Discussion

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1 Linear firing rate

Here we consider a special case when the firing rate $R(y,t) = \frac{\psi(y,t)}{1+\omega\sum_y \psi(y,t)}$ and apply the projection method

$$\begin{aligned}
u_{m0} =& \frac{\rho J_{rc}}{D_m} u_{m0} + \frac{\rho J_{rp}}{D_m} u_{\bar{m}0} + \frac{I_m}{2\pi} + I_b, \\
u_{m1} =& \frac{\rho J_{rc}}{D_m} B_1(a_0) u_{m1} + \frac{\rho J_{rp}}{D_m} B_1(a_0) u_{\bar{m}1} \cos(s_{\bar{m}} - s_m) + \frac{I_m B_1(a_0/2)}{\pi} \cos(x_m - s_m), \\
0 =& \frac{\rho J_{rp}}{D_m} B_1(a_0) u_{\bar{m}1} \sin(s_{\bar{m}} - s_m) + \frac{I_m B_1(a_0/2)}{\pi} \sin(x_m - s_m),
\end{aligned}$$

where $B_n(k) \equiv I_n(k)/I_0(k)$ and $D_m \equiv 1 + \omega \sum_y \psi(y,t) = 1 + \omega N u_{m0}$. Note that we ignore the high-order terms (i.e., $\psi_m(y,t) = u_{m0} + u_{m1}\cos(y - s_m)$) for simplicity. Consider the symmetric solution when $I_m = I_{\bar{m}} = I$ and ignore the index of the module m

$$u_1 = HJ_{rc} + HJ_{rp}\cos(s_{\bar{m}} - s_m) + \frac{F}{u_1}\cos(s_{\bar{m}} - s_m), \tag{1}$$

$$0 = HJ_{rp}\sin(s_{\bar{m}} - s_m) + \frac{F}{u_1}\sin(s_{\bar{m}} - s_m), \tag{2}$$

where $H = \frac{\rho B_1(a_0)}{D}$ and $F = \frac{I B_1(a_0/2)}{\pi}$. Note that the result is similar to the calculation of the case $R(y,t) = \frac{\psi^2(y,t)}{1+\omega\sum_y \psi^2(y,t)}$ except that $H = \frac{\rho}{D}(u_0 + 2u_2)B_1(a_0)$ is different due to the square of ψ. Hence, the following computation should be the same, that is, $T \propto \frac{1}{u_1^2}$ still holds for constant noise. Meanwhile, $\hat{A} \propto \mod(\sum_y R(y,t)e^{jy}) \propto u_1$ is still true for output-dependent noise. In a word, the output-dependent noise is still necessary to improve the accuracy of the prediction.

2 Bayesian inference

2.1 Correlated prior

We consider the correlated prior $p(s_1, s_2) = \frac{1}{2\pi} V(s_1 - s_2, \kappa_s)$. The posterior probability $p(s_1, s_2|x_1, x_2)$ is given by

$$p(s_1, s_2|x_1, x_2) = \frac{p(x_1|s_1)p(x_2|s_2)p(s_1, s_2)}{p(x_1, x_2)}. \tag{3}$$

Hence, marginalizing $s_{\bar{m}}$ yields

$$p(s_m|x_1, x_2) = \frac{p(x_m|s_m) \int ds_{\bar{m}} p(x_{\bar{m}}|s_{\bar{m}}) p(s_m, s_{\bar{m}})}{\int ds_m ds_{\bar{m}} p(x_m|s_m) p(x_{\bar{m}}|s_{\bar{m}}) p(s_m, s_{\bar{m}})} \approx V(s_m - x_m^c, \kappa_m^c), \tag{4}$$

where $\kappa_m^c e^{jx_m^c} = \kappa_m e^{jx_m} + \kappa_{\bar{m}s} e^{jx_{\bar{m}}}$ and $A(\kappa_{\bar{m}s}) = A(\kappa_{\bar{m}})A(\kappa_s)$. We estimate $\hat{s}_m$ by integrating $p(s_m|x_1, x_2)e^{js_m}$ over s_m

$$e^{j\hat{s}_m} \equiv \int ds_m p(s_m|x_1, x_2) e^{js_m} = \frac{I_1(\kappa_m^c)}{I_0(\kappa_m^c)} e^{jx_m^c}. \tag{5}$$

