# Notes for multisensory integration and segregation

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## 1 Noisy dynamics of two layers

In the first and the second layer, the dynamics of congruent groups of neurons in the presence of noise can by approximated by

$$\begin{aligned}\tau\frac{\partial\psi_m(y,t)}{\partial t} =& -\psi_m(y,t) + \sum_{y'=-\pi}^{\pi} J_{rc}V(y-y',a_0)R(y',t) + \sum_{y'=-\pi}^{\pi} J_{rp}V(y-y',a_0)\bar{R}(y',t)\\ &+ I_mV(y-x,\frac{a_0}{2}) + I_b + \sqrt{F_0\hat{A}_m}\xi_m(y,t), \qquad (1)\\ \tau\frac{\partial\psi_{2m}(y)}{\partial t} =& -\psi_{2m}(y) + \sum_{y'=-\pi}^{\pi} J_{rc}V(y-y',a_0)R_{2m}(y',t) + p_0\sum_{y'=-\pi}^{\pi} c_k\cos(y-y')R_m(y',t)\\ &+ (1-p_0)I_mV(y-x,\frac{a_0}{2}) + I_b + \sqrt{F_0\hat{A}_{2m}}\epsilon(y,t), \qquad (2)\end{aligned}$$

where $c_k = \frac{J_{rc}+J_{rp}}{1-\sqrt{1-4\rho(J_{rc}+J_{rp})}}$, $F_0$ is the Fano factor and $\hat{A}$ is length of the population vector, $\xi_m$ and $\epsilon$ are Gaussian white noise of zero mean and variance satisfying $\langle\xi_m(y,t),\xi_{m'}(y',t')\rangle = \delta_{mm'}\delta(y-y')\delta(t-t')$ and $\langle\epsilon(y,t),\epsilon(y',t')\rangle = \delta(y-y')\delta(t-t')$. Consider the dynamics of displacement mode and multiply both sides by $\sin(y-s)$, integrate over $y$

$$\begin{aligned}&\tau\frac{\partial}{\partial t}\delta s_m = -\delta s_m + \frac{\rho J_{rc}}{D_m u_{m1}}B_1(a_0)\left[2u_{m0}u_{m1} + u_{m1}u_{m2}\right]\delta s_m\\ &+ \frac{\rho J_{rp}}{D_{\bar{m}}u_{m1}}B_1(a_0)\left[(2u_{\bar{m}0}u_{\bar{m}1} + u_{\bar{m}1}u_{\bar{m}2})\cos(s_{\bar{m}}-s_m) - u_{\bar{m}1}u_{\bar{m}3}\sin(s_{\bar{m}}-s_m)\right]\delta s_{\bar{m}}\\ &+ \frac{\sqrt{F_0\hat{A}_m}}{\pi u_{m1}}\int\sin(y_m-s_m)\xi_m dy_m, \qquad (3)\\ &\tau\frac{\partial}{\partial t}\delta s_{2m} = -\delta s_{2m} + \frac{\rho J_{rc}}{D_{2m}u_{2m1}}B_1(a_0)\left[2u_{2m0}u_{2m1} + u_{2m1}u_{2m2}\right]\delta s_{2m}\\ &+ \frac{p_0c_kN}{2D_mu_{2m1}}\left[(2u_{m0}u_{m1} + u_{m1}u_{m2})\cos(s_m-s_{2m}) - u_{m1}u_{m3}\sin(s_m-s_{2m})\right]\delta s_m\\ &+ \frac{\sqrt{F_0\hat{A}_{2m}}}{\pi u_{2m1}}\int\sin(y_{2m}-s_{2m})\epsilon dy_m, \qquad (4)\end{aligned}$$

where $B_n(k) = \frac{I_n(k)}{I_0(k)}$. Together with the dynamics of independent module,

$$\tau \frac{\partial}{\partial t} \begin{bmatrix} \delta s_m \\ \delta s_{\bar{m}} \\ \delta s_{2m} \end{bmatrix} = - \begin{bmatrix} g_{11} & g_{12} & 0 \\ g_{21} & g_{22} & 0 \\ g_{31} & 0 & g_{33} \end{bmatrix} \begin{bmatrix} \delta s_m \\ \delta s_{\bar{m}} \\ \delta s_{2m} \end{bmatrix} + \begin{bmatrix} \xi_m \\ \xi_{\bar{m}} \\ \epsilon \end{bmatrix}. \tag{5}$$

