# Notes for gated network

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## 1 Bayesian model for the gated network

When the prior consists of the correlated and independent part, the Bayes factor is often defined as follows:

$$B(s) \equiv \frac{p_{seg}(s|x_1, x_2)}{p_{int}(s|x_1, x_2)}, \tag{1}$$

which is used to determine whether the network should rebuild or cut off the connections between two modules. For instance, we show how $B(s_1)$ is calculated.

$$B(s_1) = \frac{p_{seg}(s_1|x_1, x_2)}{p_{int}(s_1|x_1, x_2)} = \frac{\langle p_i p(x_2|s_2)\rangle_{s_2} p(x_1|s_1)}{\langle p_c(s_1, s_2)p(x_2|s_2)\rangle_{s_2} p(x_1|s_1)} = \frac{p_i}{\langle p_c(s_1, s_2)p(x_2|s_2)\rangle_{s_2}}, \tag{2}$$

where $p_i = \frac{1-p_0}{(2\pi)^2}$, $p_c(s_1, s_2) = \frac{p_0}{2\pi} V(s_1 - s_2, \kappa_s)$ and $p(x_2|s_2) = V(x_2 - s_2, \kappa_2)$. Using the approximation, we will get

$$\int ds_2 p_c(s_1, s_2)p(x_2|s_2) \simeq \frac{p_0}{2\pi} V(x_2 - s_1, \kappa_{2s}), \tag{3}$$

where $\kappa_{2s} = A^{-1}[A(\kappa_2)A(\kappa_s)]$. Based on our previous work, it could be implemented by the opposite group of neurons for sure.

However, $\kappa_{2s}$ is associated with the strength of reciprocal connections $J_{rp}$, which shows that $J_{rp}$ is more independent of the angular disparity $\Delta x \equiv |x_2 - x_1|$.

### 1.1 Network architecture

The neuronal dynamics of the congruent group is given by

$$\begin{aligned}\tau \frac{\partial \psi_m(y,t)}{\partial t} =& -\psi_m(y,t) + \sum_{y'=-\pi}^{\pi} J_{rc} V(y-y', a_0) R_m(y',t) \\ &+ \sum_{y'=-\pi}^{\pi} J_{rp} V(y-y', a_0) R_{\bar{m}}(y',t) + I_m^{ext}(y,t),\end{aligned} \tag{4}$$

where $J_{rc}$ and $J_{rp}$ represent the strengths of the recurrent and reciprocal couplings respectively. The firing rate is given by $R_m(y,t) \equiv \psi_m^2(y,t)/D_m$, where

$$D_m \equiv 1 + \omega \sum_y \psi_m^2(y,t) \tag{5}$$

is the global inhibition acting on the congruent group in module $m$. Project Eq. (4) onto the height mode and position mode, simplify the notation

$$1 = HJ_{rc} + HJ_{rp}\cos\Delta s + \frac{F}{u_1}\cos(x_1 - s_1), \tag{6}$$

$$0 = HJ_{rp}\sin\Delta s + \frac{F}{u_1}\sin(x_1 - s_1), \tag{7}$$

where $\Delta s \equiv s_2 - s_1$, $H = \frac{\rho}{D}(2u_0 + u_2)B_1(a_0)$, $F = \frac{IB_1(a_0/2)}{\pi}$. However, the opposite groups are connected in different manner

$$\tau\frac{\partial\bar{\psi}_m(y,t)}{\partial t} = -\bar{\psi}_m(y,t) + \sum_{y'=-\pi}^{\pi} J_{rc}V(y-y',a_0)\bar{R}_m(y',t) + \sum_{y'=-\pi}^{\pi} \bar{J}_{rp}V(y-y'+\pi,a_0)R_m(y',t) + I_m^{ext}(y,t), \tag{8}$$

where $\bar{J}_{rp}$ is the strength of the couplings from congruent neurons to opposite neurons. Similarly,

$$\bar{u}_1 = \frac{-u_1 H\bar{J}_{rp}\cos(s_1 - \bar{s}_1)}{1 - \bar{H}J_{rc}} + \frac{F\cos(x_1 - \bar{s}_1)}{1 - \bar{H}J_{rc}}, \tag{9}$$

$$0 = -u_1 H\bar{J}_{rp}\sin(s_1 - \bar{s}_1) + F\sin(x_1 - \bar{s}_1), \tag{10}$$

where $\bar{H} = \frac{\rho}{D}(2\bar{u}_0 + \bar{u}_2)B_1(a_0)$. in proportion to $u_1$. This is a vector sum because we already know

$$u_1 e^{js_1} \approx \frac{F}{1 - HJ_{rc}}e^{jx_1} + \frac{FHJ_{rp}}{(1 - HJ_{rc})^2}e^{jx_2}, \tag{11}$$

