

Notes for multisensory integration and segregation

Xiangyu Ma

1 Noisy dynamics of two layers

In the first and the second layer, the dynamics of congruent groups of neurons in the presence of noise can be approximated by

$$\begin{aligned} \tau \frac{\partial \psi_m(y, t)}{\partial t} = & -\psi_m(y, t) + \sum_{y'=-\pi}^{\pi} J_{rc} V(y - y', a_0) R(y', t) + \sum_{y'=-\pi}^{\pi} J_{rp} V(y - y', a_0) \bar{R}(y', t) \\ & + I_m V(y - x, \frac{a_0}{2}) + I_b + \sqrt{F_0 \hat{A}_m} \xi_m(y, t), \end{aligned} \quad (1)$$

$$\begin{aligned} \tau \frac{\partial \psi_{2m}(y)}{\partial t} = & -\psi_{2m}(y) + \sum_{y'=-\pi}^{\pi} J_{rc} V(y - y', a_0) R_{2m}(y', t) + p_0 \sum_{y'=-\pi}^{\pi} c_k \cos(y - y') R_m(y', t) \\ & + (1 - p_0) I_m V(y - x, \frac{a_0}{2}) + I_b + \sqrt{F_0 \hat{A}_{2m}} \epsilon(y, t), \end{aligned} \quad (2)$$

where $c_k = \frac{J_{rc} + J_{rp}}{1 - \sqrt{1 - 4\rho(J_{rc} + J_{rp})}}$, F_0 is the Fano factor and \hat{A} is length of the population vector, ξ_m and ϵ are Gaussian white noise of zero mean and variance satisfying $\langle \xi_m(y, t), \xi_{m'}(y', t') \rangle = \delta_{mm'} \delta(y - y') \delta(t - t')$ and $\langle \epsilon(y, t), \epsilon(y', t') \rangle = \delta(y - y') \delta(t - t')$. Consider the dynamics of displacement mode and multiply both sides by $\sin(y - s)$, integrate over y

$$\begin{aligned} \tau \frac{\partial}{\partial t} \delta s_m = & -\delta s_m + \frac{\rho J_{rc}}{D_m u_{m1}} B_1(a_0) [2u_{m0} u_{m1} + u_{m1} u_{m2}] \delta s_m \\ & + \frac{\rho J_{rp}}{D_{\bar{m}} u_{m1}} B_1(a_0) [(2u_{\bar{m}0} u_{\bar{m}1} + u_{\bar{m}1} u_{\bar{m}2}) \cos(s_{\bar{m}} - s_m) - u_{\bar{m}1} u_{\bar{m}3} \sin(s_{\bar{m}} - s_m)] \delta s_{\bar{m}} \\ & + \frac{\sqrt{F_0 \hat{A}_m}}{\pi u_{m1}} \int \sin(y_m - s_m) \xi_m dy_m, \end{aligned} \quad (3)$$

$$\begin{aligned} \tau \frac{\partial}{\partial t} \delta s_{2m} = & -\delta s_{2m} + \frac{\rho J_{rc}}{D_{2m} u_{2m1}} B_1(a_0) [2u_{2m0} u_{2m1} + u_{2m1} u_{2m2}] \delta s_{2m} \\ & + \frac{p_0 c_k N}{2 D_m u_{2m1}} [(2u_{m0} u_{m1} + u_{m1} u_{m2}) \cos(s_m - s_{2m}) - u_{m1} u_{m3} \sin(s_m - s_{2m})] \delta s_m \\ & + \frac{\sqrt{F_0 \hat{A}_{2m}}}{\pi u_{2m1}} \int \sin(y_{2m} - s_{2m}) \epsilon dy_m, \end{aligned} \quad (4)$$

where $B_n(k) = \frac{I_n(k)}{I_0(k)}$. Together with the dynamics of independent module,

$$\tau \frac{\partial}{\partial t} \begin{bmatrix} \delta s_m \\ \delta s_{\bar{m}} \\ \delta s_{2m} \end{bmatrix} = - \begin{bmatrix} g_{11} & g_{12} & 0 \\ g_{21} & g_{22} & 0 \\ g_{31} & 0 & g_{33} \end{bmatrix} \begin{bmatrix} \delta s_m \\ \delta s_{\bar{m}} \\ \delta s_{2m} \end{bmatrix} + \begin{bmatrix} \xi_m \\ \xi_{\bar{m}} \\ \epsilon \end{bmatrix}. \quad (5)$$

