Math 1552

Section 8.4:

Trigonometric Substitution

NOT covered on quiz 2

"trig subs"

Math 1552 lecture slides adapted from the course materials
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Quizz: on Thursday (last 20 mins. of studio):

- FTC (know how to apply it)
- integration by substitution
- IBP
- trig powers and product type integrals
- area between curves (*)

- **NO** trig subs. on this quiz
- Make you know your special trig angles in **ALL** quadrants, e.g., $\cos/\sin\left(\frac{5\pi}{4}\right)$ or $\cos/\sin\left(\frac{7\pi}{4}\right)$

Today's Learning Goals

- Identify which types of integrals can be solved with a trigonometric substitution (a.k.a., trig subs)
- Learn which substitution matches which general form
- Evaluate integrals using the method of trigonometric substitution

Trigonometric Substitutions

Recall the challenge problem:

$$\int_0^1 \sqrt{1-x^2}\,dx = \frac{\pi}{4}$$

We use a trig substitution when no other integration method will work, and when the integral contains one of these terms:

$$a^2 - x^2 \qquad \sqrt{a^2-x^2}$$

OR $x^2 - a^2$ ″

OR $a^2 + x^2$ ″

$a > 0$ some constant

Rules to Trig Substitutions

- Begin by replacing x with a trig function.

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- Don't forget to also replace *dx* with the appropriate trig function.

(important)

$$dx = \frac{d}{d\theta}\left[x(\theta)\right] * d\theta$$

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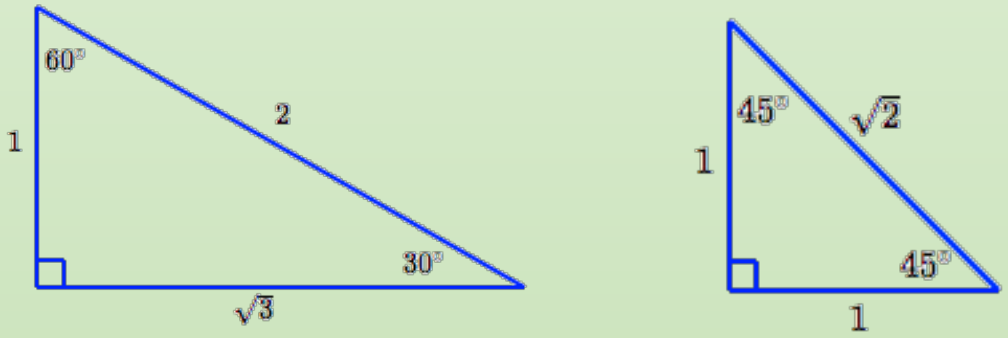
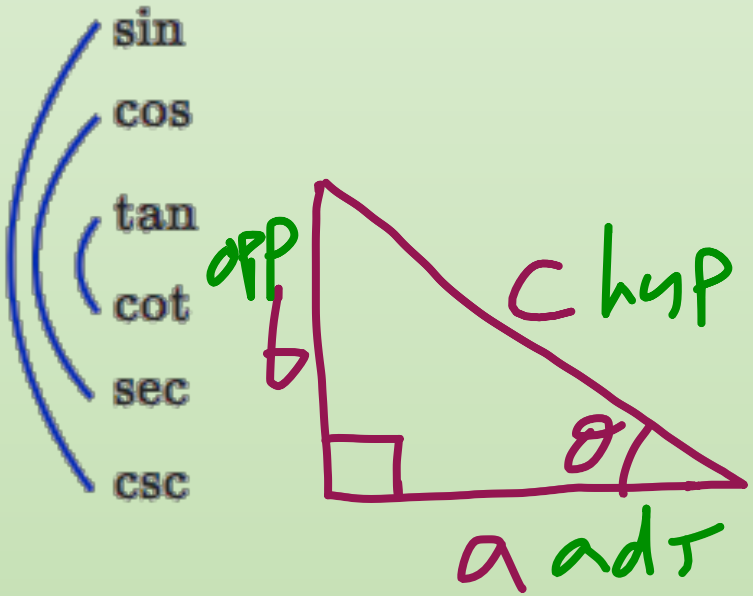
- Don't forget to also replace *dx* with the appropriate trig function.
- Use trig identities to solve the resulting integral.
- Be sure to rewrite your final answer in terms of *x*. (*)
- *Know how to derive the corresponding right triangle in each of the three cases we consider below without memorizing them*

