Probability Comprehensive Exam
January 18, 2017

Student Number:

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Show that if X_n and Y_n are independent for $n = 1, 2, ...$ and $X_n \to^w X$, $Y_n \to^w Y$, where X and Y are independent, then $X_n + Y_n \to^w X + Y$.

Solution: Since X and Y are independent, the characteristic function of $X + Y$ evaluated at $t \in \mathbb{R}$ is

$$\phi_{X+Y}(t) = \mathbb{E}e^{it(X+Y)} = \mathbb{E}e^{itX}e^{itY} = \mathbb{E}e^{itX}\mathbb{E}e^{itY} = \phi_X(t)\phi_Y(t).$$

On the other hand, since $X_n \to^w X$ and $Y_n \to^w Y$, one has $\phi_{X_n}(t) \to \phi_X(t)$ and $\phi_{Y_n}(t) \to \phi_Y(t)$ for each t. Again using independence of X_n and Y_n,

$$\phi_{X_n+Y_n}(t) = \phi_{X_n}(t)\phi_{Y_n}(t) \to \phi_X(t)\phi_Y(t) = \phi_{X+Y}(t).$$

Since the characteristic function of $X_n + Y_n$ converges pointwise to that of $X + Y$, we conclude that $X_n + Y_n \to^w X + Y$.

2. Let X be a random variable with mean zero and finite variance σ^2. Prove that for every $c > 0$,

$$P(X > c) \leq \frac{\sigma^2}{\sigma^2 + c^2}.$$

Hint: Combine the inequality $\mathbb{E}(c - X) \leq \mathbb{E}\left((c - X)\mathbf{1}_{\{X<c\}}\right)$ with the Cauchy-Schwartz inequality.

Solution: By Cauchy-Schwarz,

$$c = \mathbb{E}(c - X) \leq \mathbb{E}(c - X)\mathbf{1}_{\{X<c\}} \leq \sqrt{\mathbb{E}(c - X)^2\mathbb{P}(X < c)}.$$

However

$$\mathbb{E}(c - X)^2 = \mathbb{E}(c^2 - 2cX + X^2) = c^2 + \sigma^2,$$

so

$$c \leq \sqrt{(c^2 + \sigma^2)\mathbb{P}(X < c)},$$

or

$$\mathbb{P}(X < c) \geq \frac{c^2}{c^2 + \sigma^2}.$$

3. Let $X_1, X_2, ...$ be i.i.d. random variables uniformly distributed on $[0, 1]$. Show that with probability 1,

$$\lim_{n\to\infty} (X_1 \cdot \cdots \cdot X_n)^{\frac{1}{n}}$$

exists and compute its value.

Solution: Define $Y_n = \log X_n$, so that the quantity we are considering is

$$\lim_{n\to\infty} \exp\left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right).$$

We can compute $\mathbb{E}Y_n$ as

$$\mathbb{E}Y_n = \int_0^1 \log x \, \mathrm{d}x = -1,$$

so since (Y_n) is an i.i.d. sequence with entries of mean -1, the strong law of large numbers gives $\frac{1}{n}\sum_{i=1}^{n} Y_i \to -1$ a.s. Since $x \mapsto e^x$ is continuous, we obtain a.s.

$$\lim_{n\to\infty} (X_1 \cdot \cdots \cdot X_n)^{1/n} = \exp\left(\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} Y_i\right) = e^{-1}.$$

4. Let X and Y be independent and suppose that each has a uniform distribution on $(0,1)$. Let $Z = \min\{X, Y\}$. Find the density $f_Z(z)$ for Z.

Solution: Let $z \in (0,1)$. By independence,

$$\mathbb{P}(Z > z) = \mathbb{P}(X > z \text{ and } Y > z) = \mathbb{P}(X > z)\mathbb{P}(Y > z) = \mathbb{P}(X > z)^2 = (1-z)^2.$$

Therefore the distribution function of Z is $F(z) = \mathbb{P}(Z \le z) = 1 - (1-z)^2$ when $z \in (0,1)$. It is easy to see that $F(z) = 0$ if $z \le 0$ and $F(z) = 1$ if $z \ge 1$. To compute the density, we take the derivative:

$$f_Z(z) = \frac{\mathrm{d}}{\mathrm{d}z} F(z) = 2(1-z),$$

whenever $z \in (0,1)$, and zero otherwise.

