Universität Augsburg
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Master's thesis

Classification of meromorphic connections, using Stokes structures and the representations of the corresponding groups

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Introduction

The basic problem, from which this masters-thesis arises, is that there are differential equations $(\frac{d}{dt} - A)\widehat{v} = w$ with coefficients in convergent powers series, which have no solution with converging entries. One can use the theory of meromorphic connections, to look at this problem and the interesting object is then the classifying set (cf. Section 2.5).

We will only be interested in local information and thus only in local classification of meromorphic connections. This classification can be splittet in the coarse formal classification and the fine meromorphic classification.

The formal classification problem was solved by the Levelt-Turittin Theorem (cf. Theorem 2.27). It states that in every formal equivalence class are some meromorphic connections of special form, which will be called models. These models are meromorphic connections, which are defined to be isomorphic to some direct sum of elementary meromorphic connections and elementary meromorphic connections are well understood.

Starting with a formal equivalence class corresponding to some model A^0, we can apply the meromorphic classification. The obtained set of meromorphic classes will be called the classifying set $\mathcal{H}(A^0)$ (cf. Section 2.5). The tool to describe the classifying set will be the Stokes structures which turn out to deliver exactly the needed information. This idea is formulated in the Malgrange-Sibuya Theorem (cf. Theorem 3.3), where the Stokes structures appear as the first cohomology $H^1(S^1; \Lambda(A^0))$ of the Stokes sheaf $\Lambda(A^0)$ on S^1 (cf. Definition 3.1).

The Malgrange-Sibuya theorem can be improved by showing that in each element in the $H^1(S^1; \Lambda(A^0))$ contains a unique cocycle called the Stokes cocycle (cf. Definition 3.29) of spezial form. These Stokes cocycles are given by the elements in the product over some spezial directions $\theta \in \mathbb{A} \subset S^1$ determined by A^0 (cf. Definition 3.12) of groups $\mathrm{Sto}_\theta(A^0) \subset \Lambda_\theta(A^0)$ (cf. Definition 3.14). We will see that $\mathrm{Sto}_\theta(A^0)$ has faithful representation $\mathbb{S}\mathrm{to}_\theta(A^0)$ (cf. Proposition 3.16). The elements of $\mathbb{S}\mathrm{to}_\theta(A^0)$ are the so-called Stokes matrices and it is easy to see that they are nilpotent. This characteristic can be used to define the structure of a nilpotent Lie group on the classifying set $\mathcal{H}(A^0)$.

Stokes matrices provide exactly the required information in order to describe a meromorphic class of a meromorphic connection, and since we know which restrictions hold for these matrices, we can explicitly give an isomorphism of pointed sets, $\mathbb{C}^n \to H^1(S^1; \Lambda(A^0))$, for a convenient n (cf. Section 3.3.3).

List of used symbols

1. Asymptotic analysis

$\mathfrak{s}_{a,b}(r)$	the sector $\{t \in \mathbb{C} \mid a < \arg(t) < b, 0 < \lvert t \rvert < r\}$
$\mathfrak{s}_I(r)$	the sector $\mathfrak{s}_{a,b}(r)$ for $I = (a, b)$ or the arc I
$\bar{\mathfrak{s}}_{a,b}(r)$	the closure of $\mathfrak{s}_{a,b}(r)$ in $\mathbb{C}^*$.
(θ, θ')	the arc from $\theta \in S^1$ to θ'
$U(\theta, \varepsilon)$	the arc $\left(\theta - \frac{\varepsilon}{2}, \theta + \frac{\varepsilon}{2}\right) \subset S^1$ resp. the corresponding sector
$\mathfrak{s} \Subset \mathfrak{s}'$	$\mathfrak{s}$ is proper sub-sector of $\mathfrak{s}'$
$\widehat{f}$	the asymptotic expanson of f (if not used in another way)
$f \sim_{\mathfrak{s}} \widehat{f}$	f has $\widehat{f}$ as asymptotic expanson on the sector $\mathfrak{s}$
$f \sim_I \widehat{f}$	f has $\widehat{f}$ as asymptotic expanson on the arc I
$f \sim_\theta \widehat{f}$	f has $\widehat{f}$ as asymptotic expanson in the direction θ
$\mathcal{A}$	the sheaf of asymptotic expansions
T_I	the Taylor map on the arc I
$T_{\mathfrak{s}}$	the Taylor map on the sector $\mathfrak{s}$
$\mathcal{A}^{<0}(I)$	$\ker(T_I)$; the functions asymptotic to 0 on I
$\mathcal{A}^{<0}(\mathfrak{s})$	$\ker(T_{\mathfrak{s}})$; the functions asymptotic to 0 on $\mathfrak{s}$

2. Systems and meromorphic connections

M	a riemanian surface
Z	a effective divisor on M
$\mathscr{M}$	a holomorphic bundle over M
$\mathscr{M}$; $(\mathscr{M}, \nabla)$	a meromorphic connection on $\mathscr{M}$ with poles on Z
$\mathcal{M}$; $(\mathcal{M}, \nabla)$	germ of a meromorphic connection $(\mathscr{M}, \nabla)$ at $0 \in M$
$\triangle$	a differential operator
$[A]$	the system corresponding to the connection matrix A
$G[t]$	$\mathrm{GL}_n(\mathbb{C}[t])$
$G\{t\}$	$\mathrm{GL}_n(\mathbb{C}\{t\})$; the analytic transformations
$G(\{t\})$	$\mathrm{GL}_n\left(\mathbb{C}\{t\}[t^{-1}]\right)$; the meromorphic transformations
$G[\![t]\!]$	$\mathrm{GL}_n\left(\mathbb{C}[\![t]\!]\right)$; the maybe not applicable formal transformations
$G(\!(t)\!)$	$\mathrm{GL}_n\left(\mathbb{C}[\![t]\!][t^{-1}]\right)$; the maybe not applicable formal meromorphic transformations
${}^F\!A$	$(dF)F^{-1} + FAF^{-1}$; the transforamtion of A by F
$\widehat{G}(A)$	the set of all (applicable) formal transformations
$[A, B]$	the linear differential system $\frac{dF}{dt} = BF - FA$
$[\mathrm{End}\, A]$	$[A^0, A^0]$
$G_0(A)$	the set of all isotropies of A
$\mathcal{Y}$	a fundamental solution

$\mathcal{E}^{\varphi}$	the germ $(\mathbb{C}(\{t\}), d - \varphi')$
$\mathcal{N}_{\alpha,0}$	the germ $\left(\mathbb{C}(\{t\}), d + \frac{\alpha}{t}\right)$
$\mathcal{N}_{\alpha,d}$	an elementary regular model
$(\mathcal{E}^{\varphi}, \nabla) \otimes (\mathcal{R}, \nabla)$	a elementary meromorphic connection
$(\mathcal{M}^{nf}, \nabla^{nf})$	a model
λ	the isomorphism from definition 2.26
$Q(t^{-1})$	the irregular part of a system $[A]$
$L \in \mathrm{GL}_n(\mathbb{C})$	the matrix of formal monodromy of $[A]$
${}^0C(\mathcal{M}^{nf}, \nabla^{nf})$	all isomorphism classes of meromorphic connections, which are formally isomorphic to $(\mathcal{M}^{nf}, \nabla^{nf})$
${}^0C(A^0)$	the system variant of ${}^0C(\mathcal{M}^{nf}, \nabla^{nf})$
$\mathcal{H}(\mathcal{M}^{nf}, \nabla^{nf})$	the space of marked (meromorphic) pairs
$\mathrm{Syst}_m(A^0)$	the systems formally meromorphic equivalent to A^0
$\mathcal{H}(A^0)$	the system version of $\mathcal{H}(\mathcal{M}^{nf}, \nabla^{nf})$
$\widehat{\mathrm{Syst}}_m(A^0)$	the marked pairs corresponding to A^0

3. Stokes Structures

$\Lambda(A^0)$	the Stokes sheaf (on S^1)
$\mathcal{S}t(A^0)$	$H^1(S^1; \Lambda(A^0))$; the first non abelien cohomology of the Stokes sheaf
exp	the Malgrange-Sibuya isomorphism
$\exp_A$	the Malgrange-Sibuya isomorphism corresponding to the system $[A]$
$\mathcal{G}$	the sheaf of flat functions
Θ	$G[\![t]\!]/G\{t\} \rightarrow H^1(S^1; \mathcal{G})$; the isomorphism from the other Malgrange-Sibuya theorem
$\mathcal{Q}(A^0)$	the set of all determining polynomials of $[\mathrm{End}\, A^0]$
$a_{jl} \in \mathbb{C}\backslash\{0\}$	the leading factor of $q_j - q_l \in \mathcal{Q}(A^0)$
$q_{jl}(t^{-1})$	the leading coefficient of $q_j - q_l \in \mathcal{Q}(A^0)$
k_{jl}	the degree of $q_j - q_l \in \mathcal{Q}(A^0)$
$\mathcal{K}$	$\{k_1 < \cdots < k_r\}$; the set of all levels of a system.
$q_j \underset{\theta}{\prec} q_l$	the first relation from definition 3.10
$q_j \underset{\theta}{\preccurlyeq} q_l$	the second relation from definition 3.10
$\Re(z)$	the real part of a complex number z
$\Im(z)$	the imaginary part of a complex number z
$\arg(z)$	the argument of a complex number z
$\mathbb{A}$	$\{\alpha_1, \ldots, \alpha_\nu\}$; the set of all anti-Stokes directions
$\mathbb{S}$	the set of all Stokes directions
$\mathrm{Sto}_\theta(A^0)$	the Stokes group of A^0 in direction θ whose elements are Stokes germs
ϑ_θ	
$\boldsymbol{\delta}_{jl}$	a block matrix version of Kronecker's delta, corresponding to the structure of Q
$\mathbb{S}\mathrm{to}_\theta(A^0)$	the group of all Stokes matrices of A^0 in direction θ

$\widehat{\mathbb{S}\mathrm{to}}_\theta(A^0)$	
$\widehat{\vartheta}_\theta$	
ρ_θ	$\mathrm{Sto}_\theta(A^0) \to \mathbb{S}\mathrm{to}_\theta(A^0)$; the map from proposition 3.16
C_{φ_θ}	$\rho_\theta(\varphi_\theta)$; the Stokes matrix corresponding to φ_θ
$\Lambda^k(A^0)$	the subsheaf of $\Lambda(A^0)$ of all germs, which are generated by k-germs
$\Lambda^{\leq k}(A^0)$	the subsheaf of $\Lambda(A^0)$ generated by k'-germs for all $k' \leq k$
$\Lambda^{<k}(A^0)$	the subsheaf of $\Lambda(A^0)$ generated by k'-germs for all $k' < k$
$\Lambda^{\geq k}(A^0)$	the subsheaf of $\Lambda(A^0)$ generated by k'-germs for all $k' geqk$
$\mathrm{Sto}^\star_\theta(A^0)$	$\mathrm{Sto}_\theta(A^0) \cap \Lambda^\star_\theta(A^0)$; the restriction of the Stokes sheaf for $\star \in \{k, <k, \leq k, \dots\}$
$\mathbb{S}\mathrm{to}^\star_\theta(A^0)$	the groups of representations, which correspond to elements of $\mathrm{Sto}^\star_\theta(A^0)$
$\mathbb{A}^k$	the set of anti-Stokes directions bearing the level k
$\mathbb{A}^{\leq k}$	$\bigcup_{k' \leq k} \mathbb{A}^{k'}$
$\mathbb{A}^{<k}$	$\bigcup_{k' < k} \mathbb{A}^{k'}$
$\mathbb{A}^{\geq k}$	$\bigcup_{k' \geq k} \mathbb{A}^{k'}$
$\mathcal{K}_\alpha$	the set of levels beared by $\alpha \in \mathbb{A}$
i_α	$\mathrm{Sto}_\alpha(A^0) \to \prod_{k \in \mathcal{K}_\alpha} \mathrm{Sto}^k_\alpha(A^0)$; the isomorphism corresponding to a factorization by a given order in a semidirect product
ρ^k_α	$\mathrm{Sto}^k_\alpha(A^0) \to \mathbb{S}\mathrm{to}^k_\alpha(A^0)$; the restriction of the map ρ_α to the level k.
i_α	$\mathbb{S}\mathrm{to}_\alpha(A^0) \to \prod_{k \in \mathcal{K}_\alpha} \mathbb{S}\mathrm{to}^k_\alpha(A^0)$; the Stokes matrix version of the i_α above
$\dot{\mathcal{U}}$	the nerve of the covering $\mathcal{U}$
h	$\prod_{\alpha \in \mathbb{A}} \mathrm{Sto}_\alpha(A^0) \longrightarrow \mathcal{S}t(A^0)$; the isomorphism from theorem 3.30
$\mathcal{U}^\star$	for $\star \in \{k, <k, \leq k\}$ the adequate coverings defined in section 3.3.2
$\Gamma(\dot{U}^\star; \Lambda^\star(A^0))$	$\prod_{\alpha \in \mathbb{A}^\star} \Gamma(\dot{U}^\star_\alpha; \Lambda^\star(A^0))$ for every $\star \in \{k, <k, \leq k, \dots\}$
s^k	$\Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A^0)) \to H^1(S^1; \Lambda^k(A^0))$; the quotient map
σ^k	$\Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A^0)) \to \Gamma(\dot{\mathcal{U}}^{\leq k}; \Lambda^{\leq k}(A^0))$; an injective map defined in definition 3.40
$\sigma^{<k}$	$\Gamma(\dot{\mathcal{U}}^{<k}; \Lambda^{<k}(A^0)) \to \Gamma(\dot{\mathcal{U}}^{\leq k}; \Lambda^{\leq k}(A^0))$; an injective map defined in definition 3.40
$\mathfrak{S}^{\leq k}$	$\Gamma(\dot{\mathcal{U}}^{<k}; \Lambda^{<k}(A^0)) \times \Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A^0)) \to \Gamma(\dot{\mathcal{U}}^{\leq k}; \Lambda^{\leq k}(A^0))$; the product map of cocycles
α^+	the nearest anti-Stokes direction on the right of α
τ	$\prod_{k \in \mathcal{K}} \Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A^0)) \to \Gamma(\dot{\mathcal{U}}; \Lambda(A^0))$; the product map of single-leveled cocycles
$\mathcal{T}$	$\prod_{k \in \mathcal{K}} \Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A^0)) \to H^1(\mathcal{U}; \Lambda(A^0))$; the isomorphism induced by τ on the cohomology
$\mathfrak{T}$	a map introduced in the proof of theorem 3.30

1 Poincarè asymptotic expansions

In this chapter, we want to look at Poincarè asymptotic expansions (at t_0), or for short, asymptotic expansions (at t_0). There are various literature references for asymptotic expansions. We will mostly refer to Loday-Richaud's book [Lod14, chapter 2], [HS12, pp. XI-1-13] from Hsieh and Sibuya and the book [PS03, chapter 7] written by van der Put and Singer although this topic is scratched in many publications. We will assume that $t_0 = 0$ and this is without loss of generality, since asymptotic expansions at $t_0 \in \mathbb{C}$ reduce to asymptotic expansions at 0 after the change of variable $t \mapsto s = t - t_0$ and asymptotic expansions at $t_0 = \infty$ are obtained by $t \mapsto s = \frac{1}{t}$.

We introduce the following notations:

Notations 1.1

We denote by

- $\mathfrak{s}_{a,b}(r)$, $a, b \in \mathbb{S}^1$ and $r \in \mathbb{R}_{>0}$ the open sector

$$\mathfrak{s}_{a,b}(r) := \{t \in \mathbb{C} \mid a < \arg(t) < b, 0 < |t| < r\}$$

  of all points $t \in \mathbb{C}$ satisfying $a < \arg(t) < b$ and $0 < |t| < r$;
- $\mathfrak{s}_I(r) = \mathfrak{s}_{a,b}(r)$ for an open arc[a] $I = (a, b)$;
- $\bar{\mathfrak{s}}_{a,b}(r)$ the closure of $\mathfrak{s}_{a,b}(r)$ in $\mathbb{C}^* := \mathbb{C}\backslash\{0\}$.

The notion of sectors can easily be extended to sectors on a q-sheet cover or sectors on the Universal cover of $\mathbb{C}^*$, but we will not use that.

[a] i.e. an open interval of S^1

Definition 1.2

We will say that *a sector* $\mathfrak{s}_I(r)$ *contains* $\theta \in S^1$ if $\theta \in I$. In the same way we will say that $\mathfrak{s}_I(r)$ contains $U \subset S^1$ if $U \subset I$ or that $U \subset S^1$ contains a sector $\mathfrak{s}_I(r)$ if $I \subset U$.

Remark 1.3

Since the explicit value r of the radius does not matter as long as it is small enough, we shall simply speak of an *arc* or *interval* I as $\mathfrak{s}_I(r)$ for small enough r.
It will be sometimes convenient to talk of arcs

$$U(\theta, \varepsilon) := \left(\theta - \frac{\varepsilon}{2}, \theta + \frac{\varepsilon}{2}\right) \subset S^1$$

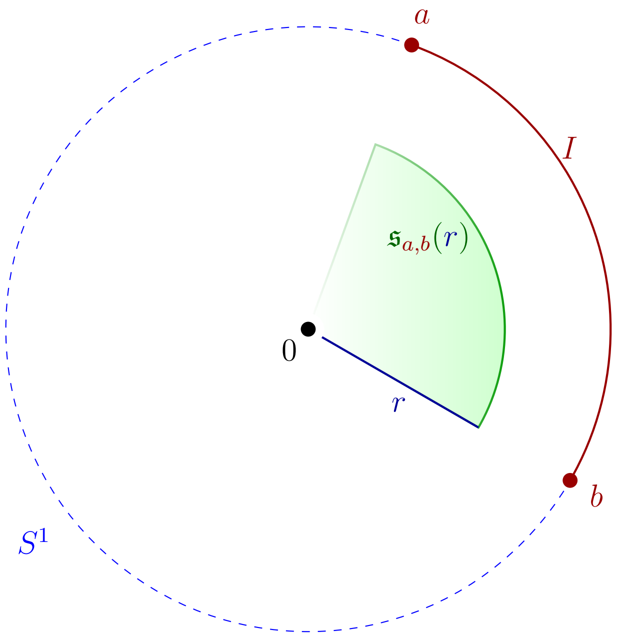

Figure 1.1: An exemplary arc $\mathfrak{s}_{a,b}(r)$ spanning from a to b with radius r.

defined by the midpoint together with the radius.

Definition 1.4

A sector $\mathfrak{s}_{a',b'}(r')$ is said to be a proper sub-sector of the sector $\mathfrak{s}_{a,b}(r)$, $\mathfrak{s}_{a',b'}(r') \Subset \mathfrak{s}_{a,b}(r)$, if its closure $\bar{\mathfrak{s}}_{a',b'}(r')$ is included in $\mathfrak{s}_{a,b}(r)$.

Remark 1.4.1

$\mathfrak{s}_{a',b'}(r') \Subset \mathfrak{s}_{a,b}(r)$ if and only if $a < a' < b' < b$ and $r' < r$.

Definition 1.5

A function $f \in \mathcal{O}(\mathfrak{s})$ is said to have the formal[a] Laurent series $\widehat{g} = \sum_{n \geq n_0} a_n t^n$ as asymptotic expansion (or *to be asymptotic to the series*) on the sector $\mathfrak{s}$ if

- for all proper sub-sectors $\mathfrak{s}' \Subset \mathfrak{s}$ and
- all $\mathbf{N} \in \mathbb{N}$,

there exists a constant $C(\mathbf{N}, \mathfrak{s}')$ such that the following estimate[b] holds

$$\left| f(x) - \sum_{n_0 \leq n \leq \mathbf{N}-1} c_n t^n \right| \leq C(\mathbf{N}, \mathfrak{s}')|t|^{\mathbf{N}} \qquad \text{for all } t \in \mathfrak{s}' .$$

We will use the following notations:

1. We will say that f *has the formal Laurent series $\widehat{g}$ as asymptotic expansion on the interval* $I \subset S^1$ if there exists a $r \in \mathbb{R}_{>0}$ such that f has the formal Laurent series $\widehat{g}$ as asymptotic expansion on $\mathfrak{s}_I(r)$.
2. We will say that f *has the formal Laurent series $\widehat{g}$ as asymptotic expansion in the direction* θ if there exists an interval $\theta \in I \subset S^1$ such that f has formal Laurent series $\widehat{g}$ as asymptotic expansion on the interval I.

[a]The term "formal" emphasizes that we do not restrict the coefficients $a_n \in \mathbb{C}$ in any way.
[b]Sometimes, for example in [Sab90], this condition is written as

$$\lim_{z \to 0, z \in \mathfrak{s}'} |t|^{-(\mathbf{N}-1)} \left| f(x) - \sum_{n_0 \le n \le \mathbf{N}-1} c_n t^n \right| = 0 \qquad \text{for all } t \in \mathfrak{s}'.$$

If f has $\widehat{f}$ as asymptotic expansion on the sector $\mathfrak{s}$ (or on the interval $I \subset S^1$, or in the direction θ), we denote that by $f \sim_{\mathfrak{s}} \widehat{f}$ (or $f \sim_I \widehat{f}$, or $f \sim_\theta \widehat{f}$).

Definition 1.6

1. Define $\mathcal{A}(\mathfrak{s})$ as the set of holomorphic functions $f \in \mathcal{O}(\mathfrak{s})$ which admit an asymptotic expansion at 0 on $\mathfrak{s}$.

   Remark 1.6.1

   a) $\mathcal{A}(\mathfrak{s})$ is a subring of $\mathcal{O}(\mathfrak{s})$.

   b) $\mathcal{A}(\mathfrak{s})$ contains $\mathbb{C}(\{t\})$ as a subfield.

2. For $(a,b) = I$ define $\mathcal{A}(a,b) = \mathcal{A}(I)$ as the limit $\varinjlim_{r \to 0} \mathcal{A}(\mathfrak{s}_{a,b}(r))$[a].

[a]In more detail: the elements of $\mathcal{A}(a,b)$ are pairs $(f, \mathfrak{s}_{a,b}(r))$ mit $f \in \mathcal{A}(\mathfrak{s}_{a,b}(r))$. The equivalence relation is given by $(f_1, \mathfrak{s}_{a,b}(r_1)) \sim (f_2, \mathfrak{s}_{a,b}(r_2))$ if there is a pair $(f_3, \mathfrak{s}_{a,b}(r_3))$ such that $r_3 < \min(r_1, r_2)$ and $f_3 = f_1 = f_2$ on $\mathfrak{s}_{a,b}(r_3)$.

It can easily be seen that $\mathcal{A} : U \to \mathcal{A}(U)$ defines a sheaf on S^1.

Definition 1.7

- One writes

$$\begin{aligned} T_I : \mathcal{A}(I) &\to \mathbb{C}((t)) \qquad (\text{resp.} \qquad T_{\mathfrak{s}} : \mathcal{A}(\mathfrak{s}) \to \mathbb{C}((t)) \qquad) \\ f &\mapsto \widehat{f} \end{aligned}$$

  for the mapping, called *Taylor map*, which associates to each function $f \in \mathcal{A}(I)$ (resp. $f \in \mathcal{A}(\mathfrak{s})$) its asymptotic expansion.
- A function f which is asymptotic to the identity, i.e. with $T_{\mathfrak{s}}(f) = \widehat{f} = \mathrm{id}_{\mathfrak{s}}$, is called *flat (on $\mathfrak{s}$)*.

- Denote by $\mathcal{A}^{<0}(I)$ (resp. $\mathcal{A}^{<0}(\mathfrak{s})$) the set $\ker(T_I) \subset \mathcal{A}(I)$ (resp. $\ker(T_{\mathfrak{s}}) \subset \mathcal{A}(\mathfrak{s})$) of functions asymptotic to zero at 0.

Remark 1.7.1

The kernel $\mathcal{A}^{<0}(I)$ is not zero in general. Let $I = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, then $\mathcal{A}^{<0}(I)$ contains the element $e^{-\frac{1}{\sqrt{t}}}$ where $\sqrt{t}$ stands for the principal determination of $x^{\frac{1}{2}}$.

Example 1.8: Trivial example

If f is an analytic function on $D = \{t \mid |t| < r\}$ then f is asymptotic to its Taylor series at 0 on $D^* = D \backslash \{0\}$. Reciprocally, if f is analytic on D^* and has an asymptotic expansion at 0 on D^* then, f is bounded near 0 and according to the removable singularity theorem, f is analytic on D.

Remark 1.8.1

This implies that $\mathcal{A}(S^1) \cong \mathbb{C}(\{t\})$.

For many more extensively discussed examples, like the Euler function or the exponential integral, see Loday-Richaud's book [Lod14, Sec.2.2].

Proposition 1.9

For every arc $I \subset S^1$ is $\mathcal{A}(I)$ is stable under derivation, i.e. $f'(t)$ is on I asymptotic to $\widehat{f}'(t)$ for every function f asymptotic to $\widehat{f}$ on I.

Remark 1.9.1

Let f_1 and f_2 be functions which satisfying $f_1 \sim_I \widehat{f}_1$ and $f_2 \sim_I \widehat{f}_2$. We have also the rules:

1. $f_1(t)+f_2(t)$ is on I asymptotic to $\widehat{f}_1(t)+\widehat{f}_2(t)$ (cf. Theorem [Bal00, 4.5.Thm.13] or [HS12, pp. XI-1-6]).
2. $f_1(t)f_2(t)$ is on I asymptotic to $\widehat{f}_1(t)\widehat{f}_2(t)$ (cf. Theorem [Bal00, 4.5.Thm.14] or [HS12, pp. XI-1-6]).
3. $\int_0^t f(s)\,ds$ is on I asymptotic to $\int_0^t \widehat{f}(t)\,ds$ (cf. Theorem [Bal00, 4.5.Thm.20]).
4. If $f \in \mathcal{A}(I) \backslash \mathcal{A}^{<0}(I)$ then $f^{-1}(t)$ is on I asymptotic to $\widehat{f}^{-1}(t)$ (cf. Theorem [Bal00, 4.5.Thm.21] or [HS12, pp. XI-1-9]).

