28 May 2018

Bump Height Dependence of Estimates

1. Equal Stimulus Strength

In the model of vonMises7, the decoded concentrations of the congruent and opposite modules are effectively the same and independent of the disparity. This does not agree with the simulation results in Figs. 6F to I. Since it was assumed that the bump heights of the congruent and opposite groups of neurons are the same, here we consider whether the predictions can be improved by taking into consideration the different bump heights of the congruent and opposite groups of neurons.

We first focus on the case that the stimulus strengths are equal, that is, $A_1 = A_2$. In this case, $u_1 = u_2$, $\bar{u}_1 = \bar{u}_2$, implying $g_{11} = g_{22}$ and $g_{12} = g_{21}$. The solution is symmetric with respect to the perception displacement, that is, $s_1 - x_1 = x_2 - s_2$. Then

$$s_1 - x_1 = (x_2 - x_1) - (s_2 - s_1) - (x_2 - s_2) = \frac{1}{2}(x_2 - x_1) - \frac{1}{2}(s_2 - s_1).$$

However, in the previous approximation, we have to satisfy both of the following equations

$$\begin{cases} g_{11}e^{js_1} - g_{12}e^{js_2} = A_1 e^{jx_1}, \\ g_{11} - g_{12} = A_1. \end{cases}$$

Since the equations can only be satisfied with $s_1 = s_2 = x_1$, we have to take into account the existence of higher order modes, in which case the second equation is invalid after including the higher order modes. We need to use the projection method. The steady state of the dynamics of the congruent group of neurons in module 1 can be approximated by

$$\psi(y_1) = \frac{\rho J_{rc}}{D_1}\int dy_2 V(y_1 - y_2, a)\psi(y_2)^2 + \frac{\rho J_{rp}}{D_1}\int dy_2 V(y_1 - y_2, a)\bar{\psi}(y_2)^2 + I_1 V\left(y_1 - x_1, \frac{a}{2}\right) + I_b.$$

Let $\psi(y_1) = u_1 V(y_1 - s_1, \kappa_{1u})$ + higher order modes. Then

$$u_1 V(y_1 - s_1, \kappa_{1u}) \approx \rho J_{rc}\frac{u_1^2}{D_1}\frac{I_0(2\kappa_{1u})}{2\pi I_0(\kappa_{1u})^2}V(y_1 - s_1, \kappa_{1r}) + \rho J_{rp}\frac{u_2^2}{D_2}\frac{I_0(2\kappa_{2u})}{2\pi I_0(\kappa_{2u})^2}V(y_1 - s_2, \kappa_{2r}) + I_1 V\left(y_1 - x_1, \frac{a}{2}\right) + I_b,$$

where $I_1 = \alpha_1 e^{-a/2} 2\pi I_0(a/2)$. Approximating $\kappa_{1u} \approx \kappa_{2u} \approx a/2$ and $\kappa_{1r} \approx \kappa_{2r} \approx a_r = A^{-1}(A(a)^2)$, multiplying both sides by $V(y_1 - s_1, b)$ and integrating over y_1, we obtain

$$u_1 V(0, b_u) \approx \rho J_{rc}\frac{u_1^2}{D_1}\frac{I_0(a)}{2\pi I_0\left(\frac{a}{2}\right)^2}V(0, b_r) + \rho J_{rp}\frac{u_2^2}{D_2}\frac{I_0(a)}{2\pi I_0\left(\frac{a}{2}\right)^2}V(s_2 - s_1, b_r) + I_1 V(x_1 - s_1, b_u) + I_b,$$

where $b_u = A^{-1}(A(a/2)A(b))$ and $b_r = A^{-1}(A(a_r)A(b)) = A^{-1}\left(A(a)^2A(b)\right)$. The global inhibition factors are given by

$$D_n = 1 + \omega\sum_x \left[u_n^2 V(x, a/2)^2 + J_{\text{int}} u_{\bar{n}}^2 V(x, a/2)^2\right] = 1 + \frac{I_0(a)\omega\rho(u_n^2 + J_{\text{int}}\bar{u}_n^2)}{2\pi I_0(a/2)^2}.$$

For comparison, Wenhao introduced a network assuming that the synaptic inputs of the opposite group of neurons are the same as those of the congruent group.

$$u_1 \approx \rho J_{rc}\frac{u_1^2}{1 + \frac{I_0(a)\omega\rho(1+J_{\text{int}})}{2\pi I_0(a/2)^2}u_1^2}\frac{I_0(a)}{2\pi I_0\left(\frac{a}{2}\right)^2} + \rho J_{rp}\frac{u_2^2}{1 + \frac{I_0(a)\omega\rho(1+J_{\text{int}})}{2\pi I_0(a/2)^2}u_2^2}\frac{I_0(a)}{2\pi I_0\left(\frac{a}{2}\right)^2} + I_1.$$

In the absence of stimuli, we have $u_1 = u_2$. Thus

$$\omega\rho(1+J_{\text{int}})u_1^2 - \rho(J_{rp} + J_{rc})u_1 + \frac{2\pi I_0(a/2)^2}{I_0(a)} = 0.$$

$$u_1 = \frac{J_{rp} + J_{rc}}{2\omega(1+J_{\text{int}})} \pm \sqrt{\left[\frac{J_{rp} + J_{rc}}{2\omega(1+J_{\text{int}})}\right]^2 - \frac{2\pi I_0(a/2)^2}{I_0(a)\omega\rho(1+J_{\text{int}})}}.$$

The critical coupling of $J_{rp} + J_{rc}$ and the critical synaptic input are given by

$$J_c = \sqrt{\frac{8\pi I_0(a/2)^2\omega(1+J_{\text{int}})}{I_0(a)\rho}}.$$

$$u_c = \sqrt{\frac{2\pi I_0(a/2)^2}{I_0(a)\rho\omega(1+J_{\text{int}})}}.$$

They satisfy the equation

$$\frac{u_1^2}{u_c^2} - 2\left(\frac{J_{rp} + J_{rc}}{J_c}\right)\frac{u_1}{u_c} + 1 = 0.$$

