

Characterizing the summatory functions of invertible multiplicative functions f by strong additivity, f -partitions of the integers, and limiting Erdős-Kac type central limit theorems

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Last Revised: Tuesday 9th February, 2021 @ 22:17:55 – Compiled with L^AT_EX2e

Abstract

TODO...

Keywords and Phrases: *TODO.*

Math Subject Classifications (MSC 2010): *TODO.*

1 Introduction

Suppose that $f \geq 1$ is an integer-valued multiplicative function defined such that $f(1) = 1$. The inverse f^{-1} of f with respect to the Dirichlet convolution operator $*$ exists and satisfies $f^{-1} * f = f * f^{-1} = \varepsilon$ where $\varepsilon(n) = \delta_{n,1}$ is the multiplicative identity with respect to $*$. The Dirichlet inverse of f is computed recursively, or exactly, given the following two identities for $n \geq 2$ where $f^{-1}(1) = 1/f(1)$ [1, §TODO] [?]:

$$\begin{aligned} f^{-1}(n) &= \text{TODO}, n \geq 2 \\ f^{-1}(n) &= \text{TODO}, n \geq 1. \end{aligned}$$

We are interested in evaluating bounds on the summatory function of f^{-1} defined for any integers $x \geq 1$ as

$$F^{-1}(x) := \sum_{n \leq x} f^{-1}(n).$$

We can prove that for all $n \geq 1$, $\text{sgn}(f^{-1}(n)) = (-1)^{\omega(n)}$ where $\omega(n) = \sum_{p|n} 1$ counts the number of primes dividing $n \geq 2$ without multiplicity and with $\omega(1) = 0$. Thus the sums $F^{-1}(x)$ are signed and oscillatory ^A

Definition 1.1. For a fixed multiplicative arithmetic function $f \geq 1$, let its set of values over the integers $n \geq 1$ be defined by

$$\mathcal{A}_f^* := \{f(n) : n \geq 1\}.$$

Let the set of all multiples of $f(n)$ be defined by

$$\mathcal{A}_f := \left(\bigcup_{k \geq 1} (\mathcal{A}_f^*)^k \right) \setminus \{1\}.$$

We denote by χ_f the indicator function of \mathcal{A}_f . Let the strongly additive function $\omega_f(n)$ be defined by

$$\omega_f(n) := \begin{cases} 0, & \text{if } n = 1; \\ \sum_{\substack{d|n \\ d \in \mathcal{A}_f}} 1, & \text{if } n \geq 2. \end{cases}$$

For $k \geq 1$ and $x \geq 2$, the summatory functions

$$\pi_{f,k}(x) := \#\{2 \leq n \leq x : \omega_f(n) = k\}.$$

Similarly, we define $\Omega_f(n)$ to be the function that counts TODO. The analog to the *Liouville lambda function* corresponding to this definition is defined by $\lambda_f(n) := (-1)^{\Omega_f(n)}$ for $n \geq 1$. We define the \mathcal{A}_f -counting function by

$$\pi_f(x) := \sum_{n \leq x} \chi_f(n).$$

1.1 Applications and open questions about signed summatory functions

Let $V(f, Y)$ denote the number of sign changes of f on the interval $(0, Y]$ for real $Y > 0$. More precisely, we have that

$$V(f, Y) := \sup \left\{ N : \exists \{x_i\}_{i=1}^N, 0 < x_1 < \dots < x_N \leq Y, f(x_i) \neq 0, \text{sgn}(f(x_i)) \neq \text{sgn}(f(x_{i+1})), \forall 1 \leq i < N \right\}.$$

^ATODO: Need the result for sign changes ...

It is known that the analytic properties, poles, and zeros of the Dirichlet generating function, $D_f(s)$, or DGF, of f provide key insights into the sign changes of these functions [3]. For example, if the DGF of f is analytic on some half-plane, subject to certain restrictions, then Landau showed in 1905 that the summatory function of f , $S_f(x)$, changes signs infinitely often as we let x tend to infinity. In particular, we obtain the following theorem of Pólya, which extends Landau's result to provide a statement concerning the frequency of the sign changes of $S_f(x)$:

Theorem 1.2 (Pólya on Sign Changes of the Summatory Function of an Arithmetic Function). *Suppose that $S_f(x)$ is real-valued for all $x \geq x_0$, and define the function $\hat{F}_f(s)$ by the Mellin transform at $-s$ as*

$$\hat{F}_f(s) := \int_{x_0}^{\infty} \frac{S_f(x)}{x^{s+1}} dx.$$

Suppose that $\hat{F}_f(s)$ is analytic for all $\operatorname{Re}(s) > \theta$, but is not analytic in any half-plane $\operatorname{Re}(s) > \theta - \varepsilon$ for $\varepsilon > 0$. Furthermore, suppose that $\hat{F}_f(s)$ is meromorphic in some half-plane $\operatorname{Re}(s) > \theta - c_0$ for some $c_0 > 0$. Let

$$\gamma_f := \begin{cases} \inf\{|t| : \hat{F}_f(s) \text{ is not analytic at } s = \theta + it\}, & \text{if } f \text{ is not analytic at } \operatorname{Re}(s) = \theta; \\ \infty, & \text{otherwise.} \end{cases}$$

Then

$$\limsup_{Y \rightarrow \infty} \left\{ \frac{V(S_f, Y)}{\log Y} \right\} \geq \frac{\gamma_f}{\pi}.$$

Remark 1.3 (Open questions about signed summatory functions). TODO ...

Example 1.4 (The Mertens function). TODO ...

Example 1.5 (The Liouville lambda function). We have that $\lambda(n) = |\mu(n)|^{-1}$. Hence, the summatory function $L(x) := \sum_{n \leq x} \lambda(n)$ satisfies

$$\frac{\zeta(2s)}{s\zeta(s)} = \int_1^{\infty} \frac{L(x)}{x^{s+1}} dx, \operatorname{Re}(s) > 1.$$

TODO ...

Open questions and known results ...

1.2 Main new results

Primary new theorems ...

1.3 Significance of the new results

Notice by the way we have constructed $\omega_f = f * \chi_f$, we can write the convolution for this function in terms of the restricted convolutions with divisors (and input indices) $n, d \in \mathcal{A}_f \cup \{1\}$. That is, we have that $\omega_f = f *_{\mathcal{A}_f} 1$:

$$\omega_f(n) = \begin{cases} \sum_{\substack{d|n \\ d \in \mathcal{A}_f}} f(d), & n \geq 2 \wedge n \in \mathcal{A}_f; \\ 0, & \text{otherwise.} \end{cases}$$

Now suppose that $n, m \geq 1$ are such that $(n, m) = 1$. If $n \notin \mathcal{A}_f$ or $m \notin \mathcal{A}_f$, then the convolution above is zero. On the other hand, if both $n, m \in \mathcal{A}_f$ and these two integers are relatively prime, then all of the f -divisors of nm either belong to n or to m . Thus

$$\omega_f(nm) = \sum_{\substack{d|n \\ d \in \mathcal{A}_f}} f(d) + \sum_{\substack{d|m \\ d \in \mathcal{A}_f}} f(d) = \omega_f(n) + \omega_f(m).$$

Since this relation holds for any positive integers n, m , it follows that the function $\omega_f(n)$ is *strongly additive*.

