General Model

Circular Data Analysis

Our previous work reveals that the heading direction $s_m,\ m=1,2$ is subject to the normal distribution, that is, $s_m \sim \mathcal{N}(\mu, \sigma^2)$, where μ this the *mean direction* of s_m. The *mean resultant length* ρ is defined as

$$\rho = \mathbb{E}[\cos(s_m - \mu)]. \tag{1}$$

Assuming $\sigma \ll \pi$ ($\kappa > 1$), the *mean resultant length* is calculated by

$$\rho = \int_{-\pi}^{\pi} \cos(s_m) \frac{\exp[-\frac{(s_m-\mu)^2}{2\sigma^2}]}{\sigma\sqrt{2\pi}} ds_m \approx \int_{-\infty}^{\infty} \cos(s_m) \frac{\exp[-\frac{(s_m-\mu)^2}{2\sigma^2}]}{\sigma\sqrt{2\pi}} ds_m = e^{-\frac{\sigma^2}{2}}. \tag{2}$$

Definition of Circular Variance

If the *circular variance* $\hat{\sigma}^2$ is defined as

$$\hat{\sigma}^2 \equiv -2\ln\rho, \tag{3}$$

then $\hat{\sigma}^2$ is equivalent to σ^2. We are using $\frac{1}{\hat{\sigma}^2}$ to represent the *concentration parameter* $\hat{\kappa}$.

Zhang et al. assume s_m can be represented by a von Mises distribution, that is, $s_m \sim \mathcal{M}(\hat{s}_m, \hat{\kappa})$. The maximum likelihood of $\hat{\kappa}$ is the solution to

$$\rho = A(\hat{\kappa}), \tag{4}$$

where $A(x) \equiv I_1(x)/I_0(x)$. This solution is not easy to find for a large $\hat{\kappa}$ unless we are using the following approximation (see Fisher (1993) page 88 and Mardia & Jupp (2000) pages 85-86)

$$\hat{\kappa} = \begin{cases} 2\rho + \rho^3 + \frac{5\rho^5}{6}, & \rho < 0.53 \\ -0.4 + 1.39\rho + \frac{0.43}{1-\rho}, & 0.53 \le \rho < 0.85 \\ \frac{1}{3\rho - 4\rho^2 + \rho^3}, & \rho \ge 0.85. \end{cases} \tag{5}$$

Consider the Taylor expansion of $\hat{\kappa}$ centered at $\rho = 1$ for a large $\hat{\kappa}$, then $\hat{\kappa} \approx \frac{1}{3\rho-4\rho^2+\rho^3} \approx \frac{1}{2(1-\rho)} = \frac{1}{2(1-e^{-\sigma^2/2})} \approx \frac{1}{\sigma^2}$ is fine for $\rho \to 1$. That is reason why we still get right answer while using the Gaussian noise. However, after introducing the output-dependent noise, the concentration parameter might be smaller so that we can no longer use the von Mises distribution to approximate a normal distribution. In that case, we shall use Eq. (3) to calculate the variance of circular data.

Bayesian Inference

According to Bayes rule, the posterior $p(s_1, s_2|x_1, x_2)$ is given by

$$p(s_1, s_2|x_1, x_2) = \frac{p(x_1|s_1)p(x_2|s_2)p(s_1, s_2)}{p(x_1)p(x_2)}. \tag{6}$$

$p(x_m)$ and $p(s_m)$ are uniform distributions. Take the derivative of $p(s_1, s_2|x_1, x_2)$ with respect to s_1 to be zero

$$\kappa_s \sin(s_2 - s_1) + \kappa_1 \sin(x_1 - s_1) = 0, \tag{7}$$

that will allow us to find the peak position. We project the dynamic equation to the position mode, then we get

$$HJ_{rp} \sin(s_2 - s_1) + \frac{F}{u_1} \sin(x_1 - s_1) = 0. \tag{8}$$

Compared with Eq. (7), our network will encode the prior information in the following way

$$\frac{\kappa_s}{\kappa_1} = \frac{HJ_{rp}}{F} u_1. \tag{9}$$

Discussion

If we project the dynamic equation to the height mode $\cos(y_m - s_m)$ and position mode $\sin(y_m - s_m)$, these two equations can be combined together