The eigenvalues are $\lambda_\pm = \frac{1}{2}(g_{11} + g_{22} \pm \sqrt{D}),\ g_{33}$, where $D = (g_{22} - g_{11})^2 + 4g_{12}g_{21}$. Diagonalize the matrix $\mathbf{G}$, $\mathbf{G} = \mathbf{S}\mathrm{diag}(\lambda_+, \lambda_-, g_{33})\mathbf{S^{-1}}$. The block matrix $\mathbf{S}$ will be

$$\mathbf{S} = \begin{bmatrix} -g_{12} & -g_{12} & 0 \\ \frac{1}{2}(g_{11} - g_{22} - \sqrt{D}) & \frac{1}{2}(g_{11} - g_{22} + \sqrt{D}) & 0 \\ \frac{-g_{12}g_{31}}{\frac{1}{2}(g_{11}+g_{22}+\sqrt{D})-g_{33}} & \frac{-g_{12}g_{31}}{\frac{1}{2}(g_{11}+g_{22}-\sqrt{D})-g_{33}} & 1 \end{bmatrix}, \tag{6}$$

with $\det(\mathbf{S}) = -g_{12}\sqrt{D}$, find the inverse $\mathbf{S^{-1}}$ of that matrix

$$\mathbf{S^{-1}} = \begin{bmatrix} -\frac{g_{11}-g_{22}+\sqrt{D}}{2g_{12}\sqrt{D}} & -\frac{1}{\sqrt{D}} & 0 \\ \frac{g_{11}-g_{22}-\sqrt{D}}{2g_{12}\sqrt{D}} & \frac{1}{\sqrt{D}} & 0 \\ \frac{g_{31}(g_{33}-g_{22})}{g_{33}^2-(g_{11}+g_{22})g_{33}+g_{11}g_{22}-g_{12}g_{21}} & \frac{g_{12}g_{31}}{g_{33}^2-(g_{11}+g_{22})g_{33}+g_{11}g_{22}-g_{12}g_{21}} & 1 \end{bmatrix}. \tag{7}$$

The equivalent noise temperature is given by

$$T_m = \frac{F_0\hat{A}_m}{2\pi^2\rho u_{m1}^2} \int \sin^2(y_m - s_m)dy_m = \frac{F_0\sqrt{(2u_{m0} + u_{m2})^2 + u_{m3}^2}}{2ND_m u_{m1}}, \tag{8}$$

$$T_{2m} = \frac{F_0\hat{A}_{2m}}{2\pi^2\rho u_{2m1}^2} \int \sin^2(y_{2m} - s_{2m})dy_m = \frac{F_0\sqrt{(2u_{2m0} + u_{2m2})^2 + u_{2m3}^2}}{2ND'_m u_{2m1}}. \tag{9}$$

Consider the steady-state solutions of LTI SDEs of the form

$$\tau d\mathbf{x} = -\mathbf{G}\mathbf{x}dt + \mathbf{I}d\beta, \tag{10}$$

where $\beta$ is the Brownian motion and $\mathbf{I}$ is the identity matrix. The diffusion matrix $\mathbf{Q}$ can be found according to the rule

$$d\beta d\beta^T = \mathbf{Q}dt. \tag{11}$$

At steady state, the time derivative of covariance $\mathbf{P}$ should be zero, that is

$$\mathbf{GP} + \mathbf{PG}^T = \frac{1}{\tau}\mathbf{Q}, \tag{12}$$

where $\mathbf{Q} = \mathrm{diag}(2T_m, 2T_{\bar{m}}, 2T_{2m})$. This is so called the Lyapunov equation, and the trick to calculate $\mathbf{P}$ is to find $\tilde{\mathbf{Q}} = \mathbf{S^{-1}Q(S^{-1})}^T$ first, then compare two matrices element by element

$$\tilde{p}_{ij} = \frac{\tilde{q}_{ij}}{\lambda_i + \lambda_j}, \tag{13}$$

where $\lambda_i,\ i = 1, 2, 3$ are eigenvalues of $\mathbf{G}$. Finally, we get the covariance $\mathbf{P} = \mathbf{S}\tilde{\mathbf{P}}\mathbf{S}^T$.