and for opposite neurons

$$\bar{u}_1 e^{j\bar{s}_1} = -\frac{H\bar{J}_{rp}}{1 - \bar{H}J_{rc}}u_1 e^{js_1} + \frac{F}{1 - \bar{H}J_{rc}}e^{jx_1}, \tag{12}$$

eliminate $u_1$, then

$$\bar{u}_1 e^{j\bar{s}_1} \approx \frac{F(1 - HJ_{rc} - HJ_{rp})}{(1 - HJ_{rc})(1 - \bar{H}J_{rc})}e^{jx_1} - \frac{FH^2 J_{rp}\bar{J}_{rp}}{(1 - HJ_{rc})^2(1 - \bar{H}J_{rc})}e^{jx_2}. \tag{13}$$

That is, $\bar{u}_1$ increases monotonically as the disparity $\Delta x$ increases from 0 to $\pi$. However, I don't know what this means exactly.

The output-dependent noise will generate the concentration $\kappa_1$ in proportion to $u_1$.

## 1.2 Bayesian model

According to Bayes' rule, there exists two vector, the first one is already known

$$p(s_1|x_1,x_2) \propto p(x_1|s_1)\int ds_2 p(x_2|s_2)p_c(s_1,s_2) = V(s_1 - x_1, \kappa_1)V(s_1 - x_2, \kappa_{2s}), \tag{14}$$

and this will yield a vector sum

$$\kappa_1^c e^{js_1} = \kappa_1 e^{jx_1} + \kappa_{2s} e^{jx_2}, \tag{15}$$

which makes it possible to achieve a perfect match between the network architecture ($C_1$ and $C_2$ are only necessary) and the Bayesian model (correlated prior). However, consider the prior with independent component, and integrate $e^{js_1}p(s_1|x_1,x_2)$, this will generate another vector sum

$$e^{j\hat{s}_1} \propto p_c\alpha_1 e^{js_1} + (1-p_c)e^{jx_1}, \tag{16}$$

where $\alpha_1$ is a coefficient.

Recall that when dealing with the second layer, we project the steady state equation onto height and position modes similarly. Consider the dynamics of the congruent group in module 1,

$$1 = H' J_{rc} + (1-p_0)\frac{F}{u_1'}\cos(x_1 - s_1') + p_0\frac{\pi c_k \rho u_1}{D u_1'}\left[(2u_0+u_2)\cos(s_1-s_1') - u_3\sin(s_1-s_1')\right], \tag{17}$$

$$0 = (1-p_0)\frac{F}{u_1'}\sin(x_1 - s_1') + p_0\frac{\pi c_k \rho u_1}{D u_1'}\left[(2u_0+u_2)\sin(s_1-s_1') + u_3\cos(s_1-s_1')\right], \tag{18}$$

where $H' = \frac{\rho J_{rc}}{D'}(2u_0' + u_2')B_1(a_0)$, $F = \frac{IB_1(a_0/2)}{\pi}$, consider the weak input limit, we know the solution is

$$u_1' = (1-p_0)\frac{F}{1-H'J_{rc}}\cos(x_1 - s_1') + p_0\frac{\pi c_k H u_1}{J_{rc}B_1(a_0)(1-H'J_{rc})}\cos(s_1 - s_1'), \tag{19}$$

$$0 = (1-p_0)\frac{F}{1-H'J_{rc}}\sin(x_1 - s_1') + p_0\frac{\pi c_k H u_1}{J_{rc}B_1(a_0)(1-H'J_{rc})}\sin(s_1 - s_1'). \tag{20}$$

That is, for combined case $u_1' e^{js_1'} = (1-p_0)\frac{F}{1-H'J_{rc}}e^{jx_1} + p_0\frac{\pi c_k H u_1}{J_{rc}B_1(a_0)(1-H'J_{rc})}e^{js_1}$, which is similar to the last vector sum of Bayesian model. I must confess that I haven't thought about that before. The last one is more reasonable and elegant although it has a different network structure compared with the first case.

It's clear that $C_1$ and $C_2$ can only calculate the correlated prior. $p_0$ (or $p_c$) may served as the weight. If so, the external input and the feedforward input will be rescaled by $1-p_0$ and $p_0$ respectively, that is, $p_0$ will not be stored in gated mechanism to avoid nonlinear effects.

However, implementing $p(s_1|x_1,x_2)$ with correlated and independent prior is not necessary. Note that in the past, we used

$$\begin{aligned}\ln[p(s_1'|x_1,x_2)] \approx & p_0[\kappa_{2s}\cos(s_1'-x_2) + \kappa_1\cos(s_1'-x_1)] \\ & + (1-p_0)\kappa_1\cos(s_1'-x_1) - \ln[2\pi I_0(\kappa_1)],\end{aligned} \tag{21}$$

to change the summation to multiplication, which means if we select the parameters carefully, the architecture could be simplify (only $p_0$ exists).