Much like the canonical prime factor counting functions, $\omega(n)$ and $\Omega(n)$, strong additivity is a very expressive property. We know that any strongly additive function has a central limit type theorem, or *Erdős-Kac theorem analog*, that describes the distribution of its values $n \leq x$ as $x \rightarrow \infty$ succinctly. In particular, suppose that $h(n)$ is a strongly additive arithmetic function. We set the mean and variance analogs for h to be respectively

$$A_h(x) := \sum_{p \leq x} h(p)p^{-1},$$

$$B_h(x) := \sum_{p \leq x} h^2(p)p^{-1}.$$

Then there is a limiting probability measure, or equivalently a CDF $\nu_h(x, z)$, such that each of the following limits exists and is finite as $x \rightarrow \infty$ [?, §1.7]:

$$\nu_h(x, z) := \frac{1}{x} \cdot \# \left\{ n \leq x : \frac{h(n) - A_h(x)}{B_h(x)} \leq z \right\}$$

Given any fixed $Y > 0$, we can usually show that the convergence of the above CDF formula is uniform for any $-Y \leq z \leq Y$.

The property of strong additivity actually implies more about the distribution of the underlying function h and partial sums of this function. Namely, let

$$E_h(x) := \sum_{p^\alpha \leq x} h(p^\alpha)p^{-\alpha}(1 - p^{-1}), \quad D_h(x) := \sqrt{\sum_{p^\alpha \leq x} |h(p^\alpha)|^2 p^{-\alpha}}.$$

Then the summatory functions of h satisfy

$$\sum_{n \leq x} h(n) = xE_h(x) + O(\sqrt{x}D_h(x)),$$

$$\sum_{n \leq x} h^2(n) = xE_h^2(x) + O(xD_h^2(x)).$$

Moreover, whenever h is strongly additive

$$\sum_{n \leq x} |h(n) - E_h(x)|^2 \ll xD_h^2(x).$$

The reason we dwell on the special characterizations of functions like our newly defined $\omega_f(n)$ through strong additivity is that the limiting probability distributions of the values of this function, and composite functions created by convolutions of $\omega_f(n)$ with itself, lend to constructions that are not only regular, but have a very central normal tendency towards the average order of the function. That is to say that, if we seek to form weighted sums of functions that exhibit strong additivity underpinnings (say as convolutions

of additive functions), we can almost always assume that the function described in this way is within a small bounded additive factor of the mean of the ω_f .

More to the point, the results we obtain in the next sections provide centrally normal tending distributions of unsigned summands whose signed weighted values comprise the summatory functions $F^{-1}(x)$ we are interested in bounding. In this way, we have generalized the model construction for the Mertens function summatory function corresponding to taking $f(n) \equiv 1$ outlined in the reference [?]. Then for every invertible positive integer-valued multiplicative function f , we have identified an associated strongly additive function ω_f upon which we can build up limiting probability distributions for auxiliary summatory functions that characterize the sums $F^{-1}(x)$ exactly.

2 Combinatorial properties of the component sequences

2.1 Several observations and lemmas on the generalized construction

Proposition 2.1 (The f -factor Möbius function is f^{-1} under \mathcal{A}_f -convolution). *For integers $n \geq 1$, let the f -Möbius function be defined such that $\mu_f^2(n)$ is the indicator function of the f -squarefree integers:*

$$\mu_f(n) := \begin{cases} 1, & n = 1; \\ (-1)^{\omega_f(n)}, & \omega_f(n) = \Omega_f(n) \wedge n \geq 2; \\ 0, & \text{otherwise.} \end{cases}$$

Recall that we define the \mathcal{A}_f -convolution operation as an analog to Dirichlet convolution generated by divisors of n as

$$(g *_{\mathcal{A}_f} h)(n) := \begin{cases} \sum_{\substack{d|n \\ d \in \mathcal{A}_f \cup \{1\}}} g(d)h(n/d), & n \in \mathcal{A}_f \cup \{1\}; \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, an arithmetic function h is invertible with respect to $*_{\mathcal{A}_f}$ if and only if $h(1) \neq 0$. We can identify a special inverse function with $\mu_f(n)$. Namely, the inverse of f , denoted by $f^{-1}|_{\mathcal{A}_f}$ to distinguish notation, is unique and equal to $\mu_f(n)$ for all $n \geq 1$.

Proof. To distinguish notation between the Dirichlet inverse, f^{-1} , of f and the inverse of f with respect to the restricted \mathcal{A}_f -convolutions, let $\hat{f}^{-1}(n) := f^{-1}|_{\mathcal{A}_f}(n)$. We use the identity that for $n \geq 2$ and $n \in \mathcal{A}_f$,

$$\chi_f(n) = \sum_{\substack{d|n \\ d \in \mathcal{A}_f \cup \{1\}}} \hat{f}^{-1}(d) \omega_f\left(\frac{n}{d}\right).$$

The characteristic function χ_f is f -multiplicative in so much as if $n = f(m_1)^{\alpha_1} \cdots f(m_r)^{\alpha_r}$ with $m_i \geq 2$ and $f(m_i) \neq f(m_j)$ for all $i \neq j$, then

$$\chi_f(n) = \prod_{i=1}^r \chi_f(f(m_i)^{\alpha_i}).$$

Then it suffices to evaluate this function at positive f -powers of the form $f(m)^\alpha$ for some $m \geq 2$ and $\alpha \geq 1$. We argue that $\text{sgn}(\hat{f}^{-1}(n)) = (-1)^{\omega_f(n)}$. It follows that since $\chi_f(n) = 1$, and this condition must hold for any positive integers $\alpha \geq 2$, the next two equations are satisfied:

$$\begin{aligned} \chi_f(f(m)^\alpha) &= \sum_{i=0}^{\alpha-1} (-1)^{\omega_f(f(m)^i)} |\hat{f}^{-1}(f(m)^i)| (\alpha + 1 - i) & \implies \\ \hat{f}^{-1}(1) - \sum_{i=1}^{\alpha-1} |\hat{f}^{-1}(f(m)^i)| &= 0 \wedge \sum_{i=1}^{\alpha-1} i \cdot |\hat{f}^{-1}(f(m)^i)| = 1. \end{aligned}$$

Since $|\hat{f}| \geq 0$, this shows that necessarily we must have $|\hat{f}^{-1}(f(m)^i)| = 0$ for all $2 \leq i \leq \alpha$ and that $\hat{f}^{-1}(1) = \hat{f}^{-1}(f(m)) = 1$. \square