In the presence of stimuli $I_m = A_m u_c$, the equations are modified. Let $\bar{u}_1$ be the synaptic input of the opposite group of neurons in module 1 and $I_b = A_b u_c$. Then

$$\frac{u_1}{u_c}V(0, b_u) \approx \frac{J_{rc}}{J_c}\frac{2u_1^2}{u_c^2 + \frac{u_1^2 + J_{\text{int}}\bar{u}_1^2}{1+J_{\text{int}}}}V(0, b_r) + \frac{J_{rp}}{J_c}\frac{2u_2^2}{u_c^2 + \frac{u_2^2 + J_{\text{int}}\bar{u}_2^2}{1+J_{\text{int}}}}V(s_2 - s_1, b_r) + A_1 V(x_1 - s_1, b_u) + A_b.$$

$$g_{11}V(0, b_u) - g_{12}V(s_2 - s_1, b_r) \approx A_1 V(x_1 - s_1, b_u) + A_b.$$

Here

$$g_{11} \equiv \frac{u_1}{u_c} - \frac{I_0(b_u)}{I_0(b_r)}\frac{J_{rc}}{J_c}\frac{2u_1^2/u_c^2}{1+\frac{u_1^2+J_{\text{int}}\bar{u}_1^2}{(1+J_{\text{int}})u_c^2}} \text{ and } g_{12} \equiv \frac{J_{rp}}{J_c}\frac{2u_2^2/u_c^2}{1+\frac{u_2^2+J_{\text{int}}\bar{u}_2^2}{(1+J_{\text{int}})u_c^2}}.$$

The steady state equation is also simplified,

$$\frac{u_1}{u_c}V\left(y_1 - s_1, \frac{a}{2}\right) \approx \frac{J_{rc}}{J_c}\frac{2u_1^2/u_c^2}{1+\frac{u_1^2+J_{\text{int}}\bar{u}_1^2}{(1+J_{\text{int}})u_c^2}}V(y_1 - s_1, a_r) + \frac{J_{rp}}{J_c}\frac{2u_2^2/u_c^2}{1+\frac{u_2^2+J_{\text{int}}\bar{u}_2^2}{(1+J_{\text{int}})u_c^2}}V(y_1 - s_2, a_r)$$
$$+A_1 V\left(y_1 - x_1, \frac{a}{2}\right) + A_b.$$

For the projection on to the displacement mode, consider

$$\int_{-\pi}^{\pi} dy_1 V(y_1 - s_1, a_1)\sin(y_1 - s_1)\,V(y_1 - s_2, a_2) = \frac{1}{a_1}\frac{\partial}{\partial s_1}\int_{-\pi}^{\pi} dy_1 V(y_1 - s_1, a_1)V(y_1 - s_2, a_2)$$
$$= \frac{1}{a_1}\frac{\partial}{\partial s_1}V(s_2 - s_1, a_{12}) = V(s_2 - s_1, a_{12})\sin(s_2 - s_1)\frac{a_{12}}{a_1}$$

Hence the projection of $V(y_1 - s_2, a_2)$ to the displacement mode $V(y_1 - s_1, a_1)\sin(y_1 - s_1)$ is $\frac{a_{12}}{a_1}V(s_2 - s_1, a_{12})\sin(s_2 - s_1)$. Multiplying both sides of the steady state equation by $V(y_1 - s_1, b)\sin(y_1 - s_1)$ and integrating over y_1, we obtain

$$-\frac{b_r}{b_u}g_{12}V(s_2 - s_1, b_r)\sin(s_2 - s_1) = A_1 V(x_1 - s_1, b_u)\sin(x_1 - s_1).$$

Procedure for finding $u_1 = u_2$ and $\bar{u}_1 = \bar{u}_2$:

For a given $s_2 - s_1$, the steady state equations will let us determine g_{11} and g_{12}:

$$s_1 - x_1 = \frac{1}{2}(x_2 - x_1) - \frac{1}{2}(s_2 - s_1),$$
$$g_{12} = \frac{A_1 b_u V(s_1 - x_1, b_u)\sin(s_1 - x_1)}{b_r V(s_2 - s_1, b_r)\sin(s_2 - s_1)},$$
$$g_{11} = \frac{g_{12}V(s_2 - s_1, b_r) + A_1 V(s_1 - x_1, b_u) + A_b}{V(0, b_u)}.$$

This enables us to find u_1 and $\bar{u}_1$:

$$\frac{u_1}{u_c} = g_{11} + \frac{I_0(b_u)}{I_0(b_r)}\frac{J_{rc}}{J_{rp}}g_{12}.$$
$$\frac{\bar{u}_2}{u_c} = \sqrt{\frac{1+J_{\text{int}}}{J_{\text{int}}}\left[\left(\frac{2J_{rp}}{g_{12}J_c} - \frac{1}{1+J_{\text{int}}}\right)\frac{u_2^2}{u_c^2} - 1\right]}.$$

These can be applied to the opposite group of neurons to obtain

$$\bar{g}_{11} \equiv \frac{\bar{u}_1}{u_c} - \frac{I_0(b_u)}{I_0(b_r)}\frac{J_{rc}}{J_c}\frac{2\bar{u}_1^2/u_c^2}{1+\frac{\bar{u}_1^2+J_{\text{int}}u_1^2}{(1+J_{\text{int}})u_c^2}}.$$

$$\bar{g}_{12} \equiv \frac{J_{rp}}{J_c} \frac{2\bar{u}_2^2/u_c^2}{1+\frac{\bar{u}_2^2+J_{\text{int}}u_2^2}{(1+J_{\text{int}})u_c^2}}.$$

Hence $s_2 - s_1$ and $\bar{s}_2 - \bar{s}_1$ has to be found such that they satisfy both

$$\bar{g}_{11}V(0,b_u) - \bar{g}_{12}V(\bar{s}_2 - \bar{s}_1 + \pi, b_r) - A_1 V(\bar{s}_1 - x_1, b_u) - A_b = 0,$$

$$\frac{b_r}{b_u}\bar{g}_{12}V(\bar{s}_2 - \bar{s}_1 + \pi, b_r)\sin(\bar{s}_2 - \bar{s}_1) + A_1 V(\bar{s}_1 - x_1, b_u)\sin(\bar{s}_1 - x_1) = 0,$$

where $\bar{s}_1 - x_1 = \frac{1}{2}(x_2 - x_1) - \frac{1}{2}(\bar{s}_2 - \bar{s}_1)$.

