### 3.3.2 Proof of Theorem 3.30

We will only look at the unramified case, for which we refer to [Lod94, Sec.II.3]. The proof in the ramified case can be found in [Lod94, Sec.II.4]. We first have to introduce adequate coverings, which will be used in the proof.

#### Adequate coverings

##### Definition 3.33

Let  $\star \in \{k, < k, \leq k, \dots\}$ . A covering  $\mathcal{U}$  beyond which the inductive limit  $\varinjlim_{\mathcal{U}} H^1(\mathcal{U}; \Lambda^\star(A^0))$  is stationary is said to be *adequate* to describe  $H^1(S^1; \Lambda^\star(A^0))$  or *adequate* to  $\Lambda^\star(A^0)$ .

In other words is a covering  $\mathcal{U}$  adequate, if and only if the canonical map

$$\Gamma(\mathcal{U}; \Lambda^\star(A^0)) \longrightarrow H^1(S^1; \Lambda^\star(A^0))$$

is surjective.

The following proposition is in Loday-Richauds paper [Lod94] given as Proposition II.1.7. It contains a simple characterization, which will be used to see, that our defined coverings are adequate.

##### Proposition 3.34

Let  $k \in \mathcal{K}_\alpha$ .

##### Definition 3.34.1

Let  $\alpha \in \mathbb{A}^k$ . An arc  $U(\alpha, \frac{\pi}{k})$  is called a *Stokes arc of level  $k$  at  $\alpha$* .

A cyclic covering  $\mathcal{U} = (U_j)_{j \in J}$ , which satisfies

for every  $\alpha \in \mathbb{A}^k$  contains the Stokes arc  $U(\alpha, \frac{\pi}{k})$  at least one arc  $\dot{U}_j$  from the nerve  $\dot{\mathcal{U}}$  of  $\mathcal{U}$

is adequate to  $\Lambda^k(A^0)$ .

The covering  $\mathcal{U}$  is adequate to  $\Lambda^{\leq k}(A^0)$  (resp.  $\Lambda^{< k}(A^0)$ ) if it is adequate to  $\Lambda^{k'}(A^0)$  for every  $k' \leq k$  (resp.  $k' < k$ ).

Let  $k \in \mathcal{K}$ . We want to define the three cyclic coverings  $\mathcal{U}^k$ ,  $\mathcal{U}^{\leq k}$  and  $\mathcal{U}^{< k}$  which will be adequate to  $\Lambda^k(A^0)$ ,  $\Lambda^{\leq k}(A^0)$  and  $\Lambda^{< k}(A^0)$ . Furthermore will the coverings be comparable at the different levels.

1. The first covering  $\mathcal{U}^k = \{\dot{U}_\alpha^k \mid \alpha \in \mathbb{A}^k\}$  is the cyclic covering with nerve

$$\dot{\mathcal{U}}^k := \left\{ \dot{U}_\alpha^k = U\left(\alpha, \frac{\pi}{k}\right) \mid \alpha \in \mathbb{A}^k \right\}$$

consisting of all Stokes arcs of level  $k$  for anti-Stokes directions bearing the level  $k$ .

### Remark 3.35

Boalch introduces in his publications [Boa01, p. 19] and [Boa99, Def.1.23] the notion of *supersectors*  $\widehat{\text{Sect}}$ , they are in the case of a single level  $k$ , defined as follows:

write the anti-Stokes directions as  $\mathbb{A} = \{\alpha_1, \dots, \alpha_\nu\}$  arranged according to the clockwise ordering, then is the  $i$ -th supersector defined as the arc

$$\widehat{\text{Sect}}_i := \left( \alpha_i - \frac{\pi}{2k}, \alpha_{i+1} + \frac{\pi}{2k} \right).$$

This yields a cyclic covering  $(\widehat{\text{Sect}}_i)_{i \in \{1, \dots, \nu\}}$  whose nerve is exactly  $\dot{\mathcal{U}}^k$  defined above.

If we extend to several levels, the set  $\bigcup_k \left\{ U\left(\alpha, \frac{\pi}{k}\right) \mid \alpha \in \mathbb{A}^k \right\}$  is no longer a nerve. Hence we have to define the coverings  $\mathcal{U}^{\leq k}$  and  $\mathcal{U}^{< k}$  in a different way. Denote by

$$\{K_1 < \dots < K_s = k\} = \left\{ \max(\mathcal{K}_\alpha \cap [0, k]) \mid \alpha \in \mathbb{A}^{\leq k} \right\}$$

the set of all  $k$ -maximum levels for  $\alpha \in \mathbb{A}^{\leq k}$ .

2. The cyclic covering  $\mathcal{U}^{\leq k} = \{U_\alpha^{\leq k} \mid \alpha \in \mathbb{A}^{\leq k}\}$  will be defined by induction. Let us assume that

the  $\dot{U}_\alpha^{\leq k}$  are defined for all  $\alpha \in \mathbb{A}^{\leq k}$  with  $k$ -maximum level greater than  $K_i$  such that their complete family is a nerve.

Let

- $\alpha$  be a anti-Stokes direction with  $k$ -maximum level  $K_i$  and
- $\alpha^-$  (resp.  $\alpha^+$ ) be the next anti-Stokes direction with  $k$ -maximum level greater then  $K_i$  on the left (resp. on the right) and define  $\dot{U}_{\alpha^-, \alpha^+}$  as the clockwise hull of the arcs  $\dot{U}_{\alpha^-}^{\leq k}$  and  $\dot{U}_{\alpha^+}^{\leq k}$  already defined by induction. If there are no anti-Stokes directions with  $k$ -maximum level greater then  $K_i$  we set  $\dot{U}_{\alpha^-, \alpha^+} = S^1$ .

We then set

$$\dot{U}_\alpha^{\leq k} := U\left(\alpha, \frac{\pi}{K_i}\right) \cap \dot{U}_{\alpha^-, \alpha^+}$$

and the family of all  $\dot{U}_\alpha^{\leq k}$  is a nerve.

**Remark 3.36**

If  $\alpha$  has a  $k$ -maximum level equal to  $k$  then is  $\dot{U}_\alpha^{\leq k}$  equal to the Stokes arc  $U\left(\alpha, \frac{\pi}{k}\right) = \dot{U}_\alpha^k$ .

3. The last cyclic covering,  $\mathcal{U}^{<k} = \left\{U_\alpha^k \mid \alpha \in \mathbb{A}^{<k}\right\}$ , is defined as  $\mathcal{U}^{<k} := \mathcal{U}^{\leq k'}$  where  $k' := \max\{k'' \in \mathcal{K} \mid k'' < k\}$ .

**Remark 3.37**

The coverings  $\mathcal{U}^k$ ,  $\mathcal{U}^{\leq k}$  and  $\mathcal{U}^{<k}$  depend only on  $\mathcal{Q}(A^0)$ . Hence they depend only on the determining polynomials.

It is obvious, that for every  $k \in \mathcal{K}$  the covering  $\mathcal{U}^{\leq k}$  refines  $\mathcal{U}^k$  and  $\mathcal{U}^{<k}$ . Furthermore are the coverings defined, such that they satisfy the condition in Proposition 3.34. Thus the first property in the following proposition is satisfied. The other two can be found at [Lod94, Prop.II.3.1 (iv)]

**Proposition 3.38**

Let  $k \in \mathcal{K}$ , then

1. the coverings  $\mathcal{U}^k$ ,  $\mathcal{U}^{\leq k}$  and  $\mathcal{U}^{<k}$  are adequate to  $\Lambda^k(A^0)$ ,  $\Lambda^{\leq k}(A^0)$  and  $\Lambda^{<k}(A^0)$ ;
2. there exists no 0-cochain in  $\Lambda^k(A^0)$  on  $\mathcal{U}^k$ ;
3. on  $\mathcal{U}^{\leq k}$  there is no 0-cochain in  $\Lambda^{\leq 0}(A^0)$  of level  $k$ , i.e. all 0-cochains of  $\Lambda^{\leq k}(A^0)$  belong to  $\Lambda^{<k}(A^0)$ .

To have a shorter notation, we denote the product  $\prod_{\alpha \in \mathbb{A}^\star} \Gamma(\dot{U}_\alpha^\star; \Lambda^\star(A^0))$  by  $\Gamma(\dot{U}^\star; \Lambda^\star(A^0))$  for every  $\star \in \{k, <k, \leq k, \dots\}$ .

### The case of a unique level

First we will proof theorem 3.30 in the case of a unique level. This means that

- either  $\Lambda(A^0)$  has only one level  $k$ , thus
  - $\Lambda(A^0) = \Lambda^k(A^0)$  and



