Midterm Exam Solutions (Math 1552 A – Summer 2021)

Note: You can add +15% to the raw percent score of the exam as a curve.

The midterm consisted of:

- 9 questions at 5 raw points per Q
- 1 question at 6 raw points
- one bonus question worth a maximum of +4

Q1: Evaluate the definite integral

$$I = \int_0^1 (1-\sqrt{x})^{p-1}\,dx, \; p>1$$

- Note that since $p>1$, $p-1>0$.
- use the u-sub:

$$u = 1-\sqrt{x}, \; du = -\frac{1}{2\sqrt{x}}\,dx,$$

$$u(0) = 1, \; u(1) = 0$$

- then

$$I = \int_0^1 \frac{-2\sqrt{x}}{-2\sqrt{x}} (1-\sqrt{x})^{p-1}\,dx$$

$$= -2\int_1^0 (1-u)\cdot u^{p-1}\,du$$

$$= 2\int_0^1 \left[u^{p-1} - u^p\right] du$$

$$= 2\left(\frac{u^p}{p} - \frac{u^{p+1}}{p+1}\right)\Bigg|_0^1$$

$$= 2\left[\left(\frac{1}{p} - \frac{1}{p+1}\right) - (0-0)\right]$$

$$= 2\left(\frac{1}{p} - \frac{1}{p+1}\right)$$

$$= \frac{2}{p(p+1)}\left[p+1-p\right],$$

taking a common denominator

$$= \frac{2}{p(p+1)}.$$

Q2: $I = \int_{-2}^{3} w^5 \sqrt{1+2w^3}\, dw$

Find an integral that is the same as I after performing the substitution $u = 2w^3$.

- $u = 2w^3$, $du = 6w^2 dw$,
  $w^5\, dw = \frac{1}{12} u \cdot du$
- change the limits of integration:
  $u(3) = 2 \cdot 3^3 = 54$
  $u(-2) = 2 \cdot (-2)^3 = -16$
- Then $I = \frac{1}{12} \int_{-16}^{54} u \sqrt{1+u}\, du$

Q3: Evaluate the definite integral $I = \int_{\pi/16}^{\pi/8} \sin^2(2x)\,dx$.

Solution(1):

- we have that
  $\sin^2(t) = \frac{1}{2}(1-\cos(2t))$, for all real t
- So $I = \frac{1}{2}\int_{\frac{\pi}{16}}^{\pi/8} (1-\cos(4x))dx$

$$= \frac{1}{2}\left(x - \frac{1}{4}\sin(4x)\right)\Big|_{\pi/16}^{\pi/8}$$

$$= \frac{1}{2}\left(\frac{\pi}{8} - \frac{\pi}{16}\right) - \frac{1}{8}\left(\sin(\tfrac{\pi}{2}) - \sin(\pi/4)\right)$$

$$= \frac{\pi}{32} - \frac{1}{8} + \frac{\sqrt{2}}{16}$$

Solution (2) - alternate:

- for all real t we have

$$\sin(2t) = 2\sin(t)\cos(t)$$

$$\sin^2(t) = \frac{1}{2}(1-\cos(2t))$$

$$\cos^2(t) = \frac{1}{2}(1+\cos(2t))$$

- $$I = 4\int_{\pi/16}^{\pi/8} \sin^2(x)\cos^2(x)\,dx$$

$$= \int_{\pi/16}^{\pi/8} (1-\cos(2x))(1+\cos(2x))\,dx$$

$$= \int_{\pi/16}^{\pi/8} (1-\cos^2(2x))\,dx$$

$$= \frac{1}{2}\int_{\pi/16}^{\pi/8} (1-\cos(4x))\,dx$$

Now integrate and apply the FTC as in the first solution to get that

$$I = \frac{\pi}{32} + \frac{\sqrt{2}}{16} - \frac{1}{8}$$

Q4: Evaluate the indefinite integral

$$I = \int \frac{\sec^2(\theta)\tan(\theta)}{\tan^2(\theta) - 4\tan(\theta) + 3} \, d\theta$$