Lemma 2.2. *Let*

$$D_f := \sum_{n \geq 1} \frac{\mu(n)}{f(n)^2} \in (0, +\infty).$$

Then we have that the number of f -free integers $n \leq x$ is given by

$$Q_f(x) := \sum_{n \leq x} \mu_f^2(n) = D_f x + O(\sqrt{x}).$$

Proof. This argument follows from a standard classical proof that provides asymptotics for the number of squarefree integers $n \leq x$ [?, §18.6]. Fix any $x \geq 1$. We can arrange all of the positive integers $n \leq x^2$ into disjoint sets $\{S_{f,d}\}_{d \geq 1}$ such that for $d \geq 1$, the set S_d contains just those positive integers n whose largest f -square factor is $f(d)^2$. The number of $n \leq x^2$ such that $n \in S_{f,d}$ is given by $Q_f\left(\frac{x^2}{f(d)^2}\right)$. Hence, we can sum over all such sets to see that

$$x^2 = \sum_{d \leq x^{1/\sigma_f}} Q_f\left(\frac{x^2}{f(d)^2}\right).$$

By routine inversion of summatory functions, we then obtain that

$$\begin{aligned} Q_f(x) &= \sum_{d \leq \sqrt{x}} \mu(d) \left\lfloor \frac{x}{f(d)^2} \right\rfloor = \sum_{d \leq \sqrt{x}} \mu(d) \left(\frac{x}{f(d)^2} + O(1) \right) \\ &= x \times \sum_{d \leq \sqrt{x}} \frac{\mu(d)}{f(d)^2} + O(\sqrt{x}) \\ &= D_f x + O\left(\frac{x}{x^{2/\sigma_f}}\right) + O(\sqrt{x}). \end{aligned} \quad \square$$

We will use Lemma 2.2 in the next sections to establish a key formula for the mean, or average order expected value of $g_f^{-1}(n)$ as $n \rightarrow \infty$.

Lemma 2.3 (DGFs of f -factor related arithmetic functions). *For $\operatorname{Re}(s) > \sigma_f$ and f a multiplicative arithmetic function such that $f(1) \neq 0$, we have that*

$$\operatorname{DGF}[\chi_f + \varepsilon](s) = \mathcal{F}_*(s) = \prod_{n \geq 2} (1 - f(n)^{-s})^{-1} = \prod_p \left(1 + \sum_{r \geq 1} \frac{1}{f(p^r)^s} \right)^{-1}, \quad (\text{A})$$

$$\operatorname{DGF}[\omega_f](s) = \mathcal{F}(s)(\mathcal{F}_*(s) - 1), \quad (\text{B})$$

$$\operatorname{DGF}[(\omega_f + f)^{-1}](s) = \frac{1}{\mathcal{F}(s)(1 + \mathcal{F}_*(s))}, \quad (\text{C})$$

$$\operatorname{DGF}[2^{\omega_f} \chi_f](s) = \frac{\mathcal{F}_*(s)^2}{\mathcal{F}_*(2s)}, \quad (\text{D})$$

$$\operatorname{DGF}[\lambda_f \chi_f](s) = \frac{\mathcal{F}_*(2s)}{\mathcal{F}_*(s)}. \quad (\text{E})$$

Proof. The proof of these results follows immediately from their definitions. The only subtle property is in formulating the Dirichlet generating function of $\lambda_f(n)$ restricted to the set $\mathcal{A}_f \cup \{1\}$. We notice by a counting argument with products that

$$\begin{aligned} 1 + \sum_{n \in \mathcal{A}_f} \frac{2^{\omega_f(n)}}{f(n)^s} &= \prod_{n \geq 1} \left(1 + \frac{zf(n)^{-s}}{1 - f(n)^{-s}} \right)^{-1} \Big|_{z=2} \\ &= \prod_{n \geq 2} \left(\frac{f(n)^s + 1}{f(n)^s - 1} \right) = \frac{\mathcal{F}_*(s)^2}{\mathcal{F}_*(2s)}. \end{aligned}$$

We notice that the f -product analogs to the typical Euler product constructions over the primes leads to very similar factorizations in terms of the \mathcal{A}_f -set generator DGF, $\mathcal{F}_*(s)$. We also get an analog to the Dirichlet divisor sum identities where $2^\omega = 1 * \mu^2$. Here, we obtain in analogous form that

$$2^{\omega_f} = 1 *_{\mathcal{A}_f} \mu_f^2,$$

where $\lambda_f = |\mu_f|^{-1}$ with respect to \mathcal{A}_f -convolution. Following through the DGF arithmetic restricted to the f -factorized set $\mathcal{A}_f \cup \{1\}$ implies the last DGF identity. \square

We briefly consider the asymptotics of the summatory function $L_f(x) := \sum_{n \leq x} \lambda_f(n)$ in the last section of the article. In general, bounding these sums for large x in the ordinary zeta function DGF flavors is an extremely challenging problem. We pose the topic of finding new asymptotic bounds for $L_f(x)$ based on these new DGF identities and Mellin inversion when we cover applications below in Section 5.

Remark 2.4 (Elementary consequences of the DGF expansions). We will be brief and not dwell too much on the elementary combinatorics that motivate the analytic flavored results we prove in the next section. A couple of properties that are justified by the structure of the DGF formulas in the last lemma are apparent, useful, and will be required for reference in the next sections.

The same construction behind the proof we gave that $\text{sgn}((\omega + 1)^{-1}(n)) = \lambda(n)$ in the reference [?] shows easily that $\text{sgn}((\omega_f + f)^{-1}(n)) = \lambda_f(n)$. This argument essentially boils down to an expansion by geometric series of $(\mathcal{F}_*(s) + 1)^{-1}$ in which the coefficient of the convolutions for the resulting DGF powers are non-zero precisely when the number of f -factors of n is the power in the expansion. We also can easily see as in the references [?, ?] that

$$\frac{1}{\mathcal{F}_*(s) + 1} = \sum_{n \geq 1} C_{\Omega_f(n)}(n) n^{-s} = 1 + \sum_{n \geq 2} \left(\prod_{f(n)^\beta \parallel_{\mathcal{A}_f} n} (\beta!)^{-1} \right) \times (\Omega_f(n))! n^{-s}.$$

2.2 Exact formulas for $F^{-1}(x)$

Proposition 2.5. *We have that for all $n \geq 1$*

$$g_f^{-1}(n) = \begin{cases} 1, & \text{if } n = 1; \\ \lambda_f(n) \times \sum_{\substack{d|n \\ d \in \mathcal{A}_f \cup \{1\}}} \mu_f^2\left(\frac{n}{d}\right) C_{\Omega_f(n)}(n), & \text{if } n \geq 2 \wedge n \in \mathcal{A}_f; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Since we compute the inverses $(\omega_f + f)^{-1}(n)$ with respect to f -convolution, or with n and the divisors $d|n$ restricted to the set $\mathcal{A}_f \cup \{1\}$, we see that $g_f^{-1}(1) = 1$, and whenever $n \geq 2$ and $n \notin \mathcal{A}_f$, $g_f^{-1}(n) = 0$. Suppose that $n \geq 2$ and that $n \in \mathcal{A}_f$. Then by the standard recursive inversion formula for Dirichlet convolution, we see that

$$(\omega_f + f)^{-1}(n) = - \sum_{\substack{d|n \\ d \in \mathcal{A}_f}} g_f^{-1}\left(\frac{n}{d}\right) (\omega_f(d) + f(d)).$$

It follows that

$$(g_f^{-1} *_{\mathcal{A}_f} f)(n) = - \sum_{\substack{d|n \\ d \in \mathcal{A}_f}} g_f^{-1}\left(\frac{n}{d}\right) \omega_f(d) = \lambda_f(n) C_{\Omega_f(n)}(n),$$

where the second equation above follows by recursively expanding the inverse formula to a maximum depth of $\Omega_f(n)$ factors.