However, $\frac{I_1(\kappa)}{I_0(\kappa)} \to 1$ for large concentration κ. We rewrite the result as a vector sum

$$\kappa_m^c e^{j\hat{s}_m} \approx \kappa_m e^{jx_m} + \kappa_{\bar{m}s} e^{jx_{\bar{m}}}, \tag{6}$$

then eliminate $x_{\bar{m}}$

$$\kappa_m^c e^{j\hat{s}_m} \approx \frac{\kappa_{\bar{m}s}\kappa_{\bar{m}}^c}{\kappa_{\bar{m}}} e^{j\hat{s}_{\bar{m}}} + (\kappa_m - \frac{\kappa_{ms}\kappa_{\bar{m}s}}{\kappa_{\bar{m}}}) e^{jx_m}. \tag{7}$$

However, consider the symmetric $\kappa_m = \kappa_{\bar{m}}$ and $\kappa_m^c \approx \kappa_m \gg \kappa_{ms}$, hence

$$\kappa_m^c e^{j\hat{s}_m} \approx \kappa_m e^{jx_m} + \kappa_{\bar{m}s} e^{js_{\bar{m}}}. \tag{8}$$

3 Two-component prior

Similarly, for two-component prior $p(s_1, s_2) = \frac{p_c}{2\pi} V(s_1 - s_2, \kappa_s) + \frac{1-p_c}{4\pi^2}$, the posterior probability is given as

$$p(s_1, s_2|x_1, x_2) \propto V(s_1 - x_1, \kappa_1) V(s_2 - x_2, \kappa_2) [\frac{p_c}{2\pi} V(s_1 - s_2, \kappa_s) + \frac{1 - p_c}{4\pi^2}]. \tag{9}$$

Marginalize $s_{\bar{m}}$

$$p(s_m|x_1, x_2) \approx \frac{p_c \alpha_m V(s_m - x_m^c, \kappa_m^c) + (1 - p_c) V(s_m - x_m, \kappa_m)}{p_c(\alpha_m - 1) + 1}, \tag{10}$$

where $\alpha_m \equiv \frac{I_0(\kappa_m^c)}{I_0(\kappa_m) I_0(\kappa_{\bar{m}s})}$. Then we can estimate $\hat{s}_m$

$$e^{j\hat{s}_m} \equiv \int ds_m p(s_m|x_1, x_2) e^{js_m} = \frac{p_c \alpha_m \frac{I_1(\kappa_m^c)}{I_0(\kappa_m^c)} e^{jx_m^c} + (1 - p_c) \frac{I_1(\kappa_m)}{I_0(\kappa_m)} e^{jx_m}}{p_c(\alpha_m - 1) + 1}. \tag{11}$$

Thus, the result can be presented as a vector sum

$$[p_c(\alpha_m - 1) + 1]\kappa_m^c e^{j\hat{s}_m} \approx [(1-p_c)\kappa_m^c + p_c\alpha_m\kappa_m]e^{jx_m} + p_c\alpha_m\kappa_{\bar{m}s}e^{jx_{\bar{m}}}. \quad (12)$$

Eliminating $x_{\bar{m}}$ yields

$$\begin{aligned}[p_c(\alpha_m - 1) + 1]\kappa_m^c e^{j\hat{s}_m} \approx & [(1-p_c)\kappa_m^c + p_c\alpha_m\kappa_m - \frac{p_c^2\alpha_m\alpha_{\bar{m}}\kappa_{ms}\kappa_{\bar{m}s}}{[(1-p_c)k_{\bar{m}}^c + p_c\alpha_{\bar{m}}\kappa_{\bar{m}}]}]e^{jx_m} \\ & + \frac{[p_c(\alpha_{\bar{m}} - 1) + 1]p_c\alpha_m\kappa_{\bar{m}}^c\kappa_{\bar{m}s}}{[(1-p_c)\kappa_{\bar{m}}^c + p_c\alpha_{\bar{m}}\kappa_{\bar{m}}]}. \quad (13)\end{aligned}$$

Consider the symmetry $\alpha_m = \alpha_{\bar{m}} = \alpha$ and simplify the expression

$$\kappa_m^c e^{j\hat{s}_m} \approx \kappa_m e^{jx_m} + \frac{p_c\alpha}{p_c(\alpha - 1) + 1}\kappa_{\bar{m}s}e^{j\hat{s}_{\bar{m}}}, \quad (14)$$

where $\alpha \approx 1/I_0(\kappa_s)$.