Contents

1 Introduction

TODO: see [PS03] for better text Let t be a local coordinate around 0 in $\mathbb{C}$. The basic fact, from which this master's thesis arises, is that there are differential equations $(\frac{d}{dt} - A)\widehat{v} = w$ with coefficients in convergent powers series, which have no solution with converging entries. By looking at asymptotic solutions on sectors at 0 and how they can be extended to larger arcs, one discovers the Stokes phenomenon. It describes that there are some directions, called Stokes directions, beyond which some solutions can not be extended. To overlapping sectors one obtains matrices, which describe how solutions on the corresponding sectors correlate. Some germs of these matrices, which satisfy some condition, will be enough to classify corresponding to the differential equation up to convergent transformation. Instead of differential equations, one can use the language of meromorphic connections and the Stokes structures, which will be encoded in the first homology of the Stokes sheaf, will be used to describe the classifying set.

In the first chapter we will define the notion of asymptotic expansions. We will also state the important Borel-Ritt Lemma. The second chapter is dedicated to the theory of meromorphic connections and the equivalent language of systems. We will talk about local expression of meromorphic connections and the formal classification of such objects.

The classification of meromorphic connections can be splitted in the coarse formal classification and the fine meromorphic classification. The formal classification problem was solved by the Levelt-Turittin Theorem (cf. Theorem 3.26). It states that in every formal equivalence class are some meromorphic connections of special form, which will be called models and that such models are unique up to meromorphic equivalence. These models are meromorphic connections, which are defined to be isomorphic to some direct sum of elementary meromorphic connections and elementary meromorphic connections are well understood. On the level of systems there are normal forms, which correspond to models.

Starting with a formal equivalence class corresponding to some normal form A^0, we can apply the meromorphic classification to the corresponding set of ambassadors. The obtained set of meromorphic classes will be called the classifying set. But instead of the classifying set we will look at the slightly larger space $\mathcal{H}(A^0)$ of meromorphic pairs which also handles the information, how some meromorphic connection is related to it's model (cf. Section 3.5). The classifying set can be obtained from the set of meromorphic pairs in a rather simple way (cf. Corollary 3.40.1).

We have already said that the tool to describe the set of meromorphic pairs will be the Stokes structures. This idea is formulated in the Malgrange-Sibuya Theorem (cf. Theorem 4.4), where the Stokes structures appear as the first cohomology $H^1(S^1; \Lambda(A^0))$ of the Stokes sheaf $\Lambda(A^0)$ on S^1 (cf. Definition 4.1).

The Malgrange-Sibuya theorem can be improved by showing that in each element in the $H^1(S^1; \Lambda(A^0))$ contains a unique cocycle called the Stokes cocycle (cf. Definition 4.33) of special form. This goes back to Loday-Richaud's work and especially her Paper [Lod94]. These Stokes cocycles are given by the elements in the product over some special directions $\theta \in \mathbb{A} \subset S^1$ determined by A^0 of Stokes groups $\mathrm{Sto}_\theta(A^0) \subset \Lambda_\theta(A^0)$ (cf. Section 4.2). We will see that $\mathrm{Sto}_\theta(A^0)$ has the faithful representation $\mathbb{S}\mathrm{to}_\theta(A^0)$, which has a rather simple definition (cf. Section 4.2.2). The elements of $\mathbb{S}\mathrm{to}_\theta(A^0)$ are the so-called Stokes matrices and it is easy to see that they are nilpotent. Since the Stokes matrices also provide the required information to describe a meromorphic class of a meromorphic connection one obtains a structure of a nilpotent Lie group on the set of isomorphism classes of meromorphic pairs $\mathcal{H}(A^0)$ and an isomorphism $\mathbb{C}^N \to H^1(S^1; \Lambda(A^0))$, where N is the irregularity of A^0 (cf. Section 4.3.3).

In the Section 4.6 we will improve this even further by collecting the data of multiple Stokes groups into one group which is more stable under deformations on the chosen model. In the single-leveled case is this extensively used by Boalch in his Publications [Boa01; Boa99]. After that we will mention some ideas from the summability theory, just to give a rough view of the concept. In the last Section 4.6 we will draw a diagram which will contain many isomorphisms and objects which were defined in the Chapter . It will also clearify, how some morphisms were composed.

List of used symbols

1. Poincaré asymptotic expansions

$\mathfrak{s}_{a,b}(r)$	the sector $\{t \in \mathbb{C} \mid a < \arg(t) < b, 0 < \|t\| < r\}$
$\mathfrak{s}_I(r)$	the sector $\mathfrak{s}_{a,b}(r)$ for $I = (a,b)$ or the arc I
$\bar{\mathfrak{s}}_{a,b}(r)$	the closure of $\mathfrak{s}_{a,b}(r)$ in $\mathbb{C}^*$.
(θ, θ')	the arc from $\theta \in S^1$ to θ'
$U(\theta, \varepsilon)$	the arc $\left(\theta - \frac{\varepsilon}{2}, \theta + \frac{\varepsilon}{2}\right) \subset S^1$ resp. the corresponding sector
$\mathfrak{s} \Subset \mathfrak{s}'$	$\mathfrak{s}$ is proper sub-sector of $\mathfrak{s}'$
$\widehat{f}$	the asymptotic expanson of f (if not used in another way)
$f \sim_{\mathfrak{s}} \widehat{f}$	f has $\widehat{f}$ as asymptotic expanson on the sector $\mathfrak{s}$
$f \sim_I \widehat{f}$	f has $\widehat{f}$ as asymptotic expanson on the arc I
$\mathcal{A}$	the sheaf of asymptotic expansions
T_I	the Taylor map on the arc I
$\mathcal{A}^{<0}(I)$	$\ker(T_I)$; the functions asymptotic to 0 on I
$\mathcal{A}^{<0}(\mathfrak{s})$	$\ker(T_{\mathfrak{s}})$; the functions asymptotic to 0 on $\mathfrak{s}$

2. Systems and meromorphic connections

M	a riemanian surface
Z	a effective divisor on M
$\mathscr{M}$	a holomorphic bundle over M
$\mathscr{M}$; $(\mathscr{M}, \nabla)$	a meromorphic connection on $\mathscr{M}$ with poles on Z
$\mathcal{M}$; $(\mathcal{M}, \nabla)$	germ of a meromorphic connection $(\mathscr{M}, \nabla)$ at $0 \in M$
$\triangle$	a differential operator
$[A]$	the system corresponding to the connection matrix A
$G(\{t\})$	$\mathrm{GL}_n\left(\mathbb{C}\{t\}[t^{-1}]\right)$; the meromorphic transformations
$G((t))$	$\mathrm{GL}_n\left(\mathbb{C}[\![t]\!][t^{-1}]\right)$; the maybe not applicable formal meromorphic transformations
${}^FA$	$(dF)F^{-1} + FAF^{-1}$; the transforamtion of A by F
$\widehat{G}(A)$	the set of all (applicable) formal transformations
$[A, B]$	the linear differential system $\frac{dF}{dt} = BF - FA$
$[\mathrm{End}\, A]$	$[A^0, A^0]$
$G_0(A)$	the set of all isotropies of A
$\mathcal{Y}$	a fundamental solution
$\mathcal{E}^{\varphi}$	the germ $(\mathbb{C}(\{t\}), d - \varphi')$
$\mathcal{N}_{\alpha,0}$	the germ $\left(\mathbb{C}(\{t\}), d + \frac{\alpha}{t}\right)$
$\mathcal{N}_{\alpha,d}$	an elementary regular model
$(\mathcal{E}^{\varphi}, \nabla) \otimes (\mathcal{R}, \nabla)$	a elementary meromorphic connection
$(\mathcal{M}^{nf}, \nabla^{nf})$	a model
λ	the isomorphism from Definition 3.25
$Q(t^{-1})$	the irregular part of a system $[A]$

$L \in \mathrm{GL}_n(\mathbb{C})$	the matrix of formal monodromy of $[A]$
${}^0C(\mathcal{M}^{nf}, \nabla^{nf})$	all isomorphism classes of meromorphic connections, which are formally isomorphic to $(\mathcal{M}^{nf}, \nabla^{nf})$
${}^0C(A^0)$	the system variant of ${}^0C(\mathcal{M}^{nf}, \nabla^{nf})$
$\mathcal{H}(\mathcal{M}^{nf}, \nabla^{nf})$	the space of isomorphism classes of marked (meromorphic) pairs
$\mathrm{Syst}_m(A^0)$	the systems formally meromorphic equivalent to A^0
$\mathcal{H}(A^0)$	the system version of $\mathcal{H}(\mathcal{M}^{nf}, \nabla^{nf})$
$\widehat{\mathrm{Syst}}_m(A^0)$	the marked pairs corresponding to A^0

3. Stokes Structures

$\Lambda(A^0)$	the Stokes sheaf (on S^1)
$\mathcal{S}t(A^0)$	$H^1(S^1; \Lambda(A^0))$; the first non abelian cohomology of the Stokes sheaf
$\exp$	the Malgrange-Sibuya isomorphism
$\exp_A$	the Malgrange-Sibuya isomorphism corresponding to the system $[A]$
$\mathcal{G}$	the sheaf of flat functions
Θ	$G[\![t]\!]/G\{t\} \to H^1(S^1; \mathcal{G})$; the isomorphism from the other Malgrange-Sibuya theorem
$\mathcal{Q}(A^0)$	the set of all determining polynomials of $[\mathrm{End}\, A^0]$
$a_{jl} \in \mathbb{C}\backslash\{0\}$	the leading factor of $q_j - q_l \in \mathcal{Q}(A^0)$
$q_{jl}(t^{-1})$	the leading coefficient of $q_j - q_l \in \mathcal{Q}(A^0)$
k_{jl}	the degree of $q_j - q_l \in \mathcal{Q}(A^0)$
$\mathcal{K}$	$\{k_1 < \cdots < k_r\}$; the set of all levels of a system.
$q_j \underset{\theta}{\prec} q_l$	the first relation from Definition 4.11
$q_j \underset{\theta}{\prec\!\!\!\prec} q_l$	the second relation from Definition 4.11
$\Re(z)$	the real part of a complex number z
$\Im(z)$	the imaginary part of a complex number z
$\arg(z)$	the argument of a complex number z
$\mathbb{A}$	$\{\alpha_1, \ldots, \alpha_\nu\}$; the set of all anti-Stokes directions
$\mathrm{Sto}_\theta(A^0)$	the Stokes group of A^0 in direction θ whose elements are Stokes germs
ϑ_α	The map from Definition 4.16
$\boldsymbol{\delta}_{jl}$	a block matrix version of Kronecker's delta, corresponding to the structure of Q
$\mathbb{S}\mathrm{to}_\theta(A^0)$	the group of all Stokes matrices of A^0 in direction θ
$\widehat{\mathbb{S}\mathrm{to}}_\theta(A^0)$	The space of collected Stokes matrices
$\widehat{\vartheta}_\alpha$	The map from Remark 4.16.1
ρ_θ	$\mathrm{Sto}_\theta(A^0) \to \mathbb{S}\mathrm{to}_\theta(A^0)$; the map from Proposition 4.17
C_{φ_θ}	$\rho_\theta(\varphi_\theta)$; the Stokes matrix corresponding to φ_θ
$\Lambda^k(A^0)$	the subsheaf of $\Lambda(A^0)$ of all germs, which are generated by k-germs
$\Lambda^{\leq k}(A^0)$	the subsheaf of $\Lambda(A^0)$ generated by k'-germs for all $k' \leq k$
$\Lambda^{<k}(A^0)$	the subsheaf of $\Lambda(A^0)$ generated by k'-germs for all $k' < k$

$\Lambda^{\geq k}(A^0)$	the subsheaf of $\Lambda(A^0)$ generated by k'-germs for all $k' \geq k$
$\mathrm{Sto}^{\star}_{\theta}(A^0)$	$\mathrm{Sto}_{\theta}(A^0) \cap \Lambda^{\star}_{\theta}(A^0)$; the restriction of the Stokes sheaf for $\star \in \{k, < k, \leq k, \dots\}$
$\mathbb{S}\mathrm{to}^{\star}_{\theta}(A^0)$	the groups of representations, which correspond to elements of $\mathrm{Sto}^{\star}_{\theta}(A^0)$
$\mathbb{A}^k$	the set of anti-Stokes directions bearing the level k
$\mathbb{A}^{\leq k}$	$\bigcup_{k' \leq k} \mathbb{A}^{k'}$
$\mathbb{A}^{< k}$	$\bigcup_{k' < k} \mathbb{A}^{k'}$
$\mathbb{A}^{\geq k}$	$\bigcup_{k' \geq k} \mathbb{A}^{k'}$
$\mathcal{K}_{\alpha}$	the set of levels beared by $\alpha \in \mathbb{A}$
i_{α}	$\mathrm{Sto}_{\alpha}(A^0) \to \prod_{k \in \mathcal{K}_{\alpha}} \mathrm{Sto}^{k}_{\alpha}(A^0)$; the isomorphism corresponding to a factorization by a given order in a semidirect product
ρ^k_{α}	$\mathrm{Sto}^{k}_{\alpha}(A^0) \to \mathbb{S}\mathrm{to}^{k}_{\alpha}(A^0)$; the restriction of the map ρ_{α} to the level k.
i_{α}	$\mathbb{S}\mathrm{to}_{\alpha}(A^0) \to \prod_{k \in \mathcal{K}_{\alpha}} \mathbb{S}\mathrm{to}^{k}_{\alpha}(A^0)$; the Stokes matrix version of the i_{α} above
$\dot{\mathcal{U}}$	the nerve of the covering $\mathcal{U}$
h	$\prod_{\alpha \in \mathbb{A}} \mathrm{Sto}_{\alpha}(A^0) \longrightarrow \mathcal{S}t(A^0)$; the isomorphism from Theorem 4.34
$\mathcal{U}^{\star}$	for $\star \in \{k, < k, \leq k\}$ the adequate coverings defined in Section 4.3.2
$\Gamma(\dot{U}^{\star}; \Lambda^{\star}(A^0))$	$\prod_{\alpha \in \mathbb{A}^{\star}} \Gamma(\dot{U}^{\star}_{\alpha}; \Lambda^{\star}(A^0))$ for every $\star \in \{k, < k, \leq k, \dots\}$
s^k	$\Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A^0)) \to H^1(S^1; \Lambda^k(A^0))$; the quotient map
σ^k	$\Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A^0)) \to \Gamma(\dot{\mathcal{U}}^{\leq k}; \Lambda^{\leq k}(A^0))$; an injective map defined in Definition 4.43
$\sigma^{<k}$	$\Gamma(\dot{\mathcal{U}}^{<k}; \Lambda^{<k}(A^0)) \to \Gamma(\dot{\mathcal{U}}^{\leq k}; \Lambda^{\leq k}(A^0))$; an injective map defined in Definition 4.43
$\mathfrak{S}^{\leq k}$	$\Gamma(\dot{\mathcal{U}}^{<k}; \Lambda^{<k}(A^0)) \times \Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A^0)) \to \Gamma(\dot{\mathcal{U}}^{\leq k}; \Lambda^{\leq k}(A^0))$; the product map of cocycles
α^+	the nearest anti-Stokes direction on the right of α
τ	$\prod_{k \in \mathcal{K}} \Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A^0)) \to \Gamma(\dot{\mathcal{U}}; \Lambda(A^0))$; the product map of single-leveled cocycles
$\mathcal{T}$	$\prod_{k \in \mathcal{K}} \Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A^0)) \to H^1(\mathcal{U}; \Lambda(A^0))$; the isomorphism induced by τ on the cohomology
$\mathfrak{T}$	a map introduced in the proof of Theorem 4.34

2 Poincaré asymptotic expansions

´classical'

- [Sab90, p. 60] Chapter II.2.2
  - [Sab00]
- **Van der Put: [PS03, Chapter 7]: Exact Asymptotics**
- [Maj84]
- [Bal00, Sec.4.4]
- [Lod94]
- **[Lod14] Chapter 2**
- [HS12]

´sheafical'

- **[Sab07, p. II.5]**

TODO: [Sib90] Appendix A.3

In this chapter, we want to look at Poincaré asymptotic expansions (at t_0), or for short, asymptotic expansions. There are various literature references for asymptotic expansions. We will mostly refer to Loday-Richaud's book [Lod14, chapter 2], [HS12, pp. XI-1-13] from Hsieh and Sibuya and the book [PS03, chapter 7] written by van der Put and Singer although this topic is scratched in many publications. We will assume that $t_0 = 0$ and this is without loss of generality, since asymptotic expansions at $t_0 \in \mathbb{C}$ reduce to asymptotic expansions at 0 after the change of variable $t \mapsto s = t - t_0$ and asymptotic expansions at $t_0 = \infty$ are obtained by $t \mapsto s = \frac{1}{t}$.

We introduce the following terminology:

Definition 2.1

We denote by

- $\mathfrak{s}_{a,b}(r)$, $a, b \in \mathbb{S}^1$ and $r \in \mathbb{R}_{>0}$ the open sector

$$\mathfrak{s}_{a,b}(r) := \{t \in \mathbb{C} \mid a < \arg(t) < b, 0 < |t| < r\}$$

  of all points $t \in \mathbb{C}$ satisfying $a < \arg(t) < b$ and $0 < |t| < r$;
- $\mathfrak{s}_I(r) = \mathfrak{s}_{a,b}(r)$ for an open arc[a] $I = (a, b)$;
- $\bar{\mathfrak{s}}_{a,b}(r)$ the closure of $\mathfrak{s}_{a,b}(r)$ in $\mathbb{C}^* := \mathbb{C}\backslash\{0\}$.

The notion of sectors can easily be extended to sectors on a q-sheet cover or sectors on the Universal cover of $\mathbb{C}^*$, but we will not use them.

[a]i.e. an open interval of S^1

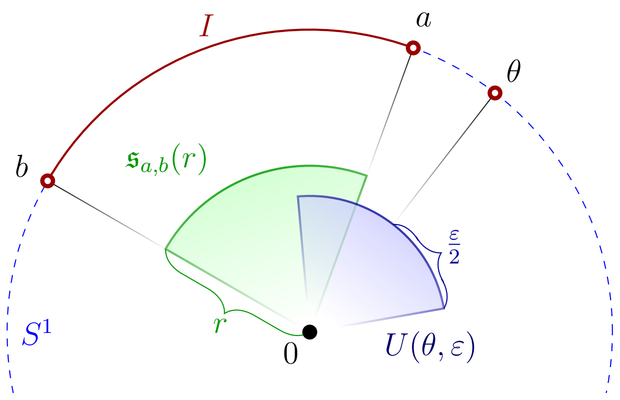


Figure 2.1: An exemplary arc $\mathfrak{s}_{a,b}(r)$ spanning from a to b with radius r corresponding to the arc I and an arc $U(\theta, \varepsilon)$ for some θ and ε.

We will say that *a sector* $\mathfrak{s}_I(r)$ *contains* $\theta \in S^1$ if $\theta \in I$. In the same way we will say that $\mathfrak{s}_I(r)$ contains $U \subset S^1$ if $U \subset I$ or that $U \subset S^1$ contains a sector $\mathfrak{s}_I(r)$ if $I \subset U$.

Remark 2.2

Since the explicit value r of the radius does not matter as long as it is small enough, we shall simply speak of an *arc* or *interval* I as $\mathfrak{s}_I(r)$ for a small enough r. It will be sometimes convenient to talk of arcs

$$U(\theta, \varepsilon) := \left(\theta - \frac{\varepsilon}{2}, \theta + \frac{\varepsilon}{2}\right) \subset S^1$$

defined by the midpoint together with the opening.

Definition 2.3

A sector $\mathfrak{s}_{a',b'}(r')$ is said to be a proper sub-sector of the sector $\mathfrak{s}_{a,b}(r)$, $\mathfrak{s}_{a',b'}(r') \Subset \mathfrak{s}_{a,b}(r)$, if its closure $\bar{\mathfrak{s}}_{a',b'}(r')$ is included in $\mathfrak{s}_{a,b}(r)$.

Remark 2.3.1

$\mathfrak{s}_{a',b'}(r') \Subset \mathfrak{s}_{a,b}(r)$ if and only if (a', b') is a real sub-interval of (a, b) and $r' < r$.

2.1 Definition and basic properties

Let us now give the definition of asymptotic expansions.

Definition 2.4

A function $f \in \mathcal{O}(\mathfrak{s})$ is said to have the formal[a] Laurent series $\widehat{g} = \sum_{n \geq n_0} a_n t^n$ as asymptotic expansion (or *to be asymptotic to the series*) on the sector $\mathfrak{s}$ if [PS03, Defn.7.1]

- for all proper sub-sectors $\mathfrak{s}' \Subset \mathfrak{s}$ and
- all $N \in \mathbb{N}$,

there exists a constant $C(N, \mathfrak{s}')$ such that the following estimate[b] holds

$$\Big| f(x) - \sum_{n_0 \leq n \leq N-1} a_n t^n \Big| \leq C(N, \mathfrak{s}')|t|^N \qquad \text{for all } t \in \mathfrak{s}' .$$

[a] The term "formal" emphasizes that we do not restrict the coefficients $a_n \in \mathbb{C}$ in any way.

[b] Sometimes, for example in [Sab90], this condition is written as

$$\lim_{z \to 0, z \in \mathfrak{s}'} |t|^{-(N-1)} \Big| f(x) - \sum_{n_0 \leq n \leq N-1} a_n t^n \Big| = 0 \qquad \text{for all } t \in \mathfrak{s}' .$$

We will say that f *has the formal Laurent series* $\widehat{g}$ *as asymptotic expansion on the interval* $I \subset S^1$ if there exists a radius $r \in \mathbb{R}_{>0}$ such that f has the formal Laurent series $\widehat{g}$ as asymptotic expansion on $\mathfrak{s}_I(r)$.

If f has $\widehat{g}$ as asymptotic expansion on the sector $\mathfrak{s}$ (resp. on the interval $I \subset S^1$), we denote that by $f \sim_{\mathfrak{s}} \widehat{g}$ (resp. $f \sim_I \widehat{g}$).

Theorem 2.5

For $f \in \mathcal{O}(\mathfrak{s})$, there is at most one formal Laurent series $\widehat{f}$ which satisfies $f \sim_{\mathfrak{s}} \widehat{f}$ (cf. Theorem [HS12, Thm.XI-1-5]).

The following proposition gives equivalent characterizations of asymptotic expansions.

Proposition 2.6

Let $f \in \mathcal{O}(\mathfrak{s})$ then are the statements

1. $f \sim_{\mathfrak{s}} \widehat{f} = \sum_{n \geq n_0} a_n t^n$,
2. the function f is infinitely often differential at the origin and $f^{(n)}(0) = n! a_n$ for $n \geq 0$ and
3. all derivatives $f^{(n)}(t)$ are continuous at the origin and $\lim_{\substack{t \to 0 \\ t \in \mathfrak{s}}} f^{(n)}(t) = n! a_n$

equivalent (cf. [Bal00, 4.4.Prop.8]).

Definition 2.7

1. Define $\mathcal{A}(\mathfrak{s})$ as the set of holomorphic functions $f \in \mathcal{O}(\mathfrak{s})$ which admit an asymptotic expansion at 0 on $\mathfrak{s}$.

Remark 2.7.1

[Sab90, p. II.2.2.4] a) $\mathcal{A}(\mathfrak{s})$ is a subring of $\mathcal{O}(\mathfrak{s})$.

[Sab90, p. II.2.2.4] b) $\mathcal{A}(\mathfrak{s})$ contains $\mathbb{C}(\{t\})$ as a subfield.

2. For $(a,b) = I$ define $\mathcal{A}(a,b) = \mathcal{A}(I)$ as the limit $\varinjlim_{r\to 0} \mathcal{A}(\mathfrak{s}_{a,b}(r))$[a].

[a] In more detail: the elements of $\mathcal{A}(a,b)$ are pairs $(f, \mathfrak{s}_{a,b}(r))$ with $f \in \mathcal{A}(\mathfrak{s}_{a,b}(r))$. The equivalence relation is given by $(f_1, \mathfrak{s}_{a,b}(r_1)) \sim (f_2, \mathfrak{s}_{a,b}(r_2))$ if there is a pair $(f_3, \mathfrak{s}_{a,b}(r_3))$ such that $r_3 < \min(r_1, r_2)$ and $f_3 = f_1 = f_2$ on $\mathfrak{s}_{a,b}(r_3)$.

It can easily be seen that $\mathcal{A} : U \to \mathcal{A}(U)$ defines a sheaf on S^1.

One writes

$$\begin{aligned} T_I : \mathcal{A}(I) &\to \mathbb{C}(\!(t)\!) \\ f &\mapsto \widehat{f} \end{aligned}$$

for the so-called *Taylor map* which associates to each function $f \in \mathcal{A}(I)$ its asymptotic expansion. If $\widehat{f}$ is not defined to be something else, we write $\widehat{f}$ for the image of f under T_I.

Definition 2.8

A function f which is asymptotic to the identity, i.e. with $T_I(f) = \widehat{f} = \mathrm{id}_{\mathfrak{s}_I}$, is called *flat (on I)*.

The kernel of T_I is not zero in general. For example is the function $e^{-\frac{1}{t}}$ asymptotic to zero on every $I \subset \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. TODO: changed from $e^{-\frac{1}{\sqrt{t}}}$ after hint from giovanni

Definition 2.9

Denote by $\mathcal{A}^{<0}(I)$ the kernel $\ker(T_I) \subset \mathcal{A}(I)$ of functions asymptotic to zero at 0. This does also define a sheaf $\mathcal{A}^{<0}$.

Example 2.10: Trivial example

[Lod14, Exmp.2.2.3] If f is a holomorphic function on $D = \{t \mid |t| < r\}$ then f is asymptotic to its Taylor series at 0 on every arc $\mathfrak{s} \subset \mathfrak{s}_{S^1}(r) = D \backslash \{0\}$.

Reciprocally, if f is holomorphic on $\mathfrak{s}_{S^1}(r)$ and has an asymptotic expansion at 0 on $\mathfrak{s}_{S^1}(r)$ then, f is bounded near 0 and according to the removable singularity theorem, f is meromorphic on D.

Remark 2.10.1

This implies that $\mathcal{A}(S^1) \cong \mathbb{C}(\{t\})$.

For many more extensively discussed examples, like the Euler function or the exponential integral, see Loday-Richaud's book [Lod14, Sec.2.2].

Proposition 2.11

For every arc $I \subset S^1$ is $\mathcal{A}(I)$ is stable under derivation, i.e. $f'(t)$ is on I asymptotic to $\widehat{f}'(t)$ for every function f asymptotic to $\widehat{f}$ on I. [Sab90, p. II.2.2.4]i ,[HS12, pp. XI-1-11]

Proof. Let $f \in \mathcal{A}(I)$ with asymptotic expansion $\widehat{f} = \sum_{-n_0 \leq n} a_n x^n$ its asymptotic expansion. One has for each $m \geq 0$

$$f(t) = \sum_{n=0}^{m} a_n t^n + R_m(t) t^m$$

with $\lim_{\substack{t \to 0 \\ t \in \mathfrak{s}_I(r)}} R_m(t) = 0$. This implies that R_m is holomorphic in $\mathfrak{s}_I(r)$. Thus one has

$$f'(t) = \sum_{n=0}^{m} n a_n t^{n-1} + m R_m(t) t^{m-1} + R'_m(t) t^m$$

Let $C_{x,\rho}$ be the circle at x with radius ρ contained in $\mathfrak{s}_I(r)$. The Cauchy theorem implies then that

$$|R'_m(t)| \leq \frac{1}{\rho} \max_{s \in \mathbb{C}_{x,\rho}} |R(s)| \,.$$

Let J be a relatively compact open set in I, then there exists a positive number α such that for all $x \in \mathfrak{s}_I(r)$ one has $C_{x,\alpha|x|} \subset \mathfrak{s}_I(r)$. TODO This proves that $\widehat{f}'$ is an asymptotic expansion for f' in $\mathfrak{s}_I(r)$. □

Other properties can be found in the books [Bal00] from Balser and [HS12] from Hsieh and Sibuya. Some are stated in the following remark.

Remark 2.12

Let f_1 and f_2 be functions which satisfying $f_1 \sim_I \widehat{f}_1$ and $f_2 \sim_I \widehat{f}_2$. We have also the rules:

1. $f_1(t) + f_2(t)$ is on I asymptotic to $\widehat{f}_1(t) + \widehat{f}_2(t)$ (cf. Theorem [Bal00, 4.5.Thm.13] or [HS12, pp. XI-1-6]).
2. $f_1(t) f_2(t)$ is on I asymptotic to $\widehat{f}_1(t)\widehat{f}_2(t)$ (cf. Theorem [Bal00, 4.5.Thm.14] or [HS12, pp. XI-1-6]).
3. $\int_0^t f(s)\,ds$ is on I asymptotic to $\int_0^t \widehat{f}(t)\,ds$ (cf. Theorem [Bal00, 4.5.Thm.20]).
4. If $f \in \mathcal{A}(I) \backslash \mathcal{A}^{<0}(I)$ then $f^{-1}(t)$ is on I asymptotic to $\widehat{f}^{-1}(t)$ (cf. Theorem [Bal00, 4.5.Thm.21] or [HS12, pp. XI-1-9]).

2.2 Borel-Ritt Lemma

- [Sab90, Lem.II.2.2.5]: T_I
- **[PS03, Th.7.3]**: $T_I = T_{(a,b)}$
- [Lod14, Th.2.4.1]: $T_{\mathfrak{s}}$
- [Bal00, 4.4.Thm.16]
- [HS12, pp. XI-1-13]

The most important theorem here is the Borel-Ritt Lemma.

Theorem 2.13: Borel-Ritt

[Lod14, Thm.2.4.1]

Let $I \subsetneq S^1$ be an open sub-arc, i.e. an arc with a opening less than 2π, then is the Taylor map $T_I : \mathcal{A}(I) \twoheadrightarrow \mathbb{C}(\!(t)\!)$ onto.

[Lod14, p. 111], [HS12, pp. XI-1-13]

In other words does this theorem say, that to each formal Laurent series $\widehat{f} \in \mathbb{C}(\!(t)\!)$ and every arc $I \subsetneq S^1$ there are functions f asymptotic to $\widehat{f}$ on I. There are multiple approaches to obtain such a function $f \in \mathcal{A}(I)$ asymptotic to a given function $\widehat{f} \in \mathbb{C}(\!(t)\!)$ in a canonical way. For example (multi-)multisummability which is in depth described in Loday-Richaud's book [Lod14]. This topic is also discussed in the paper [MR91] from Martinet and Ramis.

However, when $\widehat{f}$ satisfies an equation, these asymptotic functions do not necessarily satisfy the same equation in general. This gap will be filled in Section 3.4 by the main asymptotic existence theorem for small enough sectors.

Let us now proof the Borel-Ritt Lemma.

Proof of Theorem 2.13. Let $I := (-\pi, \pi)$ and $R \in \mathbb{R}_{>0}$. We will prove this for the sector

$$\mathfrak{s} := \mathfrak{s}_I(R) = \{t \in \mathbb{C} \mid |\arg(t)| < \pi, 0 < |t| < R\}$$

in which, after rotation and for large enough R, every sector $\mathfrak{s}'' \subsetneq \mathbb{C}^*$ lies. Let $\sum a_n z^n$ be a formal Laurent series. We look for a function $f \in \mathcal{A}(I)$ with Taylor series $T_I f = \sum a_n z^n$. By subtracting the principal part TODO: ? we may assume that this series has no terms with negative degree. Let b_n be a sequence, which satisfies

the series $\sum |a_n| b_n R^{n-\frac{1}{2}}$ is convergent.

For example TODO: realy?, one may set

$$b_n = \begin{cases} 0 & \text{when } n = 0 \\ \frac{1}{n!}|a_n| & \text{when } n > 0 \end{cases}.$$

Let $\sqrt{t}$ be the branch of the square root function that satisfies $|\arg(\sqrt{t})| < \frac{\pi}{2}$[a] for all $t \in \mathfrak{s}$. For any real number b_n, the function $\beta_n(t) := 1 - e^{-\frac{b_n}{\sqrt{t}}}$ satisfies

(a) $|\beta_n(t)| \leq \frac{b_n}{\sqrt{|t|}}$ since $1 - e^t = -\int_0^t e^s \, ds$ implies that $|1 - e^t| < |t|$ for $\Re(t) < 0$ and

(b) β_n has asymptotic expansion 1 on $\mathfrak{s}$ (thus $\beta_n - 1$ has asymptotic expansion 0 on $\mathfrak{s}$).

Define $f(t) := \sum a_n \beta_n(t) t^n$. Since

$$|a_n \beta_n(t) t^n| \leq |a_n| b_n |z|^{n-\frac{1}{2}} \leq |a_n| b_n R^{n-\frac{1}{2}},$$

the series $\sum a_n \beta_n(t) t^n$ converges and its sum $f(t)$ is in $\mathcal{O}(\mathfrak{s})$.

Consider a proper sub-sector $\mathfrak{s}' \Subset \mathfrak{s}$ and $t \in \mathfrak{s}'$. Then, for every $N > 0$

$$\begin{aligned}\left| f(t) - \sum_{n=0}^{N-1} a_n t^n \right| &\leq \left| \sum_{n=0}^{N-1} a_n (\beta_n(t) - 1) t^n \right| + |t|^N \sum_{n \geq N} \left| a_n \beta_n(t) t^{n-N} \right| \\ &= \left| \sum_{n=0}^{N-1} a_n e^{-\frac{b_n}{\sqrt{t}}} t^n \right| + |t|^N \sum_{n \geq N} \left| a_n \beta_n(t) t^{n-N} \right|\end{aligned}$$

The first summand is a finite sum of terms, which are all asymptotic to 0 and thus is majorized by $C'|t|^N$ for a convenient constant C'. The second summand is majorized by

$$|t|^N \left(2|a_n| + \sum_{n \geq N+1} |a_n| b_n R^{n-\frac{1}{2}-N} \right).$$

By setting $C = C' + 2|a_n| + \sum_{n \geq N+1} |a_n| b_n R^{n-\frac{1}{2}-N}$ we obtain a positive constant C, which depends on N and the radius of $\mathfrak{s}' < R$, such that

$$\left| f(t) - \sum_{n=0}^{N-1} a_n t^n \right| \leq C|t|^N \qquad \text{for all } t \in \mathfrak{s}'.$$

TODO: The GENERAL CASE: see [Lod14] page 28 □

[a] For $t = re^{i\varphi}$ is $\sqrt{t} = \sqrt{r} e^{i\frac{\varphi}{2}}$ with $-\pi < \varphi < \pi$.

3 Systems and meromorphic connections

Here we will start with the definition of meromorphic connections on holomorphic bundles although we will only be interested in local information. For local description we will use **germs of meromorphic connections**, the coordinate dependent **systems** and **connection matrices**. Other coordinate independent approaches arise for example from the theory of (localized holonomic) $\mathcal{D}$-modules.

Meromorphic connections are introduced and discussed in many resources. A good starting point are Sabbah's lecture notes [Sab90]. More advanced resources are for example Sabbah's book [Sab07], Varadarajan's book [Var96] or the book [HTT08] from Hotta et al. The necessary facts about meromorphic connections are also stated in Boalch's paper [Boa01] (resp. his thesis [Boa99]), and Loday-Richaud's paper [Lod94].

Although the language of meromorphic connections is often preferred, we will use the language of systems most of the time. Systems are, for example, discussed in the book [HTT08] from Hotta et al, Loday-Richaud's paper [Lod94] and her book [Lod14] and Boalch's publications [Boa01; Boa99]. Another resource might Remy's paper [Rem14] be.

We will use all of the above mentioned resources in this chapter.

3.1 (Global) meromorphic connections

Let M be a riemanian surface and let $Z = k_1(a_1) + \cdots + k_m(a_m) > 0$ be an effective divisor[1] on M. It is sufficient to think $M = \mathbb{P}^1$ and $0 \in |Z|$[2], since we will only be interested in local information (at 0).

Let $\mathscr{M}$ be a holomorphic bundle over M i.e. a locally free $\mathcal{O}_M$-module of rank n. A global meromorphic connection, i.e. a meromorphic connection on $\mathscr{M}$, is then defined as follows.

Definition 3.1

A *meromorphic connection* $(\mathscr{M}, \nabla)$ *on* $\mathscr{M}$ *with poles on* Z is is defined by a $\mathbb{C}$-linear morphism of sheaves

$$\nabla : \mathscr{M} \to \Omega^1_M(*Z) \otimes \mathscr{M}$$

[1]The $a_i \in M$ are distinct points and the k_i are positive integers.
[2]If $Z = k_1(a_1) + \cdots + k_m(a_m)$ then $|Z| := \{a_1, \ldots, a_m\}$.

satisfying, for each open subset $U \underset{\text{op.}}{\subset} M$, the *Leibniz rule*

$$\nabla(fs) = f\nabla s + (df) \otimes s$$

for $s \in \Gamma(U, \mathscr{M})$ and $f \in \mathcal{O}_M(U)$. The *rank* of the meromorphic connection $(\mathscr{M}, \nabla)$ is defined to be the rank of the Bundle $\mathscr{M}$. It will usually denoted by the letter n.

Remark 3.1.1

Some authors use the factors k_i of the divisor Z to limit the pole orders at the points a_i. Since we do not need this restriction, we allow arbitrary pole orders. Denoted is this by the $*$ in $\Omega_M^1(*Z)$.

The *sheaf of meromorphic differential* 1*-forms on* M $\Omega_M^1(*Z)$ is defined as

$$\Omega_M^1(*Z) := \mathcal{O}_M(*Z) \underset{\mathcal{O}_M}{\otimes} \Omega_M^1 ,$$

where $\mathcal{O}_M(*Z)$ is the sheaf of functions, which are meromorphic along Z. A detailed definition of these sheaves can be found in Sabbah's book [Sab07] in Section 0.8 and Section 0.9.b

The word "global" used to emphasize that the connection is "on $\mathscr{M}$", i.e. not only a germ at some point of M, like it will be used later (cf. Remark 3.4).

In the definition above is the variant 'holomorphic bundle with meromorphic connection' chosen, like in Boalch's paper [Boa01]. There is also the interchanged description 'meromorphic bundle with holomorphic connection' which is for example used in Sabbah's book [Sab07].

By choosing a lattice of a meromorphic bundle (cf. [Sab07, Def.0.8.3]), one gets a holomorphic bundle but if the meromorphic bundle had a holomorphic connection, the induced connection on the lattice is no longer guaranteed to be holomorphic. Thus we obtain a meromorphic connection on a holomorphic bundle in our sense.

We will only look at unramified connections. Some of the following statements are only valid in the unramified case although we will not always mention this restriction.

Definition 3.2

[Sab07, p. 0.12.2] The connection $\nabla : \mathscr{M} \to \Omega_M^1 \otimes_{\mathcal{O}_M} \mathscr{M}$ is said to be *integrable* or *flat*, if its curvature vanishes, i.e. $R_\nabla \equiv 0$ where $R_\nabla := \nabla \circ \nabla : \mathcal{E} \to \Omega_M^2 \otimes_{\mathcal{O}_M} \mathcal{E}$ is a $\mathcal{O}_M$-linear morphism.

Remark 3.2.1

Here are all connections flat, since all connections will be of Dimension one.

3.2 Local expression of meromorphic connections and systems

We will only be interested in local information of meromorphic connections. This means that we look at a connection in a neighbourhood of 0 which has its unique singularity at 0. There are several ways of expressing the local information without the need of an fixed neighbourhood. We will either talk about germs (at 0) of meromorphic connections or systems, which depend on the choice of a trivialization. [Sab07, p. 28], [Boa99, p. 2] and [BV89, p. 11]

Proposition 3.3

A germ of a meromorphic connection $(\mathscr{M}, \nabla)$ is the sheaf-theoretic germ (at $t = 0$) and thus is given by a tuple $(\mathcal{M}, \nabla)$ where **[HTT08, Rem.5.2.4]** [Lod14, Def.4.2.1]

- $\mathcal{M}$ is the germ at 0 of the holomorphic bundle $\mathscr{M}$ and thus a $\mathbb{C}(\{t\})$-vectorspace of dimension n, since the ring of germs of meromorphic functions with poles at 0 is the ring $\mathbb{C}(\{t\})$, and
- $\nabla : \mathcal{M} \to \mathcal{M}$ is a $\mathbb{C}$-linear map, which satisfies the *Leibniz rule*

$$\nabla(fm) = \frac{d}{dt} f \cdot m + f\nabla(m)$$

for all $f \in \mathbb{C}(\{t\})$ and $m \in \mathcal{M}$.

Remark 3.3.1

Loday-Richaud calls the germs of meromorphic connections in her book [Lod14, Def.4.2.1] *differential modules.*

Remark 3.4

From now on we will only talk about **germs of** meromorphic connections $(\mathcal{M}, \nabla)$ and we will call them meromorphic connection. If we want to talk about meromorphic connections in the sense of Definition 3.1 we will emphasize this by the word 'global' or by talking about a meromorphic connection **on** M.

It will occasionally be convenient to omit the ∇ and simply call $\mathcal{M}$ the meromorphic connection.

Definition 3.5

A *(iso-)morphism of meromorphic connections* $\Phi : (\mathcal{M}, \nabla) \xrightarrow{\sim} (\mathcal{M}', \nabla')$ is a (iso-)morphism of $\mathbb{C}(\{t\})$-vectorspaces $\Phi : \mathcal{M} \xrightarrow{\sim} \mathcal{M}'$ which commutes with the connections, i.e. which satisfies $\nabla \circ \varphi = (\mathrm{id} \otimes \varphi) \circ \nabla$. [HTT08, Def.5.2.1]

Choose a $\mathbb{C}(\{t\})$-basis $\underline{e} = (e_1, e_2, \dots, e_n)$ of $\mathcal{M}$. Let $A = (a_{jk})_{j,k\in\{1,\dots,n\}}$ be a $n \times n$ matrix with entries in $\mathbb{C}(\{t\})$ such that the it describes the action of ∇ corresponding to the chosen basis, i.e. satisfies $\nabla e_k = -\sum_{1\le j\le n} a_{jk}(t) e_j$ for every $k \in \{1, \dots, n\}$. [Lod14, p. 65], [HTT08, p. 129]

Definition 3.6

The matrix A is called a *connection matrix* of $(\mathcal{M}, \nabla)$.

The connection ∇ is fully determined by A. Indeed, let $x = \sum_{0 \le j \le n} x_j e_j = \underline{e} \cdot X$ be an arbitrary element of $\mathcal{M}$, where $X = {}^t(x_1, x_2, \dots, x_n)$ is a column matrix. Applying ∇ to x using the Leibniz rule, yields

$$\begin{aligned} \nabla x &= \nabla \left(\underline{e} \cdot X \right) \\ &= \underline{e} \cdot dX + \nabla \underline{e} \cdot X \\ &= \underline{e} \left(dX - AX \right). \end{aligned}$$

Such that horizontal sections of $(\mathcal{M}, \nabla)$, i.e. sections which satisfy $\nabla x = 0$, correspond to solutions of

$$\frac{d}{dt} x = Ax \,. \tag{3.1}$$

Thus, with the connection ∇ and the chosen $\mathbb{C}(\{t\})$-basis $\underline{e}$, is naturally the differential operator $\triangle = d - A$ associated, which has order one and dimension n.

Definition 3.7

[MR91, Sec.4] We call (3.1), determined by the differential operator $\triangle = d - A$, a *germ of a meromorphic linear differential system*[3] of rank n, or just a *system*.

Proposition 3.7.1

Thus, the set of systems is isomorphic to the set

$$\operatorname{End}(E) \otimes \mathbb{C}(\{t\}) = \mathrm{gl}_n(\mathbb{C}(\{t\}))$$

of all connection matrices.

Such a system will be denoted by $[A] = d - A$ and we will call A the connection matrix of the system $[A]$.

TODO: possibly multivalued solutions $(\widetilde{K})$? [HTT08] on page 128

If we start with a system $[A]$ and we want a meromorphic connection $(\mathcal{M}, \nabla)$ which has A as connection matrix we can do this in the following way.

Proposition 3.8

[3]Martinet and Ramis call them in [MR91] *germs of meromorphic differential operators*.

If we start with either a system $[A]$ of rank n, or a connection matrix $A \in \mathrm{gl}_n(\mathbb{C}(\{t\}))$, we get a germ of a meromorphic connection via

$$(\mathcal{M}_A, \nabla_A) = (\mathbb{C}(\{t\})^n, d - A)$$

which has A as its connection matrix.

Proposition 3.9

Let $(\mathcal{M}_1, \nabla_1)$ and $(\mathcal{M}_2, \nabla_2)$ be two meromorphic connections with the connection matrices A_1 and A_2. A connection matrix of $(\mathcal{M}_1, \nabla_1) \oplus (\mathcal{M}_1, \nabla_1)$ is then given by the block-diagonal matrix $\mathrm{diag}(A_1, A_2)$.

Corollary 3.9.1

Let $(\mathcal{M}, \nabla)$ be an meromorphic connection, whose connection matrix can be written as $\mathrm{diag}(A_1, A_2)$. Then one can decompose the meromorphic connection corresponding to the block structure of A, i.e. $(\mathcal{M}, \nabla)$ is then isomorphic to the direct sum $(\mathcal{M}_{A^1}, \nabla_{A^1}) \oplus (\mathcal{M}_{A^2}, \nabla_{A^2})$

Proof. Use Proposition 3.8 to write the connection as

$$(\mathbb{C}(\{t\})^{n_1}, d - A_1) \oplus (\mathbb{C}(\{t\})^{n_2}, d - A_2)$$

and we want to show that it is isomorphic to

$$\left(\mathbb{C}(\{t\})^{n_1+n_2}, d - \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}\right) .$$

Denote by $A := \mathrm{diag}(A_1, A_2)$ the block-diagonal matrix build from A_1 and A_2. For every $j \in \{1, 2\}$ we have the inclusion $i_j : \mathbb{C}(\{t\})^{n_j} \hookrightarrow \mathbb{C}(\{t\})^{n_1+n_2}$ and the diagram

$$\begin{tikzcd}
\mathbb{C}(\{t\})^{n_j} \arrow[r, "d-A_j"] \arrow[d, "i_j"] & \mathbb{C}(\{t\})^{n_j} \arrow[d, "i_j"] \\
\mathbb{C}(\{t\})^{n_1+n_2} \arrow[r, "d-A"] & \mathbb{C}(\{t\})^{n_1+n_2}
\end{tikzcd}$$

which commutes, since the derivation commutes with the inclusion and the matrix A is build in the correct way, to satisfy $i_j(A_j x) = A_j(i_j(x))$:

$$\begin{aligned}
i_j(dx - A_j x) &= i_j(dx) - i_j(A_j x) \\
&= d(i_j(x)) - A_j(i_j(x)) \\
&= ((d - A) \circ i_j)(x) .
\end{aligned}$$

□

Remark 3.10

Let $(\mathcal{M}_1, \nabla_1)$ and $(\mathcal{M}_2, \nabla_2)$ be meromorphic connections. Then $\mathcal{M}_1 \otimes \mathcal{M}_2$ is endowed with the structure of a meromorphic connection by

$$\nabla(u_1 \otimes u_2) = \nabla_1 u_1 \otimes u_2 + u_1 \otimes \nabla_2 u_2$$

where $u_i \in \mathcal{M}_i$ (cf. [HTT08, p. 129]).

$\mathrm{Hom}_{\mathbb{C}(\{t\})}(\mathcal{M}_1, \mathcal{M}_2)$ is endowed with the structure of a meromorphic connection by

$$(\nabla\varphi)(u_1) = \nabla_2(\varphi(u_1)) - \varphi(\nabla_1 u_1)$$

where $\varphi \in \mathrm{Hom}_{\mathbb{C}(\{t\})}(\mathcal{M}_1, \mathcal{M}_2)$ and $u_i \in \mathcal{M}_i$.

Lemma 3.11

PROBLEM:how looks the connection matrix? Simply the sum? Let $(\mathcal{M}_1, \nabla_1)$ and $(\mathcal{M}_2, \nabla_2)$ be meromorphic connections with connection matrices A_1 and A_2. The connection matrix of $(\mathcal{M}_1 \otimes \mathcal{M}_2, \nabla)$ is then given by $A_1 + A_2$.