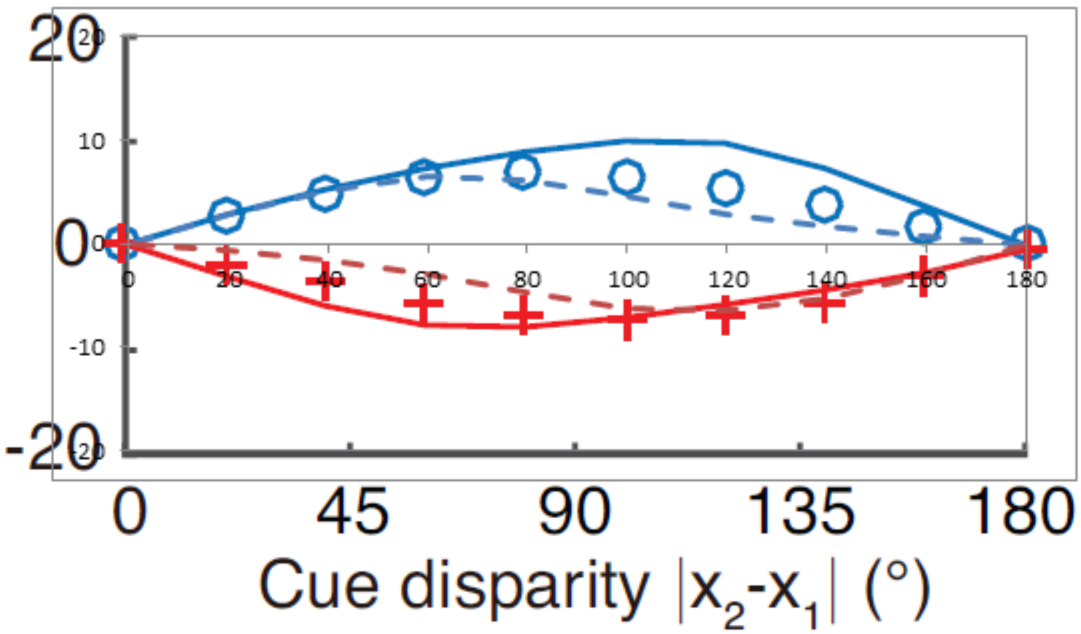

Figure 1: The decoded direction of the congruent and opposite group of neurons as a function of the disparity using projections on to the height and displacement modes of concentration *a*/2.

As shown in Fig. 1, the prediction of the projection method using the height and displacement modes of concentration $b = a/2$ does not agree well with the simulation results. However, we found that the prediction is rather sensitive to the choice of b. Figure 2 shows the result of a smaller value of b. It produces an almost perfect agreement with the simulation results. The correct value of b should be determined by considering the projection to the width mode, but this has not been done yet.

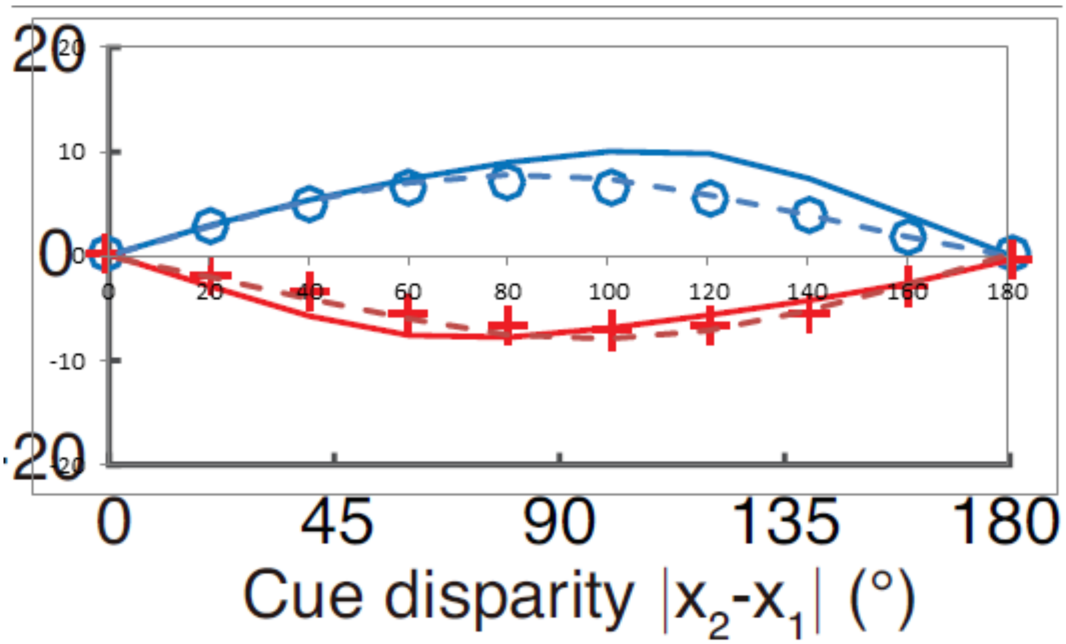

Figure 2: The decoded direction of the congruent and opposite group of neurons as a function of the disparity using projections on to the height and displacement modes of concentration $b = 0.6$.