By the inversion result we proved in Proposition 2.1, we have that for $n \geq 2$ such that $n \in \mathcal{A}_f$,

$$g_f^{-1}(n) = \sum_{\substack{d|n \\ d \in \mathcal{A}_f \cup \{1\}}} \mu_f\left(\frac{n}{d}\right) \lambda_f(d) C_{\Omega_f(d)}(d) = \lambda_f(n) \times \sum_{\substack{d|n \\ d \in \mathcal{A}_f \cup \{1\}}} \mu_f^2\left(\frac{n}{d}\right) C_{\Omega_f(d)}(d).$$

The last equation is justified by noting that since $\lambda_f(n)$ is completely multiplicative, for any f -divisors d of n , we have that $\lambda_f(n/d)\lambda_f(d) = \lambda_f(n)$. The f -Möbius function variant, $\mu_f(n)$ is non-zero only for n which are f -squarefree, so that whenever $\mu_f(n) \neq 0$, we have $\mu_f(n) = (-1)^{\omega_f(n)} = (-1)^{\Omega_f(n)} = \lambda_f(n)$. \square

Theorem 2.6. *We have that for all $x \geq 1$*

$$F^{-1}(x) = G_f^{-1}(x) + \sum_{k=1}^x g_f^{-1}(k) \pi_f\left(\left\lfloor \frac{x}{k} \right\rfloor\right), \quad (\text{A})$$

$$F^{-1}(x) = G_f^{-1}(x) + \sum_{\substack{a \leq x \\ a \in \mathcal{A}_f}} G_f^{-1}\left(\left\lfloor \frac{x}{a} \right\rfloor\right). \quad (\text{B})$$

Proof. To prove the first result in (A), observe that by our construction of ω_f , we have

$$\pi_f(x) + 1 = \sum_{n \leq x} (\omega_f + f) *_{\mathcal{A}_f} f^{-1}(n) = \sum_{\substack{n \leq x \\ n \in \mathcal{A}_f \cup \{1\}}} (\omega_f + f)(n) F^{-1}\left(\left\lfloor \frac{x}{n} \right\rfloor\right).$$

Then by ordinary inversion rules for summatory functions weighted by an invertible arithmetic function [?, cf. §TODO], we get that

$$F^{-1}(x) = \sum_{n \leq x} g_f^{-1}(n) \left[\pi_f\left(\left\lfloor \frac{x}{n} \right\rfloor\right) + 1 \right].$$

To prove (B), we see that taking the backward difference of the last sum yields that

$$f^{-1}(n) - g_f^{-1}(n) = \sum_{\substack{d|n \\ d \in \mathcal{A}_f}} g_f^{-1}(d).$$

Then summing each side of the previous equation over $n \leq x$ shows that

$$F^{-1}(x) = G_f^{-1}(x) + \sum_{\substack{n \leq x \\ n \in \mathcal{A}_f}} G_f^{-1}\left(\left\lfloor \frac{x}{n} \right\rfloor\right). \quad \square$$

2.3 Combinatorial identities and relations to the distribution of f -powers over the integers

We have the following properties of $g_f^{-1}(n)$ over the integers $n \geq 2$ such that $n \in \mathcal{A}_f$ [?, cf. §TODO]:

- (1) If n is f -squarefree, e.g., if $\mu_f^2(n) = 1$, then

$$|g_f^{-1}(n)| = \sum_{m=0}^{\omega_f(n)} \binom{\omega_f(n)}{m} m!.$$

(2) If $\Omega_f(n) = k$, then

$$2 \leq |g_f^{-1}(n)| \leq \sum_{j=0}^k \binom{k}{j} j!.$$

(3) We have symmetry in the action of the *distinct exponents* of the irreducible f -factor powers of any $n \in \mathcal{A}_f$. That is, suppose that $n_1 = f(m_{1,1})^{\alpha_1} \cdots f(m_{1,r})^{\alpha_r}$ and $n_2 = f(m_{2,1})^{\beta_1} \cdots f(m_{2,r})^{\beta_r}$ where $f(m_{i,j}) \neq f(m_{i,k})$ for $i = 1, 2$ and all $j \neq k$, and where $\alpha_i, \beta_i \geq 1$ for all $1 \leq i \leq r$. If $\{\alpha_1, \dots, \alpha_r\} \equiv \{\beta_1, \dots, \beta_r\}$ are equal as multisets of the f -factorization exponents, then $g_f^{-1}(n_1) = g_f^{-1}(n_2)$.

3 Analytic constructions of preliminary bounds and asymptotic formulas

We now restrict ourselves to the cases where $\mathcal{F}(s)$ has a simple pole with residue A_f at $s := \sigma_f$ so that for some absolute constant B_f we have

$$\mathcal{F}(s) = \frac{A_f s}{(s - \sigma_f)} + B_f + O(|s - \sigma_f|).$$

Theorem 3.1. *Suppose that $\mathcal{F}(s)$ is the DGF of f which converges for all $\operatorname{Re}(s) > \sigma_f$ where $2 > \sigma_f > 1$ is the abscissa of convergence for the DGF. Suppose that for any non-zero $z \in \mathbb{C}$, we have that*

$$\mathcal{F}(s)^z = \sum_{n \geq 1} \frac{\widehat{f}_z(n)}{n^s}, \operatorname{Re}(s) > \sigma_f.$$

Let the summatory function $\widehat{F}_z(x) := \sum_{n \leq x} \widehat{f}_z(n)$, and let $R > 0$ be any positive real number. If $x \geq 2$, then

$$\widehat{F}_z(x) = \frac{x^{\sigma_f} A_f^z (\log x)^{z-1}}{\Gamma(z)} + O_R \left(x^{\sigma_f} (\log x)^{\operatorname{Re}(z)-2} \right),$$

uniformly for $|z| \leq R$.

Proof. Let $a := \sigma_f + \frac{1}{\log x}$. Then we have that

$$\widehat{F}_z(x) - \frac{1}{2\pi i} \int_{a-iT}^{a+iT} \mathcal{F}(s)^z \frac{x^s}{s} ds \ll \sum_{\frac{x}{2} < n < 2x} |\widehat{f}_z(n)| \min \left(1, \frac{x}{T|x-n|} \right) + \frac{x^a}{T} \times \sum_{n \geq 1} |\widehat{f}_z(n)| n^{-a}. \quad (1)$$

Without loss of generality, we assume that R is integer-valued. We notice that $|\widehat{f}_z(n)| \leq \widehat{f}_{|z|}(n) \leq \widehat{f}_R(n)$ for all $n \geq 1$. Suppose that $\mathbb{E}[f(n)] \sim e_f(n)$. Then by the hyperbola method for bounding sums of convolutions of functions and induction, we can see that

$$\widehat{F}_R(x) = x \cdot P_R(e_f(x)) + O_R(x^{1-1/R}),$$

where P_R is a polynomial of degree $R - 1$.