Proof. Let $u_1 \otimes u_2 \in \mathcal{M}_1 \otimes \mathcal{M}_2$.

$$\begin{aligned}\nabla(u_1 \otimes u_2) &= \nabla_1 u_1 \otimes u_2 + u_1 \otimes \nabla_2 u_2 \\ &= (\nabla_1 u_1) \otimes u_2 + u_1 \otimes \nabla_2 u_2\end{aligned}$$

□

Formalization

Let $[A]$ be a system. We view it as a formal system, by allowing formal solutions. TODO

Let $(\mathcal{M}, \nabla)$ be a meromorphic connection. The connection ∇ naturally extends to $\widehat{\mathcal{M}} := \mathcal{M} \otimes \mathbb{C}((t))$ and $\widetilde{\mathcal{M}}_\theta := \mathcal{M} \otimes \mathcal{A}_\theta$. TODO

As differential operator

[Lod14, Sec.4.2]

From the theory of ordinary differential equations we know that to (3.1) there is a equivalent ordinary differential equation of order n which can be written as

$$\underbrace{(a_n \partial_t^n + a_{n-1}\partial_t^{n-1} + \cdots a_1 \partial_t + a_0)}_{=:P} \cdot v = 0$$

where $a_i \in \mathbb{C}(\{t\})$. This leads to the theory of $\mathcal{D}$-modules.

See [BV83, Sec.1.4] for **ode of rank** n **to system**.

3.2.1 Transformation of systems

By *meromorphic*[4] *transformation*, or just *transformation*, of a system we mean a $\mathbb{C}(\{t\})$-linear change of the trivialization. Such a change is given by a matrix F in $G(\{t\}) := \mathrm{GL}_n\left(\mathbb{C}\{t\}[t^{-1}]\right)$ and the transformed connection matrix ${}^F\!A$ is obtained through the Gauge transformation [Lod14, Sec.4.3.1]

$$ {}^F\!A = (dF)F^{-1} + FAF^{-1} \,. \tag{3.2} $$

Let $B := {}^F\!A$ be the transformed matrix and rewrite the equation (3.2) to obtain the linear differential equation

$$ \frac{dF}{dt} = BF - FA $$

which will be denoted by $[A, B]$.

Remark 3.12

The system matrix B is obtained from A by transformation F if and only if F solves the linear differential system $[A, B]$.

If F is formal, i.e. $F \in G(\!(t)\!) := \mathrm{GL}_n\left(\mathbb{C}[\![t]\!][t^{-1}]\right)$, it will usually be denoted by $\widehat{F}$. The transformation of A by $\widehat{F} \in \mathrm{G}(\!(\mathrm{t})\!)$ is not guaranteed to have convergent entries and thus is no system. We denote by $\widehat{G}(A)$ the set of all *(applicable) formal transformations*

$$ \widehat{G}(A) := \left\{ \widehat{F} \in G(\!(t)\!) \mid {}^{\widehat{F}}\!A \text{ has convergent entries, i.e. } {}^{\widehat{F}}\!A \in G(\{t\}) \right\}. $$

Let $\widehat{F'} \in \widehat{G}(A)$ and $A' := {}^{\widehat{F'}}\!A$, then are the sets $\widehat{G}(A)$ and $\widehat{G}(A')$ related by

$$ \widehat{G}(A') = \widehat{G}(A)\widehat{F'}^{-1} = \left\{ \widehat{F} \in G(\{t\}) \mid \widehat{F}\widehat{F'} \in \widehat{G}(A) \right\}. $$

Remark 3.13

The Gauge transformations arise naturally from an isomorphism of meromorphic connections in the following way. If we start with

- two base choices $\mathcal{M} \xrightarrow{\sim} \mathbb{C}(\{t\})^n$ and $\mathcal{M}' \xrightarrow{\sim} \mathbb{C}(\{t\})^n$ and
- an isomorphism $\Phi : (\mathcal{M}, \nabla) \xrightarrow{\sim} (\mathcal{M}', \nabla')$ together with the corresponding base change $F \in G(\{t\})$

[4] We use the term meromorphic in the sens of convergent meromorphic. Otherwise we say formal meromorphic.

we have the following commutative diagram:

$$\begin{tikzcd}
\mathbb{C}(\{t\})^n \arrow[rrr, "F"] \arrow[ddd, "d-A"'] & & & \mathbb{C}(\{t\})^n \arrow[ddd, "d-B"] \\
& \mathcal{M} \arrow[ul] \arrow[r, "\Phi"] \arrow[d, "\nabla"] & \mathcal{M}' \arrow[ur] \arrow[d, "\nabla'"] & \\
& \mathcal{M} \arrow[dl] \arrow[r, "\Phi"] & \mathcal{M}' \arrow[dr] & \\
\mathbb{C}(\{t\})^n \arrow[rrr, "F"] & & & \mathbb{C}(\{t\})^n
\end{tikzcd}$$

Thus the commutation property for the outer rectangle reads

$$(d - B) \circ F = F \circ (d - A)$$

which is equivalent to

$$\frac{dF}{dt} = BF - FA.$$

Remark 3.14

The simple but useful rule

$$^{(F_2F_1)}A = {}^{F_2}\left({}^{F_1}A\right)$$

can be seen by calculation:

$$\begin{aligned}
^{(F_2F_1)}A &= d(F_2F_1)(F_2F_1)^{-1} + F_2F_1A(F_2F_1)^{-1} \\
&= \left(\left(dF_2\right) F_1 + F_2 \left(dF_1\right) \right) F_1^{-1}F_2^{-1} + F_2F_1AF_1^{-1}F_2^{-1} \\
&= \left(dF_2\right) F_2^{-1} + F_2 \left(dF_1\right) F_1^{-1}F_2^{-1} + F_2 \left({}^{F_1}A - \left(dF_1\right) F^{-1}\right) F_2^{-1} \\
&= \left(dF_2\right) F_2^{-1} + F_2{}^{F_1}AF_2^{-1} \\
&= {}^{F_2}\left({}^{F_1}A\right) .
\end{aligned}$$

Let us now introduce equivalence relations on systems, which will correspond to the isomorphism relations on meromorphic connections.

Definition 3.15

We define the *(formal) equivalence relation on the connection matrices (resp. on the corresponding systems)* as

$\boldsymbol{A}$ is (formally) equivalent to $\boldsymbol{B}$

if and only if

$\boldsymbol{B}$ is obtained from $\boldsymbol{A}$ by (formal) transformation.

The *class of a connection matrix* is the orbit under the Gauge transformation by $G(\{t\})$. The *formal class* is the orbit by $\widehat{G}(A)$.

Remark 3.15.1

Thus A is (formally) equivalent to B if and only if there is a (formal) solution of $[A, B]$.

The defined equivalence relations are compatible with the isomorphisms relations on meromorphic connections (cf. [HTT08, Lem.5.1.3]), i.e. the following proposition holds.

Proposition 3.16

Two germs of meromorphic connections are (formally) isomorphic if and only if their corresponding connection matrices are (formally) equivalent.

Definition 3.17

An *isotropy* of A^0 or of $[A^0]$ is a transformation $\widehat{F}$ which satisfies ${}^{\widehat{F}}A^0 = A^0$. Thus, the isotropies are the solutions of the system $[\operatorname{End} A^0] := [A^0, A^0]$. [Lod94, p. 853]

Let $G_0(A^0)$ denote the set of all isotropies of A^0.

Remark 3.17.1

The isotropies are, a priori, formal transformations. Loday-Richaud mentions in her paper [Lod94, p. 853], that $G_0(A^0)$ is actually a subgroup of $\mathrm{GL}_n(\mathbb{C}[1/t, t])$.

Lemma 3.17.1

Two formal transformations $\widehat{F}_1$ and $\widehat{F}_2$ take A^0 into equivalent matrices ${}^{\widehat{F}_1}A^0$ and ${}^{\widehat{F}_2}A^0$ if and only if there exists isotropy $f_0 \in G_0(A^0)$ such that $\widehat{F}_1 = \widehat{F}_2 f_0$ (cf. [Lod94, p. 854]). [Lod94, p. 854]

The meromorphic connections are distinguished into regular and irregular meromorphic connections. TODO: more filltext! [HTT08, Defn.5.1.6]

Definition 3.18

A connection with connection matrix A has *regular singularity* at 0 if there exists a convergent transformation, by which A is obtained from a matrix with at most a simple pole at $t = 0$. Otherwise, the singularity is called *irregular*. [Sab07, p. 86]

Remark 3.18.1

This implies that, if A has irregular (resp. regular) singularity, then also all meromorphic equivalent matrices ${}^{F}A$ have irregular (resp. regular) singularity.

Theorem 3.19

Let $(\mathcal{M}, \nabla)$ be a regular singular meromorphic connection and A its connection matrix. Then there exists a formal matrix $F \in G(\{t\})$ such that after transformation by F the matrix $B = {}^F A$ is constant i.e. ${}^F A \in \mathrm{GL}_n(\mathbb{C})$ (cf. [Sab07, Thm.II.2.8] or [HTT08, Sec.5.1.2]).

3.2.2 Fundamental solutions and the monodromy of a system

[Lod14, Sec.4.3.2], [HTT08, p. 130], **[BJL79, p. 50]** ,[Sib90, p. 26]

It is well known in the theory of linear ODEs that the solutions to a system like (3.1) form a vector space of dimension n over $\mathbb{C}$, i.e. if $x_0(t)$ and $x_{00}(t)$ are two solution of (3.1) and c_1, $c_2 \in \mathbb{C}$, then is also $c_1 x_0(t) + c_2 x_{00}(t)$ a solution of the same system.

Definition 3.20

[Boa99, 4f]

A *fundamental matrix of (formal) solutions* or *(formal) fundamental solution* $\mathcal{Y}$ on a sector $\mathfrak{s}$ of the system $[A]$ is an invertible $n \times n$ matrix (with formal entries), which solves $[0, A]$ on $\mathfrak{s}$.

This means that the columns of $\mathcal{Y}$ have to be n $\mathbb{C}$-linearly independent solutions of the system $[A]$ on $\mathfrak{s}$.

Remark 3.20.1

- Some authors PROBLEM:? introduce multi-valued solutions, to avoid the restriction to sectors.

  Multi-valued are functions, which are not single-valued and *single-valued* is a function f, which satisfies

  $$f(t) = f(t \exp(2\pi i)) \qquad \text{whenever both sides are defined.}$$

  The function $t \to t^\alpha$, for example, is single-valued whenever $\alpha \in \mathbb{Z}$.

The Stokes phenomenon, which will be discussed in the Chapter 4, results from the fact that there is always a formal fundamental solution, which solves a system on the full arc S^1. But holomorphic solutions, which are asymptotic to the formal solution, may exist only on small sub-sectors of S^1. The Stokes structures are then roughly the information about the relation between the holomorphic solutions on the overlap of two sectors.

Remark 3.21

If the trivialization is changed by F (resp. $\widehat{F}$) the fundamental solution $\mathcal{Y} \in G(\{t\})$ changes to $F\mathcal{Y}$ (resp. $\widehat{F}\mathcal{Y}$). (cf. [Lod14, Thm.4.3.1] or [FJ09, p. 2.1.3]) [FJ09, p. 2.1.3]

Choose a fundamental solution $\mathcal{Y}$. Then analytic continuation along a closed path γ in $M\backslash Z$ provides another fundamental solution. [HTT08, p. 130], [Heu10, p. 6]

Definition 3.22

Let $[A]$ be a system with fundamental solution $\mathcal{Y}$. The analytic continuation of $\mathcal{Y}$ along a small circle around $t = 0$ yields the fundamental solution

$$\lim_{s \to 2\pi} \mathcal{Y}(e^{\sqrt{-1}s}t) = \mathcal{Y}(t)L$$

where $L \in \mathrm{GL}_n(\mathbb{C})$ is called the TODO: (formal)? *monodromy matrix* of $[A]$. PROBLEM: wie passt das zu normal

3.3 Formal classification

In every formal equivalence class of meromorphic connections, there are some meromorphic connections of special form, which we will call models. They are not unique but all of them, which are formally isomorphic to a given meromorphic connection, lie in the same convergent equivalence class. In fact, every element of this convergent equivalence class will be a model in our definition. [Lod14, Thm.4.3.1]

The models will be used to classify meromorphic connections up to formal isomorphism. Two meromorphic connections are formally isomorphic, if they are isomorphic to the (up to convergent isomorphism) same model. In the language of systems will normal forms be the objects, which correspond to models.

This is essentially given by the Levelt-Turittin Theorem, which states that each meromorphic connection is, after potentially needed ramification, formally isomorphic to such a model. Thus the Levelt-Turittin Theorem solves the formal classification problem.

3.3.1 In the language of meromorphic connections: models

We will start by defining the models in the language of meromorphic connections. This approach is discussed by Sabbah in [Sab90] and in a more general context in [Sab07, Sec.II.5].

Definition 3.23

For $\alpha \in \mathbb{C}$, define the *elementary regular meromorphic connection of rank one* $\mathcal{N}_{\alpha,0}$ as the germ

$$(\mathcal{N}_{\alpha,0}, \nabla) := \left(\mathbb{C}(\{t\}), d + \frac{\alpha}{t}\right).$$

This corresponds to the system satisfied by $t^{-\alpha}$.

[Sab07, Defn.II.2.5]

An *elementary regular model of arbitrary rank* is a meromorphic connection which has a basis, in which the connection Matrix can be written as

$$\frac{1}{t}(\alpha \operatorname{id} + N)$$

where $N \in \mathrm{gl}_{d+1}(\mathbb{C})$ is a nilpotent matrix. If $\alpha \operatorname{id} + N$ is a single Jordan Block, we denote the corresponding connection by $\mathcal{N}_{\alpha,d}$.

The Corollary [Sab07, Cor.II.2.9] in Sabbah's book states, that every germ of a regular meromorphic connection $(\mathcal{R}, \nabla)$ is isomorphic to some direct sum

$$(\mathcal{R}, \nabla) \cong \bigoplus_{\alpha,d} (\mathcal{N}_{\alpha,d}, \nabla).$$

of elementary regular meromorphic connections. The rough idea to proof this is to use Theorem 3.19 and assume that the connection matrix A of $(\mathcal{R}, \nabla)$ is constant. Since the Gauge transformation by a constant matrix $F \in \mathrm{GL}(\mathbb{C})$ is given by ${}^F A = FAF^{-1}$, i.e. a similarity transformation, one can find a transformation F such that ${}^F A$ is in Jordan normal form. One can then use Corollary 3.9.1 to decompose the corresponding to the structure of the Jordan normal form and obtain the result. This solves the classification problem, for regular meromorphic connections. For a detailed analysis of regular meromorphic connections see Sabbah's book [Sab07, Sec.II.2] or the book [HTT08, Sec.5.2] from Hotta et al.

The simplest irregular meromorphic connections will be called elementary and are defined below.

Definition 3.24

For a $\varphi \in \mathbb{C}(\{t\})$ we use $\mathcal{E}^\varphi$ to denote the meromorphic connection

$$(\mathcal{E}^\varphi, \nabla) := (\mathbb{C}(\{t\}), d - \varphi').$$

This corresponds to the system satisfied by the function e^φ, since $(d - \varphi')e^\varphi = 0$.

Proposition 3.24.1

$\mathcal{E}^\varphi$ is determined by the class of φ in $\mathbb{C}(\{t\})/\mathbb{C}\{t\} \cong t^{-1}\mathbb{C}[t^{-1}]$. In the following, we will only assume, that φ is always given by the unique ambassador which has no holomorphic part.

[Sab07, Def.II.5.2]

A germ $(\mathcal{M}, \nabla)$ is called *elementary* if it is isomorphic to some germ $(\mathcal{E}^\varphi, \nabla) \otimes (\mathcal{R}, \nabla)$ where $(\mathcal{R}, \nabla)$ has regular singularity at $\{0\}$, i.e. is isomorphic to a direct sum of regular elementary meromorphic connections. [Sab07, II.2.f]

Definition 3.25

A germ $(\mathcal{M}', \nabla')$ is a *model* if there exists after ramification $\mathcal{M} = \pi^*\mathcal{M}'$ by π a isomorphism

$$\lambda : (\mathcal{M}, \nabla) \xrightarrow{\cong} \bigoplus_{\varphi} \mathcal{E}^\varphi \otimes \mathcal{R}_\varphi$$

to a direct sum of elementary meromorphic connections where the $\varphi \in t^{-1}\mathbb{C}[t^{-1}]$ are pairwise distinct and the $\mathcal{R}_\varphi$ have regular singularity.

Remark 3.25.1

Here it is not necessary to understand ramification, since we restrict to the unramified case.

The important theorem here is the Levelt-Turittin Theorem, which solves the formal classification problem.

Theorem 3.26: Levelt-Turittin decomposition

To each germ $(\mathcal{M}', \nabla')$ of a meromorphic connection there exists, after potentially needed pullback $\pi^*\mathcal{M}' =: \mathcal{M}$ by some suitable ramification $t = z^q$ of order $q \geq 1$, a **formal** isomorphism

$$\widehat{\lambda} : \widehat{\mathcal{M}} \xrightarrow{\cong} \widehat{\mathcal{M}}^{nf} := \widehat{\mathcal{O}}_M \otimes \mathcal{M}^{nf}$$

to a model $\mathcal{M}^{nf}$. We then call $\mathcal{M}^{nf}$ a *formal decomposition* or *formal model* of $\mathcal{M}$ or $\mathcal{M}'$.

Remark 3.26.1

But there is in general **no** lift of the isomorphism $\widehat{\lambda}$, i.e. there is no isomorphism making the diagram

$$\begin{tikzcd} \mathcal{M} \arrow[r, dotted, "?"] \arrow[d] & \mathcal{M}^{nf} \arrow[d] \\ \widehat{\mathcal{M}} \arrow[r, "\widehat{\lambda}"] & \widehat{\mathcal{M}}^{nf} \end{tikzcd}$$

commutative. But sectorwise, there are lifts given by the main asymptotic existence theorem (cf. Theorem 3.36).

Proofs of this theorem can be found in multiple places, for example in [Sab90, Thm.5.4.7] or in [Hub13].

3.3.2 In the language of systems: normal forms

[Rem14] ,[Sib90, p. 146] Here we will see that the fundamental solution of a model can be written in a special form and that this property characterizes models. This will be used to define normal forms which will be models on the level of systems.

Definition 3.27

The fundamental solution corresponding to a model (resp. a normal form, see Definition 3.32) will be called a *normal solution*.

Let $(\mathcal{M}, \nabla)$ be an unramified meromorphic connection which is via some $\widehat{\lambda}$ formally isomorphic to some model $\mathcal{M}^{nf} = \bigoplus_{j=1}^{s} \mathcal{E}^{\varphi_j} \otimes \mathcal{R}_j$, where we assume that $\varphi_j \in t^{-1}\mathbb{C}[t^{-1}]$ (cf. Corollary 3.24.1). For every j is a connection matrix, corresponding to $\mathcal{R}_j$ given by $\frac{1}{t}L_j$, where L is in Jordan normal form. A connection matrix to $\mathcal{E}^{\varphi_j} \otimes \mathcal{R}_j$ is then by $q_j'(t^{-1}) \cdot \mathrm{id}_{n_j} + \frac{1}{t}L_j$ given TODO: (cf. ???), where $q_j(t^{-1}) = \varphi_j(t)$ PROBLEM:repl. all φ by q? is a polynomial in t^{-1} without constant term and $q_j'(t^{-1}) = \frac{d}{dt}\varphi_j(t)$[5]. A connection matrix for $\mathcal{M}^{nf}$ is then obtained by

$$\begin{aligned} A^0 &= \bigoplus_{j=1}^{s} \left(q_j'(t^{-1}) \cdot \mathrm{id}_{n_j} + \frac{1}{t} L_j \right) \\ &= \bigoplus_{j=1}^{s} q_j'(t^{-1}) \cdot \mathrm{id}_{n_j} + \frac{1}{t} \bigoplus_{j=1}^{s} L_j \\ &= Q'(t^{-1}) + \frac{1}{t} L \,, \end{aligned}$$

where $Q(t^{-1}) := \bigoplus_{j=1}^{s} q_j(t^{-1}) \cdot \mathrm{id}_{n_j}$ and $L := \bigoplus_{j=1}^{s} L_j$.

Definition 3.28

Let L be a block diagonal matrix $L = \bigoplus_{j=1}^{s} L_j$, where the L_j are of size $n_j \times n_j$ and let Q be a diagonal matrix. We will say that *the block structure of L is finer then the structure of Q* if there are q_j's such that $Q = \bigoplus_{j=1} q_j \cdot \mathrm{id}_{n_j}$.

Remark 3.28.1

1. The matrices Q and L defined above clearly satisfy this condition.
2. When this condition is satisfied do L and Q commute.

Proposition 3.29

Let $L \in \mathrm{GL}_n(\mathbb{C})$ be constant and $Q(t^{-1}) = \mathrm{diag}(q_1(t^{-1}), \dots, q_n(t^{-1}))$ a diagonal matrix of polynomials in t^{-1} where L is in Jordan normal form and its block structure is finer then the structure of the derivative $Q(t^{-1})$.

[5]By abuse of notation we denote by $q_j'(t^{-1})$ the derivation $\frac{d}{dt}(q_j(t^{-1})) = \left(\left(\frac{d}{dt}q_j\right) \circ t^{-1}\right) \cdot \frac{d}{dt}t^{-1}$. The same does apply for $Q'(t^{-1})$.

The matrix $\mathcal{Y}_0 := t^L e^{Q(t^{-1})}$ is then a fundamental solution of the system determined by the matrix

$$A^0 = Q'(t^{-1}) + L\frac{1}{t}\,.$$

Proof. It is easy to see (or cf. [Bal00, Appendix C.1]) that the block structure of t^L is finer then the structure of $Q'(t^{-1})$ and thus $t^L Q'(t^{-1})t^{-L} = Q'(t^{-1})$, since the block structure is preserved under the exponential and derivation. Thus we can prove that $\mathcal{Y}_0$ is a matrix consisting of solutions:

$$\begin{aligned}
\frac{d}{dt}\mathcal{Y}_0 &= \frac{d}{dt}\left(t^L e^{Q(t^{-1})}\right)\\
&= t^L \frac{d}{dt} e^{Q(t^{-1})} + \frac{d}{dt} t^L e^{Q(t^{-1})}\\
&= t^L Q'(t^{-1}) e^{Q(t^{-1})} + L\frac{1}{t} t^L e^{Q(t^{-1})}\\
&= \left(t^L Q'(t^{-1}) t^{-L} + L\frac{1}{t}\right) t^L e^{Q(t^{-1})}\\
&= \left(Q'(t^{-1}) + L\frac{1}{t}\right) t^L e^{Q(t^{-1})}\\
&= A^0 \mathcal{Y}_0\,.
\end{aligned}$$

The invertability condition is clear. □

Remark 3.30

If one starts without the assumption on the structure, one only obtains

$$A = t^L Q'(t^{-1}) t^{-L} + L\frac{1}{t}$$

as possible system matrix. But the obtained matrix may not define a system, since it could contain entries, which are not in $\mathbb{C}(\{t\})$.

A simple example where this happens is given by

- $L = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$ and
- $Q = \begin{pmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{pmatrix}$,

with $\varphi_1 \neq \varphi_2$. The matrix A is then

$$\begin{aligned}
A &= t^L Q'(t^{-1}) t^{-L} + L\frac{1}{t}\\
&= \begin{pmatrix} \varphi_1' & (-\varphi_1' + \varphi_2')\ln(t) \\ 0 & \varphi_2' \end{pmatrix} + \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}\frac{1}{t} \notin \mathrm{gl}_n(\mathbb{C}(\{t\}))\,.
\end{aligned}$$

To $\widehat{\lambda}$ and the (implicitly) used choices of bases corresponds a formal transformation $\widehat{F}$ and thus we have

- a fundamental solution $\widehat{F}t^L e^{Q(t^{-1})}$ (cf. Remark 3.21) and
- a connection matrix ${}^{\widehat{F}}A^0$

of the meromorphic connection $(\mathcal{M}, \nabla)$, which was defined at the beginning of this subsection.

Corollary 3.31

From the Levelt-Turittin theorem we deduce that every meromorphic connection $(\mathcal{M}, \nabla)$ and thus every system $[A]$ has a fundamental solution of the form

$$\mathcal{Y} = \widehat{F}t^L e^{Q(t^{-1})}$$

where $\widehat{F}$ is a formal transformation, solving the differential equation corresponding to the formal isomorphism $(\mathcal{M}^{nf}, \nabla^{nf}) \to (\mathcal{M}, \nabla)$. On the other hand is a system, which has $\mathcal{Y}$ as fundamental solution, given by the connection matrix

$$\bar{A} = {}^{\widehat{F}}\left(Q'(t^{-1}) + L\frac{1}{t}\right) .$$

Remark 3.31.1

It is always possible to permutate the columns of a fundamental solution by

$$P^{-1}\mathcal{Y}P = \widehat{F}t^{P^{-1}LP}e^{P^{-1}Q(t^{-1})P}$$

with a permutation matrix P and obtain another fundamental solution for the same system (cf. [Lod14, p. 73]).

In the special case when $\widehat{\lambda}$ is a convergent isomorphism, i.e. in the case, when A is already a model, we see that it has a fundamental solution of the form $Ft^L e^{Q(t^{-1})}$, where F is a convergent transformation corresponding to $\widehat{\lambda}$. In fact are the models uniquely characterized as the meromorphic connections, with a fundamental solution which can be written as $\mathcal{Y}_0 = Ft^L e^{Q(t^{-1})}$ and we will use this fact to say when a system is something corresponding to a model.

Definition 3.32

Let $[A]$ be a system. We call $[A]$ (or A) a *normal form* if a fundamental solution of $[A]$ can be written as

$$\mathcal{Y}_0(t) = Ft^L e^{Q(t^{-1})}$$

with

- an *irregular part* $e^{Q(t^{-1})}$ of $\mathcal{Y}_0$ determined by

$$Q(t^{-1}) = \bigoplus_{j=1}^{s} q_j(t^{-1})\,\mathrm{id}_{n_j} = \mathrm{diag}(\underbrace{q_1,\dots,q_1}_{n_1\text{-times}}, q_2,\dots,q_s)$$

  where the $q_i(t^{-1})$ are polynomials in $\frac{1}{t}$ (or in a fractional power $\frac{1}{s} = \frac{1}{t^{1/p}}$ of $\frac{1}{t}$ for the ramified case) such that $q_j(0) = 0$, i.e. without constant term,

- a constant matrix $L \in \mathrm{gl}_n(\mathbb{C})$ called the *matrix of formal monodromy*, where t^L means $e^{L\ln t}$ and

- a (convergent) transformation $F \in G(\{t\})$.

  Remark 3.32.1

  By changing the basis via F^{-1} we obtain from $[A^0]$ the equivalent system $[{}^{F^{-1}}A^0]$ with the normal solution $F^{-1}\mathcal{Y}_0(t) = t^L e^{Q(t^{-1})}$. Thus it is always possible to assume that the transformation matrix F is trivial.

The normal forms will often be denoted A^0. If A is formally equivalent to a normal form A^0 we say that A^0 *is a normal form for* A and for $[A]$.

Corollary 3.33

Since we only look at unramified connections, we are able to assume that L is in Jordan normal form and that it has a block structure, which is finer then the structure of $Q = \bigoplus_{j=1}^{s} q_j(t^{-1}) \cdot \mathrm{id}_{n_j}$ (cf. [Rem14, Sec.1] or [MR91, Sec.4]).

The following corollary enables us to use normal forms in place of models.

Corollary 3.34

A meromorphic connection $(\mathcal{M}, \nabla)$ is a model if and only if its connection matrix A is a normal form.

3.4 The main asymptotic existence theorem (M.A.E.T)

Classical:

- [Lod14, Thm.4.4.1]
- [PS03, Thm.7.10][PS03, Thm.7.12]
- [Was02, Thm.IV.12.1]
- [Var96, 5.3.Thm.1]
- [Bal00, p. 207]: Some historical remarks
- [BJL79, Thm.A]

Sheafical:

- [Sab90, Thm.2.3.1]
- [Lod14, Sec.4.4]

[Bal00, p. 207]

Here we want to state the main asymptotic expansion theorem (or often M.A.E.T.) which is essentially a deduction from the Borel-Ritt Lemma. It states that to every formal solution of a system of meromorphic differential equations and every sector with sufficiently small opening, one can find a holomorphic solution of the system having the formal one as its asymptotic expansion.

Definition 3.35

[Lod94, p. 855]

Let A be via $\widehat{F}$ formally equivalent to A^0. We call F a *lift of* $\widehat{F}$ *on* $I \subset S^1$ if

- $F \sim_I \widehat{F}$ (cf. Page 11) and
- F satisfies the same system $[A^0, A]$ as $\widehat{F}$.

The following theorem, often called the main asymptotic existence theorem, can be for example found in as Theorem A in the paper [BJL79] from Balser, Jurkat and Lutz, Theorem 3.1 in Boalch's paper [Boa01] or Theorem 4.4.1 in Loday-Richaud's Book [Lod14].

Theorem 3.36: M.A.E.T

To every $\widehat{F} \in G((t))$ and to every small enough arc $I \subsetneq S^1$ there exists a lift F on I.

Remark 3.36.1

[Lod14, Thm.4.4.1]

If we write the system $[A^0, A]$ as a differential operator $\triangle$. The theorem then states that $\triangle$ acts surjectively on the sheaf $\mathcal{A}^{<0}$, i.e. the sequence

$$\mathcal{A}^{<0} \xrightarrow{\triangle} \mathcal{A}^{<0} \longrightarrow 0$$

is an exact sequences of sheaves of $\mathbb{C}$-vector spaces.

(cf. [Mal91, App.1;Thm.1])

Remark 3.37

PROBLEM: Think about it? In the language of meromorphic connections is this theorem sometimes called *sectorial decomposition* and stated for example in [Sab07, Thm.II.5.12] and [Sab90, Sec.II.2.4]:

Theorem 3.37.1: Sectorial decomposition

PROBLEM: needs more defns Let $(\mathcal{M}, \nabla)$ be a meromorphic connection and let $\widehat{\lambda} : \widehat{\mathcal{M}} \to \widehat{\mathcal{M}}^{nf}$ be the isomorphism given by Theorem 3.26 together with the model $\mathcal{M}^{nf}$. There exists then, for any $e^{i\theta^0} \in S^1$, an isomorphism $\widetilde{\lambda}_{\theta^0} : \widetilde{\mathcal{M}}_{\theta^0} = \mathcal{A}_{\theta^0} \otimes \mathcal{M} \to \widetilde{\mathcal{M}}^{nf}_{\theta^0}$ lift $\widehat{\lambda}$, that is, such that the following diagram

$$\begin{array}{ccc} \widetilde{\mathcal{M}}_{\theta^o} & \xrightarrow{\widetilde{\lambda}_{\theta^o}} & \widetilde{\mathcal{M}}^{nf}_{\theta^o} \\ \downarrow & & \downarrow \\ \widehat{\mathcal{M}} & \xrightarrow{\widehat{\lambda}} & \widehat{\mathcal{M}}^{nf} \end{array}$$

commutes

3.5 The classifying set

We want to understand the Set $\left\{ \left[(\mathcal{M}, \nabla) \right] \right\}$ of the (convergent) isomorphism classes of all meromorphic connections. Since we can use the formal classification (cf. Section 3.3) and we know that the convergent classification is finer than the formal classification, we can reduce the problem by fixing a model $(\mathcal{M}^{nf}, \nabla^{nf})$ with the corresponding normal form A^0. Thus we can restrict to the subset

$$\begin{aligned} {}^0C(\mathcal{M}^{nf}, \nabla^{nf}) = \Big\{ \left[(\mathcal{M}, \nabla) \right] \mid & \text{ there exists a formal isomorphism} \\ & \widehat{f} : (\widehat{\mathcal{M}}, \widehat{\nabla}) \xrightarrow{\sim} (\widehat{\mathcal{M}}^{nf}, \widehat{\nabla}^{nf}) \Big\} \end{aligned}$$

of all isomorphism classes of meromorphic connections, which are formally isomorphic to $(\mathcal{M}^{nf}, \nabla^{nf})$. This is the set that we will be calling the *classifying set (to* $(\mathcal{M}^{nf}, \nabla^{nf})$*)* and we will also denote it also by ${}^0C(A^0)$, if we are using the language of systems.

Remark 3.38

Note that we classify

> meromorphic connections within fixed **formal meromorphic classes, modulo meromorphic equivalence**.

Whereas for example Boalch in [Boa01; Boa99] classifies

> meromorphic connections within fixed **formal analytic classes, modulo analytic equivalence**

as it was done in the older literature. This makes no difference, since the resulting classifying sets are isomorphic (cf. [Boa99] or [BV89]).

This distinction relates to the difference between **'regular singular'** connections and **'logarithmic'** connections (cf. [Boa99, Rem.1.41]).

[Boa99, p. 6] ([Boa01, p. 19]) [Lod94, p. 852] [Sab07, p. 111] [BV83]

It is convenient to look at the slightly larger space of isomorphism classes of *marked (meromorphic) pairs*

$$\mathcal{H}(\mathcal{M}^{nf},\nabla^{nf}) = \left\{\left[(\mathcal{M},\nabla,\widehat{f})\right] \mid \widehat{f} : (\widehat{\mathcal{M}},\widehat{\nabla}) \xrightarrow{\sim} (\widehat{\mathcal{M}}^{nf},\widehat{\nabla}^{nf})\right\}$$

in which we also handle the additional information of the formal isomorphism by which the meromorphic connection is isomorphic to the model. The isomorphisms of marked pairs are defined as expected:

Definition 3.39

Two germs $(\mathcal{M},\nabla,\widehat{f})$ and $(\mathcal{M}',\nabla',\widehat{f}')$ are isomorphic if there exists an isomorphism $g : (\mathcal{M},\nabla) \xrightarrow{\sim} (\mathcal{M}',\nabla')$ such that $\widehat{f} = \widehat{f}' \circ \widehat{g}$.

Remark 3.39.1

Sabbah states in [Sab07, p. 111] that such an isomorphism is then unique.

Equivalently, one can talk in terms of systems. We then denote by

$$\mathrm{Syst}_m(A^0) := \left\{[A] \mid A = {}^{\widehat{F}}A^0 \text{ for some } \widehat{F} \in G(\!(t)\!)\right\}$$

the set of systems formally meromorphic equivalent to A^0. Since we use meromorphic equivalences, in contrast to [Boa01; Boa99], we denote that in Syst_m by the subscript $_m$. Thus ${}^0C(\mathcal{M}^{nf},\nabla^{nf})$ corresponds to the set ${}^0C(A^0) := \mathrm{Syst}_m(A^0)/G(\{t\})$ of meromorphic classes which are formally equivalent to A^0. Analogous, $\mathcal{H}(\mathcal{M}^{nf},\nabla^{nf})$ corresponds to the set $\mathcal{H}(A^0)$ of equivalence classes, i.e. orbits of $G(\{t\})$, in

$$\widehat{\mathrm{Syst}}_m(A^0) := \left\{\left(A,\widehat{F}\right) \mid A = {}^{\widehat{F}}A^0 \text{ for some } \widehat{F} \in G(\!(t)\!)\right\}.$$

The isomorphisms defined in Definition 3.39 translate into the following equivalence relation:

two marked pairs $(A,\widehat{F})$ and $(A',\widehat{F}')$ are equivalent, if and only if there is a base change H such that PROBLEM: $H \in G(\{t\})$? or asymptotic functions?

1. $A' = {}^{H}A$, i.e. H is a solution of $[A, A']$, and
2. $\widehat{F}' = \widehat{H}\widehat{F}$ (cf. [BV89, p. 71]).

In the following diagram the first condition is equivalent to the commutativity of the green square. The second property corresponds to the commutation property of the top (resp. bottom) triangle on the level of asymptotic expansions.

$$\begin{tikzcd}
 & & \mathbb{C}(\{t\})^n \arrow[dr, "H"] \arrow[dd, "d-A"] & \\
\mathbb{C}(\{t\})^n \arrow[urr, "\widehat{F}"] \arrow[rrr, "\widehat{F}'"] \arrow[dd, "d-A^0"] & & & \mathbb{C}(\{t\})^n \arrow[ddd, "d-A'"] \\
 & & \mathbb{C}(\{t\})^n \arrow[ddr, "H"] & \\
\mathbb{C}(\{t\})^n \arrow[urr, "\widehat{F}"] \arrow[drrr, "\widehat{F}'"] & & & \\
 & & & \mathbb{C}(\{t\})^n
\end{tikzcd}$$

Lemma 3.40

Since $G_0(A^0)$ is by definition the stabilizer of A^0 (cf. Definition 3.17) and $\mathrm{Syst}_m(A^0)$ is the corresponding orbit we can use the Proposition 3.1 of the book [Wie64] from Wielandt and deduce http://mathworld.wolfram.com/Stabilizer.html

$$\mathrm{Syst}_m(A^0) \cong \widehat{G}(A^0)/G_0(A^0)\,.$$

Corollary 3.40.1

Thus the *set of meromorphic classes of systems formally equivalent to* A^0 are just the orbits of $G(\{t\})$ in $\mathrm{Syst}_m(A^0)$ that is [BV83, Eq.1.9b]

$${}^0C(A^0) \cong G(\{t\})\backslash\widehat{G}(A^0)/G_0(A^0)$$

whereas the *set of meromorphic classes of marked pairs* $\mathcal{H}(A^0)$ *of* $[A^0]$ is canonically isomorphic to the left quotient $G(\{t\})\backslash\widehat{G}(A^0)$ (cf. [Boa99, Lem.1.17]).

The group $G_0(A^0)$ is easy to compute and is often trivial. In fact are the elements of $G_0(A^0)$ block-diagonal corresponding to the structure of Q, see [Lod14, p. 77]. Thus the structure of ${}^0C(A^0)$ is easily deduced from the structure of $G(\{t\})\backslash\widehat{G}(A^0)$.

4 Stokes Structures

- Characterized completely

Let $(\mathcal{M}^{nf}, \nabla^{nf})$ be a fixed model with the corresponding normal form A^0 and let us also fix a normal solution $\mathcal{Y}_0$ of A^0. Here we want to introduce the Stokes structures, which will characterize the isomorphism classes of meromorphic connections uniquely, i.e. an space which is isomorphic to the set $\mathcal{H}(A^0)$ of isomorphism classes of marked pairs. A great overview of this topic is given by Varadarajan in [Var96].

One of the most important theorems, which is fundamental for the whole chapter, is the Malgrange-Sibuya Theorem. It states that the classifying set $\mathcal{H}(A^0)$ is via an map exp isomorphic to the first non abelian cohomology space $H^1(S^1; \Lambda(A^0)) =: \mathcal{S}t(A^0)$ of the Stokes sheaf $\Lambda(A^0)$ and will be proven in the first section. In Section 4.3 we will improve the Malgrange-Sibuya Theorem by showing that each 1-cohomology class in $\mathcal{S}t(A^0)$ contains a unique 1-cocycle of a special form called *the Stokes cocycle*. We will further show that such cocycles can be identified with their germs at some special directions, i.e. anti-Stokes directions. These germs are called Stokes germs and for an anti-Stokes direction α do these germs form the Stokes groups $\mathrm{Sto}_\alpha(A^0)$ (cf. Section 4.2). The morphism, which maps each product of Stokes germs to its corresponding 1-cocycle will be denoted by h. This will be further improved in Section 4.6, where we will collect multiple Stokes germs to their product to obtain a more robust version of the Stokes space.

If one introduces the map g, which arises from the theory of summation[1] and takes an equivalence class and returns the corresponding Stokes cocycle in a canonically way one obtains the following commutative diagram.

$$\begin{tikzcd}
\mathcal{H}(A^0) \arrow[r, "\exp"] \arrow[d, "g"'] & \mathcal{S}t(A^0) \\
\prod_{\theta \in \mathbb{A}} \mathrm{Sto}_\theta(A^0) \arrow[ur, "h"'] &
\end{tikzcd}$$

This diagram will be enhanced in Section 4.6 by adding a couple of isomorphisms.

In the first section of this chapter we will use Sabbah's book [Sab07, section II] as its main resource together with [BV89] for the main proof. In the Sections 4.2 and 4.3 will Loday-Richaud's paper [Lod94] and her book [Lod14, Sec.4] be useful. Stokes groups are also discussed Boalch's paper [Boa01] (resp. his thesis [Boa99]) which looks only at

[1] We will not use the theory of summation and only think of it as a black box, although we will roughly mention some facts in the Section 4.5.

the single-leveled case or the paper [MR91] from Martinet and Ramis. Another very important paper is [BJL79] from Balser, Jurkat and Lutz. One may also want to have a look at Chapter 9 of [PS03] from van der Put and Singer.

4.1 Stokes structures: Malgrange-Sibuya isomorphism

[Lod94, Thm.I.2.1], [Lod14, Thm.4.3.9], [Sab07, Thm.II.6.2]

Here we will look at the set $\mathcal{H}(A^0)$ of isomorphism classes of marked pairs and we will proof that it is isomorphic TODO: as... to the first non abelian cohomology set $H^1(S^1; \Lambda(A^0))$ which will be denoted as $\mathcal{S}t(A^0)$. If we talk about cocycles or cochains, we will in the following always mean 1-cocycles or 1-cochains.

Let us first define the Stokes sheaf $\Lambda(A^0)$ on S^1 as the sheaf of flat isotropies of $[A^0]$.

Definition 4.1

The Stokes sheaf $\Lambda(A^0)$ of A^0 is the sheaf of groups defined on S^1 whose stalk at any point $\theta \in S^1$ is the group of germs of $f \in \mathrm{GL}_n(\mathcal{O}(\mathfrak{s}))$, $\mathfrak{s}$ a sector containing θ, satisfying the conditions:

1. Flatness: $\lim\limits_{\substack{x \to 0 \\ x \in \mathfrak{s}}} f(x) = \mathrm{id}$ and $f \sim_{\mathfrak{s}} 1$;
2. Isotropy of A^0: ${}^f A^0 = A^0$.

The Stokes sheaf $\Lambda(A^0)$ of A^0 is defined as the subsheaf of $\mathrm{GL}_n(\mathcal{A})$ in the following way. For some $\theta \in S^1$ is the stalk at θ the subgroup of $\mathrm{GL}_n(\mathcal{A})_\theta$ of elements f which satisfy

1. Multiplicatively flatness: f is asymptotic to the identity, i.e. $f \sim_{\mathfrak{s}} 1$ for some $\mathfrak{s}$ containing θ;
2. Isotropy of A^0: ${}^f A^0 = A^0$.

Remark 4.1.1

This definition makes also sense as $\Lambda(A)$ where A stands for a systems wich is not in normal form. The elements of $\Lambda(A)$ then have to be isotropies of a normal form A^0 of A.

Since the sheaf by definition only contains solutions of the differential equation [End A^0] we obtain the following proposition.

Proposition 4.2

The sheaf $\Lambda(A^0)$ is piecewise-constant (cf. [Lod04, Prop.2.1]).

4.1.1 The theorem

Let us now state the Malgrange-Sibuya Theorem. We will first give it in the language of meromorphic connections and after that we will give the same theorem in the language of systems. The second formulation of this theorem will be proven in the next section.

In the language of meromorphic connections is the map, of the Malgrange-Sibuya Theorem below, described as follows.

Let $(\mathcal{M}, \nabla, \widehat{f})$ be a marked germ of a meromorphic connection. By Theorem 3.37.1 there exists an open covering $\mathcal{U} = (U_j)_{j \in J}$ and for every open set, an isomorphism

$$f_j : (\widetilde{\mathcal{M}}, \widetilde{\nabla})_{|U_j} \longrightarrow (\widetilde{\mathcal{M}}^{nf}, \widetilde{\nabla}^{nf})_{|U_j}$$

such that $f_j \sim_{U_j} \widehat{f}$. By $(f_k f_j^{-1})_{jk}$ is then a cocycle of the sheaf $\mathcal{S}t(A^0)$, relative to the covering $\mathcal{U}$, defined. For other lifts f_j' of $\widehat{f}$ on W_j, $(f_j' f_j^{-1})$ is a 0-cochain of $\mathrm{Sto}(A^0)$ relative to $\mathcal{U}$. Thus the associated cochians to (f_j) and (f_j') are equivalent. One can also check that, if $(\mathcal{M}, \nabla, \widehat{f})$ and $(\mathcal{M}', \nabla', \widehat{f}')$ are isomorphic, the corresponding cocycles define the same cohomology class. This defines a mapping of pointed sets

$$\exp : \mathcal{H}(\mathcal{M}^{nf}, \nabla^{nf}) \longrightarrow H^1(S^1; \Lambda(A^0))$$

to the first non abelian cohomology of $\Lambda(A^0)$, which sends the class of $(\mathcal{M}^{nf}, \nabla^{nf}, \widehat{\mathrm{id}})$ to that of id, i.e. the trivial cohomology class.

Theorem 4.3: Malgrange-Sibuya

The homomorphism

$$\exp : \mathcal{H}(\mathcal{M}^{nf}, \nabla^{nf}) \longrightarrow \mathcal{S}t(A^0) := H^1(S^1; \Lambda(A^0))$$

is an isomorphism of pointed sets.

[BV89, Thm.I.4.5.1], [Mal83, Thm.3.4], [MR91, Thm.13]

The theorem (system version)

Since the language of meromorphic connections is equivalent to the one of systems, there is also the translated version of the Malgrange-Sibuya isomorphism to the language of systems. The corresponding map is then build as follows.

Let $(A, \widehat{F})$ be a marked pair, thus $\widehat{F}$ solves $[A^0, A]$. By the M.A.E.T (Theorem 3.36) there exists an open covering $\mathcal{U} = (U_j)_{j \in J}$ together with for every open set U_j a lift $F_j \in \mathrm{GL}_n(\mathcal{A}(U_j))$ of $\widehat{F}$ (cf. Definition 3.35), which solves $[A, A^0]$. By the cocycle $(F_l^{-1} F_j)_{jl} \in \Gamma(\mathcal{U}; \Lambda(A^0))$ is then a cohomology class in $\mathcal{S}t(A^0)$ relative to the covering $\mathcal{U}$ [Lod94, p. 855] determined. For other lifts F'_j of $\widehat{F}$ on U_j is $(G_j = F_j^{-1}F'_j)$ a 0-cochain of $\Lambda(A^0)$ relative to $\mathcal{U}$, which satisfies

$$F_k^{-1}F_j = G_k F_k'^{-1} F'_j G_j^{-1} .$$

Thus the cochians associated to (F_j) and (F'_j) determine the same cohomology class in $\mathcal{S}t(A^0)$. One can also check that, if $(A, \widehat{F})$ and $(A', \widehat{F}')$ are equivalent, the corresponding cocycles define the same cohomology class. This defines a welldefined mapping of pointed sets

$$\mathcal{H}(A^0) \to H^1(S^1; \Lambda(A^0))$$

to the first non abelian cohomology of $\Lambda(A^0)$, which we call exp. It maps the class of $([A^0], \widehat{\mathrm{id}})$ to that of id, i.e. the trivial cohomology class.

[BV89, Theorem 4.5.1]

Theorem 4.4: Malgrange-Sibuya (system version)

The homomorphism

$$\exp : \mathcal{H}(A^0) \longrightarrow \mathcal{S}t(A^0) := H^1(S^1; \Lambda(A^0))$$

is an isomorphism of pointed sets.

Remark 4.5

The Theorem [BV89, Thm.III.1.1.2] in the book from Babbitt and Varadarajan, states that $\mathcal{S}t(A^0)$ is actually a local moduli space for marked pairs, which are formally isomorphic to a given system $[A^0]$.

This means that

- the morphism property,
- the criterion of equivalence and
- the existence of universal families

are satisfied (cf. [BV89, p. 169]).

In fact is the whole third part of [BV89] dedicated to this topic.

Since the morphism exp depends on the choice of the normal form, we will denote that, if it is not clear, by $\exp_{A^0} = \exp$ (In Loday-Richaud's paper is this denoted as $\exp_{\mu_0}$).

