8 April 2018

Theoretical Results of the Bimodular Network

1. Steady State of the Congruent Group of Neurons in the Absence of Noise

We consider whether the simulation results of the decoded directions in Figs. 6F to I agree with the theoretical predictions. The steady state of the dynamics of the congruent group of neurons in module 1 can be approximated by

$$u_1 V(y_1 - s_1, \kappa_{1u}) \approx \rho J_{rc} \frac{u_1^2}{D_1} \frac{I_0(2\kappa_{1u})}{2\pi I_0(\kappa_{1u})^2} V(y_1 - s_1, \kappa_{1ar})$$
$$+ \rho J_{rp} \frac{u_2^2}{D_2} \frac{I_0(2\kappa_{2u})}{2\pi I_0(\kappa_{2u})^2} V(y_1 - s_2, \kappa_{2ar}) + I_1 V\left(y_1 - x_1, \frac{a}{2}\right),$$

where $I_1 = \alpha_1 e^{-a/2} 2\pi I_0(a/2)$. Approximating $\kappa_{1u} \approx \kappa_{1ar} \approx \kappa_{2u} \approx \kappa_{2ar} \approx a/2$ and integrating over y_1, we obtain

$$u_1 \approx \rho J_{rc} \frac{u_1^2}{D_1} \frac{I_0(a)}{2\pi I_0(a/2)^2} + \rho J_{rp} \frac{u_2^2}{D_2} \frac{I_0(a)}{2\pi I_0(a/2)^2} + I_1.$$

The global inhibition factors are given by

$$D_n = 1 + \omega \sum\nolimits_x \left[u_n^2 V(x, a/2)^2 + J_{\text{int}} u_{\bar{n}}^2 V(x, a/2)^2\right] = 1 + \frac{I_0(a)\omega\rho(u_n^2 + J_{\text{int}} \bar{u}_n^2)}{2\pi I_0(a/2)^2}.$$

For comparison, Wenhao introduced a network assuming that the synaptic inputs of the opposite group of neurons are the same as those of the congruent group.

$$u_1 \approx \rho J_{rc} \frac{u_1^2}{1 + \frac{I_0(a)\omega\rho(1 + J_{\text{int}})}{2\pi I_0(a/2)^2} u_1^2} \frac{I_0(a)}{2\pi I_0\left(\frac{a}{2}\right)^2} + \rho J_{rp} \frac{u_2^2}{1 + \frac{I_0(a)\omega\rho(1 + J_{\text{int}})}{2\pi I_0(a/2)^2} u_2^2} \frac{I_0(a)}{2\pi I_0\left(\frac{a}{2}\right)^2} + I_1.$$

In the absence of stimuli, we have $u_1 = u_2$. Thus

$$\omega\rho(1 + J_{\text{int}}) u_1^2 - \rho(J_{rp} + J_{rc}) u_1 + \frac{2\pi I_0(a/2)^2}{I_0(a)} = 0.$$

$$u_1 = \frac{J_{rp} + J_{rc}}{2\omega(1 + J_{\text{int}})} \pm \sqrt{\left[\frac{J_{rp} + J_{rc}}{2\omega(1 + J_{\text{int}})}\right]^2 - \frac{2\pi I_0(a/2)^2}{I_0(a)\omega\rho(1 + J_{\text{int}})}}.$$

The critical coupling of $J_{rp} + J_{rc}$ and the critical synaptic input are given by

$$J_c = \sqrt{\frac{8\pi I_0(a/2)^2\omega(1+J_{\text{int}})}{I_0(a)\rho}}.$$

$$u_c = \sqrt{\frac{2\pi I_0(a/2)^2}{I_0(a)\rho\omega(1+J_{\text{int}})}}.$$

They satisfy the equation

$$\frac{u_1^2}{u_c^2} - 2\left(\frac{J_{rp}+J_{rc}}{J_c}\right)\frac{u_1}{u_c} + 1 = 0.$$

In the presence of stimuli $I_m = A_m u_c$, the equations are modified. Let $\bar{u}_1$ be the synaptic input of the opposite group of neurons in module 1. Then

$$\frac{u_1}{u_c} \approx \frac{J_{rc}}{J_c}\frac{2u_1^2/u_c^2}{1+\frac{u_1^2+J_{\text{int}}\bar{u}_1^2}{(1+J_{\text{int}})u_c^2}} + \frac{J_{rp}}{J_c}\frac{2u_2^2/u_c^2}{1+\frac{u_2^2+J_{\text{int}}\bar{u}_2^2}{(1+J_{\text{int}})u_c^2}} + A_1.$$

The steady state equation of the profile is also simplified,

$$\frac{u_1}{u_c}V\left(y_1-s_1,\frac{a}{2}\right) \approx \frac{J_{rc}}{J_c}\frac{2u_1^2/u_c^2}{1+\frac{u_1^2+J_{\text{int}}\bar{u}_1^2}{(1+J_{\text{int}})u_c^2}}V\left(y_1-s_1,\frac{a}{2}\right)$$
$$+\frac{J_{rp}}{J_c}\frac{2u_2^2/u_c^2}{1+\frac{u_2^2+J_{\text{int}}\bar{u}_2^2}{(1+J_{\text{int}})u_c^2}}V\left(y_1-s_2,\frac{a}{2}\right) + A_1V\left(y_1-x_1,\frac{a}{2}\right).$$

Multiplying both sides of the steady state equation bye^{jy_1} and integrating over y_1,

$$\left(\frac{J_{rc}}{J_c}\frac{2u_1^2/u_c^2}{1+\frac{u_1^2+J_{\text{int}}\bar{u}_1^2}{(1+J_{\text{int}})u_c^2}} - \frac{u_1}{u_c}\right)e^{js_1} + \frac{J_{rp}}{J_c}\frac{2u_2^2/u_c^2}{1+\frac{u_2^2+J_{\text{int}}\bar{u}_2^2}{(1+J_{\text{int}})u_c^2}}e^{js_2} + A_1e^{jx_1} = 0.$$

Together with the steady state in module 2, the matrix equation becomes

$$\begin{pmatrix} g_{11} & -g_{12} \\ -g_{21} & g_{22} \end{pmatrix}\begin{pmatrix} e^{js_1} \\ e^{js_2} \end{pmatrix} = \begin{pmatrix} A_1e^{jx_1} \\ A_2e^{jx_2} \end{pmatrix},$$

where $g_{12} \equiv \frac{J_{rp}}{J_c}\frac{2u_2^2/u_c^2}{1+\frac{u_2^2+J_{\text{int}}\bar{u}_2^2}{(1+J_{\text{int}})u_c^2}}$ and $g_{11} \equiv \frac{u_1}{u_c} - \frac{J_{rc}}{J_c}\frac{2u_1^2/u_c^2}{1+\frac{u_1^2+J_{\text{int}}\bar{u}_1^2}{(1+J_{\text{int}})u_c^2}} = A_1 + g_{12}$. Inverting,

$$\begin{pmatrix} e^{js_1} \\ e^{js_2} \end{pmatrix} = \frac{1}{g_{11}g_{22} - g_{12}g_{21}} \begin{pmatrix} g_{22} & g_{12} \\ g_{21} & g_{11} \end{pmatrix} \begin{pmatrix} A_1 e^{jx_1} \\ A_2 e^{jx_2} \end{pmatrix}.$$

$$\left(A_1 + \frac{A_2 g_{12}}{A_2 + g_{21}}\right) e^{js_1} = A_1 e^{jx_1} + \frac{A_2 g_{12}}{A_2 + g_{21}} e^{jx_2}.$$

Note that this equation only has the solution $s_1 = x_1 = x_2$ and only when $x_1 = x_2$. This is due to the assumption that the widths of the stimuli and the bumps are all equal. In practice, when the stimuli are located in different directions, the bumps should be broader. Higher order equations will be required.

The decoded result is given by

$$\hat{s}_c = \arctan \frac{A_1 \sin x_1 + A_{2s} \sin x_2}{A_1 \cos x_1 + A_{2s} \cos x_2},$$

$$\hat{\kappa}_c = \frac{\sqrt{A_1^2 + A_{2s}^2 + 2A_1 A_{2s} \cos(x_1 - x_2)}}{A_1 + A_{2s}},$$

where $A_{2s} \equiv A_2 g_{12}/(A_2 + g_{21})$.

2. Discussion on Wenhao's Approach

We compare the above approach with Wenhao's. In his work, a set of dynamical equations from (S50) to (S56) in the presence of noise was derived using the projection method. (The projection method will give similar results to the above approach, so there are other more significant differences.) Equations (S50) and (S51) are then reorganized to become (S57) and (S58). He obtained the eigenvalue equations given by Eqs. (S59) and (S60),

$$\frac{dz_1^c}{dt} = \lambda_1 \sin\left(\frac{\langle z_1^c \rangle - z_1^c}{2}\right) + \beta_1 \eta_1(t),$$

$$\frac{dz_2^c}{dt} = \lambda_2 \sin\left(\frac{\langle z_2^c \rangle - z_2^c}{2}\right) + \beta_2 \eta_2(t),$$

where the eigenvalues should be

$$\lambda_1^c = h_1^c \cos\left(\frac{x_1 - \langle z_1^c \rangle}{2}\right) + g_{12}^c \cos\left(\frac{\langle z_2^c \rangle - \langle z_1^c \rangle}{2}\right),$$

$$\lambda_2^c = h_1^c \cos\left(\frac{x_2 - \langle z_2^c \rangle}{2}\right) + g_{21}^c \cos\left(\frac{\langle z_1^c \rangle - \langle z_2^c \rangle}{2}\right),$$

but there was a step jumped ahead to

$$\lambda_1^c e^{j\langle z_1^c \rangle/2} = h_1^c e^{jx_1/2} + g_{12}^c e^{jz_2^c/2},$$

$$\lambda_1^c e^{j\langle z_1^c\rangle/2} = h_1^c e^{jx_2/2} + g_{12}^c e^{jz_1^c/2}.$$

Together with the steady state in module 2, the matrix equation (S61) was obtained as

$$\begin{pmatrix} e^{j\langle z_1^c\rangle/2} \\ e^{j\langle z_2^c\rangle/2} \end{pmatrix} = \frac{1}{\lambda_1^c\lambda_2^c - g_{12}^c g_{21}^c}\begin{pmatrix} \lambda_2^c & g_{12}^c \\ g_{21}^c & \lambda_1^c \end{pmatrix}\begin{pmatrix} h_1^c e^{j\langle x_1\rangle/2} \\ h_1^c e^{j\langle x_2\rangle/2} \end{pmatrix}.$$

With direct intuition, a further approximation $\lambda_1 = h_1 + g_{12}$ and $\lambda_2 = h_2 + g_{21}$ was adopted in Eq. (S67), yielding Eq. (S68),

$$\left(h_1^c + \frac{g^c h_2^c}{g^c + h_2^c}\right) e^{j\langle z_1^c\rangle/2} = h_1^c e^{jx_1/2} + \frac{g^c h_2^c}{g^c + h_2^c} e^{jx_2/2}.$$

The next step, probably the most controversial one, is that the factors of 2 in each of the 3 exponents are canceled, resulting in Eq. (S69). The analogy was made that $h_1 \propto \kappa_1$ and $g_{12} \propto \kappa_s$.

