Approximation

Actually the approximation of Von Mises convolution may not as accurate as we expect. When angular disparity is comparatively small, that approximation could fit the Bessel function very well, regardless of which κ_1 and κ_2 we choose. That approximation is approximated by a wrapped normal distribution, after that he used a Von Mises distribution to fit the wrapped normal distribution. It works perfectly when κ_1 and κ_2 are approaching infinity.

We know:

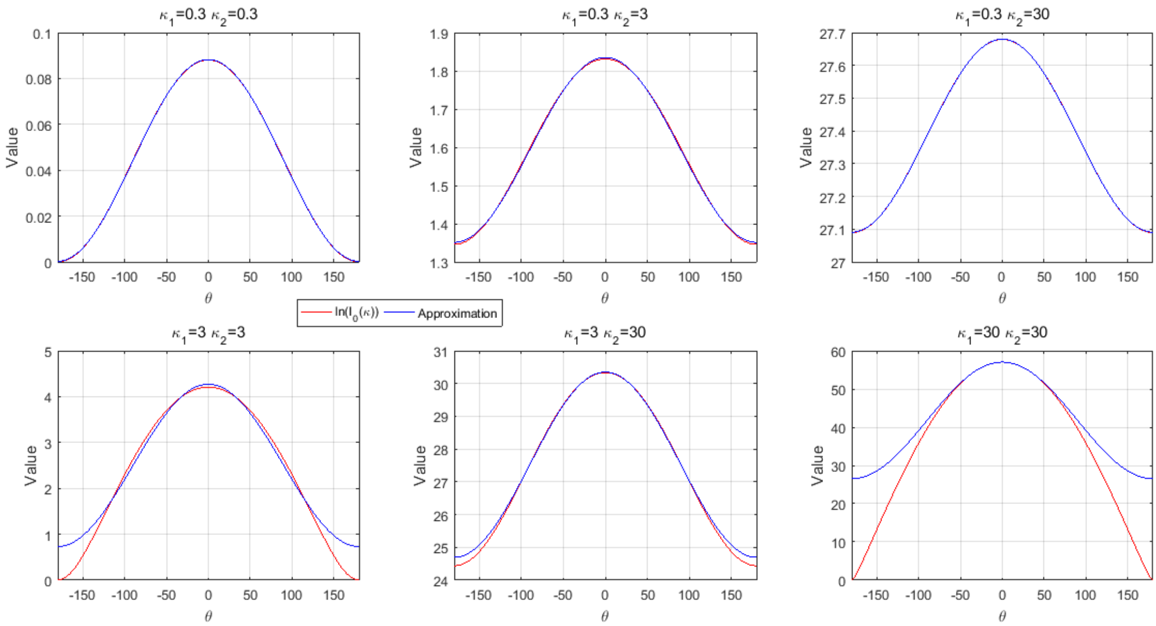
$$\int_{-\pi}^{\pi} e^{k_1 cos(\theta-\theta')} e^{k_2 cos\theta} d\theta' = 2\pi I_0(\sqrt{k_1^2 + k_2^2 + 2k_1k_2cos\theta})$$

Previously, the Bessel function is approximated by:

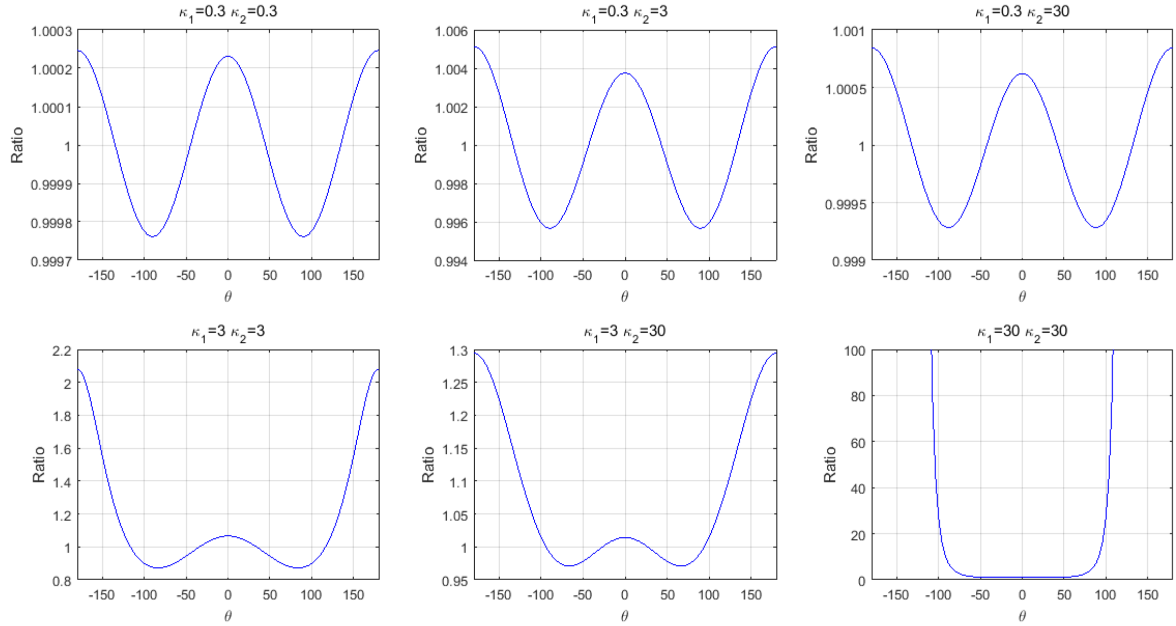
$$I_0(\sqrt{k_1^2 + k_2^2 + 2k_1k_2cos\theta}) \approx \frac{I_0(k_1)I_0(k_2)}{I_0(k_3)} e^{k_3 cos\theta} \quad where\ k_3 = A^{-1}(A(k_1)A(k_2))$$

For convenience, we take the log of both sides. Note $k = \sqrt{k_1^2 + k_2^2 + 2k_1k_2cos\theta}$. In the following chapter we will compare our approximation with $ln(I_0(k))$.

Previous Approximation



Note $Ratio = \frac{I_0(k_1)I_0(k_2)}{I_0(k_3)I_0(k)} e^{k_3 cos\theta}$.



New Approach

We focus on $ln(I_0(k))$, of course we can expand this function into a Fourier cosine series. When κ_1 and κ_2 are sufficiently large, we have to add high order terms. That is to say, we no longer have a Von Mises form approximation. In this case, suppose $ln(I_0(k))$ could be approximated by $d + hcos(f\theta)$.

We expand $ln(I_0(k))$, consider first three terms only.

$$ln(I_0(k)) \approx \frac{b_0}{2} + b_1 cos\theta + b_2 cos2\theta$$

Where $b_i = \frac{1}{\pi}\langle ln(I_0(k)), cos(i\theta)\rangle \quad i = 0, 1, 2$