2. Noisy Dynamics

Next we consider whether the simulation results of the decoded concentrations in Figs. 6F to I agree with the theoretical predictions. In the manuscript, the concentrations are determined by the distribution of the decoded directions in the presence of noise. Hence we have to consider the steady state of the dynamics of the congruent group of neurons in module 1 in the presence of noise, which can be approximated by

$$\tau\frac{\partial}{\partial t}u_1V(y_1-s_1,\kappa_{1u}) = -u_1V(y_1-s_1,\kappa_{1u}) + \rho J_{rc}\frac{u_1^2}{D_1}\frac{I_0(2\kappa_{1u})}{2\pi I_0(\kappa_{1u})^2}V(y_1-s_1,\kappa_{1ar})$$
$$+\rho J_{rp}\frac{u_2^2}{D_2}\frac{I_0(2\kappa_{2u})}{2\pi I_0(\kappa_{2u})^2}V(y_1-s_2,\kappa_{2ar}) + I_1V\left(y_1-x_1,\frac{a}{2}\right) + \sqrt{FI_1V(y_1-x_1,\kappa_{1u})}\xi_1(y_1,t) + I_b$$
$$+\sqrt{FI_b}\epsilon_1^c(y_1,t).$$

Considering the dynamics of the displacement mode only,

$$\tau u_1V\left(y_1-s_1,\frac{a}{2}\right)\frac{a}{2}\sin(y_1-s_1)\frac{\partial}{\partial t}\delta s_1 \approx -u_1V\left(y_1-s_1,\frac{a}{2}\right)\frac{a}{2}\sin(y_1-s_1)\,\delta s_1$$
$$+\rho J_{rc}\frac{u_1^2}{D_1}\frac{I_0(a)}{2\pi I_0\left(\frac{a}{2}\right)^2}V(y_1-s_1,a_r)a_r\sin(y_1-s_1)\,\delta s_1$$
$$+\rho J_{rp}\frac{u_2^2}{D_2}\frac{I_0(a)}{2\pi I_0\left(\frac{a}{2}\right)^2}V(y_1-s_2,a_r)a_r\sin(y_1-s_2)\,\delta s_2$$

$$+\sqrt{FI_1 V\left(y_1 - x_1, \frac{a}{2}\right)}\,\xi_1(y_1,t) + \sqrt{FI_b}\epsilon_1^c(y_1,t).$$

To consider projection on to the displacement mode, we note that

$$\begin{aligned}
&\int dy_1 V(y_1 - s_1, a_1)\sin(y_1 - s_1)\, V(y_1 - s_2, a_2)\sin(y_1 - s_2)\\
&= \frac{1}{a_1 a_2}\frac{\partial^2}{\partial s_1 \partial s_2}\int dy_1 V(y_1 - s_1, a_1)V(y_1 - s_2, a_2) = \frac{1}{a_1 a_2}\frac{\partial^2}{\partial s_1 \partial s_2}V(s_2 - s_1, a_{12})\\
&= -\frac{a_{12}}{a_1 a_2}\frac{\partial}{\partial s_1}[V(s_2 - s_1, a_{12})\sin(s_2 - s_1)].\\
&= \frac{a_{12}}{a_1 a_2}V(s_2 - s_1, a_{12})[\cos(s_2 - s_1) - a_{12}\sin^2(s_2 - s_1)]\\
&\int dy_1 V(y_1 - s_1, a_1)\sin(y_1 - s_1)\, V(y_1 - s_2, a_2)\\
&= \frac{1}{a_1}\frac{\partial}{\partial s_1}\int dy_1 V(y_1 - s_1, a_1)V(y_1 - s_2, a_2) = \frac{1}{a_1}\frac{\partial}{\partial s_1}V(s_2 - s_1, a_{12})\\
&= \frac{a_{12}}{a_1}V(s_2 - s_1, a_{12})\sin(s_2 - s_1).
\end{aligned}$$

Multiplying both sides by the position mode $V(y_1 - s_1, b)\sin(y_1 - s_1)$ and integrating over y_1,

$$\begin{aligned}
&\tau u_1 \frac{b_u}{b}V(0,b_u)\frac{\partial}{\partial t}\delta s_1 \approx -u_1\frac{b_u}{b}V(0,b_u)\delta s_1 + \rho J_{rc}\frac{u_1^2}{D_1}\frac{I_0(a)}{2\pi I_0\left(\frac{a}{2}\right)^2}\frac{b_r}{b}V(0,b_r)\delta s_1\\
&+\rho J_{rp}\frac{u_2^2}{D_2}\frac{I_0(a)}{2\pi I_0\left(\frac{a}{2}\right)^2}\frac{b_r}{b}V(s_2 - s_1, b_r)[\cos(s_2 - s_1) - b_r\sin^2(s_2 - s_1)]\delta s_2\\
&+\sqrt{FI_1}\int dy_1\sqrt{V\left(y_1 - x_1, \frac{a}{2}\right)}\,V(y_1 - s_1, b)\sin(y_1 - s_1)\,\xi_1(y_1,t)\\
&+\sqrt{FI_b}\int dy_1 V(y_1 - s_1, b)\sin(y_1 - s_1)\,\epsilon_1^c(y_1,t).
\end{aligned}$$