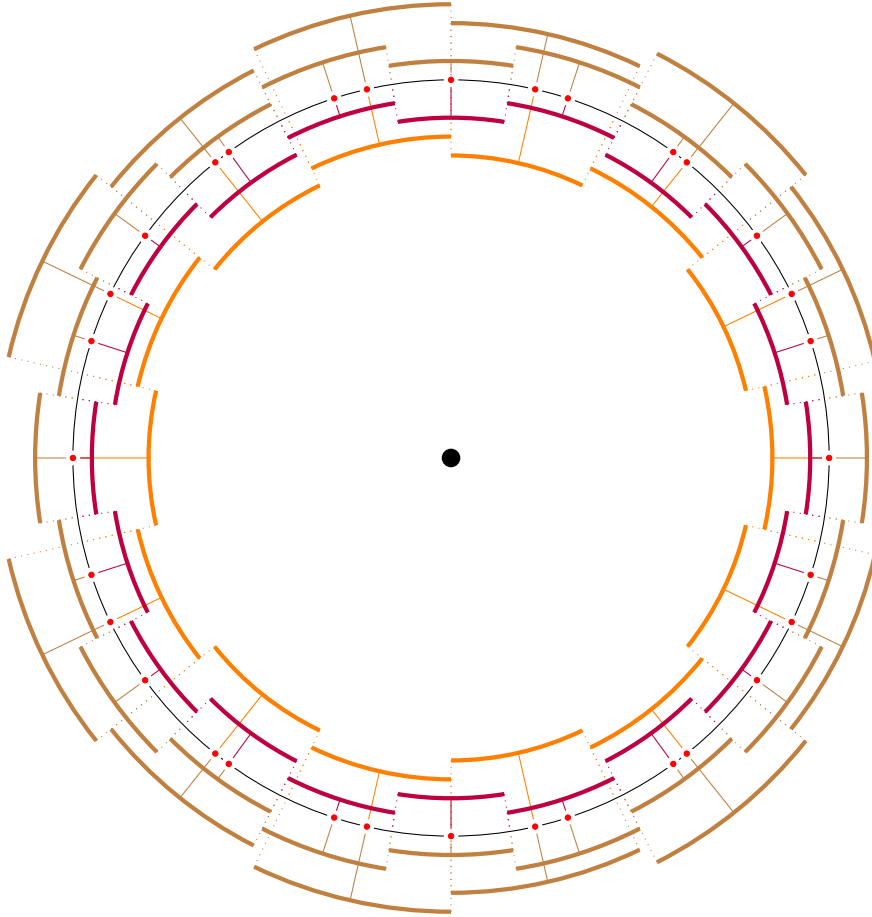


Figure 3.2: The adequate coverings for an example with  $\mathcal{K} = \{7, 10\}$  and  $\mathbb{A} = \left\{ \frac{j \cdot \pi}{k} \mid k \in \mathcal{K}, j \in \mathbb{N} \right\}$ . The anti-Stokes directions are marked by the red dots. The arcs of  $\dot{\mathcal{U}}^7 = \dot{\mathcal{U}}^{\leq 7}$  are orange, the arcs of  $\dot{\mathcal{U}}^{10}$  are purple and the arcs of  $\dot{\mathcal{U}}^{\leq 10} = \dot{\mathcal{U}}$  are brown.

- $\text{Sto}_\theta(A^0) = \text{Sto}_\theta^k(A^0)$  for every  $\theta$ ,
- or we restrict to a given level  $k \in \mathcal{K}$ .

**Lemma 3.39**

Let  $k \in \mathcal{K}$ . The morphism  $h$  from Theorem 3.30 is in the case of an unique level build as

$$\begin{array}{ccc} \prod_{\alpha \in \mathbb{A}} \text{Sto}_\alpha^k(A^0) & \xrightarrow{i^k} & \Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A^0)) \xrightarrow{s^k} H^1(S^1; \Lambda^k(A^0)) \\ \parallel & & \parallel \\ \prod_{\alpha \in \mathbb{A}} \text{Sto}_\alpha(A^0) & \xrightarrow{h} & H^1(S^1; \Lambda(A^0)) \end{array}$$

from

- the canonical injective map

$$i^k : \prod_{\alpha \in \mathbb{A}} \text{Sto}_\alpha^k(A^0) \longrightarrow \Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A^0)),$$

i.e. the map which is the canonical extension of germs to their natural arc of definition, and

- the quotient map

$$s^k : \Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A^0)) \longrightarrow H^1(S^1; \Lambda^k(A^0))$$

which are both isomorphisms.

*Proof.* 1. The map

$$i^k : \prod_{\alpha \in \mathbb{A}} \text{Sto}_\alpha^k(A^0) \longrightarrow \underbrace{\prod_{\alpha \in \mathbb{A}^k} \Gamma(\dot{\mathcal{U}}_\alpha^k; \Lambda^k(A^0))}_{\parallel \Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A^0))}$$

is welldefined, since the sections of  $\Lambda^k(A^0)$  are solution of the system  $[A^0, A^0]$  and it is very well known from the theory of differential equations that an element  $f_\alpha \in \Gamma(\dot{\mathcal{U}}_\alpha^k; \Lambda^k(A^0))$  is uniquely determined as the extension of its germ at some point  $\alpha$ .

It is also a group isomorphism (cf. [Lod94]).

2. The second map

$$s^k : \overbrace{\prod_{\alpha \in \mathbb{A}^k} \Gamma(\dot{\mathcal{U}}_\alpha^k; \Lambda^k(A^0))}^{\Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A^0))} \longrightarrow H^1(S^1; \Lambda^k(A^0))$$

is a bijection, since from proposition 3.38 we know that it is

- **surjective**, since  $\mathcal{U}^k$  is adequate to  $\Lambda^k(A^0)$  and
- **injective**, since on  $\mathcal{U}^k$  there is no 0-cochain in  $\Lambda^k(A^0)$ .

□

### The case of several levels

In the proof of the case of several levels, we will still use Loday-Richaud's paper [Lod94] as reference.

#### Definition 3.40

Here we want to define a *product map of cocycles*  $\mathfrak{S}^{\leq k}$ . This map will be composed from the following injective maps:

1. The first map is defined as

$$\begin{aligned} \sigma^k : \Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A^0)) &\longrightarrow \Gamma(\dot{\mathcal{U}}^{\leq k}; \Lambda^{\leq k}(A^0)) \\ \dot{f} = (\dot{f}_\alpha)_{\alpha \in \mathbb{A}^k} &\longmapsto (\dot{G}_\alpha)_{\alpha \in \mathbb{A}^{\leq k}} \end{aligned}$$

where

$$\bullet \quad \dot{G}_\alpha = \begin{cases} \dot{f}_\alpha \text{ restricted to } \dot{\mathcal{U}}_\alpha^{\leq k} \text{ and seen as being in } \Lambda^{\leq k}(A^0) & \text{when } \alpha \in \mathbb{A}^k \\ \text{id (the identity)} & \text{when } \alpha \notin \mathbb{A}^k \end{cases}$$

2. and the second map

$$\begin{aligned} \sigma^{< k} : \Gamma(\dot{\mathcal{U}}^{< k}; \Lambda^{< k}(A^0)) &\longrightarrow \Gamma(\dot{\mathcal{U}}^{\leq k}; \Lambda^{\leq k}(A^0)) \\ \dot{f} = (\dot{f}_\alpha)_{\alpha \in \mathbb{A}^{< k}} &\longmapsto (\dot{F}_\alpha)_{\alpha \in \mathbb{A}^{\leq k}} \end{aligned}$$

is defined, in a similar way, as

$$\bullet \quad \dot{F}_\alpha = \begin{cases} \dot{f}_\alpha \text{ restricted to } \dot{\mathcal{U}}_\alpha^{\leq k} \text{ and seen as being in } \Lambda^{\leq k}(A^0) & \text{when } \alpha \in \mathbb{A}^{< k} \\ \text{id (the identity)} & \text{when } \alpha \notin \mathbb{A}^{< k} \end{cases}$$

Thus we can define

$$\begin{aligned} \mathfrak{S}^{\leq k} : \Gamma(\dot{\mathcal{U}}^{<k}; \Lambda^{<k}(A^0)) \times \Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A^0)) &\longrightarrow \Gamma(\dot{\mathcal{U}}^{\leq k}; \Lambda^{\leq k}(A^0)) \\ (\dot{f}, \dot{g}) &\longmapsto (\dot{F}_\alpha \dot{G}_\alpha)_{\alpha \in \mathbb{A}^{\leq k}} \end{aligned}$$

where  $(\dot{F}_\alpha)_{\alpha \in \mathbb{A}^{\leq k}} = \sigma^{<k}(\dot{f})$  and  $(\dot{G}_\alpha)_{\alpha \in \mathbb{A}^{\leq k}} = \sigma^k(\dot{g})$  are defined as above.

**Remark 3.40.1**

This map  $\mathfrak{S}^{\leq k}$  is injective, since injectivity for germs implies injectivity for sections.

**Lemma 3.41**

Let  $k \in \mathcal{K}$ .

1. If the cocycles  $\mathfrak{S}^{\leq k}(\dot{f}, \dot{g})$  and  $\mathfrak{S}^{\leq k}(\dot{f}', \dot{g}')$  are cohomologous in  $\Gamma(\dot{\mathcal{U}}^{\leq k}; \Lambda^{\leq k}(A^0))$  then  $\dot{f}$  and  $\dot{f}'$  are cohomologous in  $\Gamma(\dot{\mathcal{U}}^{<k}; \Lambda^{<k}(A^0))$ .
2. Any cocycle in  $\Gamma(\dot{\mathcal{U}}^{\leq k}; \Lambda^{\leq k}(A^0))$  is cohomologous to a cocycle in the range of  $\mathfrak{S}^{\leq k}$ .

*Proof.* 1. Denote by  $\alpha^+$  the nearest anti-Stokes direction in  $\mathbb{A}^{\leq k}$  on the right<sup>a</sup> of  $\alpha$ . The cocycles  $\mathfrak{S}^{\leq k}(\dot{f}, \dot{g})$  and  $\mathfrak{S}^{\leq k}(\dot{f}', \dot{g}')$  are cohomologous if and only if there is a 0-cochain  $c = (c_\alpha)_{\alpha \in \mathbb{A}^{\leq k}} \in \Gamma(\dot{\mathcal{U}}^{\leq k}; \Lambda^{\leq k}(A^0))$  such that

$$\dot{F}_\alpha \dot{G}_\alpha = c_\alpha^{-1} \dot{F}'_\alpha \dot{G}'_\alpha c_{\alpha^+} \quad (3.3)$$

for every  $\alpha \in \mathbb{A}$ . From Proposition 3.38 follows, that  $c$  is with values in  $\Lambda^{<k}(A^0)$ . The fact that  $\Lambda^k(A^0)$  is normal in  $\Lambda^{\leq k}(A^0)$  in Proposition 3.21, can be used to see that  $c_{\alpha^+}^{-1} \dot{G}'_\alpha c_{\alpha^+} \in \Gamma(\dot{\mathcal{U}}^{\leq k}; \Lambda^k(A^0))$ . Thus, we rewrite the relation (3.3) to

$$\dot{F}_\alpha \dot{G}_\alpha = (c_\alpha^{-1} \dot{F}'_\alpha c_{\alpha^+})(c_{\alpha^+}^{-1} \dot{G}'_\alpha c_{\alpha^+}), \quad \text{for } \alpha \in \mathbb{A}^{\leq k}.$$

Since Corollary 3.22 tells us, that the factorization into the factors of the semidirect product are unique, we get for all  $\alpha \in \mathbb{A}^k$

$$\dot{F}_\alpha = c_\alpha^{-1} \dot{F}'_\alpha c_{\alpha^+} \quad \text{and} \quad \dot{G}_\alpha = c_{\alpha^+}^{-1} \dot{G}'_\alpha c_{\alpha^+}.$$

The former relation implies that  $(\dot{F}_\alpha)$  and  $(\dot{F}'_\alpha)$  are cohomologous with values in  $\Lambda^{<k}(A^0)$  on  $\dot{\mathcal{U}}^{\leq k}$ . Since  $\dot{\mathcal{U}}^{<k}$  is already adequate to  $\Lambda^{<k}(A^0)$ , are  $(\dot{F}_\alpha)$  and  $(\dot{F}'_\alpha)$  already on  $\dot{\mathcal{U}}^{<k}$ , i.e. in  $\Gamma(\dot{\mathcal{U}}^{<k}; \Lambda^{<k}(A^0))$ , cohomologous.