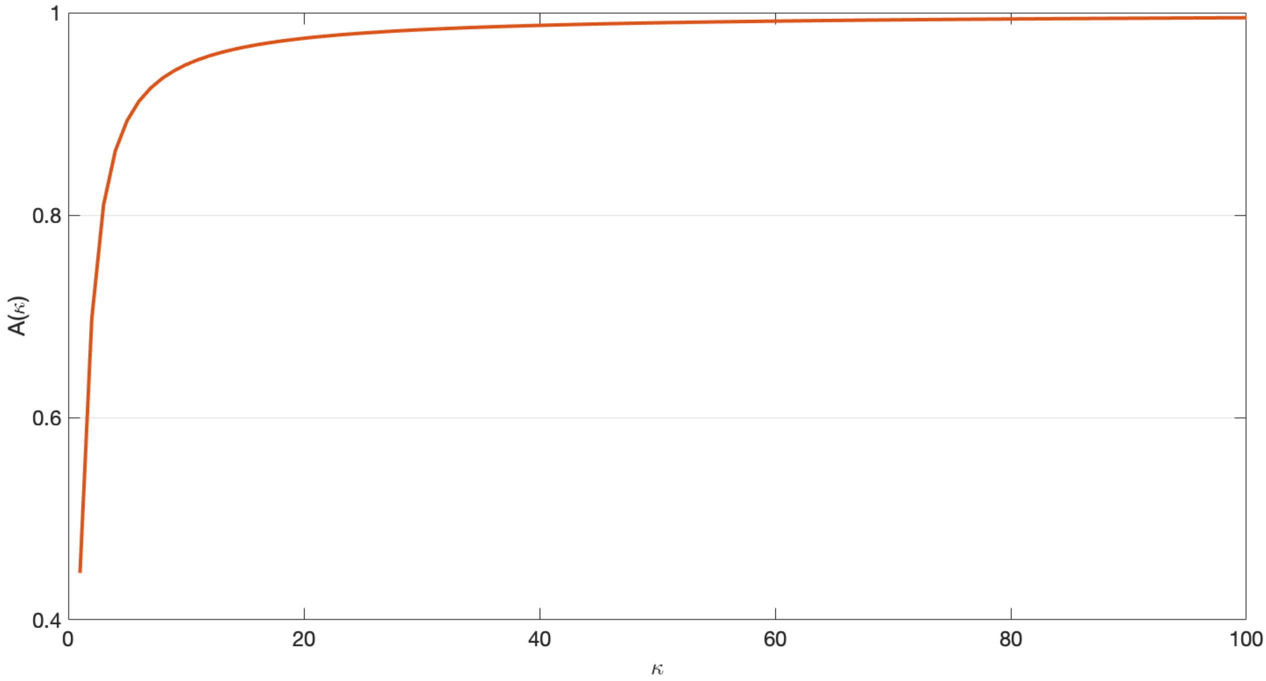
$$u_1 e^{js_1} = u_1 \frac{HJ_{rp}}{1 - HJ_{rc}} e^{js_2} + \frac{F}{1 - HJ_{rc}} e^{jx_1}, \tag{10}$$

where j is the imaginary unit. If we assume u_1 on the right hand side to be $\frac{F}{1-HJ_{rc}} e^{jx_1}$ and $s_2 \approx x_2$, we have $\hat{\kappa}_1 e^{js_1}|_{I_1, I_2} = \hat{\kappa}_1 e^{jx_1}|_{I_1} + \hat{\kappa}_{2s} e^{jx_2}$ in our CCN paper.

The ratio will be

$$\frac{\kappa_s}{\kappa_1} \approx \frac{HJ_{rp}}{1 - HJ_{rc}}. \tag{11}$$

However, the bias $\frac{\hat{s}_1 - x_1}{x_2 - x_1}|_{x_2 \to 0}$ and $\frac{\hat{\kappa}_{2s}}{\hat{\kappa}_1}$ are also approaching $\frac{HJ_{rp}}{1-HJ_{rc}}$. We maginalize s_2 then posterior will give us $\kappa_{2s} = A^{-1}[A(\kappa_2)A(\kappa_s)]$. When κ_2 is large ($\kappa_2 > 50$), we have $\kappa_2 \approx \kappa_{2s}$.



We project the dynamic equation to the height mode

$$1 = HJ_{rc} + HJ_{rp}\cos(\Delta s) + \frac{F}{u_1}\cos(x_1 - s_1). \tag{12}$$

If we take derivative of both sides with respect to s_1, that will give us Eq. (8). We maginalize s_2

$$p(s_1|x_1, x_2) \propto p(s_1|x_1)p(s_1|x_2), \tag{13}$$

according to vector diagram, we have

$$\kappa_1|_{I_1,I_2} = \kappa_{12}\cos(x_2 - s_1) + \kappa_1|_{I_1}\cos(x_1 - s_1). \tag{14}$$