In summary, the intuition in arriving at the approximate conclusions is highly appreciated, but for publication, a more rigorous approach is needed.

3. Steady State of the Opposite Group of Neurons in the Absence of Noise

For the opposite group of neurons, we have

$$\frac{\bar{u}_1}{u_c} \approx \frac{J_{rc}}{J_c}\frac{2\bar{u}_1^2/u_c^2}{1 + \frac{\bar{u}_1^2 + J_{\text{int}}u_1^2}{(1+J_{\text{int}})u_c^2}} + \frac{J_{rp}}{J_c}\frac{2\bar{u}_2^2/u_c^2}{1 + \frac{\bar{u}_2^2 + J_{\text{int}}u_2^2}{(1+J_{\text{int}})u_c^2}} + A_1,$$

and

$$\frac{\bar{u}_1}{u_c} V\left(y_1 - \bar{s}_1, \frac{a}{2}\right) \approx \frac{J_{rc}}{J_c}\frac{2\bar{u}_1^2/u_c^2}{1 + \frac{\bar{u}_1^2 + J_{\text{int}}u_1^2}{(1+J_{\text{int}})u_c^2}} V\left(y_1 - \bar{s}_1, \frac{a}{2}\right)$$
$$+ \frac{J_{rp}}{J_c}\frac{2\bar{u}_2^2/u_c^2}{1 + \frac{\bar{u}_2^2 + J_{\text{int}}u_2^2}{(1+J_{\text{int}})u_c^2}} V\left(y_1 - \bar{s}_2 + \pi, \frac{a}{2}\right) + A_1 V\left(y_1 - x_1, \frac{a}{2}\right).$$

Multiplying both sides of the steady state equation by e^{jy_1} and integrating over y_1,

$$\left(\frac{J_{rc}}{J_c}\frac{2\bar{u}_1^2/u_c^2}{1 + \frac{\bar{u}_1^2 + J_{\text{int}}u_1^2}{(1+J_{\text{int}})u_c^2}} - \frac{\bar{u}_1}{u_c}\right) e^{j\bar{s}_1} - \frac{J_{rp}}{J_c}\frac{2\bar{u}_2^2/u_c^2}{1 + \frac{\bar{u}_2^2 + J_{\text{int}}u_2^2}{(1+J_{\text{int}})u_c^2}} e^{j\bar{s}_2} + A_1 e^{jx_1} = 0.$$

Together with the steady state in module 2, the matrix equation becomes

$$\begin{pmatrix} \bar{g}_{11} & \bar{g}_{12} \\ \bar{g}_{21} & \bar{g}_{22} \end{pmatrix} \begin{pmatrix} e^{js_1} \\ e^{js_2} \end{pmatrix} = \begin{pmatrix} A_1 e^{jx_1} \\ A_2 e^{jx_2} \end{pmatrix},$$

where $\bar{g}_{12} \equiv \frac{J_{rp}}{J_c} \frac{2\bar{u}_2^2/u_c^2}{1+\frac{\bar{u}_2^2+J_{\rm int}u_2^2}{(1+J_{\rm int})u_c^2}}$ and $\bar{g}_{11} \equiv \frac{\bar{u}_1}{u_c} - \frac{J_{rc}}{J_c} \frac{2\bar{u}_1^2/u_c^2}{1+\frac{\bar{u}_1^2+J_{\rm int}u_1^2}{(1+J_{\rm int})u_c^2}} = A_1 + \bar{g}_{12}$. Inverting,

$$\begin{pmatrix} e^{js_1} \\ e^{js_2} \end{pmatrix} = \frac{1}{\bar{g}_{11}\bar{g}_{22} - \bar{g}_{12}\bar{g}_{21}} \begin{pmatrix} \bar{g}_{22} & -\bar{g}_{12} \\ -\bar{g}_{21} & \bar{g}_{11} \end{pmatrix} \begin{pmatrix} A_1 e^{jx_1} \\ A_2 e^{jx_2} \end{pmatrix}.$$

$$\left(A_1 + \frac{A_2\bar{g}_{12}}{A_2 + \bar{g}_{21}}\right) e^{js_1} = \alpha_1 e^{jx_1} - \frac{A_2\bar{g}_{12}}{A_2 + \bar{g}_{21}} e^{jx_2}.$$

The decoded result is given by

$$\hat{s}_o = \arctan \frac{A_1 \sin x_1 - \bar{A}_{2s} \sin x_2}{A_1 \cos x_1 - \bar{A}_{2s} \cos x_2},$$

$$\hat{\kappa}_o = \frac{\sqrt{\alpha_1^2 + \bar{\alpha}_{2s}^2 - 2\alpha_1\bar{\alpha}_{2s} \cos(x_1 - x_2)}}{A_1 + \bar{A}_{2s}},$$

where $\bar{A}_{2s} \equiv A_2\bar{g}_{12}/(A_2 + \bar{g}_{21})$.

4. Comparison with Simulation Results

4.1. The Case of Unequal Stimulus Strength

We first compare the theoretical prediction with the decoded direction in Fig. 6F. Since the stimulus strengths are not equal, the four equations are

$$\frac{u_1}{u_c} = \frac{J_{rc}}{J_c} \frac{2u_1^2/u_c^2}{1 + \frac{u_1^2 + J_{\rm int}\bar{u}_1^2}{(1 + J_{\rm int})u_c^2}} + \frac{J_{rp}}{J_c} \frac{2u_2^2/u_c^2}{1 + \frac{u_2^2 + J_{\rm int}\bar{u}_2^2}{(1 + J_{\rm int})u_c^2}} + A_1.$$

$$\frac{u_2}{u_c} = \frac{J_{rc}}{J_c} \frac{2u_2^2/u_c^2}{1 + \frac{u_2^2 + J_{\rm int}\bar{u}_2^2}{(1 + J_{\rm int})u_c^2}} + \frac{J_{rp}}{J_c} \frac{2u_1^2/u_c^2}{1 + \frac{u_1^2 + J_{\rm int}\bar{u}_1^2}{(1 + J_{\rm int})u_c^2}} + A_2.$$

$$\frac{\bar{u}_1}{u_c} = \frac{J_{rc}}{J_c} \frac{2\bar{u}_1^2/u_c^2}{1 + \frac{\bar{u}_1^2 + J_{\rm int}u_1^2}{(1 + J_{\rm int})u_c^2}} + \frac{J_{rp}}{J_c} \frac{2\bar{u}_2^2/u_c^2}{1 + \frac{\bar{u}_2^2 + J_{\rm int}u_2^2}{(1 + J_{\rm int})u_c^2}} + A_1.$$

$$\frac{\bar{u}_2}{u_c} = \frac{J_{rc}}{J_c} \frac{2\bar{u}_2^2/u_c^2}{1 + \frac{\bar{u}_2^2 + J_{\rm int}u_2^2}{(1 + J_{\rm int})u_c^2}} + \frac{J_{rp}}{J_c} \frac{2\bar{u}_1^2/u_c^2}{1 + \frac{\bar{u}_1^2 + J_{\rm int}u_1^2}{(1 + J_{\rm int})u_c^2}} + A_2.$$

Inverting the equations,

$$\bar{u}_1 = \sqrt{\frac{1+J_{\text{int}}}{J_{\text{int}}}\left[2u_1^2\frac{J_{rc}}{J_c}\left(\frac{u_1 - A_1 u_c}{u_c} - \frac{J_{rp}}{J_c}\frac{2u_2^2/u_c^2}{1+\frac{u_2^2 + J_{\text{int}}\bar{u}_2^2}{(1+J_{\text{int}})u_c^2}}\right)^{-1} - u_c^2\right] - \frac{u_1^2}{J_{\text{int}}}}.$$

$$u_1 = \sqrt{\frac{1+J_{\text{int}}}{J_{\text{int}}}\left[2\bar{u}_1^2\frac{J_{rc}}{J_c}\left(\frac{\bar{u}_1 - A_1 u_c}{u_c} - \frac{J_{rp}}{J_c}\frac{\frac{2\bar{u}_2^2}{u_c^2}}{1+\frac{\bar{u}_2^2 + J_{\text{int}}u_2^2}{(1+J_{\text{int}})u_c^2}}\right)^{-1} - u_c^2\right] - \frac{\bar{u}_1^2}{J_{\text{int}}}}.$$

$$\bar{u}_2 = \sqrt{\frac{1+J_{\text{int}}}{J_{\text{int}}}\left[2u_2^2\frac{J_{rc}}{J_c}\left(\frac{u_2 - A_2 u_c}{u_c} - \frac{J_{rp}}{J_c}\frac{2u_1^2/u_c^2}{1+\frac{u_1^2 + J_{\text{int}}\bar{u}_1^2}{(1+J_{\text{int}})u_c^2}}\right)^{-1} - u_c^2\right] - \frac{u_2^2}{J_{\text{int}}}}.$$

$$u_2 = \sqrt{\frac{1+J_{\text{int}}}{J_{\text{int}}}\left[2\bar{u}_2^2\frac{J_{rc}}{J_c}\left(\frac{\bar{u}_2 - A_2 u_c}{u_c} - \frac{J_{rp}}{J_c}\frac{\frac{2\bar{u}_1^2}{u_c^2}}{1+\frac{\bar{u}_1^2 + J_{\text{int}}u_1^2}{(1+J_{\text{int}})u_c^2}}\right)^{-1} - u_c^2\right] - \frac{\bar{u}_2^2}{J_{\text{int}}}}.$$

Assume $u_1 = \bar{u}_1$ and $u_2 = \bar{u}_2$. Then

$$u_1 = \sqrt{\frac{1+J_{\text{int}}}{J_{\text{int}}}\left[2u_1^2\frac{J_{rc}}{J_c}\left(\frac{u_1 - A_1 u_c}{u_c} - \frac{J_{rp}}{J_c}\frac{2u_2^2}{u_c^2+u_2^2}\right)^{-1} - u_c^2\right] - \frac{u_1^2}{J_{\text{int}}}}.$$
$$u_2 = \sqrt{\frac{1+J_{\text{int}}}{J_{\text{int}}}\left[2u_2^2\frac{J_{rc}}{J_c}\left(\frac{u_2 - A_2 u_c}{u_c} - \frac{J_{rp}}{J_c}\frac{2u_1^2}{u_c^2+u_1^2}\right)^{-1} - u_c^2\right] - \frac{u_2^2}{J_{\text{int}}}}.$$

In summary, results can be obtained from the following steps.