2. The proof of part 2. (together with a proof of part 1.) can be found in Loday-Richaud's paper [Lod94, Proof of Lem.II.3.3].



where the products are following an ascending order of levels and the maps  $\tau_k$  are defined as

$$\begin{aligned} \tau^k : \Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A^0)) &\xrightarrow{\sigma^k} \Gamma(\dot{\mathcal{U}}^{\leq k}; \Lambda^{\leq k}(A^0)) \longrightarrow \Gamma(\dot{\mathcal{U}}; \Lambda(A^0)) \\ (\dot{f}_\alpha)_{\alpha \in \mathbb{A}^k} &\longmapsto (\dot{F}_\alpha)_{\alpha \in \mathbb{A}} \end{aligned}$$

with

$$\dot{F}_\alpha = \begin{cases} \dot{f}_\alpha \text{ restricted to } \dot{U}_\alpha \text{ and seen as being in } \Lambda(A^0) & \text{when } \alpha \in \mathbb{A}^k \\ \text{id (the identity on } \dot{U}_\alpha) & \text{when } \alpha \notin \mathbb{A}^k \end{cases}$$

The defined map  $\tau$  is clearly injective and it can be extended to an arbitrary order of levels (cf. Remark [Lod94, Rem.II.3.5]).

### Corollary 3.43

The product map of single-leveled cocycles  $\tau$  induces on the cohomology a bijective and natural map

$$\begin{aligned} \mathcal{T} : \prod_{k \in \mathcal{K}} \Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A^0)) &\longrightarrow H^1(\mathcal{U}; \Lambda(A^0)) . \\ \prod_{k \in \mathcal{K}} \Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A^0)) &\xrightarrow{\cong} H^1(S^1; \Lambda^k(A^0)) \xrightarrow{\cong} H^1(S^1; \Lambda(A^0)) \end{aligned}$$

**Composing functions to obtain  $h$**  We have the ingredients to define the function  $h$  from Theorem 3.30 by composition of already bijective maps.

*Proof of theorem 3.30.* Let  $i_\alpha : \text{Sto}_\alpha(A^0) \rightarrow \prod_{k \in \mathcal{K}} \text{Sto}_\alpha^k(A^0)$  be the map which corresponds to the filtration from proposition 3.24 and denote the composition

$$\begin{array}{ccc} \prod_{\alpha \in \mathbb{A}} \text{Sto}_\alpha(A^0) & \xrightarrow{\prod_{\alpha \in \mathbb{A}} i_\alpha} & \prod_{\alpha \in \mathbb{A}} \prod_{k \in \mathcal{K}} \text{Sto}_\alpha^k(A^0) \\ & & \parallel \\ & & \prod_{k \in \mathcal{K}} \prod_{\alpha \in \mathbb{A}} \text{Sto}_\alpha^k(A^0) \xrightarrow{\prod_{k \in \mathcal{K}} i^k} \prod_{k \in \mathcal{K}} \Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A^0)) \\ & \searrow \tau & \nearrow \end{array}$$

by  $\mathfrak{T}$ . The bijection  $h$  is then obtained as

$$\mathcal{T} \circ \mathfrak{T} : \prod_{\alpha \in \mathbb{A}} \text{Sto}_{\alpha}(A^0) \longrightarrow H^1(\mathcal{U}; \Lambda(A^0)).$$

□

### 3.4 Further improvements

The set  $\prod_{\alpha \in \mathbb{A}} \text{Sto}_{\alpha}(A^0)$  and thus  $\prod_{\alpha \in \mathbb{A}} \text{Sto}_{\alpha}(A^0)$  has some bad properties, when small deformations are applied to  $[A^0]$ , since under arbitrary small changes, one Stokes ray can split into two. In the prove of theorem 3.30 we have seen, that not only  $\mathcal{H}(A^0) \cong \prod_{\alpha \in \mathbb{A}} \text{Sto}_{\alpha}(A^0)$  but also that

$$\begin{aligned} \mathcal{H}(A^0) &\cong \prod_{k \in \mathcal{K}} \prod_{\alpha \in \mathbb{A}} \text{Sto}_{\alpha}^k(A^0) \\ &\cong \prod_{k \in \mathcal{K}} \prod_{\alpha \in \mathbb{A}^k} \text{Sto}_{\alpha}^k(A^0) \quad (\text{since } \text{Sto}_{\alpha}^k(A^0) = \{\text{id}\} \text{ when } k \notin \mathcal{K}_{\alpha}.) \\ &\cong \prod_{k \in \mathcal{K}} \prod_{\alpha \in \mathbb{A}^k} \text{Sto}_{\alpha}^k(A^0). \end{aligned}$$

This representation can be used to achieve further improvements, since one is able to multiply some succeeding Stokes matrices (resp. Stokes germs) of the same level without loss of information. This is based on the Corollary 3.50 stated below

Let us fix a level  $k \in \mathcal{K}$ . We want to rewrite the product  $\prod_{\alpha \in \mathbb{A}^k} \text{Sto}_{\alpha}^k(A^0)$  by collecting the information of multiple Stokes matrices into one product of these matrices.

Boalch, which looks in his publications [Boa01; Boa99] only at the single-leveled case, uses this extensively to obtain better Stokes matrices, which are stable under small deformations (cf. Remark ??).

#### Definition 3.44

A subset of  $\mathbb{A}^k$  consisting of  $\frac{\#\mathbb{A}^k}{2k}$  consecutive anti-Stokes directions of level  $k$  will be called a *half-period (of level  $k$ )*.

#### Remark 3.44.1

The set  $\mathbb{A}^k$  can clearly be split into  $2k$  distinct half-periods.

From the definition of anti-Stokes directions of level  $k$  it is clear that every arc of width  $\frac{\pi}{k}$ , which has no anti-Stokes direction of level  $k$  on its border, contains  $\frac{\#\mathbb{A}^k}{2k}$  anti-Stokes directions since

for every sector  $I = U(\theta, \frac{\pi}{k})$ , of width  $\frac{\pi}{k}$  and center  $\theta$ , which satisfies  $\theta \pm \frac{\pi}{2k} \notin \mathbb{A}^k$ , there is to any pair  $(q_j, q_l)$  in  $\mathcal{Q}(A^0)$ , which has level  $k_{jl} = k$ , exactly one direction  $\alpha \in \mathbb{A}^k \cap I$  which satisfies, that  $\Im(a_{jl}e^{-ik_{jl}\alpha}) = 0$ . At such a direction  $\alpha$ , corresponding to the pair  $(q_j, q_l)$ , is then either  $q_j \prec_{\alpha, \max} q_l$  or  $q_l \prec_{\alpha, \max} q_j$  satisfied.

This implies that for every  $\theta$ , which satisfies  $\theta \pm \frac{\pi}{2k} \notin \mathbb{A}^k$ , is  $\mathbb{A}^k \cap U(\theta, \frac{\pi}{k})$  a half-period and by  $\dot{\bigcup}_{1 \leq m \leq 2k} \mathbb{A}^k \cap U(\theta, \frac{m\pi}{k}) = \mathbb{A}^k$  is the corresponding decomposition in half-periods of  $\mathbb{A}^k$  given.

Let  $I = U(\theta, \frac{\pi}{k})$  be an arc width  $\frac{\pi}{k}$  such that  $\theta \pm \frac{\pi}{2k} \notin \mathbb{A}^k$ .

### Remark 3.45

For every  $\alpha \in \mathbb{A}^k \cap I$  we can write the Stokes matrix as

$$K^\alpha = (K_{jl}^\alpha)_{j,l \in \{1, \dots, s\}} \in \text{Sto}_\alpha^k(A^0)$$

where the  $K_{jl}^\alpha \in \mathbb{C}^{n_j \times n_l}$  are blocks corresponding to the structure of  $Q$  (cf. Definition 3.16). We then know for  $j \neq l$  that, if  $K_{jl}^\alpha \neq 0$  then

- is  $K_{lj}^\alpha = 0$  and
- for every  $\alpha' \in (\mathbb{A}^k \cap I) \setminus \{\alpha\}$  is  $K_{jl}^{\alpha'} = 0$  as well as  $K_{lj}^{\alpha'} = 0$ .

### Remark 3.46

In the situation of Remark 3.45 there exists a common (block) permutation matrix  $P \in \text{GL}_n(\mathbb{C})$  given by  $(P)_{jl} = \delta_{\pi(j)l}$  where

- $\delta_{jl}$  is the block version of Kronecker's delta which was introduced in Definition 3.16 and
- $\pi$  is the permutation of  $\{1, \dots, s\}$  corresponding to  $q_j \prec_\theta q_l \Leftrightarrow \pi(j) < \pi(l)$

such that every matrix  $P^{-1}K^\alpha P$  is upper triangular and the observation from remark 3.45 is still satisfied.

