Functional Programming and Verification

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Exercise Sheet 11 Deadline: 20.01.2019

General Information

Detailed information about the lecture, tutorials and homework assignments can be found on the lecture website[1]. Solutions have to be submitted to Moodle[2]. Make sure your uploaded documents are readable. Blurred images will be rejected. Use Piazza[3] to ask questions and discuss with your fellow students.

Big-step proofs

Unless specified otherwise, all rules used in a big-step proof tree must be annotated and all axioms ($v \Rightarrow v$) must be written down.

Assignment 11.1 (L) Big Steps

We define these functions:

```
let rec f = fun l ->
  match l with [] -> 1 | x::xs -> x + g xs
and g = fun l ->
  match l with [] -> 0 | x::xs -> x * f xs
```

Consider the following expressions. Find the values they evaluate to and construct a big-step proof for that claim.

1. `let f = fun a -> (a+1,a-1)::[] in f 7`
2. `f [3;6]`
3. `(fun x -> x 3) (fun y z -> z y) (fun w -> w + w)`

Suggested Solution 11.1

1. Big step tree:

$$\pi_0 = \text{LI}\,\cfrac{\text{TU}\,\cfrac{\text{OP}\,\cfrac{\texttt{7} \Rightarrow \texttt{7} \quad \texttt{1} \Rightarrow \texttt{1} \quad 7+1 \Rightarrow \texttt{8}}{\texttt{7+1} \Rightarrow \texttt{8}} \quad \text{OP}\,\cfrac{\texttt{7} \Rightarrow \texttt{7} \quad \texttt{1} \Rightarrow \texttt{1} \quad 7-1 \Rightarrow \texttt{6}}{\texttt{7-1} \Rightarrow \texttt{6}}}{\texttt{(7+1,7-1)} \Rightarrow \texttt{(8,6)}}}{\texttt{[(7+1,7-1)]} \Rightarrow \texttt{[(8,6)]}}$$

[1] https://www.in.tum.de/i02/lehre/wintersemester-1819/vorlesungen/functional-programming-and-verification/
[2] https://www.moodle.tum.de/course/view.php?id=44932
[3] https://piazza.com/tum.de/fall2018/in0003/home

$$\text{LD}\ \dfrac{\texttt{fun a -> [(a+1,a-1)]} \Rightarrow \texttt{fun a -> [(a+1,a-1)]} \qquad \text{APP'}\ \dfrac{\texttt{fun a -> [(a+1,a-1)]} \Rightarrow \texttt{fun a -> [(a+1,a-1)]} \quad \texttt{7} \Rightarrow \texttt{7} \quad \pi_0}{\texttt{(fun a -> [(a+1,a-1)]) 7} \Rightarrow \texttt{[(8,6)]}}}{\texttt{let f = fun a -> [(a+1,a-1)] in f 7} \Rightarrow \texttt{[(8,6)]}}$$

2. Big step tree:

$$\pi_f = \text{GD}\ \dfrac{\texttt{f} = \texttt{fun l -> match l with [] -> 1 | x::xs -> x+g xs} \quad \texttt{fun l -> match l with [] -> 1 | x::xs -> x+g xs} \Rightarrow \texttt{fun l -> match l with [] -> 1 | x::xs -> x+g xs}}{\texttt{f} \Rightarrow \texttt{fun l -> match l with [] -> 1 | x::xs -> x+g xs}}$$

$$\pi_g = \text{GD}\ \dfrac{\texttt{g} = \texttt{fun l -> match l with [] -> 0 | x::xs -> x*f xs} \quad \texttt{fun l -> match l with [] -> 0 | x::xs -> x*f xs} \Rightarrow \texttt{fun l -> match l with [] -> 0 | x::xs -> x*f xs}}{\texttt{g} \Rightarrow \texttt{fun l -> match l with [] -> 0 | x::xs -> x*f xs}}$$

$$\pi_0 = \text{PM}\ \dfrac{\texttt{[6]} \Rightarrow \texttt{[6]} \quad \text{OP}\ \dfrac{\texttt{6} \Rightarrow \texttt{6} \quad \text{APP'}\ \dfrac{\pi_f \quad \texttt{[]} \Rightarrow \texttt{[]} \quad \text{PM}\ \dfrac{\texttt{[]} \Rightarrow \texttt{[]} \quad \texttt{1} \Rightarrow \texttt{1}}{\texttt{match [] with [] -> 1 | x::xs -> x+g xs} \Rightarrow \texttt{1}}}{\texttt{f []} \Rightarrow \texttt{1}} \quad 6*1 \Rightarrow 6}{\texttt{6*f []} \Rightarrow \texttt{6}}}{\texttt{match [6] with [] -> 0 | x::xs -> x*f xs} \Rightarrow \texttt{6}}$$