Proof of Proposition 1.9. Let $f \in \mathcal{A}(I)$ with asymptotic expansion $\widehat{f} = \sum_{-n_0 \leq n} a_n x^n$ its asymptotic expansion where we assume without restriction that $n_0 = 0$. One has for each $m \geq 0$

$$f(t) = \sum_{n=0}^{m} a_n t^n + R_m(t) t^m$$

with $\lim\limits_{\substack{t \to 0 \\ t \in \mathfrak{s}_I(r)}} R_m(t) = 0$. This implies that R_m is holomorphic in $\mathfrak{s}_I(r)$. Thus one has

$$f'(t) = \sum_{n=0}^{m} n a_n t^{n-1} + m R_m(t) t^{m-1} + R_m'(t) t^m$$

Let $C_{x,\rho}$ be the circle at x with radius ρ contained in $\mathfrak{s}_I(r)$. The Cauchy theorem implies then that

$$|R_m'(t)| \leq \frac{1}{\rho} \max_{s \in \mathbb{C}_{x,\rho}} |R(s)| \,.$$

Let J be a relativly compact open set in I, then there exists a positive number α such that for all $x \in \mathfrak{s}_I(r)$ one has $C_{x,\alpha|x|} \subset \mathfrak{s}_I(r)$. This proves that $\widehat{f}'$ is an asymptotic expansion for f' in $\mathfrak{s}_I(r)$. $\square$

1.1 Borel-Ritt Lemma

The most important theorem here is the Borel-Ritt Lemma.

Theorem 1.10: Borel-Ritt

Let $I \subsetneq S^1$ be an open sub-arc, i.e. with a opening less than 2π. Then the Taylor map

$$T_I : \mathcal{A}(I) \twoheadrightarrow \mathbb{C}(\!(t)\!)$$

is onto.

Remark 1.10.1

This means that to each $\widehat{f} \in \mathbb{C}(\!(t)\!)$ and every arc $I \subsetneq S^1$ there are functions f asymptotic to $\widehat{f}$ on I. However, when $\widehat{f}$ satisfies an equation, these asymptotic functions do not necessarily satisfy the same equation in general. This gap will be filled in section 2.4 for small enough sectors.

There are multiple approaches to obtain a function $f \in \mathcal{A}(I)$ asymptotic to a given function $\widehat{f} \in \mathbb{C}(\!(t)\!)$ in a canonical way. For example (multi-)multisummability which is in depth described in Loday-Richaud's book [Lod14]. This topic is also discussed in the paper [MR91] from Martinet and Ramis.

Proof of Theorem 1.10. Let $I := (-\pi, \pi)$ and $R \in \mathbb{R}_{>0}$. We will prove this for the sector

$$\mathfrak{s} := \mathfrak{s}_I(R) = \{t \in \mathbb{C} \mid |\arg(t)| < \pi, 0 < |t| < R\}$$

in which, after rotation and for large enough R, every sector $\mathfrak{s}'' \subsetneq \mathbb{C}^*$ lies. Let $\sum a_n z^n$ be a formal Laurent series. We look for a function $f \in \mathcal{A}(I)$ with Taylor series $T_I f = \sum a_n z^n$. By subtracting the principal part we may assume that this series has no terms with negative degree. Let b_n be a sequence, which satisfies

the series $\sum |a_n| b_n R^{n-\frac{1}{2}}$ is convergent.

For example, one may set

$$b_n = \begin{cases} 0 & \text{when } n = 0 \\ \frac{1}{n!}|a_n| & \text{when } n > 0 \end{cases}$$

Let $\sqrt{t}$ be the branch of the square root function that satisfies $|\arg(\sqrt{t})| < \frac{\pi}{2}{}^a$ for all $t \in \mathfrak{s}$. For any real number b_n, the function $\beta_n(t) := 1 - e^{-\frac{b_n}{\sqrt{t}}}$ satisfies

(a) $|\beta_n(t)| \leq \frac{b_n}{\sqrt{|t|}}$ since $1 - e^t = -\int_0^t e^s\, ds$ implies that $|1 - e^t| < |t|$ for $\Re(t) < 0$ and

(b) β_n has asymptotic expansion 1 on $\mathfrak{s}$ (thus $\beta_n - 1$ has asymptotic expansion 0 on $\mathfrak{s}$).

Define $f(t) := \sum a_n \beta_n(t) t^n$. Since

$$|a_n \beta_n(t) t^n| \leq |a_n| b_n |z|^{n-\frac{1}{2}} \leq |a_n| b_n R^{n-\frac{1}{2}},$$

the series $\sum a_n \beta_n(t) t^n$ converges and its sum $f(t)$ is in $\mathcal{O}(\mathfrak{s})$.

Consider a propper sub-sector $\mathfrak{s}' \Subset \mathfrak{s}$ and $t \in \mathfrak{s}'$. Then, for every $N > 0$

$$\begin{aligned} \left| f(t) - \sum_{n=0}^{N-1} a_n t^n \right| &\leq \left| \sum_{n=0}^{N-1} a_n(\beta_n(t) - 1) t^n \right| + |t|^N \sum_{n \geq N} \left| a_n \beta_n(t) t^{n-N} \right| \\ &= \left| \sum_{n=0}^{N-1} a_n e^{-\frac{b_n}{\sqrt{t}}} t^n \right| + |t|^N \sum_{n \geq N} \left| a_n \beta_n(t) t^{n-N} \right| \end{aligned}$$

The first summand is a finite sum of terms, which are all asymptotic to 0 and then, is majorized by $C'|t|^N$ for a convenient constant C'. The second summand is majorized by

$$|t|^N \left(2|a_n| + \sum_{n \geq N+1} |a_n| b_n R^{n-\frac{1}{2}-N} \right).$$

By setting $C = C' + 2|a_n| + \sum_{n \geq N+1} |a_n| b_n R^{n-\frac{1}{2}-N}$ we obtain a positive constant C, which depends on N and the radius of $\mathfrak{s}' < R$, such that

$$\left| f(t) - \sum_{n=0}^{N-1} a_n t^n \right| \leq C|t|^N \qquad \text{for all } t \in \mathfrak{s}' .$$

$\square$

[a]For $t = re^{i\varphi}$ is $\sqrt{t} = \sqrt{r}e^{i\frac{\varphi}{2}}$ with $-\pi < \varphi < \pi$.

2 Systems and meromorphic connections

There are multiple languages, which can be used for talking about meromorphic connections. Among others there is the languages of meromorphic connections which can be used to talk about global information. For local description one can use **germs of meromorphic connections** or the coordinate dependent **systems** and **connection matrices**. Other coordinate independent approaches arise for example from the theory of (localized holonomic) $\mathcal{D}$-modules.

Meromorphic connections are introduced and discussed in many resources. A good starting point are Sabbah's lecture notes [Sab90]. More advanced resources are for example Sabbah's book [Sab07], Varadarajan's book [Var96] or the book [HTT08] from Hotta et al. The necessary facts about meromorphic connections are also stated in Boalch's paper [Boa01] (resp. his thesis [Boa99]), and Loday-Richaud's paper [Lod94].

Although the language of meromorphic connections is often preferred, we will use the language of systems most of the time. Systems are, for example, discussed in the book [HTT08] from Hotta et al, Loday-Richaud's paper [Lod94] and her book [Lod14] and Boalch's publications [Boa01; Boa99]. Another resource might Remy's paper [Rem14] be.

We will use all of the above mentioned resources in this chapter.

2.1 (Global) meromorphic connections

Let M be a riemanian surface and let $Z = k_1(a_1) + \cdots + k_m(a_m) > 0$ be an effective divisor[1] on M. It is sufficient to think $M = \mathbb{P}^1$ and $0 \in |Z|$[2], since we will only be interested in local information (at 0).

Let $\mathscr{M}$ be a holomorphic Bundle over M i.e. a locally free $\mathcal{O}_M$-module of rank n. A (global) meromorphic connection is then defined as follows.

Definition 2.1

A *meromorphic connection* $(\mathscr{M}, \nabla)$ *on* $\mathscr{M}$ *with poles on* Z is defined by a $\mathbb{C}$-linear morphism of sheaves

$$\nabla : \mathscr{M} \to \Omega^1_M(*Z) \otimes \mathscr{M}$$

[1] The $a_i \in M$ are distinct points and the k_i are positive integers.

[2] If $Z = k_1(a_1) + \cdots + k_m(a_m)$ then $|Z| := \{a_1, \ldots, a_m\}$.

satisfying, for each $U \underset{\text{op.}}{\subset} M$, the *Leibniz rule*

$$\nabla(fs) = f\nabla s + (df) \otimes s$$

for $s \in \Gamma(U, \mathscr{M})$ and $f \in \mathcal{O}_M(U)$. The *rank* of the meromorphic connection $(\mathscr{M}, \nabla)$ is defined to be the rank of the Bundle $\mathscr{M}$.

Remark 2.1.1

Some authors use the factors k_i of the divisor Z to limit the pole orders at the points a_i. Since we do not need this restriction, we allow arbitrary pole orders. Denoted is this by the $*$ in $\Omega^1_M(*Z)$. The sheaf $\mathcal{O}_M(*Z)$ of functions, which are meromorphic along Z, is defined in Sabbah's book [Sab07, Sec.0.8] and $\Omega^1_M(*Z)$ is then defined as

$$\Omega^1_M(*Z) := \mathcal{O}_M(*Z) \underset{\mathcal{O}_M}{\otimes} \Omega^1_M$$

the *sheaf of meromorphic differential* 1*-forms* (cf. [Sab07, Sec.0.9.b]).

The word "global" used to emphasize that the connection is "on M", i.e. not only a germ at some point of M, like it will be used later (cf. Remark 2.5). It will occasionally be convenient to omit the ∇ and simply call $\mathscr{M}$ the meromorphic connection.

Remark 2.2

Here, the variant 'holomorphic bundle with meromorphic connection' is chosen, like in Boalch's paper [Boa01]. There is also the transposed description 'meromorphic bundle with holomorphic connection' which is for example used in Sabbah's book [Sab07]. By choosing a lattice of a meromorphic bundle (cf. [Sab07, Def.0.8.3]), one gets a holomorphic bundle but if the meromorphic bundle had a holomorphic connection, the induced connection on the lattice is no longer guaranteed to be holomorphic. Thus we obtain a meromorphic connection on a holomorphic bundle in our sense.

Definition 2.3

The connection $\nabla : \mathcal{M} \to \Omega^1_M \otimes_{\mathcal{O}_M} \mathcal{M}$ is said to be *integrable* or *flat*, if

its curvature vanishes, i.e. $R_\nabla \equiv 0$ where

– $R_\nabla := \nabla \circ \nabla : \mathcal{E} \to \Omega^2_M \otimes_{\mathcal{O}_M} \mathcal{E}$ is a $\mathcal{O}_M$-linear morphism.

Remark 2.3.1

Here are all connections flat, since all connections will be of Dimension one.

2.2 Local expression of meromorphic connections and systems

We will only be interested in local information of meromorphic connections. This means that we look at a connection in a neighbourhood of 0 which has its unique singularity at 0. There are multiple ways of expressing the local information without the need of an fixed neigbourhood. We will either talk about germs of meromorphic connections or systems, which depend on the choice of a trivialization.

Proposition 2.4

A germ of a meromorphic connection $(\mathscr{M}, \nabla)$ is the sheaf-theoretic germ (at $t = 0$) and thus is given by a tuple $(\mathcal{M}, \nabla)$ where

- $\mathcal{M}$ is the germ at 0 of the holomorphic bundle $\mathscr{M}$ and thus a $\mathbb{C}(\{t\})$-vectorspace of dimension n, since the ring of germs of meromorphic functions with poles at 0 is the ring $\mathbb{C}(\{t\})$, and
- $\nabla : \mathcal{M} \to \mathcal{M}$ is a additive map, which satisfies the *Leibniz rule*

$$\nabla(fm) = \frac{d}{dt} f \cdot m + f\nabla(m)$$

for all $f \in \mathbb{C}(\{t\})$ and $m \in \mathcal{M}$.

Remark 2.4.1

Loday-Richaud calls the meromorphic connections in her book [Lod14, Def.4.2.1] *differential modules.*

Remark 2.5

From now on we will mostly talk about **germs of** meromorphic connections $(\mathcal{M}, \nabla)$ and we will call them meromorphic connection. If we want to talk about meromorphic connections in the sense of Definition 2.1 we will emphasize this by the word 'global' or by talking about a meromorphic connection **on** M.

Definition 2.6

A *(iso-) morphism of meromorphic connections* $\Phi : (\mathcal{M}, \nabla) \xrightarrow{\sim} (\mathcal{M}', \nabla')$ is a (iso-) morphism of $\mathbb{C}(\{t\})$-vectorspaces $\Phi : \mathcal{M} \xrightarrow{\sim} \mathcal{M}'$ which commutes with the connections, i.e. which satisfies $\nabla \circ \varphi = (\mathrm{id} \otimes \varphi) \circ \nabla$.

Choose a $\mathbb{C}(\{t\})$-basis $\underline{e} = (e_1, e_2, \ldots, e_n)$ of $\mathcal{M}$. Let $A = (a_{jk})_{j,k\in\{1,\ldots,n\}}$ be a $n \times n$ matrix with entries in $\mathbb{C}(\{t\})$ such that the it describes the action of ∇, i.e. it satisfies $\nabla e_k = -\sum_{1\leq j\leq n} a_{jk}(t)e_j$, for every $k \in \{1, \ldots, n\}$.

Definition 2.7

The matrix A is called a *connection matrix* of $(\mathcal{M}, \nabla)$.

The connection ∇ is fully determined by A. Indeed, let $x = \sum_{0\le j\le n} x_j e_j$ be an arbitrary element of $\mathcal{M}$ which is in matrix notation described as $x = \underline{e} \cdot X$ with the column matrix $X = {}^t(x_1, x_2, \ldots, x_n)$. Then, applying ∇ using the Leibniz rule, yields

$$\begin{aligned}\nabla x &= \nabla\left(\underline{e} \cdot X\right) \\ &= \underline{e} \cdot dX + \nabla \underline{e} \cdot X \\ &= \underline{e}\left(dX - AX\right).\end{aligned}$$

Such that horizontal sections of $(\mathcal{M}, \nabla)$, i.e. sections which satisfy $\nabla x = 0$, correspond to solutions of

$$\frac{d}{dt} x = Ax\,. \tag{2.1}$$

Thus, with the connection ∇ and the $\mathbb{C}(\{t\})$-basis $\underline{e}$ is naturally associate the differential operator $\triangle = d - A$, which has order one and dimension n.

Definition 2.8

We call (2.1), determined by the differential operator $\triangle = d - A$, a *germ of a meromorphic linear differential system*[a] of rank n, or just a *system*.

Proposition 2.8.1

Thus, the set of systems is isomorphic to the set

$$\mathrm{End}(E) \otimes \mathbb{C}(\{t\}) = \mathrm{gl}_n(\mathbb{C}(\{t\}))$$

of all connection matrices.

[a] Martinet and Ramis call them in [MR91] *germs of meromorphic differential operators.*

Such a system will be denoted by $[A] = d - A$ and we will call A the connection matrix of the system $[A]$.

If we start with a system $[A]$ and we want a meromorphic connection $(\mathcal{M}, \nabla)$, which has A as connection matrix, we can do this in the following way.

Proposition 2.9

If we start with either a system $[A]$ of rank n, or a connection matrix $A \in \mathrm{gl}_n(\mathbb{C}(\{t\}))$, we get a germ of a meromorphic connection via

$$(\mathcal{M}_A, \nabla_A) = (\mathbb{C}(\{t\})^n, d - A)$$

which has A as its connection matrix.

Proposition 2.10

Let $(\mathcal{M}_1, \nabla_1)$ and $(\mathcal{M}_2, \nabla_2)$ be two meromorphic connections with the connection matrices A_1 and A_2. A connection matrix of $(\mathcal{M}_1, \nabla_1) \oplus (\mathcal{M}_1, \nabla_1)$ is then given by the block-diagonal matrix $\operatorname{diag}(A_1, A_2)$.

Proof. Use Proposition 2.9 to write the connection as

$$(\mathbb{C}(\{t\})^{n_1}, d - A_1) \oplus (\mathbb{C}(\{t\})^{n_2}, d - A_2)$$

and we want to show that it is isomorphic to

$$\left(\mathbb{C}(\{t\})^{n_1+n_2}, d - \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}\right) .$$

Denote by $A := \operatorname{diag}(A_1, A_2)$ the block-diagonal matrix build from A_1 and A_2. For every $j \in \{1, 2\}$ we have the corresponding inclusion $i_j : \mathbb{C}(\{t\})^{n_j} \hookrightarrow \mathbb{C}(\{t\})^{n_1+n_2}$ and the diagram

$$\begin{array}{ccc} \mathbb{C}(\{t\})^{n_j} & \xrightarrow{d-A_j} & \mathbb{C}(\{t\})^{n_j} \\ \downarrow{\scriptstyle i_j} & & \downarrow{\scriptstyle i_j} \\ \mathbb{C}(\{t\})^{n_1+n_2} & \xrightarrow{d-A} & \mathbb{C}(\{t\})^{n_1+n_2} \end{array}$$

which commutes, since the derivation commutes with the inclusion and the matrix A is build in the correct way, to satisfy $i_j(A_j x) = A_j(i_j(x))$:

$$\begin{aligned} i_j(dx - A_j x) &= i_j(dx) - i_j(A_j x) \\ &= d(i_j(x)) - A_j(i_j(x)) \\ &= ((d - A) \circ i_j)(x) . \end{aligned}$$

$\square$

Remark 2.11

Let $(\mathcal{M}_1, \nabla_1)$ and $(\mathcal{M}_2, \nabla_2)$ be meromorphic connections. Then $\mathcal{M}_1 \otimes \mathcal{M}_2$ is endowed with the structure of a meromorphic connection by

$$\nabla(u_1 \otimes u_2) = \nabla_1 u_1 \otimes u_2 + u_1 \otimes \nabla_2 u_2$$

where $u_i \in \mathcal{M}_i$.

2.2.1 Transformation of systems

Notations 2.12

We will use the following notations

- $G[t] = \mathrm{GL}_n(\mathbb{C}[t])$;
- $G\{t\} = \mathrm{GL}_n(\mathbb{C}\{t\})$ the analytic transformations;
- $G(\{t\}) = \mathrm{GL}_n\left(\mathbb{C}\{t\}[t^{-1}]\right)$ the meromorphic[a] transformations;
- $G[\![t]\!] = \mathrm{GL}_n\left(\mathbb{C}[\![t]\!]\right)$ the maybe not applicable formal transformations;
- $G(\!(t)\!) = \mathrm{GL}_n\left(\mathbb{C}[\![t]\!][t^{-1}]\right)$ the maybe not applicable formal meromorphic transformations.

[a] We use the term meromorphic in the sens of convergent meromorphic. Otherwise we say formal meromorphic.

By *meromorphic transformation*, or just *transformation*, of a system we mean a $\mathbb{C}(\{t\})$-linear change of the trivialization. Such a change is given by a matrix $F \in G(\{t\})$ and the transformed connection matrix ${}^F\!A$ is obtained through the Gauge transformation

$$ {}^F\!A = (dF)F^{-1} + FAF^{-1} \, . $$

If F is formal i.e. $F \in G(\!(t)\!)$, it will usually be denoted by $\widehat{F}$. The transformation of A by $\widehat{F}$ is not guaranteed to have convergent entries. We denote by $\widehat{G}(A)$ the set of all *(applicable) formal transformations*

$$ \widehat{G}(A) := \left\{ \widehat{F} \in G(\!(t)\!) \mid {}^{\widehat{F}}\!A \text{ has convergent entries, i.e. } {}^{\widehat{F}}\!A \in G(\{t\}) \right\} . $$

Let $\widehat{F'} \in \widehat{G}(A^0)$ and $A' := {}^{\widehat{F'}}\!A$, then are the sets $\widehat{G}(A)$ and $\widehat{G}(A')$ related by

$$ \widehat{G}(A') = \widehat{G}(A)\widehat{F'}^{-1} = \left\{ \widehat{F} \in G(\{t\}) \mid \widehat{F}\widehat{F'} \in \widehat{G}(A) \right\} . $$

Remark 2.13

The condition

> B is obtained from A by transformation F

is clearly equivalent to

> F solves the linear differential system
>
> $$ \frac{dF}{dt} = BF - FA $$
>
> which is denoted by $[A, B]$.

Remark 2.13.1

If we start with

- two base choices $\mathcal{M} \xrightarrow{\sim} \mathbb{C}(\{t\})^n$ and $\mathcal{M}' \xrightarrow{\sim} \mathbb{C}(\{t\})^n$ and
- an isomorphism $\Phi : (\mathcal{M}, \nabla) \xrightarrow{\sim} (\mathcal{M}', \nabla')$ together with the corresponding base change $F \in G(\{t\})$

we have the following commutative diagram:

$$
\begin{tikzcd}
\mathbb{C}(\{t\})^n \arrow[rrr, "F"] \arrow[ddd, "d-A"] & & & \mathbb{C}(\{t\})^n \arrow[ddd, "d-B"] \\
& \mathcal{M} \arrow[ul] \arrow[r, "\Phi"] \arrow[d, "\nabla"] & \mathcal{M}' \arrow[ur] \arrow[d, "\nabla'"] & \\
& \mathcal{M} \arrow[dl] \arrow[r, "\Phi"] & \mathcal{M}' \arrow[dr] & \\
\mathbb{C}(\{t\})^n \arrow[rrr, "F"] & & & \mathbb{C}(\{t\})^n
\end{tikzcd}
$$

Thus the commutation property for the outer rectangle reads

$$(d - B) \circ F = F \circ (d - A)$$

which is equivalent to

$$\frac{dF}{dt} = BF - FA.$$

Remark 2.14

The simple but useful rule

$$^{(F_2F_1)}A = {}^{F_2}\left({}^{F_1}A\right)$$

can be seen by calculation:

$$
\begin{aligned}
^{(F_2F_1)}A &= d(F_2F_1)(F_2F_1)^{-1} + F_2F_1A(F_2F_1)^{-1} \\
&= ((dF_2)\, F_1 + F_2\,(dF_1)\,)\, F_1^{-1}F_2^{-1} + F_2F_1AF_1^{-1}F_2^{-1} \\
&= (dF_2)\, F_2^{-1} + F_2\,(dF_1)\, F_1^{-1}F_2^{-1} + F_2 \left({}^{F_1}A - (dF_1)\, F^{-1}\right) F_2^{-1} \\
&= (dF_2)\, F_2^{-1} + F_2{}^{F_1}AF_2^{-1} \\
&= {}^{F_2}\left({}^{F_1}A\right) .
\end{aligned}
$$

Definition 2.15

We define the *(formal) equivalence relation on the connection matrices (resp. on the corresponding systems)* as

$\boldsymbol{A}$ is (formally) equivalent to $\boldsymbol{B}$

if and only if

$\boldsymbol{B}$ is obtained from $\boldsymbol{A}$ by (formal) transformation.

The *class of a connection matrix* is the orbit under the Gauge transformation by $G(\{t\})$. The *formal class* is the orbit by $\widehat{G}(A)$.

Remark 2.15.1

Thus A is (formally) equivalent to B if and only if there is a (formal) solution of $[A, B]$.

The defined equivalence relations are compatible with the isomorphisms relations on meromorphic connections (cf. [HTT08, Lem.5.1.3]), i.e. the following proposition holds.

Proposition 2.16

Two germs of meromorphic connections are (formally) isomorphic if and only if their corresponding connection matrices are (formally) equivalent.

Definition 2.17

An *isotropy* of A^0 or of $[A^0]$ is a transformation $\widehat{F}$ which satisfies ${}^{\widehat{F}}A^0 = A^0$. Thus, the isotropies are the solutions of the system $[\operatorname{End} A^0] := [A^0, A^0]$.

Let $G_0(A^0)$ denote the set of all isotropies of A^0.

Remark 2.17.1

The isotropies are, a priori, formal transformations. Loday-Richaud mentions in her paper [Lod94, p. 853], that actually $G_0(A^0)$ is a subgroup of $\mathrm{GL}_n(\mathbb{C}[1/t, t])$.

Lemma 2.18

Two formal transformations $\widehat{F}_1$ and $\widehat{F}_2$ take A^0 into equivalent matrices ${}^{\widehat{F}_1}A^0$ and ${}^{\widehat{F}_2}A^0$ if and only if there exists isotropy $f_0 \in G_0(A^0)$ such that $\widehat{F}_1 = \widehat{F}_2 f_0$ (cf. [Lod94, p. 854]).

Regular / irregular singularities

The meromorphic connections are distinguished into regular and irregular meromorphic connections.

Definition 2.19

A connection with connection matrix A has *regular singularity* at 0 if there exists a konvergent transformation, by which A is obtained from a matrix with at most a simple pole at $t = 0$. Otherwise, the singularity is called *irregular*.

Remark 2.19.1

This implies that, if A has irregular (resp. regular) singularity, then also all meromorphic equivalent matrices ${}^{F}A$ have irregular (resp. regular) singularity.

Theorem 2.20

Let $(\mathcal{M}, \nabla)$ be a regular singular meromorphic connection and A its connection matrix. Then there exists a matrix $F \in G(\{t\})$ such that after transformation by F the matrix $B = {}^{F}A$ is constant i.e. ${}^{F}A \in \mathrm{GL}_n(\mathbb{C})$ (cf. [Sab07, Thm.II.2.8] or [HTT08, Sec.5.1.2]).

2.2.2 Fundamental solutions and the monodromy of a system

It is well known in the theory of linear ODEs that the solutions to a system like (2.1) form a vector space of dimension n over $\mathbb{C}$, i.e. if $x_0(t)$ and $x_{00}(t)$ are two solution of (2.1) and c_1, $c_2 \in \mathbb{C}$ are two constants, then is also $c_1 x_0(t) + c_2 x_{00}(t)$ a solution of the same system.

Definition 2.21

A *fundamental matrix of (formal) solutios* or *(formal) fundamental solution* $\mathcal{Y}$ on a sector $\mathfrak{s}$ of the system $[A]$ is an invertible $n \times n$ matrix (with formal entries), which solves $[0, A]$ on $\mathfrak{s}$.

Remark 2.21.1

- This means that the columns of $\mathcal{Y}$ are n $\mathbb{C}$-linearly independent solutions of the system $[A]$ on $\mathfrak{s}$.
- Some authors introduce multi-valued solutions, to avoid the restriction to sectors.