Remark 4.6

[Lod94] Remark I.2.2

To another normal form $A^1 = {}^{\Phi}A^0$ there correspond cochains, which are conjugated via $\Phi \in G(\{t\})$. This is formulated in the following commutative diagram:

$$
\begin{tikzcd}
G\backslash\widehat{G}(A^1) \arrow[rr, "\cdot\Phi"] \arrow[dd] \arrow[dr, phantom, "\ni" description] & & G\backslash\widehat{G}(A^0) \arrow[dd] \arrow[dr, phantom, "\ni" description] & \\
& \widehat{F} \arrow[rr, mapsto] \arrow[dd, mapsto, "\exp_{A^1}"'] & & \widehat{F}\Phi \arrow[dd, mapsto, "\exp_{A^0}"'] \\
H^1(S^1;\Lambda(A^1)) \arrow[rr] \arrow[dr, phantom, "\ni" description] & & H^1(S^1;\Lambda(A^0)) \arrow[dr, phantom, "\ni" description] & \\
& \exp_{A^1}(\widehat{F}) \arrow[rr, mapsto] & & \exp_{A^0}(\widehat{F}\Phi)
\end{tikzcd}
$$

where $\exp_{A^0}(\widehat{F}\Phi) = \Phi^{-1}\exp_{A^0}(\widehat{F})\Phi$.

4.1.2 Proof of Theorem 4.4

We will mainly refer to [BV89, Proof of Theorem 4.5.1] and [Sab07, Section 6.d], where a slightly more complicated case with deformation space is proven. These both resources proof the theorem using the languages of meromorphic connections whereas we will use systems.

See also [BJL79] and [BV89] although the proof goes back to work from Malgrange and Sibuya (see for example [Sib90]).

We will start by proofing the injectivity of the morphism exp.

Proof of the injectivity. Consider the two marked pairs $(A, \widehat{F})$ and $(A', \widehat{F}')$ in $\widehat{\mathrm{Syst}}_m(A^0)$, whose classes in $\mathcal{H}(A^0)$ get mapped to same element

$$\exp([(A,\widehat{F})]) = \lambda = \exp([(A',\widehat{F}')]) \in H^1(S^1;\Lambda(A^0))\,.$$

By using refined coverings, it is possible to find a common finite covering $\mathcal{U} = \{U_j; j \in J\}$ of S^1 such that λ is the class of the cocycles $(F_l^{-1}F_j)$ and $(F_l'^{-1}F_j')$, where F_j (resp. F_j') are lifts of $\widehat{F}$ (resp. $\widehat{F}'$) on $U_j \in \mathcal{U}$. From $[(F_l^{-1}F_j)] = [(F_l'^{-1}F_j')]$ follows that there exists a 0-cochain $(G_j)_{j\in J}$ of the sheaf $\Lambda(A^0)$ relative to the covering $\mathcal{U}$, such that

$$F_l'^{-1}F_j' = G_l F_l^{-1} F_j G_j^{-1} \text{ on the arc } U_j \cap U_l\,,$$

which can be rewritten to

$$F_j' G_j F_j^{-1} = F_l' G_l F_l^{-1} \text{ on the arc } U_j \cap U_l\,. \tag{4.1}$$

If we set $H_j := F_j' G_j F_j^{-1}$ on U_j, we get PROBLEM:refactor!!!

- that from equation (4.1) that the H_j glue together PROBLEM:holomorphic??,
- that H_j is a solution of $[A, A']$ on every U_j, i.e. it satisfies there ${}^{H_j}A = A'$, since

$$
\begin{aligned}
{}^{H_j}A &= {}^{F_j' G_j F_j^{-1}}A \\
&= {}^{F_j' G_j}A^0 && (\text{since } F_j' \text{ is a lift of } \widehat{F}' \text{ on } U_j)
\end{aligned}
$$

$$
\begin{aligned}
&= {}^{F_j'}A^0 && \text{(since } G_j \text{ is a is an isotropy of } A^0\text{)}\\
&= A' && \text{(since } F_j \text{ is a lift of } \widehat{F} \text{ on } U_j\text{)}
\end{aligned}
$$

and

- which satisfies $\widehat{F'} = \widehat{H}_j \widehat{F}$ on every U_j, since

$$
\begin{aligned}
\widehat{H}_j \widehat{F} &= \widehat{F_j' G_j F_j^{-1}} \widehat{F}\\
&= \widehat{F_j'} \underbrace{\widehat{G_j}}_{\substack{\parallel \\ \mathrm{id}}} \widehat{F_j^{-1}} \widehat{F} && \text{(since } G_j \text{ is flat, i.e. } \hat{G}_j = \mathrm{id}\text{)}\\
&= \widehat{F'} \underbrace{\widehat{F^{-1}} \widehat{F}}_{\substack{\parallel \\ \mathrm{id}}}\\
&= \widehat{F'}
\end{aligned}
$$

Therefore are $(A, \widehat{F})$ and $(A', \widehat{F}')$ equivalent (cf. page 36) and injectivity is proven. □

For the proof of the surjectivity we will use another result from Malgrange and Sibuya, which is also called the Malgrange-Sibuya Theorem (Theorem 4.8). It will be stated bellow but one can also be found in Babbitt and Varadarajans's book [BV89, 65ff] as Theorem 4.2.1.

Let $\widehat{F} \in G(\!(t)\!)$ be a matrix with formally meromorphic entries. By the Borel-Ritt Lemma (cf. Theorem 2.13) we then know that there exists for every arc $I \subsetneq S^1$ a holomorphic function $G : \mathfrak{s}_I \to \mathrm{GL}_n(\mathbb{C})$ which is asymptotic to $\widehat{F}$. We will denote the set of all such holomorphic functions, which are multiplicatively flat on I, i.e. are on the arc I asymptotic to $\mathrm{id} \in G(\!(t)\!)$, by

$$\mathcal{G}(I) = \{G \in \mathrm{GL}_n(\mathcal{A}(I)) \mid G \sim_I \mathrm{id}\} \ ,$$

and this defines a sheaf $\mathcal{G}$ on S^1. The statement of the (second) Malgrange-Sibuya Theorem (Theorem 4.8) is then, that the difference between formal and convergent invertible matrices is described by the first sheaf cohomology $H^1(S^1;\mathcal{G})$ of $\mathcal{G}$ via the map

$$\Theta : G(\!(t)\!)/G(\{t\}) \longrightarrow H^1(S^1;\mathcal{G})\,,$$

which will turn out to be an isomorphism. It is set up as follows:

> Let $[\widehat{F}] \in G(\!(t)\!)/G(\{t\})$ with ambassador $\widehat{F}$ and let $\mathcal{U} = \{U_j \mid j \in J\}$ be a finite covering of S^1 by open arcs. The Borel-Ritt Lemma yields for every arc $J \subsetneq S^1$ containing j a holomorphic function F_j which satisfies $F_j \sim_{U_j} \widehat{F}$. By $(F_l F_j^{-1})_{j,l\in J}$ is then a cocycle for $\mathcal{G}$ defined, and write $\Theta([\widehat{F}])$ for the corresponding cohomology class.

This construction is similar to the definition of the map of Theorem 4.4. The difference is, that we instead of M.A.E.D, to obtain lifts in the sense of Definition 3.35, we use only the Borel-Ritt Lemma to obtain only asymptotic lifts.

It can be verified, that the class $\Theta([\widehat{F}])$ does not depend on

- the choise of an ambassador $\widehat{F}$ in $[\widehat{F}] \in G((t))/G(\{t\})$ PROBLEM: proof! ,
- the choice of the covering $\mathcal{U}$ PROBLEM: proof! nor
- the choice the lifts F_j PROBLEM: proof! .

Lemma 4.7

The mapping Θ is injective.

Proof. Let $\widehat{F}$ and $\widehat{F}' \in G((t))$ such that $\Theta([\widehat{F}]) = \Theta([\widehat{F}'])$. We then can find a covering $\mathcal{U} = \{U_j \mid j \in J\}$ together with holomorphic functions F_j and F_j', which satisfy $F_j \sim_{U_j} \widehat{F}$ and $F_j' \sim_{U_j} \widehat{F}'$, such that $(F_l^{-1}F_j)_{j,l\in J}$ and $(F_l'^{-1}F_j')_{j,l\in J}$ determine the classes $\Theta([\widehat{F}])$ and $\Theta([\widehat{F}'])$. This implies that there are maps G_j, which are on U_j holomorphic and satisfy $G_j \sim_{U_j}$ id such that

$$F_l'^{-1}F_j' = G_l F_l^{-1} F_j G_j^{-1} \text{ on the arc } U_j \cap U_l$$

This equation can be rewritten to

$$F_j' G_j F_j^{-1} = F_l' G_l F_l^{-1} \text{ on the arc } U_j \cap U_l\,.$$

Since this tells us, that the functions $F_j' G_j F_j^{-1}$ coincide on the overlapping and define a holomorphic map from the arc S^1 (i.e. a punctured disc with a small radius) into $\mathrm{gl}_n(\mathbb{C})$, which will be called H. Since $F_j' G_j F_j^{-1} \sim_{U_j}$ id for all $j \in J$, we have $H \sim_{S^1}$ id. Thus the defined H meromorphic at 0 and satisfies $H = F'^{-1}F$, so that $[F] = [F']$. □

Theorem 4.8: Malgrange-Sibuya

The map $\Theta : G((t))/G(\{t\}) \to H^1(S^1; \mathcal{G})$ is an isomorphism.

This Theorem is proven in Section 4.4 of Babbitt and Varadarajan's book [BV89] or on page 371 of [MR91] from Martinet and Ramis. It is also mentioned on page 30 of [Var96].

We are now able to proof the surjectivity of the map from Theorem 4.4.

Proof of surjectivity. Let the cohomology class $\lambda \in H^1(S^1; \Lambda(A^0))$ be represented by a cocycle $(F_{jl})_{j,l\in J}$ associated with some finite covering $\mathcal{U} = \{U_j; j \in J\}$ of S^1. The sections of $\Lambda(A^0)$ are asymptotic to id and thus the cocycle $(F_{jl})_{j,l\in J}$ also determines an element $\sigma \in H^1(S^1; \mathcal{G})$. From the Theorem 4.8 we know, that there is a $\widehat{F} \in G((t))$ whose class $[\widehat{F}]$ gets via Θ mapped to σ. Thus there exists holomorphic functions $F_j : \mathfrak{s}_{U_j} \to \mathrm{gl}_n(\mathbb{C})$ with $F_j \sim_{U_j} \widehat{F}$ and $F_l^{-1}F_j = F_{jl}$ on $\mathfrak{s}_{U_j \cap U_l}$ for all $j, l \in J$. [BV89, p. 72]

Define on every arc U_j the matrix $A_j := {}^{F_j}A^0$. On the intersections $U_j \cap U_l$ we know that $A_j = A_l$, since from $F_l^{-1}F_j \in \Lambda(A^0)$ follows on $U_j \cap U_l$ that

$$A^0 = {}^{F_l^{-1}}({}^{F_j}A^0) \qquad \Longrightarrow \qquad \underbrace{{}^{F_l}A^0}_{=A_l} = \underbrace{{}^{F_j}A^0}_{=A_j}.$$

Thus the A_j glue to a section A, which satisfies ${}^{\widehat{F}}A = A_0$ by construction. We have found an element $(A, \widehat{F}) \in \mathcal{H}(A^0)$ whose image under exp is σ. □

4.2 The Stokes groups

Here we want to introduce the notion of Stokes groups. They are for example also introduced by Loday-Richaud in [Lod94; Lod14] or section 4 of [MR91] by Martinet and Ramis.

Let us recall, that the normal form A^0 can be written as $A^0 = Q'(t^{-1}) + L\frac{1}{t}$ and a normal solution is given by $\mathcal{Y}_0(t) = t^L e^{Q(t^{-1})}$ (cf. Proposition 3.29), where

- $Q(t^{-1}) = \bigoplus_{j\in\{1,\dots,s\}} q_j(t^{-1}) \cdot \mathrm{id}_{n_j}$ and
- the block structure of L is finer then the structure of Q (cf. Definition 3.28).

Let $\{q_1(t^{-1}), \dots, q_s(t^{-1})\}$ be the *set of all determining polynomials of* $[A^0]$ and denote by

$$\mathcal{Q}(A^0) := \left\{ q_j - q_l \mid q_j \text{ and } q_l \text{ determining polynomials of } [A^0],\ q_j \neq q_l \right\}$$

the *set of all determining polynomials of* $[\mathrm{End}\, A^0]$. Instead of $q_j - q_l \in \mathcal{Q}(A^0)$ we will sometimes talk of (ordered) pairs $(q_j, q_l) \in \mathcal{Q}(A^0)$.

Definition 4.9

We call

- $a_{jl} \in \mathbb{C}\backslash\{0\}$ the *leading factor*,
- $\frac{a_{jl}}{t^{k_{jl}}}$ the *leading coefficient* and
- $k_{jl} \in \mathbb{Q}$ the *degree*

of $q_j - q_l \in \mathcal{Q}(A^0)$ if

$$q_j - q_l \in \left\{ \frac{a_{jl}}{t^{k_{jl}}} + h \mid h \in o(t^{-k_{jl}}), a_{jl} \neq 0 \right\}.$$

Remark 4.9.1

1. It is obvious that $k_{jl} = k_{lj}$ and $\frac{a_{jl}}{t^{k_{jl}}} = \frac{-a_{lj}}{t^{k_{lj}}}$.
2. In Boalch's paper [Boa01] (and also in [Boa99]) are the degrees of the pairs always incremented by one. We will prefer the other notion, which is also used in Loday-Richaud's paper [Lod94].
3. In Loday-Richaud's book [Lod14, Def.4.3.6] a_{jl} is negated to be consistent with calculations at ∞. Here this is not necessary, since we use the clockwise orientation on S^1 (cf. Definition 4.13).

The degrees of the elements in $\mathcal{Q}(A^0)$ are defined to be the *levels* of A^0. The set of all levels of A^0 will be denoted by

$$\mathcal{K} = \{k_1 < \cdots < k_r\} \subset \mathbb{Q}\,.$$

Remark 4.9.2

The system $[A^0]$ is unramified if and only if $\mathcal{K} \subset \mathbb{Z}$. Since we only want to consider the unramified case, this will be always the case.

4.2.1 Anti-Stokes directions and the Stokes group

[Lod94, p. I.4]

Definition 4.10

[HTT08, p. 130], [Lod14, p. 79]

Let $k \in \mathbb{N}$ and $a \in \mathbb{C}$. We say that an exponential $e^{q(t^{-1})}$, where $q(t^{-1}) \in \frac{a}{t^k} + o(t^{-k})$, has *maximal decay in a direction* $\theta \in S^1$ if and only if $ae^{-ik\widetilde{\theta}}$ is real negative. We say that an matrix has maximal decay, if every entry has maximal decay.

$be^0 \in \mathbb{C}$ corresponding to has maximal decay if and only if PROBLEM:1 has maximal decay?

On the determining polynomials of $[A^0]$ we define the following (partial) order relations:

Definition 4.11

Let $\widetilde{\theta}$ be a determination of $\theta \in S^1$.

1. We define the relation $\boldsymbol{q_j \underset{\widetilde{\theta}}{\prec} q_l}$ to be equivalent to the condition

   $e^{(q_j-g_l)(t^{-1})}$ is flat at 0 in a neighbourhood of the direction $\widetilde{\theta}$, i.e. if and only if $\Re(a_{jl}e^{-ik_{jl}\widetilde{\theta}}) < 0$.

2. Let us define another relation $\boldsymbol{q_j \underset{\widetilde{\theta}}{\prec\!\!\!\cdot} q_l}$ equivalent to

   $e^{(q_j-g_l)(t^{-1})}$ is of maximal decay in the direction $\widetilde{\theta}$,

   which by itself is equivalent to

   $a_{jl}e^{-ik_{jl}\widetilde{\theta}}$ is a real negative number, i.e. $q_j \underset{\widetilde{\theta}}{\prec} q_l$ and $\Im(a_{jl}e^{-ik_{jl}\widetilde{\theta}}) = 0$.

Remark 4.11.1

In the unramified case do these relations not depend on the determination $\widetilde{\theta}$ of θ. As a consequence we will only write $\underset{\theta}{\prec}$ and $\underset{\theta}{\prec\!\!\!\cdot}$.

To understand the previous definition better, it is convenient to look closer at functions of the form $f : \theta \mapsto ae^{-ik\theta}$, $k \in \mathbb{Z}$, corresponding to some pair (q_j, q_l). Write a as $a = |a|e^{i\arg(a)}$, thus the function f writes as

$$\begin{aligned} f(\theta) &= |a|e^{i(\arg(a)-k\theta)} \\ &= |a|(\cos(\arg(a)-k\theta) + i\sin(\arg(a)-k\theta)) \,. \end{aligned}$$

In the Figure 4.1, we illustrate the real and the imaginary part of f.

The graphs, corresponding to the flipped pair (q_l, q_j) are then obtained by the transformation $\arg(a) \to \arg(-a) = \arg(a) + \pi$, i.e. the shift by $\frac{\pi}{k}$ to the right. This $\frac{\pi}{k}$ is exactly a half period, thus the new graphs are also obtained by mirroring at the line $t = 0$.

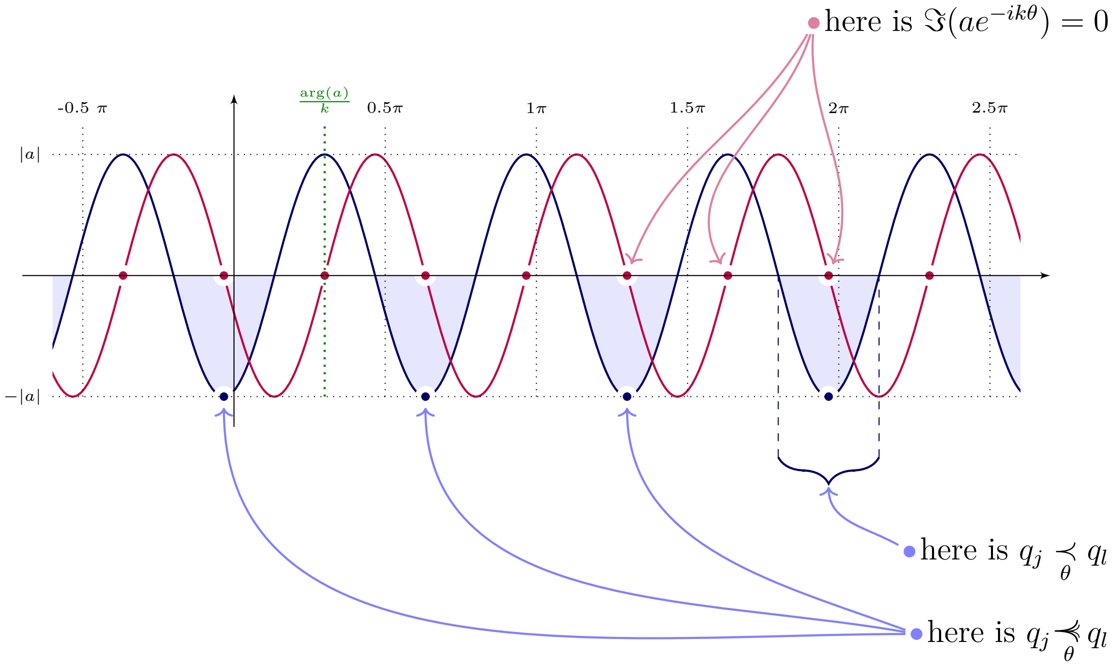


Figure 4.1: In this plot is the real part of $f(\theta) = ae^{-ik\theta}$, corresponding to some pair (q_j, q_l), in blue and the imaginary part in purple sketched.

Remark 4.12

Let k_{jl} be the degree of $q_j - q_l$. It is easy to see (cf. Figure 4.1), that the condition $q_j \underset{\theta}{\prec} q_l$ is equivalent to

there is a $\theta' \in U(\theta, \frac{\pi}{k_{jl}})$ such that $q_j \underset{\theta}{\prec\!\!\!\prec} q_l$.

Let us now use the defined relations from Definition 4.11 to say, which are the interesting directions of S^1.

Definition 4.13

Let $\theta \in S^1$ be an direction.

1. θ is an *anti-Stokes direction* if there is at least one pair (q_j, q_l) in $\mathcal{Q}(A^0)$, which satisfies $q_j \underset{\theta}{\prec\!\!\!\prec} q_l$.

   Let $\mathbb{A} = \{\alpha_1, \ldots, \alpha_\nu\}$ denote the set of all anti-Stokes directions in a clockwise ordering. For a uniform notation later, define $\mathbb{A}$ to contain a single, arbitrary direction if $\mathcal{K} = \{0\}$.

2. The direction θ is a *Stokes direction* if there is at least one pair (q_j, q_l) in $\mathcal{Q}(A^0)$, which satisfies neither $q_j \underset{\theta}{\prec} q_l$ nor $q_l \underset{\theta}{\prec} q_j$.

   Let $\mathbb{S} = \{\sigma_1 < \cdots < \sigma_\mu\}$ be the set of Stokes directions.

If one starts with a fundamental solution on a sector, one is able to extend this analytic to an larger sector up to some barrier. The Stokes directions are exactly the directions, beyond which some fundamental directions can't be extended.

The clockwise ordering is chosen, similar to Loday-Richaud's paper [Lod94], since the calculations are then compatible with the calculations, which look at ∞ and take a counterclockwise ordering. Boalch uses in [Boa01] and [Boa99] the inverse ordering, but looks also at 0, thus there might be some incompatibilities. In Loday-Richaud's book [Lod14] this problem is solved by an additional minus sign for some coefficient.

We will use the Greek letter α whenever we want to emphasize that a direction is an anti-Stokes direction. For generic directions, we will use θ. In fact will most of the following definitions and constructions work for every $\theta \in S^1$, but the Stokes group (cf. Definition 4.15) for example will be trivial for every $\theta \notin \mathbb{A}$. Thus the interesting directions are only the anti-Stokes directions $\alpha \in \mathbb{A}$.

Lemma 4.14

Let $\alpha \in \mathbb{A}$ together with a pair $(q_j - q_l)(t^{-1}) \in \mathcal{Q}(A^0)$ of degree k_{jl}, such that $q_j \underset{\alpha}{\prec} q_l$ be given. We then know for every $m \in \mathbb{N}$ that

$$\underbrace{\alpha + m\frac{\pi}{k_{jl}}}_{=:\alpha'} \in \mathbb{A}\,.$$

Especially is either $q_j \underset{\alpha'}{\prec} q_l$ (in the case, when m is even) or $q_l \underset{\alpha'}{\prec} q_j$ (when m is uneven) satisfied (see Figure 4.1).

Corollary 4.14.1

It follows that in the case $\mathcal{K} = \{k\}$, the set $\mathbb{A}$ has $\frac{\pi}{k}$-rotational symmetry.

[Boa99, p. 8] *Proof.* Let (j,l) be a pair such that $q_j \underset{\alpha}{\prec} q_l$, i.e. such that $a_{jl}e^{-ik_{jl}\alpha} \in \mathbb{R}_{<0}$. Hence for $m \in \mathbb{N}$

$$a_{jl}e^{-ik_{jl}\left(\alpha + m\frac{\pi}{k_{jl}}\right)} = a_{jl}e^{-ik_{jl}\alpha}e^{-im\pi} = \begin{cases} a_{jl}e^{-ik_{jl}\alpha} \in \mathbb{R}_{<0} & \text{, if } m \text{ is even} \\ -a_{jl}e^{-ik_{jl}\alpha} \in \mathbb{R}_{>0} & \text{, if } m \text{ is uneven} \end{cases}$$

is in the case when m is even, also real and negative. In the other case, when n is uneven, we use that $a_{jl} = -a_{lj}$ and $k_{jl} = k_{lj}$ to obtain $a_{lj}e^{-ik_{lj}\left(\alpha + m\frac{\pi}{k_{lj}}\right)} \in \mathbb{R}_{<0}$.

Thus, for $\alpha' := \alpha + m\frac{\pi}{k_{jl}}$, we have $\alpha' \in \mathbb{A}$ since

- $q_j \underset{\alpha'}{\prec} q_l$ when m is even or
- $q_l \underset{\alpha'}{\prec} q_j$ when m is uneven.

□

As a subgroup of the stalk at θ of the in Definition 4.1 defined Stokes sheaf $\Lambda(A^0)$ we define the Stokes group as follows.

Definition 4.15

Define the *Stokes group*

$$\operatorname{Sto}_\theta(A^0) := \left\{\varphi_\theta \in \Lambda_\theta(A^0) \mid \varphi_\theta \text{ has maximal decay at } \theta\right\}$$

whose elements are called *Stokes germs*. TODO: This is in fact a group, since...

Remark 4.15.1

For $\theta \notin \mathbb{A}$ the group $\operatorname{Sto}_\theta(A^0)$ is trivial, since at θ no flat isotropy has maximal decay, but the identity.

4.2.2 Stokes matrices

Stokes matrices, which Wasow calls in his book [Was02] Stokes multipliers and Boalch calls Stokes factors in [Boa01; Boa99], arise either [Boa99, 9f], [Lod94, ??]

> as faithful representations of Stokes germs

or, if one starts by comparing the actual fundamental solutions on arcs, as

> the matrices describing the blending between two adjacent fundamental solutions, with some additional assumptions (cf. Definition [Lod14, p. 80]).

Definition 4.16

Let us use

$$\boldsymbol{\delta}_{jl} := \begin{cases} 0 \in \mathbb{C}^{n_j \times n_l} & \text{, if } j \neq l \\ \mathrm{id} \in \mathbb{C}^{n_j \times n_l} & \text{, if } j = l \end{cases}$$

as a block version of Kronecker's delta corresponding to the structure of the normal solution $\mathcal{Y}_0$, which was fixed. Define the group

$$\mathbb{S}\mathrm{to}_\theta(A^0) = \left\{K = (K_{jl})_{j,l\in\{1,\ldots,s\}} \in \mathrm{GL}_n(\mathbb{C}) \mid K_{jl} = \boldsymbol{\delta}_{jl} \text{ unless } q_j \underset{\theta}{\preccurlyeq} q_l\right\}$$

of all *Stokes matrices* of A^0 in the direction θ. They will arise as a faithful representation (cf. [Hal03, Def.4.1]) of $\operatorname{Sto}_\theta(A^0)$.

Remark 4.16.1

There is obviously a bijection $\vartheta_\theta : \prod_{q_j \underset{\theta}{\preccurlyeq} q_l} \mathbb{C}^{n_j \cdot n_l} \xrightarrow{\cong} \mathbb{S}\mathrm{to}_\theta(A^0)$.

Proposition 4.17

[Lod94, Def.I.4.7]
[Lod14, 78f]

In this situation is

$$\begin{aligned}\rho_\theta : \mathrm{Sto}_\theta(A^0) &\longrightarrow \mathbb{S}\mathrm{to}_\theta(A^0)\\ \varphi_\theta &\longmapsto C_{\varphi_\theta} := \mathcal{Y}_0\varphi_\theta\mathcal{Y}_0^{-1}\end{aligned}$$

Boalch uses $C_{\varphi_\theta} := \mathcal{Y}_0^{-1}\varphi_\theta\mathcal{Y}_0$

an isomorphism which maps a germ of $\mathrm{Sto}_\theta(A^0)$ to the corresponding Stokes matrix C_{φ_θ} such that

$$\varphi_\theta(t)\mathcal{Y}_0(t) = \mathcal{Y}_0(t)C_{\varphi_\theta} \tag{4.2}$$

near θ. The matrix C_{φ_θ} is then called a *representation of* φ_θ.

Remark 4.17.1

1. In the ramified case does this morphism depend on the choice of the determination $\widetilde{\theta}$ of θ and the corresponding choice of a realization of the fundamental solution with that determination of the argument near the direction θ (cf. [Lod94] or [Lod14, 78f]).
2. [Lod94, Defn.I.4.7] This construction defines also a morphism, which takes a germ $\varphi_\theta \in \Lambda_\theta(A^0)$ into its unique representation matrix

$$\underset{\mathbb{S}\mathrm{to}_\theta(A^0)}{C_{\varphi_\theta} \in \underset{\cup}{\widehat{\mathbb{S}\mathrm{to}}_\theta(A^0)}} := \left\{(K_{jl})_{j,l\in\{1,\dots,s\}} \in \mathrm{GL}_n(\mathbb{C}) \mid K_{jl} = \boldsymbol{\delta}_{jl} \text{ unless } q_j \underset{\theta}{\prec} q_l\right\}$$

and there is a bijection $\widehat{\vartheta}_\theta : \prod_{q_j \underset{\theta}{\prec} q_l} \mathbb{C}^{n_j\cdot n_l} \xrightarrow{\cong} \widehat{\mathbb{S}\mathrm{to}}_\theta(A^0)$.

Does this define a local-constant sheaf

$$I \mapsto \widehat{\mathbb{S}\mathrm{to}}_I(A^0) := \left\{(K_{jl})_{j,l\in\{1,\dots,s\}} \in \mathrm{GL}_n(\mathbb{C}) \mid K_{jl} = \boldsymbol{\delta}_{jl} \text{ unless } q_j \underset{\theta}{\prec} q_l \text{ for some } \theta \in I\right\}$$

and a skyscraper sheaf

$$I \mapsto \mathbb{S}\mathrm{to}_I(A^0)\,.$$

PROBLEM

Proof. It is well known (cf. [Boa99, p. 10]), that the morphism ρ_θ, i.e. conjugation by the fundamental solution, relates solutions φ_θ of $[\mathrm{End}(A^0)] = [A^0, A^0]$ to solutions of $[0,0]$ which are the constant matrices $\mathrm{GL}_n(\mathbb{C})$. Thus we have to show, that the image of $\mathrm{Sto}_\theta(A^0)$ under ρ_θ is $\mathbb{S}\mathrm{to}_\theta(A^0)$.

To see that the obtained matrix has the necessary zeros, to lie in $\mathbb{S}\mathrm{to}_\theta(A^0)$ we look at Equation (4.2) and deduce

$$\varphi_\theta(t) = t^L e^{Q(t^{-1})} C_{\varphi_\theta} e^{-Q(t^{-1})} t^{-L} \tag{4.3}$$

with the given choice of the argument near θ. After decomposing C_{φ_θ} into

$$
\begin{aligned}
C_{\varphi_\theta} &= 1_n + \begin{pmatrix} c_{(1,1)} & c_{(1,2)} & \cdots & \\ c_{(2,1} & \ddots & & \\ \vdots & & & \\ & & & c_{(s,s)} \end{pmatrix} \\
&= 1_n + \underbrace{\begin{pmatrix} c_{(1,1)} & 0 & \cdots \\ 0 & & \\ \vdots & & \\ & & \end{pmatrix}}_{C_{\varphi_\theta}^{(1,1)}} + \underbrace{\begin{pmatrix} 0 & c_{(1,2)} & 0 & \cdots \\ & 0 & & \\ & \vdots & & \\ & & & \end{pmatrix}}_{C_{\varphi_\theta}^{(1,2)}} + \cdots + \underbrace{\begin{pmatrix} & & \vdots \\ & & 0 \\ \cdots & 0 & c_{(s,s)} \end{pmatrix}}_{C_{\varphi_\theta}^{(s,s)}} \\
&= 1_n + \sum_{(l,j)} C_{\varphi_\theta}^{(l,j)}
\end{aligned}
$$

where the $c_{(j,l)}$ are blocks of size $n_j \times n_l$ which correspond to the structure of Q. After rewriting the Equation (4.3) we get

One can ignore the block structure by using 1×1 sized blocks. But one looses the uniqueness of the q_j's.

$$
\varphi_\theta = t^L \left(1_n + \sum_{(l,j)} C_{\varphi_\theta}^{(l,j)} e^{(q_l - q_j)(t^{-1})} \right) t^{-L} .
$$

$$
\begin{aligned}
\varphi_\theta(t) &= t^L e^{Q(t^{-1})} \left(1_n + C_{\varphi_\theta}\right) e^{-Q(t^{-1})} t^{-L} \\
&= t^L e^{Q(t^{-1})} \left(1_n + \sum_{(l,j)} C_{\varphi_\theta}^{(l,j)} \right) e^{-Q(t^{-1})} t^{-L} \\
&= t^L \left(1_n + \sum_{(l,j)} e^{Q(t^{-1})} C_{\varphi_\theta}^{(l,j)} e^{-Q(t^{-1})} \right) t^{-L} \\
&= t^L \left(1_n + \sum_{(l,j)} C_{\varphi_\theta}^{(l,j)} e^{(q_l - q_j)(t^{-1})} \right) t^{-L} .
\end{aligned}
$$

Thus, for φ_θ to be flat in direction θ, it is necessary and sufficient that if $e^{(q_l - q_j)(t^{-1})}$ does not have maximal decay in direction θ the corresponding block $C_{\varphi_\theta}^{(l,j)}$ vanishes. Thus we have seen, that C_{φ_θ} is an element of $\mathbb{S}\mathrm{to}_\theta(A^0)$.

The **surjectivity** can now be seen easily since every constant matrix with zeros at the necessary positions characterizes a unique element of $\mathrm{Sto}_\theta(A^0)$:

> Let $C = 1_n + \sum_{(l,j) | q_j \underset{\theta}{\not<} q_l} C^{(l,j)}$ be an element of $\mathbb{S}\mathrm{to}_\theta(A^0)$. Then is a pre-image of C given by $t^L e^{Q(t^{-1})} C e^{-Q(t^{-1})} t^{-L}$ which lies in $\mathrm{Sto}_\theta(A^0)$, since it satisfies the condition discussed above.

The map $\rho_{\widetilde{\theta}}$ is also **injective**, since it is the conjugation by an invertible matrix. $\square$

From the calculations in the proof it is clear that

1. for $j = l$ the (diagonal) blocks $C^{(l,j)}_{\varphi_\theta}$ vanish since $q_l - q_j = 0$ does not have maximal decay and
2. if $e^{q_j - q_l}$ has has maximal decay, then $e^{q_l - q_j}$ has not. Thus if $C^{(l,j)}_{\varphi_\theta}$ is not equal to zero, the block $C^{(j,l)}_{\varphi_\theta}$ is necessarily zero.

This implies that the matrix C_{φ_θ} is unipotent, and hence is $\mathrm{Sto}_\theta(A^0)$ is a unipotent Lie group.

Proposition 4.18

Some of the above results are also true for the groups $\widehat{\mathbb{S}\mathrm{to}}_\theta(A^0)$. One could use these, to see that the sheaf $\Lambda(A^0)$ is a piecewise-constant sheaf of non-Abelian unipotent Lie groups (cf. [Lod04, Prop.2.1]).

One can use the Stokes matrices to give an alternative characterization of Stokes germs:

> a germ $\varphi_\theta \in \Lambda_\theta(A^0)$ is in $\mathrm{Sto}_\theta(A^0)$ if and only if there exists a $C \in \mathbb{S}\mathrm{to}_\theta(A^0)$ such that $\varphi_\theta = \mathcal{Y}_0 C \mathcal{Y}_0^{-1}$.

Formulated is this in the following corollary.

Corollary 4.19

[Lod94, Def.I.4.12] A germ $\varphi_\theta \in \Lambda_\theta(A^0)$ is a Stokes germ, i.e. an element in $\mathrm{Sto}_\theta(A^0)$, if and only if it has a representation C_{φ_θ} where

$$C_{\varphi_\theta} = 1_n + \sum_{(l,j) \mid q_j \underset{\theta}{\not\prec} q_l} C^{(l,j)}_{\varphi_\theta}$$

and the $C^{(l,j)}_{\varphi_\theta}$ have the necessary block structure, i.e. it is in $\mathbb{S}\mathrm{to}_\theta(A^0)$.

Remark 4.19.1

In Loday-Richaud's book [Lod14, p. 78] are the elements of $\mathrm{Sto}_\theta(A^0)$ actually characterized as the flat transformations, such that Equation (4.2) is satisfied for some unique constant invertible matrix $C \in \mathbb{S}\mathrm{to}_\theta(A^0)$.

Definition 4.20

We denote the set of *levels of the germ* $\varphi_\theta \in \Lambda_\theta(A^0)$ by

$$\mathcal{K}(\varphi_\theta) := \left\{ \deg(q_j - q_l) \mid C^{(l,j)}_{\varphi_\theta} \neq 0 \text{ in some representation of } \varphi_\theta \right\} \subset \mathcal{K}\,.$$

A germ φ_θ is called a *k-germ* when $\mathcal{K}(\varphi_\theta) \subset \{k\}$, i.e. it has at most the level k.

PROBLEM: This would be good

Lemma 4.22

Every k-germ in direction α can be extended to the Stokes arc $U(\frac{\pi}{k}, \alpha)$.

Corollary 4.22

Every germ $\varphi_\alpha \in \mathrm{Sto}_\alpha(A^0)$ can be extended to the arc $U(\frac{\pi}{\max \mathcal{K}(\varphi_\alpha)}, \alpha)$, i.e. there is a section $\varphi \in \Gamma\left(U(\frac{\pi}{\max \mathcal{K}(\varphi_\alpha)}, \alpha), \Lambda(A^0)\right)$ which has φ_α as its germ at α.

Let φ_α be a **simple** k-germ in the sense that it is build from a single block, i.e.

$$\varphi_\theta = t^L \left(1_n + C_{\varphi_\theta}^{(l,j)} e^{(q_l - q_j)(t^{-1})}\right) t^{-L} .$$

for some pair (j, l). Assume also that the block has size 1×1.

Let φ be the extension of φ_α around alpha, i.e. the matrix which has germ φ_α at α and which solves [End A^0] and is multiplicatively flat.

The system [End A^0] is

$$\frac{dF}{dt} = A^0 F - F A^0 .$$

Question: Which form has the extension φ around α of the germ φ_α, does it retain the structure?

1. Look at a diagonal element $\varphi^{j,j}$ an (j, j):
   - it satisfies $\varphi_\alpha^{j,j} = 1$ and
   - is satisfies some complicated equation **Question:** Is it constantly 1

     hopefully yes
2. Look at an off-diagonal position at (j, l):

   case a: $\varphi_\alpha^{j,l} \neq 0$

   case b: $\varphi_\alpha^{j,l} = 0$

4.2.3 Decomposition of the Stokes group by levels

The goal of this section is, to introduce a filtration of $\Lambda(A^0)$, which will be restricted to $\mathrm{Sto}_\theta(A^0)$. This leads to a decomposition of $\mathrm{Sto}_\theta(A^0)$ into a semidirect product (cf. Proposition 4.28). [Lod94], [MR91, 362ff]

Let us introduce a couple of notations and definitions, which coincide with the notations used in Loday-Richaud's paper [Lod94]. Another good resource, which uses similar notations, is for example the paper [MR91, 362f] from Martinet and Ramis.

Notations 4.23

[Lod94, Not.I.4.15], [MR91, p. 362]

For every level $k \in \mathcal{K}$ and direction $\theta \in S^1$ we set

- $\Lambda^k(A^0)$ as the subsheaf of $\Lambda(A^0)$ of all germs, which are generated by k-germs;
- $\Lambda^{\leq k}(A^0)$ (resp. $\Lambda^{<k}(A^0)$ or $\Lambda^{\geq k}(A^0)$) as the subsheaf of $\Lambda(A^0)$ generated by k'-germs for all $k' \leq k$ (resp. $k' < k$ or $k' \geq k$).

Let $\star \in \{k, < k, \leq k, \dots\}$. The restrictions to Sto_θ yield the groups

$$\mathrm{Sto}_\theta^\star(A^0) := \mathrm{Sto}_\theta(A^0) \cap \Lambda_\theta^\star(A^0)$$

and let us also define $\mathbb{S}\mathrm{to}_\theta^\star(A^0)$ as the groups of representations, which correspond to elements of $\mathrm{Sto}_\theta^\star(A^0)$.

Corresponding to the definitions above, one can define $\mathbb{A}^\star := \{\alpha \in \mathbb{A} \mid \mathrm{Sto}_\alpha^\star(A^0) \neq \{\mathrm{id}\}\}$ for $\star \in \{k, < k, \leq k, \dots\}$.

Remark 4.24

It is clear that for every $k \in \mathcal{K}$ we have the canonical inclusions $\mathbb{A}^k \hookrightarrow \mathbb{A}^{\leq k}$ and $\mathbb{A}^{<k} \hookrightarrow \mathbb{A}^{\leq k}$.

We say that α *is bearing the level* k if $\alpha \in \mathbb{A}^k$ and denote the *set of levels beared by an direction* $\alpha \in \mathbb{A}$ by

$$\mathcal{K}_\alpha := \left\{k \in \mathcal{K} \mid \mathrm{Sto}_\alpha^k(A^0) \neq \{\mathrm{id}\}\right\} .$$

Corollary 4.25

The Lemma 4.14 implies that from $k \in \mathcal{K}_\alpha$ follows that $k \in \mathcal{K}_{\alpha + m\frac{\pi}{k}}$ for $m \in \mathbb{N}$.

Let us now study the sheaves $\Lambda^\star(A^0)$ and discuss how they correlate and how they can be composed from the others. The following proposition can be found as [Lod94, Prop.I.5.1] and the key-statement is also given in [MR91, Prop.4.10].

Proposition 4.26

[Lod94, Prop.I.5.1]

For any level $k \in \mathcal{K}$ one has that

1. $\Lambda^k(A^0)$, $\Lambda^{\leq k}(A^0)$ and $\Lambda^{<k}(A^0)$ are sheaves of subgroups of $\Lambda(A^0)$,
2. the sheaf $\Lambda^k(A^0)$ is normal in $\Lambda^{\leq k}(A^0)$ and
3. if one defines [MR91, Proposition 10]
   - $i : \Lambda^k(A^0) \hookrightarrow \Lambda^{\leq k}(A^0)$ to be the canonical inclusion and
   - $p : \Lambda^{\leq k}(A^0) \twoheadrightarrow \Lambda^{<k}(A^0)$ to be the truncation to terms of levels $< k$

does the exact sequence of sheaves

$$1 \longrightarrow \Lambda^k(A^0) \xrightarrow{i} \Lambda^{\leq k}(A^0) \xrightarrow{p} \Lambda^{<k}(A^0) \longrightarrow 1\,,$$

split.

From the splitting of the sequence, we obtain immediately the following decomposition into a semidirect product (cf. [Rob03, p. 75]).

Corollary 4.27

For any $k \in \mathcal{K}$, there are the two following ways of factoring $\Lambda^{\leq k}(A^0)$ in a semidirect product: [Lod94, Cor.I.5.2]

$$\begin{aligned}\Lambda^{\leq k}(A^0) &\cong \Lambda^{<k}(A^0) \ltimes \Lambda^k(A^0)\\ &\cong \Lambda^k(A^0) \ltimes \Lambda^{<k}(A^0)\,.\end{aligned}$$

This means that any germ $f^{\leq k} \in \Lambda^{\leq k}(A^0)$ can be uniquely written as

- $f^{\leq k} = f^{<k} g^k$, where $f^{<k} \in \Lambda^{<k}$ and $g^k \in \Lambda^k$, or
- $f^{\leq k} = f^k f^{<k}$, where $f^k \in \Lambda^k$ and $f^{<k} \in \Lambda^{<k}$.

Remark 4.27.1

We can get the factor $f^{<k}$ common to both factorizations by truncation of $f^{\leq k}$ to terms of level $< k$, i.e. by applying the map p from Proposition 4.26. This truncation can explicitly be achieved, in terms of Stokes matrices, by keeping in the representations $1 + \sum C^{(j,l)}$ of $f^{\leq k}$ only the blocks $C^{(j,l)}$ such that $\deg(q_j - q_l) < k$. [Lod94, Cor.I.5.2(ii)]

A factorization algorithm could then be:

> get the factor $f^{<k}$ common to both factorizations by truncation of $f^{\leq k}$ to terms of level $< k$ and set $g^k := (f^{<k})^{-1} f^{\leq k}$ and $f^k := f^{\leq k}(f^{<k})^{-1}$.

This decomposition in a semidirect product can be extended to all levels, since $\Lambda^{<k}(A^0) = \Lambda^{\leq \max\{k' \in \mathcal{K} | k' < k\}}$. Thus

$$\Lambda(A^0) \cong \ltimes_{k \in \mathcal{K}} \Lambda^k(A^0)\,,$$

where the semidirect product is taken in an ascending or descending order of levels.

Loday-Richaud states in her paper [Lod94, Prop.I.5.3] the following proposition, which is a more general version of Proposition 4.26.

Proposition 4.27.1

For any levels $k,k' \in \mathcal{K}$ with $k' < k$ one has: [Lod94, Prop.I.5.3]

1. the sheaf $\Lambda^{\geq k'}(A^0) \cap \Lambda^{\leq k'}(A^0)$ is normal in $\Lambda^{\leq k}(A^0)$;

2. the exact sequence of sheaves

$$1 \longrightarrow \Lambda^{\geq k'}(A^0) \cap \Lambda^{\leq k}(A^0) \xrightarrow{i} \Lambda^{\leq k}(A^0) \xrightarrow{p} \Lambda^{<k'}(A^0) \longrightarrow 1\,,$$

where

- i is the canonical inclusion and
- p is the truncation to terms of levels $< k'$,

splits.

TODO: is $\Lambda^{\geq k'}(A^0) \cap \Lambda^{\leq k}(A^0) = \Lambda^k(A^0)$ and thus the first proposition a corollary of this?

We can use this proposition to follow (cf. [Lod94, Cor.I.5.4]) that

1. the filtration

$$\Lambda^{k_r}(A^0) = \Lambda^{\geq k_r}(A^0) \subset \Lambda^{\geq k_{r-1}}(A^0) \subset \cdots \subset \Lambda^{\geq k_1}(A^0) = \Lambda(A^0)$$

is normal and

2. we can use this to achieve the decomposition

$$\Lambda(A^0) \cong \bigltimes_{k \in \mathcal{K}} \Lambda^k(A^0)$$

taken in an arbitrary order. In fact, one can also extend the algorithm from Remark 4.27.1 to an arbitrary order of levels.

The important statement, which we will use later, is then the following Proposition. It is stated by Loday-Richaud in her Paper [Lod94] as Proposition I.5.5 or in the Paper [MR91, Thm.4.8] by Martinet and Ramis.

Proposition 4.28

The results from above can be restricted to the Stokes groups. Thus, for $\alpha \in \mathbb{A}$, one has

$$\mathrm{Sto}_\alpha(A^0) \cong \bigltimes_{k \in \mathcal{K}_\alpha} \mathrm{Sto}_\alpha^k(A^0)$$

the semidirect product being taken in an arbitrary order (we will only be interested in the ascending order).

Definition 4.28.1

We will denote the map which gives the factors of this factorization by

$$i_\alpha : \mathrm{Sto}_\alpha(A^0) \xrightarrow{\cong} \prod_{k \in \mathcal{K}_\alpha} \mathrm{Sto}_\alpha^k(A^0)\,,$$

where the factorization is taken in ascending order.

Write $\rho_\alpha^k : \mathrm{Sto}_\alpha^k(A^0) \to \mathbb{S}\mathrm{to}_\alpha^k(A^0)$ for the restriction of the map ρ_α (cf. Proposition 4.17) to the level k. By abuse of notation we denote the induced decomposition also by

$$i_\alpha : \mathbb{S}\mathrm{to}_\alpha(A^0) \xrightarrow{\cong} \prod_{k\in\mathcal{K}_\alpha} \mathbb{S}\mathrm{to}_\alpha^k(A^0)$$

and the corresponding diagram

$$\begin{tikzcd} \mathrm{Sto}_\alpha(A^0) \arrow[r, "i_\alpha"] \arrow[d, "\rho_\alpha"] & \prod_{k\in\mathcal{K}_\alpha} \mathrm{Sto}_\alpha^k(A^0)\,, \arrow[d, "\prod_{k\in\mathcal{K}} \rho_\alpha^k"] \\ \mathbb{S}\mathrm{to}_\alpha(A^0) \arrow[r, "i_\alpha"] & \prod_{k\in\mathcal{K}_\alpha} \mathbb{S}\mathrm{to}_\alpha^k(A^0)\,, \end{tikzcd}$$

commutes.