#### Remark 3.46.1

After moving the sector  $U(\theta, \frac{\pi}{k})$  to  $U(\theta, \frac{\pi}{k}) + \frac{\pi}{k} = U(\theta + \frac{\pi}{k}, \frac{\pi}{k})$  we obtain also a corresponding permutation  $\pi'$  which is inverse to  $\pi$ . The corresponding permutation matrix  $P'$  is then given by  $P' = P^{-1}$ .

Such that the permutation  $P$  transforms Stokes matrices on the sector  $I + \frac{\pi}{k}$  to lower triangular matrices



In the single leveled case of this phenomenon is discussed on page 18 of Boalch's paper [Boa01].

In the paper [BJL79] from Balser, Jurkat and Lutz is the following Lemma stated as Lemma 2 on page 75.

**Lemma 3.47**

Let  $T \subset \{1, \dots, n\} \times \{1, \dots, n\}$  be a position set, which satisfies the *completeness property*:

if  $(j, k)$  and  $(k, l) \in T$  then is also  $(j, l) \in T$ .

Choose a indexing  $i : \{1, \dots, \mu\} \xrightarrow[\#T]{\cong} T$  of the position set and denote by  $\delta_{jl} \in \mathbb{C}$  the ordinary Kronecker's delta, then there exists for every

$$K \in \left\{ K = (K_{jl})_{j,l \in \{1, \dots, n\}} \in \text{GL}_n(\mathbb{C}) \mid K_{jl} = \delta_{jl} \text{ unless } (j, l) \in T \right\}$$

unique scalars  $t_j \in \mathbb{C}$  such that

$$K = (\text{id} + t_1 E_{i(1)}) \cdots (\text{id} + t_\mu E_{i(\mu)}).$$

**Remark 3.47.1**

The completeness property is reasonable, since for example  $S_{\{(2,3),(3,2)\}}$ , corresponding to the not complete set  $\{(2,3), (3,2)\}$ , is not stable under the product:

$$\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a & ab \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \notin S_{\{(2,3),(3,2)\}}.$$

**Lemma 3.48**

Every position set, corresponding to some block, is complete.

*Proof.* Such a position set is given by

$$T := \{(j, l) \mid j_1 \leq j \leq j_2, l_1 \leq l \leq l_2\}$$

for some  $j_1, j_2, l_1$  and  $l_2$  in  $\{1, \dots, n\}$ .

Let  $(j, k), (k, l) \in T$  be two positions. It is obvious that  $(j, l) \in T$ , since  $j_1 \leq j \leq j_2$  and  $l_1 \leq l \leq l_2$  are satisfied.  $\square$

**Corollary 3.49**

We can write the lemma 3.47 in block form, corresponding to the structure of  $Q$ . Let  $T \subset \{1, \dots, s\} \times \{1, \dots, s\}$  be a position set and choose a indexing

$$i : \{1, \dots, \mu\} \xrightarrow[\#T]{\cong} T$$

of  $T$ .

**Definition 3.49.1**

Define

$$S_T := \left\{ K = (K_{jl})_{j,l \in \{1, \dots, n\}} \in \text{GL}_n(\mathbb{C}) \mid K_{jl} = \delta_{jl} \text{ unless } (j, l) \in T \right\},$$

where  $\delta_{jl}$  is the block version of Kronecker's delta, corresponding to the structure of  $Q$ .

From the lemma 3.47 then follows that for every  $K \in S_T$  is the decomposition

$$K = K_1 \cdot K_2 \cdots K_\mu,$$

where  $K_j \in S_{\{i(j)\}}$ , is unique.

From the previous corollary we deduce the following corollary.

**Corollary 3.50**

Let  $T_1, \dots, T_r \subset \{1, \dots, s\} \times \{1, \dots, s\}$  be a distinct position sets, such that

$$T := T_1 \dot{\cup} \dots \dot{\cup} T_r \text{ as well as every } T_m \text{ satisfy the completeness property.}$$

Then is by

$$\begin{aligned} S_{T_1} \times \dots \times S_{T_m} &\longrightarrow S_T \\ (K_1, \dots, K_m) &\longmapsto K_1 \cdots K_m \end{aligned}$$

an isomorphism defined.

We can apply the previous corollary in our situation. This yields the following theorem.

**Theorem 3.51**

Let  $\theta \in S^1$  be a fixed direction which satisfies  $\theta \pm \frac{\pi}{2k} \notin \mathbb{A}^k$ , then

$$\mathcal{H}(A^0) \cong \prod_{k \in \mathcal{K}} \prod_{j \in \{1, \dots, 2k\}} \widehat{\text{Sto}}_{\theta + j \frac{\pi}{k}}^k(A^0),$$

where

$$\begin{aligned} \widehat{\text{Sto}}_{\theta}^k(A^0) &:= \left\{ K = (K_{jl})_{j,l \in \{1, \dots, s\}} \in \text{GL}_n(\mathbb{C}) \mid K_{jl} = \delta_{jl} \text{ unless } q_j \prec_{\theta} q_l \right\} \\ &= S_{\{(j,l) \mid q_j \prec_{\theta} q_l\}} \end{aligned}$$

and

$$\widehat{\text{Sto}}_{\theta}^k(A^0) := \{\rho_{\theta}^{-1}(K) \mid K \in \widehat{\text{Sto}}_{\theta}^k(A^0)\}.$$

### Remark 3.51.1

This means that the information of all Stokes-matrices on every level  $k$  can be grouped into  $2k$  matrices, which are products of the corresponding Stokes matrices.

*Proof.* We have an isomorphism

$$\eta_{\theta}^k : \prod_{\alpha \in \mathbb{A}^k \cap U(\theta, \frac{\pi}{k})} \underbrace{\text{Sto}_{\alpha}^k(A^0)}_{\substack{\parallel \\ S_{\{(j,l) \mid q_j \prec_{\alpha, \max} q_l\}}}} \xrightarrow{\cong} \underbrace{\widehat{\text{Sto}}_{\theta}^k(A^0)}_{\substack{\parallel \\ S_{\{(j,l) \mid q_j \prec_{\theta} q_l\}}}.$$

from Corollary 3.50, since

- for every  $\alpha \in \mathbb{A}$  satisfies the set  $\{(j, l) \mid q_j \prec_{\alpha, \max} q_l\}$  the completeness property, since it is defined via a transitive relation, and
- the union of all  $\{(j, l) \mid q_j \prec_{\alpha, \max} q_l\}$  for  $\alpha \in I \cap \mathbb{A}^k$  is then

$$\begin{aligned} \bigcup_{\alpha \in I \cap \mathbb{A}} \{(j, l) \mid q_j \prec_{\alpha, \max} q_l\} &= \{(j, l) \mid q_j \prec_{\alpha, \max} q_l \text{ for some } \alpha \in \mathbb{A}^k \cap U(\theta, \frac{\pi}{k})\} \\ &= \{(j, l) \mid q_j \prec_{\theta} q_l\} \quad (\text{cf. Remark 3.12}) \end{aligned}$$

and is also complete, since  $\prec_{\theta}$  is also a transitive relation.

The isomorphism of the theorem is then the Stokes germ version of

$$\underbrace{\prod_{k \in \mathcal{K}} \prod_{j \in \{1, \dots, 2k\}} \eta_{\theta+j\frac{\pi}{k}}^k}_{\substack{!! \\ \eta}} : \underbrace{\prod_{k \in \mathcal{K}} \prod_{\alpha \in \mathbb{A}^k} \text{Sto}_{\alpha}^k(A^0)}_{\substack{!! \\ \mathcal{H}(A^0)}} \xrightarrow{\cong} \prod_{k \in \mathcal{K}} \prod_{j \in \{1, \dots, 2k\}} \widehat{\text{Sto}}_{\theta+j\frac{\pi}{k}}^k(A^0).$$

□

### 3.5 The complete Diagram

If we start with the diagram from page 29, we can add all the defined isomorphisms can be written into one big diagram.

$$\begin{array}{ccc}
 \mathcal{H}(A^0) & \xrightarrow{\exp_{A^0}} & \mathcal{St}(A^0) \\
 \downarrow g & & \nearrow \mathcal{T} \\
 & & \prod_{k \in \mathcal{K}} \Gamma(\mathcal{U}^k; \Lambda^k(A^0)) \\
 & \nearrow \prod_{k \in \mathcal{K}} i^k & \\
 & \prod_{k \in \mathcal{K}} \prod_{\alpha \in \mathbb{A}} \text{Sto}_{\alpha}^k(A^0) & \\
 \chi \circ \prod_{\alpha \in \mathbb{A}} i_{\alpha} \nearrow & \downarrow \prod_{k \in \mathcal{K}} \prod_{\alpha \in \mathbb{A}} \rho_{\alpha}^k & \searrow \\
 \prod_{\alpha \in \mathbb{A}} \text{Sto}_{\alpha}(A^0) & \prod_{k \in \mathcal{K}} \prod_{\alpha \in \mathbb{A}} \text{Sto}_{\alpha}^k(A^0) & \prod_{k \in \mathcal{K}} \prod_{j \in \{1, \dots, 2k\}} \widehat{\text{Sto}}_{\theta+j\frac{\pi}{k}}^k(A^0) \\
 \downarrow \prod_{\alpha \in \mathbb{A}} \rho_{\alpha} & \nearrow \chi \circ \prod_{\alpha \in \mathbb{A}} i_{\alpha} & \searrow \eta \\
 \prod_{\alpha \in \mathbb{A}} \text{Sto}_{\alpha}(A^0) & & \prod_{k \in \mathcal{K}} \prod_{j \in \{1, \dots, 2k\}} \widehat{\text{Sto}}_{\theta+j\frac{\pi}{k}}^k(A^0) \\
 & \searrow \text{Cor.3.31} & \downarrow \text{(induced by } \rho) \\
 & \prod_{\alpha \in \mathbb{A}} \mathbb{C}^{\sum_{q_j \prec_{\alpha, \max} q_l} \deg(q_j - q_l) \cdot n_j \cdot n_l} & \prod_{k \in \mathcal{K}} \prod_{j \in \{1, \dots, 2k\}} \widehat{\text{Sto}}_{\theta+j\frac{\pi}{k}}^k(A^0)
 \end{array}$$

Where

- the **purple path** is the isomorphism  $h$  from theorem 3.30 and
- we denote

$$- \chi : \prod_{\alpha \in \mathbb{A}} \prod_{k \in \mathcal{K}} \text{Sto}_{\alpha}^k(A^0) \equiv \prod_{k \in \mathcal{K}} \prod_{\alpha \in \mathbb{A}} \text{Sto}_{\alpha}^k(A^0) \text{ the reordering and}$$

$$- \text{by abuse of notation we also denote the Stokes matrix version in the same way } \chi : \prod_{\alpha \in \mathbb{A}} \prod_{k \in \mathcal{K}} \text{Sto}_{\alpha}^k(A^0) \equiv \prod_{k \in \mathcal{K}} \prod_{\alpha \in \mathbb{A}} \text{Sto}_{\alpha}^k(A^0).$$

## 4 Structure of Stokes germs and Stokes matrices

Here we want to discuss, which information is required to describe the Stokes cocycle corresponding to a multileveled system in more depth. We will start by looking at a single-leveled system determined by a normal form  $A^0 \in \mathrm{GL}_3(\mathbb{C}(\{t\}))$  with exactly 2 levels. We will apply the techniques developed in the previous chapter in a rather explicit way.

