1. (Applying the Definite Integral) A marketing company is trying a new campaign. The campaign lasts for three weeks, and during this time, the company finds that it gains customers as a function of time according to the formula:

$$C(t) = 10t - 4t^2 + 5$$

where t is time in weeks and the number of customers is given in thousands.

(a) Using the following form of the definite integral,

$$\int_a^b f(t)dt = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f(a + \frac{(b-a)i}{n})$$

calculate the average number of customers gained during the three-week campaign.

Solution. We consider t to be number of weeks.

$$\int_0^3 (10t - 4t^2 + 5)dt = \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left(10\left(\frac{3i}{n}\right) - 4\left(\frac{3i}{n}\right)^2 + 5\right)$$
$$= \lim_{n\to\infty} \frac{90}{n^2} \sum_{i=1}^{n} i - \frac{108}{n^3} \sum_{i=1}^{n} i^2 + \frac{15}{n} \sum_{i=1}^{n} 1 = \frac{90}{2} - \frac{108}{3} + 15 = 24.$$

(b) Also compute the average number of customers using the form

$$\int_a^b f(t)dt = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f(a + \frac{(b-a)(i-1)}{n})$$

Compare the results. What can you confirm from this comparison?

Solution. The result will be the same, and it confirms that x_i^* can be any point in the interval $[x_{i-1}, x_i]$.

2. Explain why the following property is true:

$$\left|\int_a^b f(t)dt\right| \le \int_a^b |f(t)|dt$$

Can you find an example where the inequality is strict?

Solution. The LHS is the net area, while the RHS is the total area. Please draw a picture and explain it. No need for proof.

3. Determine if each statement below is true or false.

(a) We always set x_i^* to be the right-hand endpoint of the ith interval.

Solution. False. x_i^* can be any point in the interval $[x_{i-1}, x_i]$.

(b)

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Solution. True. Please help them to prove it using induction.

(c) If $f(x) \geq 0$ on $[a, b]$, then $\int_a^b f(t)dt$ represents the total area bounded by f, $x = a$, $x = b$ and the x-axis.

Solution. True. Please help them understand the necessity of condition $f \geq 0$.