4.3 Stokes structures: using Stokes groups

The goal in this section is to prove that there is a bijective and natural map [Lod94], [Lod14, Thm.4.3.11] and [Boa01; Boa99] and [BV89] and [BJL79] and [MR91, Chapter 4]

$$h : \prod_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha(A^0) \longrightarrow \mathcal{S}t(A^0)$$

which endows $\mathcal{S}t(A^0)$ with the structure of a unipotent Lie group. And since $\mathrm{Sto}_\alpha(A^0)$ has $\mathbb{S}\mathrm{to}_\alpha(A^0)$ as a faithful representation, we also get the isomorphism $\prod_{\alpha\in\mathbb{A}} \mathbb{S}\mathrm{to}_\alpha(A^0) \cong \mathcal{S}t(A^0)$ as a corollary. TODO: This goes back to [BJL79]?

Let us recall, that $\mathcal{S}t(A^0)$ is defined to be $H^1(S^1; \Lambda(A^0))$ (cf. Section 4.1). The elements of $\prod_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha(A^0)$ define in a canonical way cocycles of the sheaf $\Lambda(A^0)$ (cf. Equation (4.4)), called Stokes cocycles (cf. Definition 4.33). In fact, will h map such cocycles to the cohomology class, to which they correspond. Thus the statement, that h is a bijection, is equivalent to the statement that

> in each cohomology class of $\mathcal{S}t(A^0)$ is a unique 1-cocycle, which is a Stokes cocycle.

Cyclic coverings

To formulate the theorem in the next section, we use the notion of cyclic coverings and nerves of such coverings, which are defined as follows.

Definition 4.29

Let J be a finite set, identified to $\{1, \dots, p\} \subset \mathbb{Z}$. [Lod94, Sec.II.1] and [Lod94, Sec.II.3.1]

1. A *cyclic covering* of S^1 is a finite covering $\mathcal{U} = \left(U_j := U(\theta_j, \varepsilon_j)\right)_{j\in J}$ consisting of arcs, which satisfies that
   a) $\widetilde{\theta}_j \geq \widetilde{\theta}_{j+1}$ for $j \in \{1, \dots, p-1\}$, i.e. the center points are ordered in ascending order with respect to the clockwise orientation of S^1 and

b) $\widetilde{\theta}_j + \frac{\varepsilon_j}{2} \geq \widetilde{\theta}_{j+1} + \frac{\varepsilon_{j+1}}{2}$ for $j \in \{1, \ldots, p-1\}$ and $\widetilde{\theta}_p + \frac{\varepsilon_p}{2} \geq \widetilde{\theta}_1 - 2\pi + \frac{\varepsilon_1}{2}$, i.e. the arcs are not encased by another arc,

where the $\widetilde{\theta}_j \in [0, 2\pi[$ are determinations of the $\theta_j \in S^1$.

a) the θ_j are in ascending order with respect to the clockwise orientation of S^1;

b) the $U_j \cap U_{j+1}$ have only one connected component when $\#J > 2$;

c) the U_j are not encased by another arc, this means that the open sets $U_j \backslash U_l$ are connected for all $j, l \in J$.

2. The *nerve* of a cyclic covering $\mathcal{U} = \{U_j; j \in J\}$ is the family $\dot{\mathcal{U}} = \{\dot{U}_j; j \in J\}$ defined by:
   - $\dot{U}_j = U_j \cap U_{j+1}$ when $\#J > 2$,
   - $\dot{U}_1$ and $\dot{U}_2$ the connected components of $U_1 \cap U_2$ when $\#J = 2$.

Remark 4.29.1

The nerve of the cyclic covering $\mathcal{U} = (U(\theta_j, \varepsilon_j))_{j \in J}$ is explicitly given by

$$\dot{\mathcal{U}} = \left(\left(\theta_j - \frac{\varepsilon_j}{2}, \theta_{j+1} + \frac{\varepsilon_{j+1}}{2}\right)\right)_{j \in J}.$$

The cyclic coverings correspond one-to-one to nerves of cyclic coverings. If one starts with a nerve $\{\dot{U}_j \mid j \in J\}$, one obtains a cyclic covering as $\mathcal{U} = \{U_j \mid j \in J\}$ where the arc U_j are the connected clockwise hulls from $\dot{U}_{j-1}$ to $\dot{U}_j$.

Definition 4.30

A covering $\mathcal{V}$ is said to *refine* a covering $\mathcal{U}$ if, to each open set $V \in \mathcal{V}$ there is at least one $U \in \mathcal{U}$ with $V \subset U$.

Each refined covering of $\mathcal{U}$ is obtained by successively

1. narrowing an arc $U \in \mathcal{U}$ to a smaller arc $\widetilde{U} \subset U$ or
2. splitting an arc $U \in \mathcal{U}$ into two smaller arcs U' and U'' satisfying $U = U' \cup U''$.

This can be used to see the following proposition (cf. [Lod94, Prop.II.1.3]).

Proposition 4.31

The covering $\mathcal{V}$ refines $\mathcal{U}$ if and only if the corresponding nerves $\dot{\mathcal{U}} = \{\dot{U}_j\}$ and $\dot{\mathcal{V}} = \{\dot{V}_l\}$ satisfy

each $\dot{U}_j$ contains at least one $\dot{V}_l$.

The cyclic coverings and especially the nerves of such coverings will be useful, since we have the following proposition (cf. [Lod04, Prop.2.6]).

Proposition 4.32

The set of 1-cocycles of $\mathcal{U}$ is canonically isomorphic to the set of 1-cochains restricted to $\dot{\mathcal{U}} = \left(\dot{U}_j\right)_{j\in J}$ without any cocycle condition, i.e. the product $\prod_{j\in J}\Gamma(\dot{U}_j, \Lambda(A^0))$.

4.3.1 The theorem

Let $\{\theta_j \mid j \in J\} \subset S^1$ be a finite set and $\dot{\varphi} = (\dot{\varphi}_{\theta_j})_{j\in J} \in \prod_{j\in J}\Lambda_{\theta_j}(A^0)$ be a finite family of germs. Let $\dot{\varphi}_j$ be the function representing the germ $\dot{\varphi}_{\theta_j}$ on its (maximal) arc of definition Ω_j around θ_j. In the following way, one can associate a cohomology class in $\mathcal{S}t(A^0)$ to $\dot{\varphi}$: [Lod94, p. 868]

> for every cyclic covering $\mathcal{U} = (U_j)_{j\in J}$, which satisfies $\dot{U}_j \subset \Omega_j$ for all $j \in J$, one can define the 1-cocycle $(\dot{\varphi}_{j|\dot{U}_j})_{j\in J} \in \Gamma(\dot{\mathcal{U}}; \Lambda(A^0))$.

To a different cyclic covering, satisfying the condition above, this construction yields a cohomologous 1-cocycle, thus the induced map

$$\prod_{j\in J}\Lambda_{\theta_j}(A^0) \longrightarrow H^1(S^1; \Lambda(A^0)) = \mathcal{S}t(A^0) \tag{4.4}$$

is welldefined (cf. [Lod94, p. 868]).

Recall that $\nu = \#\mathbb{A}$ is the number of all anti-Stokes directions and the set of all anti-Stokes directions is denoted by $\mathbb{A} = \{\alpha_1, \dots, \alpha_\nu\}$.

Definition 4.33

A *Stokes cocycle* is a 1-cocycle $(\varphi_j)_{j\in\{1,\dots,\nu\}} \in \prod_{j\in\{1,\dots,\nu\}}\Gamma(U_j; \Lambda(A^0))$ corresponding to some cyclic covering with nerve $\dot{\mathcal{U}} = (\dot{U}_j)_{j\in\{1,\dots,\nu\}}$, which satisfies for every $j \in \{1, \dots, \nu\}$ [Lod94, Def.II.1.8] , [Lod14, p. 4.3.10] , [MR91, Defn 6 on p 374]

- $\alpha_j \in \dot{U}_j$ and
- the germ $\varphi_{\alpha_j} := \varphi_{j,\alpha_j}$ of φ_j at α_j is an element of $\mathrm{Sto}_{\alpha_j}(A^0)$.

Remark 4.33.1

PROBLEM: refactor!remove? The sections $\Gamma(\dot{U}_j; \Lambda(A^0))$ are uniquely determined as the extension of the germ at α_j, since the sheaf $\Lambda(A^0)$ defined via the system $[A^0, A^0]$ (cf. Definition 4.1). We thus have an injective map

$$\prod_{j\in\{1,\dots,\nu\}} \Gamma(\dot{U}_j; \Lambda(A^0)) \hookrightarrow \prod_{j\in\{1,\dots,\nu\}} \mathrm{Sto}_{\alpha_j}(A^0)\,,$$

which takes an Stokes cocycle and yields the corresponding Stokes germs. For a fine enough covering $\mathcal{U}$, i.e. a covering $\mathcal{U}$ with a nerve $\dot{\mathcal{U}}$ which consists of small enough arcs satisfying the conditions above, is this map a bijection.

We will use this fact implicitly and assume that the covering is always fine enough to call elements of $\prod_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha(A^0)$ Stokes cocycles.

We can use the mapping (4.4), corresponding to the construction at the beginning of this section, to obtain a mapping

$$h : \textstyle\prod_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha(A^0) \longhookrightarrow \prod_{\alpha\in\mathbb{A}} \Lambda_\alpha(A^0) \xrightarrow{(4.4)} \mathcal{S}t(A^0),$$

which takes a complete set of Stokes germs to its corresponding cohomology class, given by a Stokes cocycle.

Theorem 4.34

The map

$$h : \textstyle\prod_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha(A^0) \longrightarrow \mathcal{S}t(A^0)$$

is a bijection and natural.

[Lod94, p. 869],[Lod94, Sec.III.3.3]

Remark 4.34.1

Natural means that h commutes to isomorphisms and constructions over systems or connections they represent.

To define the inverse map of h, one has to find in each cocycle in $\mathcal{S}t(A^0)$ the Stokes cocycle and take the germs. Loday-Richaud gives an algorithm in Section II.3.4 of her paper [Lod94], which takes a cocycle over an arbitrary cyclic covering and outputs cohomologous Stokes cocycle and thus solves this problem. This means, that the inverse of h is constructible.

Corollary 4.35

PROBLEM: mentioned twice PROBLEM: remove? Using the isomorphisms $\mathrm{Sto}_\theta(A^0) \cong \mathbb{S}\mathrm{to}_\theta(A^0)$ from Proposition 4.17 we obtain

$$\mathcal{S}t(A^0) \cong \prod_{\alpha \in \mathbb{A}} \mathbb{S}\mathrm{to}_\alpha(A^0)\,.$$

Since the product of Lie groups is in an obvious way again a Lie group, this endows $\mathcal{S}t(A^0)$ with the structure of an unipotent Lie group with the finite complex dimension $N := \dim_{\mathbb{C}} \mathcal{S}t(A^0)$ (cf. [Lod94, Sec.III.1]).

Remark 4.36

This number N is known to be the *irregularity* of $[\mathrm{End}\, A^0]$ and can be rewritten in the following way:

$$N = \sum_{\alpha \in \mathbb{A}} \dim_{\mathbb{C}} \mathbb{S}\mathrm{to}_\alpha(A^0) = \sum_{\alpha \in \mathbb{A}} \sum_{q_j \underset{\alpha}{\prec} q_l} n_j \cdot n_l = \sum_{\substack{1 \le j,l \le n \\ j<l}} 2 \cdot \deg(q_j - q_l) \cdot n_j \cdot n_l\,.$$

One can also define the structure of a linear affine variety on the set $\mathcal{S}t(A^0)$. This was for example done in Section [BV89, Sec.II.3] or in [Lod94, Sec.III.1], where actually multiple structures of linear affine varieties on $\mathcal{S}t(A^0)$ are defined. In [Var96, 35ff] is mentioned that one can also define a scheme structure on $\mathcal{S}t(A^0)$.

4.3.2 Proof of Theorem 4.34

We will only look at the unramified case, for which we refer to [Lod94, Sec.II.3]. The proof in the ramified case can be found in Section II.4 of Loday-Richaud's Paper [Lod94] and a sketch of the complete proof is also in [Lod04]. We first have to introduce the notion of adequate coverings, which will be used in the proof.

Adequate coverings

TODO: Adequate is acyclic in Loday2004?

1. [MR91, p. 371] defines adapted coverings
2. [Lod04, p. 5269] defines acyclic coverings
   - uses the **theorem of Leray** `https://de.wikipedia.org/wiki/Satz_von_Leray`

Definition 4.37

Let $\star \in \{k, <k, \le k, \dots\}$. A covering $\mathcal{U}$ beyond which the inductive limit

$$\varinjlim_{\mathcal{U}} H^1(\mathcal{U}; \Lambda^\star(A^0))$$

is stationary is said to be *adequate* to describe $H^1(S^1; \Lambda^\star(A^0))$ or *adequate* to $\Lambda^\star(A^0)$.

A covering $\mathcal{U}$ is said to be *adequate* to describe $H^1(S^1; \Lambda^\star(A^0))$ or *adequate* to $\Lambda^\star(A^0)$ if for every element in $\varinjlim_{\mathcal{U}} H^1(\mathcal{U}; \Lambda^\star(A^0))$ given by some covering $\mathcal{U}'$ and an element of $\Gamma(\mathcal{U}'; \Lambda^\star(A^0))$ there exists

- an element in $\Gamma(\mathcal{U}; \Lambda^\star(A^0))$ and
- an common refinement of $\mathcal{U}$ and $\mathcal{U}'$

such that PROBLEM:the elements are ?? on the refined covering.

In other words is a covering $\mathcal{U}$ adequate to $\Lambda^\star(A^0)$, if and only if the quotient map

$$\Gamma(\mathcal{U}; \Lambda^\star(A^0)) \longrightarrow H^1(S^1; \Lambda^\star(A^0))$$

is surjective. TODO: Proof? check??

[MR91, p. 371] introduces the following definition

Definition 4.37.1

A covering $\mathcal{U}$ is *adapted* if every anti-Stokes direction is contained in exactly one element of the nerve $\dot{\mathcal{U}}$.

Adequate coverings are sometimes called acyclic, for example in [Lod04], or adapted, for example in [MR91].

The following proposition is in Loday-Richaud's paper [Lod94] given as Proposition II.1.7. It contains a simple characterization, which will be used to see, that our defined coverings are adequate.

Proposition 4.38

[Lod94, Prop.II.1.7]

Let $k \in \mathcal{K}_\alpha$.

Definition 4.38.1

Let $\alpha \in \mathbb{A}^k$. An arc $U\left(\alpha, \frac{\pi}{k}\right)$ is called a *Stokes arc of level k at α*.

If $(q_j, q_l) \in \mathcal{Q}(A^0)$ is a pair, such that $q_j \underset{\alpha}{\prec} q_l$, is the arc $U\left(\alpha, \frac{\pi}{k}\right)$ exactly the arc of decay of the exponential $e^{q_j - q_l}$ (cf. [Lod04, p. 5269] and Remark 4.12).

A cyclic covering $\mathcal{U} = (U_j)_{j \in J}$ which satisfies

> for every $\alpha \in \mathbb{A}^k$ contains the Stokes arc $U\left(\alpha, \frac{\pi}{k}\right)$ at least one arc $\dot{U}_j$ from the nerve $\dot{\mathcal{U}}$ of $\mathcal{U}$

is adequate to $\Lambda^k(A^0)$.

The covering $\mathcal{U}$ is adequate to $\Lambda^{\leq k}(A^0)$ (resp. $\Lambda^{<k}(A^0)$) if it is adequate to $\Lambda^{k'}(A^0)$ for every $k' \leq k$ (resp. $k' < k$).

Proof. Show that for every $U_j, U_l \in \mathcal{U}$ is

$$H^1(U_j \cap U_l; \Lambda^k(A^0)) = 0\,.$$

Then the theorem of Leray implies that $H^1(S^0; \Lambda^k(A^0)) = H^1(\mathcal{U}; \Lambda^k(A^0))$ □

Let $k \in \mathcal{K}$. We want to define the three cyclic coverings $\mathcal{U}^k$, $\mathcal{U}^{\leq k}$ and $\mathcal{U}^{<k}$ which will be adequate to $\Lambda^k(A^0)$, $\Lambda^{\leq k}(A^0)$ and $\Lambda^{<k}(A^0)$,respectively . Furthermore will the coverings be comparable at the different levels.

1. The first covering $\mathcal{U}^k = \{\dot{U}^k_\alpha \mid \alpha \in \mathbb{A}^k\}$ is the cyclic covering determined by the nerve

$$\dot{\mathcal{U}}^k := \left\{\dot{U}^k_\alpha = U\left(\alpha, \frac{\pi}{k}\right) \mid \alpha \in \mathbb{A}^k\right\}$$

consisting of all Stokes arcs of level k for anti-Stokes directions bearing the level k.

Since $\dot{\mathcal{U}}^k$ consists only of arcs with equal opening is this canonically a nerve.

Remark 4.39

Boalch, who looks only at the single-leveled case, introduces in his publications [Boa01, p. 19] and [Boa99, Def.1.23] the notion of supersectors which are defined as follows:

> write the anti-Stokes directions as $\mathbb{A} = \{\alpha_1, \ldots, \alpha_\nu\}$ arranged according to the clockwise ordering, then is the i-th *supersector* defined as the arc
>
> $$\widehat{\mathrm{Sect}}^k_i := \left(\alpha_{i+1} - \frac{\pi}{2k}, \alpha_i + \frac{\pi}{2k}\right)\,.$$

This yields a cyclic covering $(\widehat{\mathrm{Sect}}^k_i)_{i\in\{1,\ldots,\nu\}}$ whose nerve is exactly $\dot{\mathcal{U}}^k$, which was defined above.

If we extend to a subset J of $\mathcal{K}$ containing more then one level level, $\#J > 1$, the set

$$\bigcup_{k\in J}\left\{U\left(\alpha, \frac{\pi}{k}\right) \mid \alpha \in \mathbb{A}^k\right\} = \bigcup_{k\in J}\dot{\mathcal{U}}^k$$

is no longer guaranteed to be a nerve. Hence we have to define the coverings $\mathcal{U}^{\leq k}$ and $\mathcal{U}^{<k}$ in a different way. We will start by adding the arcs corresponding to the largest degree first and continue by adding the arcs corresponding to smaller degrees successively as described below.

Denote by $\{K_1 < \cdots < K_s = k\} = \left\{\max\left(\mathcal{K}_\alpha \cap [0,k]\right) \mid \alpha \in \mathbb{A}^{\leq k}\right\}$ the set of all k-*maximum levels* for all $\alpha \in \mathbb{A}^{\leq k}$.

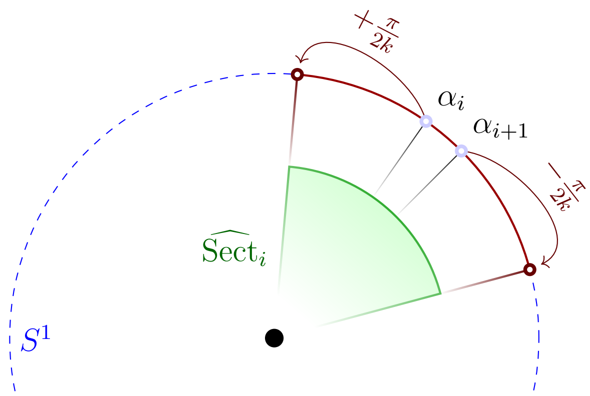


Figure 4.2: An exemplary supersector $\widehat{\mathrm{Sect}}_i$ corresponding to the anti-Stokes directions α_i and α_{i+1}.

2. The cyclic covering $\mathcal{U}^{\leq k} = \left\{U_\alpha^{\leq k} \mid \alpha \in \mathbb{A}^{\leq k}\right\}$ will be defined by decreasing induction on the levels. Let us assume that

> the $\dot{U}_\alpha^{\leq k}$ are defined for all $\alpha \in \mathbb{A}^{\leq k}$ with k-maximum level greater than K_i such that their complete family is a nerve.

For every anti-Stokes direction α with k-maximum level K_i let α^- (resp. α^+) be the next anti-Stokes direction with k-maximum level greater then K_i on the left (resp. on the right).

Define $\dot{U}_{\alpha^-,\alpha^+}$ as the clockwise hull of the arcs $\dot{U}_{\alpha^-}^{\leq k}$ and $\dot{U}_{\alpha^+}^{\leq k}$ already defined by induction. If there are no anti-Stokes directions with k-maximum level greater then K_i we set $\dot{U}_{\alpha^-,\alpha^+} = S^1$.

We then add for every anti-Stokes direction α with k-maximum level K_i the arc

$$\dot{U}_\alpha^{\leq k} := U\Big(\alpha, \frac{\pi}{K_i}\Big) \cap \dot{U}_{\alpha^-,\alpha^+}$$

to $\dot{\mathcal{U}}^{\leq k}$ and the received family is still a nerve.

3. The last cyclic covering, $\mathcal{U}^{<k} = \left\{U_\alpha^{<k} \mid \alpha \in \mathbb{A}^{<k}\right\}$, is defined as $\mathcal{U}^{<k} := \mathcal{U}^{\leq k'}$ where $k' := \max\{k'' \in \mathcal{K} \mid k'' < k\}$.

Remark 4.40

The coverings $\mathcal{U}^k$, $\mathcal{U}^{\leq k}$ and $\mathcal{U}^{<k}$ depend only on $\mathcal{Q}(A^0)$. Hence they depend only on the determining polynomials.

For every $k' \leq k \in \mathcal{K}$ such that $k' \leq k$ and every $\alpha \in \mathbb{A}^{k'} \subset \mathbb{A}^{\leq k}$ (resp. $\alpha \in \mathbb{A}^{<k'} \subset \mathbb{A}^{\leq k}$) is $\dot{U}_\alpha^{\leq k} \subset \dot{U}_\alpha^{k'}$ (resp. $\dot{U}_\alpha^{\leq k} \subset \dot{U}_\alpha^{<k'}$), thus the covering $\mathcal{U}^{\leq k}$ refines $\mathcal{U}^{k'}$ and $\mathcal{U}^{<k'}$. Thus the coverings are comparable on the different levels. Furthermore are the coverings defined, such that they satisfy the condition in Proposition 4.38, such that the first property in the following proposition is satisfied. The other two can be found at [Lod94, Prop.II.3.1 (iv)] or on page 5269 of [Lod04].

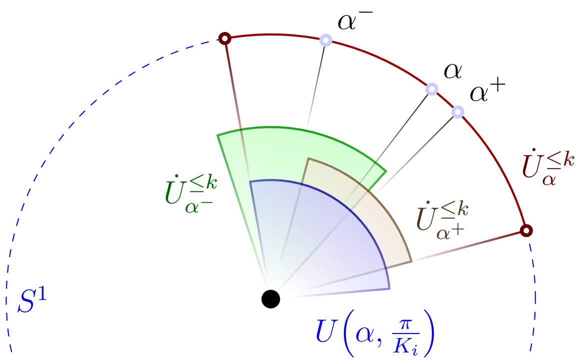


Figure 4.3: An exemplary construction of an arc $\dot{U}_{\alpha}^{\leq k}$ corresponding (in red) to the anti-Stokes direction α. The green arc is $\dot{U}_{\alpha^-}^{\leq k}$, the brown arc is $\dot{U}_{\alpha^+}^{\leq k}$ and the blue arc is the Stokes arc of level K_i at α, where K_i is the k-maximum level of α.

Proposition 4.41

PROBLEM: Was bedeutet das? Let $k \in \mathcal{K}$, then [Lod94, Prop.II.3.1]

1. the coverings $\mathcal{U}^k$, $\mathcal{U}^{\leq k}$ and $\mathcal{U}^{<k}$ are adequate to $\Lambda^k(A^0)$, $\Lambda^{\leq k}(A^0)$ and $\Lambda^{<k}(A^0)$, respectively;
2. there exists no 0-cochain in $\Lambda^k(A^0)$ on $\mathcal{U}^k$;
3. on $\mathcal{U}^{\leq k}$ there is no 0-cochain in $\Lambda^{\leq k}(A^0)$ of level k, i.e. all 0-cochains of $\Lambda^{\leq k}(A^0)$ belong to $\Lambda^{<k}(A^0)$.

To have a shorter notation, we denote the product $\prod_{\alpha \in \mathbb{A}^\star} \Gamma(\dot{U}_{\alpha}^{\star}; \Lambda^\star(A^0))$ by $\Gamma(\dot{U}^\star; \Lambda^\star(A^0))$ for every $\star \in \{k, <k, \leq k, \dots\}$. Let us also denote $\mathcal{U} := \mathcal{U}^k$ where k is the maximal degree, i.e. $k = \max \mathcal{K}$.

The case of a unique level

First we will proof Theorem 4.34 in the case of a unique level. This means that [Lod94, p. II.3.2]

- either $\Lambda(A^0)$ has only one level k, thus
$$\Lambda(A^0) = \Lambda^k(A^0) \text{ and } \mathrm{Sto}_\theta(A^0) = \mathrm{Sto}_\theta^k(A^0) \text{ for every } \theta,$$
- or we restrict to a given level $k \in \mathcal{K}$.

The following lemma, which will also be required for the multi-leveled case, solves the case of a unique level.

Lemma 4.42

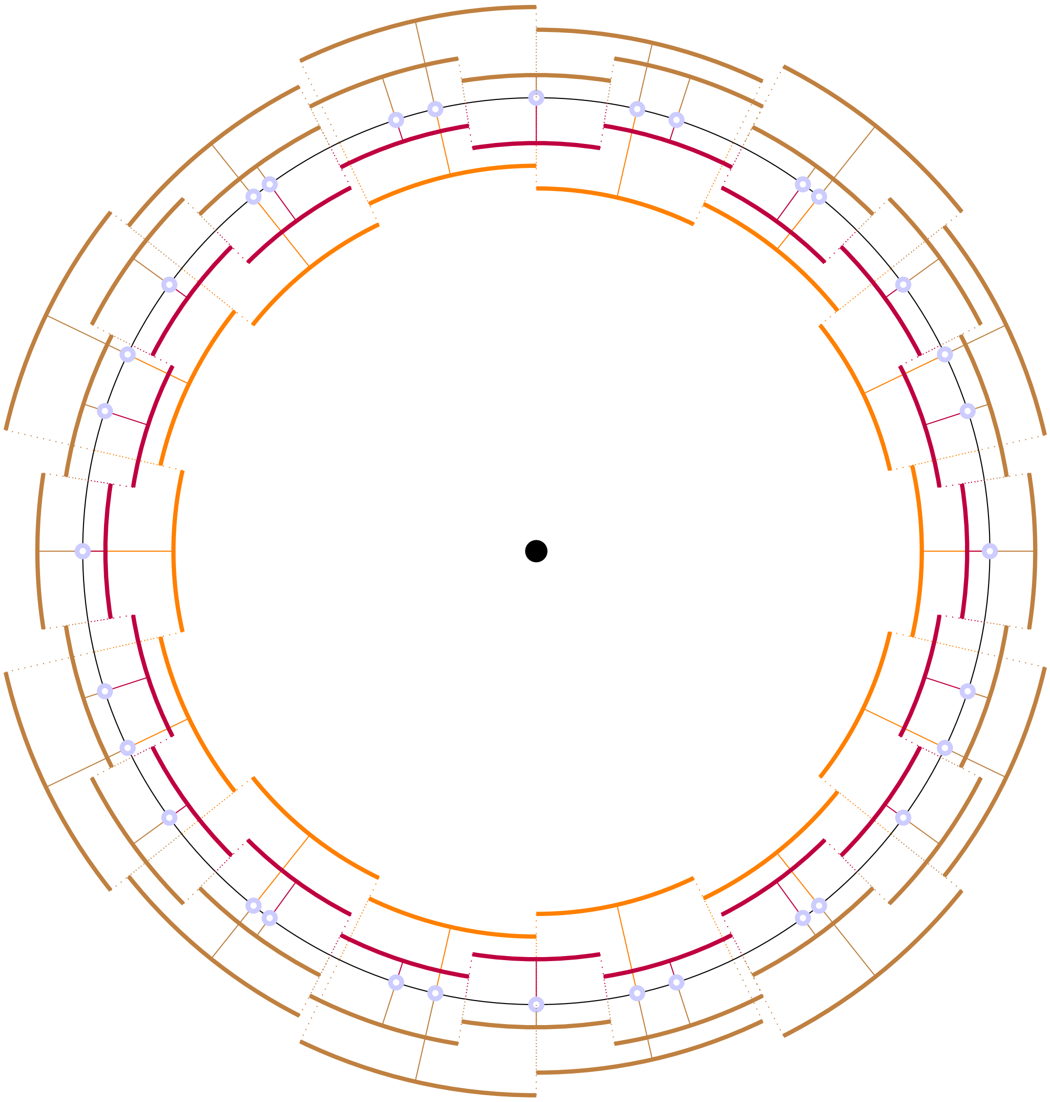


Figure 4.4: The adequate coverings for an example with $\mathcal{K} = \{7, 10\}$ and $\mathbb{A} = \left\{\frac{j\cdot\pi}{k} \mid k \in \mathcal{K},\, j \in \mathbb{N}\right\}$. The anti-Stokes directions are marked by the blue circles. The arcs of $\dot{\mathcal{U}}^{7} = \dot{\mathcal{U}}^{\leq 7}$ are orange, the arcs of $\dot{\mathcal{U}}^{10}$ are purple and the arcs of $\dot{\mathcal{U}}^{\leq 10} = \dot{\mathcal{U}}$ are brown.

Let $k \in \mathcal{K}$. The morphism h from Theorem 4.34 is in the case of a unique level build as

$$\begin{tikzcd}
\prod\limits_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha^k(A^0) \arrow[r, "i^k"] \arrow[d, equal] & \Gamma(\dot{\mathcal{U}}^k;\Lambda^k(A^0)) \arrow[r, "s^k"] & H^1(S^1;\Lambda^k(A^0)) \arrow[d, equal] \\
\prod\limits_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha(A^0) \arrow[rr, "h"] & & H^1(S^1;\Lambda(A^0))
\end{tikzcd}$$

only in the single-leveled case

from

- the canonical injective map i^k i.e. the map which is the canonical extension of germs to their natural arc of definition, and
- the quotient map s^k

which are both isomorphisms.

Proof. 1. The map

$$i^k : \prod_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha^k(A^0) \longrightarrow \textstyle\prod_{\alpha\in\mathbb{A}^k} \Gamma(\dot{\mathcal{U}}_\alpha^k;\Lambda^k(A^0))$$

is welldefined, i.e. the germs at α can be uniquely extended to the arc $\dot{\mathcal{U}}_\alpha^k$, since

- the sheaf $\Lambda(A^0)$, and thus $\Lambda^k(A^0)$, is a piecewise-constant sheaf, thus we know that the sections on arcs are uniquely determined by their germs and
- arcs of $\dot{\mathcal{U}}_\alpha^k$ are the natural domains of existence of the corresponding Stokes germs.

The second fact can be found in Loday-Richaud's paper [Lod04] on page 5269. In [Lod04, p. 5269] is also stated, that only the extensions of Stokes germs at α are sections of $\Lambda(A^0)$ on $\dot{U}_\alpha^k$ and this implies surjectivity.

2. The second map

$$s^k : \textstyle\prod_{\alpha\in\mathbb{A}^k} \Gamma(\dot{\mathcal{U}}_\alpha^k;\Lambda^k(A^0)) \longrightarrow H^1(S^1;\Lambda^k(A^0))$$

is a bijection, since from Proposition 4.41 we know that it is

- **surjective**, since $\mathcal{U}^k$ is adequate to $\Lambda^k(A^0)$ and
- **injective**, since on $\mathcal{U}^k$ there is no 0-cochain in $\Lambda^k(A^0)$.

PROBLEM: Show that this is the correct h PROBLEM: Naturality? □

The case of several levels

Let us now proof the Theorem 4.34 in the case of several levels, We will begin by defining some maps, which will be composed to obtain the isomorphism h from the theorem.

Definition 4.43

Begin with the *product map of cocycles* $\mathfrak{S}^{\leq k}$. This map will be composed from the following injective maps:

1. The first map is defined as

$$\begin{aligned} \sigma^k : \Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A^0)) &\longrightarrow \Gamma(\dot{\mathcal{U}}^{\leq k}; \Lambda^{\leq k}(A^0)) \\ \dot{f} = (\dot{f}_\alpha)_{\alpha \in \mathbb{A}^k} &\longmapsto (\dot{G}_\alpha)_{\alpha \in \mathbb{A}^{\leq k}} \end{aligned}$$

where

$$\dot{G}_\alpha = \begin{cases} \dot{f}_\alpha \text{ restricted to } \dot{U}_\alpha^{\leq k} \text{ and seen as being in } \Lambda^{\leq k}(A^0) & \text{when } \alpha \in \mathbb{A}^k \\ \mathrm{id} \ \ \text{(the identity)} & \text{when } \alpha \notin \mathbb{A}^k \end{cases}$$

2. and the second map

$$\begin{aligned} \sigma^{<k} : \Gamma(\dot{\mathcal{U}}^{<k}; \Lambda^{<k}(A^0)) &\longrightarrow \Gamma(\dot{\mathcal{U}}^{\leq k}; \Lambda^{\leq k}(A^0)) \\ \dot{f} = (\dot{f}_\alpha)_{\alpha \in \mathbb{A}^{<k}} &\longmapsto (\dot{F}_\alpha)_{\alpha \in \mathbb{A}^{\leq k}} \end{aligned}$$

is defined, in a similar way, as

$$\dot{F}_\alpha = \begin{cases} \dot{f}_\alpha \text{ restricted to } \dot{U}_\alpha^{\leq k} \text{ and seen as being in } \Lambda^{\leq k}(A^0) & \text{when } \alpha \in \mathbb{A}^{<k} \\ \mathrm{id} \ \ \text{(the identity)} & \text{when } \alpha \notin \mathbb{A}^{<k} \end{cases}$$

Thus we can define

$$\begin{aligned} \mathfrak{S}^{\leq k} : \Gamma(\dot{\mathcal{U}}^{<k}; \Lambda^{<k}(A^0)) \times \Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A^0)) &\longrightarrow \Gamma(\dot{\mathcal{U}}^{\leq k}; \Lambda^{\leq k}(A^0)) \\ (\dot{f}, \dot{g}) &\longmapsto (\dot{F}_\alpha \dot{G}_\alpha)_{\alpha \in \mathbb{A}^{\leq k}} \end{aligned}$$

where $(\dot{F}_\alpha)_{\alpha \in \mathbb{A}^{\leq k}} = \sigma^{<k}(\dot{f})$ and $(\dot{G}_\alpha)_{\alpha \in \mathbb{A}^{\leq k}} = \sigma^k(\dot{g})$ are defined as above.

Remark 4.43.1

This map $\mathfrak{S}^{\leq k}$ is injective, since injectivity for germs implies injectivity for sections.

Lemma 4.44

[Lod94, Lem.II.3.3] Let $k \in \mathcal{K}$.

1. If the cocycles $\mathfrak{S}^{\leq k}(\dot{f}, \dot{g})$ and $\mathfrak{S}^{\leq k}(\dot{f}', \dot{g}')$ are cohomologous in $\Gamma(\dot{\mathcal{U}}^{\leq k}; \Lambda^{\leq k}(A^0))$ then $\dot{f}$ and $\dot{f}'$ are cohomologous in $\Gamma(\dot{\mathcal{U}}^{<k}; \Lambda^{<k}(A^0))$.

2. Any cocycle in $\Gamma(\dot{\mathcal{U}}^{\leq k}; \Lambda^{\leq k}(A^0))$ is cohomologous to a cocycle in the range of $\mathfrak{S}^{\leq k}$.

Proof. 1. Denote by α^+ the nearest anti-Stokes direction in $\mathbb{A}^{\leq k}$ on the right of α. The cocycles $\mathfrak{S}^{\leq k}(\dot{f}, \dot{g})$ and $\mathfrak{S}^{\leq k}(\dot{f}', \dot{g}')$ are cohomologous if and only if there is a 0-cochain $c = (c_\alpha)_{\alpha \in \mathbb{A}^{\leq k}} \in \Gamma(\mathcal{U}^{\leq k}, \Lambda^{\leq k}(A^0)$ such that

$$\dot{F}_\alpha \dot{G}_\alpha = c_\alpha^{-1} \dot{F}'_\alpha \dot{G}'_\alpha c_{\alpha^+} \tag{4.5}$$

for every $\alpha \in \mathbb{A}$. From Proposition 4.41 follows, that c is with values in $\Lambda^{<k}(A^0)$. The fact that $\Lambda^k(A^0)$ is normal in $\Lambda^{\leq k}(A^0)$ in Proposition 4.26, can be used to see that $c_{\alpha^+}^{-1} G'_\alpha c_{\alpha^+} \in \Gamma(\mathcal{U}^{\leq k}; \Lambda^k(A^0))$. Thus, we rewrite the relation (4.5) to

$$\dot{F}_\alpha \dot{G}_\alpha = (c_\alpha^{-1} \dot{F}'_\alpha c_{\alpha^+})(c_{\alpha^+}^{-1} G'_\alpha c_{\alpha^+}) \,, \qquad \text{for } \alpha \in \mathbb{A}^{\leq k} \,.$$

Since Corollary 4.27 tells us that this factorization corresponds to a semidirect product and thus is unique we obtain

$$\dot{F}_\alpha = c_\alpha^{-1} \dot{F}'_\alpha c_{\alpha^+} \qquad \text{and} \qquad \dot{G}_\alpha = c_{\alpha^+}^{-1} \dot{G}'_\alpha c_{\alpha^+}$$

for all $\alpha \in \mathbb{A}^{\leq k}$. The former relation implies that $(\dot{F}_\alpha)$ and $(\dot{F}'_\alpha)$ are cohomologous with values in $\Lambda^{<k}(A^0)$ on $\mathcal{U}^{\leq k}$. Since the coarser covering $\mathcal{U}^{<k}$ is already adequate to $\Lambda^{<k}(A^0)$ are $(\dot{F}_\alpha)$ and $(\dot{F}'_\alpha)$ already in $\Gamma(\dot{\mathcal{U}}^{<k}; \Lambda^{<k}(A^0))$ cohomologous.

2. The proof of part 2. (together with a proof of part 1.) can be found in Loday-Richaud's paper [Lod94, Proof of Lem.II.3.3].

□

Let $k \in \mathcal{K} = \{k_1 < k_2 < \cdots < k_r\}$ and $k' = \max\{k' \in \mathcal{K} \mid k' < k\}$. We then know by definition that $\mathcal{U}^{<k} = \mathcal{U}^{\leq k'}$ as well as $\Lambda^{<k}(A^0) = \Lambda^{\leq k'}(A^0)$ and thus $\Gamma(\dot{\mathcal{U}}^{<k}; \Lambda^{<k}(A^0)) = \Gamma(\dot{\mathcal{U}}^{\leq k'}; \Lambda^{\leq k'}(A^0))$ and we obtain the following proposition.

Proposition 4.45

By applying $\mathfrak{S}^{\leq k}$ successively for different k's in decending order, one obtains the *product map of single-leveled cocycles* τ in the following way

$$\underbrace{\Gamma(\dot{\mathcal{U}}^{<k_r};\Lambda^{<k_r}(A^0))} \times \Gamma(\dot{\mathcal{U}}^{k_r};\Lambda^{k_r}(A^0)) \xrightarrow{\mathfrak{S}^{\leq k_r}} \begin{array}{c}\scriptstyle \Gamma(\dot{\mathcal{U}};\Lambda(A^0)) \\ \scriptstyle \parallel \\ \Gamma(\dot{\mathcal{U}}^{\leq k_r};\Lambda^{\leq k_r}(A^0))\end{array}$$

$$\cong$$

$$\underbrace{\Gamma(\dot{\mathcal{U}}^{<k_{r-1}};\Lambda^{<k_{r-1}}(A^0))} \times \Gamma(\dot{\mathcal{U}}^{k_{r-1}};\Lambda^{k_{r-1}}(A^0)) \xrightarrow{\mathfrak{S}^{\leq k_{r-1}}} \Gamma(\dot{\mathcal{U}}^{\leq k_{r-1}};\Lambda^{\leq k_{r-1}}(A^0))$$

$$\cong$$

$$\cdots \xrightarrow{\mathfrak{S}^{\leq k_{r-2}}} \Gamma(\dot{\mathcal{U}}^{\leq k_{r-2}};\Lambda^{\leq k_{r-2}}(A^0))$$

$$\underbrace{\Gamma(\dot{\mathcal{U}}^{<k_3};\Lambda^{<k_3}(A^0))} \times \Gamma(\dot{\mathcal{U}}^{k_3};\Lambda^{k_3}(A^0)) \xrightarrow{\mathfrak{S}^{\leq k_3}} \cdots$$

$$\cong$$

$$\Gamma(\dot{\mathcal{U}}^{k_1};\Lambda^{k_1}(A^0)) \times \Gamma(\dot{\mathcal{U}}^{k_2};\Lambda^{k_2}(A^0)) \xrightarrow{\mathfrak{S}^{\leq k_2}} \Gamma(\dot{\mathcal{U}}^{\leq k_2};\Lambda^{\leq k_2}(A^0))$$

which can be written in the following compact form

$$\begin{aligned} \tau : \prod_{k\in\mathcal{K}} \Gamma(\dot{\mathcal{U}}^k;\Lambda^k(A^0)) &\longrightarrow \Gamma(\dot{\mathcal{U}};\Lambda(A^0)) \\ (\dot{f}^k)_{k\in\mathcal{K}} &\longmapsto \prod_{k\in\mathcal{K}} \tau^k(\dot{f}^k) \end{aligned}$$

where the product is following an ascending order of levels and the maps τ_k are defined as

$$\begin{aligned} \tau^k : \Gamma(\dot{\mathcal{U}}^k;\Lambda^k(A^0)) \xrightarrow{\sigma^k} \Gamma(\dot{\mathcal{U}}^{\leq k};\Lambda^{\leq k}(A^0)) &\longrightarrow \Gamma(\dot{\mathcal{U}};\Lambda(A^0)) \\ (\dot{f}_\alpha)_{\alpha\in\mathbb{A}^k} &\longmapsto (\dot{G}_\alpha)_{\alpha\in\mathbb{A}} \end{aligned}$$

with

$$\dot{G}_\alpha = \begin{cases} \dot{f}_\alpha \text{ restricted to } \dot{U}_\alpha \text{ and seen as being in } \Lambda(A^0) & \text{when } \alpha\in\mathbb{A}^k \\ \mathrm{id} \ \ (\text{the identity on } \dot{U}_\alpha) & \text{when } \alpha\notin\mathbb{A}^k \end{cases}$$

The defined map τ is clearly injective and it can be extended to an arbitrary order of levels (cf. Remark [Lod94, Rem.II.3.5]).

From the previous statements one obtains the following Corollary (cf. [Lod94, Prop.II.3.4]).

Corollary 4.46

The product map of single-leveled cocycles τ induces on the cohomology a bijective and natural map

$$\begin{tikzcd}
\mathcal{T} : \prod_{k\in\mathcal{K}} \Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A^0)) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & H^1(\mathcal{U}; \Lambda(A^0))\,. \arrow[d, phantom, "\cong" rotate=90] \\
\prod\limits_{k\in\mathcal{K}} H^1(S^1;\Lambda^k(A^0)) & H^1(S^1;\Lambda(A^0))
\end{tikzcd}$$

Composing functions to obtain h We have the ingredients to define the function h from Theorem 4.34 by composition of already bijective maps.

Proof of Theorem 4.34. Let $i_\alpha : \mathrm{Sto}_\alpha(A^0) \to \prod_{k\in\mathcal{K}} \mathrm{Sto}^k_\alpha(A^0)$ be the map which corresponds to the filtration from Proposition 4.28 and denote the composition

$$\begin{tikzcd}
\prod\limits_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha(A^0) \arrow[r, "\prod_{\alpha\in\mathbb{A}} i_\alpha"] \arrow[drr, bend right, "\mathfrak{T}"'] & \prod\limits_{\alpha\in\mathbb{A}} \prod\limits_{k\in\mathcal{K}} \mathrm{Sto}^k_\alpha(A^0) \arrow[d, phantom, "\cong" rotate=90] & \\
& \prod\limits_{k\in\mathcal{K}} \prod\limits_{\alpha\in\mathbb{A}} \mathrm{Sto}^k_\alpha(A^0) \arrow[r, "\prod_{k\in\mathcal{K}} i^k"] & \prod\limits_{k\in\mathcal{K}} \Gamma(\dot{\mathcal{U}}^k; \Lambda^k(A^0))
\end{tikzcd}$$

by $\mathfrak{T}$, where i^k was defined in Lemma 4.42. The bijection h is then obtained as the composition

$$\mathcal{T} \circ \mathfrak{T} : \prod_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha(A^0) \longrightarrow H^1(\mathcal{U}; \Lambda(A^0))\,.$$

PROBLEM:naturality (is obvious?) PROBLEM:Show that this is the correct h $\square$

4.3.3 Some exemplary calculations

Here we want to discuss which information is required to describe the Stokes cocycle corresponding to a multi-leveled system in more depth. We will look at the simplest case of multi-leveled systems, i.e. systems corresponding to a normal form in $\mathrm{GL}_3(\mathbb{C}(\{t\}))$ with exactly two levels, and will apply the techniques developed in the previous sections in an rather explicit way.

Let A^0 be a normal form in $\mathrm{GL}_3(\mathbb{C}(\{t\}))$ with two levels $\mathcal{K} = \{k_1 < k_2\}$ and assume that there is at least one anti-Stokes direction θ which is beared by both levels. This assumption implies that the set of anti-Stokes directions is uniquely determined by the levels and θ. Let $q_j(t^{-1})$ be the determining polynomials and let k_{jl} be the degrees of $(q_j - q_l)(t^{-1})$.

Lemma 4.47

Up to permutation we know that the leading coefficients of $(q_1 - q_2)(t^{-1})$ and $(q_1 - q_3)(t^{-1})$ are equal.

Proof. Let q_1, q_2, q_3 be sorted, such that

$$\deg(q_1 - q_2) =: k_2 > k_1 := \deg(q_2 - q_3),$$

i.e. such that the leading coefficients $\frac{a_{1,2}}{t^{k_2}}$ and $\frac{a_{2,3}}{t^{k_1}}$ are of different degree, seen as polynomials in t^{-1}. To find the leading coefficient of $q_1 - q_3$ we have to distinguish different cases:

1. $\deg(q_1) = \deg(q_2) = k_2$: in this case has q_3 to be of degree k_2. Even further, q_2 and q_3 have to have the same leading coefficients, since they have to cancel out.
2. $\deg(q_1) < k_2$: then has q_2 to be of degree k_2 and we can continue similar to the first case.
3. $\deg(q_2) < k_2$: then can't q_3 be of degree $k_2 = \deg(q_1)$.

Thus we know for both, q_2 and q_3, that the coefficients of degree k_2 are either equal or both zero. □

From the previous lemma follows that

- up to permutation is $k_2 = k_{1,2} = k_{1,3}$ and $k_1 = k_{2,3}$, i.e. the larger degree appears twice, and
- $q_1 \underset{\alpha}{\prec} q_2$ (resp. $q_2 \underset{\alpha}{\prec} q_1$) if and only if $q_1 \underset{\alpha}{\prec} q_3$ (resp. $q_3 \underset{\alpha}{\prec} q_1$) and thus do the pairs (q_1, q_2) and (q_1, q_3) determine the same anti-Stokes directions.

The set of all anti-Stokes directions is then given as

$$\mathbb{A} = \left\{ \theta + \frac{\pi}{k} \cdot j \mid k \in \mathcal{K},\, j \in \mathbb{N} \right\} .$$

Denote by $\mathcal{Y}_0(t)$ a normal solution of $[A^0]$.

Let us start by looking at a single germ in depth. The Proposition 4.17 states that every Stokes germ φ_α can be written as its matrix representation conjugated by the normal solution, i.e. as $\varphi_\alpha = \mathcal{Y}_0 C_{\varphi_\alpha} \mathcal{Y}_0^{-1} = \rho_\alpha^{-1}(C_{\varphi_\alpha})$.

Look at an example in which we will demonstrate, from which relations on the determining polynomials which restriction on the form of the Stokes matrices arise.