$$\text{APP'}\ \dfrac{\pi_f \quad \texttt{[3;6]} \Rightarrow \texttt{[3;6]} \quad \text{PM}\ \dfrac{\texttt{[3;6]} \Rightarrow \texttt{[3;6]} \quad \text{OP}\ \dfrac{\texttt{3} \Rightarrow \texttt{3} \quad \text{APP'}\ \dfrac{\pi_g \quad \texttt{[6]} \Rightarrow \texttt{[6]} \quad \pi_0}{\texttt{g [6]} \Rightarrow \texttt{6}} \quad 3+6 \Rightarrow 9}{\texttt{3+g [6]} \Rightarrow \texttt{9}}}{\texttt{match [3;6] with [] -> 1 | x::xs -> x+g xs} \Rightarrow \texttt{9}}}{\texttt{f [3;6]} \Rightarrow \texttt{9}}$$

3. Big step tree:

$$\pi_0 = \text{APP'}\ \dfrac{\texttt{fun x -> x 3} \Rightarrow \texttt{fun x -> x 3} \quad \texttt{fun y z -> z y} \Rightarrow \texttt{fun y z -> z y} \quad \text{APP'}\ \dfrac{\texttt{fun y z -> z y} \Rightarrow \texttt{fun y z -> z y} \quad \texttt{3} \Rightarrow \texttt{3} \quad \texttt{fun z -> z 3} \Rightarrow \texttt{fun z -> z 3}}{\texttt{(fun y z -> z y) 3} \Rightarrow \texttt{fun z -> z 3}}}{\texttt{(fun x -> x 3) (fun y z -> z y)} \Rightarrow \texttt{fun z -> z 3}}$$

$$\text{APP'}\ \dfrac{\pi_0 \quad \texttt{fun w -> w+w} \Rightarrow \texttt{fun w -> w+w} \quad \text{APP'}\ \dfrac{\texttt{fun w -> w+w} \Rightarrow \texttt{fun w -> w+w} \quad \texttt{3} \Rightarrow \texttt{3} \quad \text{OP}\ \dfrac{\texttt{3} \Rightarrow \texttt{3} \quad \texttt{3} \Rightarrow \texttt{3} \quad 3+3 \Rightarrow 6}{\texttt{3+3} \Rightarrow \texttt{6}}}{\texttt{(fun w -> w+w) 3} \Rightarrow \texttt{6}}}{\texttt{(fun x -> x 3) (fun y z -> z y) (fun w -> w+w)} \Rightarrow \texttt{6}}$$

Assignment 11.2 (L) Multiplication

Prove that the function

```
let rec mul a b =
  match a with 0 -> 0 | _ -> b + mul (a-1) b
```

terminates for all inputs $a, b \geq 0$.

Suggested Solution 11.2

We prove by induction on `a` that `mul a b` terminates with $a * b$:

- Base case: `a = 0`:

$$\text{APP}\ \dfrac{\pi_{mul} \quad \text{PM}\ \dfrac{}{\texttt{match 0 with 0 -> 0 | \_ -> b + mul (-1) b} \Rightarrow \texttt{0}}}{\texttt{mul 0 b} \Rightarrow \texttt{0}}$$

- Inductive case: Assume `mul a b` terminates for an $a \geq 0$. Now, we show that it also terminates for $a + 1$:

$$\text{APP}\;\dfrac{\pi_{mul} \quad \text{PM}\;\dfrac{\text{OP}\;\dfrac{\text{APP}\;\dfrac{\text{by I.H.}}{\texttt{mul (a+1-1) b} \Rightarrow a*b} \quad b + (a*b) \Rightarrow (a+1)*b}{\texttt{b + mul (a+1-1) b} \Rightarrow (a+1)*b}}{\texttt{match a+1 with 0 -> 0 | \_ -> b + mul (a+1-1) b} \Rightarrow (a+1)*b}}{\texttt{mul (a+1) b} \Rightarrow (a+1)*b}$$

Here π_{mul} is the GD-tree of `mul`. Note one important thing here: When reducing to the induction hypothesis, we do not apply the operator rule for the `a+1-1` term, since $a+1$ is not really an OCaml expression, but the successor of a. We silently simplify $a+1-1$ to a and apply the induction hypothesis.