- first apply the substitution $t = \tan(\theta)$, $dt = \sec^2(\theta) d\theta$, so that

$$I = \int \frac{t}{t^2 - 4t + 3} \, dt$$

$$= \int \frac{t}{(t-3)(t-1)} \, dt$$

- Now apply the method of partial fractions to write

$$\frac{t}{(t-3)(t-1)} = \frac{A}{t-3} + \frac{B}{t-1}, \text{ for}$$

some constants A, B

- Solve for A, B:

$$t = A(t-1) + B(t-3) \quad (*)$$

$\rightarrow$ when $t=1$ in $(*)$:

$$1 = -2B \longleftrightarrow B = -\frac{1}{2}$$

$\rightarrow$ when $t=3$ in $(*)$:

$$3 = 2A \longleftrightarrow A = \frac{3}{2}$$

- So

$$I = \frac{3}{2}\int \frac{dt}{t-3} - \frac{1}{2}\int \frac{dt}{t-1}$$

$$= \frac{3}{2}\ln|t-3| - \frac{1}{2}\ln|t-1| + C$$

$$= \frac{3}{2}\ln|\tan\theta-3| - \frac{1}{2}\ln|\tan\theta-1| + C$$

Q5: Evaluate the definite integral

$$I = \int_0^1 \frac{\ln(1+x)}{(1+x)^2}\,dx$$

- first perform the substitution:

$$s = \ln(1+x), \quad ds = \frac{dx}{1+x},$$

$$\frac{dx}{(1+x)^2} = e^{-s}\,ds,$$

$$s(0) = \ln(1) = 0,$$

$$s(1) = \ln(2),$$

So that

$$I = \int_0^{\ln(2)} s \cdot e^{-s}\,ds$$

- Now evaluate the resulting integral by parts:

$$\left\{ \int u\,dv = uv - \int v\,du \quad \begin{array}{ll} u = s & dv = e^{-s}\,ds \\ du = ds & v = -e^{-s} \end{array} \right\}$$

$$I = -se^{-s}\Big|_0^{\ln(2)} + \int_0^{\ln(2)} e^{-s}\,ds$$

$$= -\ln(2)e^{-\ln(2)} - e^{-s}\Big|_0^{\ln(2)}$$

$$= -\frac{1}{2}\ln(2) - \left(\frac{1}{2} - 1\right)$$

$$= \frac{1}{2}\left(1 - \ln(2)\right)$$

Q6: Evaluate the definite integral

$$I = \int_1^3 \frac{e^{\frac{1}{x}}}{x^2}\,dx$$

- apply a substitution:

$u = \frac{1}{x}$, $du = -\frac{dx}{x^2}$,

$u(3) = 1/3$, $u(1) = 1$,

So that

$$I = -\int_1^{1/3} e^u\,du = \int_{1/3}^1 e^u\,du$$

$$= e^u \Big|_{1/3}^{1} = e - \sqrt[3]{e}$$

Q7: Evaluate the limit

$$L = \lim_{t \to -1} \frac{\cos\left(\frac{\pi t}{2}\right)}{t+1}$$

- we have that

(1) $\lim_{t \to -1} \cos\left(\frac{\pi t}{2}\right) = 0$

(2) $\lim_{t \to -1} t+1 = 0$,

So the limit (L) leads to an initial indeterminate form of $\frac{0}{0}$.