Review of Trigonometry

Special right triangles (ratio of sides):



Trig function inverse relationships diagram:



Rules to compute trig functions of right triangles:

SOHCAHTOA

Sine **O**pposite **H**ypotenuse | **C**osine **A**djacent **H**ypotenuse | **T**angent **O**pposite **A**djacent |

$$a^2 + b^2 = c^2$$

ex: $\sin(\theta) = b/c$

$\sec(\theta) = c/a$

$\tan(\theta) = b/a$

Credits for figures: https://www.onemathematicalcat.org/Math/Precalculus_obj/trigValuesSpecialAngles.htm

Form 1: When the integral contains a term of the form

$$a^2 - x^2,$$

use the substitution:

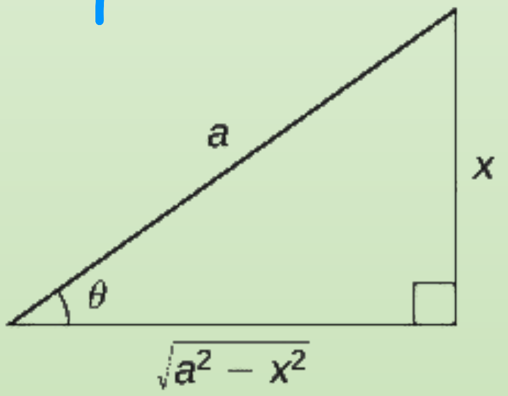
$$\boxed{x = a\sin\theta}$$

$$\frac{dx}{d\theta} = a\cos\theta$$

$$dx = a \cdot \cos\theta \, d\theta$$

$$\rightarrow a^2 - x^2 = a^2(1 - \sin^2\theta)$$
$$= a^2 \cdot \cos^2\theta$$

$$\sin\theta = \frac{x}{a}$$



Know how to get this triangle

Credits for figure: https://math.libretexts.org/Bookshelves/Calculus
(*Book:* OpenStax -> Techniques of Integration -> Trigonometric Substitution – Section 7.3)

Example 1: Evaluate the integral: $\int \boxed{\sqrt{4-x^2}}\,dx = I$ → "integrand"

→ NOTE: integrand function contains a term a^2-x^2, where $a=2$ (so we will use a trig sub to evaluate I)

→ $x = 2\sin\theta$, $dx = 2\cos\theta\, d\theta$

$$\sqrt{4-x^2} = \sqrt{4-4\sin^2\theta} = \sqrt{4(1-\sin^2\theta)}$$

$$= \sqrt{4\cos^2\theta} = 2\cos\theta$$

$$\to I = \int \overbrace{(2\cdot\cos\theta)}^{\sqrt{4-x^2}}\overbrace{(2\cos\theta\, d\theta)}^{dx}$$

$$= 4\int \cos^2\theta \, d\theta$$

$$\frac{1}{2}\sin(2\theta)$$

$$= 2\int (1+\cos(2\theta))\, d\theta$$

$$= 2\theta + \sin(2\theta) + C$$

(<u>recall</u>:

$\cos^2\theta = \frac{1}{2}(1+\cos(2\theta))$

(double angle formula))

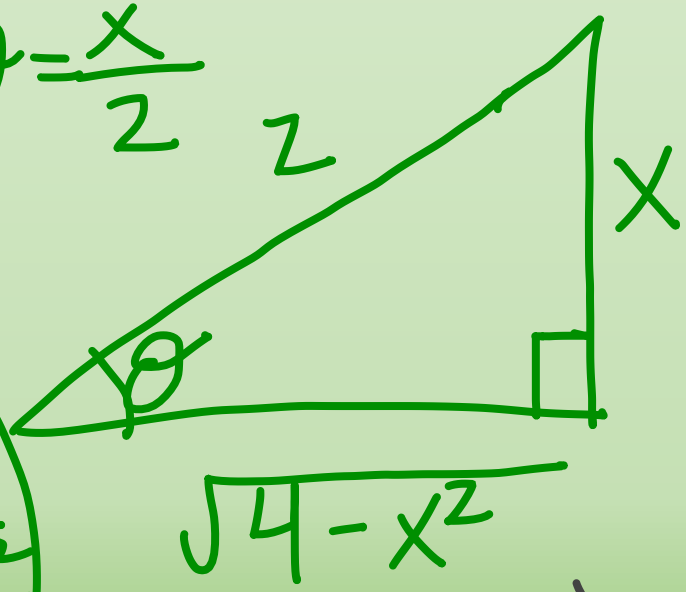
$$I = 2\theta + 2\cos\theta \cdot \sin(\theta) + C$$