  Multi-valued are functions, which are not single-valued and *single-valued* is a function f, which satisfies

  $$f(t) = f(t \exp(2\pi i)) \qquad \text{whenever both sides are defined.}$$

  The function $t \to t^{\alpha}$, for example, is single-valued whenever $\alpha \in \mathbb{Z}$.

The Stokes phenomenon, which will be discussed in the chapter 3, results from the fact that there is always a formal fundamental solution, which solves a system on the full arc S^1. But holomorphic solutions, which are asymptotic to the formal solution, may exist only on small sub-sectors of S^1.

Remark 2.22

If the trivialization is changed by F (resp. $\widehat{F}$) the fundamental solution $\mathcal{Y} \in G(\{t\})$ changes to $F\mathcal{Y}$ (resp. $\widehat{F}\mathcal{Y}$). (cf. [Lod14, Thm.4.3.1] or [FJ09, p. 2.1.3])

Choose a fundamental solution $\mathcal{Y}$. Then analytic continuation along a closed path γ in $M\backslash Z$ provides another fundamental solution $\mathcal{Y}' = \rho^{-1}(\gamma)\mathcal{Y}$ where $\rho(\gamma)$ is called the *monodromy along the path* γ.

Definition 2.23

Let $[A]$ be a system with fundamental solution $\mathcal{Y}$. The analytic continuation of $\mathcal{Y}$ along a small circle around $t = 0$ yields the fundamental solution

$$\lim_{s \to 2\pi} \mathcal{Y}(e^{\sqrt{-1}s}t) = \mathcal{Y}(t)L$$

where $L \in \mathrm{GL}_n(\mathbb{C})$ is called the *monodromy matrix* of $[A]$.

2.3 Formal classification

In every formal equivalence class of meromorphic connections, there are some meromorphic connections of special form, which we will call models. They are not unique but all of them, which are formally isomorphic to a given meromorphic connection, lie in the same convergent equivalence class. In fact, every element of this convergent equivalence class will be a model in our definition.

The models will be used to classify meromorphic connections up to formal isomorphism. Two meromorphic connections are formally isomorphic, if they are isomorphic to the (up to convergent isomorphism) same model.

The first part is given by the Levelt-Turittin Theorem, which says that each meromorphic connection is, after potentially needed ramification, formally isomorphic to such a model. Thus the Levelt-Turittin Theorem solves the *formal classification problem*.

2.3.1 In the language of meromorphic connections: models

This approach is discussed in [Sab90] and in a more general context in [Sab07, Sec.II.5].

Definition 2.24

- For a $\varphi \in \mathbb{C}(\{t\})$ we use $\mathcal{E}^\varphi$ to denote the germ

  $$(\mathcal{E}^\varphi, \nabla) = (\mathbb{C}(\{t\}), d - \varphi') .$$

  This corresponds to the system satisfied by the function e^φ, since $(d - \varphi')e^\varphi = 0$.

  Corollary 2.24.1

  $\mathcal{E}^\varphi$ is determined by the class of φ in $\mathbb{C}(\{t\})/\mathbb{C}\{t\} \cong t^{-1}\mathbb{C}[t^{-1}]$. In the following, we will only consider the unique ambassador φ in each class which has no holomorphic part.

- For $\alpha \in \mathbb{C}$, define the *elementary regular meromorphic connection of rank one* $\mathcal{N}_{\alpha,0}$ as the germ

  $$(\mathcal{N}_{\alpha,0}, \nabla) = \left(\mathbb{C}(\{t\}), d + \frac{\alpha}{t}\right) .$$

  This corresponds to the system satisfied by $t^{-\alpha}$.

  An *elementary regular model of arbitrary rank* is a meromorphic connection which has a basis, in which the connection Matrix can be written as

  $$\frac{1}{t}(\alpha \operatorname{id} + N)$$

  where $N \in \mathrm{gl}_{d+1}(\mathbb{C})$ is a nilpotent matrix. If $\alpha \operatorname{id} + N$ is a single Jordan Block, we denote the corresponding connection by $\mathcal{N}_{\alpha,d}$.

  Remark 2.24.1

  The Corollary [Sab07, Cor.II.2.9] states, that every germ of a regular meromorphic connection $(\mathcal{R}, \nabla)$ is isomorphic to some direct sum

  $$(\mathcal{R}, \nabla) = \bigoplus_{\alpha,d} (\mathcal{N}_{\alpha,d}, \nabla) .$$

  of elementary regular meromorphic connections. This solves the classification problem, for regular meromorphic connections. For a detailed analysis of regular meromorphic connections see Sabbah's book [Sab07, Sec.II.2] or the book [HTT08, Sec.5.2] from Hotta et al.

Definition 2.25

A germ $(\mathcal{M}, \nabla)$ is called *elementary* if it is isomorphic to some germ $(\mathcal{E}^\varphi, \nabla) \otimes (\mathcal{R}, \nabla)$ where

- $(\mathcal{E}^\varphi, \nabla)$ is defined as in definition 2.24 and
- $(\mathcal{R}, \nabla)$ has regular singularity at $\{0\}$, i.e. is isomorphic to a direct sum of regular elementary meromorphic connections.

Definition 2.26

A germ $(\mathcal{M}', \nabla')$ is a *model* if there exists, after ramification $\mathcal{M} = \pi^*\mathcal{M}'$ by π, a isomorphism to a direct sum of elementary meromorphic connections:

$$\lambda : (\mathcal{M}, \nabla) \xrightarrow{\cong} \bigoplus_{\varphi} \mathcal{E}^\varphi \otimes \mathcal{R}_\varphi \,.$$

- is irregular singular
- has regular singularity at $\{0\}$
- $\varphi \in t^{-1}\mathbb{C}[t^{-1}]$ pairwise distinct

Remark 2.26.1

Here it is not necessary to understand ramification, since we restrict to the unramified case.

The important theorem here is the Levelt-Turittin Theorem, which solves the formal classification problem.

Theorem 2.27: Levelt-Turittin

To each germ $(\mathcal{M}', \nabla')$ of a meromorphic connection there exists, after potentially needed pullback $\pi^*\mathcal{M}' =: \mathcal{M}$ by some suitable ramification $t = z^q$ of order $q \geq 1$, a **formal** isomorphism

$$\widehat{\lambda} : \widehat{\mathcal{M}} \xrightarrow{\cong} \widehat{\mathcal{M}}^{nf} := \widehat{\mathcal{O}}_M \otimes \mathcal{M}^{nf}$$

to a model $\mathcal{M}^{nf}$. We then call $\mathcal{M}^{nf}$ a *formal decomposition* or *formal model* of $\mathcal{M}$ or $\mathcal{M}'$.

Remark 2.27.1

But there is in general **no** lift of the isomorphism $\widehat{\lambda}$, i.e. there is no isomorphism making the diagram

$$\begin{tikzcd}
\mathcal{M} \arrow[r, dotted, "?"] \arrow[d] & \mathcal{M}^{nf} \arrow[d] \\
\widehat{\mathcal{M}} \arrow[r, "\widehat{\lambda}"] & \widehat{\mathcal{M}}^{nf}
\end{tikzcd}$$

kommutative. But sectorwise, there are lifts given by the main asymptotic existence theorem (cf. Theorem 2.37).

Proofs of this theorem can be found in multiple places, for example in [Sab90, Thm.5.4.7].

2.3.2 In the language of systems: normal forms

Here we will see, that the fundamental solution of a model can be written in a special form and that this property characterizes models.

Definition 2.28

The fundamental solution corresponding to a model (resp. a normal form, see Definition 2.33) will be called a *normal solution*.

Let $(\mathcal{M}, \nabla)$ be an unramified meromorphic connection which is via some $\widehat{\lambda}$ formally isomorphic to some model $\mathcal{M}^{nf} = \bigoplus_{j=1}^{s} \mathcal{E}^{\varphi_j} \otimes \mathcal{R}_j$, where we assume that $\varphi_j \in t^{-1}\mathbb{C}[t^{-1}]$ (cf. Corollary 2.24.1). For every j is a connection matrix, corresponding to $\mathcal{R}_j$ given by $\frac{1}{t}L_j$, where L is in Jordan normal form. A connection matrix to $\mathcal{E}^{\varphi_j} \otimes \mathcal{R}_j$ is then by $q_j'(t^{-1}) \cdot \mathrm{id}_{n_j} + \frac{1}{t}L_j$ given, where $q_j(t^{-1}) = \varphi_j(t)$ is a polynomial in t^{-1} without constant term and $q_j'(t^{-1}) = \frac{d}{dt}\varphi_j(t)$[3]. A connection matrix for $\mathcal{M}^{nf}$ is then obtained by

$$\begin{aligned} A^0 &= \bigoplus_{j=1}^{s} \left(q_j'(t^{-1}) \cdot \mathrm{id}_{n_j} + \frac{1}{t} L_j \right) \\ &= \bigoplus_{j=1}^{s} q_j'(t^{-1}) \cdot \mathrm{id}_{n_j} + \frac{1}{t} \bigoplus_{j=1}^{s} L_j \\ &= Q'(t^{-1}) + \frac{1}{t} L \,, \end{aligned}$$

where $Q(t^{-1}) := \bigoplus_{j=1}^{s} q_j(t^{-1}) \cdot \mathrm{id}_{n_j}$ and $L := \bigoplus_{j=1}^{s} L_j$.

Definition 2.29

Let L be a block diagonal matrix $L = \bigoplus_{j=1}^{s} L_j$, where the L_j are of size $n_j \times n_j$ and Q be a diagonal matrix. We will say, that *the block structure of L is finer then the structure of Q* if there are q_j's such that $Q = \bigoplus_{j=1} q_j \cdot \mathrm{id}_{n_j}$.

Remark 2.29.1

1. The matrices Q and L defined above clearly satisfy this condition.
2. When this condition is satisfied, do L and Q commute.

Proposition 2.30

[3]By abuse of notation we denote by $q_j'(t^{-1})$ the derivation $\frac{d}{dt}(q_j(t^{-1})) = \left(\left(\frac{d}{dt}q_j\right) \circ t^{-1}\right) \cdot \frac{d}{dt}t^{-1}$. The same does apply for $Q'(t^{-1})$.

Let $L \in \mathrm{GL}_n(\mathbb{C})$ be constant and $Q(t^{-1}) = \operatorname{diag}(q_1(t^{-1}), \ldots, q_n(t^{-1}))$ a diagonal matrix of polynomials in t^{-1} where L is in Jordan normal form and its block structure is finer then the structure of $Q(t^{-1})$.

The matrix $\mathcal{Y}_0 := t^L e^{Q(t^{-1})}$ is then a fundamental solution of the system determined by the matrix

$$A^0 = Q'(t^{-1}) + L\frac{1}{t}\,.$$

Proof. It is easy to see that

- the block structure of t^L is finer then the structure of $Q'(t^{-1})$ and thus $t^L Q'(t^{-1}) t^{-L} = Q'(t^{-1})$, since
  - $t^L = t^{\bigoplus_{j=1}^s L_j} = \bigoplus_{j=1}^s t^{L_j}$ and
  - $Q'(t^{-1}) = \frac{d}{dt} \bigoplus_{j=1} q_j(t^{-1}) \cdot \mathrm{id}_{n_j} = \bigoplus_{j=1} q_j'(t^{-1}) \cdot \mathrm{id}_{n_j}$,
- $\frac{d}{dt} e^{Q(t^{-1})} = Q'(t^{-1}) e^{Q(t^{-1})}$ and $\frac{d}{dt} t^L = L\frac{1}{t} t^L$

thus we can prove that $\mathcal{Y}_0$ is a matrix consisting of solutions:

$$\begin{aligned}
\frac{d}{dt}\mathcal{Y}_0 &= \frac{d}{dt}\left(t^L e^{Q(t^{-1})}\right) \\
&= t^L \frac{d}{dt} e^{Q(t^{-1})} + \frac{d}{dt} t^L e^{Q(t^{-1})} \\
&= t^L Q'(t^{-1}) e^{Q(t^{-1})} + L\frac{1}{t} t^L e^{Q(t^{-1})} \\
&= \left(t^L Q'(t^{-1}) t^{-L} + L\frac{1}{t}\right) t^L e^{Q(t^{-1})} \\
&= \left(Q'(t^{-1}) + L\frac{1}{t}\right) t^L e^{Q(t^{-1})} && \text{(using Remark 2.29.1)} \\
&= A^0 \mathcal{Y}_0\,.
\end{aligned}$$

The invertability condition is clear. □

Remark 2.31

If one starts without the assumption on the structure, one only obtains

$$A = t^L Q'(t^{-1}) t^{-L} + L\frac{1}{t}$$

as possible system matrix. But the obtained matrix may not define a matrix, since it could contain entrys, which are not in $\mathbb{C}(\{t\})$.

A simple example is: $L = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$ and $Q = \begin{pmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{pmatrix}$, with $\varphi_1 \neq \varphi_2$. The matrix A is then

$$\begin{aligned} A &= t^L Q'(t^{-1}) t^{-L} + L\frac{1}{t} \\ &= \begin{pmatrix} \varphi_1' & (-\varphi_1' + \varphi_2')\ln(t) \\ 0 & \varphi_2' \end{pmatrix} + \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix} \frac{1}{t} \notin \mathrm{gl}_n(\mathbb{C}(\{t\}))\,. \end{aligned}$$

To $\widehat{\lambda}$ and the (implicitly) used choices of bases corresponds a formal transformation $\widehat{F}$ and thus we have

- a fundamental solution $\widehat{F} t^L e^{Q(t^{-1})}$ and
- a connection matrix ${}^{\widehat{F}}A^0$

of $(\mathcal{M}, \nabla)$, which was defined at the beginning of this subsection.

Corollary 2.32

From the Levelt-Turittin theorem we deduce that every meromorphic connection $(\mathcal{M}, \nabla)$ and thus every system $[A]$ has a fundamental solution of the form

$$\mathcal{Y} = \widehat{F} t^L e^{Q(t^{-1})}$$

where $\widehat{F}$ is a formal transformation, solving the differentia equation corresponding to the formal isomorphism $(\mathcal{M}^{nf}, \nabla^{nf}) \to (\mathcal{M}, \nabla)$. In the other way is a system, which has $\mathcal{Y}$ as fundamental solution is given by

$$\bar{A} = {}^{\widehat{F}}\left(Q'(t^{-1}) + L\frac{1}{t}\right)\,.$$

In the special case when $\widehat{\lambda}$ is a convergent isomorphism, i.e. in the case, when A is already a model, we see that it has a fundamental solution of the form $Ft^L e^{Q(t^{-1})}$, where F is a convergent transformation corresponding to $\widehat{\lambda}$. In fact are the models uniquely characterized as the meromorphic connections, with a fundamental solution which can be written as $\mathcal{Y}_0 = Ft^L e^{Q(t^{-1})}$ and we will use this fact, to say when a system is something like a model.

Definition 2.33

Let $[A]$ be a system. We call $[A]$ (or A) a *normal form* if a fundamental solution of $[A]$ can be written as

$$\mathcal{Y}_0(t) = Ft^L e^{Q(t^{-1})}$$

with

- an *irregular part* $e^{Q(t^{-1})}$ of $\mathcal{Y}_0$ determined by

$$Q(t^{-1}) = \bigoplus_{j=1}^{s} q_j(t^{-1})\,\mathrm{id}_{n_j} = \mathrm{diag}(\underbrace{q_1, \ldots, q_1}_{n_1\text{-times}}, q_2, \ldots, q_s)$$

  where the $q_i(t^{-1})$ are polynomials in $\frac{1}{t}$ (or in a fractional power $\frac{1}{s} = \frac{1}{t^{1/p}}$ of $\frac{1}{t}$ for the ramified case) such that $q_j(0) = 0$, i.e. without constant term,

- a constant matrix $L \in \mathrm{gl}_n(\mathbb{C})$ called the *matrix of formal monodromy*, where t^L means $e^{L \ln t}$ and

- a (convergent) transformation $F \in G(\{t\})$.

Remark 2.33.1

By changing the basis via F^{-1} we obtain from $[A^0]$ the equivalent system $[{}^{F^{-1}}A^0]$ with the normal solution

$$F^{-1}\mathcal{Y}_0(t) = t^L e^{Q(t^{-1})}\,.$$

Thus it is always possible to assume that the transformation matrix F is trivial.

The normal forms will often be denoted A^0. If A is formally equivalent to a normal form A^0 we say that A^0 *is a normal form for* A and for $[A]$.

Corollary 2.34

Since we only look at unramified connections, by the Levelt-Turittin Theorem we are able to assume that L is in Jordan normal form and that it has a block structure, which is finer then the structure of $Q = \bigoplus_{j=1}^{s} q_j(t^{-1}) \cdot \mathrm{id}_{n_j}$ (cf. [Rem14, Sec.1] or [MR91, Sec.4]).

The following corollary enables us to use normal forms in place of models.

Corollary 2.35

A meromorphic connection $(\mathcal{M}, \nabla)$ is a model if and only if its connection matrix A is a normal form.

2.4 The main asymptotic existence theorem (M.A.E.T)

Here we want to state the main asymptotic expansion theorem (or often M.A.E.T.) which is essentially a deduction from the Borel-Ritt Lemma. It states that to every formal solution of a system of meromorphic differential equations and every sector with sufficiently small opening, one can find a holomorphic solution of the system having the formal one as its asymptotic expansion.

Definition 2.36

Let A be via $\widehat{F}$ formally equivalent to A^0. We call F a *lift of* $\widehat{F}$ *on* $I \subset S^1$ if

- $F \sim_I \widehat{F}$ (cf. Page 3) and
- F satisfies the same system $[A^0, A]$ as $\widehat{F}$.

The following theorem, often called the main asymptotic existence theorem, can be for example found in as Theorem A in the paper [BJL79] from Balser, Jurkat and Lutz, Theorem 3.1 in Boalch's paper [Boa01] or Theorem 4.4.1 in Loday-Richaud's Book [Lod14].

Theorem 2.37: M.A.E.T

To every $\widehat{F} \in G((t))$ and to every small enough arc $I \subsetneq S^1$ there exists a lift F on I.

Remark 2.37.1

If we write the system $[A^0, A]$ as a differential operator D. The theorem then states that D acts linearly and surjectively on the sheaf $\mathcal{A}^{<0}$, i.e. the sequence

$$\mathcal{A}^{<0} \xrightarrow{D} \mathcal{A}^{<0} \longrightarrow 0$$

are exact sequences of sheaves of $\mathbb{C}$-vector spaces.

Remark 2.38

In the language of meromorphic connections is this theorem sometimes called *sectorial decomposition* and stated for example in [Sab07, Thm.II.5.12] and [Sab90, Sec.II.2.4]:

Theorem 2.38.1: Sectorial decomposition

Let $(\mathcal{M}, \nabla)$ be a meromorphic connection and let $\widehat{\lambda} : \widehat{\mathcal{M}} \to \widehat{\mathcal{M}}^{nf}$ be the isomorphism given by Theorem 2.27 together with the model $\mathcal{M}^{nf}$. There exists then, for any $e^{i\theta^0} \in S^1$, an isomorphism $\widetilde{\lambda}_{\theta^0} : \widetilde{\mathcal{M}}_{\theta^0} = \mathcal{A}_{\theta^0} \otimes \mathcal{M} \to \widetilde{\mathcal{M}}^{nf}_{\theta^0}$ lifting $\widehat{\lambda}$ that is, such that the following diagram

$$\begin{tikzcd}
\widetilde{\mathcal{M}}_{\theta^o} \arrow[r, "\widetilde{\lambda}_{\theta^o}"] \arrow[d] & \widetilde{\mathcal{M}}^{nf}_{\theta^o} \arrow[d] \\
\widehat{\mathcal{M}} \arrow[r, "\widehat{\lambda}"] & \widehat{\mathcal{M}}^{nf}
\end{tikzcd}$$

commutes

This is exactly the solution to the problem stated in Remark 2.27.1.

2.5 The classifying set

We want to understand the Set $\left\{\left[(\mathcal{M},\nabla)\right]\right\}$ of the (convergent) isomorphism classes of all meromorphic connections. Since we can use the formal classification (cf. Section 2.3) and we know that all elements in a convergent isomorphism class lie in the same formal isomorphism class (In other words: the convergent classification is finer than the formal classification) we can reduce the problem by fixing a model $(\mathcal{M}^{nf},\nabla^{nf})$ with the corresponding normal form A^0. Thus we can restrict ourself to the subset

$$ {}^0C(\mathcal{M}^{nf},\nabla^{nf}) = \Big\{\left[(\mathcal{M},\nabla)\right] \mid \text{there exists a formal isomorphism } \hat{f} : (\widehat{\mathcal{M}},\widehat{\nabla}) \xrightarrow{\sim} (\widehat{\mathcal{M}}^{nf},\widehat{\nabla}^{nf})\Big\} $$

of all isomorphism classes of meromorphic connections, which are formally isomorphic to $(\mathcal{M}^{nf},\nabla^{nf})$. This is the set that we will be calling the *classifying set (to $(\mathcal{M}^{nf},\nabla^{nf})$)* and we will also denote it also by ${}^0C(A^0)$, if we are using the language of systems.

Remark 2.39

Note that we classify

> meromorphic connections within fixed **formal meromorphic classes, modulo meromorphic equivalence**.

Whereas for example Boalch in [Boa01] and [Boa99] classifies

> meromorphic connections within fixed **formal analytic classes, modulo analytic equivalence**

as it was done in the older literature. This makes no difference, since the resulting classifying sets are isomorphic (cf. [Boa99] or [BV89]).

This distinction relates to the difference between **'regular singular'** connections and **'logarithmic'** connections.

It is convenient to look at the slightly larger space of isomorphism classes of *marked (meromorphic) pairs*

$$ \mathcal{H}(\mathcal{M}^{nf},\nabla^{nf}) = \left\{\left[(\mathcal{M},\nabla,\hat{f})\right] \mid \hat{f} : (\widehat{\mathcal{M}},\widehat{\nabla}) \xrightarrow{\sim} (\widehat{\mathcal{M}}^{nf},\widehat{\nabla}^{nf})\right\} $$

in which we also handle the additional information, of the formal isomorphism, by which the meromorphic connection is isomorphic to the model. The isomorphisms of marked pairs are defined as follows:

Definition 2.40

Two germs $(\mathcal{M}, \nabla, \widehat{f})$ and $(\mathcal{M}', \nabla', \widehat{f}')$ are isomorphic if there exists an isomorphism $g : (\mathcal{M}, \nabla) \xrightarrow{\sim} (\mathcal{M}', \nabla')$ such that $\widehat{f} = \widehat{f}' \circ \widehat{g}$.

Remark 2.40.1

Sabbah states in [Sab07, p. 111] that such an isomorphism is then unique.

Equivalently, one can talk in terms of systems. We then denote by

$$\mathrm{Syst}_m(A^0) := \{[A] \mid A = {}^{\widehat{F}}A^0 \text{ for some } \widehat{F} \in G((t))\}$$

the set of systems formally meromorphic equivalent to A^0. Since we use meromorphic equivalences, in contrast to [Boa01; Boa99], we denote that in Syst_m by the subscript $_m$. Thus ${}^0C(\mathcal{M}^{nf}, \nabla^{nf})$ corresponds to the set ${}^0C(A^0) := \mathrm{Syst}_m(A^0)/G(\{t\})$ of meromorphic classes which are formally equivalent to A^0. Analogous, $\mathcal{H}(\mathcal{M}^{nf}, \nabla^{nf})$ corresponds to the set $\mathcal{H}(A^0)$ of equivalence classes, i.e. orbits of $G(\{t\})$, in

$$\widehat{\mathrm{Syst}}_m(A^0) := \left\{ \left(A, \widehat{F}\right) \mid A = {}^{\widehat{F}}A^0 \text{ for some } \widehat{F} \in G((t)) \right\} .$$

Especially follows, that two marked pairs $(A, \widehat{F})$ and $(A', \widehat{F}')$ are equivalent, if and only if there is a base change H such that

- $A' = {}^{H}A$, i.e. H is a solution of $[A, A']$, and
- $\widehat{F}' = \widehat{H}\widehat{F}$ (cf. [BV89, p. 71]).

In the following diagram is the first condition equivalent to the commutation property of the green square. The second property corresponds to the commutation property of the top (resp. bottom) triangle on the level of asymptotic expansions.

$$\begin{tikzcd}
 & \mathbb{C}(\{t\})^n \arrow[ddr, "H"] \arrow[ddd, "d-A"] & \\
\mathbb{C}(\{t\})^n \arrow[ur, "\widehat{F}"] \arrow[drr, "\widehat{F}'"] \arrow[ddd, "d-A^0"] & & \\
 & & \mathbb{C}(\{t\})^n \arrow[ddd, "d-A'"] \\
 & \mathbb{C}(\{t\})^n \arrow[ddr, "H"] & \\
\mathbb{C}(\{t\})^n \arrow[ur, "\widehat{F}"] \arrow[drr, "\widehat{F}'"] & & \\
 & & \mathbb{C}(\{t\})^n
\end{tikzcd}$$

Lemma 2.41

Since $G_0(A^0)$ is by definition the stabilizer of A^0 (cf. definition 2.17) and $\mathrm{Syst}_m(A^0)$ is the corresponding orbit we can use the Proposition 3.1 in the book [Wie64] from Wielandt and deduce

$$\mathrm{Syst}_m(A^0) \cong \widehat{G}(A^0)/G_0(A^0)\,.$$

Corollary 2.41.1

Thus the *set of meromorphic classas of systems formally equivalent to* A^0 are just the orbits of $G(\{t\})$ in $\mathrm{Syst}_m(A^0)$ that is

$${}^0C(A^0) \cong G(\{t\})\backslash\widehat{G}(A^0)/G_0(A^0)$$

whereas the *set of meromorphic classes of marked pairs* $\mathcal{H}(A^0)$ *of* $[A^0]$ is canonically isomorphic to the left quotient $G(\{t\})\backslash\widehat{G}(A^0)$ (cf. [Boa99, Lem.1.17]).

The group $G_0(A^0)$ is easy to compute and is often trivial. In fact, the elements are block-diagonal corresponding to the structure of Q, see [Lod14, p. 77]. Thus the structure of ${}^0C(A^0)$ is easily deduced from the structure of $G(\{t\})\backslash\widehat{G}(A^0)$.