The case of unequal stimulus strength
1) For given values of J_{rc}/J_c, J_{rp}/J_c and α, solve u_1/u_c and u_2/u_c from $$\frac{u_1}{u_c} = \frac{J_{rc}}{J_c}\frac{2u_1^2}{u_c^2+u_1^2} + \frac{J_{rp}}{J_c}\frac{2u_2^2}{u_c^2+u_2^2} + A_1.$$ $$\frac{u_2}{u_c} = \frac{J_{rc}}{J_c}\frac{2u_2^2}{u_c^2+u_2^2} + \frac{J_{rp}}{J_c}\frac{2u_1^2}{u_c^2+u_1^2} + A_2.$$

2) For given values of x_1 and x_2, calculate $g_{12} = \frac{J_{rp}}{J_c}\frac{2u_2^2}{u_c^2+u_2^2}$ and $g_{21} = \frac{J_{rp}}{J_c}\frac{2u_1^2}{u_c^2+u_1^2}$. Then $A_{2s} = \frac{A_2 g_{12}}{A_2+g_{21}}$. For the congruent group of neurons in module 1,

$$\hat{s}_c = \arctan\frac{A_1\sin x_1 + A_{2s}\sin x_2}{A_1\cos x_1 + A_{2s}\cos x_2},$$

$$\hat{\kappa}_c = \frac{\sqrt{A_1^2 + A_{2s}^2 + 2A_1A_{2s}\cos(x_1 - x_2)}}{A_1 + A_{2s}}.$$

For the opposite group of neurons,

$$\hat{s}_o = \arctan\frac{A_1\sin x_1 - A_{2s}\sin x_2}{A_1\cos x_1 - A_{2s}\cos x_2},$$

$$\hat{\kappa}_o = \frac{\sqrt{A_1^2 + A_{2s}^2 - 2A_1A_{2s}\cos(x_1 - x_2)}}{A_1 + A_{2s}}.$$

As shown in Fig. 1, the fit to the simulation results of the network mean is comparable to that of Wenhao's approach in Fig. 6F, although the agreement for the opposite group of neurons is less impressive.

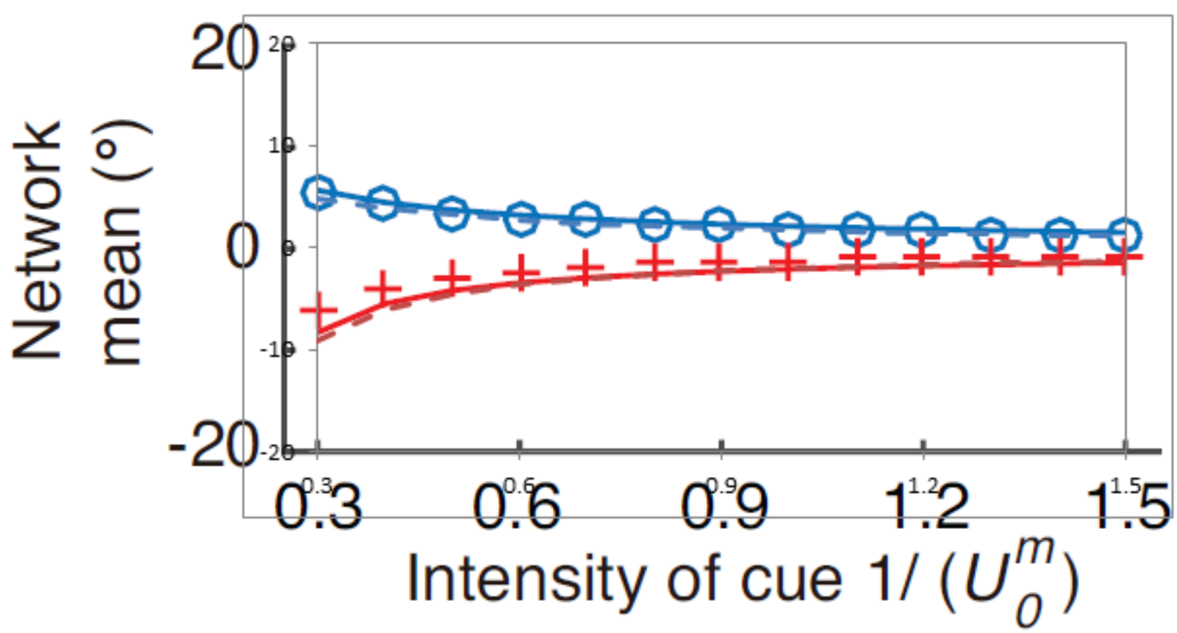


Fig. 1: Comparison of the results of the network mean (dashed line) and the results in Fig. 6F of the manuscript (solid line).

4.2. The Case of Equal Stimulus Strength

In Figs. 6G to 6I, $\alpha_1 = \alpha_2 = \alpha$. The solution becomes $u_1 = u_2 = u$, and , $\bar{u}_1 = \bar{u}_2 = \bar{u}$, Then

$$\frac{u}{u_c} \approx \frac{J_{rc} + J_{rp}}{J_c} \frac{2u^2/u_c^2}{1 + \frac{u^2 + J_{\text{int}}\bar{u}^2}{(1 + J_{\text{int}})u_c^2}} + A.$$
$$\frac{\bar{u}}{u_c} \approx \frac{J_{rc} + J_{rp}}{J_c} \frac{2\bar{u}^2/u_c^2}{1 + \frac{\bar{u}^2 + J_{\text{int}}u^2}{(1 + J_{\text{int}})u_c^2}} + A.$$

Inverting the relations,

$$\bar{u} = \sqrt{\left(\frac{1 + J_{\text{int}}}{J_{\text{int}}}\right)\left[\left(\frac{J_{rc} + J_{rp}}{J_c}\right)\frac{2u_c}{u - Au_c}u^2 - u_c^2\right] - \frac{u^2}{J_{\text{int}}}}.$$
$$u = \sqrt{\left(\frac{1 + J_{\text{int}}}{J_{\text{int}}}\right)\left[\left(\frac{J_{rc} + J_{rp}}{J_c}\right)\frac{2u_c}{\bar{u} - Au_c}\bar{u}^2 - u_c^2\right] - \frac{\bar{u}^2}{J_{\text{int}}}}.$$

For the cases we studied, we only found the symmetric solution with $u = \bar{u}$.

In summary, results can be obtained from the following steps.

The case of equal stimulus strength
1) For given values of $J_{rc}/J_c, J_{rp}/J_c$ and α, solve u/u_c from $$\frac{u}{u_c} \approx \frac{J_{rc} + J_{rp}}{J_c} \frac{2u^2}{u_c^2 + u^2} + A.$$ 2) For given values of x_1 and x_2, calculate $g \equiv \frac{J_{rp}}{J_c}\frac{2u^2}{u_c^2+u^2}$. Then $A_s \equiv Ag/(A + g)$. For the congruent group of neurons, $$\hat{s}_c = \arctan\frac{A\sin x_1 + A_s\sin x_2}{A\cos x_1 + A_s\cos x_2},$$ $$\hat{\kappa}_c = \frac{\sqrt{A^2 + A_s^2 + 2AA_s\cos(x_1 - x_2)}}{A + A_s}.$$ For the opposite group of neurons, $$\hat{s}_o = \arctan\frac{A\sin x_1 - A_s\sin x_2}{A\cos x_1 - A_s\cos x_2},$$ $$\hat{\kappa}_o = \frac{\sqrt{A^2 + A_s^2 - 2AA_s\cos(x_1 - x_2)}}{A + A_s}.$$

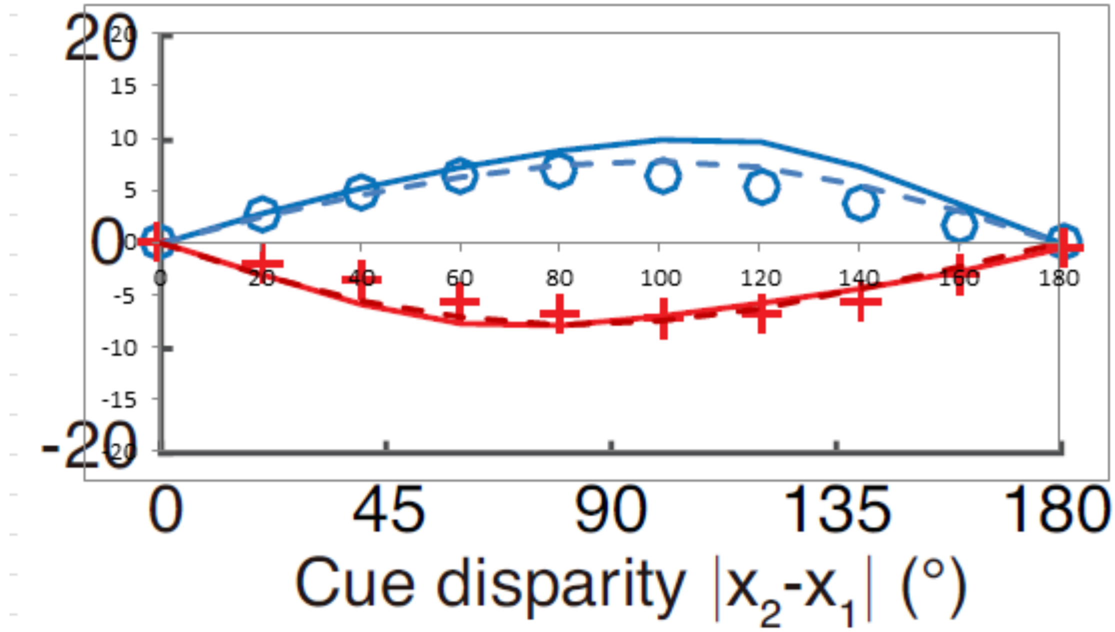


Fig. 2: Comparison of the results of the network mean (dashed line) and the results in Fig. 6G of the manuscript (solid line).