- Hence, we can apply L'Hopital's rule to see that

$$L = \lim_{t \to -1} \frac{\frac{d}{dt}\left[\cos\left(\frac{\pi t}{2}\right)\right]}{\frac{d}{dt}\left[t+1\right]}$$

$$= \lim_{t \to -1} \frac{-\frac{\pi}{2}\sin\left(\frac{\pi t}{2}\right)}{1}$$

$$= \left(-\frac{\pi}{2}\right)\sin\left(-\frac{\pi}{2}\right) = +\frac{\pi}{2}.$$

Q8: Evaluate the improper integral

$$I = \int_0^{2e} \ln(x)\,dx$$

- Since $\lim_{x \to 0^+} \ln(x) = -\infty$, the integral is improper. So we rewrite I as:

$$I = \lim_{a \to 0^+} \int_a^{2e} \ln(x)\,dx \quad (*)$$

- find the antiderivative of the indefinite integral by parts:

$$\left(\begin{array}{ll} u = \ln(x) & dv = dx \\ du = \dfrac{dx}{x} & v = x \end{array} \right)$$

$$\int \ln(x)\,dx = x\ln(x) - x + C$$

- Then by (*) we have that

$$I = 2e(\ln(2e) - 1) - \lim_{a\to 0^+}(a\ln(a) - a)$$

- we need the value of the limit

$$\lim_{x\to 0^+} x\cdot\ln(x) = \lim_{x\to 0^+}\frac{\ln(x)}{\frac{1}{x}},$$

which gives the indeterminate form $-\frac{\infty}{\infty}$ so we can apply L'Hopital's rule as:

$$\lim_{x\to 0^+}\frac{\ln(x)}{\frac{1}{x}} = \lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}}$$

$$= \lim_{x\to 0^+} -x = 0$$

- So in conclusion, we have that

$$I = 2e(\ln(2e) - 1) - (0 - 0)$$
$$= 2e(\ln(2) + \ln(e) - 1)$$
$$= 2e \cdot \ln(2).$$

Q9: Which trigonometric substitution (trig. sub.) do we apply to evaluate

$$I = \int \sqrt{1 + \frac{3x^2}{4}}\, dx \, ?$$

Solution (1):

- we have a term of the form $a^2 + t^2$ when $a = 1$ and $t = \sqrt{\frac{3}{4}} \cdot x$.

- use $t = \frac{\sqrt{3}}{2} x = a \cdot \tan(\theta)$

$$\longleftrightarrow \quad x = \frac{2}{\sqrt{3}} \tan(\theta)$$

$$= \frac{2\sqrt{3}}{3} \tan(\theta)$$

Solution(2): factor first as

$$I = \int \sqrt{\frac{3}{4}\left(\frac{4}{3}+x^2\right)}\,dx.$$

Then we have a term of the form a^2+x^2 when $a=\sqrt{\frac{4}{3}}$.