3 Stokes Structures

Stokes structures are contain exactly necessary information to classify meromorphic classification, i.e. with the Stokes structures we are able to construct a space, which is isomorphic to the classifying set.

A great overview of this topic is given by Varadarajan in [Var96]. Other resources we will use are for example Sabbah's book [Sab07, section II] for Section 3.1. For the Sections 3.2 and 3.3 will Loday-Richaud's paper [Lod94] and the book [Lod14, Sec.4] be useful. Also useful was Boalch's paper [Boa01] (resp. his thesis [Boa99]) which looks only at the single leveled case or the paper [MR91, Thm.13] from Martinet and Ramis.

Let $(\mathcal{M}^{nf}, \nabla^{nf})$ be a fixed model with the corresponding normal form A^0. Let us also fix a normal solution $\mathcal{Y}_0$ of A^0. The purpose of the next section (Section 3.1) is, to proof the Malgrange-Sibuya Theorem. It states that the classifying set $\mathcal{H}(A^0)$ is via an map exp isomorphic to the first non abelian cohomology $H^1(S^1; \Lambda(A^0))$ of the Stokes sheaf $\Lambda(A^0)$. It will be denoted by $\mathcal{S}t(A^0)$. In Section 3.3 we will improve the Malgrange-Sibuya Theorem by showing that each 1-cohomology class in $\mathcal{S}t(A^0)$ contains a unique 1-cocycle of a special form called *the Stokes cocycle* (cf. Section 3.2). The morphism, which maps each Stokes cocycle to its corresponding 1-cocycle will be denoted by h. This will be further improved in Section 3.4.

If one introduces the map g, which arises from the theory of summation and takes an equivalence class (resp. an ambassador of such a class) and returns a corresponding Stokes cocycle in an canonically way (cf. Appendix B where the theory of summation will be roughly discussed), as a black-box one can write the following commutative diagram.

$$\begin{tikzcd}
\mathcal{H}(A^0) \arrow[r, "\exp"] \arrow[d, "g"'] & \mathcal{S}t(A^0) \\
\prod_{\theta \in \mathbb{A}} \mathrm{Sto}_\theta(A^0) \arrow[ur, "h"'] &
\end{tikzcd}$$

This diagram will be enhanced in Section 3.5 by adding a couple of isomorphisms.

3.1 Stokes structures: Malgrange-Sibuya isomorphism

Here we will look at the classifying set and we will proof that it is isomorphic to the first non abelian cohomology set $H^1(S^1; \Lambda(A^0))$, which will be denoted as $\mathcal{S}t(A^0)$. If we talk about cocycles or cochains, we will in the following always mean 1-cocycles or 1-cochains.

Let us first define the Stokes sheaf $\Lambda(A^0)$ on S^1, as the sheaf of flat isotropies of $[A^0]$.

Definition 3.1

The Stokes sheaf $\Lambda(A^0)$ of A^0, is defined as the subsheaf of $\mathrm{GL}_n(\mathcal{A})$, in the following way. For some $\theta \in S^1$ is the stalk at θ the subgroup of $\mathrm{GL}_n(\mathcal{A})_\theta$ of elements f which satisfy

1. Multiplicatively flatness: f is asymptotic to the identity, i.e. $f \sim_{\mathfrak{s}} 1$;
2. Isotropy of A^0: ${}^f A^0 = A^0$.

Remark 3.1.1

This definition makes also sense as $\Lambda(A)$ where A stands for a systems wich is not in normal form. The elements of $\Lambda(A)$ then have to be isotropies of a normal form A^0 of A.

3.1.1 The theorem

Here we want to state the Malgrange-Sibuya Theorem. We will first give it in the language of meromorphic connections and after that we will give the same theorem in the language of systems. The second variant of this theorem will be proven in the next section.

In the language of meromorphic connections is the map, in the Malgrange-Sibuya Theorem bellow, described as follows.

Let $(\mathcal{M}, \nabla, \widehat{f})$ be a marked germ of a meromorphic connection. By Theorem 2.38.1 there exists an open covering $\mathcal{U} = (U_j)_{j\in J}$ and for every open set, an isomorphism

$$f_j : (\widetilde{\mathcal{M}}, \widetilde{\nabla})_{|U_j} \to (\widetilde{\mathcal{M}}^{nf}, \widetilde{\nabla}^{nf})_{|U_j}$$

such that $f_j \sim_{U_j} \widehat{f}$. By $(f_k f_j^{-1})_{jk}$ is then a cocycle of the sheaf $\mathcal{S}t(A^0)$, relative to the covering $\mathcal{U}$, defined. This defines a mapping of pointed sets

$$\exp : \mathcal{H}(\mathcal{M}^{nf}, \nabla^{nf}) \longrightarrow H^1(S^1; \Lambda(A^0))$$

to the first non abelian cohomology of $\Lambda(A^0)$, which sends the class of $(\mathcal{M}^{nf}, \nabla^{nf}, \widehat{\mathrm{id}})$ to that of id, i.e. the trivial cohomology class.

Theorem 3.2: Malgrange-Sibuya

The homomorphism

$$\exp : \mathcal{H}(\mathcal{M}^{nf}, \nabla^{nf}) \longrightarrow \mathcal{S}t(A^0) := H^1(S^1; \Lambda(A^0))$$

is an isomorphism of pointed sets.

The theorem (system version)

Since the language of meromorphic connections is equivalent to the one of systems, there is also the translated version of the Malgrange-Sibuya isomorphism to the language of systems. The corresponding map is then build as follows.

Let $(A, \widehat{F})$ be a marked pair, thus $\widehat{F}$ solves $[A^0, A]$. By the M.A.E.T (Theorem 2.37) there exists an open covering $\mathcal{U} = (U_j)_{j \in J}$ together with, for every open set U_j a lift $F_j \in \mathrm{GL}_n(\mathcal{A}(U_j))$ of $\widehat{F}$ (cf. Definition 2.36), which solves $[A, A^0]$. By the cocycle $(F_l^{-1} F_j)_{jl} \in \Gamma(\dot{\mathcal{U}}; \Lambda(A^0))$ is then a cohomology class in $\mathcal{S}t(A^0)$, relative to the covering $\mathcal{U}$, determined. For other lifts F'_j of $\widehat{F}$ on U_j is $(G_j = F_j^{-1} F'_j)$ a 0-cochain of $\Lambda(A^0)$ relative to $\mathcal{U}$, which satisfies

$$F_k^{-1} F_j = G_k F_k'^{-1} F'_j G_j^{-1} .$$

Thus the cochians associated to (F_j) and (F'_j) determine the same cohomology class in $\mathcal{S}t(A^0)$. One can also check that, if $(A, \widehat{F})$ and $(A', \widehat{F}')$ are equivalent, the corresponding cocycles define the same cohomology class. This defines a welldefined mapping of pointed sets

$$\mathcal{H}(A^0) \to H^1(S^1; \Lambda(A^0))$$

to the first non abelian cohomology of $\Lambda(A^0)$, which we call exp. It maps the class of $(A^0, \widehat{\mathrm{id}})$ to that of id, i.e. the trivial cohomology class.

Theorem 3.3: Malgrange-Sibuya (system version)

The homomorphism

$$\exp : \mathcal{H}(A^0) \longrightarrow \mathcal{S}t(A^0) := H^1(S^1; \Lambda(A^0))$$

is an isomorphism of pointed sets.

Remark 3.4

The Theorem [BV89, Thm.III.1.1.2], in the book from Babbitt and Varadarajan, states that $\mathcal{S}t(A^0)$ is actually a local moduli space for marked pairs, which are formally isomorphic to a given system $[A^0]$. In fact is the whole third part of [BV89] dedicated to this topic.

Since the morphism exp depends on the choice of the normal form, we will denote that, if it is not clear, by $\exp_{A^0} = \exp$.

Remark 3.5

To another normal form $A^1 = {}^{\Phi}A^0$ there correspond cochains, which are conjugated via $\Phi \in G(\{t\})$. We especially get the following commutative diagram:

$$\begin{tikzcd}
G\backslash\widehat{G}(A^1) \arrow[rr, "\cdot\Phi"] \arrow[dd] \arrow[dr, phantom, "\ni"] & & G\backslash\widehat{G}(A^0) \arrow[dd] \arrow[dr, phantom, "\ni"] & \\
& \widehat{F} \arrow[rr, mapsto] \arrow[dd, mapsto, "\exp_{A^1}"'] & & \widehat{F}\Phi \arrow[dd, mapsto, "\exp_{A^0}"'] \\
H^1(S^1;\Lambda(A^1)) \arrow[rr] \arrow[dr, phantom, "\ni"] & & H^1(S^1;\Lambda(A^0)) \arrow[dr, phantom, "\ni"] & \\
& \exp_{A^1}(\widehat{F}) \arrow[rr, mapsto] & & \exp_{A^0}(\widehat{F}\Phi)
\end{tikzcd}$$

where $\exp_{A^0}(\widehat{F}\Phi) = \Phi^{-1}\exp_{A^0}(\widehat{F})\Phi$.

3.1.2 Proof of Theorem 3.3

We will mainly refer to [BV89, Proof of Theorem 4.5.1] and [Sab07, Section 6.d], where a slightly more complicated case with deformation space is proven. These both resources proof the theorem using the languages of meromorphic connections whereas we will use systems.

We will start by proofing the injectivity of the morphism exp.

Proof of the injectivity. Consider the two marked pairs $(A, \widehat{F})$ and $(A', \widehat{F}')$ in $\widehat{\mathrm{Syst}}_m(A^0)$, whose classes in $\mathcal{H}(A^0)$ get mapped to same element

$$\exp([(A, \widehat{F})]) = \lambda = \exp([(A', \widehat{F}')]) \in H^1(S^1; \Lambda(A^0))\,.$$

By using refined coverings, it is possible to find a common finite covering $\mathcal{U} = \{U_j; j \in J\}$ of S^1 such that λ is the class of the cocycles $(F_l^{-1}F_j)$ and $(F_l'^{-1}F_j')$, where F_j (resp. F_j') are lifts of $\widehat{F}$ (resp. $\widehat{F}'$) on $U_j \in \mathcal{U}$. From $[(F_l^{-1}F_j)] = [(F_l'^{-1}F_j')]$ follows that there exists a 0-cochain $(G_j)_{j\in J}$ of the sheaf $\Lambda(A^0)$ relative to the covering $\mathcal{U}$, such that

$$F_l'^{-1}F_j' = G_l F_l^{-1} F_j G_j^{-1} \text{ on the arc } U_j \cap U_l\,,$$

which can be rewritten to

$$F_j' G_j F_j^{-1} = F_l' G_l F_l^{-1} \text{ on the arc } U_j \cap U_l\,. \tag{3.1}$$

If we set $H_j := F_j' G_j F_j^{-1}$ on U_j, we get

- that from equation (**??**) that the H_j glue together and yield ,
- that H_j is a solution of $[A, A']$ on every U_j, i.e. it satisfies there ${}^{H_j}A = A'$, since

$$\begin{aligned} {}^{H_j}A &= {}^{F_j' G_j F_j^{-1}}A \\ &= {}^{F_j' G_j}A^0 && \text{(since } F_j' \text{ is a lift of } \widehat{F}' \text{ on } U_j) \\ &= {}^{F_j'}A^0 && \text{(since } G_j \text{ is a is an isotropy of } A^0) \\ &= A' && \text{(since } F_j \text{ is a lift of } \widehat{F} \text{ on } U_j) \end{aligned}$$

  and

- which satisfies $\widehat{F}' = \widehat{H}_j \widehat{F}$ on every U_j, since

$$\begin{aligned} \widehat{H}_j \widehat{F} &= \widehat{F_j' G_j F_j^{-1}} \widehat{F} \\ &= \widehat{F_j'} \underbrace{\widehat{G_j}}_{\substack{\| \\ \mathrm{id}}} \widehat{F_j^{-1}} \widehat{F} && \text{(since } G_j \text{ is flat, i.e. } \widehat{G}_j = \mathrm{id}) \\ &= \widehat{F'} \underbrace{\widehat{F^{-1}} \widehat{F}}_{\substack{\| \\ \mathrm{id}}} \\ &= \widehat{F'} \end{aligned}$$

Therefore are $(A, \widehat{F})$ and $(A, \widehat{F}')$ equivalent (cf. page 26) and injectivity is proven. $\square$

For the proof of the surjectivity we will use another result from Malgrange and Sibuya, which is also called the Malgrange-Sibuya Theorem (Theorem 3.7). It can for example be found in Babbitt and Varadarajans's book [BV89, 65ff] as Theorem 4.2.1.

Let $\widehat{F} \in G((t))$ be a matrix with formally meromorphic entries. By the Borel-Ritt Lemma (cf. Theorem 1.10) we then know, that there exists for every sector $\mathfrak{s} \subsetneq S^1$ a holomorphic function $G : \mathfrak{s} \to \mathrm{GL}_n(\mathbb{C})$ which is asymptotic to $\widehat{F}$. We will denote the set of all such holomorphic functions, which are on the arc I asymptotic to $\mathrm{id} \in G((t))$ by

$$\mathcal{G}(I) = \{G \in \mathrm{GL}_n(\mathcal{A}(I)) \mid g \sim_I \mathrm{id}\}\ ,$$

and this defines a sheaf $\mathcal{G}$ on S^1. The statement of the (second) Malgrange-Sibuya Theorem (Theorem 3.7) is then, that the difference between formal and konvergent invertible matrices is described by the first sheaf cohomology $H^1(S^1;\mathcal{G})$ of $\mathcal{G}$ via the map

$$\Theta : G(\!(t)\!)/G(\{t\}) \longrightarrow H^1(S^1;\mathcal{G})\,,$$

which will turn out to be an isomorphism. It is set up as follows:

> Let $[\widehat{F}] \in G(\!(t)\!)/G(\{t\})$ with ambassador $\widehat{F}$ and let $\mathcal{U} = \{U_j \mid j \in J\}$ be a finite covering of S^1 by open arcs. The Borel-Ritt Lemma yields for every arc $j \in J \subsetneq S^1$ a holomorphic function F_j which satisfies $F_j \sim_{U_j} \widehat{F}$. By $(F_l F_j^{-1})_{j,l\in J}$ is then a cocycle for $\mathcal{G}$ defined, and write $\Theta([\widehat{F}])$ for the corresponding cohomology class.

This construction is similar to the definition of the map of Theorem 3.3. The difference is, that we instead of M.A.E.D, to obtain lifts in the sense of Definition 2.36, we use only the Borel-Ritt Lemma to obtain only asymptotic lifts.

It can be verified, that the class $\Theta([\widehat{F}])$ does not depend on

- the choise of an ambassador $\widehat{F}$ in $[\widehat{F}] \in G(\!(t)\!)/G(\{t\})$,
- the choice of the covering $\mathcal{U}$ nor
- the choice the F_j.

Lemma 3.6

The mapping Θ is injective.

Proof. Let $\widehat{F}$ and $\widehat{F}' \in G(\!(t)\!)$ such that $\Theta([\widehat{F}]) = \Theta([\widehat{F}'])$. We then can find a covering $\mathcal{U} = \{U_j \mid j \in J\}$ together with holomorphic functions F_j and F'_j, which satisfy $F_j \sim_{U_j} \widehat{F}$ and $F'_j \sim_{U_j} \widehat{F}'$, such that $(F_l^{-1}F_j)_{j,l\in J}$ and $(F_l'^{-1}F'_j)_{j,l\in J}$ determine the classes $\Theta([\widehat{F}])$ and $\Theta([\widehat{F}'])$. This implies that there are maps G_j, which are on U_j holomorphic and satisfy $G_j \sim_{U_j} \mathrm{id}$ such that

$$F_l'^{-1}F'_j = G_l F_l^{-1} F_j G_j^{-1} \text{ on the arc } U_j \cap U_l$$

This equation can be rewritten to

$$F'_j G_j F_j^{-1} = F'_l G_l F_l^{-1} \text{ on the arc } U_j \cap U_l\,.$$

Since this tells us, that the functions $F'_j G_j F_j^{-1}$ coincide on the overlapping and define a holomorphic map from the arc S^1 (i.e. a punctured disc with a small radius) into $\mathrm{gl}_n(\mathbb{C})$, which will be called G. Since $F'_j G_j F_j^{-1} \sim_{U_j} \mathrm{id}$ for all $j \in J$, we have $G \sim_{S^1} \mathrm{id}$. Thus the defined G meromorphic at 0 and satisfies $G = F'^{-1}F$, so that $[F] = [F']$. $\square$

Theorem 3.7: Malgrange-Sibuya

The map $\Theta : G((t))/G(\{t\}) \to H^1(S^1;\mathcal{G})$ is an isomorphism.

This Theorem is proven in Section 4.4 of Babbitt and Varadarajan's book [BV89] or on page 371 of [MR91].

We are now able to proof the surjectivity of the map from Theorem 3.3.

Proof of surjectivity. Let the cohomology class $\lambda \in H^1(S^1;\Lambda(A^0))$ be represented by a cocycle $(F_{jl})_{j,l\in J}$ associated with some finite covering $\mathcal{U} = \{U_j; j \in J\}$ of S^1. We especially know, that

- F_{jl} is on $U_j \cap U_l$ asymptotic to id and
- it is a isotropy, i.e. ${}^{F_{jl}}A^0 = A^0$.

The cocycle $(F_{jl})_{j,l\in J}$ also determines an element in $\sigma \in H^1(S^1;\mathcal{G})$. From the Theorem 3.7 we know, that there is a $\widehat{F} \in G[\![t]\!] \subset G((t))$ whose class $[\widehat{F}]$ gets via Θ mapped to σ. Thus there exists holomorphic functions $F_j : \mathfrak{s}_{U_j} \to \mathrm{gl}_n(\mathbb{C})$ with $F_j \sim_{U_j} \widehat{F}$ and $F_l^{-1}F_j = F_{jl}$ on $\mathfrak{s}_{U_j\cap U_l}$ for all $j,l \in J$.

Define on every arc U_j the matrix $A_j := {}^{F_j}A^0$. On the intersections $U_j \cap U_l$ we know that $A_j = A_l$, since from $F_l^{-1}F_j \in \Lambda(A^0)$ follows on $U_j \cap U_l$ that

$$A^0 = {}^{F_l^{-1}}({}^{F_j}A^0) \qquad \Longrightarrow \qquad \underbrace{{}^{F_l}A^0}_{=A_l} = \underbrace{{}^{F_j}A^0}_{=A_j}.$$

Thus the A_j glue to a section A, which satisfies ${}^{\widehat{F}}A = A_0$ by construction. We have found an element $(A, \widehat{F}) \in \mathcal{H}(A^0)$ whose image under exp is σ. □

3.2 The Stokes groups

Here we want to introduce the notion of Stokes groups. They are for example also introduced by Loday-Richaud in [Lod94; Lod14] or section 4 of [MR91] by Martinet and Ramis.

Let us recall, that the normal form A^0 can be written as $A^0 = Q'(t^{-1}) + L\frac{1}{t}$ and a normal solution is given by $\mathcal{Y}_0(t) = t^L e^{Q(t^{-1})}$ (cf. Proposition 2.30), where

- $Q(t^{-1}) = \bigoplus_{j\in\{1,\ldots,s\}} q_j(t^{-1}) \cdot \mathrm{id}_{n_j}$ and
- the block structure of L is finer then the structure of Q (cf. Definition 2.29).

Let $\{q_1(t^{-1}), \ldots, q_s(t^{-1})\}$ be the *set of all determining polynomials of* $[A^0]$ and denote by

$$\mathcal{Q}(A^0) := \left\{q_j - q_l \mid q_j \text{ and } q_l \text{ determining polynomials of } [A^0],\ q_j \neq q_l\right\}$$

the *set of all determining polynomials of* [End A^0]. Instead of $q_j - q_l \in \mathcal{Q}(A^0)$ we will sometimes talk of (ordered) pairs $(q_j, q_l) \in \mathcal{Q}(A^0)$.

Definition 3.8

We call

- $a_{jl} \in \mathbb{C}\backslash\{0\}$ the *leading factor*,
- $\frac{a_{jl}}{t^{k_{jl}}}$ the *leading coefficient* and
- $k_{jl} \in \mathbb{Q}$ the *degree*

of $q_j - q_l \in \mathcal{Q}(A^0)$ if

$$q_j - q_l \in \left\{ \frac{a_{jl}}{t^{k_{jl}}} + h \mid h \in o(t^{-k_{jl}}), a_{jl} \neq 0 \right\} .$$

Remark 3.8.1

1. It is obvious that $k_{jl} = k_{lj}$ and $\frac{a_{jl}}{t^{k_{jl}}} = \frac{-a_{lj}}{t^{k_{lj}}}$.
2. In Boalch's paper [Boa01] (and also in [Boa99]) are the degrees of the pairs always incremented by one. We will prefer the other notion, which is also used in Loday-Richaud's paper [Lod94].
3. In Loday-Richaud's book [Lod14, Def.4.3.6] a_{jl} is negated to be consistend with calculations at ∞. Here this is not necessary, since we use the clockwise orientation on S^1 (cf. Definition 3.12).

The degrees of the elements in $\mathcal{Q}(A^0)$ are defined to be the *levels* of A^0. The set of all levels of A^0 will be denoted by

$$\mathcal{K} = \{k_1 < \cdots < k_r\} \subset \mathbb{Q}\,.$$

Remark 3.8.2

The system $[A^0]$ is unramified if and only if $\mathcal{K} \subset \mathbb{Z}$. Since we only want to consider the unramified case, this will be always the case.

3.2.1 Anti-Stokes directions and the Stokes group

Definition 3.9

Let $k \in \mathbb{N}$ and $a \in \mathbb{C}$. We say that an exponential $e^{q(t^{-1})}$, where $q(t^{-1}) \in \frac{a}{t^k} + o(t^{-k})$, has *maximal decay in a direction* $\theta \in S^1$ if and only if $ae^{-ik\widetilde{\theta}}$ is real negative. We say that an matrix has maximal decay, if every entry has maximal decay.

On the determining polynomials of $[A^0]$ we define the following (partial) order relations:

Definition 3.10

Let $\widetilde{\theta}$ be a determination of θ.

- We define the relation $\boldsymbol{q_j} \underset{\widetilde{\boldsymbol{\theta}}}{\prec} \boldsymbol{q_l}$ to be equivalent to the condition

  $$e^{(q_j - g_l)(t^{-1})} \text{ is flat at 0 in a neighbourhood of the direction } \widetilde{\theta}.$$

- Let us define another relation $\boldsymbol{q_j} \underset{\widetilde{\boldsymbol{\theta}}}{\prec\!\!\!\cdot} \boldsymbol{q_l}$ equivalent to

  $$e^{(q_j - g_l)(t^{-1})} \text{ is of maximal decay in the direction } \widetilde{\theta}.$$

Remark 3.10.1

In the unramified case do these relations not depend on the determination $\widetilde{\theta}$ of θ. As a consequence we will only write $\underset{\theta}{\prec}$ and $\underset{\theta}{\prec\!\!\!\cdot}$.

We know that

1. the condition $q_j \underset{\theta}{\prec} q_l$ is satisfied if and only if $\Re(a_{jl}e^{-ik_{jl}\theta}) < 0$ and
2. the condition $q_j \underset{\theta}{\prec\!\!\!\cdot} q_l$ is equivalent to

   $$a_{jl}e^{-ik_{jl}\theta} \text{ is a real negative number, i.e. } q_j \underset{\theta}{\prec} q_l \text{ and } \Im(a_{jl}e^{-ik_{jl}\theta}) = 0.$$

Thus it is convenient to look closer at functions of the form $f : \theta \mapsto ae^{-ik\theta}$, $k \in \mathbb{Z}$, corresponding to some pair (q_j, q_l). Write a as $a = |a|e^{i \arg(a)}$, thus the function writes as

$$\begin{aligned} f(\theta) &= |a|e^{i(\arg(a) - k\theta)} \\ &= |a|(\cos(\arg(a) - k\theta) + i\sin(\arg(a) - k\theta)) \,. \end{aligned}$$

In the Figure 3.1, we illustrate the real and the imaginary part of f.

The graphs, corresponding to the flipped pair (q_l, q_j) are then obtained by the transformation $\arg(a) \to \arg(-a) = \arg(a) + \pi$, i.e. the shift by $\frac{\pi}{k}$ to the right. This $\frac{\pi}{k}$ is exactly a half period, thus the new graphs are obtained by mirroring at the line $t = 0$.

Remark 3.11

Let k_{jl} be the degree of $q_j - q_l$. It is easy to see (cf. Figure 3.1), that the condition

$$q_j \underset{\theta}{\prec} q_l$$

is equivalent to

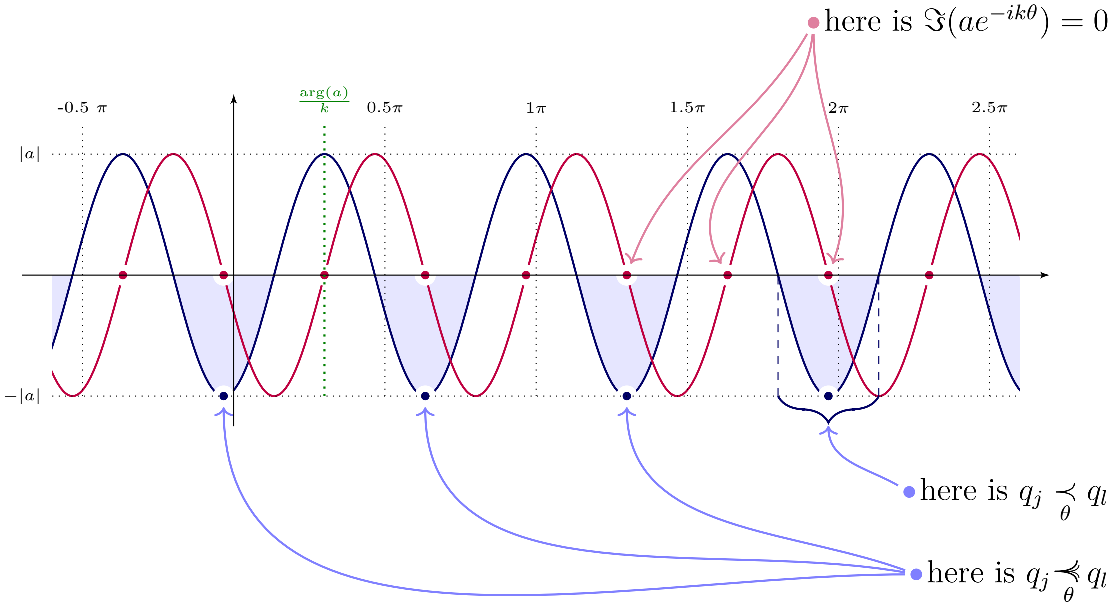

Figure 3.1: In this plot is the real part of $f(\theta) = ae^{-ik\theta}$, corresponding to some pair (q_j, q_l), in blue and the imaginary part in purple sketched.

there is a $\theta' \in U(\theta, \frac{\pi}{k_{jl}})$ such that $q_j \underset{\theta}{\prec\!\!\!\prec} q_l$.