Later we will look at a more general multileveled case.

### 4.1 Simplest, multileveled case

Let  $A^0$  be a normal form with dimension  $n = 3$  and two levels  $\mathcal{K} = \{k_1 < k_2\}$ , which satisfies that there is at least one anti-Stokes direction  $\theta$  which is beared by both levels. Let  $q_j(t^{-1})$  be the determining polynomials and let  $k_{jl}$  be the degrees of  $(q_j - q_l)(t^{-1})$ . Since it is the only way to get two levels in dimension 3 we have, up to permutation:  $k_1 = k_{1,2} \geq k_{1,3}$  and  $k_2 = k_{2,3}$ . We know also that the leading terms of  $(q_1 - q_2)(t^{-1})$  and  $(q_1 - q_3)(t^{-1})$  are equal and thus determine the same anti-Stokes directions. The set of all anti-Stokes directions is then given as

$$\mathbb{A} = \left\{ \theta + \frac{\pi}{k} \cdot j \mid k \in \mathcal{K}, j \in \mathbb{N} \right\} =: \underbrace{\{\alpha_1, \dots, \alpha_\nu\}}_{\theta}.$$

Denote by  $\mathcal{Y}_0(t)$  a normal solution of  $[A^0]$ .

#### 4.1.1 What do the Stokes germs $\varphi_\alpha$ look like?

First we want to use the Stokes matrices of Stokes germs, discussed in Section 3.2.2, to obtain some statements about the structure of the Stokes germs in our situation.

The Proposition 3.17 states that every Stokes germ  $\varphi_\alpha$  can be written as its matrix representation conjugated by the normal solution. That means that there is a Stokes matrix in  $\mathrm{Sto}_\alpha(A^0)$ , which gets via

$$\begin{aligned} \rho_\alpha^{-1} : \mathrm{Sto}_\alpha(A^0) &\xrightarrow{\cong} \mathrm{Sto}_\alpha(A^0) \\ C_{\varphi_\alpha} &\longmapsto \mathcal{Y}_0 C_{\varphi_\alpha} \mathcal{Y}_0^{-1}, \end{aligned}$$

mapped to the Stokes germ  $\varphi_\alpha$ .

**Using the structure of  $\text{Sto}_\alpha(A^0)$**  First look at an example in which we will explain, from which relations on the determining polynomials which restriction on the form of the Stokes matrices follow.

### Example 4.1

Let  $\alpha \in \mathbb{A}$  be an anti-Stokes direction. From the definition of  $\text{Sto}_\alpha(A^0)$  (cf. Definition 3.16) we know that, if one has  $q_1 \prec_{\alpha, \max} q_2$ , the Stokes matrix has the form

$$\begin{pmatrix} 1 & c_1 & \star \\ 0 & 1 & \star \\ \star & \star & 1 \end{pmatrix}$$

where  $c_j \in \mathbb{C}$  and  $\star \in \mathbb{C}$ .

#### Remark 4.1.1

In our situation, we also know that

$$q_1 \prec_{\alpha, \max} q_2 \Leftrightarrow q_1 \prec_{\alpha, \max} q_3 \quad \text{respectively} \quad q_2 \prec_{\alpha, \max} q_1 \Leftrightarrow q_3 \prec_{\alpha, \max} q_1$$

since the leading terms of  $q_1 - q_2$  and  $q_1 - q_3$  are equal.

Thus the representation has the form

$$\begin{pmatrix} 1 & c_1 & c_2 \\ 0 & 1 & \star \\ 0 & \star & 1 \end{pmatrix}.$$

If we also know that neither  $q_2 \prec_{\alpha, \max} q_3$  nor  $q_3 \prec_{\alpha, \max} q_2$  it has the form

$$\begin{pmatrix} 1 & c_1 & c_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We also know that every matrix of this form is a representation to some Stokes germ. Thus we have an isomorphism

$$\begin{aligned} \vartheta_\alpha : \mathbb{C}^2 &\longrightarrow \text{Sto}_\alpha(A^0) \\ (c_1, c_2) &\longmapsto \begin{pmatrix} 1 & c_1 & c_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

In fact, the following 9 cases of Stokes matrices can arise.

	$q_2 \prec_{\alpha, \max} q_3$	$q_3 \prec_{\alpha, \max} q_2$	else
$q_1 \prec_{\alpha, \max} q_2$ and $q_1 \prec_{\alpha, \max} q_3$	$\begin{pmatrix} 1 & c_2 & c_3 \\ 0 & 1 & c_1 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & c_2 & c_3 \\ 0 & 1 & 0 \\ 0 & c_1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & c_2 & c_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
$q_2 \prec_{\alpha, \max} q_1$ and $q_3 \prec_{\alpha, \max} q_1$	$\begin{pmatrix} 1 & 0 & 0 \\ c_2 + c_1 c_3 & 1 & c_1 \\ c_3 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ c_2 & 1 & 0 \\ c_1 c_2 + c_3 & c_1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ c_2 & 1 & 0 \\ c_3 & 0 & 1 \end{pmatrix}$
else	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & c_1 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c_1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

In the **blue** cases we have  $\mathcal{K}_\alpha = \mathcal{K}$  and  $\mathbb{C}^3 \xrightarrow[\cong]{\vartheta_\alpha} \text{Sto}_\alpha(A^0)$ . In the **green** cases  $\mathcal{K}_\alpha = \{k_2\}$  and  $\mathbb{C}^2 \xrightarrow[\cong]{\vartheta_\alpha} \text{Sto}_\alpha(A^0)$  as well as in the **purple** cases  $\mathcal{K}_\alpha = \{k_1\}$  and  $\mathbb{C}^1 \xrightarrow[\cong]{\vartheta_\alpha} \text{Sto}_\alpha(A^0)$ . Thus, for every  $\alpha \in \mathbb{A}$ , we have an isomorphism  $\rho_\alpha^{-1} \circ \vartheta_\alpha$ . Two of the cases are more complicated than necessary, to be consistent with the decomposition in the next part (cf. Example 4.2).

**Decomposition by levels** In proposition 3.24 and especially remark 3.24.1 we have defined a decomposition of the Stokes group  $\text{Sto}_\alpha(A^0)$  in subgroups generated by  $k$ -germs for  $k \in \mathcal{K}$ . In our case, we have at most two level, such that this decomposition is given by

$$\varphi_\alpha = \varphi_\alpha^{k_1} \varphi_\alpha^{k_2} \xrightarrow{i_\alpha} (\varphi_\alpha^{k_1}, \varphi_\alpha^{k_2}) \in \text{Sto}_\alpha^{k_1}(A^0) \times \text{Sto}_\alpha^{k_2}(A^0),$$

where  $\varphi_\alpha^{k_1} \in \text{Sto}_\alpha^{k_1}(A^0) = \text{Sto}_\alpha^{<k_2}(A^0)$ ,  $\varphi_\alpha^{k_2} \in \text{Sto}_\alpha^{k_2}(A^0)$  and  $i_\alpha$  is the map, which gives the factors of this factorization in ascending order.

This decomposition of a germ  $\varphi_\alpha$  is trivial if  $\#\mathcal{K}(\varphi_\alpha) \leq 1$ , thus the interesting cases are the **blue** cases.

