PROBABILITY COMPREHENSIVE EXAM SPRING 2015

Problem 1. *Assume X is a symmetric random variable such that $\mathbb{E}[X^2] = 1$ and $\mathbb{E}[X^4] = 2$. Show that*

$$\mathbb{P}(X \geq 1) \leq \frac{14}{27}.$$

Solution. If we try a brute force Chebyshev inequality we obtain a trivial bound, namely that $\mathbb{P}(X \geq 1) \leq 1$, something not very useful. The idea is to use the Chebyshev's inequality in a cleverer way. Take first a positive $t \geq 0$ and then write

$$\mathbb{P}(X \geq 1) = \mathbb{P}(X + t \geq 1 + t) \leq \mathbb{P}((X + t)^4 \geq (1 + t)^4).$$

Then use Chebyshev's inequality to continue with

$$\mathbb{P}(X+t \geq 1+t) \leq \mathbb{P}((X+t)^4 \geq (1+t)^4) \leq \frac{\mathbb{E}[(X + t)^4]}{(1 + t)^4} = \frac{\mathbb{E}[X^4] + 4t\mathbb{E}[X^3] + 6t^2\mathbb{E}[X^2] + 4t^3\mathbb{E}[X] + t^4}{(1 + t)^4}.$$

Since the variable is symmetric, $\mathbb{E}[X] = 0$ and also $\mathbb{E}[X^3] = 0$. Thus,

$$\mathbb{P}(X \geq 1) \leq \frac{2 + 6t^2 + t^4}{(t + 1)^4}.$$

Now we want to take the best possible choice for t and for that matter we need to find the minimum value of

$$f(t) = \frac{2 + 6t^2 + t^4}{(t + 1)^4}$$

Taking the derivative, we get

$$f'(t) = \frac{4\left(t^3 - 3t^2 + 3t - 2\right)}{(t + 1)^5} = \frac{4(t - 2)(t^2 - t + 1)}{(t + 1)^5}.$$

Therefore, $t = 2$ is a critical point and it is also a minimum point. This means

$$\mathbb{P}(X \geq 1) \leq f(2) = \frac{14}{27}.$$

□

Problem 2. *Assume that $X_1, X_2, \dots, X_n, \dots$ is a sequence of iid random variables such that for some $\alpha < 1/2$,*

$$\frac{X_1 + X_2 + \cdots + X_n}{n^\alpha} \xrightarrow[n\to\infty]{a.s.} m$$

for some real number m (and the convergence is in the almost sure sense). Show that almost surely $X_i = 0$.

Solution. We prove in the first place that X_i are integrable. That is a standard application of Borel-Cantelli. Denote $S_n = X_1 + X_2 + \cdots + X_n$. Then

$$\frac{X_n}{n^\alpha} = \frac{S_n}{n^\alpha} - \frac{S_{n-1}}{n^\alpha} \xrightarrow[n\to\infty]{a.s.} 0.$$

Now we show that the the variable is square integrable. To do this we start with the fact that

$$\frac{X_n^2}{n^{2\alpha}} \xrightarrow[n\to\infty]{a.s.} 0.$$

thus, we have by Borel-Cantelli and the fact that X_n are independent that

$$\sum_{n\geq 1}\mathbb{P}\left(\frac{X_n^2}{n^{2\alpha}}\geq\epsilon\right)<\infty.$$

Indeed, the divergence of the series and independence would mean that $\mathbb{P}\left(\frac{X_n^2}{n^{2\alpha}}\geq\epsilon\, i.o.\right)=1$ and this would contradict the convergence of the $\frac{X_n}{n^\alpha}$ to 0.

Now, we have that

$$\sum_{n\geq 1}\mathbb{P}\left(\frac{X_n^2}{n^{2\alpha}}\geq 1\right)=\sum_{n\geq 1}\mathbb{P}\left(\frac{X_1^2}{n^{2\alpha}}\geq 1\right)=\sum_{n\geq 1}\mathbb{P}\left(X_1^2\geq n^{2\alpha}\right)\leq\sum_{n\geq 1}\mathbb{P}\left(X_1^2\geq n\right)<\infty$$

This last part implies that the variable X_1^2 is integrable. In particular X_1 is also integrable.