Simplifying,

$$\begin{aligned}
&\tau\frac{\partial}{\partial t}\delta s_1 \approx -\delta s_1 + \frac{\rho J_{rc}}{u_1}\frac{u_1^2}{D_1}\frac{I_0(a)}{2\pi I_0\left(\frac{a}{2}\right)^2}\frac{b_r}{b_u}\frac{I_0(b_u)}{I_0(b_r)}\delta s_1\\
&+\frac{\rho J_{rp}}{u_1}\frac{u_2^2}{D_2}\frac{I_0(a)}{2\pi I_0\left(\frac{a}{2}\right)^2}\frac{b_r}{b_u}\frac{V(s_2 - s_1, b_r)}{V(0,b_u)}[\cos(s_2 - s_1) - b_r\sin^2(s_2 - s_1)]\delta s_2\\
&+\frac{b\sqrt{FI_1}}{b_u u_1 V(0,b_u)}\int dy_1\sqrt{V\left(y_1 - x_1, \frac{a}{2}\right)}\,V(y_1 - s_1, b)\sin(y_1 - s_1)\,\xi_1(y_1,t)\\
&+\frac{b\sqrt{FI_b}}{b_u u_1 V(0,b_u)}\int dy_1 V(y_1 - s_1, b)\sin(y_1 - s_1)\,\epsilon_1^c(y_1,t).
\end{aligned}$$

In terms of the coupling constants,

$$\begin{aligned}
&\tau\frac{\partial}{\partial t}\delta s_1 \approx -\left[1 - \frac{b_r}{b_u}\left(1 - \frac{u_c}{u_1}g_{11}\right)\right]\delta s_1\\
&+\frac{b_r}{b_u}\frac{u_c}{u_1}g_{12}\frac{V(s_2 - s_1, b_r)}{V(0,b_u)}[\cos(s_2 - s_1) - b_r\sin^2(s_2 - s_1)]\delta s_2 + \beta_1\eta_1(t) + \gamma_1^c\varsigma_1^c(t),
\end{aligned}$$

where $\langle\eta_1(t)\rangle = 0$, $\langle\eta_1(t)\eta_1(t')\rangle = \delta(t - t')$, $\langle\varsigma_1^c(t)\rangle = 0$, $\langle\varsigma_1^c(t)\varsigma_1^c(t')\rangle = \delta(t - t')$, and

$$\beta_1\eta_1(t) \equiv \frac{b\sqrt{FI_1}}{b_u u_1 V(0,b_u)}\int dy_1\sqrt{V\left(y_1 - x_1, \frac{a}{2}\right)}\,V(y_1 - s_1, b)\sin(y_1 - s_1)\,\xi_1(y_1, t),$$

$$\gamma_1^c\varsigma_1^c(t) \equiv \frac{b\sqrt{FI_b}}{b_u u_1 V(0,b_u)}\int dy_1\,V(y_1 - s_1, b)\sin(y_1 - s_1)\,\epsilon_1^c(y_1, t).$$

In simulations, the positions of the neurons are discretized. So the integrals are replaced by summations,

$$\beta_1\eta_1(t) = \frac{b\sqrt{FI_1}}{b_u u_1 V(0,b_u)}\sum_i \Delta y\sqrt{V\left(y_i - x_1, \frac{a}{2}\right)}\,V(y_i - s_1, b)\sin(y_i - s_1)\,\xi_{1i}(t),$$

$$\gamma_1^c\varsigma_1^c(t) \equiv \frac{b\sqrt{FI_b}}{b_u u_1 V(0,b_u)}\sum_i \Delta y V(y_i - s_1, b)\sin(y_i - s_1)\,\epsilon_{1i}^c(t),$$

where $\Delta y \equiv \frac{N}{2\pi} = \frac{1}{\rho}$. The equivalent noise temperature is given by

$$T_1 = \frac{b^2 F}{2\rho[b_u u_1 V(0,b_u)]^2}\Big[I_1\int dy_1 V\left(y_1 - x_1, \frac{a}{2}\right)V(y_1 - s_1, b)^2\sin^2(y_1 - s_1)$$
$$+ I_b\int dy_1 V(y_1 - s_1, b)^2\sin^2(y_1 - s_1)\Big].$$

Eliminating the square term,

$$T_1 = \frac{b^2 F}{2\rho[b_u u_1 V(0,b_u)]^2}\frac{I_0(2b)}{2\pi I_0(b)^2}\Big\{I_1\int dy_1 V\left(y_1 - x_1, \frac{a}{2}\right)V(y_1 - s_1, 2b)\sin(y_1 - s_1)\,[\sin(y_1 - x_1)\cos(x_1 - s_1) + \cos(y_1 - x_1)\sin(x_1 - s_1)] + I_b\int dy_1 V(y_1 - s_1, 2b)\sin(y_1 - s_1)\,[\sin(y_1 - x_1)\cos(x_1 - s_1) + \cos(y_1 - x_1)\sin(x_1 - s_1)]\Big\}.$$

Eliminating the trigonometric terms,

$$T_1 = \frac{b^2 I_0(2b) F}{4\pi\rho[b_u u_1 V(0,b_u) I_0(b)]^2}\Big[I_1\cos(x_1 - s_1)\frac{1}{ab}\frac{\partial^2}{\partial x_1\partial s_1}\int dy_1 V\left(y_1 - x_1, \frac{a}{2}\right)V(y_1 - s_1, 2b)$$
$$+ I_1\sin(x_1 - s_1)\lim_{\kappa\to a/2}\frac{1}{2bI_0(\kappa)}\frac{\partial^2}{\partial\kappa\partial s_1}\int dy_1 I_0(\kappa)V(y_1 - x_1, \kappa)V(y_1 - s_1, 2b)$$
$$+ I_b\cos(x_1 - s_1)\frac{1}{2b}\frac{\partial}{\partial s_1}\int dy_1 V(y_1 - s_1, 2b)\sin(y_1 - x_1)$$
$$+ I_b\sin(x_1 - s_1)\frac{1}{2b}\frac{\partial}{\partial s_1}\int dy_1 V(y_1 - s_1, 2b)\cos(y_1 - x_1)\Big].$$