Example 4.48

Let $\alpha \in \mathbb{A}$ be an anti-Stokes direction. From the definition of $\mathbb{S}\mathrm{to}_\alpha(A^0)$ (cf. Definition 4.16) we know that, if one has $q_1 \underset{\alpha}{\prec} q_2$, the Stokes matrix has the form

$$\begin{pmatrix} 1 & \boldsymbol{c_1} & \star \\ \mathbf{0} & 1 & \star \\ \star & \star & 1 \end{pmatrix}$$

where $c_j \in \mathbb{C}$ and $\star \in \mathbb{C}$. We have seen that $q_1 \underset{\alpha}{\prec} q_2 \Rightarrow q_1 \underset{\alpha}{\prec} q_3$ (cf. Lemma 4.47) thus the representation has the structure

$$\begin{pmatrix} 1 & c_1 & \boldsymbol{c_2} \\ 0 & 1 & \star \\ \mathbf{0} & \star & 1 \end{pmatrix}$$

and if we also know that neither $q_2 \underset{\alpha}{\prec} q_3$ nor $q_3 \underset{\alpha}{\prec} q_2$ it has the form

$$\begin{pmatrix} 1 & c_1 & c_2 \\ 0 & 1 & \mathbf{0} \\ 0 & \mathbf{0} & 1 \end{pmatrix}.$$

The obvious isomorphism ϑ_α from Remark 4.16.1 can explicitly given as

$$\begin{aligned} \vartheta_\alpha : \mathbb{C}^2 &\longrightarrow \mathbb{S}\mathrm{to}_\alpha(A^0) \\ (c_1, c_2) &\longmapsto \begin{pmatrix} 1 & c_1 & c_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

The following table gives an overview of the 9 possible combinations and the corresponding forms of the Stokes matrices which can arise in our situation. The case in the lower right occurs when the direction α is not an anti-Stokes direction.

	$q_2 \underset{\alpha}{\prec} q_3$	$q_3 \underset{\alpha}{\prec} q_2$	else
$q_1 \underset{\alpha}{\prec} q_2$ and $q_1 \underset{\alpha}{\prec} q_3$	$\begin{pmatrix} 1 & c_2 & c_3 \\ 0 & 1 & c_1 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & c_2 & c_3 \\ 0 & 1 & 0 \\ 0 & c_1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & c_2 & c_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
$q_2 \underset{\alpha}{\prec} q_1$ and $q_3 \underset{\alpha}{\prec} q_1$	$\begin{pmatrix} 1 & 0 & 0 \\ c_2' & 1 & c_1 \\ c_3 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ c_2 & 1 & 0 \\ c_3' & c_1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ c_2 & 1 & 0 \\ c_3 & 0 & 1 \end{pmatrix}$
else	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & c_1 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c_1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

In the blue cases we have $\mathcal{K}_\alpha = \mathcal{K}$ and $\mathbb{C}^3 \xrightarrow[\cong]{\vartheta_\alpha} \mathbb{S}\mathrm{to}_\alpha(A^0)$. In the green cases $\mathcal{K}_\alpha = \{k_2\}$ and $\mathbb{C}^2 \xrightarrow[\cong]{\vartheta_\alpha} \mathbb{S}\mathrm{to}_\alpha(A^0)$ as well as in the purple cases $\mathcal{K}_\alpha = \{k_1\}$ and $\mathbb{C}^1 \xrightarrow[\cong]{\vartheta_\alpha} \mathbb{S}\mathrm{to}_\alpha(A^0)$. We will replace c_2' by $c_2 + c_1c_3$ and c_3' by $c_1c_2 + c_3$ to be consistent with the decomposition in the next part (cf. Example 4.50).

Corollary 4.49

The morphism $\prod_{\alpha\in\mathbb{A}} \vartheta_\alpha$ is an isomorphism of pointed sets, which maps the element only containing zeros to

$$(\mathrm{id}, \mathrm{id}, \ldots, \mathrm{id}) \in \prod_{\alpha\in\mathbb{A}} \mathbb{S}\mathrm{to}_\alpha(A^0),$$

which gets by $\prod_{\alpha\in\mathbb{A}} \rho_\alpha^{-1} \circ h$ mapped to the trivial cohomology class in $\mathcal{S}t(A^0)$.

In Proposition 4.28 we have defined a decomposition of the Stokes group $\mathrm{Sto}_\alpha(A^0)$ in subgroups generated by k-germs for $k \in \mathcal{K}$. In our case, we have at most two nontrivial factors. The map which gives the factors of this factorization in ascending order is denoted by i_α and the decomposition is then given by

$$\varphi_\alpha = \varphi_\alpha^{k_1}\varphi_\alpha^{k_2} \xmapsto{i_\alpha} \left(\varphi_\alpha^{k_1}, \varphi_\alpha^{k_2}\right) \in \mathrm{Sto}_\alpha^{k_1}(A^0) \times \mathrm{Sto}_\alpha^{k_2}(A^0)\,.$$

This decomposition applied to some germ φ_α is trivial if $\#\mathcal{K}(\varphi_\alpha) \leq 1$, thus the interesting cases are the blue cases from the table above.

The lower right blue case will be discussed in the following example in more detail.

Example 4.50

Look at an anti-Stokes direction $\alpha \in \mathbb{A}$ and assume that $q_3 \underset{\alpha}{\prec} q_2$, $q_2 \underset{\alpha}{\prec} q_1$ and $q_3 \underset{\alpha}{\prec} q_1$. We then know that $\varphi_\alpha \in \mathrm{Sto}_\alpha(A^0)$ can be written as

$$\begin{aligned}\varphi_\alpha &= \mathcal{Y}_0\vartheta_\alpha(c_1, c_2, c_3)\mathcal{Y}_0^{-1}\\ &= \mathcal{Y}_0 \begin{pmatrix} 1 & 0 & 0\\ c_2 & 1 & 0\\ c_1c_2 + c_3 & c_1 & 1\end{pmatrix} \mathcal{Y}_0^{-1}\,.\end{aligned}$$

According to Remark 4.27.1 the factor $\varphi_\alpha^{k_1} \in \mathrm{Sto}_\alpha^{k_1}(A^0)$ is obtained by truncation to terms of level k_1, i.e. as

$$\varphi_\alpha^{k_1} = \mathcal{Y}_0 \begin{pmatrix} 1 & 0 & 0\\ \mathbf{0} & 1 & 0\\ \mathbf{0} & c_1 & 1\end{pmatrix} \mathcal{Y}_0^{-1}\,.$$

The other factor $\varphi_\alpha^{k_2}$ is then obtained as

$$\varphi_\alpha^{k_2} = \left(\varphi_\alpha^{k_1}\right)^{-1}\varphi_\alpha$$

$$= \mathcal{Y}_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -c_1 & 1 \end{pmatrix} \underbrace{\mathcal{Y}_0^{-1}\mathcal{Y}_0}_{=\mathrm{id}} \begin{pmatrix} 1 & 0 & 0 \\ c_2 & 1 & 0 \\ c_1c_2 + c_3 & c_1 & 1 \end{pmatrix} \mathcal{Y}_0^{-1}$$

$$= \mathcal{Y}_0 \begin{pmatrix} 1 & 0 & 0 \\ c_2 & 1 & 0 \\ c_3 & 0 & 1 \end{pmatrix} \mathcal{Y}_0^{-1}.$$

Remark 4.51

The four nontrivial decomposition in our situation are given by:

1. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & c_1 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & c_2 & c_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & c_2 & c_3 \\ 0 & 1 & c_1 \\ 0 & 0 & 1 \end{pmatrix}$
2. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c_1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & c_2 & c_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & c_2 & c_3 \\ 0 & 1 & 0 \\ 0 & c_1 & 1 \end{pmatrix}$
3. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & c_1 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ c_2 & 1 & 0 \\ c_3 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ c_2 + c_1c_3 & 1 & c_1 \\ c_3 & 0 & 1 \end{pmatrix}$
4. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c_1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ c_2 & 1 & 0 \\ c_3 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ c_2 & 1 & 0 \\ c_1c_2 + c_3 & c_1 & 1 \end{pmatrix}$

Example 4.52

Even more explicit, we can fix the levels $k_1 = 1$ and $k_2 = 3$ together with $\theta = 0$. Assume without any restriction that $q_1 \underset{\theta}{\prec} q_2$ and $q_1 \underset{\theta}{\prec} q_3$ as well as $q_2 \underset{\theta}{\prec} q_3$. Other choices would result in reordering of the tuples below. Let the matrix L be given as $L = \mathrm{diag}(l_1, l_2, l_3) \in \mathrm{GL}_n(\mathbb{C})$.

The classification space is in this case isomorphic to $\mathbb{C}^{2\cdot(1+2\cdot3)} = \mathbb{C}^{14}$. The element $({}^1c_1, {}^2c_1, {}^1c_2, {}^1c_3, {}^2c_2, {}^2c_3, \ldots, {}^6c_2, {}^6c_3) \in \mathbb{C}^{14}$ gets, via the isomorphism $\prod_{\alpha\in\mathbb{A}} j_\alpha$, mapped to

$$\left(\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & {}^1c_1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & {}^2c_1 & 1 \end{pmatrix} \right), \right.$$
$$\left. \left(\begin{pmatrix} 1 & {}^1c_2 & {}^1c_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ {}^2c_2 & 1 & 0 \\ {}^2c_3 & 0 & 1 \end{pmatrix}, \ldots, \begin{pmatrix} 1 & 0 & 0 \\ {}^6c_2 & 1 & 0 \\ {}^6c_3 & 0 & 1 \end{pmatrix} \right) \right)$$

in $\prod_{\alpha\in\mathbb{A}^1} \mathbb{S}\mathrm{to}_c\, \alpha^1(A^0) \times \prod_{\alpha\in\mathbb{A}^3} \mathbb{S}\mathrm{to}_\alpha^3(A^0)$ and thus the element

$$\left(\left(\left(\begin{pmatrix}1&0&0\\0&1&{}^1c_1\\0&0&1\end{pmatrix}, \mathrm{id}, \mathrm{id}, \begin{pmatrix}1&0&0\\0&1&0\\0&{}^2c_1&1\end{pmatrix}, \mathrm{id}, \mathrm{id}\right), \left(\begin{pmatrix}1&{}^1c_2&{}^1c_3\\0&1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0\\{}^2c_2&1&0\\{}^2c_3&0&1\end{pmatrix}, \dots, \begin{pmatrix}1&0&0\\{}^6c_2&1&0\\{}^6c_3&0&1\end{pmatrix}\right)\right)\right)$$

in $\prod_{\alpha\in\mathbb{A}} \mathbb{S}\mathrm{to}_\alpha^1(A^0) \times \prod_{\alpha\in\mathbb{A}} \mathbb{S}\mathrm{to}_\alpha^3(A^0)$. Using the morphism $\prod_{\alpha\in\mathbb{A}} i_\alpha^{-1}$ we get a complete set of Stokes matrices as

$$\left(\begin{pmatrix}1&{}^1c_2&{}^1c_3\\0&1&{}^1c_1\\0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0\\{}^2c_2&1&0\\{}^2c_3&0&1\end{pmatrix}, \begin{pmatrix}1&{}^3c_2&{}^3c_3\\0&1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0\\{}^4c_2&1&0\\{}^2c_1{}^4c_2+{}^4c_3&{}^2c_1&1\end{pmatrix}, \begin{pmatrix}1&{}^5c_2&{}^5c_3\\0&1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0\\{}^6c_2&1&0\\{}^6c_3&0&1\end{pmatrix}\right) \in \prod_{\alpha\in\mathbb{A}} \mathbb{S}\mathrm{to}_\alpha(A^0)\,.$$

Applying the isomorphism $\prod_{\alpha\in\mathbb{A}} \rho_\alpha^{-1}$, i.e. conjugation by the fundamental solution $\mathcal{Y}_0(t) = t^L e^{Q(t^{-1})}$ (cf. Proposition 4.17), yields then the corresponding element in $\prod_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha(A^0)$.

4.4 Further improvements

The set $\prod_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha(A^0)$ and thus $\prod_{\alpha\in\mathbb{A}} \mathbb{S}\mathrm{to}_\alpha(A^0)$ has some bad properties, when small deformations are applied to $[A^0]$, since under arbitrary small changes, one Stokes ray can split into two.

In the prove of Theorem 4.34 we have seen, that not only $\mathcal{H}(A^0) \cong \prod_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha(A^0)$ but also that

$$\begin{aligned}\mathcal{H}(A^0) &\cong \prod_{k\in\mathcal{K}} \prod_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha^k(A^0)\\ &\cong \prod_{k\in\mathcal{K}} \prod_{\alpha\in\mathbb{A}^k} \mathrm{Sto}_\alpha^k(A^0) && (\text{since } \mathbb{S}\mathrm{to}_\alpha^k(A^0) = \{\mathrm{id}\} \text{ when } k \notin \mathcal{K}_\alpha.)\\ &\cong \prod_{k\in\mathcal{K}} \prod_{\alpha\in\mathbb{A}^k} \mathbb{S}\mathrm{to}_\alpha^k(A^0)\,.\end{aligned}$$

This representation can be used to achieve further improvements, since one is able to multiply some succeeding Stokes matrices (resp. Stokes germs) of the same level without loss of information. This is precisely stated in Corollary 4.61 below.

TODO: move? Boalch, who looks in his publications [Boa01; Boa99] only at the single-leveled case, uses this extensively to obtain better Stokes matrices, which are stable under small deformations. Our Stokes matrices are in his publications called *Stokes factors*.

Let us fix a level $k \in \mathcal{K}$. We want to rewrite the product $\prod_{\alpha\in\mathbb{A}^k} \mathbb{S}\mathrm{to}_\alpha^k(A^0)$ by collecting the information of a subsets of these Stokes matrices into one product of these matrices.

Definition 4.53

A subset of $\mathbb{A}^k$ consisting of $\frac{\#\mathbb{A}^k}{2k}$ consecutive anti-Stokes directions of level k will be called a *half-period (of level k)*.

From the definition of anti-Stokes directions of level k it is clear that every arc of width $\frac{\pi}{k}$, which has no anti-Stokes direction of level k on its border, contains $\frac{\#\mathbb{A}^k}{2k}$ anti-Stokes directions of level k since

> for every sector $I = U(\theta, \frac{\pi}{k})$ of width $\frac{\pi}{k}$ and center θ which satisfies $\theta \pm \frac{\pi}{2k} \notin \mathbb{A}^k$ there is to any pair (q_j, q_l) in $\mathcal{Q}(A^0)$ which has level $k_{jl} = k$ exactly one direction $\alpha \in \mathbb{A}^k \cap I$ which satisfies that $\Im(a_{jl}e^{-ik_{jl}\alpha}) = 0$. At such a direction α corresponding to the pair (q_j, q_l) is then either $q_j \underset{\alpha}{\prec} q_l$ or $q_l \underset{\alpha}{\prec} q_j$ satisfied.

This implies that for every θ, which satisfies $\theta \pm \frac{\pi}{2k} \notin \mathbb{A}^k$, is $\mathbb{A}^k \cap U(\theta, \frac{\pi}{k})$ a half-period and by $\dot{\bigcup}_{1\le m\le 2k} \mathbb{A}^k \cap U(\theta + \frac{m\pi}{k}, \frac{\pi}{k}) = \mathbb{A}^k$ is a decomposition in distinct half-periods of $\mathbb{A}^k$ given (cf. Figure 4.5).

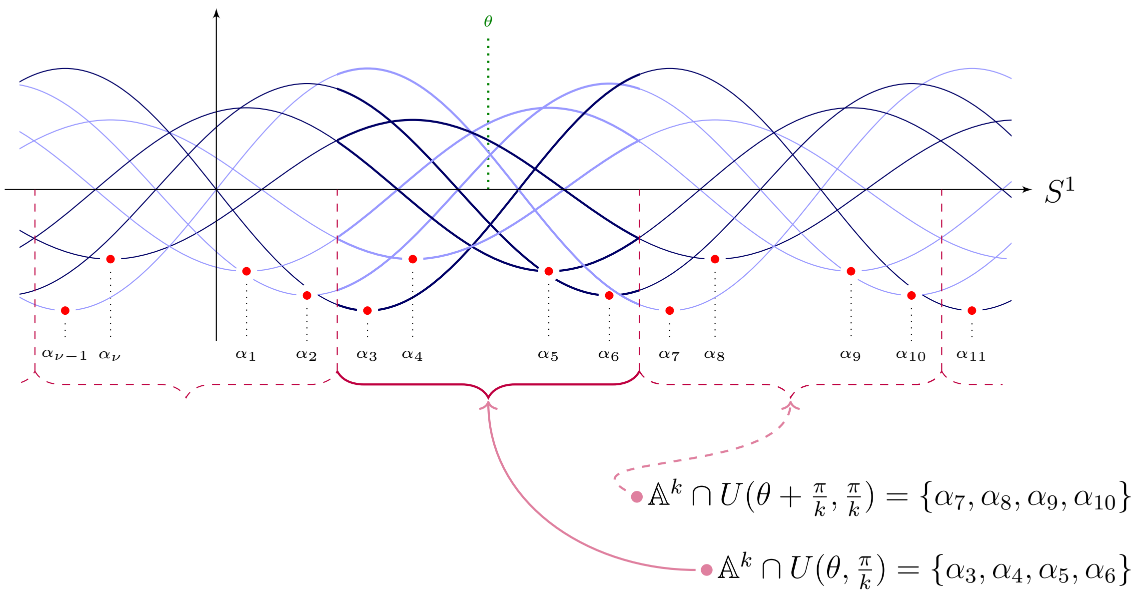


Figure 4.5: Assume that $\mathcal{Q}(A^0)$ contains exactly eight ordered pairs of degree k. Choose four of them, which are up to order different. In this picture are the real parts of the leading terms corresponding to the pairs in blue plotted. In light blue are the real parts corresponding to the flipped pairs drawn. The points, where the real part is negative and the imaginary part vanishes, i.e. where the pairs determine an anti-Stokes direction, are marked by the red dots.

Proposition 4.54

Let $I = U(\theta, \frac{\pi}{k})$ be an arc width $\frac{\pi}{k}$ such that $\theta \pm \frac{\pi}{2k} \notin \mathbb{A}^k$. For every α in the half-period $\mathbb{A}^k \cap I$ we can write the Stokes matrix as

$$K^\alpha = (K^\alpha_{jl})_{j,l \in \{1,\dots,s\}} \in \mathbb{S}\mathrm{to}^k_\alpha(A^0)$$

where the $K^\alpha_{jl} \in \mathbb{C}^{n_j \times n_l}$ are blocks corresponding to the structure of Q (cf. Definition 4.16). We then know for $j \neq l$ that, if $K^\alpha_{jl} \neq 0$ then

1. is $K^\alpha_{lj} = 0$ and
2. for every $\alpha' \in (\mathbb{A}^k \cap I) \setminus \{\alpha\}$ is $K^{\alpha'}_{jl} = 0$ as well as $K^{\alpha'}_{lj} = 0$.

Proof. The first statement was already discussed page 54. To see the second statement, look at a pair $(q_j, q_l) \in \mathcal{Q}(A^0)$ together with an anti-Stokes direction α such that $q_j \underset{\alpha}{\prec} q_l$. We then know (cf. Lemma 4.14) that the next directions α' satisfying $q_j \underset{\alpha'}{\prec} q_l$ are $\alpha + \frac{2\pi}{k}$ (on the left) and $\alpha - \frac{2\pi}{k}$ (on the right). The next directions satisfying the flipped relation $q_l \underset{\alpha'}{\prec} q_j$ are given by $\alpha + \frac{\pi}{k}$ (on the left) and $\alpha - \frac{\pi}{k}$ (on the right).

Thus there is no other anti-Stokes direction but α for the pairs (q_j, q_l) and (q_l, q_j) in every arc of width $\frac{\pi}{k}$ containing α. $\square$

Corollary 4.55

In the situation of Proposition 4.54 there exists a common (block) permutation matrix $P \in \mathrm{GL}_n(\mathbb{C})$ given by $(P)_{jl} = \boldsymbol{\delta}_{\pi(j)l}$ where

- $\boldsymbol{\delta}_{jl}$ is the block version of Kronecker's delta corresponding to the structure of Q which was introduced in Definition 4.16 and
- [Boa01, p. 18] π is the permutation of $\{1, \dots, s\}$ corresponding to $q_j \underset{\theta}{\prec} q_l \Rightarrow \pi(j) < \pi(l)$

such that every matrix $P^{-1} K^\alpha P$ is upper triangular and the observation from Proposition 4.54 is still satisfied.

Remark 4.55.1

After moving the sector $U(\theta, \frac{\pi}{k})$ to $U(\theta, \frac{\pi}{k}) + \frac{\pi}{k} = U(\theta + \frac{\pi}{k}, \frac{\pi}{k})$ we obtain also a corresponding permutation π' which is inverse to π. The corresponding permutation matrix P' is then given by $P' = P^{-1}$.

Such that the permutation P transforms Stokes matrices on the sector $I + \frac{\pi}{k}$ to lower triangular matrices

In the single-leveled case of this phenomenon is discussed on page 18 of Boalch's paper [Boa01].

In the paper [BJL79] from Balser, Jurkat and Lutz is the following Lemma stated as Lemma 2 on page 75. It is the fundamental statement, which allows us to collect the information of multiple Stokes matrices into one matrix, i.e. it states that we will not loose information.

Lemma 4.56

Let $T \subset \{1, \ldots, n\} \times \{1, \ldots, n\}$ be a position set, which satisfies the *completeness property*:

if (j, k) and $(k, l) \in T$ then is also $(j, l) \in T$.

Choose a indexing $i : \{1, 2, \ldots, \mu\} \overset{\cong}{\to} T$ of the position set and denote by $\delta_{jl} \in \mathbb{C}$ the ordinary Kronecker's delta, then there exists for every matrix

$$K \in \left\{ K = (K_{jl})_{j,l \in \{1,\ldots,n\}} \in \mathrm{GL}_n(\mathbb{C}) \mid K_{jl} = \delta_{jl} \text{ unless } (j, l) \in T \right\}$$

unique scalars $t_1, \ldots, t_\mu \in \mathbb{C}$ such that

$$K = (\mathrm{id} + t_1 E_{i(1)}) \cdots (\mathrm{id} + t_\mu E_{i(\mu)}) \,.$$

The completeness property is reasonable, since for example the matrices corresponding to the not complete set $\{(2, 3), (3, 2)\}$ are not stable under the product:

$$\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a & ab \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} .$$

The completeness property looks similar to the transitivity condition on relations, this yields the following useful proposition.

Corollary 4.57

Every position set defined via an relation $\overset{R}{<}$ as $\{(j, l) \mid j \overset{R}{<} l\}$ is complete, if and only if the relation $\overset{R}{<}$ is transitive.

Proposition 4.58

It is easy to see that

1. $\{(j, l) \mid q_j \underset{\theta}{\preccurlyeq} q_l\} = \bigcap_{\theta' \in \mathbb{A}^k \cap U(\theta, \frac{\pi}{k})} \{(j, l) \mid q_j \underset{\theta'}{\prec} q_l\}$ and
2. $\{(j, l) \mid q_j \underset{\theta}{\prec} q_l\} = \dot{\bigcup}_{\theta' \in \mathbb{A}^k \cap U(\theta, \frac{\pi}{k})} \{(j, l) \mid q_j \underset{\theta'}{\preccurlyeq} q_l\}$

and all the position sets above are complete.

Another useful fact is given by the following lemma.

Lemma 4.59

Every position set, corresponding to some block, is complete.

Proof. Such a position set, corresponding to some block, is given by

$$T = \{(j,l) \mid j_1 \le j \le j_2, l_1 \le l \le l_2\}$$

for some j_1, j_2, l_2 and l_2 in $\{1, \dots, n\}$.

Let (j,k), $(k,l) \in T$ be two positions. It is obvious that $(j,l) \in T$, since $j_1 \le j \le j_2$ and $l_1 \le l \le l_2$ are satisfied. □

Corollary 4.60

We can write the Lemma 4.56 in block form, corresponding to the structure of Q. Let s be the number of diagonal blocks of Q and $T \subset \{1, \dots, s\} \times \{1, \dots, s\}$ be a position set.

Definition 4.60.1

Define the group of matrices, corresponding to a complete position set, by

$$S_T := \left\{ K = (K_{jl})_{j,l \in \{1,\dots,n\}} \in \mathrm{GL}_n(\mathbb{C}) \mid K_{jl} = \boldsymbol{\delta}_{jl} \text{ unless } (j,l) \in T \right\} ,$$

where $\boldsymbol{\delta}_{jl}$ is the block version of Kronecker's delta, corresponding to the structure of Q.

Choose an enumeration

$$i : \{1, 2, \dots, \mu\} \xrightarrow{\cong} T$$

of T. From the Lemma 4.56 then follows that for every $K \in S_T$ there is a unique decomposition

$$K = K_1 \cdot K_2 \cdots K_\mu ,$$

where $K_j \in S_{\{i(j)\}}$.

From the previous corollary we deduce the following corollary.

Corollary 4.61

Let $T_1, \dots, T_r \subset \{1, \dots, s\} \times \{1, \dots, s\}$ be a distinct position sets, such that

$T := T_1 \dot\cup \dots \dot\cup T_r$ as well as every T_m satisfy the completeness property.

Then is by

$$\begin{aligned} S_{T_1} \times \cdots \times S_{T_m} &\longrightarrow S_T \\ (K_1, \dots, K_m) &\longmapsto K_1 \cdots K_m \end{aligned}$$

an isomorphism defined.

We can apply the previous corollary in our situation, i.e. let $\theta \in S^1$ be a fixed direction which satisfies $\theta \pm \frac{\pi}{2k} \notin \mathbb{A}^k$. Let us first define the corresponding groups.

Definition 4.62

Let us define the group

$$\begin{aligned}\widehat{\mathbb{S}\mathrm{to}}^k_\theta(A^0) &:= \left\{ K = (K_{jl})_{j,l\in\{1,\dots,s\}} \in \mathrm{GL}_n(\mathbb{C}) \mid K_{jl} = \delta_{jl} \text{ unless } q_j \underset{\theta}{\prec} q_l,\, k_{jl} = k \right\} \\ &= S_{\{(j,l) \mid q_j \underset{\theta}{\prec} q_l\}}\end{aligned}$$

which contains the matrices with the collected data. They are representation of the corresponding elements in

$$\widehat{\mathrm{Sto}}^k_\theta(A^0) := \{\mathcal{Y}_0^{-1} K \mathcal{Y}_0 \mid K \in \widehat{\mathbb{S}\mathrm{to}}^k_\theta(A^0)\}\,.$$

Denote the isomorphism

$$\begin{aligned}\widehat{\mathbb{S}\mathrm{to}}^k_\theta(A^0) &\xrightarrow{\cong} \widehat{\mathrm{Sto}}^k_\theta(A^0) \\ \varphi &\longmapsto \mathcal{Y}_0 \varphi \mathcal{Y}_0^{-1}\end{aligned}$$

by $\widehat{\rho}^k_\theta$.

The Proposition 4.58 implies that $\mathrm{Sto}^k_\theta(A^0) \cong \bigcap_{\theta' \in \mathbb{A}^k \cap U(\theta, \frac{\pi}{k})} \widehat{\mathrm{Sto}}^k_{\theta'}(A^0)$.

Theorem 4.63

Let $\theta \in S^1$ be a fixed direction which satisfies $\theta \pm \frac{\pi}{2k} \notin \mathbb{A}^k$, then there is a bijection

$$\underbrace{\prod_{k\in\mathcal{K}} \prod_{\alpha\in\mathbb{A}^k} \mathrm{Sto}^k_\alpha(A^0)}_{\substack{\cong \\ \mathcal{H}(A^0)}} \cong \prod_{k\in\mathcal{K}} \prod_{j\in\{1,\dots,2k\}} \widehat{\mathrm{Sto}}^k_{\theta + j\frac{\pi}{k}}(A^0)\,.$$

Remark 4.63.1

This means that the information of a complete set of Stokes matrices on every level $k \in \mathcal{K}$ can be grouped into $2k$ matrices, which are products of the corresponding Stokes matrices.

Proof. The product map

$$\eta^k_\theta : \prod_{\alpha \in \mathbb{A}^k \cap U(\theta, \frac{\pi}{k})} \underbrace{\mathbb{S}\mathrm{to}^k_\alpha(A^0)}_{\substack{\| \\ S_{\{(j,l) \mid q_j \underset{\alpha}{\prec} q_l\}}}} \xrightarrow{\cong} \underbrace{\widehat{\mathbb{S}\mathrm{to}}^k_\theta(A^0)}_{\substack{\| \\ S_{\{(j,l) \mid q_j \underset{\theta}{\prec} q_l\}}}}\,.$$

defined by $\eta^k_\theta(K_1, K_2, \ldots, K_m) = K_1 \cdot K_2 \cdots K_m$ is an isomorphism, since Proposition 4.58 implies the premise of Corollary 4.61.

Denote the product of the isomorphisms η^k_θ as $\eta = \prod_{k\in\mathcal{K}} \prod_{j\in\{1,\ldots,2k\}} \eta^k_{\theta+j\frac{\pi}{k}}$ and the isomorphism $\prod_{k\in\mathcal{K}} \prod_{j\in\{1,\ldots,2k\}} \widehat{\rho}^k_{\theta+j\frac{\pi}{k}}$ by $\widehat{\rho}$.

$$\begin{tikzcd}
\prod_{k\in\mathcal{K}} \prod_{\alpha\in\mathbb{A}^k} \mathrm{Sto}^k_\alpha(A^0) \arrow[r, dashed] \arrow[d, "\cong"', "\prod_{k\in\mathcal{K}} \prod_{\alpha\in\mathbb{A}} \rho^k_\alpha"] & \prod\limits_{k\in\mathcal{K}} \prod\limits_{j\in\{1,\ldots,2k\}} \widehat{\mathrm{Sto}}^k_{\theta+j\frac{\pi}{k}}(A^0) \arrow[d, "\cong"', "\widehat{\rho}"] \\
\prod_{k\in\mathcal{K}} \prod_{\alpha\in\mathbb{A}^k} \mathbb{S}\mathrm{to}^k_\alpha(A^0) \arrow[r, "\eta", "\cong"'] & \prod\limits_{k\in\mathcal{K}} \prod\limits_{j\in\{1,\ldots,2k\}} \widehat{\mathbb{S}\mathrm{to}}^k_{\theta+j\frac{\pi}{k}}(A^0)
\end{tikzcd}$$

The isomorphism, we were looking for, is then given as $\left(\prod_{k\in\mathcal{K}} \prod_{\alpha\in\mathbb{A}} \rho^k_\alpha\right) \circ \eta \circ \widehat{\rho}^{-1}$, i.e. is the isomorphism making the above diagram commute, and we will also denote it by η. □

Remark 4.64

The in the previous proof obtained isomorphism

$$\eta : \prod_{\alpha\in\mathbb{A}} \mathbb{S}\mathrm{to}_\alpha(A^0) \longrightarrow \prod_{k\in\mathcal{K}} \prod_{j\in\{1,\ldots,2k\}} \widehat{\mathbb{S}\mathrm{to}}^k_{\theta+j\frac{\pi}{k}}(A^0)$$

is in the single-leveled case with $n = s$, i.e. all diagonal elements of Q are different, given in Lemma 3.2 of Boalch's paper [Boa01, Lem.3.2]. In this case is every $\widehat{\mathbb{S}\mathrm{to}}^k_{\theta+j\frac{\pi}{k}}(A^0)$ isomorphic to U_+, i.e. the upper triangular unipotent subgroup of $\mathrm{GL}_n(\mathbb{C})$, via the map $K \mapsto PKP^{-1}$, where P is the permutation matrix mentioned in Corollary 4.55. A similar result is also found in Theorem I of [BJL79].

But there is in the multi-leveled case no obvious way to find a subset $J \in S^1$ and an isomorphism

$$\prod_{\theta\in J} \widehat{\mathrm{Sto}}_\theta(A^0) \not\rightarrow \prod_{k\in\mathcal{K}} \prod_{j\in\{1,\ldots,2k\}} \widehat{\mathrm{Sto}}^k_{\theta+j\frac{\pi}{k}}(A^0)\,,$$

where $\widehat{\mathrm{Sto}}_\theta(A^0)$ was defined in Remark 4.17.1.

4.5 A taste of summability

For the sake of completeness, let us speak a little bit about summability in this section. We will only mention some facts to give an rough idea how the map g in the diagram on page 39 arises. The understanding of this section is not necessary for the rest of this chapter. It is enough to think of the map g (resp. $\widetilde{g}$) as a black box which uses some theory to make the asymptotic expansion unique and yields an element of $\prod_{\alpha\in\mathcal{A}} \mathrm{Sto}_\alpha(A^0)$ (resp. of $\Gamma(\dot{\mathcal{U}};\Lambda(A^0))$) corresponding to some marked pair.

The aim of a theory of summation is to associate with any series an asymptotic function uniquely determined in a way as much natural as possible. A useful and extensive resources for this topic are Loday-Richaud's book [Lod14]. A resource which looks only at an special case, which naturally arises from the previous sections, is Section III of the paper [Lod94] and some of the most important statements, i.e. exactly those we need, are found in many places like Boalch's publications [Boa01; Boa99], although he looks only at the single-leveled case. [Lod14, p. 111]

The following Proposition can be found in [Lod94, Prop.III.2.1] and [Lod14, Thm.4.3.13].

Proposition 4.65

Let $\mathcal{V} = (V_j)_{j\in J}$ be a cyclic covering, let $\widehat{F} \in \widehat{G}(A^0)$ be a formal transformation and let $\dot{\varphi} = (\dot{\varphi}_j)_{j\in J} \in \Gamma(\dot{\mathcal{V}};\Lambda(A^0))$ be a 1-cocycle in the cohomology class $\exp(\widehat{F})$. [Lod94, Sec.III.2.1]

There exists a **unique** family of realizations $(F_j)_{j\in J}$ of $\widehat{F}$ over $\mathcal{V}$, i.e. matrices F_j which are analytic on V_j, satisfy $[A^0, A]$ and are asymptotic to $\widehat{F}$ on V_j, such that

$$\dot{\varphi}_j = F_{j-1}F_j^{-1} \in \Gamma(V_{j-1}\cap V_j;\Lambda(A^0))$$

for every $j \in J$.

When $\mathcal{V}$ is the, in Section 4.3.2 defined, cyclic covering $\mathcal{U}^{\leq k_r} =: \mathcal{U}$ and $\dot{\varphi}$ is in its Stokes form TODO: (cf. ??), we call the realizations F_α, which were obtained by the previous proposition, the *sums of* $\widehat{F}$.

Definition 4.66

Denote by α^+ the next anti-Stokes direction on the right of α we can define [Lod94, Defn.III.2.2]

- $S_\alpha^-(\widehat{F}) := F_\alpha \in \mathrm{GL}_n(\mathcal{A}(U_\alpha))$ as the *sum of* $\widehat{F}$ *on the left of* α and
- $S_\alpha^+(\widehat{F}) := F_{\alpha^+} \in \mathrm{GL}_n(\mathcal{A}(U_{\alpha^+}))$ as the *sum of* $\widehat{F}$ *on the right of* α.

Remark 4.67

$S_\alpha^+(\widehat{F})t^Le^{Q(t^{-1})}$ is a solution of $[A]$ on the corresponding sector. TODO: or $S_\alpha^-(\widehat{F})t^Le^{Q(t^{-1})}$? or $t^Le^{Q(t^{-1})}S_\alpha^+(\widehat{F})$? or $t^Le^{Q(}$
Using the equation (4.2) we can write

$$S_\alpha^+(\widehat{F})\mathcal{Y}_{0,\alpha}(t) = S_\alpha^-(\widehat{F})\mathcal{Y}_{0,\alpha}(t)C_{\mathcal{Y}_{0,\alpha}}$$

thus the Stokes matrix $C_{\mathcal{Y}_{0,\alpha}}$ is a matrix, which describes the blending between the two adjacent sectors. TODO: see Loday's remarks on thm.4.3.13 in [Lod14]

There are alternative and more constructive definitions, which are equivalent to Definition 4.66. Since some of them are more direct and do not use the 1-cocycle, to which they correspond, they can be used to obtain the corresponding element in $\prod_{\alpha\in\mathcal{A}} \mathrm{Sto}_\alpha(A^0)$. Many different approaches of (multi-)summation can be found in [Lod14].

Let us fix an ambassador $\widehat{F}$ in $G\backslash\widehat{G}(A^0)$ corresponding to equivalence class of marked pairs in $\mathcal{H}(A^0)$. The theory of summation says that there is a map which takes $\widehat{F}$ to a unique and natural set of functions

$$(S^-_{\alpha_1}(\widehat{F}), \underbrace{S^-_{\alpha_2}(\widehat{F})}_{\substack{\shortparallel \\ S^+_{\alpha_1}(\widehat{F})}}, \dots, S^-_{\alpha_\nu}(\widehat{F})) \in \prod_{\alpha\in\mathbb{A}} \mathrm{GL}_n(\mathcal{A}(U_\alpha))\,,$$

which have $\widehat{F}$ as asymptotic functions. The corresponding element $(\varphi_{\alpha_1}, \dots, \varphi_{\alpha_\nu})$ in $\prod_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha(A^0)$ is then found by setting $\varphi_\alpha := \left(S^-_\alpha(\widehat{F})\right)^{-1} S^+_\alpha(\widehat{F}) \in \Gamma(U_\alpha; \Lambda(A^0))$.

Definition 4.68

The corresponding map is denoted by

$$\widetilde{g} : G(\{t\})\backslash\widehat{G}(A^0) \longrightarrow \Gamma(\dot{\mathcal{U}}; \Lambda(A^0))\,.$$

The map g mentioned in the beginning of Chapter 4 is build as shown in Section 4.6.

4.6 The complete diagram

In this section we want to write the improved version of the diagram from page 39 down. It will contain nearly all of the isomorphisms which were defined in the previous sections. Additionally will isomorphisms like g be decomposed into their building blocks.

In the following diagram of **isomorphisms of pointed sets** was the isomorphism $\exp_{A^0}$ discussed in Section 4.1. The purple path, which is the isomorphism h from Theorem 4.34, together with the lower left part was discussed in Section 4.3. The lower right part was discussed in Section and the blue part on the left was discussed in the previous section.

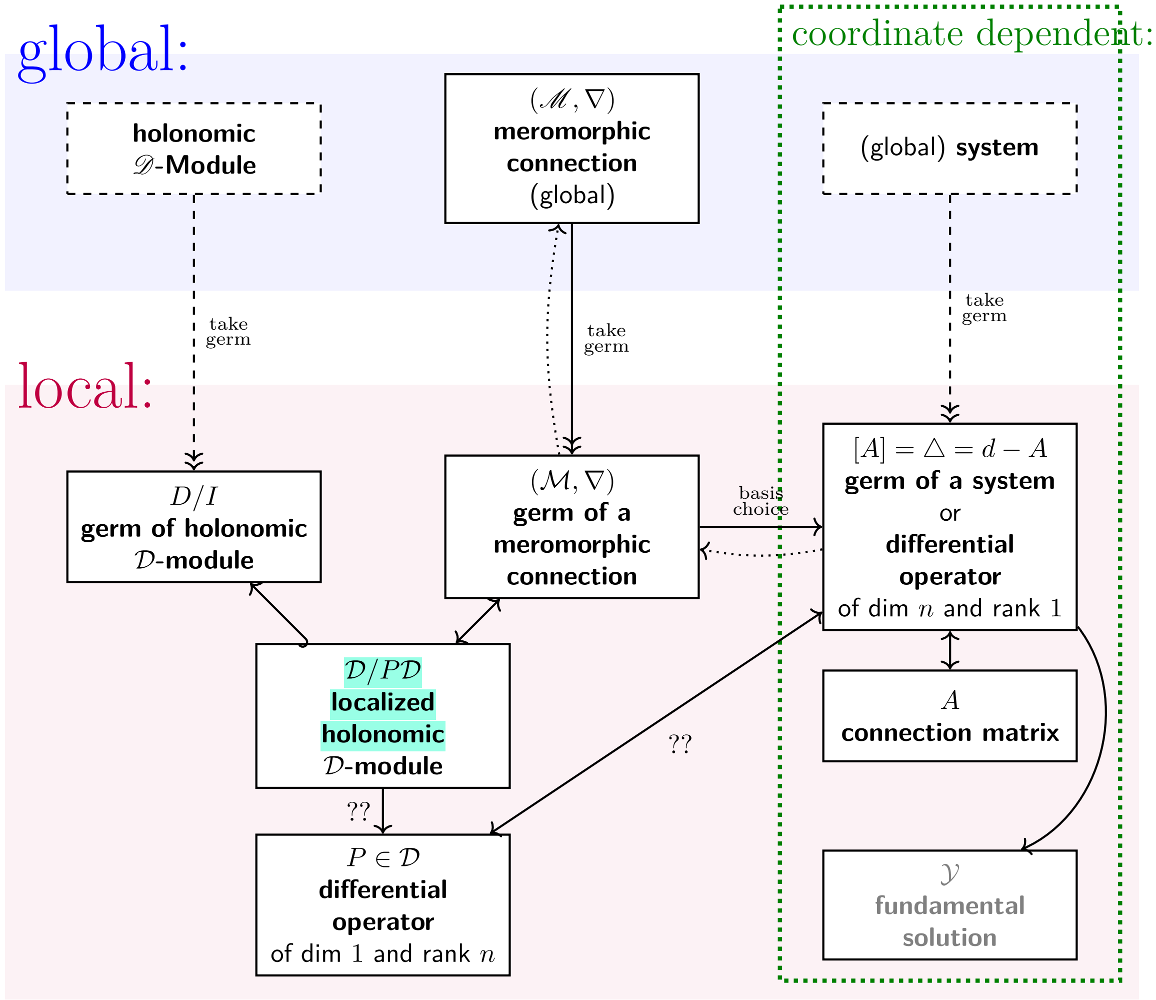
$$
\begin{tikzcd}
\mathcal{H}(A^0) \arrow[rrr, "\exp_{A^0}"] \arrow[d, shift right] & & & \mathcal{S}t(A^0) \\
G(\{t\})\backslash\widehat{G}(A^0) \arrow[u, shift right, "\text{(cf. Cor.3.40.1)}"'] \arrow[d, "\widetilde{g}"] & & \prod\limits_{k\in\mathcal{K}} \Gamma(\dot{\mathcal{U}}^k;\Lambda^k(A^0)) \arrow[ur, "\mathcal{T}"] \arrow[dll, dashed, bend right, "\tau"'] & \\
\Gamma(\dot{\mathcal{U}};\Lambda(A^0)) \arrow[dd, "\text{(take germs)}"'] \arrow[uurrr, dashed, two heads, bend left, "\text{(quotient map)}"] & \prod\limits_{k\in\mathcal{K}}\prod\limits_{\alpha\in\mathbb{A}} \mathrm{Sto}^k_\alpha(A^0) \arrow[ur, "\prod_{k\in\mathcal{K}} i^k"] \arrow[dr, "\eta"] \arrow[dd, "\prod_{k\in\mathcal{K}}\prod_{\alpha\in\mathbb{A}}\rho^k_\alpha"] & & \\
 & & \prod\limits_{k\in\mathcal{K}}\prod\limits_{j\in\{1,\dots,2k\}} \widehat{\mathrm{Sto}}^k_{\theta+j\frac{\pi}{k}}(A^0) \arrow[dd, "\widehat{\rho}"] & \\
\prod\limits_{\alpha\in\mathbb{A}} \mathrm{Sto}_\alpha(A^0) \arrow[uur, "\chi\circ\prod_{\alpha\in\mathbb{A}} i_\alpha"] \arrow[dd, "\prod_{\alpha\in\mathbb{A}}\rho_\alpha"'] \arrow[uuurr, bend left, "\mathfrak{T}"] & \prod\limits_{k\in\mathcal{K}}\prod\limits_{\alpha\in\mathbb{A}} \mathbb{S}\mathrm{to}^k_\alpha(A^0) \arrow[dr, "\eta"] & & \\
 & & \prod\limits_{k\in\mathcal{K}}\prod\limits_{j\in\{1,\dots,2k\}} \widehat{\mathbb{S}\mathrm{to}}^k_{\theta+j\frac{\pi}{k}}(A^0) & \\
\prod\limits_{\alpha\in\mathbb{A}} \mathbb{S}\mathrm{to}_\alpha(A^0) \arrow[uur, "\chi\circ\prod_{\alpha\in\mathbb{A}} i_\alpha"'] & & & \\
 & \prod\limits_{\alpha\in\mathbb{A}} \mathbb{C}^{\sum_{q_j \underset{\alpha}{\prec} q_l} \deg(q_j-q_l)\cdot n_j\cdot n_l} \arrow[ul, "\prod_{\alpha\in\mathbb{A}}\vartheta_\alpha"] & &
\end{tikzcd}
$$

Where we denote

- $\chi : \prod\limits_{\alpha\in\mathbb{A}}\prod\limits_{k\in\mathcal{K}} \mathrm{Sto}^k_\alpha(A^0) \equiv \prod\limits_{k\in\mathcal{K}}\prod\limits_{\alpha\in\mathbb{A}} \mathrm{Sto}^k_\alpha(A^0)$ the reordering and
- by abuse of notation we also denote the Stokes matrix version in the same way $\chi : \prod\limits_{\alpha\in\mathbb{A}}\prod\limits_{k\in\mathcal{K}} \mathbb{S}\mathrm{to}^k_\alpha(A^0) \equiv \prod\limits_{k\in\mathcal{K}}\prod\limits_{\alpha\in\mathbb{A}} \mathbb{S}\mathrm{to}^k_\alpha(A^0)$.

A Map of languages for TODO

Let M be a riemanian surface and let $Z = k_1(a_1) + \cdots + k_m(a_m) > 0$ be an effective divisor on M.



B Vortrag: Stokes Strukturen meromorpher Zusammenhänge

Here are the german notes of my Presentation of Stokes structures.

B.0 Motivation

Es ist eine bekannte Tatsache, dass eine irregulär singuläre Diffenentialgleichung mit konvergenten Einträgen eine formale Lösung haben kann, welche divergent ist. Deshalb will man Isomorphieklassen von (irregulär singulären) meromorphen Zusammenhängen, welche zu einem vorgegebenen 'gutem Model' formal isomorph sind, beschreiben.

Es stellt sich heraus, das dafür die formalen Invarianten, wie beispielsweise die formale Monodromie, nicht ausreichen. Die benötigte Information ist aber durch die Stokes-Struktur gegeben, welche mit Hilfe der asymptotischen Analysis entsteht.

In diesem Vortrag werden Stokes-Strukturen sowie meromorphe Zusammenhänge und alles benötigte aus der asymptotischen Analysis definiert. Kenntnisse aus der Theorie der D-Moduln werden nicht benötigt.

B.1 Meromorphe Zusammenhänge

Sei M eine **Riemann-Fläche**[1], Z ein **effektiver Divisor**[2] auf M.

Starte mit einer Diffenentialgleichung.

Über jedem Punkt von $M \backslash Z$ hat man einen endlichen Vektorraum von 'initial data' und erhalte ein holomorphes Bündel $\mathscr{M}$ auf M (durch fortsetzen).
Weiter gibt es zu jedem $x \in X$ und jedem Keim $u \in \mathscr{M}_x$ in der Faser bei x gibt es einen eindeutig bestimmten Schnitt in der Umgebung von x.

Sei $\mathscr{M}$ ein **(holomorphes) Bündel** $:\Leftrightarrow$ lokal freier $\mathcal{O}_M$-Modul.

Definition B.3

Ein *meromorpher Zusammenhang auf* $\mathscr{M}$ *mit Polen auf* Z ist eine $\mathbb{C}$-lineare Abbildung

$$\nabla : \mathscr{M} \to \Omega^1_M(*Z) \otimes \mathscr{M}$$

[1] Komplexe Mannigfaltigkeit von Dimension 1.
[2] Formale Summe von Hyperflächen mit positiven Koeffizienten.

welche für alle offenen Mengen $U \subset M$, Schnitte $s \in \Gamma(U, \mathscr{M})$ und holomorphen Funktionen $f \in \mathcal{O}(U)$ die **Leibniz Regel**

$$\nabla(fs) = f\nabla s + (df) \otimes s$$

erfüllt.