□

Assignment 11.3 (L) Threesum

Use big-step operational semantics to show that the function

```
let rec threesum = fun l ->
  match l with [] -> 0 | x::xs -> 3*x + threesum xs
```

terminates for all inputs and computes three times the sum of the input list's elements.

Suggested Solution 11.3

We define:

$$\pi_{ts} = \text{GD}\;\dfrac{\texttt{threesum = fun l -> match l with [] -> 0 | x::xs -> 3*x + threesum xs}}{\texttt{threesum} \Rightarrow \texttt{fun l -> match l with [] -> 0 | x::xs -> 3*x + threesum xs}}$$

Now, we do an induction on the length n of the list.

- Base case: $n = 0$ (`l = []`)

$$\text{APP}\;\dfrac{\pi_{ts} \quad \texttt{[]} \Rightarrow \texttt{[]} \quad \text{PM}\;\dfrac{\texttt{[]} \Rightarrow \texttt{[]} \quad 0 \Rightarrow 0}{\texttt{match [] with [] -> 0 | x::xs -> 3*x + threesum xs} \Rightarrow 0}}{\texttt{threesum []} \Rightarrow 0}$$

- Inductive step: We assume `threesum xs` terminates with $3\sum_{i=1}^{n} x_i$ for an input `xs` = $[x_n; \ldots; x_1]$ of length $n \geq 0$. Now, show that `threesum` x_{n+1}`::xs` terminates with $3\sum_{i=1}^{n+1} x_i$:

$$\text{APP}\;\dfrac{\pi_{ts} \quad x_{n+1}\texttt{::xs} \Rightarrow x_{n+1}\texttt{::xs} \quad \text{PM}\;\dfrac{x_{n+1}\texttt{::xs} \Rightarrow x_{n+1}\texttt{::xs} \quad \text{OP}\;\dfrac{\text{OP}\;\dfrac{3 \Rightarrow 3 \quad x_{n+1} \Rightarrow x_{n+1} \quad 3*x_{n+1} \Rightarrow 3x_{n+1}}{3*x_{n+1} \Rightarrow 3x_{n+1}} \quad \text{APP}\;\dfrac{\texttt{by I.H.}}{\texttt{threesum xs} \Rightarrow 3\sum_{i=1}^{n} x_i} \quad 3x_{n+1} + 3\sum_{i=1}^{n} x_i \Rightarrow 3\sum_{i=1}^{n+1} x_i}{3*x_{n+1}\texttt{ + threesum xs} \Rightarrow 3\sum_{i=1}^{n+1} x_i}}{\texttt{match } x_{n+1}\texttt{::xs with [] -> 0 | x::xs -> 3*x + threesum xs} \Rightarrow 3\sum_{i=1}^{n+1} x_i}}{\texttt{threesum } (x_{n+1}\texttt{::xs}) \Rightarrow 3\sum_{i=1}^{n+1} x_i}$$

□

Assignment 11.4 (L) Records

Let MiniOCaml++ be an extended version of MiniOCaml that comes with records. Perform these tasks:

1. Extend the operational big-step semantics of MiniOCaml for these new expressions.
2. Construct a big-step proof for the value of this expression:

```
let r = { x={ a=3+5; b=2+4::[] }; y=2*7 } in r.x.a::r.x.b
```

Suggested Solution 11.4

1. We need two new rules for the record evaluation (RE) and record access (RA):

   - $$\text{RE}\ \frac{e_1 \Rightarrow v_1 \quad \dots \quad e_n \Rightarrow v_n}{\{a_1 = e_1;\ \dots\ a_n = e_n\} \Rightarrow \{a_1 = v_1;\ \dots\ a_n = v_n\}}$$
   - $$\text{RA}\ \frac{e \Rightarrow \{\ \dots\ a = v;\ \dots\ \}}{e.a \Rightarrow v}$$