Simplifying the convolution of the von Mises functions,

$$T_1 = \frac{b^2 I_0(2b) F}{4\pi\rho[b_u u_1 V(0,b_u) I_0(b)]^2}\Big[I_1\cos(x_1 - s_1)\frac{1}{ab}\frac{\partial^2}{\partial x_1\partial s_1}V(s_1 - x_1, b_{u2})$$
$$+ I_1\sin(x_1 - s_1)\lim_{\kappa\to a/2}\frac{1}{2bI_0(\kappa)}\frac{\partial^2}{\partial\kappa\partial s_1}I_0(\kappa)V(s_1 - x_1, b_{u2}(\kappa))$$
$$+ I_b\cos(x_1 - s_1)\frac{1}{2b}\frac{\partial}{\partial s_1}A(2b)\sin(s_1 - x_1) + I_b\sin(x_1 - s_1)\frac{1}{2b}\frac{\partial}{\partial s_1}A(2b)\cos(s_1 - x_1)\Big],$$

where $b_{u2} = A^{-1}(A(a/2)A(2b))$. Performing the derivatives,

$$T_1 = \frac{b^2 I_0(2b)F}{4\pi\rho[b_u u_1 V(0,b_u) I_0(b)]^2}\Big[I_1 \cos(x_1 - s_1)\frac{(-b_{u2})}{ab}\frac{\partial}{\partial x_1}V(s_1 - x_1, b_{u2})\sin(s_1 - x_1) + I_1 \sin(x_1 - s_1)\lim_{\kappa\to a/2}\frac{(-1)}{2bI_0(\kappa)}\ \frac{\partial}{\partial\kappa}b_{u2}(\kappa)I_0(\kappa)V(s_1 - x_1, b_{u2}(\kappa))\sin(s_1 - x_1) + I_b\cos(x_1 - s_1)\frac{1}{2b}A(2b)\cos(s_1 - x_1) - I_b \sin(x_1 - s_1)\frac{1}{2b}A(2b)\sin(s_1 - x_1)\Big]$$

$$= \frac{b^2 I_0(2b)F}{4\pi\rho[b_u u_1 V(0,b_u) I_0(b)]^2}\Big\{I_1 V(s_1 - x_1, b_{u2})\Big[\cos(s_1 - x_1)\frac{b_{u2}}{ab}(\cos(s_1 - x_1) - b_{u2}\sin^2(s_1 - x_1)) + \frac{A(2b)A'(a/2)}{2bA'(b_{u2})}\sin^2(s_1 - x_1)\,(1 + b_{u2}\cos(s_1 - x_1))\Big] + I_b\frac{A(2b)}{2b}\Big\}$$

Together with the dynamics of module 2,

$$\tau\frac{\partial}{\partial t}\begin{pmatrix}\delta s_1\\ \delta s_2\end{pmatrix} = -\begin{pmatrix} 1 - \frac{b_r}{b_u}\left(1 - \frac{u_c}{u_1}g_{11}\right) & -\frac{b_r}{b_u}\frac{u_c}{u_1}g_{12}\frac{V(s_2 - s_1, b_r)}{V(0,b_u)}[\cos(s_2 - s_1) - b_r\sin^2(s_2 - s_1)] \\ -\frac{b_r}{b_u}\frac{u_c}{u_2}g_{21}\frac{V(s_2 - s_1, b_r)}{V(0,b_u)}[\cos(s_2 - s_1) - b_r\sin^2(s_2 - s_1)] & 1 - \frac{b_r}{b_u}\left(1 - \frac{u_c}{u_2}g_{11}\right)\end{pmatrix}\begin{pmatrix}\delta s_1\\ \delta s_2\end{pmatrix} + \begin{pmatrix}\xi_1(t)\\ \xi_2(t)\end{pmatrix}.$$

Consider the dynamics $\tau\frac{\partial x_i}{\partial t} = -\sum_j g_{ij}x_j + \xi_i$ where $\langle\xi_i(t)\rangle = 0$ and $\langle\xi_i(t)\xi_j(t')\rangle = 2T_i\delta_{ij}\delta(t = t')$. Diagonalizing the matrix, $g_{ij} = \sum_k S_{ik}\lambda_k S_{kj}^{-1}$. Then

$$\tau\frac{\partial}{\partial t}\sum_j S_{ij}^{-1}x_j = -\lambda_i\sum_j S_{ij}^{-1}x_j + \sum_j S_{ij}^{-1}\xi_j.$$

The solution is

$$x_i(t) = \sum_j S_{ij}\int_{-\infty}^{t}\frac{dt'}{\tau}\exp\left[-\frac{\lambda_j}{\tau}(t - t')\right]S_{jk}^{-1}\xi_k(t').$$

Consider the moments of the distribution of $x_i(t)$,

$$\langle x_i^{2n}(t)\rangle = \sum_{j_1\cdots j_{2n}k_1\cdots k_{2n}} S_{ij_1}\cdots S_{ij_{2n}}\int_{-\infty}^{t}\frac{dt_1}{\tau}\exp\left[-\frac{\lambda_{j_1}}{\tau}(t - t_1)\right]\cdots\int_{-\infty}^{t}\frac{dt_{2n}}{\tau}\exp\left[-\frac{\lambda_{j_{2n}}}{\tau}(t - t_{2n})\right]S_{j_1k_1}^{-1}\cdots S_{j_{2n}k_{2n}}^{-1}\langle\xi_{k_1}(t_1)\cdots\xi_{k_{2n}}(t_{2n})\rangle.$$

The number of ways the pairing of the noise terms can be done is $(2n)!/(2!^n\, n!) = (2n-1)!!$. Hence

$\langle x_i^{2n}(t)\rangle = (2n-1)!!\left[\sum_{j_1 j_2 k_1 k_2} S_{ij_1} S_{ij_2} \int_{-\infty}^{t} \frac{dt_1}{\tau} \exp\left[-\frac{\lambda_{j_1}}{\tau}(t-t_1)\right] \int_{-\infty}^{t} \frac{dt_2}{\tau} \exp\left[-\frac{\lambda_{j_2}}{\tau}(t-t_2)\right] S_{j_1 k_1}^{-1} S_{j_2 k_2}^{-1} 2T_{k_1} \delta_{k_1 k_2} \delta(t_1 - t_2)\right]^n.$