The eigenvalues are $\lambda_{\pm} = \frac{1}{2}(g_{11} + g_{22} \pm \sqrt{D})$, g_{33} , where $D = (g_{22} - g_{11})^2 + 4g_{12}g_{21}$. Diagonalize the matrix \mathbf{G} , $\mathbf{G} = \mathbf{S} \text{diag}(\lambda_+, \lambda_-, g_{33}) \mathbf{S}^{-1}$. The block matrix \mathbf{S} will be

$$\mathbf{S} = \begin{bmatrix} -g_{12} & -g_{12} & 0 \\ \frac{1}{2}(g_{11} - g_{22} - \sqrt{D}) & \frac{1}{2}(g_{11} - g_{22} + \sqrt{D}) & 0 \\ \frac{-g_{12}g_{31}}{\frac{1}{2}(g_{11}+g_{22}+\sqrt{D})-g_{33}} & \frac{-g_{12}g_{31}}{\frac{1}{2}(g_{11}+g_{22}-\sqrt{D})-g_{33}} & 1 \end{bmatrix}, \quad (6)$$

with $\det(\mathbf{S}) = -g_{12}\sqrt{D}$, find the inverse \mathbf{S}^{-1} of that matrix

$$\mathbf{S}^{-1} = \begin{bmatrix} -\frac{g_{11}-g_{22}+\sqrt{D}}{2g_{12}\sqrt{D}} & -\frac{1}{\sqrt{D}} & 0 \\ \frac{g_{11}-g_{22}-\sqrt{D}}{2g_{12}\sqrt{D}} & \frac{1}{\sqrt{D}} & 0 \\ \frac{g_{31}(g_{33}-g_{22})}{g_{33}^2-(g_{11}+g_{22})g_{33}+g_{11}g_{22}-g_{12}g_{21}} & \frac{g_{12}g_{31}}{g_{33}^2-(g_{11}+g_{22})g_{33}+g_{11}g_{22}-g_{12}g_{21}} & 1 \end{bmatrix}. \quad (7)$$

The equivalent noise temperature is given by

$$T_m = \frac{F_0 \hat{A}_m}{2\pi^2 \rho u_{m1}^2} \int \sin^2(y_m - s_m) dy_m = \frac{F_0 \sqrt{(2u_{m0} + u_{m2})^2 + u_{m3}^2}}{2ND_m u_{m1}}, \quad (8)$$

$$T_{2m} = \frac{F_0 \hat{A}_{2m}}{2\pi^2 \rho u_{2m1}^2} \int \sin^2(y_{2m} - s_{2m}) dy_m = \frac{F_0 \sqrt{(2u_{2m0} + u_{2m2})^2 + u_{2m3}^2}}{2ND'_m u_{2m1}}. \quad (9)$$