So use $x = a\tan(\theta)$

$$= \frac{2}{\sqrt{3}}\tan(\theta)$$

$$= \frac{2\sqrt{3}}{3}\tan(\theta).$$

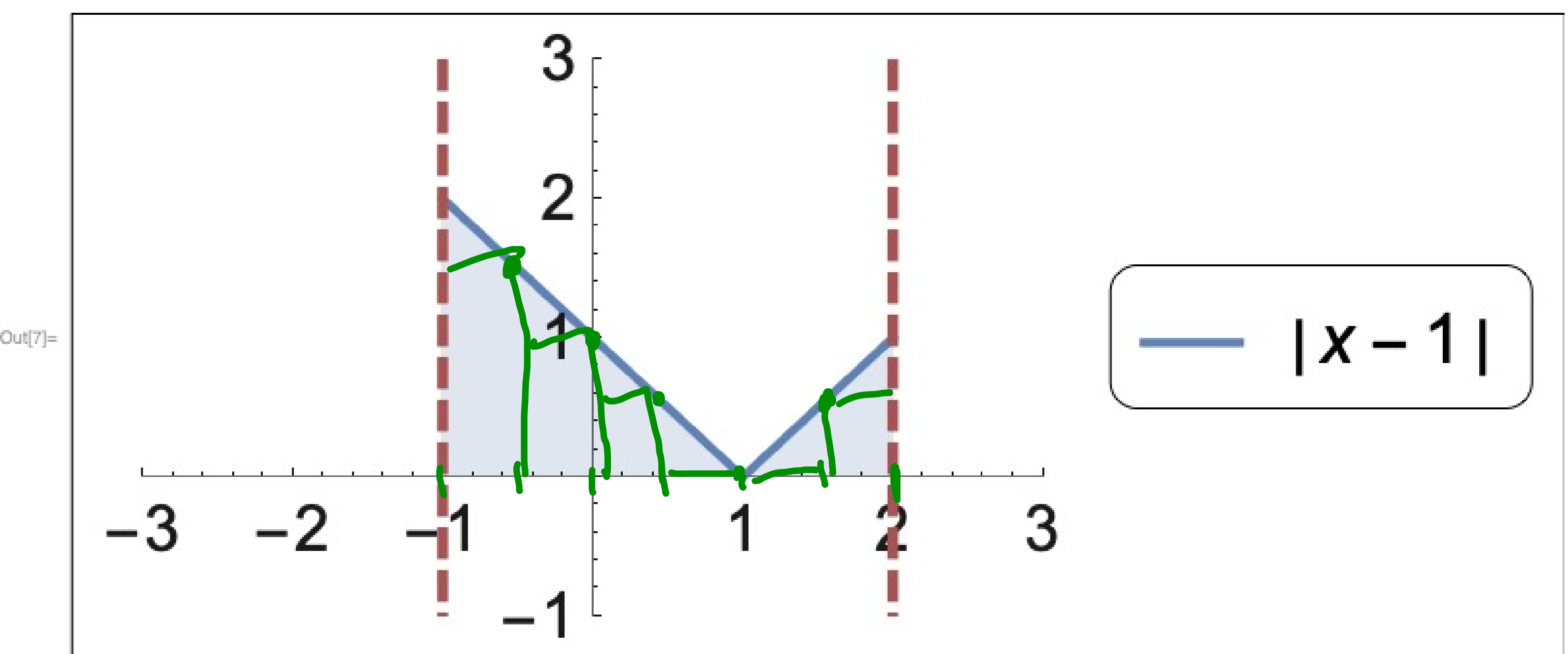
Q10: (+6 points): Numerically approximate

$$I = \int_{-1}^{2} 24|x-1|\,dx \text{ with}$$

N = 6 rectangles.

Find the lower sum estimate (L)

$$\Delta x = \frac{2-(-1)}{6} = 1/2$$



$x_0 = -1,\ x_1 = -1/2,\ x_2 = 0,\ x_3 = 1/2,$

$x_4 = 1,\ x_5 = 3/2,\ x_6 = 2$

- for $x \in [-1,1]$, the rectangle heights are the function values at the RHS endpoints, and for $x \in [1,2]$ at the LHS endpoints.
- So for $f(x) = |x-1|$,