On the other hand, since $\alpha<1/2$, we can also conclude that

$$\frac{X_1+X_2+\cdots+X_n}{n}\xrightarrow[n\to\infty]{a.s.}0$$

and thus, by the Strong Law of large numbers, the mean must be 0.

Now if we assume that $\mathbb{E}[X_1^2]=\sigma^2>0$, then the central limit theorem gives us that

$$\frac{X_1+X_2+\cdots+X_n}{n^{1/2}}\Longrightarrow N\left(0,\sigma^2\right).$$

On the other hand from the given condition ($\alpha<1/2$) we also get

$$\frac{X_1+X_2+\cdots+X_n}{n^{1/2}}\xrightarrow[n\to\infty]{a.s.}0$$

and since a.s. convergence implies the weak convergence we get that $N(0,\sigma^2)=0$, which is a contradiction.

Thus $\mathbb{E}[X_1^2]=0$ and this means that $X_1=0$ almost surely. □

Problem 3. *Assume that (X,Y) is a joint normal vector with $\mathbb{E}[X]=\mathbb{E}[Y]=0$. Show that*

$$\mathbb{E}[X^2Y^2]\geq\mathbb{E}[X^2]\mathbb{E}[Y^2]$$

with equality if and only if X and Y are independent.

Solution. For simplicity we may assume that the variance of X and Y are both equal to 1. Then the covariance matrix can be written as

$$C=\begin{bmatrix}1&\rho\\\rho&1\end{bmatrix}.$$

We learn from this that the characteristic function of (X,Y) is determined by

$$f_{X,Y}(\xi,\eta)=\mathbb{E}[e^{i\xi X+i\eta Y}]=\exp\left(-\frac{\xi^2+\eta^2+2\rho\xi\eta}{2}\right)$$

Then, the integral $\mathbb{E}[X^2Y^2]$ can be computed as the derivative of the characteristic function as

$$E[X^2Y^2]=\frac{1}{i^4}\frac{\partial^4}{\partial\xi^2\partial\eta^2}f_{X,Y}(\xi,\eta)\Big|_{\xi=\eta=0}=\frac{\partial^4}{\partial\xi^2\partial\eta^2}f_{X,Y}(\xi,\eta)\Big|_{\xi=\eta=0}.$$

This last part can be computed from the characteristic function in the form

$$\frac{\partial^4}{\partial\xi^2\partial\eta^2}f_{X,Y}(\xi,\eta)\Big|_{\xi=\eta=0}=\frac{\partial^4}{\partial\xi^2\partial\eta^2}\exp\left(-\frac{\xi^2+\eta^2+2\rho\xi\eta}{2}\right)\Big|_{\xi=\eta=0}=1+\frac{\rho^2}{2}.$$

Thus
$$\mathbb{E}[X^2Y^2] = 1 + \frac{\rho^2}{2} \geq 1 = \mathbb{E}[X^2]\mathbb{E}[Y^2]$$
with equality if and only if $\rho = 0$ which is the same as saying that X and Y are uncorrelated which because of the fact that (X, Y) is a normal vector, is equivalent to saying that X and Y are independent. □

Problem 4. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $(\mathcal{F})_{n\geq 0}$ is a filtration on it. Show that if $(M_n)_{n\geq 0}$ is a martingale such that $(M_n^4)_{n\geq 0}$ is also a martingale, then almost surely $M_n = M_0$ for any $n \geq 0$.*