Let $\mathcal{B}_{R,x} := \{n \geq 1 : |n - x| \leq x/(\log x)^{2R+1}\}$. The first error term in (1) is then given by

$$\begin{aligned} E_1(x) &= \sum_{\frac{x}{2} < n < 2x} |\widehat{f}_z(n)| \min \left(1, \frac{x}{T|x-n|} \right) \\ &\ll \sum_{n \in \mathcal{B}_{R,x}} |\widehat{f}_z(n)| + \sum_{n \notin \mathcal{B}_{R,x}} |\widehat{f}_z(n)| \min \left(1, \frac{x}{T|x-n|} \right) \\ &\ll x \cdot e_f(x)^{-(R+2)} + \frac{1}{T} \cdot e_f(x)^{2R+1} \cdot x \cdot e_f(x)^{R-1}. \end{aligned}$$

Taking $T \gg \exp(\sqrt{e_f(x)})$ ensures that $E_1(x) \ll x \cdot e_f(x)^{-(R+2)}$.

If it happens that z is a positive integer, then we can use residue calculus to extract the value of the main term for the above approximation. If on the other hand (in the general case), z is not integer valued, then $\mathcal{F}(s)^z$ has a branch point at $s := \sigma_f$. Thus we must construct a more intricate path around the contour to justify the main term. Suppose that $z \notin \mathbb{Z}^+$. Set $b := 1 - c/\log T$ where $c > 0$ is small. We then replace the initial contour from $a - iT$ to $a + iT$ with a path consisting of the following subpaths, $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$:

- (1) The path \mathcal{C}_1 is polygonal with vertices $a - iT$, $b - iT$ and $b - \iota/\log x$;
- (2) The path \mathcal{C}_2 begins with a line segment from $b - \iota/\log x$ to $\sigma_f - \iota/\log x$, then continues with the semicircle $\{1 + e^{i\theta}/\log x : -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\}$, and then finally concludes with the line segment from $\sigma_f + \iota/\log x$ to $b + \iota/\log x$;
- (3) The path \mathcal{C}_3 is polygonal with vertices $b + \iota/\log x$, $b + iT$ and $a + iT$.

For $m = 1, 2, 3$, define

$$I_m(x, T) := \frac{1}{2\pi\iota} \times \int_{\mathcal{C}_m} \mathcal{F}(s)^z \cdot \frac{x^s}{s} ds.$$

We claim that $I_1(x, T), I_2(x, T) \ll x^{\sigma_f} \cdot e_f(x)^{-(R+2)}$ as $T \rightarrow \infty$. To see this, note that $\mathcal{F}(s)^z \ll A_f^{-R} \times (\log x)^{-R}$ on this part of the new contour. Then we compute that

$$\frac{1}{2\pi\iota} \times \int_{\mathcal{C}_1 \cup \mathcal{C}_3} \mathcal{F}(s)^z \frac{x^s}{s} ds \ll x^{\sigma_f} \cdot (\log x)^{-(R+2)}.$$

It remains to bound $I_2(x, T)$ as $T \rightarrow \infty$. On \mathcal{C}_2 , we have that

$$\frac{\mathcal{F}(s)^z}{s} = A_f^z (s - \sigma_f)^{-z} (1 + O_z(|s - \sigma_f|)).$$

Hence, we see that

$$I_2(x, T) = \frac{1}{2\pi\iota} \times \int_{\mathcal{C}_2} A_f^z \cdot (s - \sigma_f)^{-z} x^s ds + O\left(\int_{\mathcal{C}_2} |s - \sigma_f|^{1-\operatorname{Re}(z)} x^\sigma |ds|\right).$$

Define the next three Hankel-type contours for $\beta := c(\log x)/\log T$:

- (1) Let $\mathcal{H}_1 := \{w = u - \iota : -\infty < u \leq -\beta\}$;
- (2) The contour \mathcal{H}_2 starts at $-\beta - \iota$, then loops around zero, and then finally ends at the point $-\beta + \iota$;
- (3) Let $\mathcal{H}_3 := \{w = u + \iota : -\infty < u \leq -\beta\}$.

If we let $\mathcal{H} := \mathcal{H}_1 \cup \mathcal{H}_2 \cup \mathcal{H}_3$, then \mathcal{H} corresponds to *Hankel's contour* for the reciprocal gamma function given by

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi\iota} \times \int_{\mathcal{H}} e^v v^{-z} dv.$$

By performing a change of variables in the form of $s = \sigma_f + w/\log x$, the main term for $I_2(x, T)$ becomes

$$\frac{x^{\sigma_f} A_f^z (\log x)^{z-1}}{2\pi\iota} \times \int_{\mathcal{H}_2} w^{-z} e^w dw = \frac{x^{\sigma_f} A_f^z (\log x)^{z-1}}{\Gamma(z)} + O_R\left(x^{\sigma_f} \cdot \exp\left(-C\sqrt{\log x}\right)\right),$$

for some absolute constant $C > 0$. The main term integral for $I_2(x, T)$ over $\mathcal{H}_1, \mathcal{H}_3$ are each $\ll_R \int_{\beta}^{\infty} e^{-u/2} du \ll_R e^{-\beta/2}$, which are each small given our choice of the limiting $T = \exp(\sqrt{x})$. On the semicircular path in \mathcal{C}_2 , we get that the contribution of I_2 is bounded above by $O(x(\log x)^{\operatorname{Re}(z)-2})$. Similarly, taking the same change of variables $s = \sigma_f + w/\log x$, we see that the linear segments in \mathcal{C}_2 contribute a bounded amount of

$$E_{L, \mathcal{C}_2}(x) \ll x^{\sigma_f} A_f^z (\log x)^{\operatorname{Re}(z)-2} \times \int_0^{\infty} (u^2 + 1)^{(R-1)/2} e^{-u} du \ll_R x^{\sigma_f} (\log x)^{\operatorname{Re}(z)-2}.$$

Thus the combined estimates provide that

$$\widehat{F}_z(x) = \frac{x^{\sigma_f} A_f^z (\log x)^{z-1}}{\Gamma(z)} + O_R\left(x^{\sigma_f} \cdot e_f(x)^{-(R+2)} + x^{\sigma_f} (\log x)^{\operatorname{Re}(z)-2}\right). \quad \square$$