$$b_0/2 + b_1 cos\theta + b_2 cos2\theta \approx d + hcos(f\theta)$$

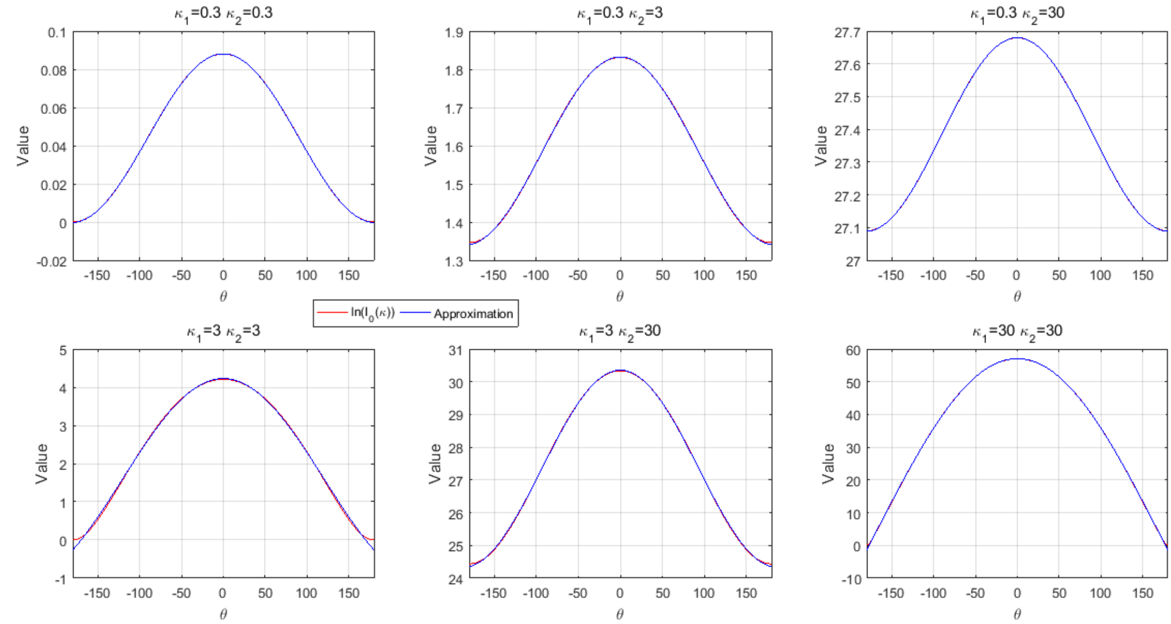
Multiply both sides by $cos(i\theta)$, integrate over θ.

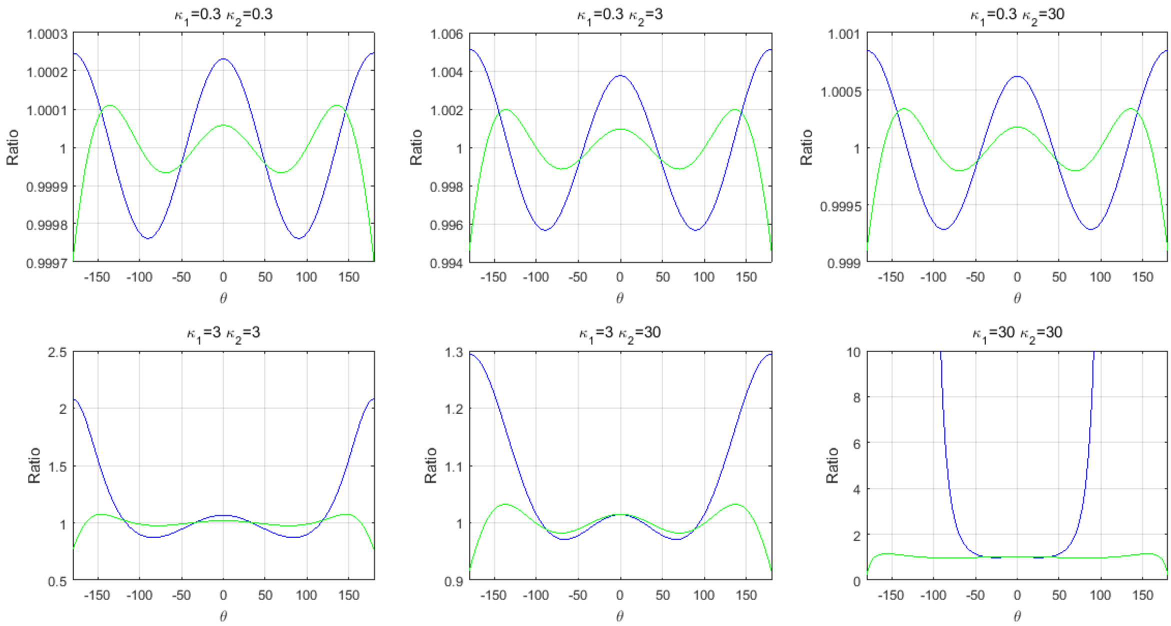
$$\pi b_0 = 2\pi d + 2h\frac{sin(\pi f)}{f}$$
$$\pi b_1 = -2h\frac{2fsin(\pi f)}{f^2 - 1}$$
$$\pi b_2 = 2h\frac{2fsin(\pi f)}{f^2 - 4}$$