Definition 3.12

Let $\theta \in S^1$ be an direction.

1. θ is an *anti-Stokes direction* if there is at least one pair (q_j, q_l) in $\mathcal{Q}(A^0)$, which satisfies $q_j \underset{\theta}{\prec\!\!\!\prec} q_l$.

   Let $\mathbb{A} = \{\alpha_1, \ldots, \alpha_\nu\}$ denote the set of all anti-Stokes directions in a clockwise ordering. For a uniform notation later, define $\mathbb{A}$ to contain a single, arbitrary direction if $\mathcal{K} = \{0\}$.

   Remark 3.12.1

   The clockwise ordering is chosen, similar to Loday-Richaud's paper [Lod94], since the calculations are then compatible with the calculations, which look at ∞ and take a counterclockwise ordering. Boalch uses in [Boa01] and [Boa99] the inverse ordering, but looks also at 0, thus there might be some incompatibilities. In Loday-Richaud's book [Lod14] this problem is solved by an additional minus sign for some koefficients.

2. θ is a *Stokes direction* if there is at least one pair (q_j, q_l) in $\mathcal{Q}(A^0)$, which satisfies neither $q_j \underset{\theta}{\prec} q_l$ nor $q_l \underset{\theta}{\prec} q_j$.

   Let $\mathbb{S} = \{\sigma_1 < \cdots < \sigma_\mu\}$ be the set of Stokes directions.

We will use the greek letter α whenever we want to emphasize that a direction is an anti-Stokes direction. For generic directions, we will use θ. In fact will most of the following definitions and constructions work for every $\theta \in S^1$, but the Stokes group (cf. Definition 3.14) for example will be trivial for every $\theta \notin \mathbb{A}$. Thus the interesting directions are only the anti-Stokes directions $\alpha \in \mathbb{A}$.

Lemma 3.13

Let $\alpha \in \mathbb{A}$ together with a pair $(q_j - q_l)(t^{-1}) \in \mathcal{Q}(A^0)$ of degree k_{jl}, such that $q_j \underset{\alpha}{\prec} q_l$ be given. We then know for every $m \in \mathbb{N}$ that

$$\underbrace{\alpha + m\frac{\pi}{k_{jl}}}_{\overset{!!}{\alpha'}} \in \mathbb{A}\,.$$

Especially is either $q_j \underset{\alpha'}{\prec} q_l$ (in the case, when m is even) or $q_l \underset{\alpha'}{\prec} q_j$ (when m is uneven) satisfied (see Figure 3.1).

Corollary 3.13.1

It follows that in the case $\mathcal{K} = \{k\}$, the set $\mathbb{A}$ has $\frac{\pi}{k}$-rotational symmetry.

Proof. Let (j, l) be a pair such that $q_j \underset{\alpha}{\prec} q_l$, i.e. such that $a_{jl}e^{-ik_{jl}\alpha} \in \mathbb{R}_{<0}$. Hence, for $m \in \mathbb{N}$,

$$a_{jl}e^{-ik_{jl}\left(\alpha + m\frac{\pi}{k_{jl}}\right)} = a_{jl}e^{-ik_{jl}\alpha}e^{-im\pi} = \begin{cases} a_{jl}e^{-ik_{jl}\alpha} \in \mathbb{R}_{<0} & \text{, if } m \text{ is even} \\ -a_{jl}e^{-ik_{jl}\alpha} \in \mathbb{R}_{>0} & \text{, if } m \text{ is uneven} \end{cases}$$

is, in the case when m is even, also real and negative. In the other case, when n is uneven, we use that $a_{jl} = -a_{lj}$ and $k_{jl} = k_{lj}$ to obtain $a_{lj}e^{-ik_{lj}\left(\alpha + m\frac{\pi}{k_{lj}}\right)} \in \mathbb{R}_{<0}$. Thus, for $\alpha' := \alpha + m\frac{\pi}{k_{jl}}$, we have $\alpha' \in \mathbb{A}$ since

- $q_j \underset{\alpha'}{\prec} q_l$ when m is even or
- $q_l \underset{\alpha'}{\prec} q_j$ when m is uneven.

$\square$

As a subgroup of the stalk at θ of the in Definition 3.1 defined Stokes sheaf $\Lambda(A^0)$ we define the Stokes group as follows.

Definition 3.14

Define the *Stokes group*

$$\mathrm{Sto}_\theta(A^0) := \left\{\varphi_\theta \in \Lambda_\theta(A^0) \mid \varphi_\theta \text{ has maximal decay at } \theta\right\}$$

whose elements are called *Stokes germs*.

Remark 3.14.1

For $\theta \notin \mathbb{A}$ the group $\mathrm{Sto}_\theta(A^0)$ is trivial, since at θ no flat isotropy has maximal decay, but the identity.

3.2.2 Stokes matrices

Stokes matrices, which Wasow calls in his book [Was02] Stokes multipliers and Boalch calls them Stokes factors in [Boa01; Boa99], arise either

as faithful representations of Stokes germs

or, if one starts by comparing the actual fundamental solutions on arcs, as

the matrices describing the blending between two adjacent fundamental solutions, with some additional assumptions (cf. Definition [Lod14, p. 80]).

Definition 3.15

Let us use

$$\boldsymbol{\delta}_{jl} := \begin{cases} 0 \in \mathbb{C}^{n_j \times n_l} & \text{, if } j \neq l \\ \mathrm{id} \in \mathbb{C}^{n_j \times n_l} & \text{, if } j = l \end{cases}$$

as a block version of Kronecker's delta corresponding to the structure of the normal solution $\mathcal{Y}_0$, which was fixed. Define the group

$$\mathbb{S}\mathrm{to}_\theta(A^0) = \left\{ K = (K_{jl})_{j,l \in \{1,\dots,s\}} \in \mathrm{GL}_n(\mathbb{C}) \mid K_{jl} = \boldsymbol{\delta}_{jl} \text{ unless } q_j \underset{\theta}{\prec\!\!\!\prec} q_l \right\}$$

of all *Stokes matrices* of A^0 in the direction θ. They will arise as a faithful representation (cf. Section A.2) of $\mathrm{Sto}_\theta(A^0)$.

Remark 3.15.1

1. In Boalch's publications [Boa01; Boa99] are our Stokes matrices called Stokes factors, since he introduces other objects, he wants to call Stokes matrices (cf. Section 3.4).
2. There is obviously a bijection $\vartheta_\theta : \prod_{q_j \underset{\theta}{\prec\!\!\!\prec} q_l} \mathbb{C}^{n_j \cdot n_l} \xrightarrow{\cong} \mathbb{S}\mathrm{to}_\theta(A^0)$.

Proposition 3.16

In this situation is the morphism

$$\begin{aligned} \rho_\theta : \mathrm{Sto}_\theta(A^0) &\longrightarrow \mathbb{S}\mathrm{to}_\theta(A^0) \\ \varphi_\theta &\longmapsto C_{\varphi_\theta} := \mathcal{Y}_0 \varphi_\theta \mathcal{Y}_0^{-1} \end{aligned}$$

an isomorphism which maps a germ of $\mathrm{Sto}_\theta(A^0)$ to the corresponding Stokes matrix C_{φ_θ} such that

$$\varphi_\theta(t)\mathcal{Y}_0(t) = \mathcal{Y}_0(t)C_{\varphi_\theta} \tag{3.2}$$

near θ. The matrix C_{φ_θ} is then called a *representation of* φ_θ.

Remark 3.16.1

1. In the unramified case does this morphism depend on the choice of the determination $\widetilde{\theta}$ of θ and the corresponding choice of a realization of the fundamental solution with that determination of the argument near the direction θ (cf. [Lod94] or [Lod14, 78f]).
2. This construction defines also a morphism, which takes a germ $\varphi_\theta \in \Lambda_\theta(A^0) \supset \mathrm{Sto}_\theta(A^0)$ into its unique representation matrix

$$C_{\varphi_\theta} \in \underset{\mathbb{S}\mathrm{to}_\theta(A^0)}{\underset{\cup}{\widehat{\mathbb{S}\mathrm{to}}_\theta(A^0)}} := \left\{ (K_{jl})_{j,l\in\{1,\ldots,s\}} \in \mathrm{GL}_n(\mathbb{C}) \mid K_{jl} = \boldsymbol{\delta}_{jl} \text{ unless } q_j \underset{\theta}{\prec} q_l \right\}$$

and there is a bijection $\widehat{\vartheta}_\theta : \prod_{q_j \underset{\theta}{\prec} q_l} \mathbb{C}^{n_j \cdot n_l} \xrightarrow{\cong} \widehat{\mathbb{S}\mathrm{to}}_\theta(A^0)$.

Proof. It is well known (cf. [Boa99, p. 10]), that the morphism ρ_θ, i.e. conjugation by the fundamental solution, relates solutions φ_θ of $[\mathrm{End}(A^0)] = [A^0, A^0]$ to solutions of $[0, 0]$ which are the constant matrices $\mathrm{GL}_n(\mathbb{C})$. Thus we have to show, that the image of $\mathrm{Sto}_\theta(A^0)$ under ρ_θ is $\mathbb{S}\mathrm{to}_\theta(A^0)$.

To see that the obtained matrix has the necessary zeros, to lie in $\mathbb{S}\mathrm{to}_\theta(A^0)$ we look at Equation (3.2) and deduce

$$\varphi_\theta(t) = t^L e^{Q(t^{-1})} C_{\varphi_\theta} e^{-Q(t^{-1})} t^{-L} \tag{3.3}$$

with the given choice of the argument near θ. After decomposing C_{φ_θ} into

$$\begin{aligned} C_{\varphi_\theta} &= 1_n + \begin{pmatrix} c_{(1,1)} & c_{(1,2)} & \cdots & \\ c_{(2,1} & \ddots & & \\ \vdots & & & \\ & & & c_{(s,s)} \end{pmatrix} \\ &= 1_n + \underbrace{\begin{pmatrix} c_{(1,1)} & 0 & \cdots \\ 0 & & \\ \vdots & & \\ & & \end{pmatrix}}_{C^{(1,1)}_{\varphi_\theta}} + \underbrace{\begin{pmatrix} 0 & c_{(1,2)} & 0 & \cdots \\ & 0 & & \\ & \vdots & & \\ & & & \end{pmatrix}}_{C^{(1,2)}_{\varphi_\theta}} + \cdots + \underbrace{\begin{pmatrix} & & \vdots \\ & & 0 \\ \cdots & 0 & c_{(s,s)} \end{pmatrix}}_{C^{(s,s)}_{\varphi_\theta}} \end{aligned}$$

$$= 1_n + \sum_{(l,j)} C_{\varphi_\theta}^{(l,j)}$$

where the $c_{(j,l)}$ are blocks of size $n_j \times n_l$ which correspond to the structure of Q. After rewriting the Equation (3.3) we get

$$\varphi_\theta = t^L \left(1_n + \sum_{(l,j)} C_{\varphi_\theta}^{(l,j)} e^{(q_l - q_j)(t^{-1})} \right) t^{-L} .$$

Thus, for φ_θ to be flat in direction θ, it is necessary and sufficient that if $e^{(q_l-q_j)(t^{-1})}$ does not have maximal decay in direction θ the corresponding block $C_{\varphi_\theta}^{(l,j)}$ vanishes. Thus we have seen, that C_{φ_θ} is an element of $\mathbb{S}\mathrm{to}_\theta(A^0)$.

The **surjectivity** can be seen easily, since every constant matrix, with zeros at the necessary positions, characterizes an unique element of $\mathrm{Sto}_\theta(A^0)$:

> Let $C = 1_n + \sum_{(l,j)|q_j \underset{\theta}{\nprec} q_l} C^{(l,j)}$ be an element of $\mathbb{S}\mathrm{to}_\theta(A^0)$. Then is a pre-image of C given by $t^L e^{Q(t^{-1})} C e^{-Q(t^{-1})} t^{-L}$ which lies in $\mathrm{Sto}_\theta(A^0)$, since it satisfies the condition discussed above.

The map $\rho_{\widetilde{\theta}}$ is also **injective**, since it is the conjugation by an invertible matrix. □

From the calculations in the proof it is clear that

1. for $j = l$ the (diagonal) blocks $C_{\varphi_\theta}^{(l,j)}$ vanish since $q_l - q_j = 0$ does not have maximal decay and
2. if $e^{q_j - q_l}$ has has maximal decay, then $e^{q_l - q_j}$ has not. Thus if $C_{\varphi_\theta}^{(l,j)}$ is not equal to zero, the block $C_{\varphi_\theta}^{(j,l)}$ is necessarily zero.

This implies that the matrix C_{φ_θ} is unipotent, and hence is $\mathrm{Sto}_\theta(A^0)$ is a unipotent Lie group.

One can use the Stokes matrices to give an alternative characterization of Stokes germs:

> a germ $\varphi_\theta \in \Lambda_\theta(A^0)$ is in $\mathrm{Sto}_\theta(A^0)$ if and only if there exists a $C \in \mathbb{S}\mathrm{to}_\theta(A^0)$ such that $\varphi_\theta = \mathcal{Y}_0 C \mathcal{Y}_0^{-1}$.

Formulated is this in the following corollary.

Corollary 3.17

A germ $\varphi_\theta \in \Lambda_\theta(A^0)$ is a Stokes germ, i.e. an element in $\mathrm{Sto}_\theta(A^0)$, if and only if it has a representation C_{φ_θ} where

$$C_{\varphi_\theta} = 1_n + \sum_{(l,j)|q_j \underset{\theta}{\nprec} q_l} C_{\varphi_\theta}^{(l,j)}$$

and the $C_{\varphi_\theta}^{(l,j)}$ have the necessary block structure.

Remark 3.17.1

In Loday-Richaud's book [Lod14, p. 78] are the elements of $\mathrm{Sto}_\theta(A^0)$ actually characterized as the flat transformations, such that Equation (3.2) is satisfied for some unique constant invertible matrix $C \in \mathbb{S}\mathrm{to}_\theta(A^0)$.

Definition 3.18

We denote the set of *levels of the germ* $\varphi_\theta \in \Lambda_\theta(A^0)$ by

$$\mathcal{K}(\varphi_\theta) := \left\{ \deg(q_j - q_l) \mid C^{(l,j)}_{\varphi_\theta} \neq 0 \text{ in some representation of } \varphi_\theta \right\} \subset \mathcal{K}\,.$$

A germ φ_θ is called a *k-germ* when $\mathcal{K}(\varphi_\theta) \subset \{k\}$, i.e. it has at most the level k.

3.2.3 Decomposition of the Stokes group by levels

The goal of this section is, to introduce a filtration of $\Lambda(A^0)$, wich will be restricted to $\mathrm{Sto}_\theta(A^0)$ and defines there a filtration. This leads to a decomposition of $\mathrm{Sto}_\theta(A^0)$ into a semidirect product (cf. Prpopsition 3.25).

Let us introduce a couple of notations and definitions, which coincide with the notations used in Loday-Richaud's paper [Lod94]. Another good resource, which uses similar notations, is for example the paper [MR91, 362f] from Martinet and Ramis.

Notations 3.19

For every level $k \in \mathcal{K}$ and direction $\theta \in S^1$ we set

- $\Lambda^k(A^0)$ as the subsheaf of $\Lambda(A^0)$ of all germs, which are generated by k-germs;
- $\Lambda^{\leq k}(A^0)$ (resp. $\Lambda^{<k}(A^0)$ or $\Lambda^{\geq k}(A^0)$) as the subsheaf of $\Lambda(A^0)$ generated by k'-germs for all $k' \leq k$ (resp. $k' < k$ or $k' \geq k$).

Let $\star \in \{k, <k, \leq k, \dots\}$. The restrictions to Sto_θ yield the groups

$$\mathrm{Sto}^\star_\theta(A^0) := \mathrm{Sto}_\theta(A^0) \cap \Lambda^\star_\theta(A^0)$$

and let us also define $\mathbb{S}\mathrm{to}^\star_\theta(A^0)$ as the groups of representations, which correspond to elements of $\mathrm{Sto}^\star_\theta(A^0)$.

Corresponding to the definitions above, one can define $\mathbb{A}^\star := \{\alpha \in \mathbb{A} \mid \mathrm{Sto}^\star_\alpha(A^0) \neq \{\mathrm{id}\}\}$ for $\star \in \{k, <k, \leq k, \dots\}$ and we say that α *is bearing the level* k if $\alpha \in \mathbb{A}^k$.

Remark 3.20

It is clear that for every $k \in \mathcal{K}$ we have the canonical inclusions $\mathbb{A}^k \hookrightarrow \mathbb{A}^{\leq k}$ and $\mathbb{A}^{<k} \hookrightarrow \mathbb{A}^{\leq k}$.

Sometimes it is also useful to talk about the *set of levels beared by an direction* $\alpha \in \mathbb{A}$:

$$\mathcal{K}_\alpha := \left\{ k \in \mathcal{K} \mid \mathrm{Sto}^k_\alpha(A^0) \neq \{\mathrm{id}\} \right\} .$$

Corollary 3.21

The Lemma 3.13 implies that from $k \in \mathcal{K}_\alpha$ follows that $k \in \mathcal{K}_{\alpha + m\frac{\pi}{k}}$ for $m \in \mathbb{N}$.

Let us now study the sheaves $\Lambda^\star(A^0)$, and discuss how they correlate and how they can be composed from the others.

The following proposition can be found as [Lod94, Prop.I.5.1] and the key-statement is also given in [MR91, Prop.4.10].

Proposition 3.22

For any level $k \in \mathcal{K}$ one has that $\Lambda^k(A^0)$, $\Lambda^{\leq k}(A^0)$ and $\Lambda^{<k}(A^0)$ are sheaves of subgroups of $\Lambda(A^0)$ and the sheaf $\Lambda^k(A^0)$ is normal in $\Lambda^{\leq k}(A^0)$.

We even know more, let

- $i : \Lambda^k(A^0) \hookrightarrow \Lambda^{\leq k}(A^0)$ be the canonical inclusion and
- $p : \Lambda^{\leq k}(A^0) \twoheadrightarrow \Lambda^{<k}(A^0)$ be the truncation to terms of levels $< k$.

Then does the exact sequence of sheaves

$$1 \longrightarrow \Lambda^k(A^0) \xrightarrow{i} \Lambda^{\leq k}(A^0) \xrightarrow{p} \Lambda^{<k}(A^0) \longrightarrow 1 ,$$

split.

From the splitting of the sequence, we obtain immediately the following decomposition into a semidirect product.

Corollary 3.23

For any $k \in \mathcal{K}$, there are the two following ways of factoring $\Lambda^{\leq k}(A^0)$ in a semidirect product:

$$\begin{aligned} \Lambda^{\leq k}(A^0) &\cong \Lambda^{<k}(A^0) \ltimes \Lambda^k(A^0) \\ &\cong \Lambda^k(A^0) \rtimes \Lambda^{<k}(A^0) . \end{aligned}$$

Thus any germ $f^{\leq k} \in \Lambda^{\leq k}(A^0)$ can be uniquely written as

- $f^{\leq k} = f^{<k} g^k$, where $f^{<k} \in \Lambda^{<k}$ and $g^k \in \Lambda^k$, or

- $f^{\leq k} = f^k f^{<k}$, where $f^k \in \Lambda^k$ and $f^{<k} \in \Lambda^{<k}$.

Remark 3.23.1

We can get the factor $f^{<k}$ common to both factorizations by truncation of $f^{\leq k}$ to terms of level $< k$, i.e. the map p from Proposition 3.22. This truncation can explicitly be achieved, in terms of Stokes matrices, by keeping in representations $1 + \sum C^{(j,l)}$ of $f^{\leq k}$ only the blocks $C^{(j,l)}$ such that $\deg(q_j - q_l) < k$.

A factorization algorithm could then be:

> get the factor $f^{<k}$ common to both factorizations by truncation of $f^{\leq k}$ to terms of level $< k$ and set $g^k := (f^{<k})^{-1} f^{\leq k}$ and $f^k := f^{\leq k}(f^{<k})^{-1}$.

This decomposition in a semidirect product can be extended to all levels, since $\Lambda^{<k}(A^0) = \Lambda^{\leq \max\{k' \in \mathcal{K} | k' < k\}}$. Thus

$$\Lambda(A^0) \cong \bigltimes_{k \in \mathcal{K}} \Lambda^k(A^0)\,,$$

where the semidirect product is taken in an ascending or descending order of levels.

Remark 3.24

Loday-Richaud states in her paper [Lod94, Prop.I.5.3] the following proposition, which is a more general version of Proposition 3.22.

Proposition 3.24.1

For any levels k,$k' \in \mathcal{K}$ with $k' < k$ one has:

1. the sheaf $\Lambda^{\geq k'}(A^0) \cap \Lambda^{\leq k'}(A^0)$ is normal in $\Lambda^{\leq k}(A^0)$;
2. the exact sequence of sheaves

$$1 \longrightarrow \Lambda^{\geq k'}(A^0) \cap \Lambda^{\leq k}(A^0) \xrightarrow{\ i\ } \Lambda^{\leq k}(A^0) \xrightarrow{\ p\ } \Lambda^{<k'}(A^0) \longrightarrow 1\,,$$

   where

   - i is the canonical inclusion and
   - p is the truncation to terms of levels $< k'$,

   splits.

We can use this proposition to follow (cf. [Lod94, Cor.I.5.4]) that

1. the filtration

$$\Lambda^{k_r}(A^0) = \Lambda^{\geq k_r}(A^0) \subset \Lambda^{\geq k_{r-1}}(A^0) \subset \cdots \subset \Lambda^{\geq k_1}(A^0) = \Lambda(A^0)$$

   is normal and

2. we can use this to achieve the decomposition

$$\Lambda(A^0) \cong \ltimes_{k\in\mathcal{K}} \Lambda^k(A^0)$$

taken in an arbitrary order. In fact, one can also extend the algorithm from Remark 3.23.1 to an arbitrary order of levels.

The important statement, which we will use later, is then the following.

Proposition 3.25

The results can be restricted to the Stokes groups (cf. [Lod94, Prop.I.5.5]). Thus, for $\alpha \in \mathbb{A}$, one has

$$\mathrm{Sto}_\alpha(A^0) \cong \ltimes_{k\in\mathcal{K}_\alpha} \mathrm{Sto}_\alpha^k(A^0)$$

the semidirect product being taken in an arbitrary order.

Definition 3.25.1

We will denote the map, wich gives the factors of this factorization by

$$i_\alpha : \mathrm{Sto}_\alpha(A^0) \xrightarrow{\cong} \prod_{k\in\mathcal{K}_\alpha} \mathrm{Sto}_\alpha^k(A^0)\,,$$

where the factorization is taken in ascending order.

Remark 3.25.1

Write $\rho_\alpha^k : \mathrm{Sto}_\alpha^k(A^0) \to \mathbb{S}\mathrm{to}_\alpha^k(A^0)$ for the restriction of the map ρ_α (cf. Proposition 3.16) to the level k. Then, one can denote the induced decomposition also by

$$i_\alpha : \mathbb{S}\mathrm{to}_\alpha(A^0) \xrightarrow{\cong} \prod_{k\in\mathcal{K}_\alpha} \mathbb{S}\mathrm{to}_\alpha^k(A^0)$$

and the corresponding diagram

$$\begin{tikzcd} \mathrm{Sto}_\alpha(A^0) \arrow[r, "i_\alpha"] \arrow[d, "\rho_\alpha"] & \prod_{k\in\mathcal{K}_\alpha} \mathrm{Sto}_\alpha^k(A^0)\,, \arrow[d, "\prod_{k\in\mathcal{K}} \rho_\alpha^k"] \\ \mathbb{S}\mathrm{to}_\alpha(A^0) \arrow[r, "i_\alpha"] & \prod_{k\in\mathcal{K}_\alpha} \mathbb{S}\mathrm{to}_\alpha^k(A^0)\,, \end{tikzcd}$$

commutes.

3.3 Stokes structures: using Stokes groups

The goal in this section is to prove that there is a bijective and natural map

$$h : \prod_{\alpha \in \mathbb{A}} \mathrm{Sto}_\alpha(A^0) \longrightarrow \mathcal{S}t(A^0) .$$

And since $\mathrm{Sto}_\alpha(A^0)$ has $\mathbb{S}\mathrm{to}_\alpha(A^0)$ as a faithful representation, we also get the isomorphism $\prod_{\alpha \in \mathbb{A}} \mathbb{S}\mathrm{to}_\alpha(A^0) \cong \mathcal{S}t(A^0)$ as a corollary.

Let us recall, that $\mathcal{S}t(A^0)$ is defined to be $H^1(S^1; \Lambda(A^0))$ (cf. Section 3.1). The elements of $\prod_{\alpha \in \mathbb{A}} \mathrm{Sto}_\alpha(A^0)$ define in a canonical way cocycles of the sheaf $\Lambda(A^0)$ (cf. Equation (3.4)), called Stokes cocycles (cf. Definition 3.29). In fact, will h map such cocycles to the cohomology class, to which they correspond. Thus the statement, that h is a bijection, is equivalent to the statement that

> in each cohomology class of $\mathcal{S}t(A^0)$ is an unique 1-cocycle, which is a Stokes cocycle.

Cyclic coverings

To formulate the following theorem, we use the notion of cyclic coverings and nerves of such coverings, which are defined as follows.

Definition 3.26

Let J be a finite set, identified to $\{1, \ldots, p\} \subset \mathbb{Z}$.