2. The big-step tree:

$$\pi_0 = \text{RE}\ \dfrac{\text{RE}\ \dfrac{\text{OP}\ \dfrac{\texttt{3} \Rightarrow \texttt{3} \quad \texttt{5} \Rightarrow \texttt{5} \quad \texttt{3+5} \Rightarrow \texttt{8}}{\texttt{3+5} \Rightarrow \texttt{8}} \quad \text{LI}\ \dfrac{\text{OP}\ \dfrac{\texttt{2} \Rightarrow \texttt{2} \quad \texttt{4} \Rightarrow \texttt{4} \quad \texttt{2+4} \Rightarrow \texttt{6}}{\texttt{2+4} \Rightarrow \texttt{6}}}{\texttt{[2+4]} \Rightarrow \texttt{[6]}}}{\texttt{\{ a=3+5; b=[2+4] \}} \Rightarrow \texttt{\{ a=8; b=[6] \}}} \quad \text{OP}\ \dfrac{\texttt{2} \Rightarrow \texttt{2} \quad \texttt{7} \Rightarrow \texttt{7} \quad \texttt{2*7} \Rightarrow \texttt{14}}{\texttt{2*7} \Rightarrow \texttt{14}}}{\texttt{\{ x=\{ a=3+5; b=[2+4] \}; y=2*7 \}} \Rightarrow \texttt{\{ x=\{ a=8; b=[6] \}; y=14 \}}}$$

$$\pi_1 = \text{RA}\ \dfrac{\text{RA}\ \dfrac{\texttt{\{ x=\{ a=8; b=[6] \}; y=14 \}} \Rightarrow \texttt{\{ x=\{ a=8; b=[6] \}; y=14 \}}}{\texttt{\{ x=\{ a=8; b=[6] \}; y=14 \}.x} \Rightarrow \texttt{\{ a=8; b=[6] \}}}}{\texttt{\{ x=\{ a=8; b=[6] \}; y=14 \}.x.a} \Rightarrow \texttt{8}}$$

$$\pi_2 = \text{RA}\ \dfrac{\text{RA}\ \dfrac{\texttt{\{ x=\{ a=8; b=[6] \}; y=14 \}} \Rightarrow \texttt{\{ x=\{ a=8; b=[6] \}; y=14 \}}}{\texttt{\{ x=\{ a=8; b=[6] \}; y=14 \}.x} \Rightarrow \texttt{\{ a=8; b=[6] \}}}}{\texttt{\{ x=\{ a=8; b=[6] \}; y=14 \}.x.b} \Rightarrow \texttt{[6]}}$$

$$\text{LD}\ \dfrac{\pi_0 \quad \text{LI}\ \dfrac{\pi_1 \quad \pi_2}{\texttt{\{ x=\{ a=8; b=[6] \}; y=14 \}.x.a::\{ x=\{ a=8; b=[6] \}; y=14 \}.x.b} \Rightarrow \texttt{[8;6]}}}{\texttt{let r = \{ x=\{ a=3+5; b=[2+4] \}; y=2*7 \} in r.x.a::r.x.b} \Rightarrow \texttt{[8;6]}}$$

Assignment 11.5 (H) More Big Steps [12 Points]

Globally defined are these functions:

```
let rec map = fun f l ->
  match l with [] -> [] | x::xs -> f x :: map f xs
and fold_left = fun f a l ->
  match l with [] -> a | x::xs -> fold_left f (f a x) xs
and comp = fun f g x -> f (g x)
and mul = fun a b -> a * b
and id = fun x -> x
```

Give big-step proofs for the following claims:

1. `fold_left mul 3 [10]` $\Rightarrow$ `30`
2. `(let a = comp (fun x -> 2 * x) in a (fun x -> x + 3)) 4` $\Rightarrow$ `14`
3. `map (id id) [8]` $\Rightarrow$ `[8]`

Suggested Solution 11.5

1.