Proposition 3.2. Suppose that $\sum_{m \geq 1} |b_z(m)| (\log m)^{2R+1} \cdot m^{-(\sigma_f+1)}$ is uniformly bounded for $|z| \leq R$. For $\operatorname{Re}(s) \geq \sigma_f + 1$, let

$$B(s, z) := \sum_{m \geq 1} \frac{b_z(m)}{m^s},$$

and let the coefficients $a_z(n)$ be defined by

$$\mathcal{F}(s)^z \cdot B(s, z) := \sum_{n \geq 1} \frac{a_z(n)}{n^s}.$$

Let the summatory function $A_z(x) := \sum_{n \leq x} a_z(n)$. Then for any $x \geq 2$ we have that

$$A_z(x) = \frac{B(\sigma_f + 1, z)}{\Gamma(z)} \cdot x^{\sigma_f} A_f^z(\log x)^{z-1} + O\left(x^{\sigma_f} (\log x)^{\operatorname{Re}(z)-2}\right).$$

Proof. □

Theorem 3.3. Suppose that $R < 1 + \sigma_f$, that

$$H_f(s, z) := \mathcal{F}(s)^{-z} \times \prod_{a \in \mathcal{A}_f} \left(1 - \frac{z}{a^s}\right)^{-1},$$

and that $G_f(z) := H_f(2, z)/\Gamma(z+1)$. Then for all $x \geq 2$

$$\pi_{f,k}(x) = G_f \left(\frac{k-1}{\log \log x} \right) \frac{x}{\sigma_f \cdot \log x} \frac{(\log \log x)^{k-1}}{(k-1)!} \left[1 + O_R \left(\frac{k}{(\log \log x)^2} \right) \right],$$

uniformly for $1 \leq k \leq R \log \log x$.

Proof. We will sketch the details to this proof. A very similar proof based on a bivariate DGF in s indexing a formal power series in z is given to prove Theorem 4.2 in the next section. The general proof method is outlined in complete detail in the reference [?, §7.4]. The result follows from the method in the reference (and below) by applying Proposition 3.2, and then separating the main and error term formulas resulting from an application of the Cauchy integral formula to invert $A_z(x)$ for its coefficients of z^k . □

Theorem 3.4 (The distribution of the exceptional values of $\Omega_f(n)$). For large $x > e$, let

$$B_f(x, r) := \#\{n \leq x : \Omega_f(n) \geq r \log \log x\}.$$

If $1 \leq r \leq R < 2$ and $x > e$, then

$$B_f(x, r) \ll_R x \cdot (\log x)^{r-1-r \log r}.$$

Remark 3.5 (Bounding signed sums over $\pi_{f,k}(x)$). It is not difficult to show using an asymptotic formula for the incomplete gamma function that [?, cf. §TODO]

$$L_{f,*}(x) := \left| \sum_{k \geq 1} (-1)^k \pi_{f,k}(x) \right| \sim \left| \sum_{1 \leq k \leq \log \log x} (-1)^k \pi_{f,k}(x) \right| \asymp \frac{x}{\sqrt{2\pi} \sigma_f} \sqrt{\log \log x}. \quad (2a)$$

Then it follows that

$$\frac{1}{L'_{f,*}(x)} \sim \sqrt{2\pi} \cdot \sqrt{\log \log x}, \text{ as } x \rightarrow \infty. \quad (2b)$$

A formal argument making this fact precise is concretely identified in the reference [?]. This bound is useful in working with the summatory function $\sum_{n \leq x} (-1)^{\omega_f(n)}$ because we have uniform bounds on $\pi_{f,k}(x)$ for $1 \leq k \leq \log \log x$ and all sufficiently large x . It happens that since the strongly additive function $\omega_f(n)$ is so near its average order asymptotic for almost every n , the contributions of this function to the signed summatory function is negligible for $\omega_f(n) \gg \log \log x$. We use the result in (2) in the proof of Corollary 4.3 in the next section.

4 The limiting distributions for the unsigned sequences

Proposition 4.1. *We have that for $\operatorname{Re}(s) > \sigma_f$*

$$\sum_{n \geq 1} \frac{(-1)^{\omega_f(n)} C_{\Omega_f(n)}(n) z^{\Omega_f(n)}}{n^s} = \frac{1}{1 + \mathcal{F}_*(s)z}.$$

Proof. By a prduct based expansion and combinatorial argument on its coefficients, we have that

$$\sum_{n \geq 1} \frac{(-1)^{\omega_f(n)} C_{\Omega_f(n)}(n) z^{\Omega_f(n)}}{(\Omega_f(n)!) \cdot n^s} = \prod_{n \geq 2} \left(1 - \sum_{r \geq 1} \frac{z^{\Omega_f(f(n)^r)}}{f(n)^{rs}} \right)^{-1} = \prod_{n \geq 2} \exp \left(-\frac{z}{f(n)^s} \right) = \exp(-z \mathcal{F}_*(s)).$$

Thus to obtain the result, we perform a Laplace transform with respect to z to obtain that

$$\int_0^\infty e^{-t} \exp(-tz \mathcal{F}_*(s)) dt = \frac{1}{1 + z \mathcal{F}_*(s)}, \operatorname{Re}(\mathcal{F}_*(s)z) > -1. \quad \square$$

Theorem 4.2. *We have for large $x > e$ uniformly for $1 \leq k \leq \log \log x$ that*

$$\widehat{C}_{f,k,*}(x) := \sum_{\substack{n \leq x \\ \Omega_f(n)=k}} (-1)^{\omega_f(n)} C_{\Omega_f(n)}(n) \asymp \frac{x}{\log x} \cdot \frac{(\log \log x)^{k-1}}{(k-1)!} \left[1 + O \left(\frac{1}{\log \log x} \right) \right].$$

Proof. Let the DGF $F(s, z)$ be defined for $\operatorname{Re}(s) \geq \sigma_f + 1$ and $|z| \leq 2$ by

$$K_f(s, z) := \frac{\mathcal{F}(s)^{-z}}{1 + z \mathcal{F}_*(s)}.$$

Define the function $\widetilde{G}_f(z) := K_f(\sigma_f + 1, z)/\Gamma(z + 1)$. We claim that uniformly for $1 \leq k \leq \log \log x$

$$\widehat{C}_{f,k,*}(x) = \widetilde{G} \left(\frac{k-1}{\log \log x} \right) \frac{x}{\sigma_f \cdot (\log x)} \cdot \frac{(\log \log x)^{k-1}}{(k-1)!} \left[1 + O \left(\frac{k}{(\log \log x)^2} \right) \right].$$

By Proposition 4.1 and Proposition 3.2, we have that for $r < 2$

$$\begin{aligned} \widehat{C}_{f,k,*}(x) &= \frac{1}{2\pi i} \int_{|z|=r} \frac{A_z(z^{1/\sigma_f})}{z^{k+1}} dz \\ &= \frac{\sigma_f x}{2\pi i \cdot (\log x)} \int_{|z|=r} \frac{\mathcal{F}(\sigma_f + 1)^{-z}}{(1 + \mathcal{F}_* z) \Gamma(z + 1)} \cdot \left(\frac{A_f(\log x)}{\sigma_f} \right)^z \cdot \frac{dz}{z^k} + O \left(\frac{1}{2\pi i} \int_{|z|=r} \frac{x(\log x)^{\operatorname{Re}(z)-2}}{z^{k+1}} dz \right). \end{aligned} \quad (3)$$

That is, we write $\widehat{C}_{f,k,*}(x) := M_{f,k}(x) + O(E_{f,k}(x))$ where the two functions correspond, respectively, to the main term and error term in our asymptotic formula.