1. A *cyclic covering* of S^1 is a finite covering $\mathcal{U} = (U_j = U(\theta_j, \varepsilon_j))_{j \in J}$ consisting of arcs, which satisfies that
   a) $\widetilde{\theta}_j \geq \widetilde{\theta}_{j+1}$ for $j \in \{1, \ldots, p-1\}$, i.e. the center points are ordered and
   b) $\widetilde{\theta}_j + \frac{\varepsilon_j}{2} \geq \widetilde{\theta}_{j+1} + \frac{\varepsilon_{j+1}}{2}$ for $j \in \{1, \ldots, p-1\}$ and $\widetilde{\theta}_p + \frac{\varepsilon_p}{2} \geq \widetilde{\theta}_1 - 2\pi + \frac{\varepsilon_1}{2}$, i.e. the arcs are not encased by another arc,

   where the $\widetilde{\theta}_j \in [0, 2\pi[$ are determinations of the $\theta_j \in S^1$.
2. The *nerve* of a cyclic covering $\mathcal{U} = \{U_j ; j \in J\}$ is the family $\dot{\mathcal{U}} = \{\dot{U}_j ; j \in J\}$ defined by:
   - $\dot{U}_j = U_j \cap U_{j+1}$ when $\#J > 2$,
   - $\dot{U}_1$ and $\dot{U}_2$ the connected components of $U_1 \cap U_2$ when $\#J = 2$.

Remark 3.26.1

The nerve of the cyclic covering $\mathcal{U} = (U(\theta_j, \varepsilon_j))_{j \in J}$ is explicitly given by

$$\dot{\mathcal{U}} = \left(\left(\theta_{j+1} - \frac{\varepsilon_{j+1}}{2}, \theta_j + \frac{\varepsilon_j}{2} \right) \right)_{j \in J} .$$

The cyclic coverings correspond one-to-one to nerves of cyclic coverings. If one starts with a nerve $\{\dot{U}_j \mid j \in J\}$, one obtains a cyclic covering as $\mathcal{U} = \{U_j \mid j \in J\}$ where the arc U_j is the connected clockwise hull from $\dot{U}_{j-1}$ to $\dot{U}_j$.

Definition 3.27

A covering $\mathcal{V}$ is said to *refine* a covering $\mathcal{U}$ if, to each open set $V \in \mathcal{V}$ there is at least one $U \in \mathcal{U}$ with $V \subset U$.

Proposition 3.28

The covering $\mathcal{V}$ refines $\mathcal{U}$ if and only if the corresponding nerves $\dot{\mathcal{U}} = \{\dot{U}_j\}$ and $\dot{\mathcal{V}} = \{\dot{V}_l\}$ satisfy

each $\dot{U}_j$ contains at least one $\dot{V}_l$.

3.3.1 The theorem

Let $\{\theta_j \mid j \in J\} \subset S^1$ be a finite set and $\dot{\varphi} = (\dot{\varphi}_{\theta_j})_{j \in J} \in \prod_{j \in J} \Lambda_{\theta_j}(A^0)$ be a finite family of germs. Let $\dot{\varphi}_j$ be the function representing the germ $\dot{\varphi}_{\theta_j}$ on its (maximal) arc of definition Ω_j around θ_j. In the following way, one can associate a cohomology class in $\mathcal{S}t(A^0)$ to $\dot{\varphi}$:

for every cyclic covering $\mathcal{U} = (U_j)_{j \in J}$ which satisfies $\dot{U}_j \subset \Omega_j$ for all $j \in J$, one can define the 1-cocycle $(\dot{\varphi}_{j|\dot{U}_j})_{j \in J} \in \Gamma(\dot{\mathcal{U}}; \Lambda(A^0))$.

To a different cyclic covering, satisfying the condition above, this construction yields a cohomologous 1-cocycle, thus the induced map

$$\prod_{j \in J} \Lambda_{\theta_j}(A^0) \longrightarrow H^1(S^1; \Lambda(A^0)) = \mathcal{S}t(A^0) \tag{3.4}$$

is welldefined (cf. [Lod94, p. 868]).

Definition 3.29

Let $\nu = \#\mathbb{A}$ the number of all anti-Stokes directions and write $\mathbb{A} = \{\alpha_1, \ldots, \alpha_\nu\}$.

A *Stokes cocycle* is a 1-cocycle $(\varphi_j)_{j \in \{1,\ldots,\nu\}} \in \prod_{j \in \{1,\ldots,\nu\}} \Gamma(U_j; \Lambda(A^0))$ corresponding to some cyclic covering with nerve $\dot{\mathcal{U}} = (\dot{U}_j)_{j \in \{1,\ldots,\nu\}}$, which satisfies for every $j \in \{1, \ldots, \nu\}$

- $\alpha_j \in \dot{U}_j$ and
- the germ $\varphi_{\alpha_j} := \varphi_{j,\alpha_j}$ of φ_j at α_j is an element of $\mathrm{Sto}_{\alpha_j}(A^0)$.

Remark 3.29.1

The sections in $\Gamma(\dot{U}_j; \Lambda(A^0))$ are uniquely determined as the extension of the germ at α_j, since the sheaf $\Lambda(A^0)$ defined via the system $[A^0, A^0]$ (cf. Definition 3.1). We thus have an injective map

$$\prod_{j\in\{1,\ldots,\nu\}} \Gamma(\dot{U}_j; \Lambda(A^0)) \hookrightarrow \prod_{j\in\{1,\ldots,\nu\}} \mathrm{Sto}_{\alpha_j}(A^0)\,,$$

which takes an Stokes cocycle and yields the corresponding Stokes germs. For a fine enough covering $\mathcal{U}$, i.e. a covering $\mathcal{U}$ with a nerve $\dot{\mathcal{U}}$ which consists of small enough arcs satisfying the conditions above, is this map a bijection.

We will use this fact implicitly and assume that the covering is always fine enough to call elements of $\prod_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha(A^0)$ Stokes cocycles.

We can use the Equation (3.4) to obtain for Stokes cocycles a mapping

$$h : \textstyle\prod_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha(A^0) \longrightarrow \prod_{\alpha\in\mathbb{A}} \Lambda_\alpha(A^0) \xrightarrow{(3.4)} \mathcal{S}t(A^0),$$

which takes a Stokes cocycle to its corresponding cohomology class.

Theorem 3.30

The map

$$h : \textstyle\prod_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha(A^0) \longrightarrow \mathcal{S}t(A^0)$$

is a bijection and natural.

Remark 3.30.1

Natural means that h commutes to isomorphisms and constructions over systems or connections they represent.

From Theorem 3.30 and Proposition 3.16 we get the following corollary.

Corollary 3.31

Using the isomorphisms $\mathrm{Sto}_\theta(A^0) \cong \mathbb{S}\mathrm{to}_\theta(A^0)$ from Proposition 3.16 we obtain

$$\mathcal{S}t(A^0) \cong \prod_{\alpha\in\mathbb{A}} \mathbb{S}\mathrm{to}_\alpha(A^0)$$

which endows $\mathcal{S}t(A^0)$ with the structure of an unipotent Lie group with the finite complex dimension $N := \dim_{\mathbb{C}} \mathcal{S}t(A^0)$ (cf. [Lod94, Sec.III.1]). This can be rewritten in the following way:

$$N = \sum_{\alpha \in \mathbb{A}} \dim_{\mathbb{C}} \mathbb{S}\mathrm{to}_\alpha(A^0) = \sum_{\alpha \in \mathbb{A}} \sum_{q_j \underset{\alpha}{\prec\!\!\prec} q_l} n_j \cdot n_l = \sum_{\substack{1 \leq j,l \leq n \\ j<l}} 2 \cdot \deg(q_j - q_l) \cdot n_j \cdot n_l \,.$$

Remark 3.31.1

This number N is known to be the *irregularity* of $[\operatorname{End} A^0]$.

Remark 3.32

To define the inverse map of h, one has to find in each cocycle in $\mathcal{S}t(A^0)$ the Stokes cocycle. Loday-Richaud gives an algorithm in section II.3.4 of her paper [Lod94], which takes a cocycle over an arbitrary cyclic covering and outputs cohomologous Stokes cocycle and thus solves this problem.

3.3.2 Proof of Theorem 3.30

We will only look at the unramified case, for which we refer to [Lod94, Sec.II.3]. The proof in the ramified case can be found in [Lod94, Sec.II.4]. We first have to introduce adequate coverings, which will be used in the proof.

Adequate coverings

Definition 3.33

Let $\star \in \{k, < k, \leq k, \dots\}$. A covering $\mathcal{U}$ beyond which the inductive limit $\varinjlim_{\mathcal{U}} H^1(\mathcal{U}; \Lambda^\star(A^0))$ is stationary is said to be *adequate* to describe $H^1(S^1; \Lambda^\star(A^0))$ or *adequate* to $\Lambda^\star(A^0)$.

In other words is a covering $\mathcal{U}$ adequate, if and only if the quotient map

$$\Gamma(\mathcal{U}; \Lambda^\star(A^0)) \longrightarrow H^1(S^1; \Lambda^\star(A^0))$$

is surjective.

The following proposition is in Loday-Richaud's paper [Lod94] given as Proposition II.1.7. It contains a simple characterization, which will be used to see, that our defined coverings are adequate.

Proposition 3.34

Let $k \in \mathcal{K}_\alpha$.

Definition 3.34.1

Let $\alpha \in \mathbb{A}^k$. An arc $U(\alpha, \frac{\pi}{k})$ is called a *Stokes arc of level k at α.*

A cyclic covering $\mathcal{U} = (U_j)_{j \in J}$, which satisfies

> for every $\alpha \in \mathbb{A}^k$ contains the Stokes arc $U(\alpha, \frac{\pi}{k})$ at least one arc $\dot{U}_j$ from the nerve $\dot{\mathcal{U}}$ of $\mathcal{U}$

is adequate to $\Lambda^k(A^0)$.

The covering $\mathcal{U}$ is adequate to $\Lambda^{\leq k}(A^0)$ (resp. $\Lambda^{<k}(A^0)$) if it is adequate to $\Lambda^{k'}(A^0)$ for every $k' \leq k$ (resp. $k' < k$).

Let $k \in \mathcal{K}$. We want to define the three cyclic coverings $\mathcal{U}^k$, $\mathcal{U}^{\leq k}$ and $\mathcal{U}^{<k}$ which will be adequate to $\Lambda^k(A^0)$, $\Lambda^{\leq k}(A^0)$ and $\Lambda^{<k}(A^0)$. Furthermore will the coverings be comparable at the different levels.

1. The first covering $\mathcal{U}^k = \{\dot{U}^k_\alpha \mid \alpha \in \mathbb{A}^k\}$ is the cyclic covering with nerve

$$\dot{\mathcal{U}}^k := \left\{\dot{U}^k_\alpha = U(\alpha, \frac{\pi}{k}) \mid \alpha \in \mathbb{A}^k\right\}$$

consisting of all Stokes arcs of level k for anti-Stokes directions bearing the level k.

Remark 3.35

Boalch introduces in his publications [Boa01, p. 19] and [Boa99, Def.1.23] the notion of *supersectors*, they are in the case of a single level k, defined as follows:

> write the anti-Stokes directions as $\mathbb{A} = \{\alpha_1, \dots, \alpha_\nu\}$ arranged according to the clockwise ordering, then is the i-th supersector defined as the arc
>
> $$\widehat{\mathrm{Sect}}^k_i := \left(\alpha_i - \frac{\pi}{2k}, \alpha_{i+1} + \frac{\pi}{2k}\right).$$

This yields a cyclic covering $(\widehat{\mathrm{Sect}}^k_i)_{i \in \{1,\dots,\nu\}}$ whose nerve is exactly $\dot{\mathcal{U}}^k$ defined above.

If we extend to more then one level level, $\#\mathcal{K} > 1$, the set $\bigcup_{k \in \mathcal{K}} \left\{U(\alpha, \frac{\pi}{k}) \mid \alpha \in \mathbb{A}^k\right\}$ is no longer a nerve. Hence we have to define the coverings $\mathcal{U}^{\leq k}$ and $\mathcal{U}^{<k}$ in a different way. Denote by

$$\{K_1 < \dots < K_s = k\} = \left\{\max\left(\mathcal{K}_\alpha \cap [0, k]\right) \mid \alpha \in \mathbb{A}^{\leq k}\right\}$$

the set of all *k-maximum levels* for $\alpha \in \mathbb{A}^{\leq k}$.

2. The cyclic covering $\mathcal{U}^{\leq k} = \left\{U^{\leq k}_\alpha \mid \alpha \in \mathbb{A}^{\leq k}\right\}$ will be defined by induction. Let us assume that

the $\dot{U}_\alpha^{\leq k}$ are defined for all $\alpha \in \mathbb{A}^{\leq k}$ with k-maximum level greater than K_i such that their complete family is a nerve.

Let

- α be a anti-Stokes direction with k-maximum level K_i and
- α^- (resp. α^+) be the next anti-Stokes direction with k-maximum level greater then K_i on the left (resp. on the right) and define $\dot{U}_{\alpha^-,\alpha^+}$ as the clockwise hull of the arcs $\dot{U}_{\alpha^-}^{\leq k}$ and $\dot{U}_{\alpha^+}^{\leq k}$ already defined by induction. If there are no anti-Stokes directions with k-maximum level greater then K_i we set $\dot{U}_{\alpha^-,\alpha^+} = S^1$.

We then set

$$\dot{U}_\alpha^{\leq k} := U\left(\alpha, \frac{\pi}{K_i}\right) \cap \dot{U}_{\alpha^-,\alpha^+}$$

and the family of all $\dot{U}_\alpha^{\leq k}$ is a nerve.

Remark 3.36

If α has a k-maximum level equal to k then is $\dot{U}_\alpha^{\leq k}$ equal to the Stokes arc $U\left(\alpha, \frac{\pi}{k}\right) = \dot{U}_\alpha^k$.

3. The last cyclic covering, $\mathcal{U}^{<k} = \left\{U_\alpha^{<k} \mid \alpha \in \mathbb{A}^{<k}\right\}$, is defined as $\mathcal{U}^{<k} := \mathcal{U}^{\leq k'}$ where $k' := \max\{k'' \in \mathcal{K} \mid k'' < k\}$.

Remark 3.37

The coverings $\mathcal{U}^k$, $\mathcal{U}^{\leq k}$ and $\mathcal{U}^{<k}$ depend only on $\mathcal{Q}(A^0)$. Hence they depend only on the determining polynomials.

It is obvious, that for every $k \in \mathcal{K}$ the covering $\mathcal{U}^{\leq k}$ refines $\mathcal{U}^k$ and $\mathcal{U}^{<k}$. Furthermore are the coverings defined, such that they satisfy the condition in Proposition 3.34. Thus the first property in the following proposition is satisfied. The other two can be found at [Lod94, Prop.II.3.1 (iv)].

Proposition 3.38

Let $k \in \mathcal{K}$, then

1. the coverings $\mathcal{U}^k$, $\mathcal{U}^{\leq k}$ and $\mathcal{U}^{<k}$ are adequate to $\Lambda^k(A^0)$, $\Lambda^{\leq k}(A^0)$ and $\Lambda^{<k}(A^0)$, respectively;
2. there exists no 0-cochain in $\Lambda^k(A^0)$ on $\mathcal{U}^k$;
3. on $\mathcal{U}^{\leq k}$ there is no 0-cochain in $\Lambda^{\leq k}(A^0)$ of level k, i.e. all 0-cochains of $\Lambda^{\leq k}(A^0)$ belong to $\Lambda^{<k}(A^0)$.

To have a shorter notation, we denote the product $\prod_{\alpha \in \mathbb{A}^\star} \Gamma(\dot{U}_\alpha^\star; \Lambda^\star(A^0))$ by $\Gamma(\dot{U}^\star; \Lambda^\star(A^0))$ for every $\star \in \{k, < k, \leq k, \dots\}$.

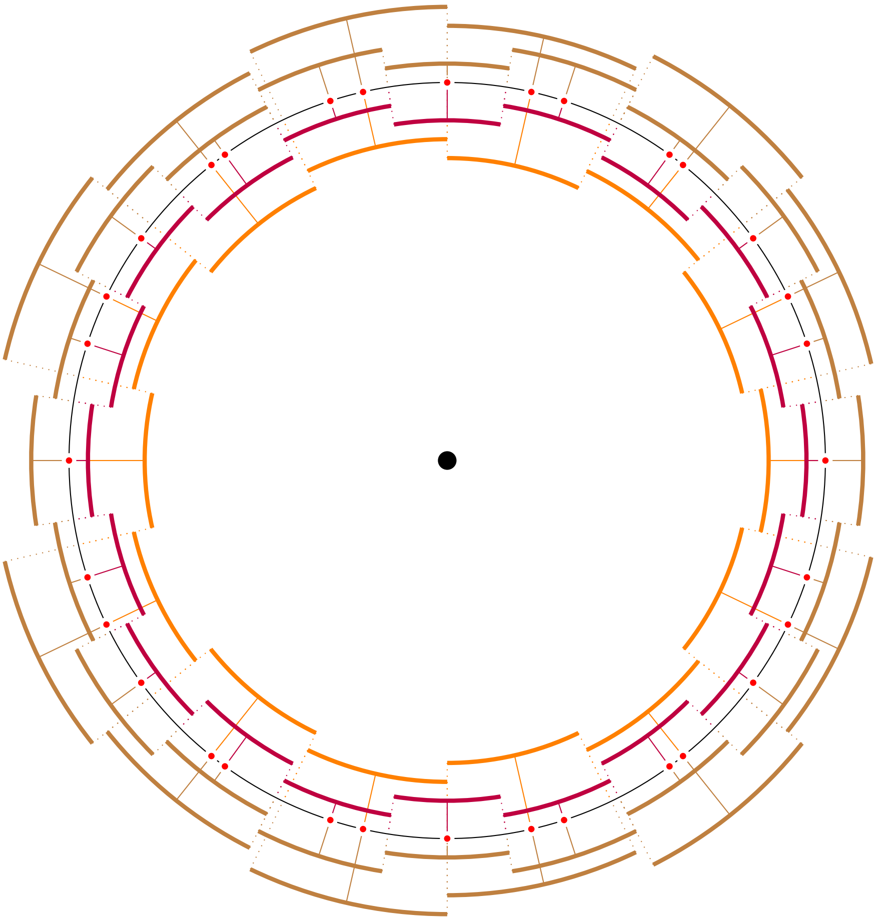

Figure 3.2: The adequate coverings for an example with $\mathcal{K} = \{7, 10\}$ and $\mathbb{A} = \left\{\frac{j\cdot\pi}{k} \mid k \in \mathcal{K},\, j \in \mathbb{N}\right\}$. The anti-Stokes directions are marked by the red dots. The arcs of $\dot{\mathcal{U}}^7 = \dot{\mathcal{U}}^{\leq 7}$ are orange, the arcs of $\dot{\mathcal{U}}^{10}$ are purple and the arcs of $\dot{\mathcal{U}}^{\leq 10} = \dot{\mathcal{U}}$ are brown.

The case of a unique level

First we will proof Theorem 3.30 in the case of a unique level. This means that

- either $\Lambda(A^0)$ has only one level k, thus
  - $\Lambda(A^0) = \Lambda^k(A^0)$ and
  - $\mathrm{Sto}_\theta(A^0) = \mathrm{Sto}_\theta^k(A^0)$ for every θ,
- or we restrict to a given level $k \in \mathcal{K}$.

Lemma 3.39

Let $k \in \mathcal{K}$. The morphism h from Theorem 3.30 is in the case of an unique level build as

$$\begin{tikzcd}
\prod_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha^k(A^0) \arrow[r, "i^k"] \arrow[d, equal] & \Gamma(\dot{\mathcal{U}}^k;\Lambda^k(A^0)) \arrow[r, "s^k"] & H^1(S^1;\Lambda^k(A^0)) \arrow[d, equal] \\
\prod_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha(A^0) \arrow[rr, "h"] & & H^1(S^1;\Lambda(A^0))
\end{tikzcd}$$

only in the single leveled case

from

- the canonical injective map

  $$i^k : \prod_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha^k(A^0) \longrightarrow \Gamma(\dot{\mathcal{U}}^k;\Lambda^k(A^0))\,,$$

  i.e. the map which is the canonical extension of germs to their natural arc of definition, and
- the quotient map

  $$s^k : \Gamma(\dot{\mathcal{U}}^k;\Lambda^k(A^0)) \longrightarrow H^1(S^1;\Lambda^k(A^0))$$

which are both isomorphisms.

Proof. 1. The map

$$i^k : \prod_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha^k(A^0) \longrightarrow \underbrace{\prod_{\alpha\in\mathbb{A}^k} \Gamma(\dot{\mathcal{U}}_\alpha^k;\Lambda^k(A^0))}_{\substack{\| \\ \Gamma(\dot{\mathcal{U}}^k;\Lambda^k(A^0))}}$$

is welldefined, since the sections of $\Lambda^k(A^0)$ are solution of the system $[A^0, A^0]$ and it is very well known from the theory of differential equations that an element $f_\alpha \in \Gamma(\dot{\mathcal{U}}^k_\alpha; \Lambda^k(A^0))$ is uniquely determined as the extension of its germ at some point α.

It is also a isomorphism groups (cf. [Lod94]).

2. The second map

$$s^k : \overbrace{\prod_{\alpha \in \mathbb{A}^k} \Gamma(\dot{\mathcal{U}}^k_\alpha; \Lambda^k(A^0))}^{\substack{\Gamma(\dot{\mathcal{U}}^k;\Lambda^k(A^0)) \\ \|}} \longrightarrow H^1(S^1; \Lambda^k(A^0))$$

is a bijection, since from Proposition 3.38 we know that it is

- **surjective**, since $\mathcal{U}^k$ is adequate to $\Lambda^k(A^0)$ and
- **injective**, since on $\mathcal{U}^k$ there is no 0-cochain in $\Lambda^k(A^0)$.

□

The case of several levels

In the proof of the case of several levels, we will still use Loday-Richaud's paper [Lod94] as reference.

Definition 3.40

Here we want to define a *product map of cocycles* $\mathfrak{S}^{\leq k}$. This map will be composed from the following injective maps:

1. The first map is defined as

$$\begin{aligned} \sigma^k : \Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A^0)) &\longrightarrow \Gamma(\dot{\mathcal{U}}^{\leq k}; \Lambda^{\leq k}(A^0)) \\ \dot{f} = (\dot{f}_\alpha)_{\alpha \in \mathbb{A}^k} &\longmapsto (\dot{G}_\alpha)_{\alpha \in \mathbb{A}^{\leq k}} \end{aligned}$$

where

$$\dot{G}_\alpha = \begin{cases} \dot{f}_\alpha \text{ restricted to } \dot{U}^{\leq k}_\alpha \text{ and seen as being in } \Lambda^{\leq k}(A^0) & \text{when } \alpha \in \mathbb{A}^k \\ \mathrm{id} \ \text{ (the identity)} & \text{when } \alpha \notin \mathbb{A}^k \end{cases}$$

2. and the second map

$$\begin{aligned} \sigma^{<k} : \Gamma(\dot{\mathcal{U}}^{<k}; \Lambda^{<k}(A^0)) &\longrightarrow \Gamma(\dot{\mathcal{U}}^{\leq k}; \Lambda^{\leq k}(A^0)) \\ \dot{f} = (\dot{f}_\alpha)_{\alpha \in \mathbb{A}^{<k}} &\longmapsto (\dot{F}_\alpha)_{\alpha \in \mathbb{A}^{\leq k}} \end{aligned}$$

is defined, in a similar way, as

$$\dot{F}_\alpha = \begin{cases} \dot{f}_\alpha \text{ restricted to } \dot{U}_\alpha^{\le k} \text{ and seen as being in } \Lambda^{\le k}(A^0) & \text{when } \alpha \in \mathbb{A}^{<k} \\ \mathrm{id} \ \text{(the identity)} & \text{when } \alpha \notin \mathbb{A}^{<k} \end{cases}$$

Thus we can define

$$\begin{aligned} \mathfrak{S}^{\le k} : \Gamma(\dot{\mathcal{U}}^{<k}; \Lambda^{<k}(A^0)) \times \Gamma(\dot{\mathcal{U}}^{k}; \Lambda^{k}(A^0)) &\longrightarrow \Gamma(\dot{\mathcal{U}}^{\le k}; \Lambda^{\le k}(A^0)) \\ (\dot{f}, \dot{g}) &\longmapsto (\dot{F}_\alpha \dot{G}_\alpha)_{\alpha \in \mathbb{A}^{\le k}} \end{aligned}$$

where $(\dot{F}_\alpha)_{\alpha\in\mathbb{A}^{\le k}} = \sigma^{<k}(\dot{f})$ and $(\dot{G}_\alpha)_{\alpha\in\mathbb{A}^{\le k}} = \sigma^{k}(\dot{g})$ are defined as above.

Remark 3.40.1

This map $\mathfrak{S}^{\le k}$ is injective, since injectivity for germs implies injectivity for sections.

Lemma 3.41

Let $k \in \mathcal{K}$.

1. If the cocycles $\mathfrak{S}^{\le k}(\dot{f}, \dot{g})$ and $\mathfrak{S}^{\le k}(\dot{f}', \dot{g}')$ are cohomologous in $\Gamma(\dot{\mathcal{U}}^{\le k}; \Lambda^{\le k}(A^0))$ then $\dot{f}$ and $\dot{f}'$ are cohomologous in $\Gamma(\dot{\mathcal{U}}^{<k}; \Lambda^{<k}(A^0))$.
2. Any cocycle in $\Gamma(\dot{\mathcal{U}}^{\le k}, \Lambda^{\le k}(A^0))$ is cohomologous to a cocycle in the range of $\mathfrak{S}^{\le k}$.