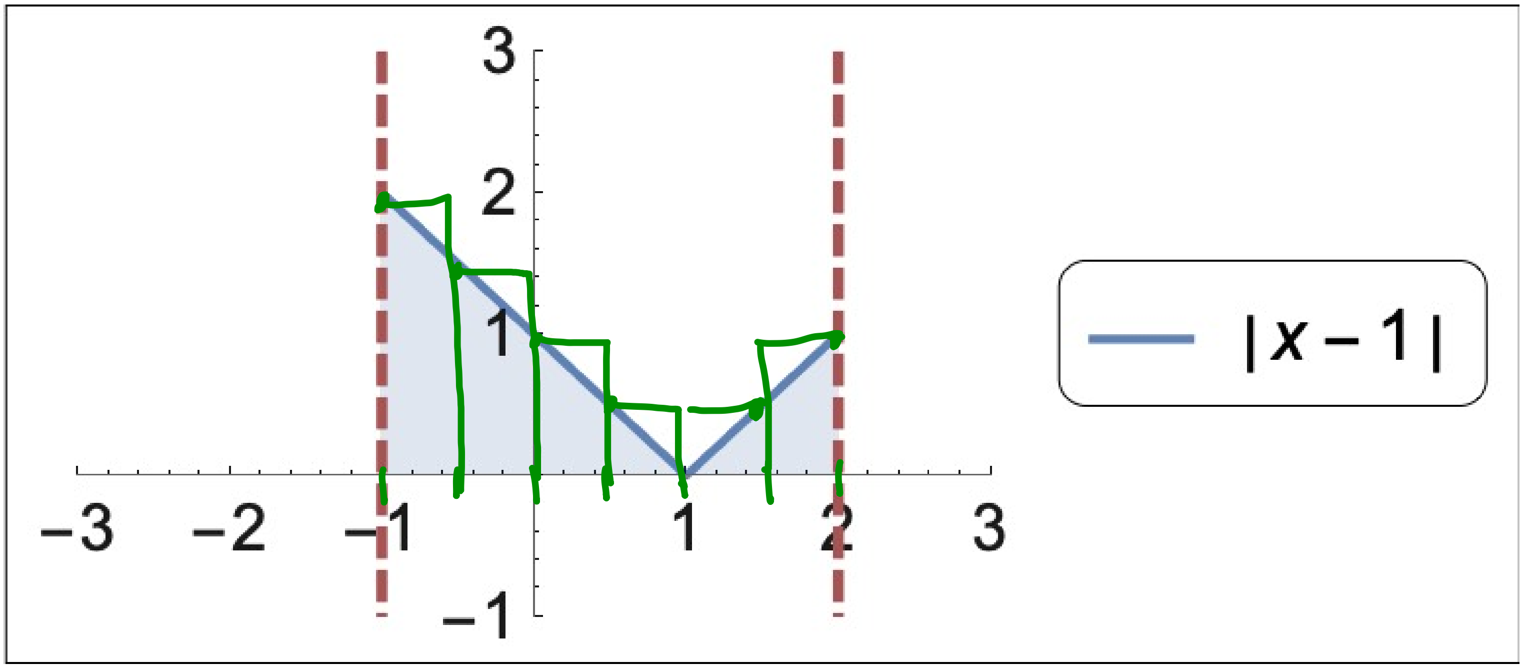
$$L = 24\Delta x\left(f(-1/2) + f(0) + f(1/2) + f(1) + f(1) + f(3/2)\right)$$