Consider the steady-state solutions of LTI SDEs of the form

$$\tau d\mathbf{x} = -\mathbf{G}\mathbf{x}dt + \mathbf{I}d\beta, \quad (10)$$

where β is the Brownian motion and \mathbf{I} is the identity matrix. The diffusion matrix \mathbf{Q} can be found according to the rule

$$d\beta d\beta^T = \mathbf{Q}dt. \quad (11)$$

At steady state, the time derivative of covariance \mathbf{P} should be zero, that is

$$\mathbf{G}\mathbf{P} + \mathbf{P}\mathbf{G}^T = \frac{1}{\tau}\mathbf{Q}, \quad (12)$$

where $\mathbf{Q} = \text{diag}(2T_m, 2T_{\bar{m}}, 2T_{2m})$. This is so called the Lyapunov equation, and the trick to calculate \mathbf{P} is to find $\tilde{\mathbf{Q}} = \mathbf{S}^{-1}\mathbf{Q}(\mathbf{S}^{-1})^T$ first, then compare two matrices element by element

$$\tilde{p}_{ij} = \frac{\tilde{q}_{ij}}{\lambda_i + \lambda_j}, \quad (13)$$

where λ_i , $i = 1, 2, 3$ are eigenvalues of \mathbf{G} . Finally, we get the covariance $\mathbf{P} = \mathbf{S}\tilde{\mathbf{P}}\mathbf{S}^T$.

The variances are derived bellow

$$\begin{aligned}
\tau\sigma_m^2 &= T_m \left[\frac{4s_{11}s_{12}s_{11}^{-1}s_{21}^{-1}}{\lambda_1 + \lambda_2} + \frac{s_{12}^2(s_{21}^{-1})^2}{\lambda_2} + \frac{s_{11}^2(s_{11}^{-1})^2}{\lambda_1} \right] \\
&\quad + T_{\bar{m}} \left[\frac{4s_{11}s_{12}s_{12}^{-1}s_{22}^{-1}}{\lambda_1 + \lambda_2} + \frac{s_{12}^2(s_{22}^{-1})^2}{\lambda_2} + \frac{s_{11}^2(s_{12}^{-1})^2}{\lambda_1} \right], \\
\tau\sigma_{\bar{m}}^2 &= T_m \left[\frac{4s_{21}s_{22}s_{11}^{-1}s_{21}^{-1}}{\lambda_1 + \lambda_2} + \frac{s_{22}^2(s_{21}^{-1})^2}{\lambda_2} + \frac{s_{21}^2(s_{11}^{-1})^2}{\lambda_1} \right] \\
&\quad + T_{\bar{m}} \left[\frac{4s_{21}s_{22}s_{12}^{-1}s_{22}^{-1}}{\lambda_1 + \lambda_2} + \frac{s_{22}^2(s_{22}^{-1})^2}{\lambda_2} + \frac{s_{21}^2(s_{12}^{-1})^2}{\lambda_1} \right], \\
\tau\sigma_{2m}^2 &= \frac{T_{2m}}{\lambda_3} + T_m \left[\frac{4s_{31}s_{32}s_{11}^{-1}s_{21}^{-1}}{\lambda_1 + \lambda_2} + \frac{4s_{31}s_{11}^{-1}s_{31}^{-1}}{\lambda_1 + \lambda_3} + \frac{4s_{32}s_{21}^{-1}s_{31}^{-1}}{\lambda_2 + \lambda_3} + \frac{(s_{31}^{-1})^2}{\lambda_3} + \frac{s_{32}^2(s_{21}^{-1})^2}{\lambda_2} + \frac{s_{31}^2(s_{11}^{-1})^2}{\lambda_1} \right] \\
&\quad + T_{\bar{m}} \left[\frac{4s_{31}s_{32}s_{12}^{-1}s_{22}^{-1}}{\lambda_1 + \lambda_2} + \frac{4s_{31}s_{12}^{-1}s_{32}^{-1}}{\lambda_1 + \lambda_3} + \frac{4s_{32}s_{22}^{-1}s_{32}^{-1}}{\lambda_2 + \lambda_3} + \frac{(s_{32}^{-1})^2}{\lambda_3} + \frac{s_{32}^2(s_{22}^{-1})^2}{\lambda_2} + \frac{s_{31}^2(s_{12}^{-1})^2}{\lambda_1} \right].
\end{aligned}$$

The concentration κ is the inverse of the variance.