Proof. 1. Denote by α^+ the nearest anti-Stokes direction in $\mathbb{A}^{\le k}$ on the right[a] of α. The cocycles $\mathfrak{S}^{\le k}(\dot{f}, \dot{g})$ and $\mathfrak{S}^{\le k}(\dot{f}', \dot{g}')$ are cohomologous if and only if there is a 0-cochain $c = (c_\alpha)_{\alpha\in\mathbb{A}^{\le k}} \in \Gamma(\mathcal{U}^{\le k}, \Lambda^{\le k}(A^0)$ such that

$$\dot{F}_\alpha \dot{G}_\alpha = c_\alpha^{-1} \dot{F}'_\alpha \dot{G}'_\alpha c_{\alpha^+} \tag{3.5}$$

for every $\alpha \in \mathbb{A}$. From Proposition 3.38 follows, that c is with values in $\Lambda^{<k}(A^0)$. The fact that $\Lambda^{k}(A^0)$ is normal in $\Lambda^{\le k}(A^0)$ in Proposition 3.22, can be used to see that $c_{\alpha^+}^{-1} G'_\alpha c_{\alpha^+} \in \Gamma(\mathcal{U}^{\le k}; \Lambda^{k}(A^0))$. Thus, we rewrite the relation (3.5) to

$$\dot{F}_\alpha \dot{G}_\alpha = (c_\alpha^{-1} \dot{F}'_\alpha c_{\alpha^+})(c_{\alpha^+}^{-1} G'_\alpha c_{\alpha^+}) , \qquad \text{for } \alpha \in \mathbb{A}^{\le k} .$$

Since Corollary 3.23 tells us, that the factorization into the factors of the semidirect product are unique, we get for all $\alpha \in \mathbb{A}^{k}$

$$\dot{F}_\alpha = c_\alpha^{-1} \dot{F}'_\alpha c_{\alpha^+} \qquad \text{and} \qquad \dot{G}_\alpha = c_{\alpha^+}^{-1} \dot{G}'_\alpha c_{\alpha^+} .$$

The former relation implies that $(\dot{F}_\alpha)$ and $(\dot{F}'_\alpha)$ are cohomologous with values in $\Lambda^{<k}(A^0)$ on $\mathcal{U}^{\le k}$. Since $\mathcal{U}^{<k}$ is already adequate to $\Lambda^{<k}(A^0)$, are $(\dot{F}_\alpha)$ and $(\dot{F}'_\alpha)$ already on $\mathcal{U}^{<k}$, i.e. in $\Gamma(\dot{\mathcal{U}}^{<k}; \Lambda^{<k}(A^0))$, cohomologous.

2. The proof of part 2. (together with a proof of part 1.) can be found in Loday-Richaud's paper [Lod94, Proof of Lem.II.3.3].

□

[a]In clockwise direction.

Let $k \in \mathcal{K}$ and $k' = \max\{k' \in \mathcal{K} \mid k' < k\}$. We then know by definition that $\mathcal{U}^{<k} = \mathcal{U}^{\leq k'}$ as well as $\Lambda^{<k}(A^0) = \Lambda^{\leq k'}(A^0)$ and thus $\Gamma(\dot{\mathcal{U}}^{<k}; \Lambda^{<k}(A^0)) = \Gamma(\dot{\mathcal{U}}^{\leq k'}; \Lambda^{\leq k'}(A^0))$ and obtain the following proposition.

Proposition 3.42

By applying $\mathfrak{S}^{\leq k}$ successively for different k's in decending order, one obtains the *product map of single leveled cocycles* τ in the following way

$$\begin{tikzcd}
\underbrace{\Gamma(\dot{\mathcal{U}}^{<k_r}; \Lambda^{<k_r}(A^0))} \times \Gamma(\dot{\mathcal{U}}^{k_r}; \Lambda^{k_r}(A^0)) \arrow[r, "\mathfrak{S}^{\leq k_r}"] \arrow[dr, bend right, "="] & \overset{\Gamma(\dot{\mathcal{U}};\Lambda(A^0))}{\overset{\shortparallel}{\Gamma(\dot{\mathcal{U}}^{\leq k_r}; \Lambda^{\leq k_r}(A^0))}} \\
\underbrace{\Gamma(\dot{\mathcal{U}}^{<k_{r-1}}; \Lambda^{<k_{r-1}}(A^0))} \times \Gamma(\dot{\mathcal{U}}^{k_{r-1}}; \Lambda^{k_{r-1}}(A^0)) \arrow[r, "\mathfrak{S}^{\leq k_{r-1}}"] \arrow[dr, bend right, "="] & \Gamma(\dot{\mathcal{U}}^{\leq k_{r-1}}; \Lambda^{\leq k_{r-1}}(A^0)) \\
\cdots \arrow[r, "\mathfrak{S}^{\leq k_{r-2}}"] & \Gamma(\dot{\mathcal{U}}^{\leq k_{r-2}}; \Lambda^{\leq k_{r-2}}(A^0)) \\
\underbrace{\Gamma(\dot{\mathcal{U}}^{<k_3}; \Lambda^{<k_3}(A^0))} \times \Gamma(\dot{\mathcal{U}}^{k_3}; \Lambda^{k_3}(A^0)) \arrow[r, "\mathfrak{S}^{\leq k_3}"] \arrow[dr, bend right, "="] & \cdots \\
\Gamma(\dot{\mathcal{U}}^{k_1}; \Lambda^{k_1}(A^0)) \times \Gamma(\dot{\mathcal{U}}^{k_2}; \Lambda^{k_2}(A^0)) \arrow[r, "\mathfrak{S}^{\leq k_2}"] & \Gamma(\dot{\mathcal{U}}^{\leq k_2}; \Lambda^{\leq k_2}(A^0))
\end{tikzcd}$$

which can be written in the following compact form

$$\begin{aligned}
\tau : \prod_{k \in \mathcal{K}} \Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A^0)) &\longrightarrow \Gamma(\dot{\mathcal{U}}; \Lambda(A^0)) \\
(\dot{f}^k)_{k \in \mathcal{K}} &\longmapsto \prod_{k \in \mathcal{K}} \tau^k(\dot{f}^k)
\end{aligned}$$

where the product is following an ascending order of levels and the maps τ_k are defined as

$$\begin{aligned}\tau^k : \Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A^0)) &\xrightarrow{\sigma^k} \Gamma(\dot{\mathcal{U}}^{\leq k}; \Lambda^{\leq k}(A^0)) \longrightarrow \Gamma(\dot{\mathcal{U}}; \Lambda(A^0)) \\ (\dot{f}_\alpha)_{\alpha\in\mathbb{A}^k} &\longmapsto (\dot{G}_\alpha)_{\alpha\in\mathbb{A}}\end{aligned}$$

with

$$\dot{G}_\alpha = \begin{cases} \dot{f}_\alpha \text{ restricted to } \dot{U}_\alpha \text{ and seen as being in } \Lambda(A^0) & \text{when } \alpha \in \mathbb{A}^k \\ \mathrm{id} \ \text{ (the identity on } \dot{U}_\alpha) & \text{when } \alpha \notin \mathbb{A}^k \end{cases}$$

The defined map τ is clearly injective and it can be extended to an arbitrary order of levels (cf. Remark [Lod94, Rem.II.3.5]).

Corollary 3.43

The product map of single-leveled cocycles τ induces on the cohomology a bijective and natural map

$$\begin{array}{ccc} \mathcal{T} : \prod_{k\in\mathcal{K}} \Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A^0)) & \longrightarrow & H^1(\mathcal{U}; \Lambda(A^0))\,. \\ \wr\| & & \wr\| \\ \prod_{k\in\mathcal{K}} H^1(S^1; \Lambda^k(A^0)) & & H^1(S^1; \Lambda(A^0)) \end{array}$$

Composing functions to obtain h We have the ingredients to define the function h from Theorem 3.30 by composition of already bijective maps.

Proof of Theorem 3.30. Let $i_\alpha : \mathrm{Sto}_\alpha(A^0) \to \prod_{k\in\mathcal{K}} \mathrm{Sto}_\alpha^k(A^0)$ be the map which corresponds to the filtration from Proposition 3.25 and denote the composition

$$\begin{tikzcd} \prod\limits_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha(A^0) \arrow[r, "\prod_{\alpha\in\mathbb{A}} i_\alpha"] \arrow[ddrr, bend right, "\mathfrak{T}"'] & \prod\limits_{\alpha\in\mathbb{A}} \prod\limits_{k\in\mathcal{K}} \mathrm{Sto}_\alpha^k(A^0) \arrow[d, phantom, "\cong"] & \\ & \prod\limits_{k\in\mathcal{K}} \prod\limits_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha^k(A^0) \arrow[r, "\prod_{k\in\mathcal{K}} i^k"] & \prod\limits_{k\in\mathcal{K}} \Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A^0)) \\ & & \end{tikzcd}$$

by $\mathfrak{T}$. The bijection h is then obtained as

$$\mathcal{T} \circ \mathfrak{T} : \prod_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha(A^0) \longrightarrow H^1(\mathcal{U}; \Lambda(A^0))\,.$$

$\square$

3.3.3 Some exemplary calculations

Here we want to discuss, which information is required to describe the Stokes cocycle corresponding to a multileveled system in more depth. We will look at a sigle-leveled system corresponding to a normal form $A^0 \in \mathrm{GL}_3(\mathbb{C}(\{t\}))$ with exactly 2 levels and will apply the techniques developed in the previous sections in an rather explicit way.

Let A^0 be a normal form with dimension $n = 3$ and two levels $\mathcal{K} = \{k_1 < k_2\}$, which satisfies that there is at least one anti-Stokes direction θ which is beared by both levels. Let $q_j(t^{-1})$ be the determining polynomials and let k_{jl} be the degrees of $(q_j - q_l)(t^{-1})$. Up to permutation, we know that in our case are the leading terms of $(q_1 - q_2)(t^{-1})$ and $(q_1 - q_3)(t^{-1})$ equal and thus

- up to permutation is $k_2 = k_{1,2} = k_{1,3}$ and $k_1 = k_{2,3}$, i.e. the larger degree appears twice, and
- $q_1 \underset{\alpha}{\prec} q_2$ (resp. $q_2 \underset{\alpha}{\prec} q_1$) if and only if $q_1 \underset{\alpha}{\prec} q_3$ (resp. $q_3 \underset{\alpha}{\prec} q_1$) and thus do they determine the same anti-Stokes directions.

The set of all anti-Stokes directions is then given as

$$\mathbb{A} = \left\{\theta + \frac{\pi}{k} \cdot j \mid k \in \mathcal{K},\, j \in \mathbb{N}\right\}$$

Denote by $\mathcal{Y}_0(t)$ a normal solution of $[A^0]$.

Let us start by looking at a single germ in depth. The Proposition 3.16 states that every Stokes germ φ_α can be written as its matrix representation conjugated by the normal solution, i.e. as $\varphi_\alpha = \mathcal{Y}_0 C_{\varphi_\alpha} \mathcal{Y}_0^{-1} = \rho_\alpha^{-1}(C_{\varphi_\alpha})$.

Look at an example in which we will demonstrate, from which relations on the determining polynomials which restriction on the form of the Stokes matrices arise.

Example 3.44

Let $\alpha \in \mathbb{A}$ be an anti-Stokes direction. From the definition of $\mathbb{S}\mathrm{to}_\alpha(A^0)$ (cf. Definition 3.15) we know that, if one has $q_1 \underset{\alpha}{\prec} q_2$, the Stokes matrix has the form

$$\begin{pmatrix} 1 & \boldsymbol{c_1} & \star \\ \boldsymbol{0} & 1 & \star \\ \star & \star & 1 \end{pmatrix}$$

where $c_j \in \mathbb{C}$ and $\star \in \mathbb{C}$.

We have seen that $q_1 \underset{\alpha}{\preccurlyeq} q_2 \Rightarrow q_1 \underset{\alpha}{\preccurlyeq} q_3$ thus the representation has the form

$$\begin{pmatrix} 1 & c_1 & \mathbf{c_2} \\ 0 & 1 & \star \\ \mathbf{0} & \star & 1 \end{pmatrix}$$

and if we also know that neither $q_2 \underset{\alpha}{\preccurlyeq} q_3$ nor $q_3 \underset{\alpha}{\preccurlyeq} q_2$ it has the form

$$\begin{pmatrix} 1 & c_1 & c_2 \\ 0 & 1 & \mathbf{0} \\ 0 & \mathbf{0} & 1 \end{pmatrix}.$$

In fact, the following 9 cases of Stokes matrices can arise:

	$q_2 \underset{\alpha}{\preccurlyeq} q_3$	$q_3 \underset{\alpha}{\preccurlyeq} q_2$	else
$q_1 \underset{\alpha}{\preccurlyeq} q_2$ and $q_1 \underset{\alpha}{\preccurlyeq} q_3$	$\begin{pmatrix} 1 & c_2 & c_3 \\ 0 & 1 & c_1 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & c_2 & c_3 \\ 0 & 1 & 0 \\ 0 & c_1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & c_2 & c_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
$q_2 \underset{\alpha}{\preccurlyeq} q_1$ and $q_3 \underset{\alpha}{\preccurlyeq} q_1$	$\begin{pmatrix} 1 & 0 & 0 \\ c_2' & 1 & c_1 \\ c_3 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ c_2 & 1 & 0 \\ c_3' & c_1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ c_2 & 1 & 0 \\ c_3 & 0 & 1 \end{pmatrix}$
else	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & c_1 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c_1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

In the blue cases we have $\mathcal{K}_\alpha = \mathcal{K}$ and $\mathbb{C}^3 \xrightarrow[\cong]{\vartheta_\alpha} \mathbb{S}\mathrm{to}_\alpha(A^0)$. In the green cases $\mathcal{K}_\alpha = \{k_2\}$ and $\mathbb{C}^2 \xrightarrow[\cong]{\vartheta_\alpha} \mathbb{S}\mathrm{to}_\alpha(A^0)$ as well as in the purple cases $\mathcal{K}_\alpha = \{k_1\}$ and $\mathbb{C}^1 \xrightarrow[\cong]{\vartheta_\alpha} \mathbb{S}\mathrm{to}_\alpha(A^0)$. We will replace c_2' by $c_2 + c_1 c_3$ and c_3' by $c_1 c_2 + c_3$ to be consistent with the decomposition in the next part (cf. Example 3.46).

Corollary 3.45

The morphism $\prod_{\alpha \in \mathbb{A}} \vartheta_\alpha$ is an isomorphism of pointed sets, which maps the element only containing zeros to

$$(\mathrm{id}, \mathrm{id}, \ldots, \mathrm{id}) \in \prod_{\alpha \in \mathbb{A}} \mathbb{S}\mathrm{to}_\alpha(A^0),$$

which gets by $\left(\prod_{\alpha \in \mathbb{A}}\right)^{-1} \circ h$ mapped to the trivial cohomology class in $\mathcal{S}t(A^0)$.

In proposition 3.25 and especially Remark 3.25.1 we have defined a decomposition of the Stokes group $\mathrm{Sto}_\alpha(A^0)$ in subgroups generated by k-germs for $k \in \mathcal{K}$. In our case, we have at most two nontrivial factors. Especially is this decomposition given by

$$\varphi_\alpha = \varphi_\alpha^{k_1}\varphi_\alpha^{k_2} \overset{i_\alpha}{\longmapsto} \left(\varphi_\alpha^{k_1}, \varphi_\alpha^{k_2}\right) \in \mathrm{Sto}_\alpha^{k_1}(A^0) \times \mathrm{Sto}_\alpha^{k_2}(A^0)\,,$$

and i_α is the map, wich gives the factors of this factorization in ascending order. This decomposition, of a germ φ_α, is trivial if $\#\mathcal{K}(\varphi_\alpha) \leq 1$, thus the interesting cases are the blue cases.

Example 3.46

Look at the example

$$\vartheta_\alpha(c_1, c_2, c_3) = \mathcal{Y}_0 \begin{pmatrix} 1 & 0 & 0 \\ c_2 & 1 & 0 \\ c_1c_2 + c_3 & c_1 & 1 \end{pmatrix} \mathcal{Y}_0^{-1} = \varphi_\alpha\,.$$

According to Remark 3.23.1 the factor $\varphi_\alpha^{k_1} \in \mathrm{Sto}_\alpha^{k_1}(A^0)$, is given by

$$\varphi_\alpha^{k_1} = \mathcal{Y}_0 \begin{pmatrix} 1 & 0 & 0 \\ \mathbf{0} & 1 & 0 \\ \mathbf{0} & c_1 & 1 \end{pmatrix} \mathcal{Y}_0^{-1}\,.$$

The other factor $\varphi_\alpha^{k_2}$ is then obtained as

$$\begin{aligned} \varphi_\alpha^{k_2} &= \left(\varphi_\alpha^{k_1}\right)^{-1} \varphi_\alpha^{k_2} \\ &= \mathcal{Y}_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -c_1 & 1 \end{pmatrix} \underbrace{\mathcal{Y}_0^{-1}\mathcal{Y}_0}_{=\mathrm{id}} \begin{pmatrix} 1 & 0 & 0 \\ c_2 & 1 & 0 \\ c_1c_2 + c_3 & c_1 & 1 \end{pmatrix} \mathcal{Y}_0^{-1} \\ &= \mathcal{Y}_0 \begin{pmatrix} 1 & 0 & 0 \\ c_2 & 1 & 0 \\ c_3 & 0 & 1 \end{pmatrix} \mathcal{Y}_0^{-1}\,. \end{aligned}$$

The four nontrivial decomposition in our situation, are given by:

1. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & c_1 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & c_2 & c_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & c_2 & c_3 \\ 0 & 1 & c_1 \\ 0 & 0 & 1 \end{pmatrix}$

2. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c_1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & c_2 & c_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & c_2 & c_3 \\ 0 & 1 & 0 \\ 0 & c_1 & 1 \end{pmatrix}$

3. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & c_1 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ c_2 & 1 & 0 \\ c_3 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ c_2 + c_1c_3 & 1 & c_1 \\ c_3 & 0 & 1 \end{pmatrix}$

4. $\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & c_1 & 1\end{pmatrix} \cdot \begin{pmatrix}1 & 0 & 0\\ c_2 & 1 & 0\\ c_3 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ c_2 & 1 & 0\\ c_1c_2 + c_3 & c_1 & 1\end{pmatrix}$

Explicit example

Even more explicit, we can fix the levels $k_1 = 1$ and $k_2 = 3$ together with $\theta = 0$. Assume without any restriction that $q_1 \underset{\theta}{\prec} q_2$ and $q_1 \underset{\theta}{\prec} q_3$ as well as $q_2 \underset{\theta}{\prec} q_3$. Other choices would result in reordering of the tuples below.

The classification space is in this case isomorphic to $\mathbb{C}^{2\cdot(1+2\cdot 3)} = \mathbb{C}^{14}$. The element

$$({}^1c_1, {}^2c_1, {}^1c_2, {}^1c_3, {}^2c_2, {}^2c_3, \dots, {}^6c_2, {}^6c_3) \in \mathbb{C}^{14}$$

gets, via the isomorphism $\prod_{\alpha\in\mathbb{A}} j_\alpha$, mapped to

$$\left(\left(\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & {}^1c_1\\ 0 & 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & {}^2c_1 & 1\end{pmatrix}\right),\right.$$
$$\left.\left(\begin{pmatrix}1 & {}^1c_2 & {}^1c_3\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0 & 0\\ {}^2c_2 & 1 & 0\\ {}^2c_3 & 0 & 1\end{pmatrix}, \dots, \begin{pmatrix}1 & 0 & 0\\ {}^6c_2 & 1 & 0\\ {}^6c_3 & 0 & 1\end{pmatrix}\right)\right)$$

in $\prod_{\alpha\in\mathbb{A}^1} \mathbb{S}\mathrm{to}^1_\alpha(A^0) \times \prod_{\alpha\in\mathbb{A}^3} \mathbb{S}\mathrm{to}^3_\alpha(A^0)$ and thus the element

$$\left(\left(\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & {}^1c_1\\ 0 & 0 & 1\end{pmatrix}, \mathrm{id}, \mathrm{id}, \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & {}^2c_1 & 1\end{pmatrix}, \mathrm{id}, \mathrm{id}\right),\right.$$
$$\left.\left(\begin{pmatrix}1 & {}^1c_2 & {}^1c_3\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0 & 0\\ {}^2c_2 & 1 & 0\\ {}^2c_3 & 0 & 1\end{pmatrix}, \dots, \begin{pmatrix}1 & 0 & 0\\ {}^6c_2 & 1 & 0\\ {}^6c_3 & 0 & 1\end{pmatrix}\right)\right)$$

in $\prod_{\alpha\in\mathbb{A}} \mathbb{S}\mathrm{to}^1_\alpha(A^0) \times \prod_{\alpha\in\mathbb{A}} \mathbb{S}\mathrm{to}^3_\alpha(A^0)$. Using the morphism $\prod_{\alpha\in\mathbb{A}} i_\alpha^{-1}$ we get a complete set of Stokes matrices as

$$\left(\begin{pmatrix}1 & {}^1c_2 & {}^1c_3\\ 0 & 1 & {}^1c_1\\ 0 & 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0 & 0\\ {}^2c_2 & 1 & 0\\ {}^2c_3 & 0 & 1\end{pmatrix}, \begin{pmatrix}1 & {}^3c_2 & {}^3c_3\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},\right.$$
$$\left.\begin{pmatrix}1 & 0 & 0\\ {}^4c_2 & 1 & 0\\ {}^2c_1{}^4c_2 + {}^4c_3 & {}^2c_1 & 1\end{pmatrix}, \begin{pmatrix}1 & {}^5c_2 & {}^5c_3\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0 & 0\\ {}^6c_2 & 1 & 0\\ {}^6c_3 & 0 & 1\end{pmatrix}\right) \in \prod_{\alpha\in\mathbb{A}} \mathbb{S}\mathrm{to}_\alpha(A^0)\,.$$

Applying the isomorphism $\prod_{\alpha\in\mathbb{A}} \rho_\alpha^{-1}$, i.e. conjugation by the fundamental solution $\mathcal{Y}_0(t) = t^L e^{Q(t^{-1})}$ (cf. Proposition 3.16), yields then the corresponding Stokes cocycle in $\prod_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha(A^0)$ and thus an element in $\mathcal{S}t(A^0)$.

3.4 Further improvements

The set $\prod_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha(A^0)$ and thus $\prod_{\alpha\in\mathbb{A}} \mathbb{S}\mathrm{to}_\alpha(A^0)$ has some bad properties, when small deformations are applied to $[A^0]$, since under arbitrary small changes, one Stokes ray can split into two.

In the prove of Theorem 3.30 we have seen, that not only $\mathcal{H}(A^0) \cong \prod_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha(A^0)$ but also that

$$\begin{aligned}\mathcal{H}(A^0) &\cong \prod_{k\in\mathcal{K}} \prod_{\alpha\in\mathbb{A}} \mathrm{Sto}^k_\alpha(A^0)\\ &\cong \prod_{k\in\mathcal{K}} \prod_{\alpha\in\mathbb{A}^k} \mathrm{Sto}^k_\alpha(A^0) \qquad\qquad (\text{since } \mathbb{S}\mathrm{to}^k_\alpha(A^0) = \{\mathrm{id}\} \text{ when } k \notin \mathcal{K}_\alpha.)\\ &\cong \prod_{k\in\mathcal{K}} \prod_{\alpha\in\mathbb{A}^k} \mathbb{S}\mathrm{to}^k_\alpha(A^0)\,.\end{aligned}$$

This representation can be used to achieve further improvements, since one is able to multiply some succeeding Stokes matrices (resp. Stokes germs) of the same level without loss of information. This is based on the Corollary 3.53 stated bellow.

Let us fix a level $k \in \mathcal{K}$. We want to rewrite the product $\prod_{\alpha\in\mathbb{A}^k} \mathbb{S}\mathrm{to}^k_\alpha(A^0)$ by collecting the information of multiple Stokes matrices into one product of these matrices.

Boalch, which looks in his publications [Boa01; Boa99] only at the single-leveled case, uses this extensively to obtain better Stokes matrices, which are stable under small deformations. Our Stokes matrices are in his publications called *Stokes factors*.

Definition 3.47

A subset of $\mathbb{A}^k$ consisting of $\frac{\#\mathbb{A}^k}{2k}$ consecutive anti-Stokes directions of level k will be called a *half-period (of level k)*.

Remark 3.47.1

The set $\mathbb{A}^k$ can clearly be split into $2k$ distinct half-periods.

From the definition of anti-Stokes directions of level k it is clear that every arc of width $\frac{\pi}{k}$, which has no anti-Stokes direction of level k on its border, contains $\frac{\#\mathbb{A}^k}{2k}$ anti-Stokes directions since

> for every sector $I = U(\theta, \frac{\pi}{k})$, of width $\frac{\pi}{k}$ and center θ, which satisfies $\theta \pm \frac{\pi}{2k} \notin \mathbb{A}^k$, there is to any pair (q_j, q_l) in $\mathcal{Q}(A^0)$, which has level $k_{jl} = k$, exactly one direction $\alpha \in \mathbb{A}^k \cap I$ which satisfies, that $\Im(a_{jl}e^{-ik_{jl}\alpha}) = 0$. At such a direction α, corresponding to the pair (q_j, q_l), is then either $q_j \underset{\alpha}{\prec} q_l$ or $q_l \underset{\alpha}{\prec} q_j$ satisfied.

This implies that for every θ, which satisfies $\theta \pm \frac{\pi}{2k} \notin \mathbb{A}^k$, is $\mathbb{A}^k \cap U(\theta, \frac{\pi}{k})$ a half-period and by $\dot{\bigcup}_{1 \leq m \leq 2k} \mathbb{A}^k \cap U(\theta, \frac{m\pi}{k}) = \mathbb{A}^k$ is the corresponding decomposition in half-periods of $\mathbb{A}^k$ given.

Let $I = U(\theta, \frac{\pi}{k})$ be an arc width $\frac{\pi}{k}$ such that $\theta \pm \frac{\pi}{2k} \notin \mathbb{A}^k$.

Remark 3.48

For every $\alpha \in \mathbb{A}^k \cap I$ we can write the Stokes matrix as

$$K^\alpha = (K^\alpha_{jl})_{j,l \in \{1,\dots,s\}} \in \mathbb{S}\mathrm{to}^k_\alpha(A^0)$$

where the $K^\alpha_{jl} \in \mathbb{C}^{n_j \times n_l}$ are blocks corresponding to the structure of Q (cf. Definition 3.15). We then know for $j \neq l$ that, if $K^\alpha_{jl} \neq 0$ then

- is $K^\alpha_{lj} = 0$ and
- for every $\alpha' \in (\mathbb{A}^k \cap I) \backslash \{\alpha\}$ is $K^{\alpha'}_{jl} = 0$ as well as $K^{\alpha'}_{lj} = 0$.