Let $B_f(x) := \log \log x + A_f - \sigma_f - \mathcal{F}(\sigma) f + 1$. Then we have that

$$\left(\frac{\log x}{\sigma_f x} \right) M_{f,k}(x) = \frac{1}{2\pi i} \int_{|z|=r} \widetilde{G}_f(z) e^{B_f(x)z} z^{-k} dz =: I_{1,k}(x) + I_{2,k}(x),$$

where we define

$$I_{1,k}(x) := \frac{G(r)}{2\pi i} \int_{|z|=r} e^{B_f(x)z} z^{-k} dz,$$

$$I_{2,k}(x) := \frac{1}{2\pi i} \int_{|z|=r} (G(z) - G(r)) e^{B_f(x)z} z^{-k} dz.$$

The main term contribution is given by the first integral, which evaluates exactly to the following form for $r := \frac{k-1}{\log \log x} < 2$:

$$I_{1,k}(x) = \frac{B_f(x)^{k-1}}{(k-1)!} \sim \frac{(\log \log x)^{k-1}}{(k-1)!}, \text{ as } x \rightarrow \infty.$$

Integrating by parts leads us to bound the second integral as

$$I_{2,k}(x) = \frac{1}{2\pi i} \int_{|z|=r} (G(z) - G(r) - G'(r)(z-r)) e^{B_f(x)z} z^{-k} dz.$$

We also have that the integrand factors in the previous equation satisfy

$$G(z) - G(r) - G'(r)(z-r) = \int_r^z (z-w) G''(w) dw \ll |z-r|^2.$$

Hence, we can parameterize the contour for $I_{2,k}(x)$ by writing $z = r e^{2\pi i \theta}$ for $\theta \in [-1/2, 1/2]$, and so

$$I_{2,k}(x) \ll r^{3-k} \times \int_{-1/2}^{1/2} (\sin \pi \theta)^2 e^{(k-1) \cos(2\pi \theta)} d\theta.$$

We observe by elementary inequalities that $|\sin x| \leq |x|$ and that $\cos(2\pi \theta) \leq 1 - 8\theta^2$ whenever $-1/2 \leq \theta \leq 1/2$. So we continue bounding the second integral as

$$\begin{aligned} I_{2,k}(x) &\ll r^{3-k} e^{k-1} \int_0^\infty \theta^2 e^{-8(k-1)\theta^2} d\theta \\ &\ll r^{3-k} e^{k-1} (k-1)^{-3/2} \ll \frac{k(\log \log x)^{k-3}}{(k-1)!}. \end{aligned}$$

We now need to evaluate the error terms $E_{f,k}(x)$. For $k \geq 2$, we have that when $r := \frac{k-1}{\log \log x}$

$$|E_{f,k}(x)| \ll x(\log x)^{r-2} r^{-k} \ll \frac{x}{(\log x)^2} \cdot \frac{e^{k-1}(\log \log x)^k}{(k-1)^k} \ll \frac{x(\log \log x)^{k-3}}{(\log x)(k-1)!}.$$

For $k := 1$, we also have that

$$\left| \frac{1}{2\pi i} \int_{|z|=r} \frac{x(\log x)^{\operatorname{Re}(z)-2}}{z^{k+1}} dz \right| \ll \frac{x}{(\log x)^2} \times \frac{d}{dm} \left[e^{m(\log \log x)} \right] \Big|_{m=0} \ll \frac{x}{(\log x)(\log \log x)^2}.$$

All combined, the estimates for $I_{1,k}(x)$, $I_{2,k}(x)$ and the last error estimates prove that (3) holds.

Finally, we see that for $1 \leq k \leq \log \log x$

$$\tilde{G}_f \left(\frac{k-1}{\log \log x} \right) \asymp \frac{1}{\Gamma(1+o(1))} \cdot \frac{\mathcal{F}(1+\sigma_f)^{\frac{1-k}{\log \log x}}}{1 + \frac{k-1}{\log \log x} \mathcal{F}_*(1+\sigma_f)} \asymp 1, \text{ as } x \rightarrow \infty.$$

This completes the proof of the result. □

Corollary 4.3. *We have that for large x that uniformly for $1 \leq k \leq \log \log x$*

$$\widehat{C}_{f,k}(x) := \sum_{\substack{n \leq x \\ \Omega_f(n)=k}} C_{\Omega_f(n)}(n) \asymp \frac{x \cdot (\log \log x)^{k-1/2}}{(2k-1)(k-1)!}.$$

Proof. For the summatory function $L_{f,*}(x)$ defined in (2), we see integrating the Abel summation formula by parts that

$$\widehat{C}_{f,k,*}(x) \asymp \int_e^x L'_{f,*}(t) C_{\Omega_f(t)}(t) [\Omega_f(t) = k]_\delta dt.$$

Now using Theorem 4.2, we have

$$\begin{aligned} C_{\Omega_f(x)}(x) [\Omega_f(x) = k]_\delta &\asymp \frac{\widehat{C}'_{f,k,*}(x)}{L'_{f,*}(x)} \\ &\asymp \frac{(\log \log x)^{k-3/2}}{(\log x)(k-1)!} + \frac{(\log \log x)^{k-5/2}}{(\log x)^2(k-2)!} - \frac{(\log \log x)^{k-3/2}}{(\log x)^2(k-1)!} =: \widehat{C}_{f,k,**}(x). \end{aligned}$$

We finally compute using integration by parts that

$$\widehat{C}_{f,k}(x) \asymp \int \widehat{C}_{f,k,**}(x) \asymp \frac{x \cdot (\log \log x)^{k-1/2}}{(2k-1)(k-1)!}. \quad \square$$

Lemma 4.4. *We have that as $n \rightarrow \infty$, the average order of $C_{\Omega_f(n)}(n)$ satisfies*

$$\mathbb{E}[C_{\Omega_f(n)}(n)] \asymp \text{TODO}.$$

Proof. □

Corollary 4.5. *We have that as $n \rightarrow \infty$, the average order of the unsigned inverse sequence $|g_f^{-1}(n)|$ satisfies*

$$\mathbb{E}|g_f^{-1}(n)| \asymp \text{TODO}.$$

Proof. □

Theorem 4.6 (Erdős-Kac theorem for $C_{\Omega_f(n)}(n)$). *Let the absolute constant*

$$C_*(f) := \lim_{n \rightarrow \infty} \frac{\mathbb{E}[C_{\Omega_f(n)}(n)]}{(\log n) \sqrt{\log \log n}}.$$

For sufficiently large $x > e$, we define

$$\mu_x(f) := \log \log x - \log \left(\frac{C_*(f)}{2} \right), \quad \sigma_x(f) := \sqrt{\log \log x}.$$

Let $Y > 0$ be fixed. Then for all large $x > e$, we have uniformly for $-Y \leq z \leq Y$ that

$$\frac{1}{x} \cdot \# \left\{ 2 \leq n \leq x : \frac{C_{\Omega_f(n)}(n) - \mu_x(f)}{\sigma_x(f)} \leq z \right\} = \Phi(z) + O\left(\frac{1}{\sqrt{\log \log x}}\right).$$