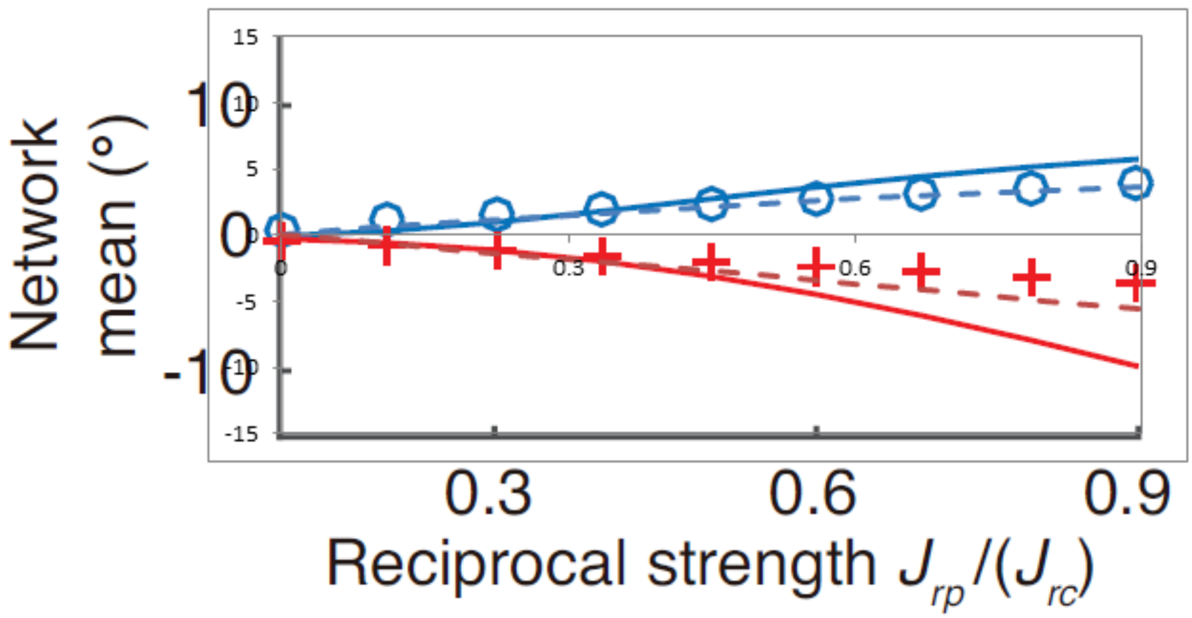


Fig. 3: Comparison of the results of my estimate (dashed line) and the results in Fig. 6H of the manuscript (solid line).

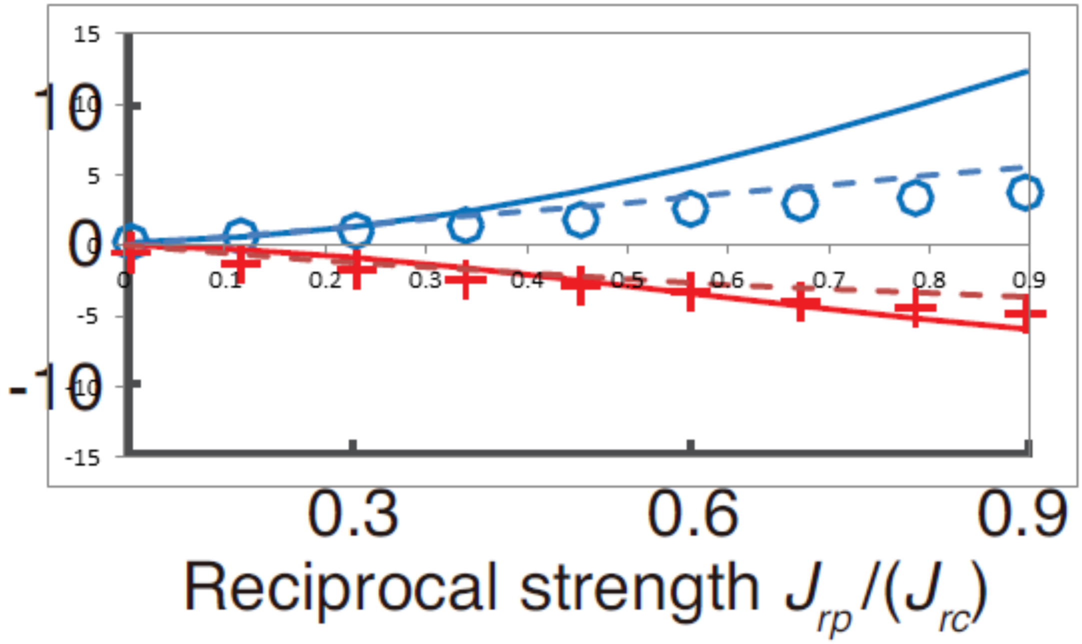


Fig. 4: Comparison of the results of my estimate (dashed line) and the results in Fig. 6I of the manuscript (solid line).

As shown in Figs. 2 to 4, and also considering Fig. 1, there is a general improvement in the fit to the simulation results when compared with Wenhao's approach, especially for the congruent module in Fig. 4.

5. Noisy Dynamics

Next we consider whether the simulation results of the decoded concentrations in Figs. 6F to I agree with the theoretical predictions. In the manuscript, the concentrations are determined by the distribution of the decoded directions in the presence of noise. Hence we have to consider the steady state of the dynamics of the congruent group of neurons in module 1 in the presence of noise, which can be approximated by

$$\tau\frac{\partial}{\partial t}u_1V(y_1-s_1,\kappa_{1u}) = -u_1V(y_1-s_1,\kappa_{1u}) + \rho J_{rc}\frac{u_1^2}{D_1}\frac{I_0(2\kappa_{1u})}{2\pi I_0(\kappa_{1u})^2}V(y_1-s_1,\kappa_{1ar})$$
$$+\rho J_{rp}\frac{u_2^2}{D_2}\frac{I_0(2\kappa_{2u})}{2\pi I_0(\kappa_{2u})^2}V(y_1-s_2,\kappa_{2ar}) + I_1V\left(y_1-x_1,\frac{a}{2}\right) + \sqrt{FI_1V(y_1-x_1,\kappa_{1u})}\xi_1(y_1,t) + I_b$$
$$+\sqrt{FI_b}\epsilon_1^c(y_1,t),$$

where, again, $I_1 = \alpha_1 e^{-a/2}2\pi I_0(a/2)$ and $A_1 = I_1/u_c$. Considering the dynamics of the position mode only,

$$\tau\frac{\partial s_1}{\partial t}\frac{a}{2}\sin(y_1-s_1)\,u_1V\left(y_1-s_1,\frac{a}{2}\right) \approx -u_1V\left(y_1-s_1,\frac{a}{2}\right) + \rho J_{rc}\frac{u_1^2}{D_1}\frac{I_0(a)}{2\pi I_0(a/2)^2}V\left(y_1-s_1,\frac{a}{2}\right)$$

$$+\rho J_{rp}\frac{u_2^2}{D_2}\frac{I_0(a)}{2\pi I_0(a/2)^2}V\left(y_1-s_2,\frac{a}{2}\right)+I_1V\left(y_1-x_1,\frac{a}{2}\right)+\sqrt{FI_1V\left(y_1-x_1,\frac{a}{2}\right)}\,\xi_1(y_1,t)+I_b$$
$$+\sqrt{FI_b}\epsilon_1^c(y_1,t).$$

Multiplying both sides by e^{jy_1} and integrating over y_1,

$$\left[1-A_2\left(\frac{a}{2}\right)\right]\frac{\tau a}{4}u_1\frac{\partial e^{js_1}}{\partial t}\approx -u_1A\left(\frac{a}{2}\right)e^{js_1}+\rho J_{rc}\frac{u_1^2}{D_1}\frac{I_0(a)}{2\pi I_0(a/2)^2}A\left(\frac{a}{2}\right)e^{js_1}$$
$$+\rho J_{rp}\frac{u_2^2}{D_2}\frac{I_0(a)}{2\pi I_0\left(\frac{a}{2}\right)^2}A\left(\frac{a}{2}\right)e^{js_2}+I_1A\left(\frac{a}{2}\right)e^{jx_1}+\beta_1\eta_1(t)+\gamma_1^c\varsigma_1^c(t),$$

where $A_2(\kappa)\equiv I_2(\kappa)/I_0(\kappa)$, $\langle\eta_1(t)\rangle=0$, $\langle\eta_1(t)\eta_1(t')\rangle=\delta\left(t-t'\right)$, $\langle\varsigma_1^c(t)\rangle=0$, $\langle\varsigma_1^c(t)\varsigma_1^c(t')\rangle=\delta\left(t-t'\right)$, and

$$\beta_1\eta_1(t)\equiv\sqrt{\frac{FI_1}{2\pi I_0(a/2)}}\int dy_1\,e^{\frac{a}{4}\cos(y_1-x_1)+jy_1}\xi_1(y_1,t),$$
$$\gamma_1^c\varsigma_1^c(t)\equiv\sqrt{FI_b}\int dy_1\,e^{jy_1}\epsilon_1^c(y_1,t).$$

In simulations, the positions of the neurons are discretized. So the integrals are replaced by summations,

$$\beta_1\eta_1(t)\equiv\sqrt{\frac{FI_1}{2\pi I_0(a/2)}}\sum_i\Delta y e^{\frac{a}{4}\cos(y_1-x_1)+jy_1}\xi_{1i}(t),$$
$$\gamma_1^c\varsigma_1^c(t)\equiv\sqrt{FI_b}\sum_i\Delta y e^{jy_1}\epsilon_{1i}^c(t),$$

where $\Delta y\equiv\frac{N}{2\pi}=\frac{1}{\rho}$. Perturbations from the steady state: Let $s_1=\langle s_1\rangle+\delta s_1$, where

$$0\approx -u_1A\left(\frac{a}{2}\right)e^{j\langle s_1\rangle}+\rho J_{rc}\frac{u_1^2}{D_1}\frac{I_0(a)}{2\pi I_0(a/2)^2}A\left(\frac{a}{2}\right)e^{j\langle s_1\rangle}+\rho J_{rp}\frac{u_2^2}{D_2}\frac{I_0(a)}{2\pi I_0\left(\frac{a}{2}\right)^2}A\left(\frac{a}{2}\right)e^{j\langle s_2\rangle}$$
$$+I_1A\left(\frac{a}{2}\right)e^{jx_1}.$$

Also note that

$I_0(x)-I_2(x)=\int_{-\pi}^{\pi}\frac{d\theta}{2\pi}\exp(x\cos\theta)\,(1-\cos 2\theta)=2\int_{-\pi}^{\pi}\frac{d\theta}{2\pi}\exp(x\cos\theta)\sin^2\theta=$
$-\frac{1}{\pi x}\int_{-\pi}^{\pi}\sin\theta\, d\exp(x\cos\theta)=\frac{2}{x}\int_{-\pi}^{\pi}\frac{d\theta}{2\pi}\exp(x\cos\theta)\cos\theta=\frac{2I_1(x)}{x}.$