$$\pi_{fl} = \texttt{fun f a l -> match l with [] -> a | x::xs -> fold\_left f (f a x) xs}$$

$$\pi_{fold\_left} = \text{GD}\,\dfrac{\texttt{fold\_left} = \pi_{fl} \quad \pi_{fl} \Rightarrow \pi_{fl}}{\texttt{fold\_left} \Rightarrow \texttt{fun f a l -> match l with [] -> a | x::xs -> fold\_left f (f a x) xs}}$$

$$\pi_{mul} = \text{GD}\,\dfrac{\texttt{mul} = \texttt{fun a b -> a*b} \quad \texttt{fun a b -> a*b} \Rightarrow \texttt{fun a b -> a*b}}{\texttt{mul} \Rightarrow \texttt{fun a b -> a*b}}$$

$$\pi_0 = \text{APP'}\,\dfrac{\texttt{fun a b -> a*b} \Rightarrow \texttt{fun a b -> a*b} \quad 3 \Rightarrow 3 \quad 10 \Rightarrow 10 \quad \text{OP}\,\dfrac{3 \Rightarrow 3 \quad 10 \Rightarrow 10 \quad 3*10 \Rightarrow 30}{3*10 \Rightarrow 30}}{\texttt{(fun a b -> a*b) 3 10} \Rightarrow 30}$$

$$\pi_1 = \text{PM}\,\dfrac{\texttt{[]} \Rightarrow \texttt{[]} \quad 30 \Rightarrow 30}{\texttt{match [] with [] -> 30 | x::xs -> fold\_left (fun a b -> a*b) ((fun a b -> a*b) 30 x) xs} \Rightarrow 30}$$

$$\pi_2 = \text{PM}\,\dfrac{\texttt{[10]} \Rightarrow \texttt{[10]} \quad \text{APP'}\,\dfrac{\pi_{fold\_left} \quad \texttt{fun a b -> a*b} \Rightarrow \texttt{fun a b -> a*b} \quad \pi_0 \quad \texttt{[]} \Rightarrow \texttt{[]} \quad \pi_1}{\texttt{fold\_left (fun a b -> a*b) ((fun a b -> a*b) 3 10) []} \Rightarrow 30}}{\texttt{match [10] with [] -> 3 | x::xs -> fold\_left (fun a b -> a*b) ((fun a b -> a*b) 3 x) xs} \Rightarrow 30}$$

$$\text{APP'}\,\dfrac{\pi_{fold\_left} \quad \pi_{mul} \quad 3 \Rightarrow 3 \quad \texttt{[10]} \Rightarrow \texttt{[10]} \quad \pi_2}{\texttt{fold\_left mul 3 [10]} \Rightarrow 30}$$

2.

$$\pi_{comp} = \text{GD}\,\dfrac{\texttt{comp} = \texttt{fun f g x -> f (g x)} \quad \texttt{fun f g x -> f (g x)} \Rightarrow \texttt{fun f g x -> f (g x)}}{\texttt{comp} \Rightarrow \texttt{fun f g x -> f (g x)}}$$

$$\pi_{fun} = \texttt{fun x -> (fun x -> 2*x) ((fun x -> x+3) x)} \Rightarrow \texttt{fun x -> (fun x -> 2*x) ((fun x -> x+3) x)}$$

$$\pi_0 = \text{APP'}\,\dfrac{\pi_{comp} \quad \texttt{fun x -> 2*x} \Rightarrow \texttt{fun x -> 2*x} \quad \texttt{fun g x -> (fun x -> 2*x) (g x)} \Rightarrow \texttt{fun g x -> (fun x -> 2*x) (g x)}}{\texttt{comp (fun x -> 2*x)} \Rightarrow \texttt{fun g x -> (fun x -> 2*x) (g x)}}$$

$$\pi_1 = \text{LD}\,\dfrac{\pi_0 \quad \text{APP'}\,\dfrac{\texttt{fun g x -> (fun x -> 2*x) (g x)} \Rightarrow \texttt{fun g x -> (fun x -> 2*x) (g x)} \quad \texttt{fun x -> x+3} \Rightarrow \texttt{fun x -> x+3} \quad \pi_{fun}}{\texttt{(fun g x -> (fun x -> 2*x) (g x)) (fun x -> x+3)} \Rightarrow \texttt{fun x -> (fun x -> 2*x) ((fun x -> x+3) x)}}}{\texttt{let a = comp (fun x -> 2*x) in a (fun x -> x+3)} \Rightarrow \texttt{fun x -> (fun x -> 2*x) ((fun x -> x+3) x)}}$$