#### Example 4.2

Look at the example

$$\vartheta_\alpha(c_1, c_2, c_3) = \mathcal{Y}_0 \begin{pmatrix} 1 & 0 & 0 \\ c_2 & 1 & 0 \\ c_1 c_2 + c_3 & c_1 & 1 \end{pmatrix} \mathcal{Y}_0^{-1} = \varphi_\alpha.$$



According to remark 3.22.1 the factor  $\varphi_\alpha^{k_1}$ , generated by the  $k_1$ -germs, is given by

$$\varphi_\alpha^{k_1} = \mathcal{Y}_0 \begin{pmatrix} 1 & 0 & 0 \\ \mathbf{0} & 1 & 0 \\ \mathbf{0} & c_1 & 1 \end{pmatrix} \mathcal{Y}_0^{-1}.$$

The other factor  $\varphi_\alpha^{k_2}$  is then obtained as

$$\begin{aligned} \varphi_\alpha^{k_2} &= (\varphi_\alpha^{k_1})^{-1} \varphi_\alpha^{k_2} \\ &= \mathcal{Y}_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -c_1 & 1 \end{pmatrix} \underbrace{\mathcal{Y}_0^{-1} \mathcal{Y}_0}_{=\text{id}} \begin{pmatrix} 1 & 0 & 0 \\ c_2 & 1 & 0 \\ c_1 c_2 + c_3 & c_1 & 1 \end{pmatrix} \mathcal{Y}_0^{-1} \\ &= \mathcal{Y}_0 \begin{pmatrix} 1 & 0 & 0 \\ c_2 & 1 & 0 \\ c_3 & 0 & 1 \end{pmatrix} \mathcal{Y}_0^{-1}. \end{aligned}$$

The four nontrivial decomposition in our situation, are given by

$$\begin{aligned} 1. \quad & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & c_1 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & c_2 & c_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & c_2 & c_3 \\ 0 & 1 & c_1 \\ 0 & 0 & 1 \end{pmatrix} \\ 2. \quad & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c_1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & c_2 & c_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & c_2 & c_3 \\ 0 & 1 & 0 \\ 0 & c_1 & 1 \end{pmatrix} \\ 3. \quad & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & c_1 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ c_2 & 1 & 0 \\ c_3 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ c_2 + c_1 c_3 & 1 & c_1 \\ c_3 & 0 & 1 \end{pmatrix} \\ 4. \quad & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c_1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ c_2 & 1 & 0 \\ c_3 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ c_2 & 1 & 0 \\ c_1 c_2 + c_3 & c_1 & 1 \end{pmatrix} \end{aligned}$$

We will use this decompositions to write the isomorphisms  $\vartheta_\alpha : \mathbb{C}^\star \rightarrow \text{Sto}_\alpha(A^0)$  as

$$\begin{array}{ccc} \mathbb{C}^\star & \xrightarrow{j_\alpha} & \text{Sto}_\alpha^{k_1}(A^0) \times \text{Sto}_\alpha^{k_2}(A^0) \xrightarrow{i_\alpha^{-1}} \text{Sto}_\alpha(A^0) \\ & \searrow & \uparrow \\ & & \vartheta_\alpha \end{array}$$

where  $\star \in \{1, 2, 3\}$ .

### 4.1.2 What do Stokes cocycles look like?

#### Proposition 4.3

By taking the product over  $\mathbb{A}$  of all  $\rho_\alpha^{-1} \circ \vartheta_\alpha$  we obtain the isomorphism

$$\prod_{\alpha \in \mathbb{A}} \rho_\alpha^{-1} \circ \vartheta_\alpha : \mathbb{C}^{k_1+2 \cdot k_2} \xrightarrow{\cong} \prod_{\alpha \in \mathbb{A}} \text{Sto}_\alpha(A^0) \xrightarrow{\cong} \prod_{\alpha \in \mathbb{A}} \text{Sto}_\alpha(A^0)$$

which can be concatenated with  $h$  to obtain an element in  $\mathcal{St}(A^0)$ . We can also define the isomorphism  $\mathbb{C}^{2 \cdot (k_1+2 \cdot k_2)} \xrightarrow{\quad} \mathcal{St}(A^0)$  which makes the following diagram commute.

$$\begin{array}{ccc} & \mathbb{C}^{2 \cdot (k_1+2 \cdot k_2)} & \\ \swarrow \text{blue} & & \searrow \text{red} \\ \prod_{\alpha \in \mathbb{A}} \text{Sto}_\alpha(A^0) & \xrightarrow{\text{green } h} & \mathcal{St}(A^0) \end{array}$$

$\prod_{\alpha \in \mathbb{A}} \rho_\alpha^{-1} \circ \vartheta_\alpha$

Using the knowledge, how the morphism were build and the fact that some of the morphism are equal, we can rewrite this diagram as follows

$$\begin{array}{ccc} & \mathbb{C}^{2 \cdot (k_1+2 \cdot k_2)} & \\ & \downarrow \text{red } \prod_{\alpha \in \mathbb{A}} j_\alpha & \\ & \prod_{k \in \{k_1, k_2\}} \prod_{\alpha \in \mathbb{A}} \text{Sto}_\alpha^k(A^0) & \\ \swarrow \text{blue } \prod_{\alpha \in \mathbb{A}} i_\alpha^{-1} & & \searrow \text{red } \prod_{k \in \{k_1, k_2\}} \prod_{\alpha \in \mathbb{A}} (\rho_\alpha^k)^{-1} \\ \prod_{\alpha \in \mathbb{A}} \text{Sto}_\alpha(A^0) & & \prod_{k \in \{k_1, k_2\}} \prod_{\alpha \in \mathbb{A}} \text{Sto}_\alpha^k(A^0) \\ \downarrow \text{blue } \prod_{\alpha \in \mathbb{A}} \rho_\alpha^{-1} & \xrightarrow{\text{green } \prod_{\alpha \in \mathbb{A}} i_\alpha} & \downarrow \text{red } \prod_{k \in \{k_1, k_2\}} i^k \\ \prod_{\alpha \in \mathbb{A}} \text{Sto}_\alpha(A^0) & & \prod_{k \in \{k_1, k_2\}} \Gamma(\mathcal{U}^k; \Lambda^k(A^0)) \\ & & \downarrow \text{red } \tau \\ & & \mathcal{St}(A^0) \end{array}$$

### Explicit example

Even more explicit, we can fix the levels  $k_1 = 1$  and  $k_2 = 3$  together with  $\theta = 0$ . Assume without any restriction that  $q_1 \prec_{\theta, \max} q_2$  and  $q_1 \prec_{\theta, \max} q_3$  as well as  $q_2 \prec_{\theta, \max} q_3$ .

The classification space is in this case isomorphic to  $\mathbb{C}^{2 \cdot (1+2 \cdot 3)} = \mathbb{C}^{14}$ . The element

$$({}^1c_1, {}^2c_1, {}^1c_2, {}^1c_3, {}^2c_2, {}^2c_3, \dots, {}^6c_2, {}^6c_3) \in \mathbb{C}^{14}$$

gets, via the morphism  $\prod_{\alpha \in \mathbb{A}} j_\alpha$ , mapped to

$$\left( \left( \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & {}^1c_1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & {}^2c_1 & 1 \end{pmatrix} \right), \right. \\ \left. \left( \begin{pmatrix} 1 & {}^1c_2 & {}^1c_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ {}^2c_2 & 1 & 0 \\ {}^2c_3 & 0 & 1 \end{pmatrix}, \dots, \begin{pmatrix} 1 & 0 & 0 \\ {}^6c_2 & 1 & 0 \\ {}^6c_3 & 0 & 1 \end{pmatrix} \right) \right)$$

in  $\prod_{\alpha \in \mathbb{A}^1} \text{Sto}_\alpha^1(A^0) \times \prod_{\alpha \in \mathbb{A}^3} \text{Sto}_\alpha^3(A^0)$  and thus the element

$$\left( \left( \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & {}^1c_1 \\ 0 & 0 & 1 \end{pmatrix}, \text{id}, \text{id}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & {}^2c_1 & 1 \end{pmatrix}, \text{id}, \text{id} \right), \right. \\ \left. \left( \begin{pmatrix} 1 & {}^1c_2 & {}^1c_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ {}^2c_2 & 1 & 0 \\ {}^2c_3 & 0 & 1 \end{pmatrix}, \dots, \begin{pmatrix} 1 & 0 & 0 \\ {}^6c_2 & 1 & 0 \\ {}^6c_3 & 0 & 1 \end{pmatrix} \right) \right)$$

in  $\prod_{\alpha \in \mathbb{A}} \text{Sto}_\alpha^1(A^0) \times \prod_{\alpha \in \mathbb{A}} \text{Sto}_\alpha^3(A^0)$ . Using the morphism  $\prod_{\alpha \in \mathbb{A}} i_\alpha^{-1}$  we get a complete set of Stokes matrices as

$$\left( \begin{pmatrix} 1 & {}^1c_2 & {}^1c_3 \\ 0 & 1 & {}^1c_1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ {}^2c_2 & 1 & 0 \\ {}^2c_3 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & {}^3c_2 & {}^3c_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \right. \\ \left. \begin{pmatrix} 1 & 0 & 0 \\ {}^4c_2 & 1 & 0 \\ {}^{2c_1}{}^4c_2 + {}^4c_3 & {}^{2c_1} & 1 \end{pmatrix}, \begin{pmatrix} 1 & {}^5c_2 & {}^5c_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ {}^6c_2 & 1 & 0 \\ {}^6c_3 & 0 & 1 \end{pmatrix} \right) \in \prod_{\alpha \in \mathbb{A}} \text{Sto}_\alpha(A^0)$$

## 4.2 In the more general multileveled context

Let us assume, that our normal form  $[A^0]$  of dimension  $n$  has the levels  $\mathcal{K}$ .

**Corollary 4.4**

We now can improve the diagram of isomorphisms from Proposition 4.3 by writing it in a more general form and adding the found isomorphism. Let  $\theta$  be a direction, which satisfies  $\theta \pm \frac{\pi}{2k} \notin \mathbb{A}^k$  for every  $k \in \mathcal{K}$ .

$$\begin{array}{c}
 \mathbb{C}^{\sum_{j \in \mathcal{K}} q_j - \sum_{l \in \mathcal{K}} q_l \deg(q_j - q_l)} \\
 \downarrow \prod_{\alpha \in \mathbb{A}} j_\alpha \\
 \prod_{k \in \mathcal{K}} \prod_{\alpha \in \mathbb{A}} \text{Sto}_\alpha^k(A^0) \longrightarrow \prod_{k \in \mathcal{K}} \prod_{j \in \{1, \dots, 2k\}} \widehat{\text{Sto}}_{\theta + j \frac{\pi}{k}}^k(A^0) \\
 \swarrow \prod_{\alpha \in \mathbb{A}} i_\alpha^{-1} \quad \searrow \prod_{k \in \mathcal{K}} \prod_{\alpha \in \mathbb{A}} (\rho_\alpha^k)^{-1} \\
 \prod_{\alpha \in \mathbb{A}} \text{Sto}_\alpha(A^0) \quad \prod_{\alpha \in \mathbb{A}} \text{Sto}_\alpha(A^0) \xrightarrow{\prod_{\alpha \in \mathbb{A}} i_\alpha} \prod_{k \in \mathcal{K}} \prod_{\alpha \in \mathbb{A}} \text{Sto}_\alpha^k(A^0) \\
 \downarrow \prod_{\alpha \in \mathbb{A}} \rho_\alpha^{-1} \quad \downarrow \prod_{k \in \mathcal{K}} i^k \\
 \prod_{\alpha \in \mathbb{A}} \text{Sto}_\alpha(A^0) \xrightarrow{\prod_{\alpha \in \mathbb{A}} i_\alpha} \prod_{k \in \mathcal{K}} \prod_{\alpha \in \mathbb{A}} \text{Sto}_\alpha^k(A^0) \xrightarrow{\prod_{k \in \mathcal{K}} i^k} \prod_{k \in \mathcal{K}} \Gamma(\mathcal{U}^k; \Lambda^k(A^0)) \\
 \downarrow \tau \\
 \text{St}(A^0)
 \end{array}$$