Definition B.3

$R_\nabla \equiv 0$

Ein Zusammenhang heißt *flach* oder *integrabel* falls seine Krümmung $R_\nabla := \nabla \circ \nabla : \mathscr{M} \to \Omega^2_M(*Z) \otimes_{\mathcal{O}_M} \mathscr{M}$ identisch verschwindet.

Remark B.3

[Sab07] Rem 0.12.5

Für $\dim(M) = 1$ ist jeder Zusammenhang flach. Also hier nicht von Bedeutung.

Remark B.4

Siehe
- [Pym12]
- [Sab07] 0.11.a

Lokal sieht $\mathscr{M}$ wie $U \times \mathbb{C}^n$ und ein $s \in \mathscr{M}$ wie $\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} =: f \in \mathcal{O}(U)^n$, aus.

Dann

$$\nabla s = df - Af$$

wobei $A \in M(n \times n, \Omega^1_M(*Z)(U))$ ist die *Zusammenhangs Matrix*.

Klassifiziere diese durch die Lösung der DGL $\nabla s = 0$.

see [Sab07] II.2.a

Ein Wechsel $F \in \mathrm{GL}_n(\mathcal{O}_X(U))$ der Trivialisierung entspricht $F[A^0] = (dF)F^{-1} + FA^0F^{-1}$.

Ein Zusammenhang is irregulär singulär, falls er einen Pol hat, der auch nach Tranformation immer noch Ordnung ≥ 1 hat.

B.1.1 Modelle und formale Zerlegung

Definition B.5

Ein Keim $(\mathcal{M}, \nabla)$ ist ein *Model* falls

$$(\mathcal{M}, \nabla) \cong \bigoplus_{\varphi} \underbrace{\mathcal{E}^{\varphi} \otimes \mathcal{R}_{\varphi}}_{\substack{\text{elementare} \\ \text{merom. Zus.}}}$$

wobei

- die $\varphi \in t^{-1}\mathbb{C}[t^{-1}]$ paarweise unterschiedlich sind,
- die $\mathcal{R}_{\varphi}$ regulär singulär sind und
- $\mathcal{E}^{\varphi}$ wird gelöst von e^{φ}.

Lemma B.6

Ist $(\mathscr{M}, \nabla)$ ein Modell, so lässt sich die Zusammenhangs Matrix schreiben als

$$A^0 = dQ + \Lambda \frac{dt}{t}$$

wobei

- $Q = \operatorname{diag}(\varphi_1, \ldots, \varphi_n)$ und
- Λ diagonal und konstant ist.

Theorem B.8: Levelt-Turittin

Zu $(\mathcal{M}, \nabla)$ gibt es, bis auf Verzweigung, immer einen **formalem** Isomorphismus

$$\widehat{\lambda} : \widehat{\mathcal{M}} \xrightarrow{\cong} \widehat{\mathcal{M}}^{good} = \widehat{\mathcal{O}} \otimes \mathcal{M}^{good}$$

zu einem **formalem Modell**.

Remark B.8

Aber keinen konvergenten lift.

$$\begin{tikzcd}
\widetilde{\mathscr{M}}_{\theta} \arrow[r, "\widetilde{\lambda}"] \arrow[d] & \widetilde{\mathscr{M}}_{\theta}^{good} \arrow[d] \\
\widehat{\mathscr{M}} \arrow[r, "\widehat{\lambda}", "\sim"'] & \widehat{\mathscr{M}}^{good}
\end{tikzcd}$$

Sektorweise aber schon, mit asymptotischer Analysis

B.2 Asymptotische Entwicklungen

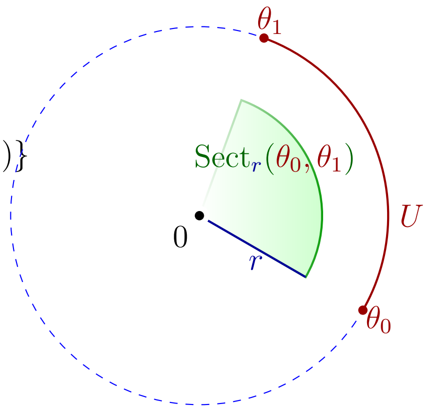
Betrachte nun $M = D = \{t \in \mathbb{C} \mid |t| < r\}$ für r beliebig klein und $Z = \{0\}$.

See [PS03]

Definition B.9

Sei $\theta_0, \theta_1 \in S^1$.

- $\mathrm{Sect}_r(\theta_0,\theta_1) := \{t = \rho e^{i\theta} \in \mathbb{C} \mid 0 < \rho < r, \theta \in (\theta_0,\theta_1)\}$
- $\mathrm{Sect}(\theta_0,\theta_1) := \mathrm{Sect}_r(\theta_0,\theta_1)$ für r klein genug.
- $\mathrm{Sect}(U) \overset{U=(\theta_0,\theta_1)}{:=} \mathrm{Sect}(\theta_0,\theta_1)$



Definition B.10

$f \in \mathcal{O}(\mathrm{Sect}(U))$ hat $\sum_{n \geq n_0} c_n t^n \in \mathbb{C}((t))$[a] als *asymptotische Entwicklung auf* $\mathrm{Sect}(U)$, falls

- $\exists\, r$ so dass $\forall\, \mathbf{N} \geq 0$ und $\forall$ abgeschlossenen Untersektoren W in $\mathrm{Sect}_r(U)$ eine Konstante $C(\mathbf{N}, W)$ existiert, so dass

$$\left| f(t) - \sum_{n_0 \leq n \leq \mathbf{N}-1} c_n t^n \right| \leq C(\mathbf{N}, W)|t|^{\mathbf{N}} \qquad \text{für alle } t \in W$$

oder äquivalent: $\lim_{z\to 0, z\in W} |t|^{-(\mathbf{N}-1)} \left| f(t) - \sum_{n_0 \leq n \leq \mathbf{N}-1} c_n t^n \right| = 0$

Erhalte die Garbe $\mathscr{A}$ auf S^1:

$$\underset{\substack{\text{off. Intervall}\\ \text{von } S^1}}{U} \mapsto \mathscr{A}(U) \subset \mathcal{O}(\mathrm{Sect}(U)) \text{ die Funktionen mit asymptotischer Entwicklung auf } \mathrm{Sect}(U)$$

[a] **formale Laurent Reihe**

Lemma B.11: Borel-Ritt

$t \mapsto e^{-\frac{1}{t}}$ hat verschwindende asy. Entw. in Sektoren um $\theta = 0$ [Var96, p. 6]

Für jedes **echte** offene Intervall U von S^1 ist die Abbildung

$$0 \to \mathscr{A}^{<0}(U) \to \mathscr{A}(U) \overset{T}{\twoheadrightarrow} \mathbb{C}((t)) \to 0 \qquad\qquad 0 \to \mathscr{A}^{<0} \to \mathscr{A} \to \pi^{-1}\widehat{\mathcal{O}}_D \to 0$$

eine surjektion.

Theorem B.12

Zu jedem $\theta \in S^1$ und jedem genügend kleinem Intervall $V \ni \theta$ gibt es einen Lift $\widetilde{\lambda}(V)$ so dass das Diagramm

$$\mathscr{A}(V) \otimes \mathscr{M} \cong: \quad \begin{array}{ccc} \widetilde{\mathscr{M}}(V) & \xrightarrow{\widetilde{\lambda}(V)} & \widetilde{\mathscr{M}}(V)^{good} \\ \downarrow & & \downarrow \\ \widehat{\mathscr{M}} & \xrightarrow{\widehat{\lambda}} & \widehat{\mathscr{M}}^{good} \end{array} \quad :\cong \mathscr{A}(V) \otimes \oplus_{\varphi}($$

kommutiert

B.3 Stokes-Strukturen

Fixiere ein **Modell** $(\mathscr{M}^{good}, \nabla^{good})$ auf D mit Pol bei $\{0\} = Z$ und somit auch eine Matrix $A^0 = dQ + \Lambda \frac{dt}{t}$.

$$\textbf{Wir sind interessiert an } \left\{(\mathscr{M}, \nabla) \mid \widehat{f} : (\widehat{\mathscr{M}, \nabla}) \xrightarrow{\cong} (\widehat{\mathscr{M}}^{good}, \widehat{\nabla}^{good})\right\} \Big/ \sim.$$

Wir betrachten dazu aber die größere die punktierte Menge

$$\mathscr{H}(\mathscr{M}^{good}) := \left\{(\mathscr{M}, \nabla, \widehat{f}) \mid \widehat{f} : (\widehat{\mathscr{M}, \nabla}) \xrightarrow{\cong} (\widehat{\mathscr{M}}^{good}, \widehat{\nabla}^{good})\right\} \Big/ \sim \qquad \ni (\mathscr{M}^{good}, \nabla^{good}, \widehat{\mathrm{Id}}) .$$

Indem wir zu Zusammenhangs Matrizen übergehen erhalten wir die isomorphen Mengen

$$\underbrace{\left\{A \mid A = \widehat{F}[A^0] \text{ für ein } \widehat{F} \in \widehat{G} := \mathrm{GL}_n(\mathbb{C}[\![t]\!])\right\}}_{=:\mathrm{Syst}(A^0)} \Big/ G\{t\}$$

und

$$\mathcal{H}(A^0) := \left\{(A, \widehat{F}) \in \mathrm{Syst}(A^0) \times \widehat{G} \mid A = \widehat{F}[A^0]\right\} \Big/ G\{t\}$$

B.3.1 Theorem 1

see:

- [Var96, p. 29]

Definition B.14

Definiere den *Stokes Raum*

$$\mathcal{S}t(\mathscr{M}^{good}) := H^1\left(S^1, \mathrm{Aut}^{<0}\left(\widetilde{\mathscr{M}}^{good}\right)\right)$$

wobei

- $\mathrm{Aut}^{<0}(\underbrace{\widetilde{\mathscr{M}}^{good}}_{\mathscr{A}_{\widetilde{D}} \otimes \mathscr{M}^{good}})$ die Garbe auf S^1 der Automorphismen welche
  - mit dem Zusammenhang kompatibel sind TODO und
  - formal äquivalent zur Identität sind.

  Die Schnitte heißen *Stokes Matrizen*.

Theorem B.14

$\mathcal{S}t(\mathscr{M}^{good})$ ist ein $\mathbb{C}$-Vektorraum.

Sei $(\mathscr{M}, \nabla, \widehat{f}) \in \mathscr{H}(\mathscr{M}^{good})$. Dann gibt es eine Überdeckung $\mathfrak{W}$ von S^1 und für jedes $W_i \in \mathfrak{W}$ einen Lift[3] von $\widehat{f}$

$$f_i : (\widetilde{\mathscr{M}}, \widetilde{\nabla})_{|W_i} \xrightarrow{\sim} (\widetilde{\mathscr{M}}^{good}, \widetilde{\nabla}^{good})_{|W_i}\,.$$

Dann ist $(f_j f_i^{-1})_{i,j}$ ein Kozykel in der Garbe $\mathrm{Aut}^{<0}(\widetilde{\mathscr{M}}^{good})$ relativ zur Überdeckung $\mathfrak{W}$. Damit haben wir eine Abbildung

$$\mathscr{H}(\mathscr{M}^{good}) \to H^1(S^1, \mathrm{Aut}^{<0}(\widetilde{\mathscr{M}}^{good})) = \mathcal{S}t(\mathscr{M}^{good})$$

Theorem B.15

Das ist ein Isomorphismus von punktierter Mengen.

[3] $\widehat{f_i} = \widehat{f}$

B.3.2 Theorem 2

Theorem B.17: Balser, Jurkat, Lutz

Es gibt einen Isomorphismus

$$\mathcal{H}(A^0) \cong (U_+ \times U_-)^{k-1} \qquad \cong \mathbb{C}^{(k-1)n(n-1)}$$
$$[(A, \widehat{F})] \mapsto \mathbf{S} = (S_1, \ldots, S_{2k-2})$$

Corollary B.17

Es gibt einen Isomorphismus

$$\mathrm{Syst}(A^0)/G\{t\} \cong (U_+ \times U_-)^{k-1}/T$$

Durch Rausteilen der Torus Wirkung: $t(\mathbf{S}) = (tS_1t^{-1}, \ldots, tS_{2k-2}t^{-1})$. $T = (\mathbb{C}^*)^n$

Definition B.18

Sei $\varphi_{ij}(z)$ der führende Term von $\varphi_i - \varphi_j$

- für $\varphi_i - \varphi_j = a/z^{k-1} + b/z^{k-2} + \ldots$ dann $\varphi_{ij} = a/z^{k-1}$.

$d \in \mathbb{A} \subset S^1 :\Leftrightarrow$ es gibt $i \neq j$ so dass $\varphi_{ij}(z) \in \mathbb{R}_{<0}$ für $z \to 0$ auf dem 'Strahl durch d'. Die Elemente in $\mathbb{A}$ heißen *anti-Stokes-Richtungen*.

Richtungen, entlang denen $e^{\varphi_i - \varphi_j}$ am schnellsten fällt

Sei $r := \#\mathbb{A}$, $l := r/(2k-2)$ und $\underline{\mathbf{d}} := (d_1, \ldots, d_l)$ *Halb-Periode*.

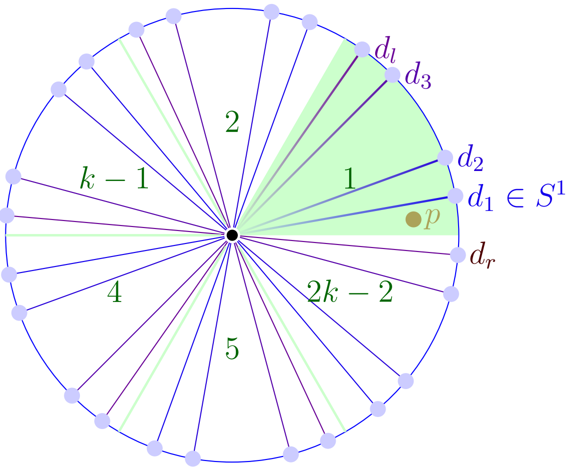
Definition B.19

Definiere die *totale Ordnung*

$\varphi_i \underset{\underline{\mathbf{d}}}{<} \varphi_j \Leftrightarrow \varphi_{ij} \in \mathbb{R}_{<0}$ entlang einem $d \in \underline{\mathbf{d}}$

und durch $\varphi_i \underset{\underline{\mathbf{d}}}{<} \varphi_j \Leftrightarrow \pi_i < \pi_j$ die

$$\textit{Permutations Matrix } (P)_{ij} = \delta_{\pi(i)j}\,.$$



Theorem B.20

Zu $(A, \widehat{F})$ gibt es auf jedem Sektor Sect (d_i, d_{i+1}) einen (kanonischen) Lift

$$\Sigma_i(\widehat{F}) \in \mathrm{GL}_n(\mathcal{O}_{\mathrm{Sect}_i})$$

eine **invertierbare** $n \times n$ Matrix von holomorphen Einträgen so dass $\Sigma_i(\widehat{F})[A^0] = A$. (Durch Borel-Summation)

$\Sigma_i(\widehat{F})$ kann analytisch auf den Supersektor $\widehat{\mathrm{Sect}}_i := \mathrm{Sect}\left(d_i - \frac{\pi}{2k-2}, d_{i+1} + \frac{\pi}{2k-2}\right)$ fortgesetzt werden und hat dort asymptotische Entwicklung $\widehat{F}$ bei 0.

Definition B.22

κ_i ist formal äquivalent zu id: $\kappa_i[A^0] = A^0$.

Die *Stokes Faktoren* zu $(A, \widehat{F})$ sind

$$K_i := e^{-Q} \cdot e^{-\Lambda} \cdot \underbrace{\Sigma_i(\widehat{F})^{-1} \cdot \Sigma_{i-1}(\widehat{F})}_{\kappa_i} \cdot e^{\Lambda} \cdot e^{Q}$$

Lemma B.22

K_i ist in der *Gruppe der Stokes Faktoren*

$$\mathbb{S}\mathrm{to}_{d_i}(A^0) := \{K \in G \mid (K)_{ij} = \delta_{ij} \text{ außer } \varphi_{ij} \in \mathbb{R}_{<0} \text{ entlang } d_i\}.$$

Lemma B.23

Sei $\mathbf{d} = (d_1, \dots, d_l)$.

1. $\prod_{d \in \mathbf{d}} \mathbb{S}\mathrm{to}_d(A^0) \cong PU_+P^{-1}; \qquad (K_1, \dots, K_l) \mapsto K_l \dots K_2 K_1 \in G$
2. $\prod_{d \in \mathbb{A}} \mathbb{S}\mathrm{to}_d(A^0) \cong (U_+ \times U_-)^{k-1}; \qquad (K_1, \dots, K_r) \mapsto (S_1, \dots, S_{2k-2})$
    - wobei $S_i := P^{-1} K_{il} \dots K_{(i-1)l+1} P \in U_{+/-}$ falls i ungerade/gerade die Stokes Matrizen.

Theorem B.24

Die Abbildung

$$\begin{aligned} \mathcal{H}(A^0) &\longrightarrow (U_+ \times U_-)^{k-1} \\ (A, \widehat{F}) &\longmapsto (S_1, \dots, S_{2k-2}) \end{aligned}$$

ist ein Isomorphismus.
Also ist insbesondere $\mathcal{H}(A^0)$ isomorph zum VR $\mathbb{C}^{(k-1)n(n-1)}$.

C Notes

C.1 Meromorphe Zusammenhänge vs. Systeme

C.2 Formal model vs. Formal Decomposition

Write A as

$$A = \left(\sum_{j=k_i}^{0} A_j \frac{dz}{z^j} \right) + A_0 dz + \dots$$

Definition C.1: [Boa01] Def 2.2

A meromorphic connection is a *nice (or generic) meromorphic connection* if

- at each a_i

the leading coefficient ${}^iA_{k_i}$ is

- diagonalisable with distinct eigenvalues and $k_i \geq 2$, or
- diagonalisable with distinct eigenvalues $\mod \mathbb{Z}$ and $k_i = 1$

Theorem C.1: Formal decomposition (II.5.7)

Let $(\mathcal{M}, \nabla)$ be a germ of meromorphic bundle with connection,

- equipped with a basis in which the matrix Ω takes the form

  $$\Omega = t^{-r} A(t) \frac{dt}{t}$$

  with $r \geq 1$, A having holomorphic entries, and $A_0 := A(0)$ being regular semisimple[a]

Then there exist

- a model $(\mathcal{M}^{good}, \nabla) = \left(\bigoplus_{\varphi} (\mathcal{E}^{\varphi} \otimes \mathcal{R}_{\varphi}) \right)$ and
- a 'formal' isomorphism

$$\hat{\mathcal{O}}_D \otimes (\mathcal{M}, \nabla) \xrightarrow{\sim} \hat{\mathcal{O}}_D \otimes \left(\bigoplus_{\varphi} (\mathcal{E}^{\varphi} \otimes \mathcal{R}_{\varphi}) \right)$$

[a] i.e. with pairwise distinct eigenvalues.

C.3 Reading text from Loday-Richaud

- **The paper:** [Lod94]
  - Stokes matrices are not all germs of isotropies, only the Sokes germs! see 4.3.13

- for **Sheaf** → **Matrix**

  Theorem C.2: II.2.1

  The map

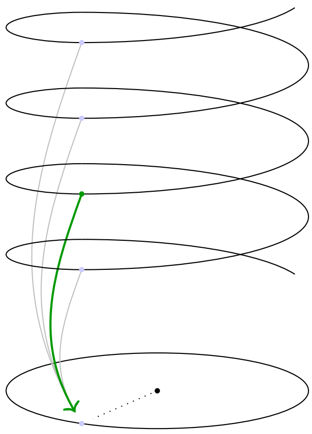
  $$h : \prod_{\alpha \in \mathbb{A}} \mathrm{Sto}_\alpha(A_0) \to H^1(S^1; \Lambda(A_0))$$

  is bijective and natural.

- **The book:** [Lod14]
  - multilevels, multisectors
- others:
  - [**LodayRichaud2004**]

$k - 1 \to k$

C.3.1 Determination $\tilde{\theta}$ of θ



C.3.2 Definition of $\Lambda_\theta(A_0)$

p. 854f [Lod94] $\Lambda_\theta(A_0) :\Leftrightarrow$ sheaf of flat isotropies over S^1

Definition C.3

A germ of $\Lambda_\theta(A_0)$ at $\theta_0 \in S^1$ is

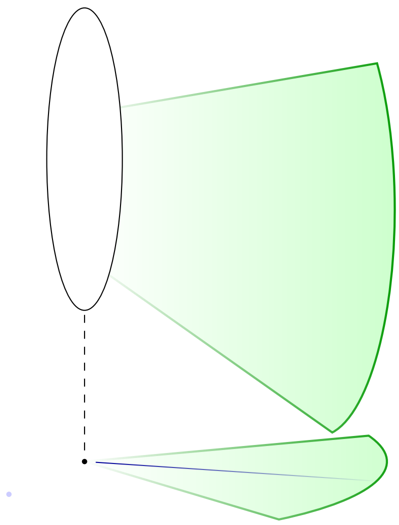
- an invertible Matrix $f \in \mathrm{GL}_n(\mathcal{O}(U))$
  - for suitable arc $U = U(\theta_0, \varepsilon, \varepsilon')$

  satisfying:

1. *Flatness:*

$$\lim_{\substack{x\to 0\\ x\in U}} f(x) = \mathrm{Id} \quad \text{and} \quad f \underset{U}{\sim} I$$

2. *Isotropy of* $[A_0]$*:* ${}^{f}A_0 = A_0$.

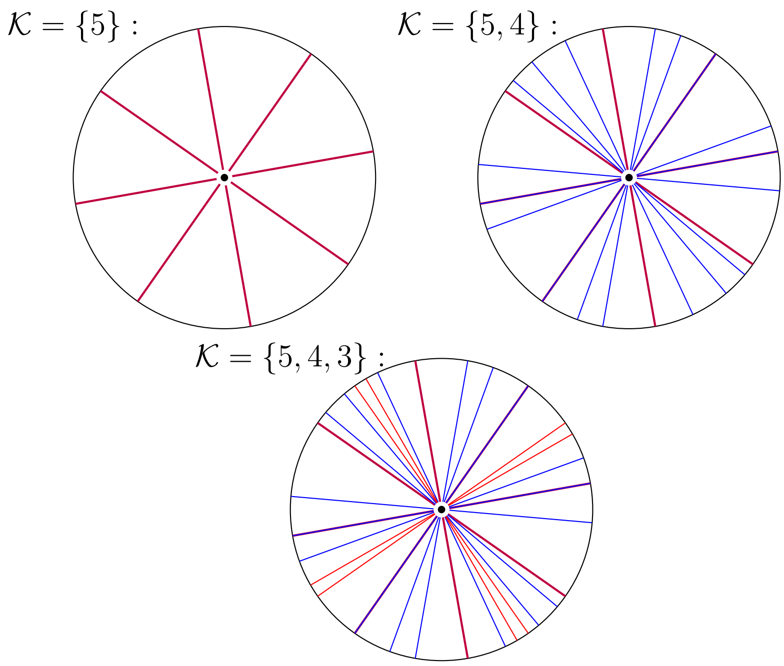


C.3.3 Definition of $\mathrm{Sto}_\alpha(A_0)$

- p. 861 in the paper
- p. 78 in the book

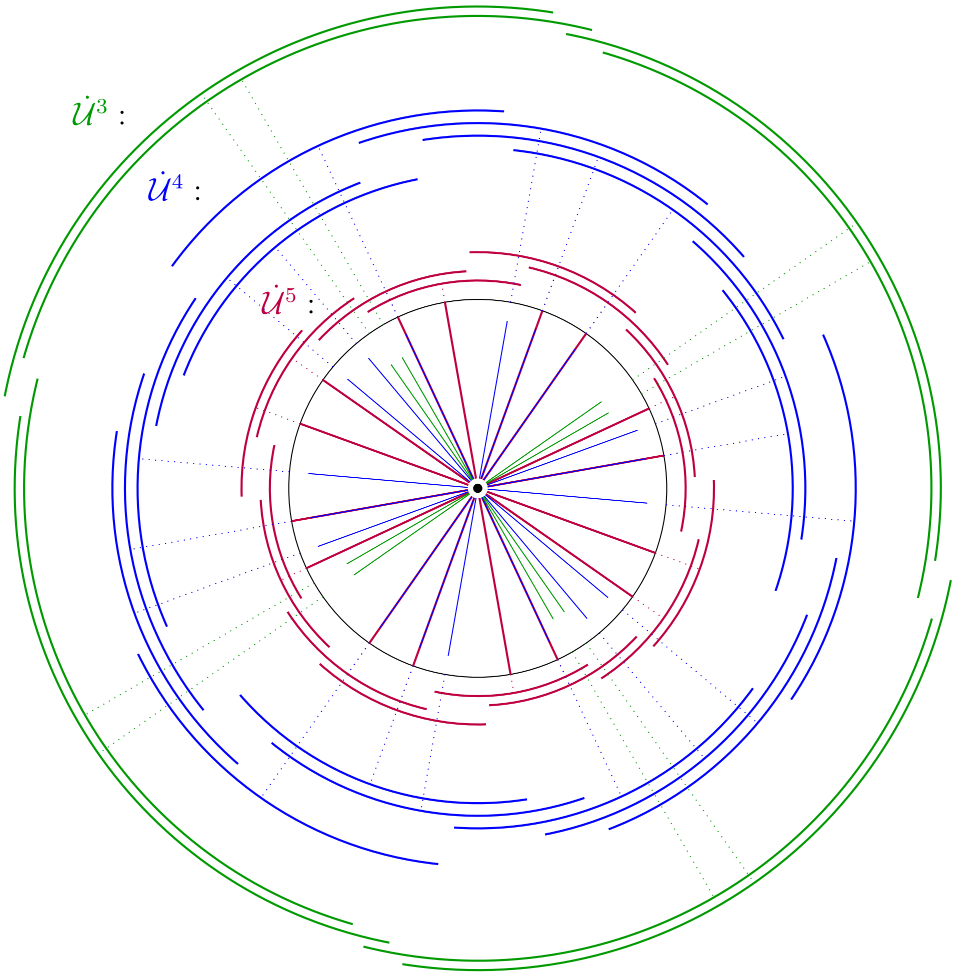
C.3.4 Definition of $\mathcal{F}$

C.3.5 Adequate coverings

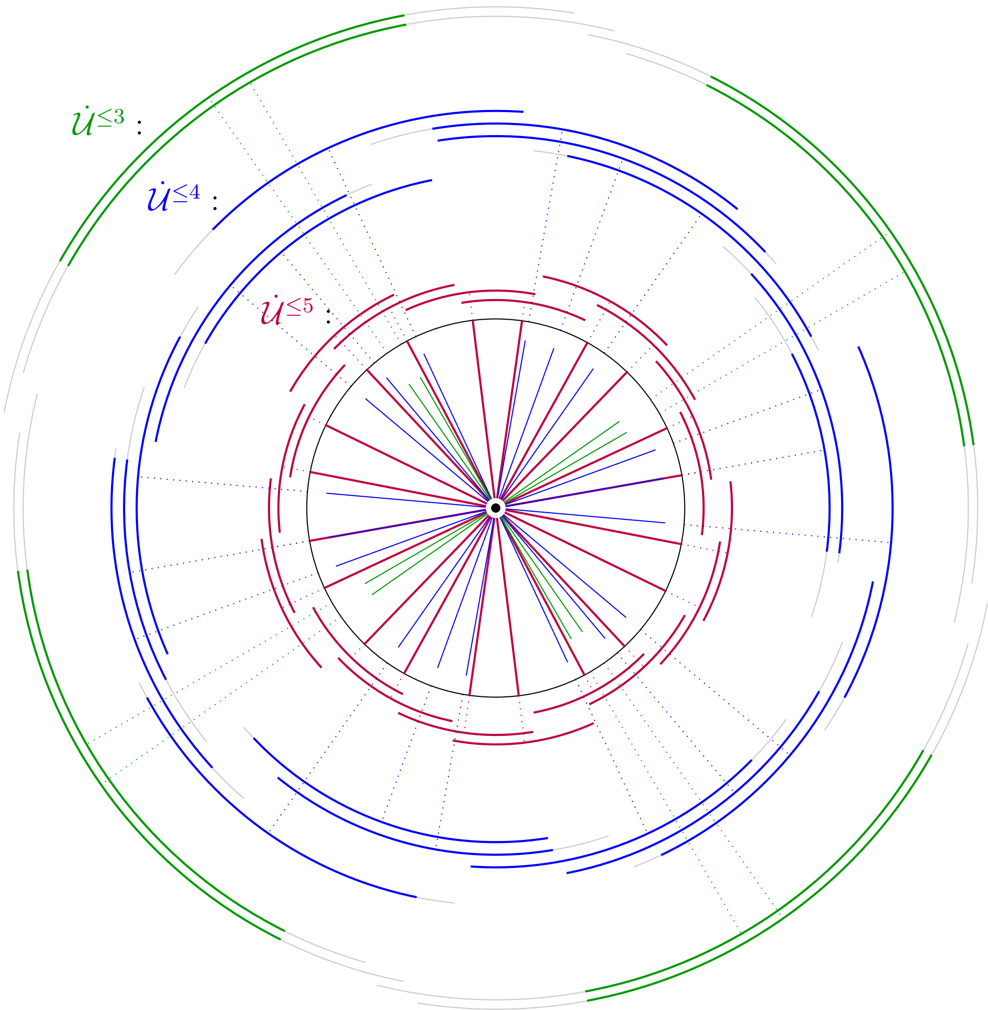


Let $k \in \mathcal{K}$.

The *cyclic covering* $\boldsymbol{\mathcal{U}^k = \{U^k_\alpha; \alpha \in \mathbb{A}^k\}}$



The *cyclic covering* $\mathcal{U}^{\leq k} = \{U^{\leq k}_{\alpha}; \alpha \in \mathbb{A}^{\leq k}\}$



C.3.6 Proof

Unique level

p. 872

Multi level

p. 872

D Meromorphe Zusammenhänge

Siehe:

- [Boa01] and [Boa99]
- [Sab07]

and

- [Var96]

D.1 From [Pym13] and [GLP13]

Consider a Riemann surface X and an effective divisor D.

Definition D.2

If E is a holomorphic vector bundle over X, a *meromorphic connection on E with poles bounded by D* is a differential operator

$$\nabla : \mathcal{E} \to \Omega^1_X(D) \otimes \mathcal{E}$$

on $\mathcal{E}$

- the sheaf of holomorphic sections on E

that satisfies the Leibniz rule

$$\nabla(fs) = f\nabla s + (df) \otimes s$$

Remark D.2

Thus, in a local trivialization of E near a point $p \in D$ of multiplicity k and a coordinate z centered at p, we may write

$$\nabla = d - \frac{1}{z^k} A(z) dz$$

for a holomorphic matrix $A(z)$.

D.2 Matrix version from [Boa01] and [Boa99]

Let

- $D = k_1(a_1) + \cdots + k_m(a_m)$ be an **effective divisor** on $\mathbb{P}^1$ so that
  - $a_1, \ldots, a_m \in \mathbb{P}^1$ are distinct points,
  - $k_1, \ldots, k_m > 0$ positive integers

  and
- $V \to \mathbb{P}^1$ be a rank n vector bundle.

Definition D.3: 2.1

A *meromorphic connection* ∇ on V with poles on D is

- a map $\nabla : V \to V \times K(D)$ where
  - from the sheaf of holomorphic sections of V
  - to the sheaf of sections of $V \otimes K(D)$[a]
- satisfying the Leibnitz rule

$$\nabla(fv) = (df) \otimes v + f\nabla v \tag{D.1}$$

  where
  - v is a local section of V,
  - f is a local holomorphic function and
  - K is the sheaf or holomorphic one-forms on $\mathbb{P}^1$

[a] for more information about $K(D)$ see [Bea96]

Remark D.4

If we choose a local coordinate z on $\mathbb{P}^1$ vanishing at a_i then

- in terms of local trivialization of V,

∇ has the form

$$\nabla = d - A = d - {}^iA^a$$

where

- $A = \left(\sum_{j=k_i}^{0} A_j \frac{dz}{z^j}\right) + A_0 dz + \ldots$
  - is a matrix of meromorphic one-forms and
  - $A_j \in \mathrm{End}(\mathbb{C}^n)$.

[a] the presuperscript is used to signify local information

Let $E = \mathbb{C}^n$ be

- a fixed complex vector space
  - with preferred basis.

Definition D.4

[Boa99, Def 1.5]. A *germ of a meromorphic linear differential system*[a] (of rank n) is

- a germ of a meromorphic connection
  - on the trivial vector bundle with fibre E.

[a] or just *system* from now on

Remark D.5

- The set of systems is isomorphic to $\mathrm{End}(E) \otimes \mathbb{C}\{z\}[z^{-1}]$
- A matrix of germs of meromorphic functions $A' \in \mathrm{End}(E) \otimes \mathbb{C}\{z\}[z^{-1}]$ determines

$$\frac{dv}{dz} = A'v$$

  - a system of equations for $v(z) \in E$
  - which corresponds to the connection germ $\nabla = d_A = d - A$
    - on E
    - where $A = A'dz$.

Definition D.6: [Boa01] Def 2.2

A meromorphic connection is a *nice (or generic) meromorphic connection* if

- at each a_i

the leading coefficient ${}^iA_{k_i}$ is

- diagonalisable with distinct eigenvalues and $k_i \geq 2$, or
- diagonalisable with distinct eigenvalues $\mod \mathbb{Z}$ and $k_i = 1$

Remark D.6

- This condition is independent of the trivialization and coordinate choice.
- We will restrict to nice connections since they are simplest yet sufficient[a] for our purposes.

[a] to describe the symplectic nature of isomonodromic deformations

Definition D.7

A *nice formal normal form* A^0 is

- a nice diagonal system with no holomorphic part.

TODO: Entspricht dies der Levelt-Turittin-Zerlegung??

Remark D.7

Such A^0 can be uniquely written as

$$A^0 := dQ + \Lambda\frac{dz}{z},$$

where

- $Q := \mathrm{diag}(q_1, \dots, q_n)$
  - $q_1, \dots, q_n \in z^{-1}C[z^{-1}]$ are polynomials
    - of degree $k-1$ in z^{-1}
    - with no constant term

  and

- $\Lambda = Res_0(A_0)$ is a constant diagonal matrix.

Local moduli spaces

Definition D.7

- $\mathrm{Syst}(A^0) := \{d - A \mid A = \widehat{F}[A^0] \text{ for some } \widehat{F} \in G[\![z]\!]\}^a$ where
  - A is a matrix of germs of meromorphic one-forms
  - $\widehat{F}[A^0] = (d\widehat{F})\widehat{F}^{-1} + \widehat{F}A^0\widehat{F}^{-1}$
  - $G[\![z]\!] := \mathrm{GL}_n(\mathbb{C}[\![z]\!])$
    * group of *formal transformations*
    * does not act on $\mathrm{Syst}(A^0)$

The group $G\{z\} := \mathrm{GL}_n(\mathbb{C}\{z\})$

- the group of *local analytic gauge transformations*

acts on $\mathrm{Syst}(A^0)$.

We are interested in $\mathrm{Syst}(\mathbf{A^0})/\mathbf{G\{z\}}^b$

[a] the set of germs at $0 \in \mathbb{C}$ of meromorphic connections on the trivial rank n vector bundle, that are formally equivalent to $d - A^0$.

[b] The set of isomorphism classes of germs of meromorphic connections formally equivalent to A^0. Note that any generic connection is formally equivalent to some such A^0.

Remark D.8

In the abelian and the simple pole case $\mathrm{Syst}(A^0)/G\{z\}$ is only a point.

Add a little bit more information

Definition D.10

A *compatible framing* at a_i of a vector bundle V with generic[a] connection ∇ is

- an isomorphism $g_0 : V_{a_i} \to \mathbb{C}^n$
  - between the firbre V_{a_i} and $\mathbb{C}^n$

  such that

  - the leading coefficient of ∇ is diagonal in any local trivialization of V extending g_0.

Remark D.10

Given a trivialization of V in a neighbourhood of a_i so that $\nabla = d - A$ as above, then

- a compatible framing is represented by a constant matrix $g_0 \in G$ such that $g_0 A_{k_i} g_0^{-1}$ is diagonal

[a] The precise notion of 'generic' is given in [Boa99] Definition 1.2 and we will refer to such connections as *nice*.

Definition D.11: 2.4

A connection (V, ∇) with compatible framing g_0 at a_i has *irregular type* ${}^iA^0$ if

- g_0 extends to a formal trivialization of V at a_i, in which ∇ differs from $d - {}^iA^0$ by a matrix of one-forms with just simple poles.

Definition D.11

The set of *applicable formal transformations* is

$$\widehat{G}(A^0) := \left\{ \widehat{F} \in \widehat{G} \mid \widehat{F}[A^0] \text{ is convergent} \right\} .$$

Definition D.12: marked pair

A *marked pair* is a pair $(A, \widehat{F})$ consisting of

- a nice system A and
- a choice of formal isomorphism $\widehat{F} \in \widehat{G}$ such that $A = \widehat{F}[A^0]$

Definition D.13

Define

- $\widehat{\mathrm{Syst}}_{cf}(A^0) :\Leftrightarrow$ the set of compatibly framed connection germs with both irregular and formal type A^0.
- $\widehat{\mathrm{Syst}}_{mp}(A^0) := \{(A, \widehat{F}) \mid A \in \mathrm{Syst}(A^0), \widehat{F} \in G[\![z]\!], A = \widehat{F}[A^0]\}$ the set of *marked pairs*.
  - $G\{z\}$ action on marked pairs: $g(A, \widehat{F}) = (g[A], g \circ \widehat{F})$

Lemma D.13

There is a canonical isomorphism

$$\widehat{\mathrm{Syst}}_{cf}(A^0) \cong \widehat{\mathrm{Syst}}_{mp}(A^0) ,$$

Let $\widehat{\mathrm{Syst}}(A^0)$ denote either of these two sets.

Definition D.14

Definition D.14

$$\mathcal{H}(A^0) := \widehat{\mathrm{Syst}}(A^0)/G\{z\}$$

Definition from [Boa99]:

$$\mathcal{H}(A^0) := G\{z\}\backslash\widehat{G}(A^0)$$

Lemma D.15

The set is canonically isomorphic to the set of isomorphism classes of compatibly framed systems having associated formal normal form A^0.

Lemma D.15

The actions of T and $G\{z\}$ on $\mathrm{Syst}(A^0)$ commute:

$$\mathrm{Syst}(A^0)/G\{z\} \cong \mathcal{H}(A^0)/T\,.$$

Moduli spaces

Let $\mathbf{a}$[1] denote the choice of

- an effective divisor $D = k_1(a_1) + \cdots + k_m(a_m)$ and
- at each point a_i
  - a germ $d - {}^iA^0$
    - of a **diagonal generic meromorphic connection**
      - on the trivial rank n vector bundle
    - write ${}^iA^0 = d({}^iQ) + {}^i\Lambda^0\frac{dz}{z}$

Definition D.16: 2.5

The *moduli space* $\mathcal{M}^*(\boldsymbol{a})$ $(\mathcal{M}(\boldsymbol{a}))$ is

- the set of isomorphism classes of pairs (V, ∇) where
  - a trivial (degree zero) rank n holomorphic vector bundle V over $\mathbb{P}^1$
  - a meromorphic connection ∇ (with poles on D) on V which is formally equivalent to $d - {}^iA^0$ at a_i for each i[a]

[a]and has no other poles

Definition D.17: 2.6

The *extended moduli space* $\widetilde{\mathcal{M}}^*(\boldsymbol{a})$ $(\widetilde{\mathcal{M}}(\boldsymbol{a}))$ is

[1]In [Boa99] denoted by $\mathbf{A}$

- the set of isomorphism classes of triples $(V, \nabla, \mathbf{g})$ where
  - a trivial (degree zero) rank n holomorphic vector bundle V over $\mathbb{P}^1$
  - a generic meromorphic connection ∇ (with poles on D) on V
  - compatible framins $\mathbf{g} = ({}^1g_0, \dots, {}^mg_0)$

  such that $(V, \nabla, \mathbf{g})$ has irregular type ${}^iA^0$ at each a_i

Since $\mathcal{M}^*(\mathbf{a})$ and $\widetilde{\mathcal{M}}^*(\mathbf{a})$ are moduli spaces of connections on trivial bundles we can obtain explicit descriptions of them. See [Boa99] page 10ff.

D.3 Definition from [Sab90]

Definition D.18

A *meromorphic connection* $\mathcal{M}_K$ is a

- K-vector space of finite dimension
- equipped with
  - a $\mathbb{C}$-linear derivation $\partial_x : \mathcal{M}_K \to \mathcal{M}_K$

  such that, for all $f \in K$ and all $m \in \mathcal{M}_K$ one has

$$\partial_x(fm) = \frac{\partial f}{\partial m} + f\partial_x m.$$

Theorem D.19

Let $\mathcal{M}_{\widehat{K}}$ be a formal meromorphic connection. There exists an integer q such that the connection $\pi^*\mathcal{M}_{\widehat{K}} = \mathcal{M}_{\widehat{L}}$ is isomorphic to a direct sum of elementary formal meromorphic connections.

D.4 Sheaf version from [Sab07]

Definition D.20: Holomorphic budles (0.3.?)

Let

- $\pi : E \to M$ be a holomorphic mapping between two complex analytic manifolds.

We will say that

- π is a *vector fibration of rank* d, or
- π makes E a *vector bundle of rank* d *on* M

if there exists a open covering...

Set $\mathscr{E}(U)$ as the set of holomorphic sections, where

- a *holomorphic section* of E on U is a holomorphic mapping $\sigma : U \to E$ which
  - is a section of the projection, i.e., which satisfies $\pi \circ \sigma = \mathrm{Id}_U$.

This defines a sheaf $\mathscr{E}$ of modules over $\mathcal{O}_M(U)$.

Definition D.21: Meromorphic bundles (0.8.?)

A *meromorphic bundle on M with poles along Z* is a locally free sheaf of $\mathcal{O}_M(*Z)$-modules of finite rank where

- Z is a smooth hypersurface in a complex analytic manifold M
- define the sheaf $\mathcal{O}_M(*Z)$ by
  - $U \mapsto$ functions which are *meromorphic along* $Z \cap U$ i.e. which
    * are holomorphic on $U \backslash Z \cap U$ and
    * for any chart V of M
      · contained in U and
      · in which $Z \cap V$ is defined by the vanishing of some coordinate z_1

      there exists
      · an integer m

      such that
      · $z_1^m f(z_1, \ldots, z_n)$ is locally bounded in the neighbourhood of any point of $Z \cap V$.

Definition D.22: Holomorphic / meromorphic connections

- A *holomorphic connection* ∇ on a holomorphic vector bundle $\pi : E \to M$ is a $\mathbb{C}$-linear homomorphism of sheaves

$$\nabla : \mathscr{E} \to \Omega^1_M \otimes_{\mathcal{O}_M} \mathscr{E}$$

  satisfying the Leibniz rule.
- A *connection* on a meromorphic bundle $\mathscr{M}$ is defined as a $\mathbb{C}$-linear homomorphism

$$\nabla : \mathscr{M} \to \Omega^1_M \otimes_{\mathcal{O}_M} \mathscr{M}$$

  satisfying
  - for
    * any open set U of M,
    * any section $s \in \Gamma(U, \mathscr{E})$ and
    * any holomorphic function $f \in \mathcal{O}(U)$

the *Leibnitz rule*:

$$\nabla(f \cdot s) = \nabla(s) + df \otimes s \in \Gamma(U, \Omega^1_M \otimes_{\mathcal{O}_M} \mathscr{E})$$

What is the difference between

- **meromorphic** bundle with **holomorphic** connection,
- **holomorphic** bundle with **meromorphic** connection and
- **meromorphic** bundle with **meromorphic** connection.

Definition D.24: Flatness (0.12.2)

The connection $\nabla : \mathcal{E} \to \Omega^1_M \otimes_{\mathcal{O}_M} \mathcal{E}$ is said to be *integrable* or *flat*, if

- its curvature $R_\nabla \equiv 0$

  where

  - $R_\nabla := \nabla \circ \nabla : \mathcal{E} \to \Omega^2_M \otimes_{\mathcal{O}_M} \mathcal{E}$ is a $\mathcal{O}_M$-linear morphism.

Proposition D.24: 0.12.4

The connection ∇ is flat if and only if, in any local basis e of $\mathcal{E}$, the connection matrix Ω satisfies

$$d\omega + \omega \wedge \omega = 0.$$

We will say that a connection on a meromorphic bundle is *integrable* or *flat* if its restriction to $M \backslash Z$ is an integrable connection on the holomorphic bundle $\mathscr{M}_{|M\backslash Z}$.

D.4.1 Models and formal decomposition of a germ

Let $\mathbf{k}$ denote $\mathbb{C}\{t\}[1/t]$.

Regular singularities

Definition D.25

An *elementary regular model* is

- a $(\mathbf{k}, \nabla)$-vector space equipped with a basis in which
  - the connection matrix is written as $\Omega(t) = (\alpha \operatorname{Id} + N)\frac{dt}{t}$ where
    - $\alpha \in \mathbb{C}$ and
    - N is a nilpotent matrix

Definition D.26

The horizontal sections of an elementary regular model have *moderate growth near the origin* if and only if for

- any horizontal section s in the neighbourhood of a closed angular sector with angel $< 2\pi$,

there exist

- an integer $n \geq 0$ and
- a constant $C > 0$

such that, on this sector,

$$\|s(t)\| \leq C|t|^{-n}.$$

Corollary D.27: II.2.9

Any $(\mathbf{k}, \nabla)$-vector space with regular singularity is isomorphic to a direct sum of elementary regular models.

Let

- $x^0 \in X$ and
- $(x_1, \ldots, x_n)$ be a system of coordinates centered at x^0.
- $\Omega = A(t,x)\frac{dt}{t} + \sum_{i=1}^n C^i(t,x)dx_i$ be the connection matrix
  - in some basis $\mathbf{e}$ of $\mathscr{M}$
  - in the neighbourhood of $(0, x^0)$.

If the matrices A and $C^{(i)}$ have holomorphic entries, $(\mathscr{M}, \nabla)$ has regular singularity in the neighbourhood of $(0, x^0)$. For more criteria see [Sab07, II.4.a]: **How to recognize a regular singularity**.

Proposition D.28: Any formal solution is convergent(II.2.18)

Let $A(t)$ be a matrix in $M_d(\mathbb{C}\{t\})$. Any vector $u(t)$ with entries in $\mathbb{C}[\![t]\!]$ which is solution of the system $tu'(t) + A(t)u(t) = 0$ has converging entries.

Definition D.29: II.2.24

A meromorphic bundle $\mathscr{M}$ on $D \times X$, with poles along $\{0\} \times X$, equipped with a flat connection ∇, has *regular singularity* if

- in the neighbourhood of any point $(0, x^0)$ of $\{0\} \times X$
  - there exists a logarithmic lattice of $(\mathscr{M}, \nabla)$.

Theorem D.30: Normal form of regular singularities with parameter (II.2.25)

There exists a basis of the germ $\mathscr{M}_{(0,x^0)}$ in which the matrix of ∇ can be written as $B\frac{dt}{t}$, where B is constant.

Irregular singularities in rank one

Let

- $\varphi \in \mathbb{C}\{t, x_1, \ldots, x_n\}[t^{-1}]$.

Denote by $\mathcal{E}^\varphi$

- the germ $(\mathcal{M}, \nabla) := (\mathbb{C}\{t, x_1, \ldots, x_n\}[t^{-1}], d - d\varphi)$
  - this is the system satisfied by the function e^φ

For $\alpha \in \mathbb{C}$, let $\mathcal{N}_{\alpha,0}$ be

- the germ $(\mathcal{M}, \nabla) := (\mathbb{C}\{t, x_1, \ldots, x_n\}[t^{-1}], d - \alpha dt/t)$.

Proposition D.31: Classification of irregular singularities in rank one (II.5.1)

Any germ $(\mathcal{M}, \nabla)$ of rank one is isomorphic to some germ $\mathcal{E}^\varphi \otimes \mathcal{N}_{\alpha,0}$.