$$\text{APP'}\,\dfrac{\pi_1 \quad 4 \Rightarrow 4 \quad \text{APP'}\,\dfrac{\texttt{fun x -> 2*x} \Rightarrow \texttt{fun x -> 2*x} \quad \text{APP'}\,\dfrac{\texttt{fun x -> x+3} \Rightarrow \texttt{fun x -> x+3} \quad 4 \Rightarrow 4 \quad \text{OP}\,\dfrac{4 \Rightarrow 4 \quad 3 \Rightarrow 3 \quad 4+3 \Rightarrow 7}{4+3 \Rightarrow 7}}{\texttt{(fun x -> x+3) 4} \Rightarrow 7} \quad \text{OP}\,\dfrac{2 \Rightarrow 2 \quad 7 \Rightarrow 7 \quad 2*7 \Rightarrow 14}{2*7 \Rightarrow 14}}{\texttt{(fun x -> 2*x) ((fun x -> x+3) 4)} \Rightarrow 14}}{\texttt{(let a = comp (fun x -> 2*x) in a (fun x -> x+3)) 4} \Rightarrow 14}$$

3.

$$\pi_{id} = \text{GD}\,\dfrac{\texttt{id} = \texttt{fun x -> x} \quad \texttt{fun x -> x} \Rightarrow \texttt{fun x -> x}}{\texttt{id} \Rightarrow \texttt{fun x -> x}}$$

$$\pi_m = \texttt{fun f l -> match l with [] -> [] | x::xs -> f x::map f xs}$$

$$\pi_{map} = \text{GD}\,\dfrac{\texttt{map} = \pi_m \quad \pi_m \Rightarrow \pi_m}{\texttt{map} \Rightarrow \texttt{fun f l -> match l with [] -> [] | x::xs -> f x::map f xs}}$$

$$\pi_0 = \text{APP'}\,\dfrac{\pi_{map} \quad \texttt{fun x -> x} \Rightarrow \texttt{fun x -> x} \quad \texttt{[]} \Rightarrow \texttt{[]} \quad \text{PM}\,\dfrac{\texttt{[]} \Rightarrow \texttt{[]} \quad \texttt{[]} \Rightarrow \texttt{[]}}{\texttt{match [] with [] -> [] | x::xs -> (fun x -> x) x::map (fun x -> x) xs} \Rightarrow \texttt{[]}}}{\texttt{map (fun x -> x) []} \Rightarrow \texttt{[]}}$$

$$\pi_1 = \text{LI}\,\dfrac{\text{APP'}\,\dfrac{\texttt{fun x -> x} \Rightarrow \texttt{fun x -> x} \quad 8 \Rightarrow 8 \quad 8 \Rightarrow 8}{\texttt{(fun x -> x) 8} \Rightarrow 8} \quad \pi_0}{\texttt{(fun x -> x) 8::map (fun x -> x) []} \Rightarrow \texttt{[8]}}$$

$$\text{APP'}\,\dfrac{\pi_{map} \quad \text{APP'}\,\dfrac{\pi_{id} \quad \pi_{id} \quad \texttt{fun x -> x} \Rightarrow \texttt{fun x -> x}}{\texttt{id id} \Rightarrow \texttt{fun x -> x}} \quad \texttt{[8]} \Rightarrow \texttt{[8]} \quad \text{PM}\,\dfrac{\texttt{[8]} \Rightarrow \texttt{[8]} \quad \pi_1}{\texttt{match [8] with [] -> [] | x::xs -> (fun x -> x) x::map (fun x -> x) xs} \Rightarrow \texttt{[8]}}}{\texttt{map (id id) [8]} \Rightarrow \texttt{[8]}}$$

Assignment 11.6 (H) Computing Zero [4 Points]

Consider the function `foo`:

```
let rec foo = fun l ->
  match l with [] -> 0
  | 0::xs -> foo xs
  | x::xs -> if x > 0 then foo (x-1::xs) else foo (x+1::xs)
```

Prove that `foo` terminates for all inputs. Axioms ($v \Rightarrow v$) may be omitted.

Suggested Solution 11.6

Let $\pi_{foo} = \text{GD}\,\dfrac{\texttt{foo = fun l -> ...} \quad \texttt{fun l -> ...} \Rightarrow \texttt{fun l -> ...}}{\texttt{foo} \Rightarrow \texttt{fun l -> ...}}$

We prove by induction on the length n of list `l`.