$\text{St}(A^0) \longrightarrow \prod_{k \in \mathcal{K}} \prod_{j \in \{1, \dots, 2k\}} \widehat{\text{Sto}}_{\theta + j \frac{\pi}{k}}^k(A^0)$

**Remark 4.4.1**

The contained isomorphism

$$\prod_{\alpha \in \mathbb{A}} \text{Sto}_\alpha(A^0) \longrightarrow \prod_{k \in \mathcal{K}} \prod_{j \in \{1, \dots, 2k\}} \widehat{\text{Sto}}_{\theta + j \frac{\pi}{k}}^k(A^0)$$

is in the single-leveled case with  $n = s$ , i.e. all diagonal elements of  $Q$  are different, given in Lemma 3.2 of Boalch's paper [Boa01, Lem.3.2]. In this case is every  $\widehat{\text{Sto}}_{\theta + j \frac{\pi}{k}}^k(A^0)$  isomorphic to some  $PU_+ P^{-1}$  where  $U_+$  is the group of all upper triangular matrices, with ones on the diagonal, and  $P$  is a permutation matrix.

# A More basics

## A.1 Semidirect products

We will refer to [Rob03, p. 75] for semidirect products. Let  $G$  be a group with a normal subgroup  $N$  ( $N \triangleleft G$ ) and a subgroup  $H$  such that

$$G = NH \quad \text{and} \quad N \cap H = 1.$$

Then  $G$  is said to be the (*internal*) *semidirect product* of  $N$  and  $H$ ,  $G = N \rtimes H$ .

### Remark A.1

1. Each element  $g \in G$  has a unique decomposition  $g = nh$  with  $n \in N$  and  $h \in H$ . Since

for  $g = n'h'$  another such expression, then  $(n')^{-1}n = h'h^{-1} \in N \cap H = 1$  such that  $n = n'$  and  $h = h'$ .

2. Conjugation in  $N$  by an element  $h \in H$  defines an automorphism of  $N$ , say  $\rho(h)$  which satisfies for  $h_i \in H$

$$\rho(h_1, h_2) = \rho(h_1)\rho(h_2)$$

and thus  $\rho : H \rightarrow \text{Aut}(N)$  is a homomorphism.

In the other direction, one can get from  $N$ ,  $H$  and a given homomorphism  $\rho : H \rightarrow \text{Aut}(N)$  the same structure back. This will be the so-called *external semidirect product*. It is defined by the underlying group

$$G := \{(n, h) \mid n \in N, h \in H\}$$

together with the multiplication

$$\begin{aligned} G \times G &\rightarrow G \\ ((n_1, h_1), (n_2, h_2)) &\mapsto (n_1\rho(h_1)(n_2), h_1h_2) \end{aligned}$$

with the identity element  $(1_N, 1_H)$  and the inverse of  $(n, h)$  given by  $(\rho(h^{-1})(n^{-1}), h^{-1})$ . In  $G$  there are the natural subgroups

$$\bar{N} = \{(n, 1_H) \mid n \in N\} \quad \text{and} \quad \bar{H} = \{(1_N, h) \mid h \in H\}$$

which are canonically isomorphic to  $N$  and  $H$  respectively. For  $n \in N$  and  $h \in H$  we have

$$(n, 1_H)(1_N, h) = (n \underbrace{\rho(1_H)}_{\text{id}_N}(1_N), h) = (n, h) \in \bar{N}\bar{H}.$$

It follows that  $G = \bar{N}\bar{H}$  and  $\bar{N} \cap \bar{H} = 1$ . To check that  $G = \bar{N} \rtimes \bar{H}$  we only have to show that  $\bar{N}$  is normal in  $G$ . Let  $n, n' \in N$  and  $h \in H$  then

$$\begin{aligned} (n, h)(n', 1_H)(n, h)^{-1} &= (n, h)(n', 1_H)(\rho(h^{-1})(n^{-1}), h^{-1}) \\ &= (n\rho(h)(n'), h)(\rho(h^{-1})(n^{-1}), h^{-1}) \\ &= (n\rho(h)(n')\rho(h)\rho(h^{-1})(n^{-1}), 1_H) \\ &= (n\rho(h)(n')n^{-1}, 1_H) \in \bar{N}. \end{aligned}$$

In the case, where  $\rho$  is the trivial homomorphism

$$\begin{aligned} \rho : H &\rightarrow \text{Aut}(N) \\ h &\mapsto \text{id}_N \end{aligned}$$

the elements of  $\bar{N}$  and  $\bar{H}$  commute, so that  $G$  becomes the direct product  $N \times H$ . Thus the semidirect product is a generalization of the direct product of two groups.

## A.2 Faithful representations

[Hal03, Def.4.1] says the following about faithful representations:

### Definition A.2

Let  $G$  be a matrix Lie group. Then a (*finite-dimensional complex*) *representation* of  $G$  is a Lie group homomorphism

$$\rho : G \rightarrow \text{GL}(V)$$

where

- $V$  is a finite-dimensional complex vector space.

If  $\rho$  is a one-to-one homomorphism, then the representation is called *faithful*.

**Remark A.2.1**

If a representation  $\rho$  is a faithful representation of a matrix Lie group  $G$  then  $\{\rho(A) \mid A \in G\}$  is a group of matrices that is isomorphic to the original group  $G$ . Thus,  $\rho$  allows us to represent  $G$  as a group of matrices.

## A.3 Multisummability

The aim of a theory of summation is to associate with any series an asymptotic function uniquely determined in a way as much natural as possible. We will take this as a black box and only use the fact, that it makes the asymptotic expansion unique.

A useful resource for this topic is Loday-Richaud's book [Lod14].

Let  $[A^0]$  be a normal form and  $\hat{F} \in \hat{G}(A^0)$  a formal transformation. Denote the transformed system by  $A = \hat{F}A^0$ .

Let  $\dot{\varphi} = (\dot{\varphi}_j)_{j \in J} \in \Gamma(\dot{\mathcal{V}}; \Lambda(A^0))$  be a 1-cocycle in the image of the equivalence class of  $\hat{F}$  by  $\exp$ .

The following Proposition can be found in [Lod94, Prop.III.2.1], [Lod14, Thm.4.3.13] and [BJL79, Thm.1].

**Proposition A.3**

There exists a **unique** family of realizations  $(F_j)_{j \in J}$  of  $\hat{F}$  over  $\mathcal{V}$ , i.e. matrices  $F_j$  which are analytic on  $V_j$ , satisfy  $[A^0, A]$  and are asymptotic to  $\hat{F}$  on  $V_j$ , such that

$$\dot{\varphi}_j = F_{j-1} F_j^{-1}$$

for every  $j \in J$ .

When  $\mathcal{V}$  is the, in section 3.3.2 defined, cyclic covering  $\mathcal{U}^{\leq k_r} =: \mathcal{U}$  and  $\dot{\varphi}$  is in its Stokes form, we call the realizations  $F_\alpha$ , the *sums of  $\hat{F}$* .

**Definition A.4**

Denote by  $\alpha^+$  the next anti-Stokes direction on the right of  $\alpha$  we can define

- $S_\alpha^-(\hat{F}) := F_\alpha$  as the *sum of  $\hat{F}$  on the left of  $\alpha$*  and
- $S_\alpha^+(\hat{F}) := F_{\alpha^+}$  as the *sum of  $\hat{F}$  on the right of  $\alpha$* .

Let  $\widehat{F}_0 \in \widehat{G}(A^0)$  and  $A^1 = \widehat{F}_0 A^0$  and let

$$\exp_{A^1}(\widehat{F}) : G \backslash \widehat{G}(A^1) \rightarrow H^1(S^1; \Lambda(A^1))$$

be the Malgrange-Sibuya isomorphism (cf. Remark 3.6).

The definition of the subsheaf  $\Lambda^{\geq k}(A^1)$  of  $\Lambda(A^1)$  needs a few additional justifications (cf. [Lod94, p. 883]).

### Definition A.5

A 1-cocycle  $\dot{\varphi} = (\dot{\varphi}_j)_{j \in J}$  is *k-summable* if, for all  $j \in J$ , it satisfies the conditions

1.  $\dot{\varphi}_j$  is of level  $\geq k$ , i.e.  $\varphi_j \in \Gamma(\dot{V}_j; \Lambda^{\geq k}(A^1))$  and
2. the opening of  $\dot{V}_j$  is  $\frac{\pi}{k}$ .

An element of  $\widehat{F} \in \widehat{G}(A^1)$  is *k-summable* when  $\exp_{A^1}(\widehat{F})$  contains a *k-summable* cocycle.

If in a cohomology class is a *k-summable* cocycle, it is unique up to extra trivial components. The realizations of such a *k-summable* cocycle define the *k-sums* of  $\widehat{F}$ . When  $[A^1]$  is meromorphically equivalent to  $[A^0]$ , i.e.  $\widehat{F} \in G(\{t\})$ , then

$$\widehat{F} \text{ is } k\text{-summable}$$

if and only if

$$\text{its Stokes cocycle } \dot{\varphi} = (\dot{\varphi}_\alpha)_{\alpha \in \mathbb{A}} \text{ belongs to } \Gamma(\mathcal{U}^k; \Lambda^k).$$

This occurs when the Stokes cocycle  $\dot{\varphi}$  of  $\widehat{F}$  bears only the level  $k$ .