Compared with Eq. (12), if we further assume $x_2 \approx s_2$, then

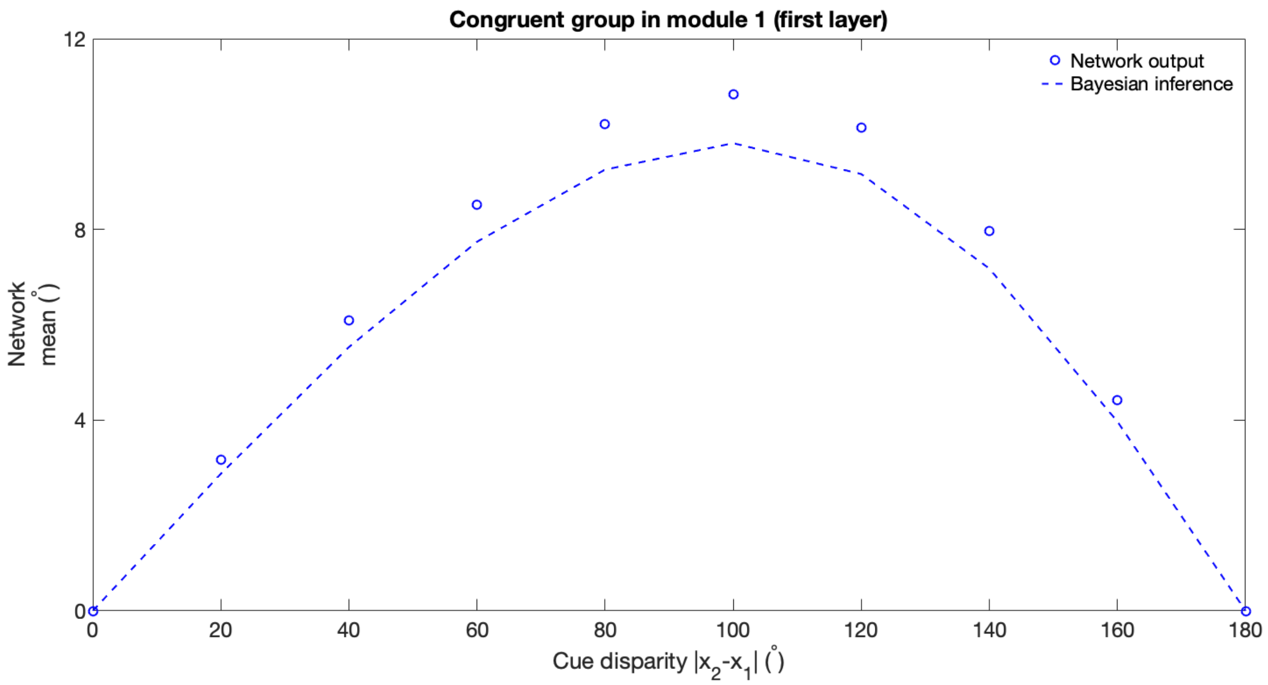
$$\begin{aligned} \hat{\kappa}_1|_{I_1,I_2} &\propto u_1, \\ \hat{\kappa}_{12} &\propto u_1\frac{HJ_{rp}}{1 - HJ_{rc}}, \\ \hat{\kappa}_1|_{I_1} &\propto \frac{F}{1 - HJ_{rc}}. \end{aligned} \tag{15}$$

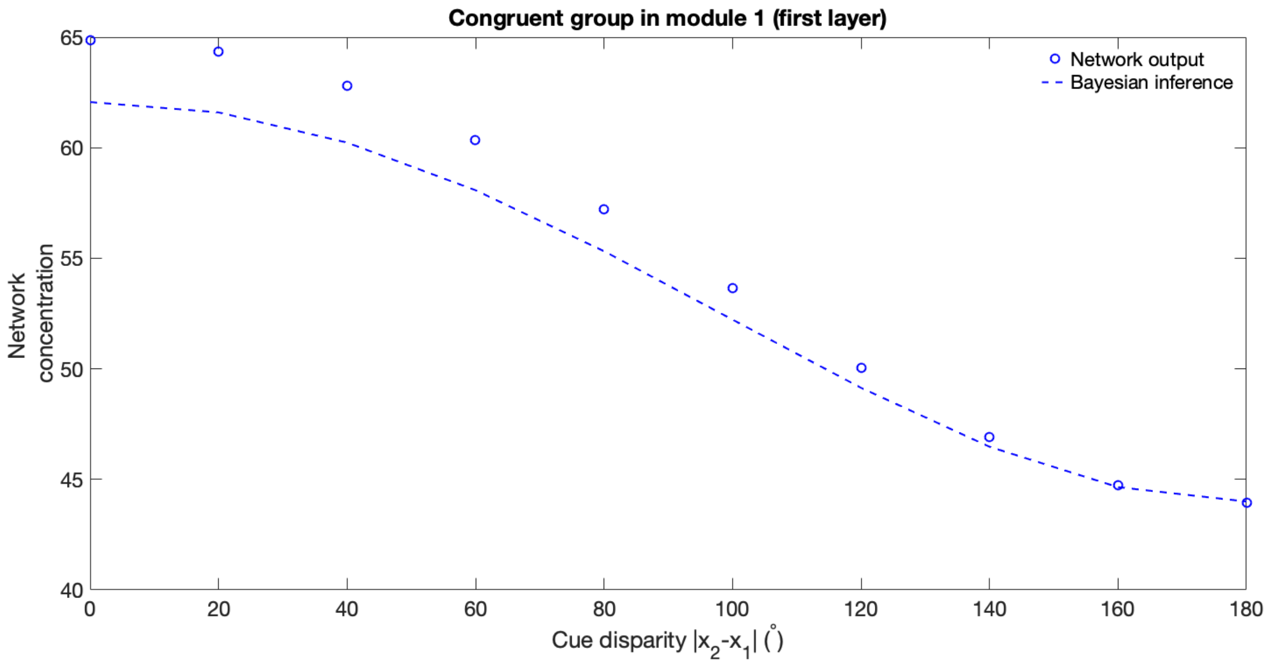
The coefficient should be $g_{mm}c_0$ based on the derivation of the noise variance