Proof. TODO ... □

Corollary 4.7 (Erdős-Kac theorem for $|g_f^{-1}(n)|$). *Suppose that $\mu_x(f)$ and $\sigma_x(f)$ are defined for large $x > e$ as in Theorem 4.6. Let $Y > 0$ be fixed. Then uniformly for any $-Y \leq z \leq Y$, we have that*

$$\frac{1}{x} \cdot \# \left\{ 2 \leq n \leq x : |g_f^{-1}(n)| - D_f \mathbb{E}|g_f^{-1}(n)| \leq z \right\} = \Phi(D_f \cdot (z + \mu_x(f)) \sigma_x(f)) + o(1), \text{ as } x \rightarrow \infty.$$

Proof. We claim that

$$|g_f^{-1}(n)| - D_f \cdot \mathbb{E}|g_f^{-1}(n)| \sim \frac{6}{\pi^2} C_{\Omega_f(n)}(n), \text{ as } n \rightarrow \infty.$$

As in the proof of Corollary 4.5, we obtain that

$$\frac{1}{x} \times \sum_{n \leq x} |g_f^{-1}(n)| = D_f \cdot \left[\mathbb{E}[C_{\Omega_f(x)}(x)] + \sum_{d < x} \frac{\mathbb{E}[C_{\Omega_f(d)}(d)]}{d} \right] + O(1).$$

Let the *backwards difference operator* with respect to x be defined for $x \geq 2$ and any arithmetic function f as $\Delta_x(f(x)) := f(x) - f(x-1)$. We see that for large n

$$\begin{aligned} |g_f^{-1}(n)| &= \Delta_n(n \cdot \mathbb{E}|g_f^{-1}(n)|) \sim \Delta_n \left(\sum_{d \leq n} D_f \cdot C_{\Omega_f(d)}(d) \cdot \frac{n}{d} \right) \\ &= D_f \cdot \left[C_{\Omega_f(n)}(n) + \sum_{d < n} C_{\Omega_f(d)}(d) \frac{n}{d} - \sum_{d < n} C_{\Omega_f(d)}(d) \frac{(n-1)}{d} \right] \\ &\sim D_f \cdot C_{\Omega_f(n)}(n) + D_f \cdot \mathbb{E}|g_f^{-1}(n-1)|, \text{ as } n \rightarrow \infty. \end{aligned}$$

Since $\mathbb{E}|g_f^{-1}(n-1)| \sim \mathbb{E}|g_f^{-1}(n)|$ for all sufficiently large n , the result finally follows by a normalization of Theorem 4.6. \square

Remark 4.8. TODO: Remarks about the probabilities ...

5 Applications

Proposition 5.1. *We have the following variants of the DGFs $\mathcal{F}_*(s)$ for the respective functions $f(n) \in \{\phi(n), J_k(n), \phi_m(n), \sigma_\alpha(n)\}$:*

$$\begin{aligned} \sum_{n \geq 1} \phi(n)^{-s} &= \zeta(s) \times \prod_p (1 - p^{-s} + (p-1)^{-s}), \operatorname{Re}(s) > 1, \\ \sum_{n \geq 1} J_k(n)^{-s} &= \zeta(ks) \times \prod_p (1 - p^{-ks} + (p^{ks} - 1)^{-s}), \operatorname{Re}(ks) > 1, \\ \sum_{n \geq 1} \phi_m(n)^{-s} &= \left(\frac{\zeta((m+1)s)}{m+1} + \frac{\zeta(ms)}{2} + \sum_{2 \leq j \leq m} \frac{B_j m^{j-1}}{j!} \zeta((m+1-j)s) \right) \times \prod_p (2 - p^k)^s, \operatorname{Re}(s) > 1, \\ \sum_{n \geq 1} \sigma_\alpha(n)^{-s} &= \prod_p \left(1 + (p^\alpha - 1)^s \times \sum_{r \geq 1} (p^{\alpha(r+1)} - 1)^{-s} \right), \operatorname{Re}(\alpha s) > 1. \end{aligned}$$

Proof. □

5.1 Problem type setup for Ernie to look at:

We are given an integer-valued multiplicative function $f \geq 1$ with $f(1) = 1$ and where we assume $f(n) \geq 2$ for all $n \geq 2$. We form the modified zeta functions over the sets of $f(n)$ as follows:

$$\mathcal{F}_{1,*}(s) := 1 + \sum_{n \geq 2} f(n)^{-s}, \operatorname{Re}(s) > \sigma_f^*.$$

Want to look at the coefficients of the following DGFs and then compute solid asymptotics for their summatory functions:

$$\mathcal{F}_*(s, z) := \left[\frac{1}{2 - \mathcal{F}_{1,*}(s)} \right]^z := \sum_{n \geq 1} \frac{\hat{f}_z(n)}{n^s}.$$

Note that

$$\mathcal{F}_*(s, z) = \prod_{n \geq 2} (1 - f(n)^{-s})^{-z}.$$

The big question I have is how to use the multiplicative structure, and say a "nice enough" (Euler product definitely) DGF for $\mathcal{F}(s)$, then use bounds on this DGF to derive a sane main and error term bound for the sums

$$\hat{F}_z(x) := \sum_{n \leq x} \hat{f}_z(n).$$

Note that the analogous summatory functions of the coefficients of the DGF $\zeta(s)^z$ are known, and derived using a somewhat intricate contour argument in Montgomery and Vaughan. The basic bound relies on an inductive hyperbola method, and then more importantly on the known special property of the zeta function that $\zeta(s) = \frac{s}{s-1} (1 + O(|s-1|))$.

N.b.: This construction is not a trinket problem. It effectively provides a generalization of the method I used to characterize the Mertens function with an underlying probability distribution (weighted by summands of $\lambda(n)$, which makes it properly still a hard problem) derived from strongly additive functions. I can do this to sum $F^{-1}(x) := \sum_{n \leq x} f^{-1}(n)$ for generally positive, Dirichlet invertible f , and then characterize these summatory functions with a limiting probability measure underneath. The proof so far is

inexact in so much as the method still needs the good bounds on the sums $\widehat{F}_z(x)$ to proceed from formalism to an exact formula for the limiting CLT-like (Erdős-Kac analog) distributions. It is nonetheless a very interesting, and definitely challenging problem that I need input on. If you feel like talking even more, I can show you the work I already have done and the specific strongly additive functions associated with any such multiplicative f .

Remark 5.2 (Remarks on constructing the f -generator DGFs $\mathcal{F}_*(s)$). Where first encountered this DGF

...

Give the DGF $\sum_{n \geq 1} \phi(n)^{-s} = \zeta(s) \times \prod_p (1 - p^{-s} + (p-1)^{-s}) \dots$

Explain how to find this formula ...

Remark 5.3 (Asymptotics of the summatory function of $\lambda_f(n)$). TODO: Remarks ??? ...

6 Conclusions

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