After integrating,

$$\langle x_i^{2n}(t)\rangle = (2n-1)!!\left[\sum_{jkl} S_{ij} S_{ik} \frac{2T_l}{(\lambda_j + \lambda_k)\tau} S_{jl}^{-1} S_{kl}^{-1}\right]^n.$$

Hence the distribution of $x_i(t)$ is a Gaussian with variance

$$\sigma_i^2 = \sum_{jkl} S_{ij} S_{ik} \frac{2T_l}{(\lambda_j + \lambda_k)\tau} S_{jl}^{-1} S_{kl}^{-1}.$$

For the matrix $\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$,

$\lambda_\pm = \frac{1}{2}\left(g_{11} + g_{22} \pm \sqrt{D}\right)$ where $D = (g_{22} - g_{11})^2 + 4g_{12}g_{21}$,

$S = \begin{pmatrix} -g_{12} & -g_{12} \\ \frac{1}{2}(g_{11} - g_{22} - \sqrt{D}) & \frac{1}{2}(g_{11} - g_{22} + \sqrt{D}) \end{pmatrix}$,

$\det S = -g_{12}\sqrt{D}$,

$S^{-1} = \frac{1}{g_{12}\sqrt{D}} \begin{pmatrix} \frac{1}{2}(g_{22} - g_{11} - \sqrt{D}) & -g_{12} \\ \frac{1}{2}(g_{11} - g_{22} - \sqrt{D}) & g_{12} \end{pmatrix}$,

$\sigma_1^2 = \left[\frac{\left(S_{11}S_{11}^{-1}\right)^2}{\lambda_1} + \frac{\left(S_{12}S_{21}^{-1}\right)^2}{\lambda_2} + \frac{4\left(S_{11}S_{11}^{-1}\right)\left(S_{12}S_{21}^{-1}\right)}{\lambda_1+\lambda_2}\right] T_1 + \left[\frac{\left(S_{11}S_{12}^{-1}\right)^2}{\lambda_1} + \frac{\left(S_{12}S_{22}^{-1}\right)^2}{\lambda_2} + \frac{4\left(S_{11}S_{12}^{-1}\right)\left(S_{12}S_{22}^{-1}\right)}{\lambda_1+\lambda_2}\right] T_2$

Coefficient of T_1/τ

$= \frac{(g_{22}-g_{11}-\sqrt{D})^2(g_{11}+g_{22}-\sqrt{D})}{8D(g_{11}g_{22}-g_{12}g_{21})} + \frac{\left((g_{22}-g_{11}+\sqrt{D})\right)^2(g_{11}+g_{22}+\sqrt{D})}{8D(g_{11}g_{22}-g_{12}g_{21})} + \frac{4g_{12}g_{21}}{D(g_{11}+g_{22})}$

$= \frac{1}{8D(g_{11}g_{22}-g_{12}g_{21})}\left\{\left[(g_{22} - g_{11})^2 - 2(g_{22} - g_{11})\sqrt{D} + D\right]\left(g_{11} + g_{22} - \sqrt{D}\right) + \left(\sqrt{D} \to -\sqrt{D}\right)\right\} + \frac{4g_{12}g_{21}}{D(g_{11}+g_{22})}$

$= \frac{4g_{22}D-4g_{12}g_{21}(g_{22}+g_{11})}{4D(g_{11}g_{22}-g_{12}g_{21})} + \frac{4g_{12}g_{21}}{D(g_{11}+g_{22})}$

$= \frac{g_{22}}{g_{11}g_{22}-g_{12}g_{21}} - \frac{g_{12}g_{21}}{(g_{11}g_{22}-g_{12}g_{21})(g_{11}+g_{22})}$

$= \frac{1}{(g_{11}+g_{22})} + \frac{g_{22}^2}{(g_{11}g_{22}-g_{12}g_{21})(g_{11}+g_{22})}$

Coefficient of T_2/τ

$= \frac{2g_{12}^2}{D\left(g_{11}+g_{22}+\sqrt{D}\right)} + \frac{2g_{12}^2}{D\left(g_{11}+g_{22}-\sqrt{D}\right)} - \frac{4g_{12}^2}{D(g_{11}+g_{22})}$

$= \frac{g_{12}^2}{(g_{11}g_{22}-g_{12}g_{21})(g_{11}+g_{22})}$

Summarizing,

$$\sigma_1^2 = \frac{T_1}{(g_{11}+g_{22})\tau} + \frac{g_{22}^2 T_1 + g_{12}^2 T_2}{(g_{11}g_{22} - g_{12}g_{21})(g_{11}+g_{22})\tau}$$

In summary, results can be obtained from the following steps.