Solution. Since M_n^4 is a martingale, we write using the properties of the conditional expectation and the fact that M_n is $\mathcal{F}_n$-measurable,
$$\begin{aligned}
\mathbb{E}[M_{n+1}^4|\mathcal{F}_n] &= \mathbb{E}[(M_{n+1} - M_n + M_n)^4|\mathcal{F}_n] \\
&= \mathbb{E}[(M_{n+1} - M_n)^4|\mathcal{F}_n] + \mathbb{E}[4(M_{n+1} - M_n)^3 M_n|\mathcal{F}_n] + 6\mathbb{E}[(M_{n+1} - M_n)^2 M_n^2|\mathcal{F}_n] \\
&\quad + 4\mathbb{E}[(M_{n+1} - M_n)M_n^3|\mathcal{F}_n] + \mathbb{E}[M_n^4|\mathcal{F}_n] \\
&= \mathbb{E}[(M_{n+1} - M_n)^2((M_{n+1} - M_n)^2 + 4(M_{n+1} - M_n)M_n + 6M_n^2)|\mathcal{F}_n] \\
&\quad + 4M_n^3\mathbb{E}[M_{n+1} - M_n|\mathcal{F}_n] + M_n^4.
\end{aligned}$$
Now, from the fact that M_n is a martingale, $\mathbb{E}[M_{n+1} - M_n|\mathcal{F}_n] = 0$ and thus we obtain that
$$\mathbb{E}[(M_{n+1} - M_n)^2((M_{n+1} - M_n)^2 + 4(M_{n+1} - M_n)M_n + 6M_n^2)|\mathcal{F}_n] = 0.$$
Simplifying this a little bit we arrive at
$$\mathbb{E}[(M_{n+1} - M_n)^2((M_{n+1} - M_n + 2M_n)^2 + 2M_n^2)|\mathcal{F}_n] = 0$$
Taking now the expectation we get
$$\mathbb{E}[(M_{n+1} - M_n)^2((M_{n+1} + M_n)^2 + 2M_n^2)] = 0$$
which yields that either $M_{n+1} = M_n$ almost surely, or $(M_{n+1} + M_n)^2 + 2M_n^2 = 0$ which again implies that $M_{n+1} = M_n$ almost surely. □

Problem 5. *For a sequence $X_1, X_2, \ldots, X_n, \ldots$ we know that*
$$\sum_{n=1}^{\infty} n\mathbb{E}[|X_n|] < \infty.$$
Show that the sequence $Y_n = X_n + X_{n+1} + \cdots + X_{10n}$, converges almost surely and in L^1 to 0.

Solution. The convergence in L^1, follows from the fact that
$$\mathbb{E}[|Y_n|] \leq \sum_{k=n+1}^{10n} \mathbb{E}[|X_k|] \leq \sum_{k=n+1}^{\infty} k\mathbb{E}[|X_k|] \xrightarrow[n\to\infty]{} 0$$

For the almost sure convergence, notice that summing over n the inequalities $\mathbb{E}[|Y_n|] \leq \sum_{k=n+1}^{\infty} \mathbb{E}[|X_k|]$, we then get
$$\sum_{n=1}^{\infty} \mathbb{E}[|Y_n|] \leq \sum_{n=1}^{\infty} (n+1)\mathbb{E}[|X_n|] \leq 2\sum_{n=1}^{\infty} n\mathbb{E}[|X_n|] < \infty$$
Therefore the almost sure convergence follows from Chebyshev's inequality and Borel-Cantelli lemma and the observation that
$$\sum_{n=1}^{\infty} \mathbb{P}(|Y_n| \geq \epsilon) \leq \frac{1}{\epsilon}\sum_{n=1}^{\infty} \mathbb{E}[|Y_n|] < \infty.$$
□

Problem 6. *Assume $\{X_n\}_{n\geq 1}$ is a sequence of iid random variables with mean 0 and variance 1. Show that*

$$Y_n = \frac{\sqrt{n}X_1 + \sqrt{n-1}X_2 + \sqrt{n-2}X_3 + \cdots + X_n}{n}$$

converges weakly (in distribution) to a normal $N(0, 1/2)$.

Solution. It is enough to show that the characteristic functions converge. Now, the characteristic function of Y_n is computed as

$$f_{Y_n}(\xi) = f_{X_1}(\sqrt{n}\xi/n)f_{X_2}(\sqrt{n-1}\xi/n)\dots f_{X_n}(\xi/n).$$

Since $X_1, X_2, \dots, X_n$ are identically distributed, we denote by $f(\xi) = f_{X_i}(\xi)$ for any i and then continue with

$$f_{Y_n}(\xi) = f(\sqrt{n}\xi/n)f(\sqrt{n-1}\xi/n)\dots f(\xi/n).$$

Since X_1 has mean 0 and variance 1, $f(\xi) = 1 - \xi^2/2 + o(\xi^2)$ which written in exponential form for small enough ξ, gives $f(\xi) = e^{-\frac{\xi^2}{2}+o(\xi^2)}$. From this we deduce that for large n and fixed ξ

$$f(\sqrt{n}\xi/n)f(\sqrt{n-1}\xi/n)\dots f(\xi/n) = \exp(-\frac{n\xi^2}{2n^2} - \frac{(n-1)\xi^2}{2n^2} - \dots - \frac{\xi^2}{2n^2} - o(1)) = \exp(-\frac{\xi^2}{4} + o(1))$$

