Math 1552

Section 10.4: Comparison Tests for Infinite Series

Math 1552 lecture slides adapted from the course materials
By Klara Grodzinsky (GA Tech, *School of Mathematics*, Summer 2021)

Recap of last class:

- *Divergence test*: if the limit is not 0, the series diverges
- *Integral test*: use with a function that has an “easy” antiderivative

$$\sum_{k=2}^{\infty} \frac{1}{k \ln k}$$

Basic Comparison Test: Part (a)

Let $\sum_{\text{k}} a_k$ be a series with $a_k \geq 0$ for all k.

If we can find a series $\sum_k c_k$ such that

$\sum_k c_k$ converges and $a_k \leq c_k$ for all but

finitely many terms, then $\sum_k a_k$ must also

converge.

Basic Comparison Test: Part (b)

Let $\sum_{k} a_k$ be a series with $a_k \geq 0$ for all k.

If we can find a series $\sum_{k} d_k$ such that $\sum_{k} d_k$ diverges and $a_k \geq d_k \geq 0$ for all but finitely many terms, then $\sum_{k} a_k$ must also diverge.

Example: Does this series converge?

$$\text{(A)} \sum_{k=1}^{\infty} \frac{1}{1+2^k}$$

Example: Does this series converge?

$$\text{(B)} \sum_{k=2}^{\infty} \frac{1}{\sqrt{k}-1}$$

Limit Comparison Test

Let $\sum_{\text{k}} a_k$ be a series with $a_k \geq 0$ for all k.

Select a series $\sum_k b_k$. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$,

then both series converge or both series diverge.

NOTE: *Use one of the series you KNOW converges or diverges (geometric, p-series, etc.).*
This test is a good alternative to the comparison test.

Example: Does the series converge?

$$(\mathrm{A})\sum_{k=1}^{\infty}\frac{k+1}{k^3+4}$$

Example: Does the series converge?

$$\text{(B)} \sum_{k=1}^{\infty} \frac{k}{\sqrt{k^3+1}}$$

<u>Challenge example:</u> Does the series converge?

$$S = \sum_{n=2}^{\infty} \frac{e^{3n}}{e^{6n} + 16}$$

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Section 10.5: The Ratio and Root Tests for Infinite Series

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Recap of last class:

- *Divergence test*: if the limit is not 0, the series diverges
- *Comparison test*: find a bigger series that converges or a smaller series that diverges
- *Integral test*: use with a function that has an “easy” antiderivative

Recap of last class:

- *Limit Comparison test*: pick a series that you know converges or diverges.

(If the limit of the ratio of terms in your series to the given series approaches a finite, positive number, then both series either converge or diverge.)

Ratio Test

Let $\sum_{k=1}^{\infty} a_k$ be a series with all positive terms.

Let $L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

(a) If $L < 1$, then $\sum_{k=1}^{\infty} a_k$ coverges.

(b) If $L > 1$, then $\sum_{k=1}^{\infty} a_k$ diverges.

(c) If $L = 1$, then the test is *INCONCLUSIVE*!!!!

Example 1:

Determine whether the next series converges or diverges.

$$\sum_{k=1}^{\infty} \frac{3^k}{k^2}$$

Example 2:

Determine whether the next series converges or diverges.

$$\sum_{k=1}^{\infty} \frac{k \cdot 3^k}{(2k)!}$$

Root Test

Let $\sum_{k=1}^{\infty} a_k$ be a series with all positive terms.

Let $R = \lim_{n\to\infty} \sqrt[n]{a_n}$.

(a) If $R < 1$, then $\sum_{k=1}^{\infty} a_k$ coverges.

(b) If $R > 1$, then $\sum_{k=1}^{\infty} a_k$ diverges.

(c) If $R = 1$, then the test is *INCONCLUSIVE*!!!!

Example:

Determine if the series converges or diverges.

$$\sum_{k=1}^{\infty}\left(1+\frac{2}{k}\right)^{k^2}$$

Tips: which test to use when?

- ALWAYS start with the divergence test.

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- ALWAYS start with the divergence test.
- Use the integral test if the function looks “easy” to integrate or can be solved with a u-substitution.
- Use the harmonic series, geometric series, or p-series in the comparison and limit comparison tests.

Tips (continued)

- If you are unsure of which way the inequality may go, use the limit comparison test instead of the comparison test.

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- Use the root test when everything is raised to the k^{th} power.

Tips (continued)

- If you are unsure of which way the inequality may go, use the limit comparison test instead of the comparison test.
- Use the root test when everything is raised to the k^{th} power.
- Use the ratio test when you have factorials, or when no other test works.

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Section 10.6: Alternating Series

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Alternating Series Test

Let $\sum_{k} a_k$ be an alternating series.

(a) If $\sum_{k} |a_k|$ converges, then the series *converges absolutely*.

Alternating Series Test (cont.)

Let $\sum_{k} a_k$ be an alternating series.

(b) If (a) fails, then if :

i) $\{|a_n|\}$ is a decreasing sequence, and

ii) $\lim_{n\to\infty}|a_n| = 0$,

then the series *converges conditionally*.

(c) Otherwise, the series *diverges*.

Example A:

Determine if the alternating series converges absolutely, converges conditionally, or diverges.

$$\sum_{k=1}^{\infty} (-1)^k \frac{1}{\sqrt{k+4}}$$

Example B:

Determine if the alternating series converges absolutely, converges conditionally, or diverges.

$$\sum_{k=1}^{\infty} (-1)^k \frac{k}{3^k}$$

Example C:

Determine if the alternating series converges absolutely, converges conditionally, or diverges.

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{k^3}{k^3+2k+1}$$

Estimating an Alternating Sum

Let $\sum_k a_k$ be a convergent alternating series with a sum of L.

Then : $|s_n - L| < |a_{n+1}|$.

Example:

Estimate the sum of the series below within an error range of 0.001.

$$\sum_{k=0}^{\infty}(-1)^k \frac{1}{(2k+1)!}$$

Rearrangements

- If an alternating series converges *absolutely*, rearranging the terms will NOT change the sum.

- If an alternating series converges *conditionally*, then the sum changes when the terms are written in a different order.

Bonus Problem 1:

If $a_n = 1 - \dfrac{(-1)^n}{n}, n \geq 1$, evaluate $\displaystyle\sum_{n=1}^{\infty} (1 - a_n)$

Bonus Problem 2:

If $a_n = 1 - \dfrac{(-1)^n}{n}, n \geq 1$, evaluate $\displaystyle\sum_{n=1}^{\infty} (1 - a_{2n})$