### Definition A.6

Let  $\widehat{F} \in \widehat{G}(A^1)$  be *k-summable* and  $(\dot{\varphi}_j)_{j \in J} \in \prod_{j \in J} \Gamma(\dot{V}_j; \Lambda(A^1))$  the corresponding *k-summable* cocycle in  $\exp_{A^1}(\widehat{F})$ .

To each  $j \in J$  with a nontrivial  $\dot{\varphi}_j$  we define the bisecting direction of  $\dot{V}_j$  as a *singular direction* for  $\widehat{F}$ .

## A.3.1 Factorization

Let  $\dot{f} \in \Gamma(\mathcal{U}; \Lambda(A^0))$  be the Stokes cocycle associated to  $\widehat{F}$  and  $k = \max(\mathcal{K}(\dot{f}))$  be the maximal level beared by  $\dot{f}$ . We then know, since  $\dot{f}^{\leq k} = \dot{f}$  and we have corollary 3.22 that there is a unique decomposition

$$\dot{f} = \dot{f}^{<k} \dot{g}^k \quad \text{and} \quad \dot{f} = \dot{f}^k \dot{f}^{<k}$$



where  $\dot{f}^k, \dot{g}^k \in \Gamma(\dot{\mathcal{U}}^{\leq k}; \Lambda^k(A^0))$  and  $\dot{f}^{<k} \in \Gamma(\dot{\mathcal{U}}^{\leq k}; \Lambda^{<k}(A^0))$ .

Since the cohomology class of  $\dot{f}^{<k}$  belongs to  $H^1(S^1; \Lambda(A^0))$  we know from the Malgrange-Sibuya theorem that there exists an, up to left meromorphic factor unique, element  $\widehat{F}^{<0}$  of  $\widehat{G}(A^0)$  such that  $\dot{f}^{<k}$  belongs to  $\exp_{A^0}(\widehat{F}^{<k})$ .

The Proposition [Lod94, Prop.III.2.6] states the following.

### Proposition A.7

We have, that

1.  $\widehat{F}^k$  is  $k$ -summable with singular directions (cf. Definition A.6) belonging to  $\mathbb{A}^k$
2. The levels in the Stokes cocycle of  $\widehat{F}^{<k}$  are  $< k$ .
3. The decomposition  $\widehat{F} = \widehat{F}^k \widehat{F}^{<k}$  with properties 1. and 2. is essentially unique, that means:

if  $\widehat{F} = \widehat{H}^k \widehat{H}^{<k}$  is another decomposition, there is a matrix  $h \in G(\{t\})$  such that  $\widehat{H}^{<k} = h \widehat{F}^{<k}$  and  $\widehat{H}^k = \widehat{F}^k h^{-1}$ .

We can use the Proposition A.7 to obtain the following factorization on transformations, which can be found in Loday's Paper [Lod94, Thm.III.2.5] or as [MR91, Thm.4.7].

### Theorem A.8

Let  $\widehat{F} \in \widehat{G}(A^0)$  be a formal transformation and  $\mathcal{K} = \{k_1 < k_2 < \dots < k_r\}$  be the set of levels of  $[A^0]$ . Then  $\widehat{F}$  can be factored in

$$\widehat{F} = \widehat{F}_r \widehat{F}_{r-1} \dots \widehat{F}_2 \widehat{F}_1$$

where the matrices  $\widehat{F}_j$  are

- $k_j$ -summable and
- with singular directions belonging to the set  $\mathcal{A}^{k_j}$ .

## A.3.2 The map $g : G \backslash \widehat{G}(A^0) \rightarrow \prod_{\alpha \in \mathcal{A}} \text{Sto}_{\alpha}(A^0)$

There are other definitions, which are equivalent to definition A.4. Since some of them are more direct and do not use the 1-cocycle, to which they correspond, they can be used to obtain the corresponding 1-cocycle. Many different approaches to define the sums can be found in [Lod14].

Let us fix an ambassador  $\hat{F}$  in  $G \backslash \hat{G}(A^0)$ . We then use proposition A.3 to obtain an 1-cocycle. An element in  $\prod_{\alpha \in \mathbb{A}} \text{Sto}_\alpha(A^0)$  corresponding to  $\hat{F}$  is found as

$$(\varphi_{\alpha_1}, \dots, \varphi_{\alpha_\nu}) \in \prod_{\alpha \in \mathbb{A}} \text{Sto}_\alpha(A^0)$$

by setting  $\varphi_\alpha := \left(S_\alpha^-(\hat{F})\right)^{-1} S_\alpha^+(\hat{F}) \in \text{Sto}_\alpha(A^0)$  (cf. Definition A.4).

**Remark A.9**

$S_\alpha^+(\hat{F})t^L e^{Q(t^{-1})}$  is a solution of  $[A]$  on the corresponding sector. Using the equation (3.1) we can write

$$S_\alpha^+(\hat{F})\mathcal{Y}_{0,\alpha}(t) = S_\alpha^-(\hat{F})\mathcal{Y}_{0,\alpha}(t)C_{\mathcal{Y}_{0,\alpha}}$$

thus the Stokes matrix  $C_{\mathcal{Y}_{0,\alpha}}$  is a matrix, which describes the blending between the two adjacent sectors.

# Bibliography

- [BJL79] W Balser, W.B Jurkat, and D.A Lutz. “Birkhoff invariants and stokes’ multipliers for meromorphic linear differential equations”. In: *Journal of Mathematical Analysis and Applications* 71 (1 1979). DOI: 10.1016/0022-247x(79)90217-8.
- [Boa01] Philip Boalch. “Symplectic Manifolds and Isomonodromic Deformations”. In: *Advances in Mathematics* 163.2 (Nov. 10, 2001), pp. 137–205. ISSN: 00018708. DOI: 10.1006/aima.2001.1998. URL: <http://dx.doi.org/10.1006/aima.2001.1998>.
- [Boa99] Philip Boalch. “Symplectic Geometry and Isomonodromic Deformations”. PhD thesis. Wadham College, Oxford, 1999. URL: <http://www.dma.ens.fr/~boalch/files/thesis.pdf>.
- [BV89] D.G. Babbitt and V.S. Varadarajan. *Local moduli for meromorphic differential equations*. Astérisque. Société mathématique de France, 1989. URL: <http://books.google.de/books?id=VAsZAQAIAAJ>.
- [Lod14] Michèle Loday-Richaud. “Divergent series and differential equations”. Part of a book. Submitted. Jan. 2014. URL: <https://hal.archives-ouvertes.fr/hal-01011050>.
- [Lod94] Michèle Loday-Richaud. “Stokes phenomenon, multisummability and differential Galois groups”. eng. In: *Annales de l’institut Fourier* 44.3 (1994), pp. 849–906. URL: <http://eudml.org/doc/75083>.
- [PS03] M. van der Put and M.F. Singer. *Galois Theory of Linear Differential Equations*. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen. Springer Berlin Heidelberg, 2003. ISBN: 9783540442288. URL: <http://books.google.de/books?id=3RjYmm6AAM4C>.
- [Sab07] Claude Sabbah. *Isomonodromic Deformations and Frobenius Manifolds: An Introduction*. Universitext. Springer, 2007. ISBN: 9781848000544. URL: <http://books.google.de/books?id=ONUZWbdtQJAC>.
- [Sab90] Claude Sabbah. *Introduction to algebraic theory of linear systems of differential equations*. Vorlesungsskript [http://www.math.polytechnique.fr/cmat/sabbah/sabbah\\_cimpa90.pdf](http://www.math.polytechnique.fr/cmat/sabbah/sabbah_cimpa90.pdf). 1990.
- [Var96] V. S. Varadarajan. “Linear meromorphic differential equations: a modern point of view”. In: *Bulletin (New Series) of the American Mathematical Society*, 33(1):1 – 42. 1996, pp. 1–42.

## Additional references

- [Bal00] Werner Balser. *Formal Power Series and Linear Systems of Meromorphic Ordinary Differential Equations*. 1st ed. Universitext. Springer, 2000. ISBN: 0387986901,9780387986906,9780387225982.
- [FJ09] El Zein Fouad and Snoussi Jawad. “Local systems and constructible sheaves”. In: (2009), pp. 111–153. URL: <http://webusers.imj-prg.fr/~fouad.elzein/elzein-snoussif.pdf>.
- [Hal03] B. Hall. *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction*. Graduate Texts in Mathematics. Springer, 2003. ISBN: 9780387401225. URL: <http://books.google.de/books?id=m1VQi8HmEwcC>.
- [HTT08] R. Hotta, K. Takeuchi, and T. Tanisaki. *D-Modules, Perverse Sheaves, and Representation Theory*. Progress in Mathematics. Birkhäuser Boston, 2008. ISBN: 9780817643638. URL: [http://books.google.de/books?id=xUHwypdk%5C\\_aAC](http://books.google.de/books?id=xUHwypdk%5C_aAC).
- [MR91] Jean Martinet and Jean-Pierre Ramis. “Elementary acceleration and multi-summability. I”. eng. In: *Annales de l’institut Henri Poincaré (A) Physique théorique* 54.4 (1991), pp. 331–401. URL: <http://eudml.org/doc/76535>.
- [Rem14] Pascal Remy. “On the highest level’s Stokes phenomenon of meromorphic linear differential systems”. 61 pages. Soumis. Nov. 2014. URL: <https://hal.archives-ouvertes.fr/hal-01081071>.
- [Rob03] Derek J. S. Robinson. *An Introduction to Abstract Algebra*. De Gruyter textbook. Walter de Gruyter, 2003. ISBN: 9783110175448,3-11-017544-4.
- [Wie64] H. Wielandt. *Finite permutation groups*. Academic paperbacks. Academic Press, 1964. URL: <http://books.google.de/books?id=a-RQAAAAAMAAJ>.