Then we obtain:

$$f = \sqrt{\frac{4b_2 + b_1}{b_2 + b_1}}$$
$$h = \frac{\pi b_1(1 - f^2)}{2fsin(\pi f)}$$
$$d = \frac{b_0}{2} - \frac{b_1(1 - f^2)}{2f^2}$$

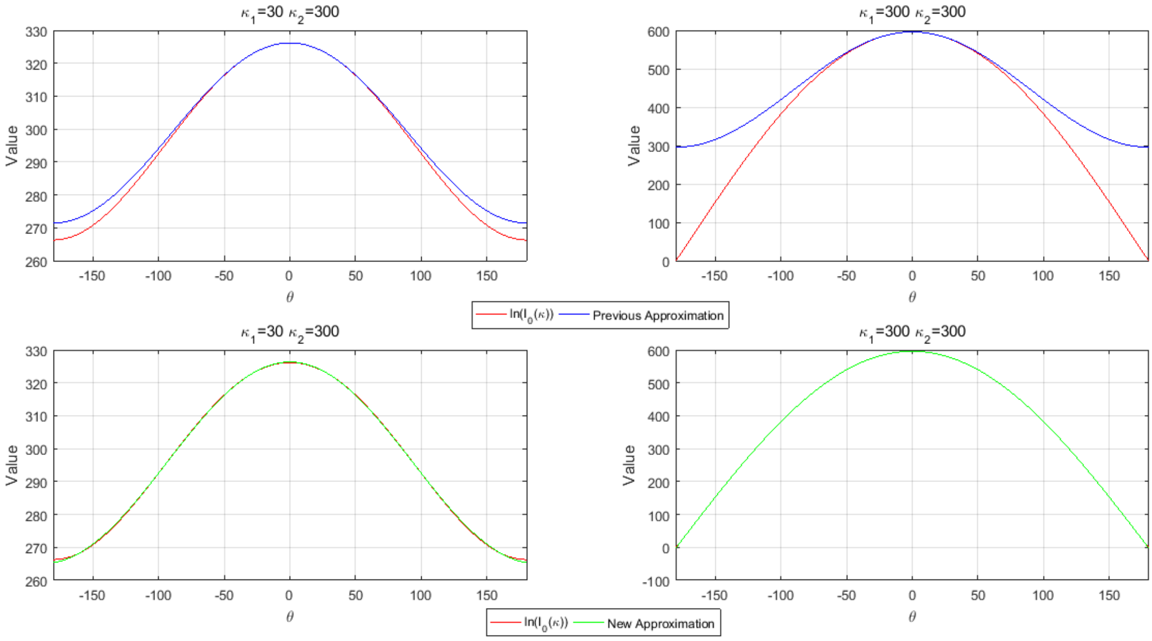
Results





Green line is new approximation, compared with previous one (blue line). When $\theta = \pm\pi$, the ratio we get from previous approximation is approaching 3.29×10^{11}.

Actually, when κ_1 and κ_2 are approaching infinity, our approximation is better than the previous one.



Discussion

We notice the deviation when angular disparity approach $\pm\pi$, that is to say we need to replace our approximation with $I_0(|k_1 - k_2|)$ when $\theta = \pm\pi$.

Summary

Given k_1 and k_2, we need first three coefficients of Fourier cosine series. Then we calculate $f\ h\ d$ respectively. We use $F = \frac{e^d}{2\pi I_0(k_1) I_0(k_2)}$ and $k_3 = h$ to represent the coefficient and concentration. So the convolution of two Von Mises function is still a Von Mises function.

$$\int_{-\pi}^{\pi} V(\theta - \theta', k_1) V(\theta', k_2) d\theta' \approx F(k_1, k_2) V(f\theta, k_3)$$