from which we get that

$$f_{Y_n}(\xi) \xrightarrow{n\to\infty} \exp(-\frac{\xi^2}{4}) = f_{N(0,1/2)}(\xi)$$

and this completes the proof. □

Problem 7. *Assume that $\{U_n\}_{n\geq 1}$ is a sequence of iid uniform random variables on $[0, 1]$. Let $V_n = \max\{U_1, U_2^2, \dots, U_n^n\}$. Show that $(1 - V_n)\ln(n)$ converges weakly (in distribution) to an exponential random variable with parameter 1.*

Solution. To do this we will compute the cumulative function of $W_n = (1 - V_n)\ln(n)$. Take a positive x and let n be large enough. Now, using independence,

$$\begin{aligned}
1 - F_{W_n}(x) &= \mathbb{P}(W_n > x) = \mathbb{P}\left(V_n < 1 - \frac{x}{\ln(n)}\right) \\
&= \mathbb{P}\left(U_1 < 1 - \frac{x}{\ln(n)}, U_2^2 < 1 - \frac{x}{\ln(n)}, \dots, U_n^n < 1 - \frac{x}{\ln(n)}\right) \\
&= \mathbb{P}\left(U_1 < 1 - \frac{x}{\ln(n)}\right)\mathbb{P}\left(U_2^2 < 1 - \frac{x}{\ln(n)}\right), \dots, \mathbb{P}\left(U_n^n < 1 - \frac{x}{\ln(n)}\right) \\
&= \mathbb{P}\left(U_1 < 1 - \frac{x}{\ln(n)}\right)\mathbb{P}\left(U_2 < \left(1 - \frac{x}{\ln(n)}\right)^{1/2}\right), \dots, \mathbb{P}\left(U_n < \left(1 - \frac{x}{\ln(n)}\right)^{1/n}\right) \\
&= \left(1 - \frac{x}{\ln(n)}\right)\left(1 - \frac{x}{\ln(n)}\right)^{1/2} \dots \left(1 - \frac{x}{\ln(n)}\right)^{1/n} \\
&= \left(1 - \frac{x}{\ln(n)}\right)^{1+1/2+\cdots+1/n} = \left(\left(1 - \frac{x}{\ln(n)}\right)^{\ln(n)}\right)^{\frac{1+1/2+\cdots+1/n}{\ln(n)}}.
\end{aligned}$$

Finally, since $(1 - x/\ln(n))^{\ln(n)}$ converges to e^{-x} as n tends to infinity and

$$1 + 1/2 + \cdots + \frac{1}{n} - (1 + 1/n) < \ln(n) < 1 + 1/2 + \cdots + 1/n$$

we obtain that
$$\frac{1 + 1/2 + \cdots + 1/n}{\ln(n)} \xrightarrow{n\to\infty} 1$$
and thus
$$1 - F_{W_n}(x) \xrightarrow{n\to\infty} e^{-x} = 1 - F_Z(x)$$
where Z is an exponential random variable with parameter 1.

□

Problem 8. *Let $\{X_n\}_{n\geq 1}$ be an iid sequence of positive random variables such that $E[X_1] < \infty$. Let*
$$N_t = \sup\{n : X_1 + X_2 + \cdots + X_n \leq t\}.$$
Show that
$$\frac{N_t}{t} \xrightarrow[t\to\infty]{a.s.} \frac{1}{\mathbb{E}[X_1]}$$
where the convergence is in almost sure sense.

Solution. Let $S_n = X_1 + X_2 + \cdots + X_n$. From the Law of Large numbers, we learn that S_n converges to $+\infty$ as n tends to ∞. In particular N_t is almost surely finite and $N(t)$ tends to infinity with t. Now for any t we obviously have
$$S_{N(t)} \leq t \leq S_{N(t)+1}$$
thus,
$$\frac{N(t)}{S_{N(t)+1}} \leq \frac{N(t)}{t} \leq \frac{N(t)}{S_{N(t)}}.$$
Now, from the Law of large numbers we have that
$$\frac{S_n}{n} \xrightarrow[n\to\infty]{a.s.} \mathbb{E}[X_1]$$
which in turn combined with the fact that N_t converges to infinity with t concludes the proof.

□