first, simplify by writing:

$$\sin(2\theta) = 2\cos\theta \cdot \sin\theta$$

- $\sin\theta = \dfrac{x}{2}$
- $\cos\theta = \dfrac{\sqrt{4-x^2}}{2}$
- $\theta = \boxed{\sin^{-1}\left(\dfrac{x}{2}\right)} = \tan^{-1}\left(\dfrac{x}{\sqrt{4-x^2}}\right)$

$\sin\theta = \dfrac{x}{2}$



$\left(\tan^{-1}\left(\dfrac{t}{\sqrt{1-t^2}}\right) = \sin^{-1}(t),\ t \text{ a real number}\right)$

So, $I = 2\sin^{-1}\left(\frac{x}{2}\right)$

$$+\, 2 \cdot \frac{\sqrt{4-x^2}}{2} \cdot \frac{x}{2} + C$$

$\rightarrow$ Simplifying:

$$I = 2\sin^{-1}\left(\frac{x}{2}\right) + \frac{x}{2}\sqrt{4-x^2} + C$$

Form 2:

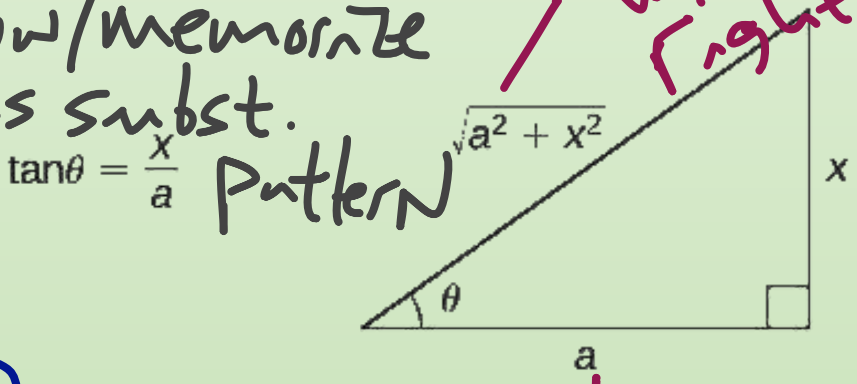
When the integral contains a term of the form

$$a^2 + x^2,$$

use the substitution:

$$\boxed{x = a\tan\theta}$$

KNOW/memorize this subst. pattern



KNOW this value by right-△ rules

(do NOT memorize)

recall:

$\tan^2\theta + 1 = \sec^2\theta$

$$dx = a \cdot \sec^2\theta \, d\theta$$

$$a^2 + x^2 = a^2 + a^2\tan^2\theta$$

$$= a^2(1 + \tan^2\theta)$$

$$= a^2 \cdot \sec^2\theta$$

Credits for figure: https://math.libretexts.org/Bookshelves/Calculus

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Example 2: Evaluate the integral: $\int \frac{1}{(9+x^2)^{3/2}}\,dx = I$

→ the integrand contains a term of the form a^2+x^2, where $a=3$

(this is a key indication to use a trig sub)

→ $x = 3\cdot\tan\theta$, $dx = 3\sec^2\theta\, d\theta$

$(9+x^2)^{3/2} = (9\sec^2\theta)^{3/2} = 27\sec^3\theta$

→ Therefore:

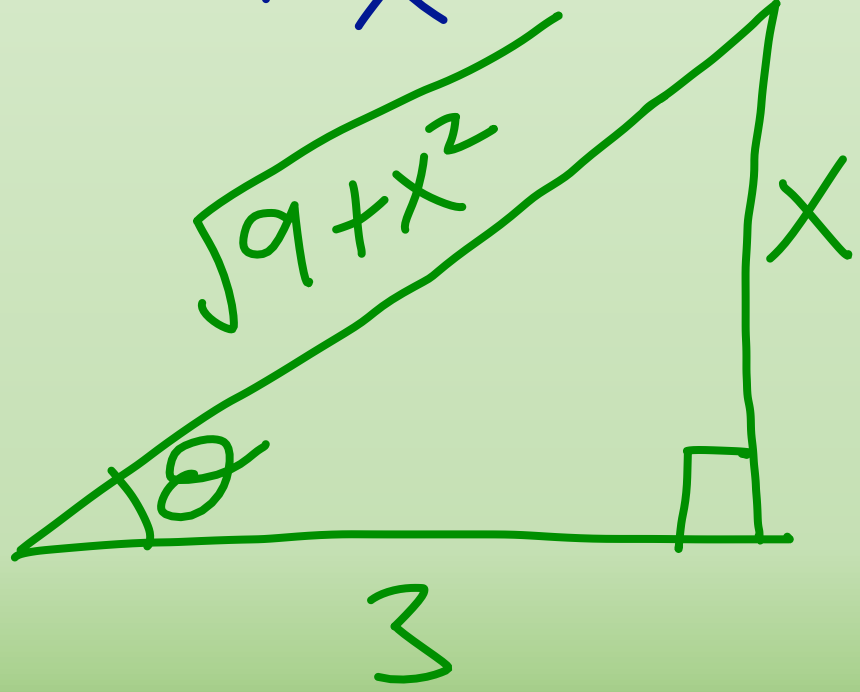
$$I = \int \frac{3 \cdot \sec^2\theta \, d\theta}{27 \sec^3\theta} = \frac{1}{9} \int \frac{d\theta}{\sec\theta}$$

$$= \frac{1}{9} \int \cos\theta \, d\theta$$

$$= \frac{1}{9} \sin\theta + C$$

→ we need to use our right-Δ trig rules to convert the antiderivative (in θ) into a function of x

$$\tan\theta = \frac{x}{3}$$



Need: $\sin\theta = \dfrac{x}{\sqrt{9+x^2}}$

→ Therefore: $I = \dfrac{x}{9\sqrt{9+x^2}} + C$

Form 3: When the integral contains a term of the form

$$x^2 - a^2,$$

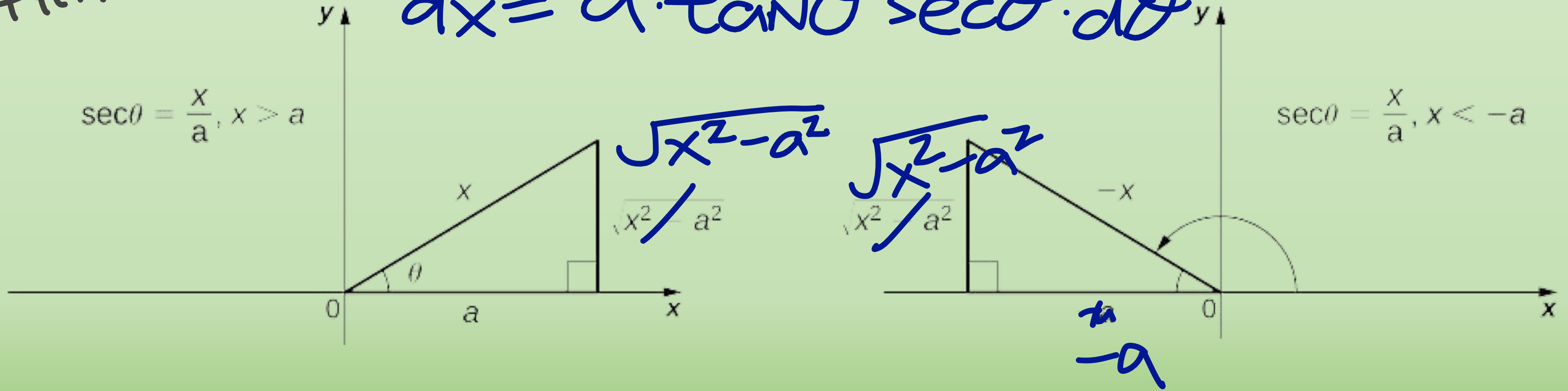
use the substitution:

$$\boxed{x = a\sec\theta}$$

know | memorize this type of subst pattern

$$x^2 - a^2 = a^2(\sec^2\theta - 1)$$
$$= a^2 \cdot \tan^2\theta$$

$$dx = a \cdot \tan\theta \sec\theta \cdot d\theta$$



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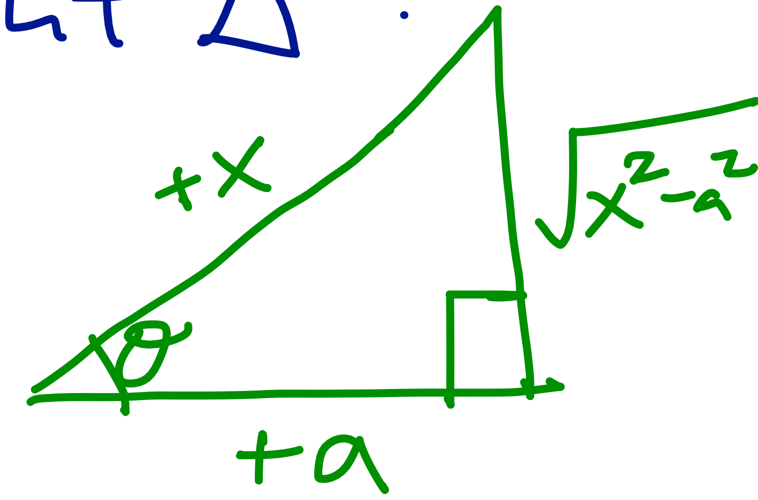
trig sub: have terms with $x^2 - a^2$, $a > 0$

$\rightarrow x = a\sec\theta$

① first case of the corresp. right Δ:

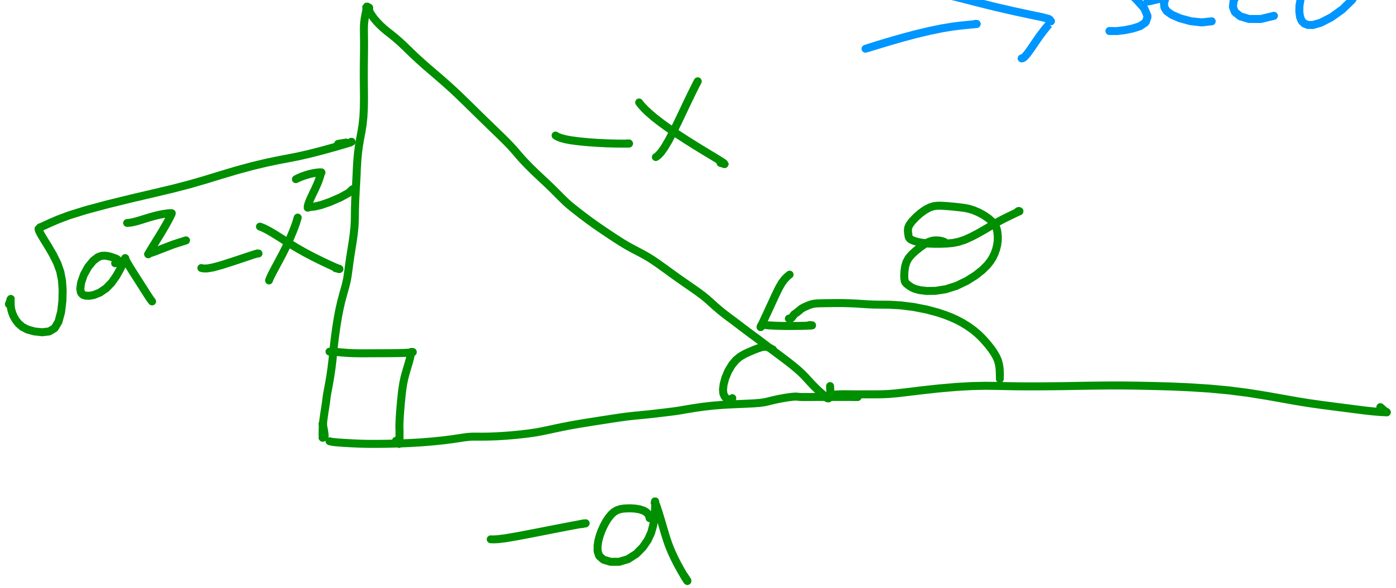
$x > a \Longleftrightarrow x^2 - a^2 > 0$