Subtracting,

$$\tau u_1 A\left(\frac{a}{2}\right)\frac{\partial}{\partial t}\delta e^{js_1} \approx -u_1 A\left(\frac{a}{2}\right)\delta e^{js_1} + \rho J_{rc}\frac{u_1^2}{D_1}\frac{I_0(a)}{2\pi I_0(a/2)^2}A\left(\frac{a}{2}\right)\delta e^{js_1}$$
$$+\rho J_{rp}\frac{u_2^2}{D_2}\frac{I_0(a)}{2\pi I_0\left(\frac{a}{2}\right)^2}A\left(\frac{a}{2}\right)\delta e^{js_2} + \beta_1\eta_1(t) + \gamma_1^c\varsigma_1^c(t).$$

We are interested in the fluctuations of the directions. Hence we have

$$\tau\frac{\partial}{\partial t}\delta s_1 \approx -\frac{u_c}{u_1}\left[\frac{u_1}{u_c} - \frac{J_{rc}}{J_c}\frac{u_1^2}{D_1}\frac{I_0(a)}{2\pi I_0(a/2)^2}\frac{\rho J_c}{u_c}\right]\delta s_1$$
$$+\frac{u_c}{u_1}\frac{J_{rp}}{J_c}\frac{u_2^2}{D_2}\frac{I_0(a)}{2\pi I_0\left(\frac{a}{2}\right)^2}\frac{\rho J_c}{u_c}e^{j(\langle s_2\rangle-\langle s_1\rangle)}\delta s_2 - \frac{je^{-j\langle s_1\rangle}}{A\left(\frac{a}{2}\right)u_1}[\beta_1\eta_1(t) + \gamma_1^c\varsigma_1^c(t)].$$

Taking the real part (the imaginary part is related to amplitude fluctuations),

$$\tau\frac{\partial}{\partial t}\delta s_1 \approx -\frac{u_c}{u_1}g_{11}\delta s_1 + \frac{u_c}{u_1}g_{12}\cos(\langle s_2\rangle - \langle s_1\rangle)\,\delta s_2 + \mathrm{Im}\frac{[\beta_1\eta_1(t) + \gamma_1^c\varsigma_1^c(t)]e^{-j\langle s_1\rangle}}{A\left(\frac{a}{2}\right)u_1}.$$

The equivalent noise temperature is given by

$$T_1 = \frac{1}{2\left[A\left(\frac{a}{2}\right)u_1\right]^2}\left[\frac{FI_1}{2\pi I_0(a/2)\rho}\int dy_1\, e^{\frac{a}{2}\cos(y_1-x_1)}\sin^2(y_1-\langle s_1\rangle) + \frac{FI_b}{\rho}\int dy_1 \sin^2(y_1-\langle s_1\rangle)\right]$$
$$= \frac{\frac{F}{u_c}A_1\left[1-A_2\left(\frac{a}{2}\right)\cos(2x_1-2\langle s_1\rangle)\right]+2\pi\frac{F}{u_c}\left(\frac{I_b}{u_c}\right)}{\rho\left[2A\left(\frac{a}{2}\right)\frac{u_1}{u_c}\right]^2}.$$

($A_b = I_b/u_c$.) Together with the dynamics of module 2,

$$\tau\frac{\partial}{\partial t}\begin{pmatrix}\delta s_1\\ \delta s_2\end{pmatrix} \approx -\begin{pmatrix}\frac{u_c}{u_1}g_{11} & -\frac{u_c}{u_1}g_{12}\cos(\langle s_2\rangle-\langle s_1\rangle)\\ -\frac{u_c}{u_2}g_{21}\cos(\langle s_2\rangle-\langle s_1\rangle) & \frac{u_c}{u_2}g_{22}\end{pmatrix}\begin{pmatrix}\delta s_1\\ \delta s_2\end{pmatrix} +$$
$$\begin{pmatrix}\mathrm{Im}\frac{[\beta_1\eta_1(t)+\gamma_1^c\varsigma_1^c(t)]e^{-j\langle s_1\rangle}}{A\left(\frac{a}{2}\right)u_1}\\ \mathrm{Im}\frac{[\beta_2\eta_2(t)+\gamma_2^c\varsigma_2^c(t)]e^{-j\langle s_2\rangle}}{A\left(\frac{a}{2}\right)u_2}\end{pmatrix}.$$

Consider the dynamics $\tau\frac{\partial x_i}{\partial t} = -\sum_j g_{ij}x_j + \xi_i$ where $\langle\xi_i(t)\rangle = 0$ and $\langle\xi_i(t)\xi_j(t')\rangle = 2T_i\delta_{ij}\delta(t-t')$. Diagonalizing the matrix, $g_{ij} = \sum_k S_{ik}\lambda_k S_{kj}^{-1}$. Then

$$\tau \frac{\partial}{\partial t}\sum_j S_{ij}^{-1}x_j = -\lambda_i \sum_j S_{ij}^{-1}x_j + \sum_j S_{ij}^{-1}\xi_j.$$

The solution is

$$x_i(t) = \sum_j S_{ij}\int_{-\infty}^{t}\frac{dt'}{\tau}\exp\left[-\frac{\lambda_j}{\tau}\left(t-t'\right)\right]S_{jk}^{-1}\xi_k(t').$$

Consider the moments of the distribution of $x_i(t)$,

$$\langle x_i^{2n}(t)\rangle = \sum_{j_1\cdots j_{2n}k_1\cdots k_{2n}} S_{ij_1}\cdots S_{ij_{2n}}\int_{-\infty}^{t}\frac{dt_1}{\tau}\exp\left[-\frac{\lambda_{j_1}}{\tau}(t-t_1)\right]\cdots\int_{-\infty}^{t}\frac{dt_{2n}}{\tau}\exp\left[-\frac{\lambda_{j_{2n}}}{\tau}(t - t_{2n})\right]S_{j_1k_1}^{-1}\cdots S_{j_{2n}k_{2n}}^{-1}\langle\xi_{k_1}(t_1)\cdots\xi_{k_{2n}}(t_{2n})\rangle.$$

The number of ways the pairing of the noise terms can be done is $(2n)!/(2!^n\, n!) = (2n-1)!!$. Hence

$$\langle x_i^{2n}(t)\rangle = (2n-1)!!\left[\sum_{j_1j_2k_1k_2} S_{ij_1}S_{ij_2}\int_{-\infty}^{t}\frac{dt_1}{\tau}\exp\left[-\frac{\lambda_{j_1}}{\tau}(t-t_1)\right]\int_{-\infty}^{t}\frac{dt_2}{\tau}\exp\left[-\frac{\lambda_{j_2}}{\tau}(t - t_2)\right]S_{j_1k_1}^{-1}S_{j_2k_2}^{-1}2T_{k_1}\delta_{k_1k_2}\delta(t_1-t_2)\right]^n.$$

After integrating,

$$\langle x_i^{2n}(t)\rangle = (2n-1)!!\left[\sum_{jkl} S_{ij}S_{ik}\frac{2T_l}{(\lambda_j+\lambda_k)\tau}S_{jl}^{-1}S_{kl}^{-1}\right]^n.$$

Hence the distribution of $x_i(t)$ is a Gaussian with variance

$$\sigma_i^2 = \sum_{jkl} S_{ij}S_{ik}\frac{2T_l}{(\lambda_j+\lambda_k)\tau}S_{jl}^{-1}S_{kl}^{-1}.$$

For the matrix $\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22}\end{pmatrix}$,

$\lambda_\pm = \frac{1}{2}\left(g_{11}+g_{22}\pm\sqrt{D}\right)$ where $D = (g_{22}-g_{11})^2 + 4g_{12}g_{21}$,
$S = \begin{pmatrix} -g_{12} & -g_{12} \\ \frac{1}{2}(g_{11}-g_{22}-\sqrt{D}) & \frac{1}{2}(g_{11}-g_{22}+\sqrt{D})\end{pmatrix}$,
$\det S = -g_{12}\sqrt{D}$,
$S^{-1} = \frac{1}{g_{12}\sqrt{D}}\begin{pmatrix} \frac{1}{2}(g_{22}-g_{11}-\sqrt{D}) & -g_{12} \\ \frac{1}{2}(g_{11}-g_{22}-\sqrt{D}) & g_{12}\end{pmatrix}$,

$$\sigma_1^2 = \left[\frac{\left(S_{11}S_{11}^{-1}\right)^2}{\lambda_1} + \frac{\left(S_{12}S_{21}^{-1}\right)^2}{\lambda_2} + \frac{4\left(S_{11}S_{11}^{-1}\right)\left(S_{12}S_{21}^{-1}\right)}{\lambda_1+\lambda_2}\right]\frac{T_1}{\tau} + \left[\frac{\left(S_{11}S_{12}^{-1}\right)^2}{\lambda_1} + \frac{\left(S_{12}S_{22}^{-1}\right)^2}{\lambda_2} + \frac{4\left(S_{11}S_{12}^{-1}\right)\left(S_{12}S_{22}^{-1}\right)}{\lambda_1+\lambda_2}\right]\frac{T_2}{\tau}$$

Coefficient of T_1/τ

$$= \frac{(g_{22}-g_{11}-\sqrt{D})^2(g_{11}+g_{22}-\sqrt{D})}{8D(g_{11}g_{22}-g_{12}g_{21})} + \frac{\left((g_{22}-g_{11}+\sqrt{D})\right)^2(g_{11}+g_{22}+\sqrt{D})}{8D(g_{11}g_{22}-g_{12}g_{21})} + \frac{4g_{12}g_{21}}{D(g_{11}+g_{22})}$$

$$= \frac{1}{8D(g_{11}g_{22}-g_{12}g_{21})}\{\left[(g_{22}-g_{11})^2 - 2(g_{22}-g_{11})\sqrt{D} + D\right]\left(g_{11}+g_{22}-\sqrt{D}\right) + \left(\sqrt{D} \to -\sqrt{D}\right)\} + \frac{4g_{12}g_{21}}{D(g_{11}+g_{22})}$$

$$= \frac{4g_{22}D - 4g_{12}g_{21}(g_{22}+g_{11})}{4D(g_{11}g_{22}-g_{12}g_{21})} + \frac{4g_{12}g_{21}}{D(g_{11}+g_{22})}$$

$$= \frac{g_{22}}{g_{11}g_{22}-g_{12}g_{21}} - \frac{g_{12}g_{21}}{(g_{11}g_{22}-g_{12}g_{21})(g_{11}+g_{22})}$$

$$= \frac{1}{g_{11}+g_{22}} + \frac{g_{22}^2}{(g_{11}g_{22}-g_{12}g_{21})(g_{11}+g_{22})}$$

