3. Optimal Predictions in Everyday Cognition

In this problem, you will recapitulate some of the computations from Optimal Predictions in Everyday Cognition *(Griffiths & Tenenbaum,* Psychological Science*, 2006). The goals of the problem are to get your hands dirty making some basic Bayesian inferences and decisions (both analytically and numerically) in a simple but ecologically relevant context. We hope this problem will take you no more than 6 hours, though we expect that if you are comfortable with Python or MATLAB, 18.01 level calculus and basic probability it should take you roughly 2 hours.*

Recall (from lecture and the paper) the basic setup for *Optimal Predictions*. An observer gets a sample t_{obs} from an interval of (unobserved) length t_{total}, drawn uniformly at random (*i.e.* nothing is special about the time of observation), so $p(t_{obs}|t_{total})$ is $\frac{1}{t_{total}}$ if $0 < t_{obs} < t_{total}$ and 0 otherwise (This is called an *anthropic* likelihood). The observer has background domain knowledge about which values of t_{total} are most likely *a priori*, captured in $p(t_{total})$. The observer is then asked to make an estimate of t_{total} based on this information. This estimate is modeled as the median of the posterior distribution

$$p(t_{total}|t_{obs}) = \frac{p(t_{total})p(t_{obs}|t_{total})}{p(t_{obs})}$$

where

$$p(t_{obs}) = \int_0^\infty p(t_{obs}|t_{total})p(t_{total})dt_{total}.$$

The posterior median is the value t^* such that half the cumulative mass of the posterior $p(t_{total}|t_{obs})$ is less than t^*, and half is greater than t^*; it is defined more formally below. We will compare the posterior median to the median of a sample of human subjects' responses.

Note that throughout this problem we will use the term *joint density* or *joint* to describe $P(H, D) = P(H)P(D|H) \propto P(H|D)$; this is consistent with standard use in Bayesian statistics. Also note that there are two integrals involved here: one in normalizing the joint density to compute the posterior and one in solving for the posterior median. The first integral is about *inference* - determining what one should believe about an unknown quantity given prior beliefs and data - while the second is about *decision making* - determining what estimate one should emit (or act on) to maximize some measure of performance. Both inference and decision making are distinct and important subproblems in rational action under uncertainty.[1]

Also note that throughout this problem set, we will freely use $p(x)$ and $Pr[X = x]$ to denote the probability density of a random variable X evaluated at value x (in some sense, 'as if' all densities were actually over fine discretizations, and therefore commensurable with probabilities of statements).

(a) Analytically determine the posterior distribution $p(t_{total}|t_{obs})$ under a power-law prior distribution with exponent γ (i.e. $p(t_{total}) \propto t_{total}^{-\gamma}$ where for convenience we will assume $\gamma \geq 1$).

(b) Analytically determine the median of the posterior distribution: t^* *s.t.* $Pr[t^* > t_{total}|t_{obs}] = 0.5$) as a function of t_{obs} and γ. Show all your work. Hint: Do this in two steps. First, write $Pr[t^* > t_{total}|t_{obs}]$ as a function of t^* and t_{obs} (this involves integrating the posterior you computed in part a). Second, set that expression equal to 0.5 and solve for t^* as a function of t_{obs}. Also submit a plot of the posterior median (as a function of t_{obs}) over a reasonable range of observed timespans t_{obs}. Use $\gamma = 2.43$ for your plots and for below.

Now we will implement the same calculations numerically. This will allow us to apply a similar analysis to cases where the posterior doesn't admit a simple closed-form expression. It will also give you some experience comparing analytical to numerical calculations, a theme that may return later on in class when we study algorithmic issues of inference.

[1] The posterior median turns out to be one of several reasonable decision procedures in this setting. A possible extension of this work could consider other decision rules and compare them with other measures of human responses besides the sample median, possibly accounting for individual differences. Feel free to ask the instructor or TAs about this possibility.

(c) Write a procedure according to the following specification:

```
[thetavals, postvals] = opt_compute_posterior(joint, theta_min, theta_max, num_steps)
```

where `thetavals` is a vector of theta (parameter) values. `postvals` is a vector of normalized posterior density values, such that `postvals(i)`$= Pr[H =$ `thetavals(i)`$|D = d]$. The first argument to `opt_compute_posterior` is a procedure `joint_density = joint(theta)` which evaluates $Pr[H = \theta, D = d]$ for some particular, pre-wired-in data value d.

`opt_compute_posterior` should work by forming a reasonable discrete approximation to

$$Z = Pr[D = d] = \int_{\theta_{min}}^{\theta_{max}} Pr[H = \theta, D = d] d\theta$$

(e.g. by a Riemann sum with `num_steps` rectangular elements, or possibly Simpson's rule[2], then returning a table with ordered entries ranging from θ_{min} to θ_{max} with values equal to `joint(theta)`$/Z$. You may use whichever language you prefer, but we have provided Python and MATLAB templates. Attach the code for your `opt_compute_posterior` function.

(d) (i) Write a procedure `opt_build_powerlaw_joint(t)` that takes as an argument the time t, where t is the time of the observed data, and returns a function, `joint(theta)`, that is suitable as the first argument of `opt_compute_posterior`. That is, `opt_build_powerlaw_joint` should return an anonymous function (function pointer) which is a function of `theta`. When evaluated on `theta` it should return the joint density of `theta` and t under the power-law prior and the anthropic likelihood. Attach your code.

(e) Write a procedure `joint = opt_build_lifespan_joint(t)` (analogous to `opt_build_powerlaw_joint` which returns a procedure, of a form suitable as an argument to `opt_compute_posterior`, that computes the joint density of a timespan `t` under a Gaussian prior (with mean 75 and standard deviation 16). Attach your code.

(f) `opt_predictions_plot(integrating_func, build_joint_func, theta_min, theta_max)`, where the first argument can take `opt_compute_posterior`[3] and the second argument can take `opt_build_powerlaw_joint`, should construct plots showing the posteriors and predictive medians with `theta_min = 0` and `theta_max = 300`.

Use your procedures from (c) - (e) along with `opt_prediction_plots` to generate plots showing posteriors and predictive medians for a range of observed lifespans. Submit these plots and comment on how the way the prediction changes as the observation approaches the mean expected lifespan compares to your own intuitions.

[2]People with numerical analysis inclinations are welcome to learn about and use predictor-corrector methods, etc, being careful of the step in the likelihood. Do **not** use a Monte Carlo scheme unless you can justify it in this 1D context - and if you can, please let the TAs know how, as there might be a paper in it. For people not so inclined, a simple left-aligned Riemann sum will get full credit. A well written version of this procedure could be useful in your future lives as Bayesians, or as a memory of better days should you turn frequentist.

[3]If you are using MATLAB, be sure to pass function handles `@opt_compute_posterior` and `@build_joint_func`.