② second case: $\underbrace{x < -a}_{}$ x is negative

$\Longleftrightarrow x^2 > a^2 \Longleftrightarrow x^2 - a^2 > 0$

Second quadrant for θ



$$\sec\theta = \frac{-x}{-a}$$

$$= \frac{x}{a}$$

Example 3: Evaluate the integral: $\int \frac{1}{x^4\sqrt{x^2-1}}\,dx = I$

→ our integrand has a term of the form x^2-a^2, for $a=1$

(we are in "trig sub land")

→ $x=\sec\theta$, $dx=\tan\theta\sec\theta\,d\theta$,

$\sqrt{x^2-1}=\sqrt{\sec^2\theta-1} = \sqrt{\tan^2\theta}$

$= \tan\theta$

$$I = \int \frac{\cancel{\tan\theta} \, \cancel{\sec\theta} \, d\theta}{\sec^{\cancel{4}\,3}\theta \cdot \cancel{\tan\theta}}$$

$$= \int \cos^3\theta \, d\theta$$

(recall that: $\cos^2\theta = 1 - \sin^2\theta$)

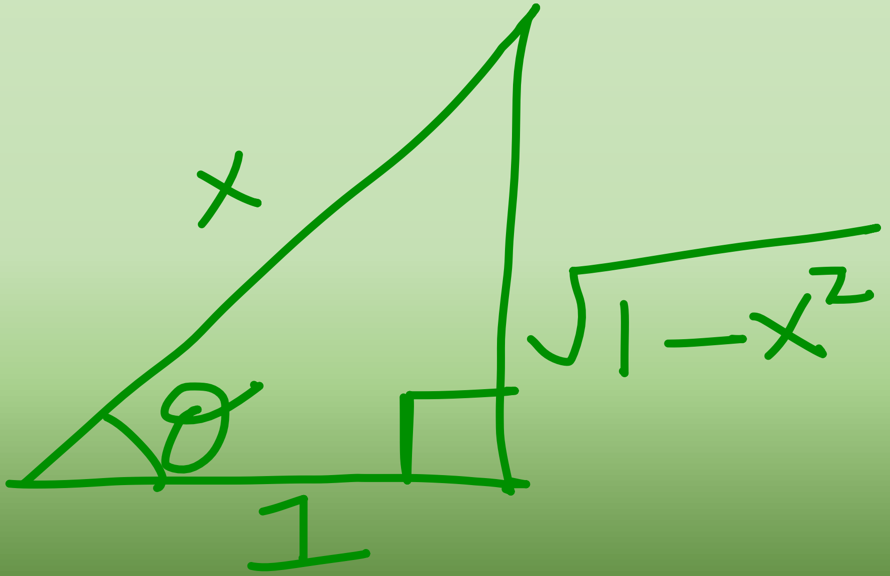
$$= \int (1 - \sin^2\theta)\cos\theta \, d\theta$$

[u-sub: $u = \sin\theta$, $du = \cos\theta \, d\theta$

$$= \int (1 - u^2) \, du$$

$$= u - \frac{u^3}{3} + C$$

$$= \sin\theta - \frac{\sin^3\theta}{3} + C$$



$$\frac{x}{1} = \sec\theta = \frac{\text{hyp}}{\text{adj}}$$

$$\sin\theta = \frac{\sqrt{1-x^2}}{x}$$

→ therefore:

$$I = \frac{\sqrt{1-x^2}}{x} - \underbrace{\frac{1}{3x^3}(1-x^2)^{3/2}}_{\frac{1}{3}\sin^3\theta} + C$$

$$= \frac{1}{3}\left(\frac{\sqrt{1-x^2}}{x}\right)^3$$