Tow such germs are corresponding to (φ_1, α_1) and (φ_2, α_2) are isomorphic iff

- $\varphi_1 - \varphi_2$ has no pole and
  - Therefore, the class of φ in $\mathbb{C}\{t, x_1, \ldots, x_n\}[t^{-1}]/\mathbb{C}\{t, x_1, \ldots, x_n\}$ determines the germ $\mathcal{E}^\varphi$.
- $\alpha_1 - \alpha_2 \in \mathbb{Z}$.

Irregular singularities in arbitrary rank

Definition D.32

Let $(\mathcal{M}, \nabla)$ be a **germ** of a meromorphic bundle with connection.

- A germ $(\mathcal{M}, \nabla)$ is *elementary* if it is isomorphic to some germ

  $$(\mathcal{E}^\varphi, \nabla) \otimes (\mathcal{R}, \nabla)$$

  where
  - $(\mathcal{R}, \nabla)$ ha regular singularity along $\{0\} \times X$

- $(\mathcal{M}, \nabla)$ is a *model* if it is isomorphic to a direct sum of elementary models, written as

$$\bigoplus_{\varphi} (\mathcal{E}^{\varphi} \otimes \mathcal{R}_{\varphi})$$

  where

  - the meromorphic bundles with connection $\mathcal{R}_{\varphi}$ have regular singularity and
  - the $\varphi \in \mathbb{C}\{t, \underbrace{x_1, \ldots, x_n}_{\text{parameter}}\}[t^{-1}]$
    * have no holomorphic part and
    * are pairwise distinct.
- We will say that a model is *good* if,
  - for all $\varphi \neq \psi$
    * such that $\mathcal{R}_{\varphi}$, $\mathcal{R}_{\psi}$ are nonzero,

    the order of the pole along $t = 0$ of $(\varphi - \psi)(t, x)$ does not depend on x being in some neighbourhood of x^o.

Theorem D.33: Formal decomposition (II.5.7)

Let $(\mathcal{M}, \nabla)$ be a germ of meromorphic bundle with connection,

- equipped with a basis
  - in which the matrix Ω takes the form

$$\Omega = t^{-r} \left[A(t, x) \frac{dt}{t} + \sum_{i=1}^{n} C^{(i)}(t, x) dx_i \right]$$

    with
    * $r \geq 1$
    * A and the $C^{(i)}$ having holomorphic entries, and
    * $A_0 := A(0, x^0)$ being regular semisimple[a]

This condition **might be** equivalent to the condition of being **nice** in [Boa99].

TODO: is this the condition, such that no ramification is necessary?

Then there exist

- a good model $(\mathcal{M}^{good}, \nabla) = \left(\bigoplus_{\varphi} (\mathcal{E}^{\varphi} \otimes \mathcal{R}_{\varphi})\right)$ and
- a 'formal' isomorphism

$$\widehat{\mathcal{O}}_{D \times X, x_0} \otimes (\mathcal{M}, \nabla) \xrightarrow{\sim} \widehat{\mathcal{O}}_{D \times X, x_0} \otimes \left(\bigoplus_{\varphi} (\mathcal{E}^{\varphi} \otimes \mathcal{R}_{\varphi}) \right)$$

[a] i.e. with pairwise distinct eigenvalues.

Remark D.34

- In the present situation, no ramification is needed. TODO: Why?
- Moreover, we will see that all the components $\mathcal{R}^{\varphi}$ occurring in the model have rank one, which is not the case in general, even when no ramification is needed.

E Asymptotic expansion

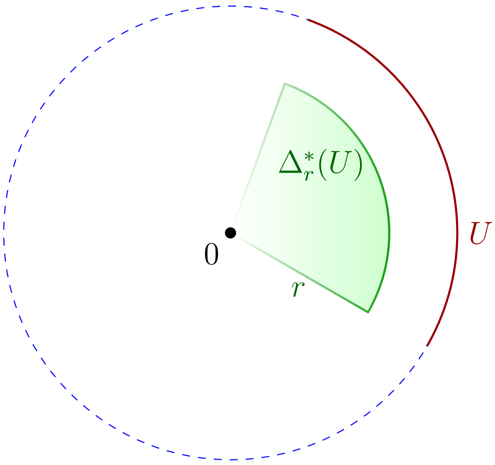
- [Sab90, p. 60] Chapter II.2.2
  - [Sab00]
- Van der Put: [PS03]: Chapter 7: Exact Asymptotics
- [Sab07, p. II.5]
- [Maj84]
- [Bal00]

E.1 Definition

Let U be an open interval in S^1

Definition E.1

- $\Delta_r^*(U) := \{z \in \Delta_r \mid z = \rho e^{i\theta}, 0 < \rho < r, \theta \in U\}$



- $\Delta_r := \Delta_r^*(S^1)$

Let $\widehat{\varphi} = \sum_{n \geq -n_0} a_n x^n$ with $a_n \in \mathbb{C}$ is as asymptotic expansion of f at 0 if for all $m \in \mathbb{N}$ one has

$$\lim_{x \to 0, x \in \Delta_r^*(U)} \left|x^{-m}\right| \left| f(x) - \sum_{-n_0 \leq n \leq m} a_n x^n \right| = 0 \tag{E.1}$$

Definition E.2

- $\bar{\mathcal{A}}(U, r) \subset \mathcal{O}(\Delta_r^*(U))$ the set of functions which admits an asymptotic power series

- $\mathcal{A}(U,r) := \bigcap_V \bar{\mathcal{A}}(V,r)$ where V are relatively compact open subsets of U
  - $\mathcal{A}(U,r) = \{\varphi \in \bar{\mathcal{A}}(U,r) \mid \varphi \in \bar{\mathcal{A}}(V,r) \forall V \text{ rel. cp. op. subset of } U\}$
- $\mathcal{A}(U) := \bigcup_r \mathcal{A}(U,r)$
  - $\mathcal{A}(U) = \{\varphi \mid \exists r\colon \varphi \in \mathcal{A}(U,r)\} = \{\varphi \mid \exists r\colon \forall V \subset U \text{ rel.cp.op.: } \varphi \in \bar{\mathcal{A}}(V,r)\}$

The mapping $U \mapsto \mathcal{A}(U)$ defines a sheaf on S^1

E.1.1 Better definition from [PS03]

Let f be a holomorphic function on $S(a,b,\rho)$, where

- $a,b \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$ and
- ρ
  - is a continous function on the open interval (a,b) and
  - has values in the positive real numbers

  and
- $S(a,b,\rho)$ are the complex numbers $z \neq 0$ satisfying
  - $\arg(z) \in (a,b)$ and
  - $|z| < \rho(\arg(z))$.

Definition E.3: 7.1

f has the **formal Laurent series** $\sum_{n\geq n_0} c_n z^n$ as asymptotic expansion if

- for every $\mathbf{N} \geq 0$ and
- every closed subsector W in $S(a,b,\rho)$

there exists a constant $C(\mathbf{N},W)$ such that

$$\left| f(x) - \sum_{n_0 \leq n \leq \mathbf{N}-1} c_n z^n \right| \leq C(\mathbf{N},W)|z|^{\mathbf{N}} \qquad \text{for all } z \in W$$

$\Leftrightarrow$

$$\lim_{z\to 0, z\in W} |z|^{-(\mathbf{N}-1)} \left| f(x) - \sum_{n_0 \leq n \leq \mathbf{N}-1} c_n z^n \right| = 0 \qquad \text{for all } z \in W$$

- One writes $J(f)$ for the formal Laurent series.
- Define $\mathcal{A}(a,b)$ as the limit of the $\mathcal{A}(S(a,b,\rho))$ where
  - $\mathcal{A}(S(a,b,\rho))$ are the functions with asymptotic expansion.

Definition E.4: 7.4

Let

- k be a positive real number and
- S be an open sector.

A function $f \in \mathcal{A}(S)$, with asymptotic expansion $J(f) = \sum_{n \geq n_0} c_n z^n$, is said to bea a *Grevrey function of order* k if:
For every closed subsector W of S there are constants

- $A > 0$ and
- $c > 0$

such that for all

- $N \geq 1$ and
- all $z \in W$ and $|z| \leq c$

one has

$$\left| f(z) - \sum_{n_0 \leq n \leq N-1} c_n z^n \right| \leq A^N \Gamma\left(1 + \frac{N}{k}\right) |z|^N$$

or equivalently

$$\left| f(z) - \sum_{n_0 \leq n \leq N-1} c_n z^n \right| \leq A^N (N!)^{\frac{1}{k}} |z|^N$$

Definition from [Lod94]

p. 854f [Lod94] $\Lambda_\theta(A_0) :\Leftrightarrow$ sheaf of flat isotropies over S^1

Definition E.5

A germ of $\Lambda_\theta(A_0)$ at $\theta_0 \in S^1$ is

- an invertible Matrix $f \in \mathrm{GL}_n(\mathcal{O}(U))$
  - for suitable arc $U = U(\theta_0, \varepsilon, \varepsilon')$

  satisfying:

  1. *Flatness:*

  $$\lim_{\substack{x \to 0 \\ x \in U}} f(x) = \mathrm{Id} \text{ and } f \underset{U}{\sim} I$$

  2. *Isotropy of* $[A_0]$*:* ${}^f A_0 = A_0$.

E.1.2 Sheaf version

Let

- D be a open disc
  - with coordinate t centered at the origin and
  - of radius $r^o > 0$,
- $\widetilde{D}$ the product $[0, r^o[\times S^1$ and
- $\pi : \widetilde{D} \to D$ the mapping $(r, e^{i\theta}) \mapsto t = re^{i\theta}$.

Define the derivations $t\frac{\partial}{\partial t}$, $\bar{t}\frac{\partial}{\partial \bar{t}}$, $\frac{\partial}{\partial x_i}$ and $\frac{\partial}{\partial \bar{x}_i}$ $(i = 1, \dots, n)$ on $\mathscr{C}^{\infty}_{\widetilde{D}\times X}$.

- On $\mathscr{C}^{\infty}_{\widetilde{D}\times X}$ we have
  - $t\frac{\partial}{\partial t} = \frac{1}{2}\left(r\frac{\partial}{\partial r} - i\frac{\partial}{\partial \theta}\right)$ and
  - $\bar{t}\frac{\partial}{\partial \bar{t}} = \frac{1}{2}\left(r\frac{\partial}{\partial r} + i\frac{\partial}{\partial \theta}\right)$.

Definition E.6: II.5.10

The sheaf of rings $\mathscr{A}_{\widetilde{D}\times X}$ is

- the subsheaf of $\mathscr{C}^{\infty}_{\widetilde{D}\times X}$
  - of germs killed by $\bar{t}\frac{\partial}{\partial \bar{t}}$ and $\frac{\partial}{\partial \bar{x}_i}$ $(i = 1, \dots, n)$.

1. On $\mathscr{A}_{\widetilde{D}\times X}$ we also have the derivation $\frac{\partial}{\partial t}$ and we can set
$$\frac{\partial}{\partial t} = e^{-i\theta}\frac{\partial}{\partial r}$$
   - by using the vanishing of $\bar{t}\frac{\partial}{\partial \bar{t}}$
2. The action of the $\frac{\partial}{\partial x_i}$ on $\mathscr{C}^{\infty}_{\widetilde{D}\times X}$ keeps $\mathscr{A}_{\widetilde{D}\times X}$ stable.
3. The sheaf $\mathscr{A}_{\widetilde{D}\times X}$ contains the subsheaf $\pi^{-1}\mathcal{O}_{D\times X}$
   - if f is holomorphic on $D \times X \Rightarrow f \circ \pi$ is a section of $\mathscr{A}_{\widetilde{D}\times X}$

Denote by $\mathscr{A}_{S^1\times X}$ the restriction of $\mathscr{A}_{\widetilde{D}\times X}$ to $\{0\} \times S^1 \times X$.

See [Sab00] appendix B.1.4 for **Sheaf** $\to$ **Matrix**.

Taylor mapping

- The Taylor expansion of
  - a C^∞ germ
    - in (θ^o, x^o)
    - along $r = 0$

  can be written as

  $$\sum_{k\geq 0} f_k(\underbrace{x_1, \dots, x_n}_{\text{parameter}}, \theta) r^k$$

  where

  - f_k are C^∞ functions on some **fixed** neighbourhood of (θ^o, x^o)

- The Taylor expansion
  - of a germ of $\mathscr{A}_{\widetilde{D}\times X}$
    - in (θ^o, x^o)
    - along $t = 0$

  thus takes the form

  $$\sum_{k\geq 0} f_k(\underbrace{x_1, \dots, x_n}_{\text{parameter}}) t^k$$

  where

  - f_k are holomorphic on some **fixed** neighbourhood of x^o.

In other words: The Taylor expansion mapping defines a homomorphism

$$T : \mathscr{A}_{S^1\times X,(\theta^o,x^o)} \to \widehat{\mathcal{O}}_{D\times X,x^o}$$

Let

- $\mathscr{A}^{<\{0\}\times X}_{S^1\times X,(\theta^o,x^o)}$ or $\mathscr{A}^{<X}_{S^1\times X,(\theta^o,x^o)}$[1] be the kernel of T.

[1]if one identifies X to the divisor $\{0\}\times X$ of $D\times X$.

E.2 Properties

1. If $\widehat{\varphi}$ is an Asymptotic expansion for f then one has

$$a_0 = \lim_{x\to 0, x\in\Delta_r^*(U)} x^{n_0} f(x)$$

and for $m > 0$,

$$a_m = \lim_{x\to 0, x\in\Delta_r^*(U)} x^{-m}\left[x^{n_0} f(x) - \sum_{0\le n\le m-1} a_n x^n\right]$$

In particular this asymptotic expansion is unique

- f admits a zero asymptotic expansion iff for all $p \in \mathbb{Z}$ one has

$$\lim_{x\to 0, x\in\Delta_r^*(U)} x^p f(x) = 0$$

2. Define $\mathcal{A}(U) \to \widehat{K}$ denoted by $f \mapsto \widehat{f}$.
   - $\mathcal{A}(U)$ is a subring of $\mathcal{O}(\Delta_r^*(U))$
   - this mapping is a morphism of rings
3. Denote by $\mathcal{A}^{<0}(U)$ the kernel of this morphism
   - $x \mapsto e^{-\frac{1}{x}}$ has zero asymptotic expansion in some sector around $\theta = 0$
4. $\mathcal{A}(U)$ is stable under derivation[2].
5. $\mathcal{A}(U)$ contains K as a subfield.

[2]Proof in [Sab90]

Lemma E.7: [Sab90]: 2.2.5

If U is a proper open interval of the unit circle the mapping

$$\mathcal{A}(U) \to \widehat{K}$$

is onto[a], thus one has an exact sequence

$$0 \to \mathcal{A}^{<0}(U) \to \mathcal{A}(U) \to \widehat{K} \to 0\,.$$

[a]1p proof at [Sab90][p63]

Lemma E.7: [Sab07]: Borel-Ritt

T is onto:

$$0 \to \mathscr{A}^{<X}_{S^1\times X,(\theta^o,x^o)} \to \mathscr{A}_{S^1\times X,(\theta^o,x^o)} \xrightarrow{T} \pi^{-1}\widehat{\mathcal{O}}_{D\times X} \to 0$$

is exact[a]

[a]where

- $\widehat{\mathcal{O}}_{D\times X}$ is a sheaf on $\{0\}\times X$, hence $\pi^{-1}\widehat{\mathcal{O}}_{D\times X}$ is a sheaf on $S^1\times X$

E.3 Main result

Let $\mathcal{M}_K$ be a meromorphic connection.

Let $\mathcal{M}$ be a germ at $(0,x^o)$ of a meromorphic bundle with poles along $\{0\}\times X$.

Theorem E.8: [Sab90]: 2.3.1

There exists

- an integer $q \geq 1$

such that,

- after the ramification $x = t^q$,

for all

- $\theta \in S^1$ and
- each sufficiently small interval V centered at θ

one has

$$\mathcal{A}_L(V) \otimes_L \mathcal{M}_L \cong \mathcal{A}_L(V) \otimes_L \left(\mathcal{F}_L^R \otimes \mathcal{G}_L\right)$$

Theorem E.8: [Sab07]: II.5.12

Let us assume that there exists

- a good model $\mathcal{M}^{good}$[a] and
- an isomorphism $\widehat{\lambda} : \widehat{\mathcal{M}} \xrightarrow{\sim} \widehat{\mathcal{M}}^{good}$.

Then,

- for any $e^{i\theta^o} \in S^1$,

there exists an isomorphism

$$\widetilde{\lambda}_{\theta^o} : \widetilde{\mathcal{M}}_{\theta^o} \xrightarrow{\sim} \widetilde{\mathcal{M}}^{good}_{\theta^o},$$

lifting $\widehat{\lambda}$.

[a]such that no ramification is needed

That is, such that the diagram

$$\begin{tikzcd}
\widetilde{\mathcal{M}}_{\theta^o} \arrow[r, "\widetilde{\lambda}_{\theta^o}"] \arrow[d] & \widetilde{\mathcal{M}}^{good}_{\theta^o} \arrow[d] \\
\widehat{\mathcal{M}} \arrow[r, "\widehat{\lambda}"] & \widehat{\mathcal{M}}^{good}
\end{tikzcd}$$

commutes.

E.3.1 More from [Sab90]

For all $\theta \in S^1$ and each **sufficiently small interval** U centered at θ:

Corollary E.9: 2.3.2

Let $\mathcal{M}_K$ be a meromorphic connection. The formal decomposition into one slope terms $\mathcal{M}_{\widehat{K}} = \bigoplus \mathcal{M}_{\widehat{K}}^{L_i}$ can be liftet into a decomposition

$$\mathcal{A}(U) \otimes_K \mathcal{M}_K \cong \bigoplus \mathcal{M}^{L_i}_{\mathcal{A}(U)}$$

Corollary E.11: 2.3.3

Let

- $\mathcal{M}_K$ and $\mathcal{M}'_K$ be two meromorphic connections and
- $\widehat{\varphi} : \mathcal{M}_{\widehat{K}} \to \mathcal{M}'_{\widehat{K}}$ a left $\mathcal{D}_{\widehat{K}}$-linear (iso-)morphism.

Then $\widehat{\varphi}$ can be liftet into a $\mathcal{D}_{\mathcal{A}(U)}$-linear (iso-)morphism

$$\varphi_U : \mathcal{A}(U) \otimes_K \mathcal{M}_K \to \mathcal{A}(U) \otimes_K \mathcal{M}'_K$$

Lemma E.11: 2.3.4

Let $\mathcal{N}_K$ be a meromorphic connection. The natural[a] mapping

$$\ker\left[\partial_x : \mathcal{N}_{\mathcal{A}(U)} \to \mathcal{N}_{\mathcal{A}(U)}\right] \to \ker\left[\partial_x : \mathcal{N}_{\widehat{K}} \to \mathcal{N}_{\widehat{K}}\right]$$

is onto.

[a]Induced by the Borel-Ritt lemma?

of 2.3.1. Let $\mathcal{M}_K$ be a meromorphic connection. Choose first a ramified covering $t \mapsto x = t^q$ in order to apply theorem I-5.4.7:

$$\mathcal{M}_{\widehat{L}} \cong \bigoplus \mathcal{F}^R_{\widehat{L}} \otimes \mathcal{G}_{\widehat{L}}$$

Theorem E.13: I-5.4.7

Let $\mathcal{M}_{\widehat{K}}$ be a formal meromorphic connection. There exists an integer q sucht that the connection $\pi^*\mathcal{M}_{\widehat{K}} = \mathcal{M}_{\widehat{L}}$ is isomorphic to a direct sum of elementary formal meromorphic connections.

Definition E.13: 5.4.4 + 5.4.5

Let $R(z) = \sum_{i=1}^k a_i z^i \in \mathbb{C}[z]_k$.

- We shall denote by $\mathcal{F}^R_{\widehat{K}}$ the following meromorphic connection:
  - The $\widehat{K}$-vector space is isomorphic to $\widehat{K}$ with a basis denoted by $e(R)$.
  - The action of $x\partial_x$ is defined by

$$x\partial_x(\varphi \cdot e(R)) = \left[x\frac{\partial\varphi}{\partial x} + \varphi x \frac{\partial R(x^{-1})}{\partial x}\right] \cdot e(R)$$

- An *elementary meromorphic connection* (over $\widehat{K}$) is a connection isomorphic to $\mathcal{F}^R_{\widehat{K}} \otimes_{\widehat{K}} \mathcal{G}_{\widehat{K}}$ where
  - $\mathcal{G}_{\widehat{K}}$ is an elementary regular meromorphic connection.

Put $\mathcal{M}_{\widehat{L}} = \mathcal{M}'_{\widehat{L}} \oplus \mathcal{M}''_{\widehat{L}}$ where

- $\mathcal{M}'_{\widehat{K}}$ is the sum of terms $\mathcal{F}^R_{\widehat{L}} \otimes \mathcal{G}_{\widehat{L}}$ for which
  - R has maximal degree k and
  - with fixed dominating coefficient $\alpha \in \mathbb{C}$
- $\mathcal{M}''_{\widehat{K}}$ the sum of the other terms.

The proof will be done **by induction**, by showing that this splitting can be liftet to $\mathcal{A}_L(V)$ when V is sufficiently small. However the terms $\mathcal{M}'_{\mathcal{A}_L(V)}$ and $\mathcal{M}''_{\mathcal{A}_L(V)}$ do not come necessarily from modules defined over L.[a]

Lemma E.14: 2.4.1

- Let $\mathcal{M}_{\mathcal{A}_L(V)}$ be a free $\mathcal{A}_L(V)$-module equipped with a connection and
- let $\mathcal{M}_{\widehat{L}}$ be its formalized module.

A splitting of $\mathcal{M}_{\widehat{L}}$ as above can be lifted to a splitting of $\mathcal{M}_{\mathcal{A}_L(V)}$ when V is sufficiently small.

[a]That is why the proof has to be done for modules defined over $\mathcal{A}_L(V)$.

For the proof of (2.4.1):
The first step is the case where $\mathcal{M}_{\widehat{L}}$ is regular. One must prove a result analogous to theorem I-5.2.2[b]:

Lemma E.16: 2.4.2

Let $\mathcal{M}_{\mathcal{A}_L(V)}$ be a free $\mathcal{A}_L(V)$-module equipped with a connection such that $\mathcal{M}_{\widehat{L}}$ is regular. There exists a $\mathcal{A}_L(V)$-basis $\mathbf{m}$ of $\mathcal{M}_{\mathcal{A}_L(V)}$ such that the matrix of $t\partial_t$ in this basis is constant.

[b]

Lemma E.15: I-5.2.2

Let $\mathcal{M}_{\widehat{K}}$ be regular formal meromorphic connection. Then there exist a basis for which the matrix $x\partial_x$ is constant.

We want to use:

- matrix of $t\partial_t$ is constant $\Rightarrow$ there exists a basis $\mathbf{m}$ such that the corresponding basis $\widehat{\mathbf{m}}$ is compatible with the splitting of $\mathcal{M}_{\widehat{L}}$?

□

F Stokes Structures

F.1 Dictionary

[Sab90]	[Sab07]	[Boa01]	[Boa99]	[Lod94]	[Lod14]
		$\mathbb{A}$ Sto_α	$\mathbb{A}$ Sto_α	A_0 $\mathbb{A}$, $\mathbb{A}^\star$ Sto_α $\Lambda(A_0)$ $[^F A]$	$\mathfrak{A}$

$\star \in \{^k, {}^{<k}, {}^{\leq k}\}$

F.2 Sheaf view: [Sab07]

Let

- X be the parameter space
  - an analytic manifold equipped with coordinates $x_1, \ldots, x_n$.
- $\mathscr{M}^{good}$ be a meromorphic bundle
  - on $D \times X$
  - with poles along $\{0\} \times X$

  equipped with **flat** connection

  $$\nabla^{good} : \mathscr{M} \to \Omega^1_{D\times X} \otimes \mathscr{M}$$

We will assume that $\left(\mathscr{M}^{good}, \nabla^{good}\right)$ is

- a good model
  - in the neighbourhood of any point $x^0 \in X$,

  that is that, for every x^0,
  - there exist pairwise distinct germs $\varphi_1, \ldots, \varphi_p \in t^{-1}\mathcal{O}_{X,x^0}[t^{-1}]$ and
  - nonzero germs of systems with regular singularity $\mathcal{R}_{\varphi_1}, \ldots, \mathcal{R}_{\varphi_p}$ along $\{0\} \times X$ such that we have, in the neighbourhood of x^0,

* an isomorphism $\mathscr{M}^{good}_{x^0} \cong \bigoplus_k (\mathcal{E}^{\varphi_k} \otimes \mathcal{R}_{\varphi_k})$.

. TODO: wo/wie kommt hier die Umgebung rein? (p110)

$k \neq l \overset{\text{II.5.6}}{\Rightarrow}$ the order of the pole with respect to t of $(\varphi_k - \varphi_l)(x,t)$ does not depend on x in a neighbourhood of x^0.

F.2.1 The sheaf $\mathrm{Aut}^{<X}(\mathscr{M}^{good})$

Definition F.1

Define $\mathrm{Aut}^{<\{0\}\times X}(\widetilde{\mathscr{M}}^{good})$ or $\mathrm{Aut}^{<X}(\widetilde{\mathscr{M}}^{good})$ as the sheaf

- on $S^1 \otimes X$
- of automorphisms of

  $$\widetilde{\mathscr{M}}^{good} := \mathscr{A}_{\widetilde{D}\times X} \otimes_{\mathcal{O}_{D\times X}} \mathscr{M}^{good}$$

  witch are

  - compatible with the connection and
  - are formally equal to the identity[a].

The sections of $\mathrm{Aut}^{<X}(\widetilde{\mathscr{M}}^{good})$ are called *Stokes matrices*.

[a] i.e., which induce the identity on $\widehat{\mathscr{M}}^{good} := \widehat{\mathcal{O}}_{D\times X} \otimes_{\mathcal{O}_{D\times X}} \mathscr{M}^{good}$

The germs φ_k of $\mathrm{Aut}^{<X}(\widetilde{\mathscr{M}}^{good})$ are polynomials in t^{-1} without constant term, with coefficients in $\mathcal{O}_{X,x^0}$.

An automorphism a of $\widetilde{\mathscr{M}}^{good}$

- on $I \times U$ where
  - I is an interval and
  - U is a sufficiently small neighbourhood of x^0

can be decomposed into blocks

$$a_{kl} : \widetilde{\mathscr{E}}^{\varphi_k} \otimes \widetilde{\mathscr{R}}_k \to \widetilde{\mathscr{E}}^{\varphi_l} \otimes \widetilde{\mathscr{R}}_l$$

The term thus takes the form $e^{\varphi_k - \varphi_l} b_{kl}$, where

- b_{kl} is a homomorphism of bundles with connection $\widetilde{\mathscr{R}}_k \to \widetilde{\mathscr{R}}_l$.

Assume that X is a ball.

Definition F.3

Define

- V_k to be the $\mathbb{C}$-vector space consisting of

- **multivalued horizontal sections** TODO: Def?
  * of $\widetilde{\mathscr{R}}_k$
  * on $D^* \times X$

Remark F.3

- V_k is finite dimensional and
- on an open set like $]0, r[\times I \times X$,
  - with $I \neq S^1$,

the **space of horizontal sections of** $\widetilde{\mathscr{R}}_{\mathbf{k}}$ can be identified[a] to $\mathbf{V}_{\mathbf{k}}$.

[a] This identification depending on the choice of a determination of the logarithm on I.

Use the identification and b_{kl} to induce a $\mathbb{C}$-linear map

$$b_{kl} : V_k \to V_l \,.$$

Remark F.4

- The matrix of b_{kl}
  - with respect to the bases of V_k, V_l

  is thus **constant**.
- The matrix of b_{kl}
  - with respect to the $\mathcal{O}_{D\times X}[t^{-1}]$-bases of $\mathscr{R}_k$, $\mathscr{R}_l$

  has **moderate growth**.

For $k \neq l$, write

$$(\varphi_k - \varphi_l)(t, x) = \frac{\psi_{kl}(x)}{t^{n_{kl}}} \cdot U_{kl}(t, x)$$

where

- $n_{kl} > 0$,
- $\psi_{kl}(x) \in \mathcal{O}^*_{X,x^0}$
  - i.e., does not vanish

  and
- $u(t, x)$
  - is holomorphic in the neighbourhood of $(0, x^0)$ and
  - $u(0, 0) = 1$.

We also choose some C^∞ determination $\eta_{kl}(x)$ of the argument of $\psi_{kl}(x)$:

$$\psi_{kl}(x) = |\psi_{kl}(x)| \cdot e^{i\eta_{kl}(x)}$$

Remark F.5

$\psi_{kl}(x) = -\psi_{lk}(x)$

Lemma F.6

Let us pick $e^{i\theta^0} \in S^1$

1. For $k \neq l$, the matrix $e^{\varphi_k - \varphi_l} b_{kl}$
   - in $\mathcal{O}_{D\times X}[t^{-1}]$-bases of $\mathscr{E}^{\varphi_k} \otimes \mathscr{R}_k$, $\mathscr{E}^{\varphi_l} \otimes \mathscr{R}_l$

   has enties in
   - $\mathscr{A}^{<X}_{S^1\times X}$
     - in the neighbourhood of $(e^{i\theta^0}, x^0)$

   if and only if $b_{kl} = 0$ or $\cos(n_{kl}\theta^0 - \eta_{kl}) < 0$.
2. The matrix $b_{kk} - \mathrm{Id}$ has entries in
   - $\mathscr{A}^{<X}_{S^1\times X}$
     - in the neighbourhood of $(e^{i\theta^0}, x^0)$

   if and only if $b_{kk} - \mathrm{Id} = 0$.

Lemma F.6

Proofsketch. For the first point, it is enough to check that $e^{\varphi_k - \varphi_l}$ belongs to $\mathscr{A}^{<X}_{S^1\times X}$ in the neighbourhood of $(e^{i\theta^0}, x^0)$ if and only if $\Re(\varphi_k - \varphi_l) < 0$ on a sufficiently small neighbourhood of $(e^{i\theta^0}, x^0)$, then to express this condition on the leading term of $\varphi_k - \varphi_l$. Then, one notices that, as b_{kl} has moderate growth, it does not affect the rapid decay or the exponential growth property. $\square$

Let

- $V = \bigoplus_k V_k$ be the $\mathbb{C}$-vector space of multivalued horizontal sections of $\bigoplus_k \mathscr{R}_k$ on $D^* \times X$.
- $\mathscr{L}$ the subsheaf of the constant sheaf $\mathrm{Aut}(V)$
  - of which the germ at $(e^{i\theta^0}, x^0)$ is the space of automorphism $\mathrm{Id} + (\oplus c_{kl})$ with
    - $c_{kk} = 0$ and
    - $c_{kl} : V_k \to V_l$ is nonzero if and only if $\cos(n_{kl}\theta^0 - \eta_{kl}) < 0$.

Remark F.7

Any Stokes matrix (local section of $\mathscr{L}$) is

- unipotent
  - if we suitably order the set of indices, we can assume that $c_{kl} = 0$ for $k < l$ and the matrix of $\oplus c_{kl}$ is strictly uppertriangular.

Corollary F.8: The Stokes martices are constant

For

- any open interval $I \neq S^1$,

the restriction to $I \times X$ of the sheaves $\mathrm{Aut}^{<\mathbf{X}}(\widetilde{\mathscr{M}}^{\mathbf{good}})$ and $\mathscr{L}$ are isomorphic:

$$\mathrm{Aut}^{<X}(\widetilde{\mathscr{M}}^{good})_{|I\times X} \cong \mathscr{L}_{|I\times X}$$

Corollary F.9: Base change for horizontal sections

Let

- $i : S^1 \times \{x^0\} \hookrightarrow S^1 \times X$ denote the inclusion

then the natural morphism

$$i^{-1}\,\mathrm{Aut}^{<X}(\widetilde{\mathscr{M}}^{good}) \to \mathrm{Aut}^{<0}(\widetilde{i^{+}\mathscr{M}}^{good})$$

is an isomorphism.

Definition F.11

The *manifold of Stokes directions*

- in $S^1 \times U$, where
  - U is a neighbourhood of x^0 on which the φ_k are defined and satisfy the goodness properties

is the union of the sets having the equation

$$\cos(n_{kl}\theta - \eta_{kl}(x)) = 0\,.$$

The restriction of this set to $S^1 \times \{x^0\}$ is a finite set of points, called the *Stokes directions at* x^0.

Remark F.11

A pair (k,l) contributes to the directions

$$\frac{\left(\eta_{kl}+\frac{\pi}{2}+j\pi\right)}{n_{kl}}$$

for $j=0,\dots,2n_{kl}-1$.

Aufgabe F.1 (Straightening the Stokes manifold ([Sab07] 6.8)).

F.2.2 The Stokes sheaf $\mathcal{S}t_X(\mathscr{M}^{good})$

Definition F.12

Define the *stokes sheaf* $\mathcal{S}t_X(\mathscr{M}^{good})$ as the sheaf (on X) associated to the presheaf

$$U \mapsto H^1\left(S^1 \times U, \mathrm{Aut}^{<X}\left(\widetilde{\mathscr{M}}^{good}\right)\right)$$

Theorem F.13: The stokes sheaf is locally constant (6.1)

The Stokes sheaf $\mathcal{S}t_X(\mathscr{M}^{good})$ is

- a locally constant

sheaf of pointed sets.

Theorem F.13: The stokes sheaf is locally constant (6.1)

Proof. [Sab07, p. 113]: paragraph 6.c. □

Proposition F.14: Base change for the Stokes sheaf

The germ at x^0 of the Stokes sheaf $\mathcal{S}t_X(\mathscr{M}^{good})$

$$\mathcal{S}t_X(\mathscr{M}^{good})_{x^0}$$

is equal to

the Stokes space[a] $H^1(S^1, \mathrm{Aut}^{<0}(\widetilde{i^+\mathscr{M}}^{good}))$ of the connection $i^+\mathscr{M}^{good}$.

[a] Is equipped with a natural structure of affine space.

F.2.3 The sheaf $\mathscr{H}_X$

Definition F.17

Let

- $(\mathscr{M}^{good}, \nabla^{good})$ be a fixed good formal model with
  - flat connection,
  - having poles along $\{0\} \times X$.

Definition F.16

$(\mathscr{M}^{good}, \nabla^{good})$ is a *good formal model* for $(\mathscr{M}, \nabla)$ if

- there exists an isomorphism

$$\widehat{f} : \underbrace{(\widehat{\mathscr{M}, \nabla})}_{\widehat{\mathcal{O}}_{D\times X}\otimes(\mathscr{M},\nabla)} \xrightarrow{\sim} \underbrace{(\widehat{\mathscr{M}^{good}}, \widehat{\nabla^{good}})}_{\widehat{\mathcal{O}}_{D\times X}\otimes(\mathscr{M}^{good},\nabla^{good})}$$

  - of sheafes of $\widehat{\mathcal{O}}_{D\times X}$-modules
  - compatible with connection.

Definition F.17

Two germs $(\mathscr{M}, \nabla, \widehat{f})$ and $(\mathscr{M}', \nabla', \widehat{f}')$ along $\{0\} \times X$ are *isomorphic*, if

- there exists an isomorphism[a] $g : (\mathscr{M}, \nabla) \to (\mathscr{M}', \nabla')$ with
  - $\widehat{f} = \widehat{f}' \circ \widehat{g}$.

[a] such an isomorphism then is unique

Define

- $\mathscr{H}_X$ be the (pre-) sheaf[b] where
  - $\mathscr{H}_X(U)$ is the (pointed) set
    - of **isomorphism classes** of germs[c] $(\mathscr{M}, \nabla, \widehat{f})$ defined on U,
      - Why germs??
      - along $\{0\} \times U$??
    - equipped with a distinguished element, namely, the class of $(\mathscr{M}^{good}, \nabla^{good}, \widehat{\mathrm{Id}})_{|U}$.

[b] Lemma 6.2: it is a sheaf.
[c] along $\{0\} \times U$

F.2.4 Main Result

Use Thoreme E.8 to construct[1] a homomorphism of sheaves of pointed sets

$$\mathscr{H}_X \to \mathcal{S}t_X(\mathscr{M}^{good})$$

- If
  - $(\mathscr{M}, \nabla, \widehat{f})$ is defined on U

  there exists
  - an open covering $\mathfrak{W}$ of $S^1 \times U$ and
  - for any W_i in $\mathfrak{W}$
    - an isomorphism $f_i : (\widetilde{\mathscr{M}}, \widetilde{\nabla})_{W_i} \xrightarrow{\sim} (\widetilde{\mathscr{M}}^{good}, \widetilde{\nabla}^{good})_{W_i}$

  such that $\widehat{f}_i = \widehat{f}$.

  Then $(f_j f_i^{-1})_{i,j}$ is a cocycle of the sheaf $\mathrm{Aut}^{<X}(\widetilde{\mathscr{M}}^{good})$ relative to the covering $\mathfrak{W}$.
  - If f_i' is another lifting of $\widehat{f}$...
  - If $(\mathscr{M}, \nabla, \widehat{f})$ and $(\mathscr{M}', \nabla', \widehat{f}')$ are isomorphic...

  We have thus defined a mapping of pointed sets

  $$\Gamma(U, \mathscr{H}_X) \to H^1(S^1 \times U, \mathrm{Aut}^{<X}(\widetilde{\mathscr{M}}^{good}))$$

  from which we deduce the homomorphism of sheaves.

Theorem F.18: The Stokes sheaf classifies meromorphic connections with fixed normal type (6.3)

The homomorphism so defined

$$\mathscr{H}_X \to St_X(\mathscr{M}^{good})$$

is an isomorphism of sheaves of pointed sets.

Corollary F.19: Analytic extension with fixed formal structure

Let

- X be 1-connected and
- $(\mathscr{M}^0, \nabla^0, \widehat{f}^0)$ be a germ of meromorphic bundle
  - on D
  - with pole at 0

[1]see: [Sab07, p. 111f]

- equipped with a formal isomorphism

$$\widehat{f}^0 : (\mathscr{M}^0, \nabla^0) \xrightarrow{\sim} i^+(\mathscr{M}^{good}, \nabla^{good})$$

where

* i denotes the inclusion $D \times \{x^0\} \hookrightarrow D \times X$

then there exists

- a meromorphic bundle with connection $(\mathscr{M}, \nabla, \widehat{f})$
  - equipped with a formal isomorphism $\widehat{f}$ to $(\mathscr{M}^{good}, \nabla^{good})$

such that

- $i^+(\mathscr{M}, \nabla, \widehat{f})$ is isomorphic to $(\mathscr{M}^0, \nabla^0, \widehat{f}^0)$.

Such an object is unique up to unique isomorphism.

F.3 Matrix view: [Boa01] and [Boa99]

Let

- z be a coordinate on $\mathbb{D}$ vanishing at 0 and
- $d - A^0$ be a diagonal meromorphic connection
  - on the trivial rank n vector bundle
    * over the unit disc $\mathbb{D} \subset \mathbb{C}$
  - with a pole of order $k \geq 2$ at 0
    * and no other poles.

  thus $A^0 = dQ + \Lambda^0 \frac{dz}{z}$ where
  - Λ^0 is a constant diagonal matrix and
  - Q is a diagonal matrix of meromorphic functions.
    * write $Q = \mathrm{diag}(g_1, \ldots, q_n)$

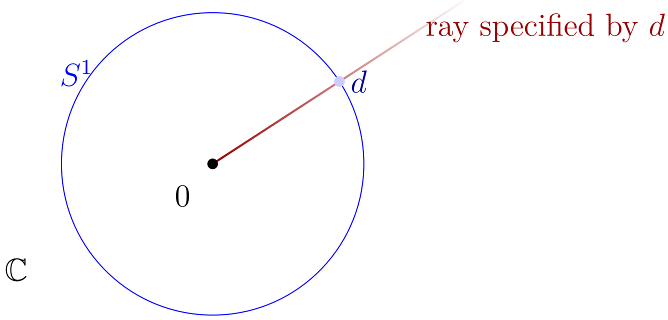
F.3.1 Basic definitions

Define $q_{ij}(z)$ to be the leading term of $q_i - q_j$.

- Thus if $q_i - q_j = a/z^{k-1} + b/z^{k-2} + \ldots$ then $q_{ij} = a/z^{k-1}$.

Definition F.20: Definition 3.2

The *anti-Stokes* directions $\mathbb{A} \subset S^1$ are the directions $d \in S^1$ such that for some $i \neq j$: $q_{ij}(z) \in \mathbb{R}_{<0}$ for z on the ray specified by d.



Remark F.21

- We have $\frac{\pi}{k-1}$ rotational symmetry
  - if $q_{ij}(z) \in \mathbb{R}_{<0}$ then $q_{ij}(z \exp(\frac{\pi\sqrt{-1}}{k-1})) \in \mathbb{R}_{<0}$
- If $q_{ij}(z) \in \mathbb{R}_{<0}$ then $q_{ji}(z) \in \mathbb{R}_{>0}$
  - in any arc $U \subset S^1$ subtending angle $\frac{\pi}{k-1}$ there are at most $\frac{n(n-1)}{2}$ anti-Stokes directions.

Definition F.23

- Define the *set of all anti stokes directions* ● $\in \mathbb{A} \subset S^1$,
  - **Labeling convention:** Choose a point[a] $p \in \mathbb{D}$. Label the first anti-Stokes ray when turning in a positive sense from p as d_1 and label the subsequent rays $d_2, \ldots, d_r$ in turn.

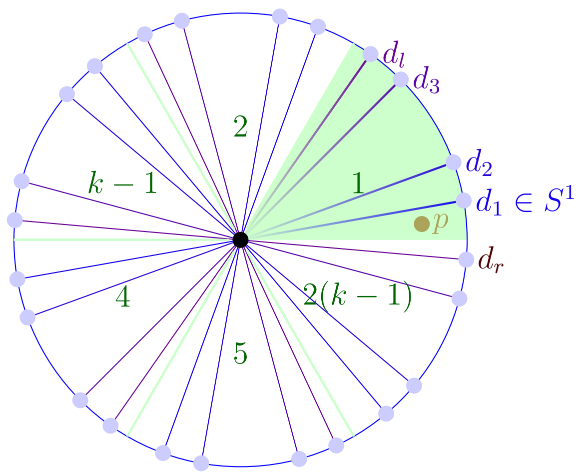
    Remark F.23

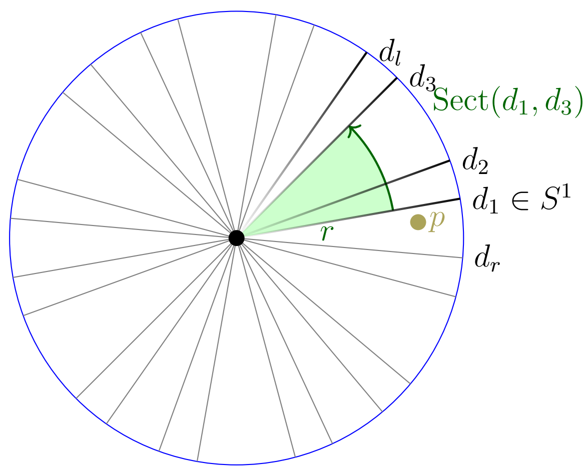
    The choice of labeling together with the choice of $\log(z)$ is given by a lift $\widetilde{p}$ of p to the universal cover of D^*.

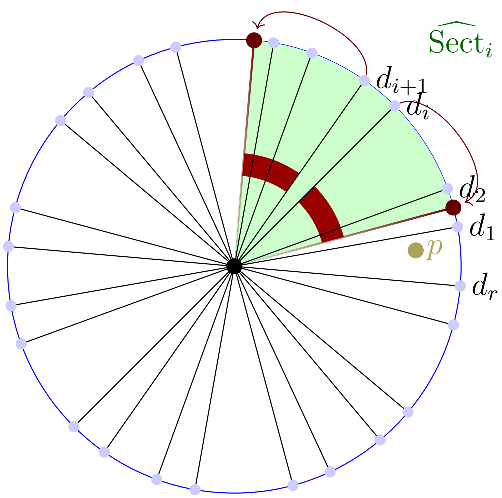
    [a] In [Boa99] a sector is chosen, which does not contain any anti Stokes direction

-
- $r := \#\mathbb{A}$
  - r is divisible by $2(k-1)$
    - define $l := \frac{r}{2k-2}$
- $\mathbf{d} := (d_i, d_{i+1}, d_{i+2}, \ldots, d_{i+l}) \subset \mathbb{A}$ is a *half-period*[b]
- Define the open sector $\mathrm{Sect}(d_i, d_j)$
  - The radius r will be taken sufficiently small when required later.

- Write $\mathrm{Sect}_i := \mathrm{Sect}(d_i, d_{i+1})$ the 'ith sector'.
  - Note $p \in \mathrm{Sect}_r = \mathrm{Sect}_0$
- $\widehat{\mathrm{Sect}}_i := \mathrm{Sect}(d_i - \frac{\pi}{2k-2}, d_{1+1} + \frac{\pi}{2k-2})$ the 'ith supersector'
  - The rays bounding these supersectors are usually refered as 'Stokes rays'.







[b] A l-tuple of **consecutive** anti-Stokes directions

Definition F.25

- $\mathrm{Roots}(d) := \{(ij) \mid g_{ij}(z) \in \mathbb{R}_{<0}$ along $d\}$
- The *multiplicity* $\mathrm{Mult}(d)$ of d is the number of roots supporting d ($\#\,\mathrm{Roots}(d)$).

> **Remark F.25**
>
> $n(n-1)/2 = \mathrm{Mult}(d_1) + \cdots + \mathrm{Mult}(d_l)$

- The group of *Stokes factors* associated to d is the group

$$\mathbb{S}\mathrm{to}_d(A^0) := \{K \in G \mid (K)_{ij} = \delta_{ij} \text{ unless } (ij) \text{ is a root of } d\}.$$

  - $i = j \Rightarrow (ij)$ is not a root of d. There are 1nes on the diagonal.
  - is a unipotent subgroup of $G = GL_n(\mathbb{C})$
    * of dimension equal to $\mathrm{Mult}(d)$

F.3.2 Main theorem: for Stokes factors

Theorem F.26: 1.22 in [Boa99]

- There is a natural isomorphism

$$\mathcal{H}(A^0) \cong \prod_{d \in \mathbb{A}} \mathrm{Sto}_d(A^0)$$

  and

- for each choice of $\log(z)$ in the direction d
  - the Stokes group $\mathrm{Sto}_d(A^0)$ has

    * a faithfull representation ρ on $\mathbb{C}^n$

  inducing an isomorphism

$$\rho : \mathrm{Sto}_d(A^0) \to \mathbb{S}\mathrm{to}_d(A^0).$$

In particular each $\mathrm{Sto}_d(A^0)$ and therefore $\mathcal{H}(A^0)$ is a unipotent Lie group and the complex dimension of $\mathcal{H}(A^0)$ is $(k-1)n(n-1)$ where

- k is the order of the pole of A^0 and
- n is the rank.

Theorem F.29: 3.1, or Prop 1.24 in [Boa99]

Let

- $\widehat{F} \in G[\![z]\!]$
  - such that $A := \widehat{F}[A_0]$ has convergent entries.

Set the radius of the sectors Sect_i, $\widehat{\mathrm{Sect}}_i$ to be less than the radius of convergence of A. Then:

1. On each sector Sect_i there is a **canonical** way to choose
   - $\Sigma_i(\widehat{F})$
     - an **invertible** $n \times n$ matrix of holomorphic functions

   such that $\Sigma_i(\widehat{F})[A^0] = A$.
2. $\Sigma_i(\widehat{F})$ can be analytically continued to the supersector $\widehat{\mathrm{Sect}}_i$ and then

   $$\Sigma_i(\widehat{F}) \qquad \text{is } \textbf{asymptotic} \text{ to} \qquad \widehat{F}$$

   - at 0
   - within $\widehat{\mathrm{Sect}}_i$.
3. If $g \in G\{z\}$ and $t \in T$ then $\Sigma_i(g \circ \widehat{F} \circ t^{-1}) = g \circ \Sigma_i(\widehat{F}) \circ t^{-1}$.