5. Show that the characteristic function φ of a random variable X is real if and only if X and $-X$ have the same distribution.

Solution: The characteristic function ϕ of $-X$ evaluated at $t \in \mathbb{R}$ is

$$\phi(t) = \mathbb{E}e^{it(-X)} = \mathbb{E}e^{-itX} = \overline{\mathbb{E}e^{itX}} = \overline{\varphi(t)}.$$

Here we have used that for a complex variable $U + iW$, one has

$$\overline{\mathbb{E}(U + iW)} = \overline{\mathbb{E}U + i\mathbb{E}W} = \mathbb{E}U - i\mathbb{E}W = \mathbb{E}(U - iW) = \mathbb{E}(\overline{U + iW}).$$

If X and $-X$ have the same distribution, then their characteristic functions are equal, so $\varphi(t) = \overline{\varphi(t)}$ for all t, meaning φ is real. Conversely, if φ is real, then $\varphi(t) = \overline{\varphi(t)}$ for all t, meaning $\phi = \varphi$. Since the characteristic functions of X and $-X$ then are equal, the variables have the same distribution.

6. Let X_i be i.i.d. random variables uniformly distributed on $[0, 2]$. Let $S_n = X_1 + \cdots + X_n$. Show that

$$\frac{3\sqrt{3}}{2} n^{\frac{1}{6}} \left(\sqrt[3]{S_n} - \sqrt[3]{n} \right) \to^w Z,$$

where Z is a standard normal random variable.

Solution: First, observe that $\mathbb{E}X_i = 1$ and $\sigma := \sqrt{\mathrm{Var}\ X_i} = \frac{2}{\sqrt{3}}$. Therefore, by the CLT, the random variable $\frac{S_n - n}{\frac{2}{\sqrt{3}}\sqrt{n}}$ converges weakly to a standard Gaussian random variable.

We estimate the probability

$$\begin{aligned}
&P\left(\frac{3\sqrt{3}}{2} n^{\frac{1}{6}} \left(\sqrt[3]{S_n} - \sqrt[3]{n} \right) \le t \right) \\
&= P\left(\sqrt[3]{S_n} \le \frac{2}{3\sqrt{3} n^{\frac{1}{6}}} t + \sqrt[3]{n} \right) \\
&= P\left(\frac{S_n - n}{\frac{2}{\sqrt{3}}\sqrt{n}} \le t + O(\frac{1}{\sqrt{n}}) \right) \\
&= P\left(\frac{S_n - n}{\frac{2}{\sqrt{3}}\sqrt{n}} \le t \right) + P\left(\frac{S_n - n}{\frac{2}{\sqrt{3}}\sqrt{n}} \in (t, t + O(\frac{1}{\sqrt{n}})) \right).
\end{aligned}$$

The second summand tends to zero as $n \to \infty$: indeed, for every $\epsilon > 0$ there exists an n large enough so that $O(\frac{1}{\sqrt{n}}) < \epsilon$, and hence

$$P\left(\frac{S_n - n}{\frac{2}{\sqrt{3}}\sqrt{n}} \in (t, t + O(\frac{1}{\sqrt{n}})) \right) \le P\left(\frac{S_n - n}{\frac{2}{\sqrt{3}}\sqrt{n}} \in (t, t + \epsilon) \right) \to \frac{1}{\sqrt{2\pi}} \int_t^{t+\epsilon} e^{-\frac{t^2}{2}} dt,$$

which can be made arbitrarily small by choosing small enough ϵ.