Coefficient of T_2/τ

$$= \frac{2g_{12}^2}{D\left(g_{11}+g_{22}+\sqrt{D}\right)} + \frac{2g_{12}^2}{D\left(g_{11}+g_{22}-\sqrt{D}\right)} - \frac{4g_{12}^2}{D(g_{11}+g_{22})}$$

$$= \frac{g_{12}^2}{(g_{11}g_{22}-g_{12}g_{21})(g_{11}+g_{22})}$$

Summarizing,

$$\sigma_1^2 = \frac{T_1}{(g_{11}+g_{22})\tau} + \frac{g_{22}^2T_1 + g_{12}^2T_2}{(g_{11}g_{22}-g_{12}g_{21})(g_{11}+g_{22})\tau}$$

Returning to the equation of the congruent group of neurons in module 1, the variance σ_1^2 is given by

$$\sigma_1^2 = \frac{T_1}{\left(\frac{u_c}{u_1}g_{11}+\frac{u_c}{u_2}g_{22}\right)\tau} + \frac{T_1\left(\frac{u_c}{u_2}g_{22}\right)^2 + T_2\left(\frac{u_c}{u_1}g_{12}\right)^2\cos^2(s_2-s_1)}{\left[\left(\frac{u_c}{u_1}g_{11}\right)\left(\frac{u_c}{u_2}g_{22}\right)-\left(\frac{u_c}{u_1}g_{12}\right)\left(\frac{u_c}{u_2}g_{21}\right)\cos^2(s_2-s_1)\right]\left(\frac{u_c}{u_1}g_{11}+\frac{u_c}{u_2}g_{22}\right)\tau}.$$

In summary, results can be obtained from the following steps.

Decoded concentration
$$T_1 = \frac{\frac{F}{u_c}A_1\left[1-A_2\left(\frac{a}{2}\right)\cos(2x_1-2s_1)\right] + 2\pi\frac{F}{u_c}\left(\frac{I_b}{u_c}\right)}{\rho\left[2A\left(\frac{a}{2}\right)\frac{u_1}{u_c}\right]^2}.$$ $$T_2 = \frac{\frac{F}{u_c}A_2\left[1-A_2\left(\frac{a}{2}\right)\cos(2x_2-2s_2)\right] + 2\pi\frac{F}{u_c}\left(\frac{I_b}{u_c}\right)}{\rho\left[2A\left(\frac{a}{2}\right)\frac{u_2}{u_c}\right]^2}.$$ $$\sigma_1^2 = \frac{T_1}{\left(\frac{u_c}{u_1}g_{11}+\frac{u_c}{u_2}g_{22}\right)\tau} + \frac{T_1\left(\frac{u_c}{u_2}g_{22}\right)^2 + T_2\left(\frac{u_c}{u_1}g_{12}\right)^2\cos^2(s_2-s_1)}{\left[\left(\frac{u_c}{u_1}g_{11}\right)\left(\frac{u_c}{u_2}g_{22}\right)-\left(\frac{u_c}{u_1}g_{12}\right)\left(\frac{u_c}{u_2}g_{21}\right)\cos^2(s_2-s_1)\right]\left(\frac{u_c}{u_1}g_{11}+\frac{u_c}{u_2}g_{22}\right)\tau}.$$ $$\kappa_1 = \frac{1}{\sigma_1^2}.$$

As shown in Fig. 5 to 8, the network concentration of the congruent neurons has similar order of magnitudes as the simulation results. However, the network concentration of the opposite group of neurons is effectively the same. In Figs. 6 to 8, the network concentrations are effectively independent of the disparity. The theory needs improvement.

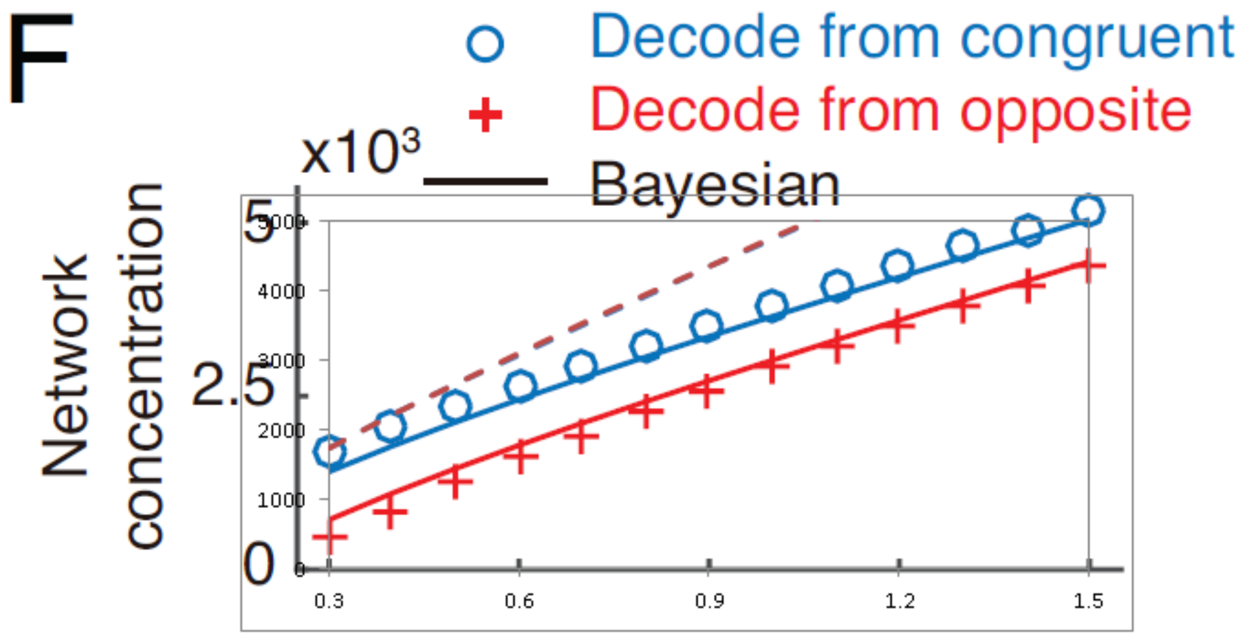


Fig. 5: Comparison of the results of the network concentration and the results in Fig. 6F of the manuscript.

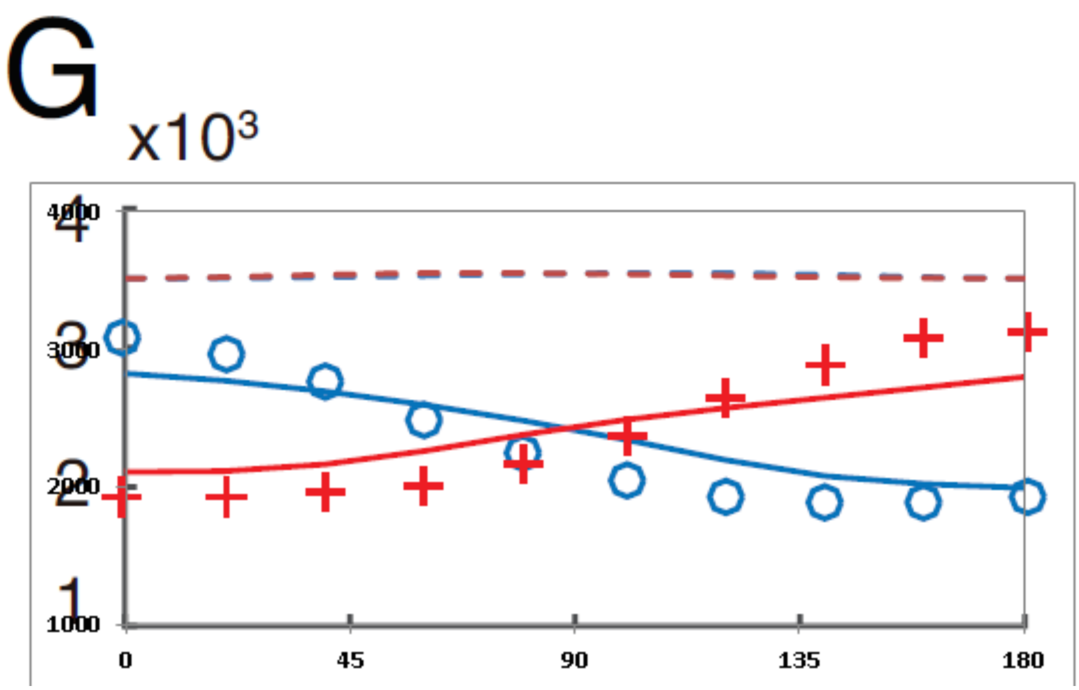


Fig. 6: Comparison of the results of the network concentration and the results in Fig. 6G of the manuscript.

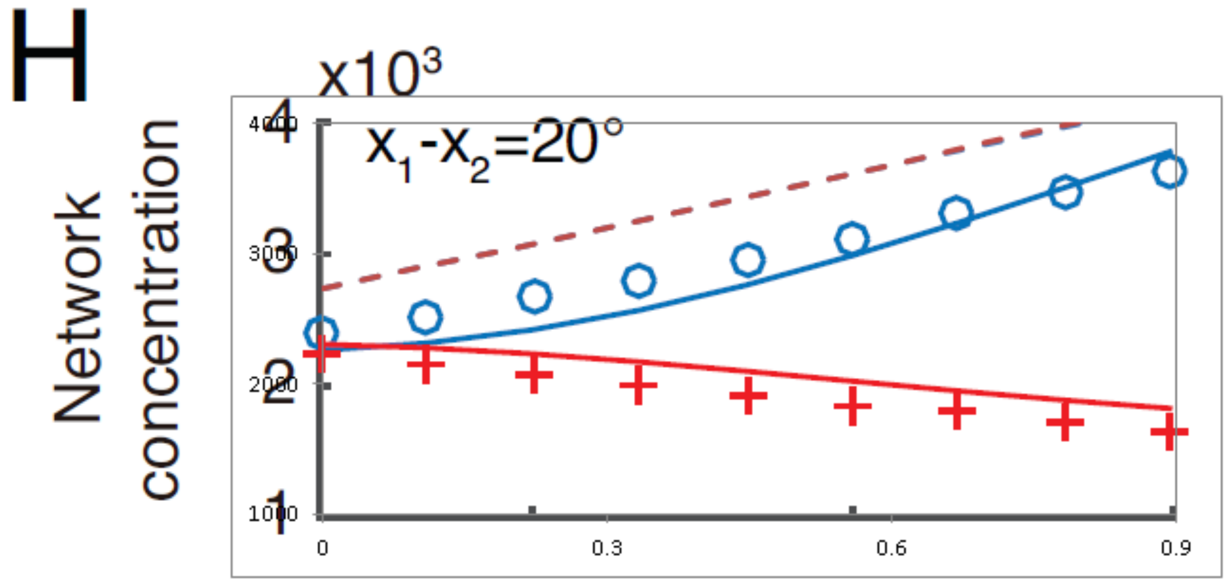


Fig. 7: Comparison of the results of the network concentration and the results in Fig. 6H of the manuscript.