Proposition F.28: Prop 1.24 in [Boa99]

If $\widehat{H} \in \widehat{G}$ is the taylor series at 0 of an analytic gauge transformation $H \in G\{z\}$ then

$$\Sigma_i(\widehat{H}\widehat{F}) = H\Sigma_i(\widehat{F}) \qquad \Sigma_i(\widehat{F}\widehat{H}) = \Sigma_i(\widehat{F})H$$

Remark F.29

- The point is that on a narrow sector there are generally many holomorphic isomorphisms between A_0 and A which are asymptotic to $\widehat{F}$ and one is being chosen in a canonical way
- $\Sigma_i(\widehat{F})$ is in fact uniquely characterised by property 2.

Now define the map $\mathcal{H}(A^0) \to \prod_{d\in\mathbb{A}} \mathrm{Sto}_d(A^0)$: choose

- $\widehat{F} \in \widehat{G}(A^0)$
  - representing an element $\mathcal{H}(A^0) \cong G\{z\}\backslash\widehat{G}(A^0)$

  and
- an anti-Stokes direction $d \in \mathbb{A}$.

The sums of $\widehat{F}$ on the two sectors adjacent to d may be analytically continued across d, and they will generally be different on the overlap. Thus to each anti-Stokes direction $d = d_i$ there is

- an associated automorphism

$$\kappa_{d_i} := (\Sigma_i(\widehat{F}))^{-1}\Sigma_{i-1}(\widehat{F})$$

  - describing how the sums of $\widehat{F}$ differ on both sides of d_i
  - it is a solution of $\mathrm{Hom}(A^0, A^0)$ asymptotic to 1 on a sectorial neighbourhood of d_i
  - The third part of the Proposition implies that the automorphism κ_d only depends on the $G\{z\}$ orbit of $\widehat{F}$.

Definition F.30: [Boa99] 1.25

The *Stokes group* $\mathrm{Sto}_d(A^0)$ of such automorphisms that arise as we vary the choice of $\widehat{F}$:

$$\mathrm{Sto}_d(A^0) := \left\{\kappa_d = (\Sigma_i(\widehat{F}))^{-1}\Sigma_{i-1}(\widehat{F}) \mid \widehat{F} \in \widehat{G}(A^0)\right\}$$

Taking all such automorphisms gives a map

$$\widehat{G}(A^0) \to \prod_{d\in\mathbb{A}} \mathrm{Sto}_d(A^0); \widehat{F} \mapsto (\kappa_{d_1}, \ldots, \kappa_{d_r})$$

inducing a well defined map

$$\mathcal{H}(A^0) \to \prod_{d\in\mathbb{A}} \mathrm{Sto}_d(A^0) .$$

Using the Proposition:

$$\begin{aligned}\kappa_{d_i} &= (\Sigma_i(\widehat{F}))^{-1}\Sigma_{i-1}(\widehat{F})\\ &= (\Sigma_i(\widehat{F}))^{-1}g^{-1}g\Sigma_{i-1}(\widehat{F})\\ &= (g\Sigma_i(\widehat{F}))^{-1}g\Sigma_{i-1}(\widehat{F})\\ &= (\Sigma_i(g\widehat{F}))^{-1}\Sigma_{i-1}(g\widehat{F})\end{aligned}$$

The faithfull representation of $\mathrm{Sto}_d(A^0)$ arises since the solutions of $\mathrm{Hom}(A^0, A^0)$ are known explicitly:
Let

- d be a direction and
- a choice of $\log(z)$ in the direction d be given.

We get a genuine fundamental solution $z^\Lambda e^Q$ of A^0 there.

The map

$$\rho : \kappa \to K := e^{-Q}z^{-\Lambda}\kappa z^\Lambda e^Q$$

relates

- solutions κ of $\mathrm{Hom}(A^0, A^0)$

to

- a konstant matrix $K \in \mathrm{End}(E)$
  - that is, to solutions of $\mathrm{Hom}(0, 0)$.

Lemma F.31

The Stokes factor K_i is in the group $\mathbb{S}\mathrm{to}_{d_i}(A^0)$.

Definition F.31: [Boa01]: def 3.5

Fix data $(A^0, z, \widetilde{\rho})$[a]. The

- *Stokes factors*
  - of a compatibly framed connection $(A, g^0) \in \widehat{\mathrm{Syst}}(A^0)$

are

$$K_i := e^{-Q}z^{-\Lambda^0}\kappa z^{\Lambda^0}e^Q$$

using

- the choice of $\log(z)$ along d_i, where $\kappa_i := \Sigma_i(\widehat{F})^{-1} \circ \Sigma_{i-1}(\widehat{F})$.

[a] as above.

Since

- $z^{\Lambda^0}e^Q$ is a fundamental solution of A^0

we have

- $d(K_i) = 0$
  - The Stokes factors are constant invertible matrices.

A usefull equivalent definition is:...

Definition F.32: 1.27

Given $\widetilde{p}$[a]. The *Stokes factors*

- of a compatibly framed system (A, g)
- with associated normal Form A^0

are

$$\mathbf{K} := (K_1, \dots, K_r) \in \prod_{d \in \mathbb{A}} \mathbb{S}\mathrm{to}_d$$

- r-tuple of matrices
- $K_i = \rho(\kappa_i) = e^{-Q} z^{-\Lambda} \kappa z^{\Lambda} e^{Q}$
  - using the choice of $\log(z)$ along d.

[a] a choice of labeling and $\log(z)$

A usefull way of thinking about these

Fix

- a choice of labelling and
- a choice of $\log(z)$.

Definition F.34

If

- (A, g) is a compatibly framed system
  - with associated formal normal form A^0

then the *cononical fundamental solution of* A on Sect_i is

$$\Phi_i := \Sigma_i(\widehat{F}) z^{\Lambda} e^{Q}$$

where

- z^{Λ} uses the given choice of $\log(z)$ on Sect_i.

Remark F.34: ???

- Φ_i is a solution of $\mathrm{Hom}(0, A)$
- Φ_i is asymptotic to $\widehat{F} z^{\Lambda} e^{Q}$
  - at 0
  - within $\widehat{\mathrm{Sect}}_i$

when continued without any winding.

Then we get:

Lemma F.35

If Φ_i is continued across the anti-Stokes ray d_{i+1} then on Sect_{i+1}:

$$\Phi_i = \Phi_{i+1} K_{i+1} \qquad \text{for } i = 1, \dots, r-1 \text{ and}$$

$$\Phi_r = \Phi_1 K_1 M_0$$

where

- $M_0 = e^{2\pi\sqrt{-1}\Lambda}$ is the formal monodromy.

Definition F.35: [Boa01]: def 3.6

If Φ_i is continued across the anti-Stokes ray d_{i+1} then on Sect_{i+1} we have:

$$K_{i+1} := \Phi_{i+1}^{-1} \circ \Phi_i$$

for all i except $K_1 := \Phi_{r+1}^{-1} \circ \Phi_r \circ M_0^{-1}$ where

- $M_0 = e^{2\pi\sqrt{-1}\Lambda}$ is the so-called 'formal monodromy'.

Lemma F.36

The Stokes factor K_i is in the group $\mathbb{S}\mathrm{to}_{d_i}(A^0)$.

Thus, in summary, a compatibly framed system (A, g) has canonical fundamental solutions Φ_i and the Stokes factors express how these differ and moreover they encode the moduli of (A, g).

F.3.3 Main theorem: for Stokes matrices

Lemma F.36: [Boa01]: 3.2

let $\mathbf{d}$ be a half-period.

1. **The product of the corresponding groups of Stokes factors** is isomorphic
   - as a variety,
   - via the product map,

   to the **P conjugate of U$_+$**^{a}:

   $$\prod_{d \in \mathbf{d}} \mathbb{S}\mathrm{to}_d(A^0) \cong P U_+ P^{-1};$$
   $$(K_1, \dots, K_l) \mapsto K_l \dots K_2 K_1 \in G$$

   where

   - $P \in G$ the permutation Matrix associated to $\mathbf{d}$ given by $(P)_{ij} = \delta_{\pi(i)j}$ where

The ordering on $\mathbf{d}+l$ is opposite to this ordering

– π is the permutation of $\{1,\dots,n\}$ corresponding to

$$\pi(i) < \pi(j) \qquad \Leftrightarrow \qquad q_i \underset{\mathbf{d}}{<} q_j$$

where

* the total ordering is defined as

$$q_i \underset{\mathbf{d}}{<} q_j \qquad :\Leftrightarrow \qquad (ij) \text{ is a root of some } d \in \mathbf{d}$$

See [Boa99, p. 13]: **Proof of Proposition 1.35**

[a]Is a subgroup of G.

Proposition F.38: [Boa99]: 1.35

let $\mathbf{d} = ((i-1)l+1,\dots,il)$ be a half-period. P is the (unique) permutation matrix, such that the multiplication map

$$\mathbb{S}\mathrm{to}_{il}(A^0) \times \cdots \times \mathbb{S}\mathrm{to}_{(i-1)l+1}(A^0) \to \mathrm{GL}_n(\mathbb{C})$$
$$(K_{il},\dots,K_{(i-1)l+1}) \mapsto P^{-1}K_{il}\cdots K_{(i-1)l+1}P$$

is

- a diffeomorphism onto $\begin{cases} U_+ & \text{if } i \text{ is odd} \\ U_- & \text{if } i \text{ is even} \end{cases}$

where

- $i = 1,\dots$ and
- all depends on the ordering $d_1,\dots,d_r$ of the anti-stokes directions.

Definition F.38

The *ith Stokes matrix*

- with respect to a fixed labelling and $\log(z)$ choice

is the following product of l Stokes factors

$$S_i := K_{il}\cdots K_{(i-1)l+1} \in PU_\pm P{-1} \subset \mathrm{GL}_n(\mathbb{C})$$

Theorem F.41: [Boa01]: Thm 3.2

Fix the data $(A^0, z, \widetilde{p})$ as above. Then

- the *'local monodromy map'*
  – taking the stokes matrices

Lemma F.39: [Boa01]: 3.2

2. The map

$$\prod_{d\in\mathbb{A}} \mathbb{S}\mathrm{to}_d(A^0) \cong (U_+ \times U_-)^{k-1};$$
$$(K_1,\dots,K_r) \mapsto (S_1,\dots,S_{2k-2})$$

induces a bijection

$$\mathcal{H}(A^0) \overset{\cong}{\to} (U_+ \times U_-)^{k-1}$$
$$[(A, g_0)] \mapsto (S_1, \ldots, S_{2k-2})$$

In particular

- $\mathcal{H}(A^0)$ is isomorphic to the vector space $\mathbb{C}^{(k-1)n(n-1)}$.

Corollary F.40

The torus action of $\mathcal{H}(A^0)$ changing the compatible framing corresponds to

- the conjugation action $t(\mathbf{S}) = (tS_1t^{-1}, \ldots, tS_{2k-2}t^{-1})$
  - on the Stokes matrices

and so, there is a bijection

$$\mathrm{Syst}(A^0)/G\{z\} \cong (U_+ \times U_-)^{k-1}/T$$

Corollary F.41

If Φ_0 is contiuned once around 0 in a positive sense, then on return to Sect_0 is will become

$$\Phi_0 \cdot PS_{2k-2} \cdots S_2 S_1 P^{-1} M_0$$

where

- $M_0 = e^{2\pi\sqrt{-1}\cdot\Lambda^0}$ is the formal monodromy.

where

- $S_i := P^{-1}K_{il} \ldots K_{(i-1)l+1}P \in U_{+/-}$ if i is odd / even

from the **product of all the groups of Stokes factors**, is

- an isomorphism of varieties.

Thus we can define the Stokes matrices of $(A, g_0) \in \widehat{\mathrm{Syst}}(A^0)$

$$S_i := P^{-1}K_{il} \ldots K_{(i-1)l+1}P \in U_{+/-} \qquad \text{if } i \text{ is odd / even}$$

where

- $i = 1, \ldots, 2k-2$ and
- P is the permutation matrix associated to the half-period $(d_1, \ldots, d_l)$.

Remark F.42: 1.41 from [Boa99] on pages 16f

Note that in most of the recent references we have used, Stokes matrices are used to classify

- meromorphic connections within fixed formal **meromorphic classes, modulo meromorphic equivalence**.

Whereas here we classify

- meromorphic connections within fixed **formal analytic classes, modulo analytic equivalence**,

as is done in the older literature. The fact is that the sets equivalence classes are the same in both cases. It is important for us to work with analytic, rather than meromorphic gauge transformations, because then the $\mathbb{C}^\infty$ viewpoint in Chapter 3 is cleaner. This distinction relates to the difference between **'regular singular'** connections and **'logarithmic'** connections.

G Aufgaben und Fragen

Diese Aufgaben haben das Ziel das Paper [Boa01] zu verstehen. Als weitere Quellen wurde vor allem [Boa99] verwendet.

vom 19.12.2013

Aufgabe G.1. *Formal sauber: Wähle Trivialisierungen, also*

- *Wähle lokale Koordinaten und*
- *erhalte daraus eine Trivialisierung*

Sei $V \to \Sigma = \mathbb{P}^1$ ein holomorphes Vektor Bündel. Sei $i \in \{1, \ldots, m\}$ fix. Hier wollen wir das wählen einer lokalen Trivialisierung beziehungsweise zunächst das wählen einer lokalen Koordinate auf $\mathbb{P}^1$ formal sauber machen, dazu verwenden wir die Proj-Konstruktion.

$$\begin{tikzcd}
 & & \mathbb{P}^1 & & \\
\left[\frac{\alpha+a_0}{a_1}:1\right] & D_+(T_0) \arrow[ur, hook] & & D_+(T_1) \arrow[ul, hook'] & \left[1:\frac{a_1}{\alpha+a_0}\right] \\
 & & \mathbb{A}^1_z \arrow[ul, dashed, "\cong"] \arrow[ur, dashed, "\cong"'] & & \\
 & & (z-\alpha) \arrow[uull, dashed, mapsto] \arrow[uurr, dashed, mapsto] & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & D_+(T_0) \cong \operatorname{Spec}\mathbb{C}[T_0,T_1]_{(T_0)} \arrow[ddr, hook, bend left] & & \\
 & & D_+(T_1) \cong \operatorname{Spec}\mathbb{C}[T_0,T_1]_{(T_1)} \arrow[dr, hook, bend left] & & \\
0 \arrow[r, phantom, "\in"] \arrow[rrrr, mapsto, bend right, "?"'] & \mathbb{A}^1_z \overset{\mathrm{def}}{=} \operatorname{Spec}\mathbb{C}[z] \arrow[uur, bend left, "\cong"] \arrow[ur, bend left, "\cong"] \arrow[dr, bend right, "\cong"'] & \cap \arrow[u, dashed] \arrow[d, dashed] & \mathbb{P}^1 \overset{\mathrm{def}}{=} \operatorname{Proj}\mathbb{C}[T_0,T_1] & a \arrow[l, phantom, "\ni"] \\
 & & D_+(f) \cong \operatorname{Spec}\mathbb{C}[T_1,T_2]_{(f)} \arrow[ur, hook, bend right] & &
\end{tikzcd}$$

Versuch 1

Wir wollen lokale Koordinaten, so dass $0 \in \mathbb{A}^1_z$ auf $a \in \mathbb{P}^1$ abgebildet wird. Sei $a = [a_0 : a_1]$ mit oBdA $a_1 \neq 0$.

$$\mathbb{A}^1_z \overset{\text{def}}{=} \operatorname{Spec} \mathbb{C}[z] \xrightarrow[\cong]{\tau_1} D_+(T_1) \cong \operatorname{Spec} \mathbb{C}[T_1, T_2]_{(T_1)} \longrightarrow \mathbb{P}^1 \overset{\text{def}}{=} \operatorname{Proj} \mathbb{C}[T_0, T_1]\,.$$

Versuch 2

Das Wählen einer lokalen Koordinate entspricht der Wahl eines homogenen $f = f_0T_0 + f_1T_1 \in \mathbb{C}[T_0, T_1]_+$ vom Grad 1 sowie eines Isomorphismus τ_1 und einer Inclusion τ_2 so dass

$$\mathbb{A}^1_z \overset{\text{def}}{=} \operatorname{Spec} \mathbb{C}[z] \xrightarrow[\cong]{\tau_1} D_+(f) \overset{(\star)}{\cong} \operatorname{Spec} \mathbb{C}[T_1, T_2]_{(f)} \xhookrightarrow{\tau_2} \mathbb{P}^1 \overset{\text{def}}{=} \operatorname{Proj} \mathbb{C}[T_0, T_1]\,.$$

Dies soll so gewählt werden, dass **die lokale Koordinate z auf $a = [a_0 : a_1] \in \mathbb{P}^1$ verschwindet,** also dass $\tau_2 \circ \tau_1(0) = a$ gilt.

Wähle f als $f = a_0T_0 + a_1T_1$. Dieses erfüllt $f(a) \neq 0$.

Um τ_1 zu finden, suche einen Isomorphismus $t_1 : \mathbb{C}[z] \to \mathbb{C}[T_0, T_1]_{(f)}$, so dass dieser nach anwenden von Spec den gesuchten Isomorphismus ergibt. Wir suchen also ein homogenes Element, auf welches wir z abbilden. Ein allgemeines solches Element ist gegeben durch $\frac{\alpha T_0 + \beta T_1}{f}$ mit $[\alpha : \beta] \in \mathbb{P}^1$.

Hier ist $\alpha = a_1$ und $\beta = -a_0$ möglich und damit wird a zu einer Nullstelle.

Wie sieht τ_2 aus?

BIS HIER HIN NEU!!

IDEE:

$$\begin{array}{ccccc} \mathbb{A}^1_z & \xrightarrow{\cong} & D_+\left(a_{i,0}T_0 + a_{i,1}T_1\right) & \xhookrightarrow{\iota} & \mathbb{P}^1 \\ 0 & \longmapsto & a_i & & \end{array}$$

Dazu:

$$\begin{aligned} D_+\left(a_{i,0}T_0 + a_{i,1}T_1\right) &= \operatorname{Proj} \mathbb{C}[T_0, T_1] \Big\backslash V_+\Big(\left(a_{i,0}T_0 + a_{i,1}T_1\right)\Big) \\ &\hookrightarrow \mathbb{P}^1 \qquad \text{durch} \qquad a \mapsto a \end{aligned}$$

Fall: $a = [1 : 0]$ dann ist $\mathbb{A}^1 \cong D_+(T_1) \hookrightarrow \mathbb{P}^1$ durch $x \mapsto (1, z)$

Weitere IDEE: Verwende eine standard Inclusion und verkette diese mit mit einer Transformation, welche z.B. $[1:0]$ auf a_i verschiebt.

$$\begin{tikzcd}
\mathbb{A}^1_z \arrow[r, "\cong"] & D_+(T_0) \arrow[r, "\cong"] & ??? \arrow[r, hook, "\iota"] & \mathbb{P}^1 \\
z \arrow[r, mapsto] & {[1:z]} \arrow[r, mapsto] & {[a_{i,0}, a_{i,1}+z]} & \\
 & {[1:0]} \arrow[r, mapsto] & a_i &
\end{tikzcd}$$

Wobei der zweite Isomorphismus durch eine invertierbare Matrix repräsentiert wird. Beispielsweise $\begin{pmatrix} a_{i,0} & 0 \\ a_{i,1} & 1 \end{pmatrix}$, diese schickt $[1:0]$ auf a_i.

Neuer Start:

Frage: Welche Automorphismen hat man auf $D_+(T_0)$ oder $\mathbb{A}^1_z$?
Antwort: $\boxed{\text{TODO}}$

Betrachte das Diagram:

$$\begin{tikzcd}
\mathbb{C}[T_0,T_1]_{(T_0)} \arrow[d, dashed, "?"] & \mathbb{C}[z] \arrow[l] \arrow[dl, dashed, "?"] \\
\mathbb{C}[T_0,T_1]_{(\alpha T_0+\beta T_1)} &
\end{tikzcd}
\overset{\text{Spec}}{\rightsquigarrow}
\begin{tikzcd}
D_+(T_0) \arrow[r] & \mathbb{A}^1_z \\
D_+(\alpha T_0+\beta T_1) \arrow[u, "\cong"] \arrow[ur] &
\end{tikzcd}$$

Weiteres Diagram:

$$\begin{tikzcd}
\operatorname{Spec}\mathbb{C}[z] \arrow[r, "\Psi"] & D_+(\alpha T_0+\beta T_1) \arrow[r, hook] & \mathbb{P}^1 & D_+(T_0) \arrow[l, hook'] & \operatorname{Spec}\mathbb{C}[t] \arrow[l, "\Phi"] \\
 & & \cup & & \\
 & & D_+(\alpha T_0+\beta T_1)\cap D_+(T_0) & & \\
\Psi^{-1}(\cap) \arrow[uuu, hook] \arrow[urr, "\Psi"] \arrow[rrrr, dashed, bend right, "?"] & & & & \Phi^{-1}(\cap) \arrow[uuu, hook] \arrow[ull, "\Phi"']
\end{tikzcd}$$

Aus dieser lokalen Koordinate erhält man (in einer Umgebung U von a_i) eine lokale Trivialisierung des Vektor Bündels V auf $\mathbb{P}^1$.

Definition G.1

Eine *lokale Trivialisierung* eines Vektorbündels $\pi : V \to \Sigma$ ist:

- eine Teilmenge U von Σ und dazu

- ein Isomorphismus $\pi^{-1}(U) \to U \times \mathbb{C}^r$ wenn r der Rang ist.

Wähle an einem Punkt und setze von dort aus fort?!?

Bezogen auf diese lokale Trivialisierung von V hat ∇ die Form $\nabla = d - A$ wobei

$$A = \underbrace{A_{k_1}\frac{dz}{z^{k_i}} + \cdots + A_1\frac{dz}{z}}_{\text{Hauptteil}} + \underbrace{A_0 dz + \ldots}_{\text{Nebenteil}}$$

eine Matrix meromorpher 1-Formen und $A_j \in \mathrm{End}(\mathbb{C}^n)$.

Definition G.2

Eine meromorphe 1-Form ist ein Element $\omega \in \Omega^1(V;\mathbb{C})$ welche sich als $\omega = udz$ mit meromorphen u schreiben lässt.

TODO

Aufgabe G.2. 1. *(von Def 2.7) Definition von Erweitertem Orbit $\widetilde{O}_i$ als top. Raum*

2. *(von Lemma 2.2)* $\widetilde{O} \cong T^*O_u \times O_B$

Es sei $n \in \mathbb{N}$ fix (Rang des Vektorbündels). Es sei $\mathrm{GL}_n(R)$ die Gruppe der $n \times n$ Matrizen mit Einträgen in R deren Determinanten eine Einheit in R sind. Es sei

$$\begin{aligned} G_k &:= \mathrm{GL}_n\left(\mathbb{C}[\xi]/\xi^k\right) \\ &= \left\{A \in \left(\mathbb{C}[\xi]/\xi^k\right)^{n\times n} \mid \det(A) \in \left(\mathbb{C}[\xi]/\xi^k\right)^\times\right\} \\ &\stackrel{?}{=} \left\{A \in \left(\mathbb{C}[\xi]/\xi^k\right)^{n\times n} \mid \det(A) \in \mathbb{C}^\times\right\} \\ &= \left(\left(\mathbb{C}[\xi]/\xi^k\right)^{n\times n}\right)^\times \end{aligned}$$

die Gruppe der $(k-1)$-Jets von Bündel Automorphismen und

$$\begin{aligned} B_k &= \left\{A \in G_k \mid A(0) = 1 \in \mathbb{C}^{n\times n}\right\} \\ &= \left\{\sum_{i=1}^{k-1} A_i\xi^i + 1 \in G_k\right\} \end{aligned}$$

die Untergruppe aller Elemente mit konstantem Term 1. Weiter sei $G := \mathrm{GL}_n(\mathbb{C}) = G_1$. Damit haben wir die kurze exakte Sequenz

$$\begin{tikzcd} 1 \arrow[r] & B_k \arrow[r, hook] & G_k \arrow[r] & G \arrow[l, bend right] \arrow[r] & 1 \\ & & A \arrow[r, mapsto] & A(0) & \end{tikzcd}$$

Diese Sequenz splitet, da G in G_k als die Untergruppe der konstanten Matritzen enthalten ist und die entsprechende Verknüpfung der Morphismen der Identität Id_G entspricht. Es folgt, dass G_k das semi-direkte Produkt $G \ltimes B_k$ ist, wobei G auf B_k durch Konjungation wirkt.

Die Kategorie der Gruppen ist nicht Abelsch, deshalb folgt nur der Zerfall in ein semidirektes Produkt. Siehe [Hat02, p.148]

Es gilt $B_k \lhd G_k$, $G_k = B_k G$ und $G \cap B_k = \{1\}$, deshalb ist $G_k = B_k \rtimes G$ bzw. $G_k = G \ltimes B_k$, wie es hier geschrieben wird.

Corollary G.3

Aus $G_k = G \ltimes B_k$ folgt, dass $G_k / B_k \cong G$.

Lemma G.4

Falls $G_k = G \ltimes B_k$ und $G_k = G \rtimes B_k$ so folgt, dass $G_k = G \times B_k$

Lemma G.5

Wenn $G = TH$ semidirektes Produkt zweier Untergruppen mit normalem T ist und $\alpha : H \to \mathrm{Aut}(T)$ durch $\alpha_h(t) = hth^{-1}$ definiert wird, dann ist die Abbildung $T \rtimes_\alpha H \to G, (t,h) \to th$ ein Gruppenisomorphismus.

$$
\begin{aligned}
G \ltimes B_k &= \mathrm{GL}_n(\mathbb{C}) \ltimes \left\{ \sum_{i=1}^{k-1} A_i \xi^i + 1 \in G_k \right\} \\
&\overset{?}{=} \left\{ g \left(\sum_{i=1}^{k-1} \cdot A_i \xi^i + 1 \right) g^{-1} \overset{?}{\in} G_k \mid g \in \mathrm{GL}_n(\mathbb{C}) \right\} \qquad \text{FALSCH!} \\
&\overset{?}{=} \left\{ \sum_{i=1}^{k-1} \left(g \cdot A_i \cdot g^{-1} \right) \xi^i + gg^{-1} \in G_k \mid g \in \mathrm{GL}_n(\mathbb{C}) \right\} \\
&\overset{?}{=} G_k \,.
\end{aligned}
$$

Sei $n = 3$ und $k = 2$. Nun suche $g \in G$ und $A_1 \xi + 1 \in B_2$ so dass

$$
g\left(A_1 \xi + 1\right) g^{-1} \overset{!}{=} \begin{pmatrix} 2 & 0 & \xi \\ \xi & 2 & 0 \\ 0 & \xi & 2 \end{pmatrix} \overset{(\star)}{\in} G_2 \qquad \text{bzw.} \qquad \underbrace{\underbrace{(gA_1 g^{-1})}_{\in \mathbb{C}^{n \times n}} \xi}_{\in \xi \mathbb{C}^{n \times n}} \overset{!}{=} \begin{pmatrix} 1 & 0 & \xi \\ \xi & 1 & 0 \\ 0 & \xi & 1 \end{pmatrix} \notin \xi \mathbb{C}^{n \times n}
$$

$(\star)$ da nach [Boa99, Chapter 2, Section 1] gilt, dass ein Element $\sum_{i=0}^{k-1} A_i \xi^i$ in G_k ist, genau dann, wenn $\det(A_0) \neq 0$ ist. Hier ist $\det(A_0) = 8$.

Ein Element A aus G_k hat die Form $A = A_0 + A_1 \xi + \cdots + A_{k-1} \xi^{k-1}$, wobei die A_i aus $\mathrm{End}(\mathbb{C}^n)$ sind. Anderstherum ist ein solches Element $A_0 + A_1 \xi + \cdots + A_{k-1} \xi^{k-1}$ in G_k genau dann, wenn $\det(A_0) \neq 0$.

Zu Zeigen: $\det(\sum_{i=0}^{k-1} A_i\xi^i) \neq 0 \Leftrightarrow \det(A_0) \neq 0$.
Denn $\det(\sum_{i=0}^{k-1} A_i\xi^i) = .\overset{?}{.}. = \det(A_0)$.

Die Lie Algebra $\mathfrak{g}_k = \mathrm{Lie}(G_k)$ zu G_k besteht aus den Elementen

$$X = X_0 + X_1\xi + \cdots + X_{k-1}\xi^{k-1}$$

mit $X_i \in \mathrm{End}(\mathbb{C}^n)$ beliebig.

Dazu: TODO

Die Elemente aus $\mathfrak{g}_k^*$, dem Vektorraum Dual von $\mathfrak{g}_k$ werden als

$$A = \left(\frac{A_k}{\xi^k} + \cdots + \frac{A_1}{\xi}\right) d\xi$$

geschrieben, wobei die A_i wieder aus $\mathrm{End}(\mathbb{C})$ beliebig sind. abusing notation. Die Elemente aus G_k sollten wohl besser g genannt werden, wie in der Diss

[Boa99, p. 22] Die Paarung zwischen $\mathfrak{g}_k^*$ und $\mathfrak{g}_k$ ist gegeben durch

$$< A, X >= \mathrm{Res}_0(\mathrm{Tr}(AX)) = \sum_{i=1}^{k} \mathrm{Tr}(A_i X_{i-1})$$

Wobei Res_0 die Residuen Abbildung ist, welche den Koeffizient vor $d\xi/\xi$ ausgibt. **Observe that the product AX is a well defined element of $\mathfrak{g}_k$, where $A \in \mathfrak{g}_k^*$ and $X \in \mathfrak{g}_k$.** Ähnlich ist das Produkt XA wohldefiniert in $\mathfrak{g}_k$. Damit ist $\mathfrak{g}_k^*$ ein Bimodul über $\mathfrak{g}_k$.

Koadjunggierte Orbiten ([Boa99, pp.23-26])

Definition G.6

- Der G_k *koadjungierte Orbit durch* $A \in \mathfrak{g}_k^*$ ist

$$O(A) := \{gAg^{-1} \mid g \in G_k\} \subset \mathfrak{g}_k^*$$

- Die *schönen* G_k koadjungierte Orbiten sind die Orbiten, die einen Erzeuger haben, dessen führender Koeffizient
  1. diagonalisierbar mit unterschiedlichen Eigenwerten ist, falls $k \geq 2$, oder
  2. diagonalisierbar mit modulo $\mathbb{Z}$ unterschiedlichen Eigenwerten, falls $k = 1$.

Diese koadjunggierten Orbiten sind homogene Räume für G_k und deshalb sind diese glatte komplexe Mannigfaltigkeiten. Nach symplektischer Geometrie haben koadjungierte Orbiten eine natürliche (Kostant-Kirillov) symplektische Struktur. they are the symplectic leaves of the (Lie) Poisson bracket on the dual of the Lie algebra. Since everything is complex here, $O(A)$ is naturally a complex symplectic manifold.

TODO

Sei X ein Element der Lie Algebra $\mathfrak{g}_k$ und setze das charakteristische Polynom

$$P_X(\lambda) := \det(\lambda 1 - X) \in \mathbb{C}[\lambda, \xi]/(\xi^k)$$

von X über dem Ring $\mathbb{C}[\xi]/(\xi)$. Dieses Polynom hat λ-Grad n mit Koeffizienten in $\mathbb{C}[\xi]/(\xi^k)$.

Lemma G.7

[Boa99, Lemma 2.5] Sei $X \in \mathfrak{g}_k$ ein Element, bei dem der Konstante Term unterschiedliche Eigenwerten hat.

1. Falls $g \in G_k$ dann ist $P_{gXg^{-1}} = P_X$
2. Falls $Y \in \mathfrak{g}_k$ und $P_Y = P_X$ dann ist $Y = gXg^{-1}$ für ein $g \in G_k$.

Und damit erhalten wir das folgende Korollar.

Corollary G.8

[Boa99, Corollary 2.6] Die schönen Orbiten sind affine algebraische Varietäten.

Corollary G.9

[Boa99, Corollary 2.7] Wir nehmen an, dass dar führende Koeffizient von $A \in \mathfrak{g}_k^*$ unterschiedliche Eigenwerte habe und das für diese bereits eine Anordnung $e_1, \dots, e_n$ gewählt wurde. Dann setzt sich die Wahl der Diagonalisierung $\mathrm{diag}(e_1, \dots, e_n)$ des führenden Koeffizienten von A eindeutig zu einer Diagonalisierung von A fort. Das bedeutet, dass es eindeutige Elemente $f_i, \dots, f_n \in \mathbb{C}[\xi]/(\xi^k)$ gibt, so dass

- $f_i(0) = e_i$ für alle i gilt und
- das entsprechende diagonale Elemente von $\mathfrak{g}_k^*$ ist in dem glechem Orbit wie A:

$$\mathrm{diag}(f_i, \dots, f_n) d\xi/\xi^k \in O(A)$$

TODO

Erwieterte Orbiten ([Boa99, pp.26-36]) Sei nun k mindestens 2 und wähle ein diagonales Element

$$A^0 := \left(\frac{A_k^0}{\xi^k} + \cdots + \frac{A_2^0}{\xi^2}\right) d\xi \in \mathfrak{b}_k^*$$

Ein diagonales Element ist ein Element, dessen Vorfaktoren alle diagonale Matritzen sind?

so dass der führende Koeffizient A_k^0 unterschiedliche Diagonaleinträge hat. Sei $O_B = O_B(A^0)$ der B_k coadjungierte Orbit durch A^0.

Definition G.10

Die *Erweiterung* oder der *erwieterte Orbit* assoziiert zu dem B_k koadjungierten Orbit O_B ist die Menge:

$$\widetilde{O} = \widetilde{O}(A^0) := \{(g_0, A) \in \mathrm{GL}_n(\mathbb{C}) \times \mathfrak{g}_k^* \mid \pi_{irr}(g_0 A g_0^{-1}) \in O_B\}$$

wobei $\pi_{irr} : \mathfrak{g}_k^* \to \mathfrak{b}_k^*$ die natürliche Projektion ist, welche das Residuum entfernt.

Warum denn **"erwieterte"** (extended) Orbiten? Sei $A \in O_B$, sei $g \in \mathrm{GL}_n(\mathbb{C})$ dann ist (id, A) und (g^{-1}, gAg^{-1}) in $\widetilde{O}$.
Es sind für alle g

$$O_B \hookrightarrow \widetilde{O}; \qquad A \mapsto (\mathrm{id}, A) \qquad \text{sowie} \qquad O_B \hookrightarrow \widetilde{O}; \qquad A \mapsto ((g^{-1}, gAg^{-1}))$$

Homöomorphismen auf ihr Bild. Klar? Nachweis?
Es ist für jedes A auch

$$\mathrm{GL}_n(\mathbb{C}) \hookrightarrow \widetilde{O}; \qquad g \mapsto ((g, g^{-1}Ag)) \qquad \text{bzw.} \qquad \mathrm{GL}_n(\mathbb{C}) \hookrightarrow \widetilde{O}; \qquad g \mapsto ((g^{-1}, gAg^{-1}))$$

ein Homöomorphismus auf sein Bild.

Symplectic aspects of the group cotangent bundles. [Boa99, p. 19] Let G be a Lie group. The **left multiplication** $L_g : G \to G; h \mapsto gh$ gives an isomrphism

$$(dL_g)_i : \mathfrak{g} = T_1 G \to T_g G$$

and induces a trivialization:

$$G \times \mathfrak{g} \cong TG; \qquad (g, X) \mapsto (g, (dL_g)_1 X)$$

which will be referred to as the *left trivialization* of TG.

By taking the duals the left trivialization of the cotangent bundle is also obtained:

$$G \times \mathfrak{g}^* \cong T^*G; \qquad (g, A) \mapsto (g, (dL_{g^{-1}})_1^\vee A)$$

where $(dL_{g^{-1}})_1^\vee$ denotes the inverse of the dual linear map to $(dV_g)_1$.

Now we can write down the natural symplectic structure on TG explicitly:

Lemma G.11

[Boa99, Lemma 1.45]

Proof. TODO □

If the right trivializations are used instead, the formula looks the same upto one sign:

Lemma G.12

[Boa99, Lemma 1.46]

Proof. TODO □

Lemma G.13

(Decoupling) [Boa99, Lemma 2.13] Die folgende Abbildung ist ein complex analytischer Isomorphismus:

$$\widetilde{O} \cong T^* \mathrm{GL}_n(\mathbb{C}) \times O_B; \qquad (g_0, A) \mapsto ((g_0, \pi_{res}(A)), \pi_{irr}(g_0 A g_0^{-1}))$$

wobei $T^* \mathrm{GL}_n(\mathbb{C}) \cong \mathrm{GL}_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C})^*$ via der links Trivialisierung und π_{irr}, π_{res} die Projektionen von $\mathfrak{g}_k^*$ nach $\mathfrak{b}_k^*$, $\mathfrak{gl}_n(\mathbb{C})^*$ passend.

Proof. Die Abbildung ist wohldefiniert.

denn??

Behauptung: Die Abbildung:

$$((g_0, S), B) \overset{\psi}{\mapsto} (g_0, g_0^{-1} B g_0 + S) \in \widetilde{O}\,,$$

mit $((g_0, S), B) \in T^* \mathrm{GL}_n(\mathbb{C}) \times O_B$, ist ein Inverses zu der oben definierten Abbildung. Sei φ die oben definierte Abbildung, dann

$$\begin{aligned} ((g_0, S), B) &\overset{\psi}{\mapsto} (g_0, g_0^{-1} B g_0 + S) \\ &\overset{\varphi}{\mapsto} ((g_0, \pi_{res}(g_0^{-1} B g_0 + S)), \pi_{irr}(g_0 (g_0^{-1} B g_0 + S) g_0^{-1})) \\ &= ((g_0, \pi_{res}(g_0^{-1} B g_0) + \pi_{res}(S)), \pi_{irr}(g_0 g_0^{-1} B g_0 g_0^{-1}) + \pi_{irr}(g_0 S g_0^{-1})) \\ &= ((g_0, S), \pi_{irr}(B) + \pi_{irr}(g_0 S g_0^{-1})) \\ &= ((g_0, S), B) \end{aligned}$$

In der Anderen Richtung gilt:

$$\begin{aligned} (g_0, A) &\overset{\varphi}{\mapsto} ((g_0, \pi_{res}(A)), \pi_{irr}(g_0 A g_0^{-1})) \\ &\overset{\psi}{\mapsto} (g_0, g_0^{-1}(\pi_{irr}(g_0 A g_0^{-1})) g_0 + \pi_{res}(A)) \\ &= (g_0, g_0^{-1} g_0 (A - \pi_{res}(A)) g_0^{-1} g_0 + \pi_{res}(A)) \end{aligned}$$

$$= (g_0, A)$$

Damit folgt die Behauptung. □

Behauptung. $\dim(\widetilde{O}) = \dim(O) + 2n$

[Boa99, p27]

$$\Theta := \left\{ A \in \mathfrak{g}_k^* \mid (g_0, A) \in \widetilde{O} \text{ für ein passendes } g_0 \in \mathrm{GL}_n(\mathbb{C}) \right\}$$

ist der Kern der Projektion

$$\widetilde{O} \to \mathfrak{g}_k^*; \qquad (g_0, A) \mapsto A$$

Corollary G.14

[Boa99, Corollary 2.15] The extended orbit $\widetilde{O}$ is a principal T bundle over θ.

Wie sieht denn die B_k Wirkung auf $T^*G_k \times O_B$ aus?
Siehe: [Boa99, Definition 1.47]

Definition G.15

[Boa99, Definition 1.47] Die *links Wirkung* von G auf T^*G ist (in terms of left trivialisation):

$$\beta(g, A) = (\beta g, A)\,.$$

Die *rechts Wirkung* ...

Lemma G.16

[Boa99, Lemma 1.48] Die links Wirkung von G auf T^*G ist Hamiltonisch mit einer equivarianten momenten Abbildung gegeben (in terms of left trivialisation) durch

$$\mu_L : G \times \mathfrak{g}^* \to \mathfrak{g}^*; \qquad (g, A) \mapsto -\mathrm{Ad}_g^*(A)$$

Die rechts Wirkung ...

Und B_k wirkt auf O_B durch Konjungation.
Damit ergibt sich zusammen die B_k Wirkung auf $T^*G_k \times O_B$ durch

$$\beta(g, A, B) = (\beta g, A, \beta B \beta^{-1})$$

Lemma G.17: Lemma 2.2 aus [Boa01]

Der erweiterte Orbit $\widetilde{O}$ ist kanonisch isomorph zu dem symplektischem quotienten $T^*G_k \times O_B /\!\!/ B_k$ wobei das Kotangentialbündel T^*G_k und der koadjungierte Orbit O_B ihre natürliche symplektische Struktur tragen.

Entspricht [Boa99, Proposition 2.19] zusammen mit [Boa99, Remark 2.20]

Proof. Die Momenten Abbildung der B_k-Wirkung auf O_B ist nur die Inklusion $O_B \to \mathfrak{b}_k^*$ und die Momenten Abbildung für die B_k-Wirkung auf T^*G_k ist die Komposition der Momenten Abbildung der links-G_k-Wirkung mit der Projektion $\pi_{irr} : \mathfrak{g}_k^* \to \mathfrak{b}_k^*$. in thboalch ist π_{irr} das π Damit ist die Momenten Abbildung der B_k-Wirkung auf dem Produkt die Summe dieser beiden Wirkungen:

$$\mu : T^*G_k \times O_B \to \mathfrak{b}_k^*; \qquad (g, A, B) \mapsto -\pi_{irr}\left(\mathrm{Ad}_g^*(A)\right) + B \tag{G.1}$$

Damit ist das Urbild von $0 \in \mathfrak{b}_k^*$ unter μ:

$$\mu^{-1}(0) = \left\{(g, A, B) \mid \pi_{irr}(gAg^{-1}) = B\right\} .$$

Sei $(g, A, B) \in T^*G_k \times O_B$ im Kern, also

$$\begin{aligned} 0 = \mu((g, A, B)) &= -\pi_{irr}\left(\mathrm{Ad}_g^*(A)\right) + B \\ &= -\pi_{irr}\left(gAg^{-1}\right) + B \qquad \text{also} \qquad \pi_{irr}\left(gAg^{-1}\right) = B \end{aligned}$$

Um den symplektischen Quotienten $T^*G_k \times O_B /\!\!/ B_k := \mu^{-1}(0)/B_k$ mit $\widetilde{O}$ zu identifizieren, betrachten wir die Abbildung:

$$\chi : \mu^{-1}(0) \to \widetilde{O}; \qquad (g, A, B) \mapsto (g(0), A) .$$

Diese Abbildung ist wohldefiniert da die Bedingung $\pi_{irr}(gAg^{-1}) = B$ aus (G.1) impliziert, dass

$$\pi_{irr}(g(0)Ag(0)^{-1}) = g(0)g^{-1}Bgg(0)^{-1}$$

$$\begin{aligned} \pi_{irr}(g(0)Ag(0)^{-1}) &= g(0)\pi_{irr}(A)g(0)^{-1} \\ &= g(0)\pi_{irr}(g^{-1}gAg^{-1}g)g(0)^{-1} \\ &\overset{?}{=} g(0)g^{-1}\pi_{irr}(gAg^{-1})gg(0)^{-1} \\ &= g(0)g^{-1}Bgg(0)^{-1} \end{aligned}$$

und dieses Element ist in dem Gleichem B_k koadjunggierten Orbit wie B, da $g(0)g^{-1} \in B_k$.

$$g(0)g^{-1} = g(0)\left(g(0) + g_i\xi^1 + \cdots + g_k\xi^k\right)^{-1}$$
$$\stackrel{?}{=} g(0)\left(g(0)^{-1} + (g_i)^{-1}\xi^1 + \cdots + (g_k)^{-1}\xi^k\right) \qquad \text{so ist das falsch?!?}$$
$$= g(0)g(0)^{-1} + g(0)(g_i)^{-1}\xi^1 + \cdots + g(0)(g_k)^{-1}\xi^k$$
$$= 1 + g(0)(g_i)^{-1}\xi^1 + \cdots + g(0)(g_k)^{-1}\xi^k \in B_k$$

ODER.

$$(g(0))^{-1} \stackrel{?}{=} g^{-1}(0)$$

Behauptung. *χ ist surjektiv und hat genau die B_k Orbiten in $\mu^{-1}(0)$ als Fasern.*

Surjektivität ist klar, da wir einen Schnitt von χ angeben können:

$$s : (g_0, A) \mapsto (g_0, A, \pi_{irr}(g_0 A g_0^{-1})) \in \mu^{-1}(0)$$

wobei $(g_0, A) \in \widetilde{O}$.

$$\chi \circ s = \left((g_0, A) \stackrel{s}{\mapsto} (g_0, A, \pi_{irr}(g_0 A g_0^{-1})) \stackrel{\chi}{\mapsto} (\underbrace{g_0(0)}_{=g_0}, A)\right) = \mathrm{id}_{\mu^{-1}(0)}$$

Nun wollen wir die Fasern betrachten, sei dazu $\chi(g, A, B) = \chi(g', A', B')$. Also ist $A = A'$, $g(0) = g'(0)$ und damit $g' = \beta g$ wobei $\beta := g' g^{-1}$ in B_k ist. Dann, erneut nach der Bedingung in (G.1), gilt:

$$B' = \pi_{irr}(g' A' g'^{-1}) = \pi_{irr}(\beta g A g^{-1} \beta^{-1}) = \beta B \beta^{-1}$$

$$B' \stackrel{(G.1)}{=} \pi_{irr}(g' A' g'^{-1}) \stackrel{A'=A}{=} \pi_{irr}\left(g'\left(g^{-1}g\right) A \left(g^{-1}g\right) g'^{-1}\right) = \pi_{irr}(\beta g A g^{-1} \beta^{-1})$$
$$= \beta \pi_{irr}(g A g^{-1}) \beta^{-1} \stackrel{(G.1)}{=} \beta B \beta^{-1}$$

Damit gilt, dass $(g', A', B') = \beta(g, A, B)$ und damit ist jede Faser von χ in einem B_k Oribt enthalten. Anderstherum ist klar dass B_k innerhalb der Fasern von χ wirkt. Dies entspricht der anderen Inclusion und damit ist die Behauptung gezeigt. **Beh.** □

Dies identifiziert den Quotienten $\mu^{-1}(0)/B_k$ mit $\widetilde{O}$ **als Mannigfaltigkeit.**

Nun wollen wir noch die symplektische Struktur identifizieren. Dazu identifizieren wir zunächst $\widetilde{O}$ mit $T^* \mathrm{GL}_n \times O_B$

TODO

□

Aufgabe G.3. *Proof (2.1)*

Proposition G.18

[Boa01, Prop 2.1]

- [Boa99, Theorem 2.35] $\mathcal{M}(\mathbf{a})$ ist isomorph zu dem komplexen symplektischen Quotienten

$$\mathcal{M}(\mathbf{a}) \cong O_1 \times \cdots \times O_m /\!\!/ G \,.$$

- [Boa99, Theorem 2.43] Analog gibt es komplexe symplektische Mannigfaltigkeiten (erweiterte Orbiten) $\widetilde{O}_i$ mit $\dim(\widetilde{O}_i) = \dim(O_i) + 2n$ und freie Hamiltonische G-Wirkungen, so dass

$$\widetilde{\mathcal{M}}(\mathbf{a}) \cong \widetilde{O}_1 \times \cdots \times \widetilde{O}_m /\!\!/ G \,.$$

- TODO: Unterpunkt 3

Proof. Wir wählen eine Koordinate z und identifizieren $\mathbb{P}^1$ mit $\mathbb{C} \cup \{\infty\}$, so dass jedes a_i endlich ist. Definiere $z_i := z - a_i$. Die gewählten Keime $d - {}^iA^0$ des meromorphen Zusammenhangs bestimmen G_{k_i} koadjunggierte Orbiten O_i und setzen sich zu erweiterten Orbiten $\widetilde{O}_i$ fort.

Setze O_i als den koadjunggierten Orbit durch den Punkt von $\mathfrak{g}_k^*$, welcher durch den Hauptteil von ${}^iA^0$ in (5) gegeben ist. Analog bestimmt der irreguläre Teil von ${}^iA^0$ einen Punkt in $\mathfrak{b}_{k_i}^*$ und $\widetilde{O}_i$ ist der erwieterte Orbit assoziiert zu dem B_{k_i} koadjunggierten Orbit durch den Punkt

□

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