$$= 12 \cdot \left(\frac{3}{2} + 1 + \frac{1}{2} + 0 + 0 + \frac{1}{2}\right)$$

$$= 12 \cdot \frac{7}{2} = 6 \cdot 7$$

$$= 42$$

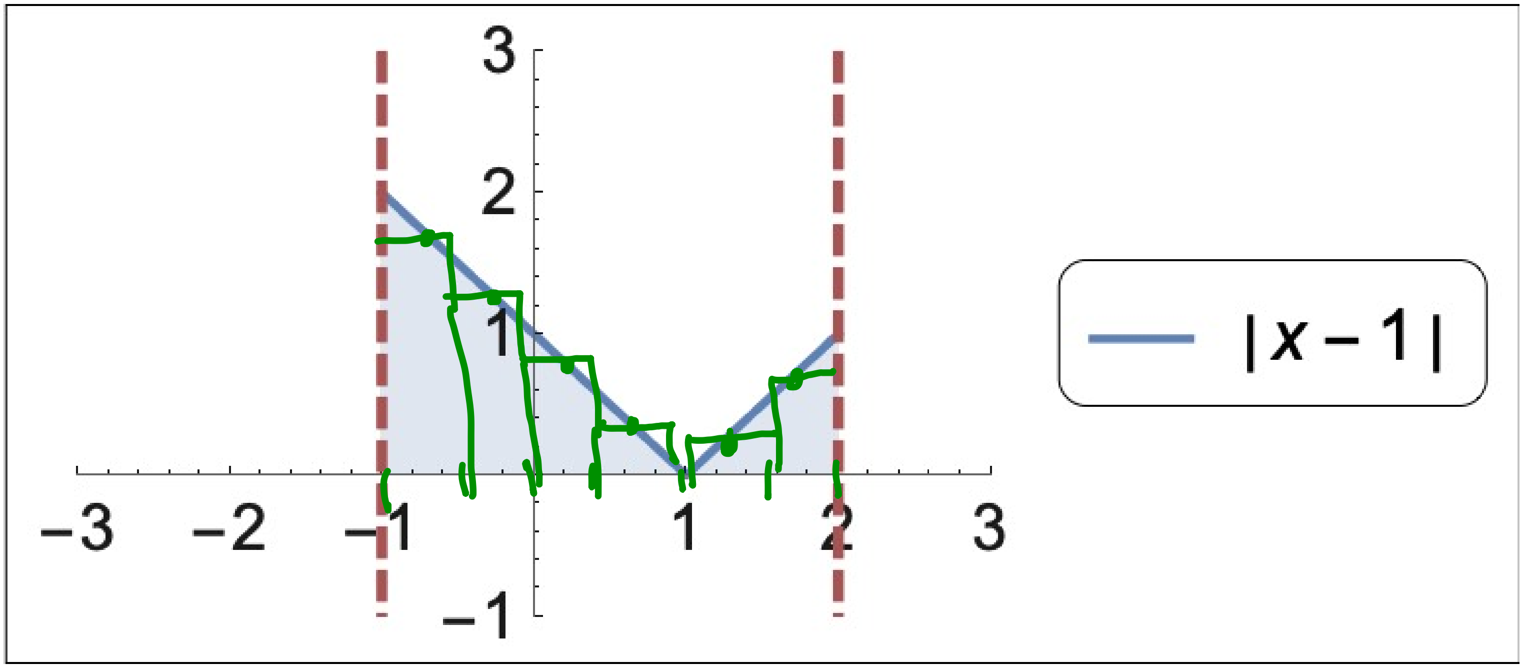
Out[7]=



Similarly, to find U:

$$U = 24\Delta x\left(2+\frac{3}{2}+1+\frac{1}{2}+\frac{1}{2}+1\right)$$

$$= 12\cdot\frac{13}{2} = 6\cdot 13 = 78$$



Let $f(x) := |x-1|$.

To evaluate M, we have:

$$M = 24\Delta x\left(f(-3/4) + f\left(-\frac{1}{4}\right) + f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right) + f\left(\frac{5}{4}\right) + f(7/4)\right)$$

$$= 12 \cdot 5 = 60$$

EC/bonus question (+4 extra points):

- write $x^4(1-x)^4 = (1+x^2)p(x) + q(x)$

By polynomial long division, we get that

$$p(x) = x^6 - 4x^5 + 5x^4 - 4x^2 + 4$$

$$q(x) = -4$$

- evaluate

$$I = \frac{1}{3}\int_0^1 \frac{x^{2/3}(1-x^{1/3})^4}{1+x^{2/3}}\,dx$$

using the u-sub

$u = x^{1/3}$, $du = \frac{1}{3x^{2/3}}dx$,

$u(0) = 0$, $u(1) = 1$,

$\frac{1}{3}x^{2/3}dx = u^4\,du$

- Hence, $I = \displaystyle\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx$

$$= \int_0^1 \left[x^6 - 4x^5 + 5x^4 - 4x^2 + 4\right] dx$$

$$- 4\int_0^1 \frac{dx}{1+x^2}$$

$$= \left(\frac{x^7}{7} - \frac{4x^6}{6} + \frac{5x^5}{5} - \frac{4x^3}{3} + 4x\right)\Bigg|_0^1 - 4\tan^{-1}(x)\Bigg|_0^1$$

$$= \frac{22}{7} - 4\left(\cancelto{\pi/4}{\tan^{-1}(1)} - \cancelto{0}{\tan^{-1}(0)}\right)$$

$$= \frac{22}{7} - \pi$$