Decoded concentration
$T_1 = \frac{b^2 I_0(2b)F}{4\pi\rho[b_u u_1 V(0,b_u) I_0(b)]^2}\left\{I_1 V(s_1 - x_1, b_{u2})\left[\cos(s_1 - x_1)\frac{b_{u2}}{ab}(\cos(s_1 - x_1) - b_{u2}\sin^2(s_1 - x_1)) + \frac{A(2b)A'(a/2)}{2bA'(b_{u2})}\sin^2(s_1 - x_1)(1 + b_{u2}\cos(s_1 - x_1))\right] + I_b\frac{A(2b)}{2b}\right\}.$
$T_2 = \frac{b^2 I_0(2b)F}{4\pi\rho[b_u u_2 V(0,b_u) I_0(b)]^2}\left\{I_1 V(s_2 - x_2, b_{u2})\left[\cos(s_2 - x_2)\frac{b_{u2}}{ab}(\cos(s_2 - x_2) - b_{u2}\sin^2(s_2 - x_2)) + \frac{A(2b)A'(a/2)}{2bA'(b_{u2})}\sin^2(s_2 - x_2)(1 + b_{u2}\cos(s_2 - x_2))\right] + I_b\frac{A(2b)}{2b}\right\}.$
$G_{11} = 1 - \frac{b_r}{b_u}\left(1 - \frac{u_c}{u_1}g_{11}\right).$
$G_{22} = 1 - \frac{b_r}{b_u}\left(1 - \frac{u_c}{u_2}g_{22}\right).$
$G_{12} = -\frac{b_r}{b_u}\frac{u_c}{u_1}g_{12}\frac{V(s_2-s_1,b_r)}{V(0,b_u)}[\cos(s_2 - s_1) - b_r\sin^2(s_2 - s_1)].$
$G_{21} = -\frac{b_r}{b_u}\frac{u_c}{u_2}g_{12}\frac{V(s_2-s_1,b_r)}{V(0,b_u)}[\cos(s_2 - s_1) - b_r\sin^2(s_2 - s_1)].$
$\sigma_1^2 = \frac{T_1}{(G_{11}+G_{22})\tau} + \frac{T_1 G_{22}^2 + T_2 G_{12}^2}{(G_{11}G_{22} - G_{12}G_{21})(G_{11}+G_{22})\tau}.$
$\kappa_1 = \frac{1}{\sigma_1^2}.$

Compared with the work in vonMises7, Figure 3 shows that the predicted network concentrations of the congruent and opposite neurons have different values and are dependent on the disparity. Hence it is useful to use the projection method, which allows the network concentrations to change with the disparity. However, the predicted values are considerably less than the simulation results. Again, we found that the predicted results are rather sensitive to the choice of b. Figure 4 shows the result of a smaller value of b. It produces concentrations in a range similar to those of the simulation results.

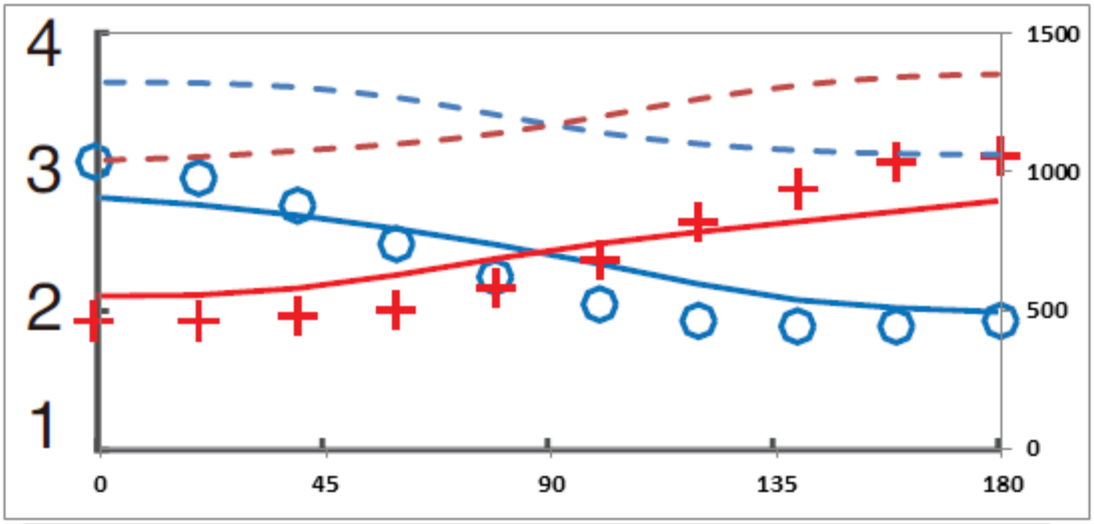

Figure 3: The decoded concentration of the congruent and opposite group of neurons as a function of the disparity using projections on to the height and displacement modes of concentration $a/2$.

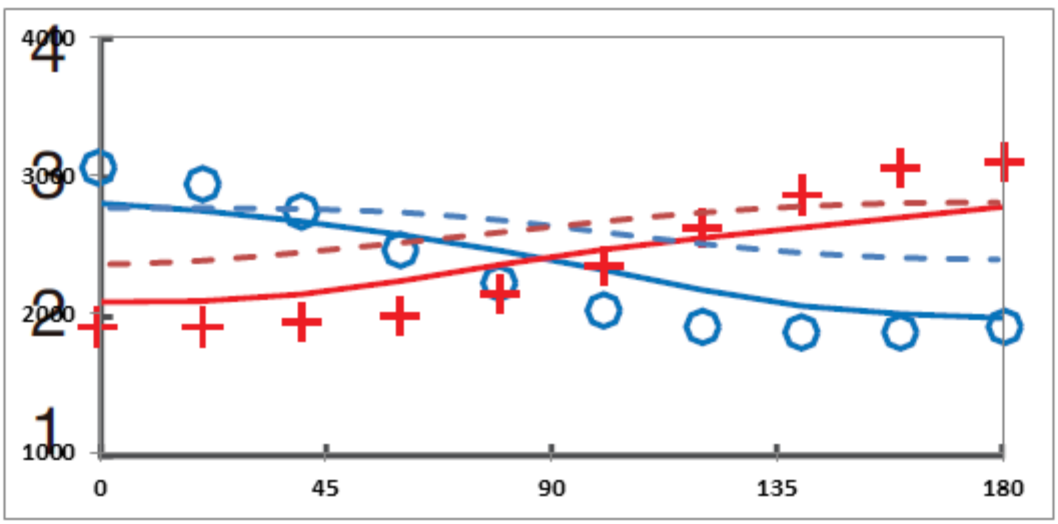

Figure 4: The decoded concentration of the congruent and opposite group of neurons as a function of the disparity using projections on to the height and displacement modes of concentration $b = 0.6$.