Note that if $\kappa_s$ is fixed, the gated mechanism works on $p_0$ rather than $J_{rp}$ since $p_0$ is the key factor to switch from the correlated state to the independent state. I have to emphasize that the corresponding architecture is completely different from the model suggested by Wenhao.

## 1.3 Implicit factors

If factors are implicit, we may leave the encoding details behind. I summarize the notes from Gate2.docx as follows:

(a) integrate $e^{js_1}p(s_1|x_1, x_2$ in terms of two-component prior, we have

$$(p_c\alpha_1 + 1 - p_c)e^{j\hat{s}_1} = p_c\alpha_1 e^{jx_1^c} + (1 - p_c)e^{jx_1}, \tag{22}$$

where $\kappa_1^c = \sqrt{\kappa_1^2 + \kappa_{2s}^2 + 2\kappa_1\kappa_{2s}\cos(x_1 - x_2)}$, $\alpha_1 = \frac{I_0(\kappa_1^c)}{I_0(\kappa_1)I_0(\kappa_{2s})}$ and $x_1^c = \mathrm{atan2}(\kappa_1 \sin x_1 + \kappa_{2s}\sin x_2, \kappa_1\cos x_1 + \kappa_{2s}\cos x_2)$;

(b) substitute $\hat{s}_2$ for $x_2$, then

$$bce^{j\hat{s}_1} = b\bar{c}e^{j\hat{s}_2} + (c^2 - \bar{c}^2)e^{jx_1}, \tag{23}$$

where $b = p_c\alpha_m + 1 - p_c$, $c = p_c\alpha_m\frac{\kappa_m}{\kappa_m^c} + 1 - p_c$ and $\bar{c} = p_c\alpha_m\frac{\kappa_{\bar{m}s}}{\kappa_m^c}$ because of the symmetry.

Go back to the network architecture, recall that

$$(\frac{F}{u_1})^2 = (1 - HJ_{rc})^2 + (HJ_{rp})^2 - 2(1 - HJ_{rc})HJ_{rp}\cos\Delta s, \tag{24}$$

$$\tan(s_1 - x_1) = \frac{HJ_{rp}\sin\Delta s}{1 - HJ_{rc} - HJ_{rp}\cos\Delta s}. \tag{25}$$

However, $HJ_{rp}$ is small, and $\sin\Delta s = \sin(s_2 - s_1) = \sin[(s_2 - x_1) - (s_1 - x_1)] \approx \sin(s_2 - x_1)$, $\cos(s_2 - s_1) \approx \cos(s_2 - x_1)$, hence

$$\tan(s_1 - x_1) \approx \frac{HJ_{rp}\sin\Delta s}{1 - HJ_{rc}} \approx \frac{HJ_{rp}\sin(s_2 - x_1)}{1 - HJ_{rc} + HJ_{rp}\cos(s_2 - x_1)}, \tag{26}$$

meanwhile

$$\begin{aligned} u_1^2 \approx& \frac{F^2}{(1 - HJ_{rc})^2} + \frac{2F^2HJ_{rp}}{1 - HJ_{rc})^3}\cos\Delta s \\ \approx& \frac{F^2}{(1 - HJ_{rc})^4}\left[(1 - HJ_{rc})^2 - 2HJ_{rp}(1 - HJ_{rc})\cos[\pi - (s_2 - x_1)] + (HJ_{rp})^2\right], \end{aligned} \tag{27}$$

that is

$$u_1 e^{js_1} \approx \frac{F}{1 - HJ_{rc}}e^{jx_1} + \frac{FHJ_{rp}}{(1 - HJ_{rc})^2}e^{js_2}. \tag{28}$$

We difine the ratio $R$

$$R \equiv \frac{b\bar{c}}{c^2 - \bar{c}^2} \approx \frac{HJ_{rp}}{1 - HJ_{rc}}, \tag{29}$$

when $R \to 0$, then $J_{rp} \to 0$ and $\bar{c} \to 0$.

The firing rate of opposite group is given by

$$\bar{r}_m = \sum_{y=-\pi}^{\pi}\frac{\Psi^2(y,t)}{1 + \omega\sum_{y'}\Psi^2(y',t)} = \frac{N[\bar{u}_{m0}^2 + \frac{\bar{u}_{m1}^2}{2} + \frac{\bar{u}_{m2}^2}{2} + \frac{\bar{u}_{m3}^2}{2}]}{1 + \omega N[\bar{u}_{m0}^2 + \frac{\bar{u}_{m1}^2}{2} + \frac{\bar{u}_{m2}^2}{2} + \frac{\bar{u}_{m3}^2}{2}]}. \tag{30}$$