The first summand tends to $P(Z \le t)$, and hence

$$P\left(\frac{3\sqrt{3}}{2}n^{\frac{1}{6}}\left(\sqrt[3]{S_n} - \sqrt[3]{n}\right) \le t\right) \to_{n\to\infty} P(Z \le t),$$

which implies weak convergence, since the distribution of Z is continuous.

7. Let $v = \left(\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}}\right)$ be a unit vector in $\mathbf{R}^n$. Consider the set A in $\mathbf{R}^n$ be given by

$$A = \left\{x \in \mathbf{R}^n : \, x_i \in \left[-\frac{1}{2}, \frac{1}{2}\right], \langle x, v\rangle \le \frac{t}{2\sqrt{3}}\right\}.$$

Prove that as the dimension $n \to \infty$,

$$Vol_n(A) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} e^{-\frac{x^2}{2}}dx + O(\frac{1}{\sqrt{n}}).$$

Solution: Consider a random vector X uniformly distributed on $[-\frac{1}{2}, \frac{1}{2}]^n$. Its coordinates are i.i.d., with $\mathbb{E}X_i = 0$, $\sqrt{\text{Var } X_i} := \sigma = \frac{1}{2\sqrt{3}}$ and $\mathbb{E}|X_i|^3 = \frac{1}{32} < +\infty$. Therefore, by Berry-Essen's theorem,

$$\left|P\left(\frac{X_1 + ... + X_n}{\sigma\sqrt{n}} \le t\right) - P\left(Z \le t\right)\right| \le O(\frac{1}{\sqrt{n}}).$$

It remains to observe that

$$Vol_n(A) = P\left(\langle X, v\rangle \le \frac{t}{2\sqrt{3}}\right) = P\left(\frac{X_1 + ... + X_n}{\sigma\sqrt{n}} \le t\right).$$

8. Assume $X_1, X_2, ..., X_n, ...$ are i.i.d. standard normal random variables. Show without using the law of the iterated logarithm that for any $\lambda > 1/2$,

$$\frac{1}{n^{\lambda}}(X_1 + ... + X_n) \to^{a.s.} 0$$

Solution: The sum of n standard normal variables is normal with mean zero and variance n, as can be seen from computing characteristic functions: the characteristic function of the sum is, by independence,

$$\mathbb{E}e^{it(X_1+\cdots+X_n)} = \left(\mathbb{E}e^{itX_1}\right)^n = \left(e^{-t^2/2}\right)^n = e^{-t^2n/2},$$

which is the characteristic function of a Gaussian with mean zero and variance n. So we can compute for $\epsilon > 0$

$$\mathbb{P}\left(\left|\frac{1}{n^\lambda}(X_1+\cdots+X_n)\right| > \epsilon\right) = \mathbb{P}(|Z_n| > \epsilon n^\lambda),$$

where Z_n is Gaussian with mean zero and variance n. If Z is a standard normal variable, then $\sqrt{n}Z$ has the same distribution as Z_n, so this probability is

$$\mathbb{P}(|Z| > \epsilon n^{\lambda-1/2}),$$

or for $\sigma = 1/(\lambda - 1/2) > 0$,

$$\mathbb{P}\left(\frac{|Z|^\sigma}{\epsilon^\sigma} > n\right).$$

However a standard Gaussian has finite moments of all orders, so we use the characterization for a nonnegative random variable Y of $\mathbb{E}Y < \infty \Leftrightarrow \sum_n \mathbb{P}(Y > n) < \infty$ to say that since $\mathbb{E}\frac{|Z|^\sigma}{\epsilon^\sigma} < \infty$, one has

$$\sum_n \mathbb{P}\left(\frac{|Z|^\sigma}{\epsilon^\sigma} > n\right) < \infty.$$

This implies

$$\sum_n \mathbb{P}\left(\left|\frac{1}{n^\lambda}(X_1+\cdots+X_n)\right| > \epsilon\right) < \infty,$$

and so by Borel-Cantelli, a.s. $\left|\frac{1}{n^\lambda}(X_1+\cdots+X_n)\right| > \epsilon$ for only finitely many n. This implies convergence to 0 a.s.