The variances are derived bellow

$$\tau\sigma_m^2 = T_m[\frac{4s_{11}s_{12}s_{11}^{-1}s_{21}^{-1}}{\lambda_1+\lambda_2} + \frac{s_{12}^2(s_{21}^{-1})^2}{\lambda_2} + \frac{s_{11}^2(s_{11}^{-1})^2}{\lambda_1}]$$
$$+ T_{\bar{m}}[\frac{4s_{11}s_{12}s_{12}^{-1}s_{22}^{-1}}{\lambda_1+\lambda_2} + \frac{s_{12}^2(s_{22}^{-1})^2}{\lambda_2} + \frac{s_{11}^2(s_{12}^{-1})^2}{\lambda_1}],$$
$$\tau\sigma_{\bar{m}}^2 = T_m[\frac{4s_{21}s_{22}s_{11}^{-1}s_{21}^{-1}}{\lambda_1+\lambda_2} + \frac{s_{22}^2(s_{21}^{-1})^2}{\lambda_2} + \frac{s_{21}^2(s_{11}^{-1})^2}{\lambda_1}]$$
$$+ T_{\bar{m}}[\frac{4s_{21}s_{22}s_{12}^{-1}s_{22}^{-1}}{\lambda_1+\lambda_2} + \frac{s_{22}^2(s_{22}^{-1})^2}{\lambda_2} + \frac{s_{21}^2(s_{12}^{-1})^2}{\lambda_1}],$$
$$\tau\sigma_{2m}^2 = T_m[\frac{4s_{31}s_{32}s_{11}^{-1}s_{21}^{-1}}{\lambda_1+\lambda_2} + \frac{4s_{31}s_{11}^{-1}s_{31}^{-1}}{\lambda_1+\lambda_3} + \frac{4s_{32}s_{21}^{-1}s_{31}^{-1}}{\lambda_2+\lambda_3} + \frac{(s_{31}^{-1})^2}{\lambda_3} + \frac{s_{32}^2(s_{21}^{-1})^2}{\lambda_2} + \frac{s_{31}^2(s_{11}^{-1})^2}{\lambda_1}]$$
$$+ T_{\bar{m}}[\frac{4s_{31}s_{32}s_{12}^{-1}s_{22}^{-1}}{\lambda_1+\lambda_2} + \frac{4s_{31}s_{12}^{-1}s_{32}^{-1}}{\lambda_1+\lambda_3} + \frac{4s_{32}s_{22}^{-1}s_{32}^{-1}}{\lambda_2+\lambda_3} + \frac{(s_{32}^{-1})^2}{\lambda_3} + \frac{s_{32}^2(s_{22}^{-1})^2}{\lambda_2} + \frac{s_{31}^2(s_{12}^{-1})^2}{\lambda_1}] + \frac{T_{2m}}{\lambda_3}.$$

The concentration $\kappa$ is the inverse of the variance.

# 2 The origin of the output-dependent noise

In order to implement Bayesian prediction, we define the length of population vector $\hat{A} \equiv \mathrm{mod}\ (\frac{1}{N}\sum_{y=-\pi}^{\pi} R_m(y)e^{jy}) = \frac{u_1}{2D}\sqrt{(2u_0+u_2)^2+u_3^2}$, where $j$ is the imaginary unit. We find that the output-dependent noise $\sqrt{F_0\hat{A}}\epsilon_m$, can efficiently improve the accuracy of the Bayesian prediction.

According to the manuscript, by taking the phase oscillations from the global inhibition into consideration, the fluctuation becomes

$$\langle\delta D_m^2\rangle = \langle\delta D_m^2\rangle_{\Delta=0}\exp{(-\Delta/2)}\int_0^\infty dt\frac{t^2}{\sqrt{t^2+\Delta}}\exp{(-t^2/2)}, \tag{14}$$

where $\Delta = \frac{\theta^2}{u_1^2\langle\delta\phi^2\rangle}$, $\theta$ is the threshold. It's known that $\langle\delta D_m^2\rangle_{\Delta=0}$ is proportional to $u_1$, hence yields the result

$$\langle\delta D_m^2\rangle = \frac{\sqrt{\pi}}{2}\exp{(-\Delta/2)}U(\frac{1}{2},0,\frac{\Delta}{2})\langle\delta D_m^2\rangle_{\Delta=0}, \tag{15}$$

where $U(a,b,x) = \frac{1}{\Gamma(a)}\int_0^\infty dt\exp{(-xt)}t^{a-1}(1+t)^{b-a-1}$ is the confluent hypergeometric function of the second kind. When $\langle\phi^2\rangle \gg \theta^2$, consider the Taylor series at $x=0$

$$\frac{\sqrt{\pi}}{2}\exp{(-\Delta/2)}U(\frac{1}{2},0,\frac{\Delta}{2}) \approx 1+\frac{\Delta}{4}(\ln{(\Delta/2)}-1.8). \tag{16}$$

In our network, with assuming $\kappa \propto u_1$, $\frac{\kappa_{max}-\kappa_{min}}{\kappa_{max}+\kappa_{min}} \approx \sin 10°$, $u_1$ varies from $0.7u_{max}$ to $u_{max}$, meanwhile, $\Delta$ varies from $\Delta_{max}$ to $0.5\Delta_{max}$. That function decays quickly with respective to $u_1$, that is, no longer have $\langle\delta D_m^2\rangle \propto u_1$.