Remark 3.49

In the situation of Remark 3.48 there exists a common (block) permutation matrix $P \in \mathrm{GL}_n(\mathbb{C})$ given by $(P)_{jl} = \boldsymbol{\delta}_{\pi(j)l}$ where

- $\boldsymbol{\delta}_{jl}$ is the block version of Kronecker's delta which was introduced in Definition 3.15 and
- π is the permutation of $\{1, \dots, s\}$ corresponding to $q_j \underset{\theta}{\prec} q_l \Leftrightarrow \pi(j) < \pi(l)$

such that every matrix $P^{-1}K^\alpha P$ is upper triangular and the observation from Remark 3.48 is still satisfied.

Remark 3.49.1

After moving the sector $U(\theta, \frac{\pi}{k})$ to $U(\theta, \frac{\pi}{k}) + \frac{\pi}{k} = U(\theta + \frac{\pi}{k}, \frac{\pi}{k})$ we obtain also a corresponding permutation π' which is inverse to π. The corresponding permutation matrix P' is then given by $P' = P^{-1}$.

Such that the permutation P transforms Stokes matrices on the sector $I + \frac{\pi}{k}$ to lower triangular matrices

In the single leveled case of this phenomenon is discussed on page 18 of Boalch's paper [Boa01].

In the paper [BJL79] from Balser, Jurkat and Lutz is the following Lemma stated as Lemma 2 on page 75.

Lemma 3.50

Let $T \subset \{1,\dots,n\} \times \{1,\dots,n\}$ be a position set, which satisfies the *completeness property*:

if (j,k) and $(k,l) \in T$ then is also $(j,l) \in T$.

Choose a indexing $i : \{1,\dots,\underset{\#T}{\underset{\parallel}{\mu}}\} \xrightarrow{\cong} T$ of the position set and denote by $\delta_{jl} \in \mathbb{C}$ the ordinary Kronecker's delta, then there exists for every

$$K \in \left\{K = (K_{jl})_{j,l\in\{1,\dots,n\}} \in \mathrm{GL}_n(\mathbb{C}) \mid K_{jl} = \delta_{jl} \text{ unless } (j,l) \in T\right\}$$

unique scalars $t_j \in \mathbb{C}$ such that

$$K = (\mathrm{id} + t_1 E_{i(1)}) \cdots (\mathrm{id} + t_\mu E_{i(\mu)})\,.$$

Remark 3.50.1

The completeness property is reasonable, since for example $S_{\{(2,3),(3,2)\}}$, corresponding to the not complete set $\{(2,3),(3,2)\}$, is not stable under the product:

$$\begin{pmatrix}1 & a & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & b\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & a & ab\\ 0 & 1 & b\\ 0 & 0 & 1\end{pmatrix} \notin S_{\{(2,3),(3,2)\}}\,.$$

Lemma 3.51

Every position set, corresponding to some block, is complete.

Proof. Such a position set, corresponding to some block, is given by

$$T = \{(j,l) \mid j_1 \le j \le j_2, l_1 \le l \le l_2\}$$

for some j_1, j_2, l_2 and l_2 in $\{1,\dots,n\}$.

Let (j,k), $(k,l) \in T$ be two positions. It is obvious that $(j,l) \in T$, since $j_1 \le j \le j_2$ and $l_1 \le l \le l_2$ are satisfied. $\square$

Corollary 3.52

We can write the Lemma 3.50 in block form, corresponding to the structure of Q. Let $T \subset \{1,\dots,s\} \times \{1,\dots,s\}$ be a position set and choose a indexing

$$i : \{1,\dots,\underset{\#T}{\underset{\parallel}{\mu}}\} \xrightarrow{\cong} T$$

of T.

Definition 3.52.1

Define the group of matrices, corresponding by a complete position set, by

$$S_T := \left\{ K = (K_{jl})_{j,l \in \{1,\dots,n\}} \in \mathrm{GL}_n(\mathbb{C}) \mid K_{jl} = \boldsymbol{\delta}_{jl} \text{ unless } (j,l) \in T \right\},$$

where $\boldsymbol{\delta}_{jl}$ is the block version of Kronecker's delta, corresponding to the structure of Q.

From the Lemma 3.50 then follows that for every $K \in S_T$ is the decomposition

$$K = K_1 \cdot K_2 \cdots K_\mu ,$$

where $K_j \in S_{\{i(j)\}}$, is unique.

From the previous corollary we deduce the following corollary.

Corollary 3.53

Let $T_1, \dots, T_r \subset \{1, \dots, s\} \times \{1, \dots, s\}$ be a distinct position sets, such that

$T := T_1 \dot\cup \dots \dot\cup T_r$ as well as every T_m satisfy the completeness property.

Then is by

$$\begin{aligned} S_{T_1} \times \cdots \times S_{T_m} &\longrightarrow S_T \\ (K_1, \dots, K_m) &\longmapsto K_1 \cdots K_m \end{aligned}$$

an isomorphism defined.

We can apply the previous corollary in our situation. This yields the following theorem.

Theorem 3.54

Let $\theta \in S^1$ be a fixed direction which satisfies $\theta \pm \frac{\pi}{2k} \notin \mathbb{A}^k$, then

$$\mathcal{H}(A^0) \cong \prod_{k \in \mathcal{K}} \prod_{j \in \{1,\dots,2k\}} \widehat{\mathrm{Sto}}^k_{\theta + j\frac{\pi}{k}}(A^0) ,$$

where

$$\begin{aligned} \widehat{\mathbb{S}\mathrm{to}}^k_\theta(A^0) &:= \left\{ K = (K_{jl})_{j,l \in \{1,\dots,s\}} \in \mathrm{GL}_n(\mathbb{C}) \mid K_{jl} = \delta_{jl} \text{ unless } q_j \underset{\theta}{\prec} q_l,\, k_{jl} = k \right\} \\ &= S_{\{(j,l) \mid q_j \underset{\theta}{\prec} q_l\}} \end{aligned}$$

and

$$\widehat{\mathrm{Sto}}^k_\theta(A^0) := \{ \rho_\theta^{-1}(K) \mid K \in \widehat{\mathbb{S}\mathrm{to}}^k_\theta(A^0) \} .$$

Remark 3.54.1

1. This means that the information of all Stokes-matrices on every level $k \in \mathcal{K}$ can be grouped into $2k$ matrices, which are products of the corresponding Stokes matrices.
2. By the definition above, it is obvious that

$$\mathrm{Sto}_\theta^k(A^0) = \bigcap_{\theta' \in \mathbb{A}^k \cap U(\theta, \frac{\pi}{k})} \widehat{\mathrm{Sto}}_{\theta'}^k(A^0)\,.$$

Proof. We have an isomorphism

$$\eta_\theta^k : \prod_{\alpha \in \mathbb{A}^k \cap U(\theta, \frac{\pi}{k})} \underbrace{\mathbb{S}\mathrm{to}_\alpha^k(A^0)}_{\substack{\| \\ S_{\{(j,l) \mid q_j \underset{\alpha}{\prec} q_l\}}}} \xrightarrow{\cong} \underbrace{\widehat{\mathbb{S}\mathrm{to}}_\theta^k(A^0)}_{\substack{\| \\ S_{\{(j,l) \mid q_j \underset{\theta}{\prec} q_l\}}}}\,.$$

from Corollary 3.53, since

- for every $\alpha \in \mathbb{A}$ satisfies the set $\{(j,l) \mid q_j \underset{\alpha}{\preccurlyeq} q_l\}$ the completeness property, since it is defined via a transitiv relation, and
- the union of all $\{(j,l) \mid q_j \underset{\alpha}{\preccurlyeq} q_l\}$ for $\alpha \in I \cap \mathbb{A}^k$ is then

$$\begin{aligned}\dot{\bigcup}_{\alpha \in I \cap \mathbb{A}} \{(j,l) \mid q_j \underset{\alpha}{\preccurlyeq} q_l\} &= \{(j,l) \mid q_j \underset{\alpha}{\preccurlyeq} q_l \text{ for some } \alpha \in \mathbb{A}^k \cap U(\theta, \frac{\pi}{k})\} \\ &= \{(j,l) \mid q_j \underset{\theta}{\prec} q_l\} \qquad \text{(cf. Remark 3.11)}\end{aligned}$$

  and is also complete, since $\underset{\theta}{\prec}$ is also a transitive relation.

The isomorphism of the theorem is then the Stokes germ version of

$$\underbrace{\prod_{k \in \mathcal{K}} \prod_{j \in \{1,\dots,2k\}} \eta_{\theta + j\frac{\pi}{k}}^k}_{\substack{\| \\ \eta}} : \underbrace{\prod_{k \in \mathcal{K}} \prod_{\alpha \in \mathbb{A}^k} \mathbb{S}\mathrm{to}_\alpha^k(A^0)}_{\substack{\cong \\ \mathcal{H}(A^0)}} \xrightarrow{\cong} \prod_{k \in \mathcal{K}} \prod_{j \in \{1,\dots,2k\}} \widehat{\mathbb{S}\mathrm{to}}_{\theta + j\frac{\pi}{k}}^k(A^0)\,.$$

$\square$

Corollary 3.55

This does also induce an isomorphism

$$\eta : \prod_{k \in \mathcal{K}} \prod_{\alpha \in \mathbb{A}} \mathrm{Sto}_\alpha^k(A^0) \xrightarrow{\cong} \prod_{k \in \mathcal{K}} \prod_{j \in \{1,\dots,2k\}} \widehat{\mathrm{Sto}}_{\theta + j\frac{\pi}{k}}^k(A^0)$$

on the level of Stokes germs instead of Stokes matrices.

Remark 3.56

The, in the proof of Theorem 3.54, obtained isomorphism

$$\eta : \prod_{\alpha \in \mathbb{A}} \mathbb{S}\mathrm{to}_\alpha(A^0) \longrightarrow \prod_{k \in \mathcal{K}} \prod_{j \in \{1,\dots,2k\}} \widehat{\mathbb{S}\mathrm{to}}^k_{\theta + j\frac{\pi}{k}}(A^0)$$

is in the single-leveled case with $n = s$, i.e. all diagonal elements of Q are different, given in Lemma 3.2 of Boalch's paper [Boa01, Lem.3.2]. In this case is every $\widehat{\mathbb{S}\mathrm{to}}^k_{\theta + j\frac{\pi}{k}}(A^0)$ isomorphic to some PU_+P^{-1} where U_+ is the group of all upper triangular matrices, with ones on the diagonal, and P is a permutation matrix.

3.5 The complete Diagram

If we start with the diagram from Page 28, we can add all the defined isomorphisms and rewrite into the following commutative diagram of **isomorphisms of pointed sets**.

$$\begin{tikzcd}
\mathcal{H}(A^0) \arrow[rrr, "\exp_{A^0}"] & & & \mathcal{S}t(A^0) \\
& & \prod_{k \in \mathcal{K}} \Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A^0)) \arrow[ur, "\mathcal{T}"] & \\
G(\{t\})\backslash \widehat{G}(A^0) \arrow[uu, "\text{Cor. 2.41.1}"'] \arrow[ddd, "g"'] & & & \\
& \prod_{k \in \mathcal{K}} \prod_{\alpha \in \mathbb{A}} \mathrm{Sto}^k_\alpha(A^0) \arrow[uur, "\prod_{k \in \mathcal{K}} i^k"] \arrow[drr, "\eta"] \arrow[dd, "\prod_{k \in \mathcal{K}} \prod_{\alpha \in \mathbb{A}} \rho^k_\alpha"] & & \\
& & & \prod_{k \in \mathcal{K}} \prod_{j \in \{1,\dots,2k\}} \widehat{\mathbb{S}\mathrm{to}}^k_{\theta + j\frac{\pi}{k}}(A^0) \arrow[dd, "\text{(induced by } \rho)"] \\
\prod_{\alpha \in \mathbb{A}} \mathbb{S}\mathrm{to}_\alpha(A^0) \arrow[uur, "\chi \circ \prod_{\alpha \in \mathbb{A}} i_\alpha"] \arrow[dd, "\prod_{\alpha \in \mathbb{A}} \rho_\alpha"'] & \prod_{k \in \mathcal{K}} \prod_{\alpha \in \mathbb{A}} \mathrm{Sto}^k_\alpha(A^0) \arrow[drr, "\eta"] & & \\
& & & \prod_{k \in \mathcal{K}} \prod_{j \in \{1,\dots,2k\}} \widehat{\mathbb{S}\mathrm{to}}^k_{\theta + j\frac{\pi}{k}}(A^0) \\
\prod_{\alpha \in \mathbb{A}} \mathbb{S}\mathrm{to}_\alpha(A^0) \arrow[uur, "\chi \circ \prod_{\alpha \in \mathbb{A}} i_\alpha"'] & & & \\
& \prod_{\alpha \in \mathbb{A}} \mathbb{C}^{\sum_{q_j \prec_\alpha q_l} \deg(q_j - q_l) \cdot n_j \cdot n_l} \arrow[ul, "\prod_{\alpha \in \mathbb{A}} \vartheta_\alpha"] & &
\end{tikzcd}$$

Where

- the map g, which arises from the theory of summation
- the purple path is the isomorphism h from Theorem 3.30 and
- we denote
  - $\chi : \prod_{\alpha\in\mathbb{A}} \prod_{k\in\mathcal{K}} \mathrm{Sto}_{\alpha}^{k}(A^0) \equiv \prod_{k\in\mathcal{K}} \prod_{\alpha\in\mathbb{A}} \mathrm{Sto}_{\alpha}^{k}(A^0)$ the reordering and
  - by abuse of notation we also denote the Stokes matrix version in the same way $\chi : \prod_{\alpha\in\mathbb{A}} \prod_{k\in\mathcal{K}} \mathbb{S}\mathrm{to}_{\alpha}^{k}(A^0) \equiv \prod_{k\in\mathcal{K}} \prod_{\alpha\in\mathbb{A}} \mathbb{S}\mathrm{to}_{\alpha}^{k}(A^0)$.

A More basics

A.1 Semidirect products

We will refer to [Rob03, p. 75] for semidirect products. Let G be a group with a normal subgroup N ($N \lhd G$) and a subgroup H such that

$$G = NH \qquad \text{and} \qquad N \cap H = 1.$$

Then G is said to be the *(internal) semidirect product* of N and H, $G = N \rtimes H$.

Remark A.1

1. Each element $g \in G$ has a unique decomposition $g = nh$ with $n \in N$ and $h \in H$. Since

   for $g = n'h'$ another such expression, then $(n')^{-1}n = h'h^{-1} \in N \cap H = 1$ such that $n = n'$ and $h = h'$.

2. Conjugation in N by an element $h \in H$ defines an automorphism of N, say $\rho(h)$ which satisfies for $h_i \in H$

   $$\rho(h_1, h_2) = \rho(h_1)\rho(h_2)$$

   and thus $\rho : H \to \mathrm{Aut}(N)$ is a homomorphism.

In the other direction, one can get from N, H and a given homomorphism $\rho : H \to \mathrm{Aut}(N)$ the same structure back. This will be the so-called *external semidirect product*. It is defined by the underlying group

$$G := \{(n, h) \mid n \in N, h \in H\}$$

together with the multiplication

$$\begin{aligned} G \times G &\to G \\ ((n_1, h_1), (n_2, h_2)) &\mapsto (n_1\rho(h_1)(n_2), h_1h_2) \end{aligned}$$

with the identity element $(1_N, 1_H)$ and the inverse of (n, h) given by $(\rho(h^{-1})(n^-1), h^-1)$. In G there are the natural subgroups

$$\bar{N} = \{(n, 1_H) \mid n \in N)\} \qquad \text{and} \qquad \bar{H} = \{(1_N, h) \mid h \in H)\}$$

which are canonically isomorphic to N and H respectively. For $n \in N$ and $h \in H$ we have

$$(n, 1_H)(1_N, h) = (n\underbrace{\rho(1_H)}_{\mathrm{id}_N}(1_N), h) = (n, h) \in \bar{N}\bar{H}\,.$$

It follows that $G = \bar{N}\bar{H}$ and $\bar{N} \cap \bar{H} = 1$. To check that $G = \bar{N} \rtimes \bar{H}$ we only have to show that $\bar{N}$ is normal in G. Let $n, n' \in N$ and $h \in H$ then

$$\begin{aligned}(n,h)(n',1_H)(n,h)^{-1} &= (n,h)(n',1_H)(\rho(h^{-1})(n^{-1}),h^{-1}) \\ &= (n\rho(h)(n'),h)(\rho(h^{-1})(n^{-1}),h^{-1}) \\ &= (n\rho(h)(n')\rho(h)\rho(h^{-1})(n^{-1}),1_H) \\ &= (n\rho(h)(n')n^{-1},1_H) \in \bar{N}\,.\end{aligned}$$

In the case, where ρ is the trivial homomorphism

$$\begin{aligned}\rho : H &\to \mathrm{Aut}(N) \\ h &\mapsto \mathrm{id}_N\end{aligned}$$

the elements of $\bar{N}$ and $\bar{H}$ commute, so that G becomes the direct product $N \times H$. Thus the semidirect product is a generalization of the direct product of two groups.

A.2 Faithful representations

[Hal03, Def.4.1] says the following about faithful representations:

Definition A.2

Let G be a matrix Lie group. Then a *(finite-dimensional complex) representation* of G is a Lie group homomorphism

$$\rho : G \to \mathrm{GL}(V)$$

where

- V is a finite-dimensional complex vector space.

If ρ is a one-to-one homomorphism, then the representation is called *faithful*.

Remark A.2.1

If a representation ρ is a faithful representation of a matrix Lie group G then $\{\rho(A) \mid A \in G\}$ is a group of matrices that is isomorphic to the original group G. Thus, ρ allows us to represent G as a group of matrices.

B Multisummability

The aim of a theory of summation is to associate with any series an asymptotic function uniquely determined in a way as much natural as possible. We will take this as a black box and only use the fact, that it makes the asymptotic expansion unique. A useful and extensive resource for this topic is Loday-Richaud's book [Lod14].

Let $[A^0]$ be a normal form and $\widehat{F} \in \widehat{G}(A^0)$ a formal transformation and denote the transformed system by $A = {}^{\widehat{F}}A^0$. Let $\dot{\varphi} = (\dot{\varphi}_j)_{j\in J} \in \Gamma(\dot{\mathcal{V}}; \Lambda(A^0))$ be a 1-cocycle in the image of the equivalence class of $\widehat{F}$ by exp.

The following Proposition can be found in [Lod94, Prop.III.2.1], [Lod14, Thm.4.3.13] and [BJL79, Thm.1].

Proposition B.1

There exists a **unique** family of realizations $(F_j)_{j\in J}$ of $\widehat{F}$ over $\mathcal{V}$, i.e. matrices F_j which are analytic on V_j, satisfy $[A^0, A]$ and are asymptotic to $\widehat{F}$ on V_j, such that

$$\dot{\varphi}_j = F_{j-1}F_j^{-1}$$

for every $j \in J$.

When $\mathcal{V}$ is the, in section 3.3.2 defined, cyclic covering $\mathcal{U}^{\leq k_r} =: \mathcal{U}$ and $\dot{\varphi}$ is in its Stokes form, we call the realizations F_α, the *sums of* $\widehat{F}$.

Definition B.2

Denote by α^+ the next anti-Stokes direction on the right of α we can define

- $S_\alpha^-(\widehat{F}) := F_\alpha$ as the *sum of* $\widehat{F}$ *on the left of* α and
- $S_\alpha^+(\widehat{F}) := F_{\alpha^+}$ as the *sum of* $\widehat{F}$ *on the right of* α.

Let $\widehat{F}_0 \in \widehat{G}(A^0)$ and $A^1 = {}^{\widehat{F}_0}A^0$ and let

$$\exp_{A^1}(\widehat{F}) : G\backslash\widehat{G}(A^1) \to H^1(S^1; \Lambda(A^1))$$

be the Malgrange-Sibuya isomorphism (cf. Remark 3.5).

The definition of the subsheaf $\Lambda^{\geq k}(A^1)$ of $\Lambda(A^1)$ needs a few additional justifications (cf. [Lod94, p. 883]).

Definition B.3

A 1-cocycle $\dot{\varphi} = (\dot{\varphi}_j)_{j\in J}$ is k*-summable* if, for all $j \in J$, it satisfies the conditions

1. $\dot{\varphi}_j$ is of level $\geq k$, i.e. $\varphi_j \in \Gamma(\dot{V}_j; \Lambda^{\geq k}(A^1))$ and
2. the opening of $\dot{V}_j$ is $\frac{\pi}{k}$.

An element of $\widehat{F} \in \widehat{G}(A^1)$ is *k-summable* when $\exp_{A^1}(\widehat{F})$ contains a k-summable cocycle.

If in a cohomology class is a k-summable cocycle, it is unique up to extra trivial components. The realizations of such a k-summable cocycle define the k-sums of $\widehat{F}$. When $[A^1]$ is meromorphically equivalent to $[A^0]$, i.e. $\widehat{F} \in G(\{t\})$, then

$\widehat{F}$ is k-summable

if and only if

its Stokes cocycle $\dot{\varphi} = (\dot{\varphi}_\alpha)_{\alpha \in \mathbb{A}}$ belongs to $\Gamma(\dot{\mathcal{U}}^k; \Lambda^k)$.

This occurs when the Stokes cocycle $\dot{\varphi}$ of $\widehat{F}$ bears only the level k.

Definition B.4

Let $\widehat{F} \in \widehat{G}(A^1)$ be k-summable and $(\dot{\varphi}_j)_{j\in J} \in \prod_{j\in J} \Gamma(\dot{V}_j; \Lambda(A^1))$ the corresponding k-summable cocycle in $\exp_{A^1}(\widehat{F})$.

To each $j \in J$ with a nontrivial $\dot{\varphi}_j$ we define the bisecting direction of $\dot{V}_j$ as a *singular direction for $\widehat{F}$*.

B.0.1 Factorization

Let $\dot{f} \in \Gamma(\dot{\mathcal{U}}; \Lambda(A^0))$ be the Stokes cocycle associated to $\widehat{F}$ and $k = \max(\mathcal{K}(\dot{f}))$ be the maximal level beared by $\dot{f}$. We then know, since $\dot{f}^{\leq k} = \dot{f}$ and we have corollary 3.23 that there is a unique decomposition

$$\dot{f} = \dot{f}^{<k}\dot{g}^k \qquad \text{and} \qquad \dot{f} = \dot{f}^k\dot{f}^{<k}$$

where $\dot{f}^k, \dot{g}^k \in \Gamma(\dot{\mathcal{U}}^{\leq k}; \Lambda^k(A^0))$ and $\dot{f}^{<k} \in \Gamma(\dot{\mathcal{U}}^{\leq k}; \Lambda^{<k}(A^0))$.

Since the cohomology class of $\dot{f}^{<k}$ belongs to $H^1(S^1; \Lambda(A^0))$ we know from the Malgrange-Sibuya theorem that there exists an, up to left meromorphic factor unique, element $\widehat{F}^{<0}$ of $\widehat{G}(A^0)$ such that $\dot{f}^{<k}$ belongs to $\exp_{A^0}(\widehat{F}^{<k})$.

The Proposition [Lod94, Prop.III.2.6] states the following.

Proposition B.5

We have, that

1. $\widehat{F}^k$ is k-summable with singular directions (cf. Definition B.4) belonging to $\mathbb{A}^k$
2. The levels in the Stokes cocycle of $\widehat{F}^{<k}$ are $< k$.
3. The decomposition $\widehat{F} = \widehat{F}^k\widehat{F}^{<k}$ with properties 1. and 2. is essentially unique, that means:

   if $\widehat{F} = \widehat{H}^k\widehat{H}^{<k}$ is another decomposition, there is a matrix $h \in G(\{t\})$ such that $\widehat{H}^{<k} = h\widehat{F}^{<k}$ and $\widehat{H}^k = \widehat{F}^k h^{-1}$.

We can use the Proposition B.5 to obtain the following factorization on transformations, which can be found an Loday's Paper [Lod94, Thm.III.2.5] or as [MR91, Thm.4.7].

Theorem B.6

Let $\widehat{F} \in \widehat{G}(A^0)$ be a formal transformation and $\mathcal{K} = \{k_1 < k_2 < \cdots < k_r\}$ be the set of levels of $[A^0]$. Then $\widehat{F}$ can be factored in

$$\widehat{F} = \widehat{F}_r \widehat{F}_{r-1} \ldots \widehat{F}_2 \widehat{F}_1$$

where the matrices $\widehat{F}_j$ are

- k_j-summable and
- with singular directions belonging to the set $\mathcal{A}^{k_j}$.

B.0.2 The map $g : G\backslash\widehat{G}(A^0) \to \prod_{\alpha\in\mathcal{A}} \mathrm{Sto}_\alpha(A^0)$

There are other definitions, which are equivalent to definition B.2. Since some of them are more direct and do not use the 1-cocycle, to which they correspond, they can be used to obtain the corresponding 1-cocycle. Many different approaches to define the sums can be found in [Lod14].

Let us fix an ambassador $\widehat{F}$ in $G\backslash\widehat{G}(A^0)$. We then use proposition B.1 to obtain an 1-cocycle. An element in $\prod_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha(A^0)$ corresponding to $\widehat{F}$ is found as

$$(\varphi_{\alpha_1}, \ldots, \varphi_{\alpha_\nu}) \in \prod_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha(A^0)$$

by setting $\varphi_\alpha := \left(S_\alpha^-(\widehat{F})\right)^{-1} S_\alpha^+(\widehat{F}) \in \mathrm{Sto}_\alpha(A^0)$ (cf. Definition B.2).

Remark B.7

$S_\alpha^+(\widehat{F}) t^L e^{Q(t^{-1})}$ is a solution of $[A]$ on the corresponding sector. Using the equation (3.2) we can write

$$S_\alpha^+(\widehat{F}) \mathcal{Y}_{0,\alpha}(t) = S_\alpha^-(\widehat{F}) \mathcal{Y}_{0,\alpha}(t) C_{\mathcal{Y}_{0,\alpha}}$$

thus the Stokes matrix $C_{\mathcal{Y}_{0,\alpha}}$ is a matrix, which describes the blending between the two adjacent sectors.

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