$$g_{mm}c_0 \approx \frac{\pi\tau ND}{F_0u_0} - \frac{2\pi\tau\rho NJ_{rc}B_1(a_0)}{F_0}. \tag{16}$$

Results

Our purpose is to encode the prior information in our network model, we compare our network outputs with Bayes' theorem.





Note that we take the average of $g_{mm}c_0$ among 10 trials. Besides, we have $\langle u_1 \rangle$ and $\langle H \rangle$ in the same way. κ_1 and κ_{12} are calculated by

$$\begin{aligned} \kappa_1 &\approx g_{mm}c_0 \frac{F}{1 - HJ_{rc}}, \\ \kappa_s/\kappa_1 &= \frac{HJ_{rp}}{F} u_1, \\ \kappa_{12} &= A^{-1}[A(\kappa_1)A(\kappa_s)]. \end{aligned} \tag{17}$$

Prior with Independent Component

The two-component prior is as follows

$$p(s_1, s_2) = \frac{p_0}{2\pi} V(s_1 - s_2, \kappa_s) + \frac{1 - p_0}{(2\pi)^2}. \tag{18}$$

The marginal posterior $p(s_1|x_1,x_2)$ is given by

$$p(s_1|x_1,x_2) = p_0 CV(s_1-x_1,\kappa_1)V(s_1-x_2,\kappa_{2s}) + (1-p_0)V(s_1-x_1,\kappa_1), \tag{19}$$

where $C=\frac{2\pi I_0(\kappa_1)I_0(\kappa_{2s})}{I_0(\sqrt{\kappa_1^2+\kappa_{2s}^2})}$, differentiation with respect to s_1 yields

$$\begin{aligned}\frac{\partial}{\partial s_1}p(s_1|x_1,x_2) =& -p_0 CV(s_1-x_1,\kappa_1)V(s_1-x_2,\kappa_{2s})[\kappa_1\sin(s_1-x_1)+\kappa_{2s}\sin(s_1-x_2)]\\ &-(1-p_0)V(s_1-x_1,\kappa_1)\sin(s_1-x_1).\end{aligned} \tag{20}$$

We project the dynamic equation of the second layer to the height mode and the position mode as below

$$u_1' = (1-p_0)\frac{F}{1-H'J_{rc}}\cos(x_1-s_1') + p_0\frac{c_k H u_1}{J_{rc}B_1(a_0)(1-H'J_{rc})}\cos(s_1-s_1'), \tag{21}$$

$$0 = (1-p_0)\frac{F}{1-H'J_{rc}}\sin(x_1-s_1') + p_0\frac{c_k H u_1}{J_{rc}B_1(a_0)(1-H'J_{rc})}\sin(s_1-s_1'). \tag{22}$$

Taking the logarithm of $p(s_1|x_1,x_2)$ yields

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\begin{equation}
  \ln[p(s_1|x_1,x_2)]=\ln[1-p_0+p_0CV(s_1-x_2,\kappa_{2s})]+\kappa_1\cos(s_1-x_1)-\ln[I_0(\kappa_1)]
\end{equation}
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$$\ln[p(s_1|x_1,x_2)] = \ln[1-p_0+p_0CV(s_1-x_2,\kappa_{2s})] + \kappa_1\cos(s_1-x_1) - \ln[I_0(\kappa_1)] \tag{1}$$

$$\ln[p(s_1|x_1,x_2)] = \ln[1-p_0+p_0CV(s_1-x_2,\kappa_{2s})] + \kappa_1\cos(s_1-x_1) - \ln[I_0(\kappa_1)] \tag{23}$$