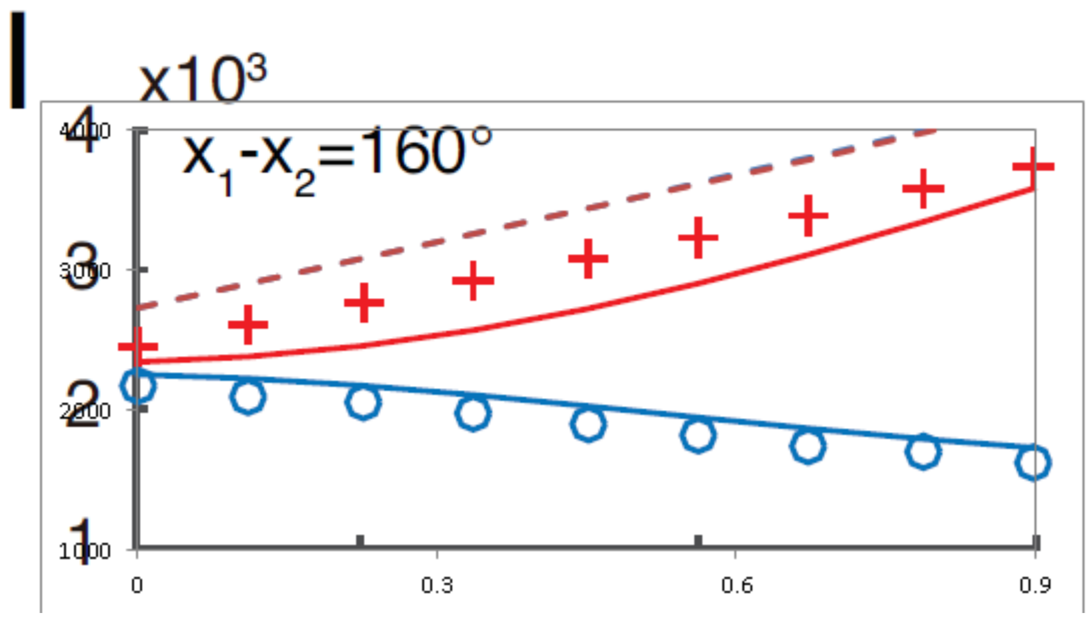


Fig. 8: Comparison of the results of the network concentration and the results in Fig. 6I of the manuscript.

6. Expectation of Simple Sum Decoding

In the manuscript, Wenhao introduced the simple sum decoding $r(x_1, x_2) = r(x_1) + r(x_2)$ for multisensory integration (page 5). The argument is based on the logarithmic relation $\ln p(s_1|x_1, x_2) = \ln p(s_1|x_1) + \ln p(s_1|x_2)$, and the similarity with a theorem proposed by Ma *et* *al*. Similarly, Wenhao introduced $r(x_1, x_2) = r(x_1) + r(x_2 + \pi)$ for multisensory segregation, based on the logarithmic relation $\ln p_d(s_1|x_1, x_2) = \ln p(s_1|x_1) + \ln p(s_1|x_2 + \pi)$. He further introduced in Fig. 7C a similar decoding to recover the posterior distribution of module 1, based on the logarithmic relation in Eq. (16).

Here we discuss the result of this simple sum decoding method, and compare it with the Bayesian result.

Consider the equation $r(x_1, x_2) = r(x_1) + r(x_2)$. Let $r_1(y_i)$ is the firing rate of the neuron (whose preferred stimulus is) y_i in the congruent group of module 1 when the input to module 1 only is centered at x_1. Assuming that the network is trained to yield the optimal output, then the distribution of its firing rate is given by the likelihood function

$$p\big(r_1(y_i)\big) = \frac{\exp[\kappa_1 \cos(y_i - x_1)]}{2\pi I_0(\kappa_1)}.$$

The decoding of $r_1(y_i)$ yields the following estimates of mean and concentration,

$$\hat{s}_1 = \arctan \frac{\sum_i r_1(y_i) \sin y_i}{\sum_i r_1(y_i) \cos y_i},$$

The expectation of the numerator is given by

$$\frac{1}{N} \sum_i r_1(y_i) \sin y_i = \int_{-\pi}^{\pi} dy_i \frac{\exp[\kappa_1 \cos(y_i - x_1)]}{2\pi I_0(\kappa_1)} \sin y_i = A(\kappa_1) \sin x_1.$$

Evaluating the denominator similarly, we obtain $\hat{s}_1 = x_1$. The expectation of the concentration is obtained by calculating

$$\sqrt{\left[\frac{\sum_i r_1(y_i) \cos y_i}{\sum_i r_1(y_i)}\right]^2 + \left[\frac{\sum_i r_1(y_i) \sin y_i}{\sum_i r_1(y_i)}\right]^2} = \sqrt{[A(\hat{\kappa}_1) \cos x_1]^2 + [A(\hat{\kappa}_1) \sin x_1]^2} = A(\hat{\kappa}_1).$$

Hence the decoded concentration is given by

$$\hat{\kappa}_1 = A^{-1} \left\{ \sqrt{\left[\frac{\sum_i r_1(y_i) \cos y_i}{\sum_i r_1(y_i)}\right]^2 + \left[\frac{\sum_i r_1(y_i) \sin y_i}{\sum_i r_1(y_i)}\right]^2} \right\}.$$

Similarly, when the input to module 2 only is centered at x_2, then the distribution of the firing rate of the neuron y_i is given by the likelihood function

$$p\big(r_2(y_i)\big) = \frac{\exp[\kappa_{2s} \cos(y_i - x_2)]}{2\pi I_0(\kappa_{2s})},$$

where $\kappa_{2s} = A^{-1}[A(\kappa_2)A(\kappa_s)]$. Then the decoded mean and concentration of $r_2(y_i)$ are given by

$$\hat{s}_1 = \arctan\frac{\sum_i r_2(y_i)\sin y_i}{\sum_i r_2(y_i)\cos y_i},$$
$$\hat{\kappa}_2 = A^{-1}\left\{\sqrt{\left[\frac{\sum_i r_2(y_i)\cos y_i}{\sum_i r_2(y_i)}\right]^2 + \left[\frac{\sum_i r_2(y_i)\sin y_i}{\sum_i r_2(y_i)}\right]^2}\right\}.$$

For the simple sum rule, the decoded direction is given by

$$\hat{s}_1 = \arctan\frac{\sum_i[r_1(y_i) + r_2(y_i)]\sin y_i}{\sum_i[r_1(y_i) + r_2(y_i)]\cos y_i},$$

Its expectation is given by

$$\hat{s}_1 = \arctan\frac{A(\kappa_1)\sin x_1 + A(\kappa_{2s})\sin x_2}{A(\kappa_1)\cos x_1 + A(\kappa_{2s})\cos x_2}.$$

The concentration of the simple sum rule is decoded by

$$\hat{\kappa}_1 = A^{-1}\left\{\sqrt{\left\{\frac{\sum_i[r_1(y_i) + r_2(y_i)]\cos y_i}{\sum_i[r_1(y_i) + r_2(y_i)]}\right\}^2 + \left\{\frac{\sum_i[r_1(y_i) + r_2(y_i)]\sin y_i}{\sum_i[r_1(y_i) + r_2(y_i)]}\right\}^2}\right\}.$$

The expectation is given by

$$\hat{\kappa}_1 = A^{-1}\left\{\frac{\sqrt{A(\kappa_1)^2 + A(\kappa_{2s})^2 + 2A(\kappa_1)A(\kappa_{2s})\cos(x_1 - x_2)}}{A(\kappa_1) + A(\kappa_{2s})}\right\}.$$

7. Comparison with Bayesian Result

In summary, the result of simple sum decoding is

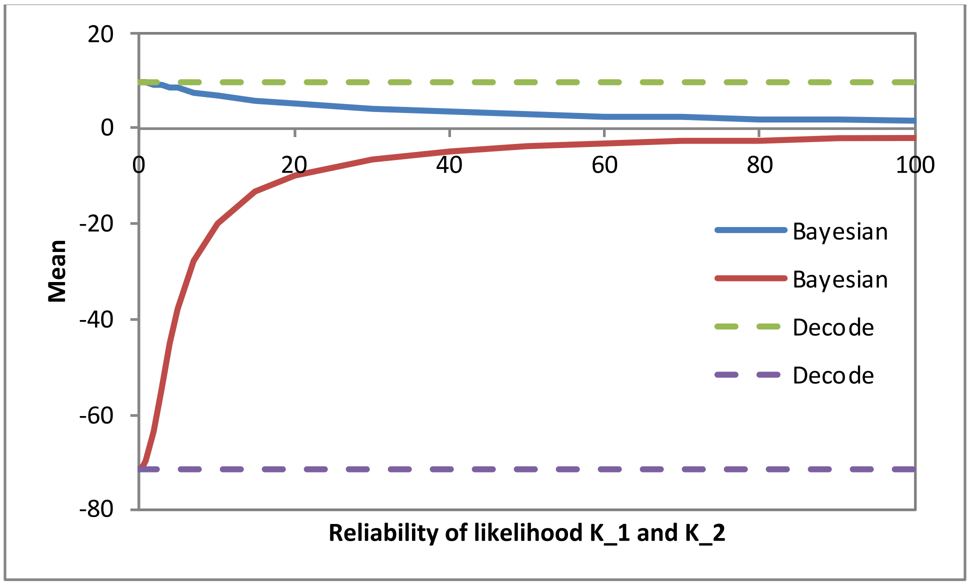
$$\hat{s}_1 = \arctan\frac{A(\kappa_1)\sin x_1 + A(\kappa_{2s})\sin x_2}{A(\kappa_1)\cos x_1 + A(\kappa_{2s})\cos x_2}.$$
$$\hat{\kappa}_1 = \sqrt{A(\kappa_1)^2 + A(\kappa_{2s})^2 + 2A(\kappa_1)A(\kappa_{2s})\cos(x_1 - x_2)}.$$

On the other hand, the result of Bayesian estimate is

$$\hat{s}_1 = \arctan\frac{\kappa_1\sin x_1 + \kappa_{2s}\sin x_2}{\kappa_1\cos x_1 + \kappa_{2s}\cos x_2}.$$

$$\hat{\kappa}_1 = A^{-1}\left\{\frac{\sqrt{A(\kappa_1)^2 + A(\kappa_{2s})^2 + 2A(\kappa_1)A(\kappa_{2s})\cos(x_1 - x_2)}}{A(\kappa_1) + A(\kappa_{2s})}\right\}.$$

Is the difference significant? To see this, we repeat the calculations in Fig. 3C-E of the manuscript. As shown in Figs. 7 to 9, the magnitudes and behaviors of the mean and concentration are very different.