- Base case: $n = 0$ (`l = []`):

$$\text{APP}\,\dfrac{\pi_{foo} \quad \text{PM}\,\texttt{match [] with [] -> 0 | ...} \Rightarrow \texttt{0}}{\texttt{foo []} \Rightarrow \texttt{0}}$$

- Inductive step: We assume `foo xs` terminates with 0 for a list `xs` of length $n \geq 0$. Now, we show that it also terminates for a list `x::xs` of length $n+1$. To do so, we do another induction on the absolute value k of the list's first element `x`:

  - Base case: $k = 0$, so $|x| = 0$ and thus $x = 0$:

$$\text{APP}\,\dfrac{\pi_{foo} \quad \text{PM}\,\dfrac{\text{APP}\,\dfrac{\text{by I.H.}}{\texttt{foo xs} \Rightarrow \texttt{0}}}{\texttt{match 0::xs with ... | 0::xs -> foo xs | ...} \Rightarrow \texttt{0}}}{\texttt{foo (0::xs)} \Rightarrow \texttt{0}}$$

  - Inductive step: We assume `foo x::xs` terminates with 0 for a value $|x| = k \geq 0$. Now, we show that `foo (x::xs)` also terminates when $|x| = k+1$. We consider two cases:

Case 1: $x > 0$

$$\text{APP}\,\dfrac{\pi_{foo} \quad \text{PM}\,\dfrac{\text{PM}\,\dfrac{\text{APP}\,\dfrac{\text{by I.H. (since } |x-1| = k)}{\texttt{foo (x-1::xs)} \Rightarrow \texttt{0}}}{\texttt{match x > 0 with true -> foo (x-1::xs) | ...} \Rightarrow \texttt{0}}}{\texttt{match x::xs with ... | x::xs -> match x > 0 with true -> foo (x-1::xs) | ...} \Rightarrow \texttt{0}}}{\texttt{foo (x::xs)} \Rightarrow \texttt{0}}$$

Case 2: $x < 0$

$$\text{APP}\,\dfrac{\pi_{foo} \quad \text{PM}\,\dfrac{\text{PM}\,\dfrac{\text{APP}\,\dfrac{\text{by I.H. (since } |x+1| = k)}{\texttt{foo (x+1::xs)} \Rightarrow \texttt{0}}}{\texttt{match x > 0 with ... | false -> foo (x+1::xs) | ...} \Rightarrow \texttt{0}}}{\texttt{match x::xs with ... | x::xs -> match x > 0 with ... | false -> foo (x+1::xs)} \Rightarrow \texttt{0}}}{\texttt{foo (x::xs)} \Rightarrow \texttt{0}}$$

□

Assignment 11.7 (H) Raise the bar! [4 Points]

Given are these definitions:

```
let rec impl = fun n a ->
  match n with 0 -> a | _ -> impl (n-1) (a * n * n)
and bar = fun n -> impl n 1
```

Prove that `bar n` terminates with $n! * n!$ for all non-negative inputs n. Axioms $(v \Rightarrow v)$ may be omitted.

Suggested Solution 11.7

We first prove that `impl n a` terminates with $a * n! * n!$: We omit the global definitions of π_{impl} and π_{bar} of `impl` and `bar` here for simplicity.

- Base case: $n = 0$:

$$\text{APP'}\ \frac{\pi_{impl} \quad \text{PM}\ \dfrac{}{\texttt{match 0 with 0 -> a | ...} \Rightarrow a}}{\texttt{impl 0 a} \Rightarrow a}$$

- Inductive step: We assume `impl n a` terminates with $a * n! * n!$ for an input $n \geq 0$. Now, we show that it terminates with $a * (n+1)! * (n+1)!$ for input $n+1$:

$$\text{APP'}\ \frac{\pi_{impl} \quad \text{PM}\ \dfrac{\text{APP'}\ \dfrac{\text{by I.H.}}{\texttt{impl (n+1-1) (a * (n+1) * (n+1))} \Rightarrow (a * (n+1) * (n+1)) * n! * n!}}{\texttt{match n+1 with ... | \_ -> impl (n+1-1) (a * (n+1) * (n+1))} \Rightarrow (a * (n+1) * (n+1)) * n! * n!}}{\texttt{impl (n+1) a} \Rightarrow a * (n+1)! * (n+1)!}$$

Now, that we have shown that `impl n a` $\Rightarrow a * n! * n!$ (1), we prove:

$$\text{APP}\ \frac{\pi_{bar} \quad \text{APP'}\ \dfrac{\text{by (1)}}{\texttt{impl n 1} \Rightarrow n! * n!}}{\texttt{bar n} \Rightarrow n! * n!}$$

□