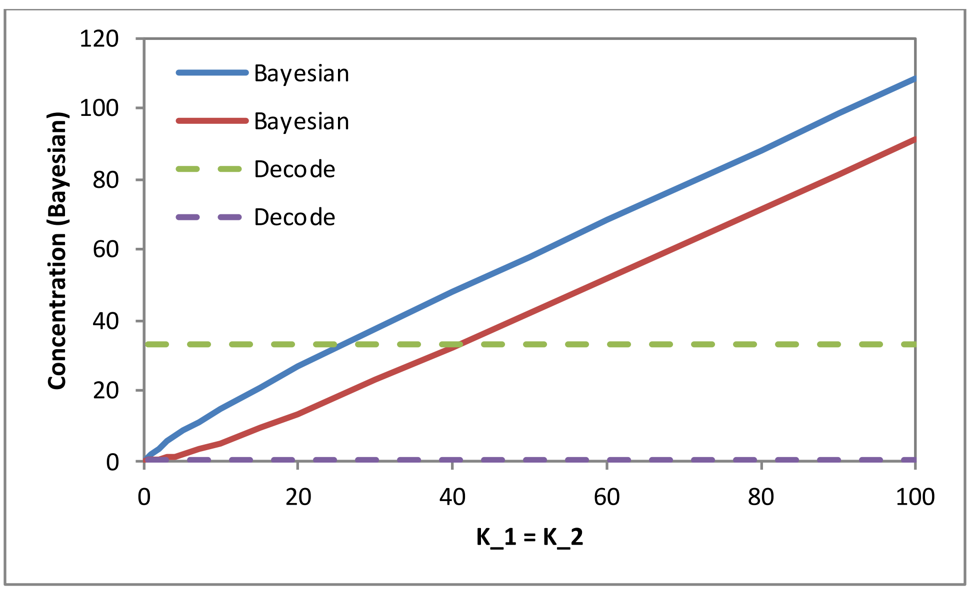
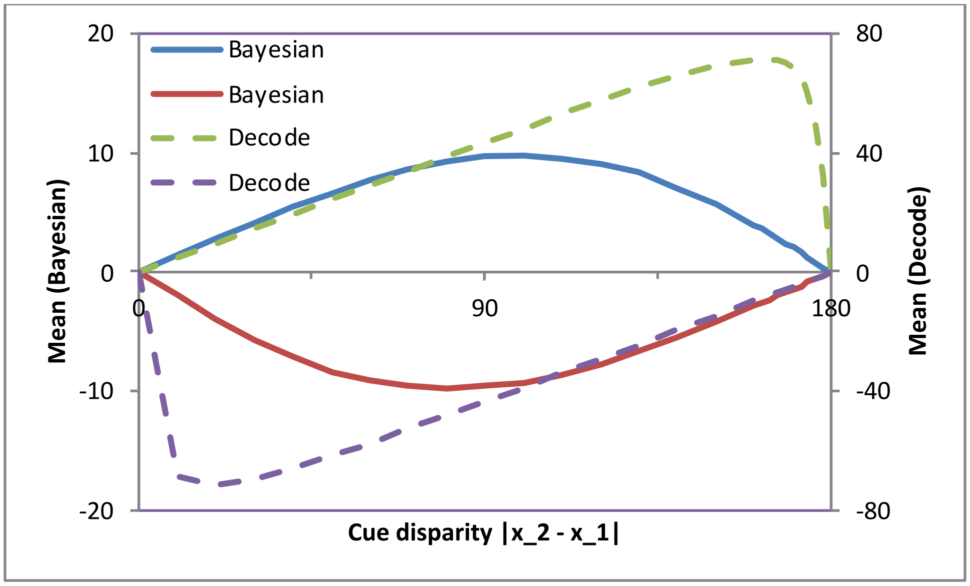


Fig. 7: Comparison of the results of the Bayesian estimate and the simple sum decoding in Fig. 3C of the manuscript.



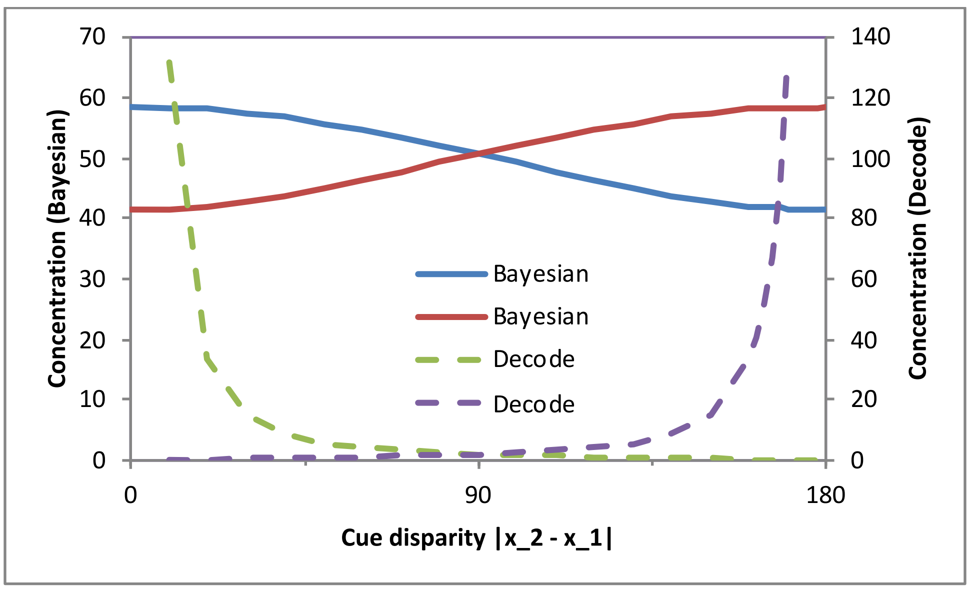
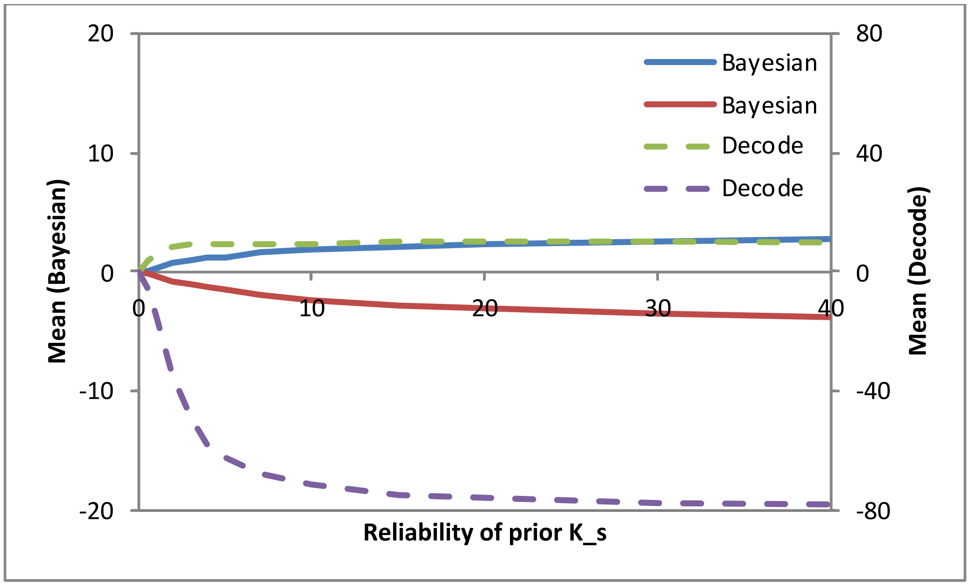


Fig. 8: Comparison of the results of the Bayesian estimate and the simple sum decoding in Fig. 3D of the manuscript.



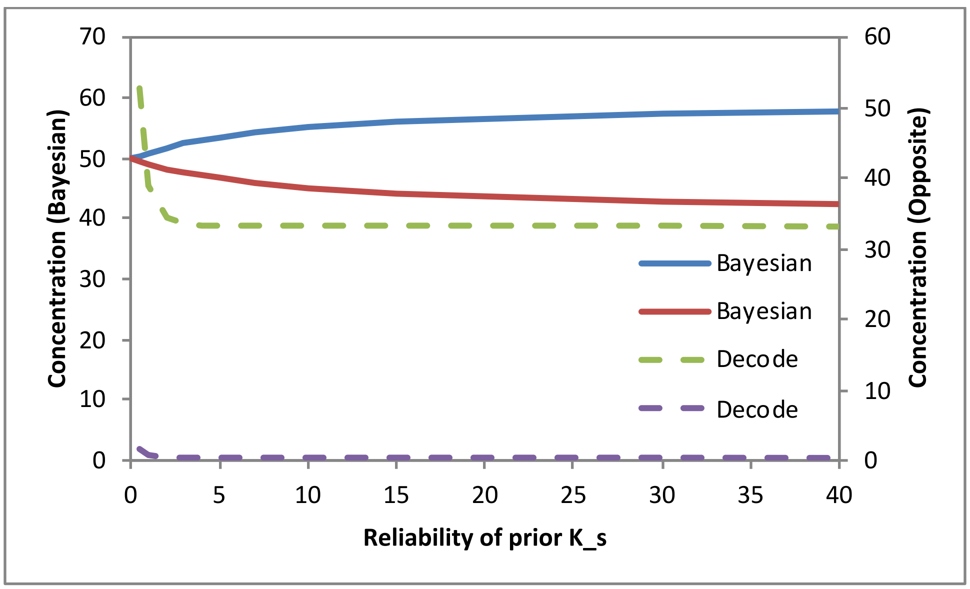


Fig. 9: Comparison of the results of the Bayesian estimate and the simple sum decoding in Fig. 3E of the manuscript.