Exact formulas for partial sums of the Möbius function expressed by partial sums of weighted Liouville functions

Maxie Dion Schmidt

Georgia Institute of Technology
School of Mathematics

maxieds@gmail.com
mschmidt34@gatech.edu

March 17, 2022

High-level overview and takeways of the talk

- Study new expressions for partial sums of a signed classical function
- Identify new unsigned sequences through which we can express these partial sums, or summatory functions
- Elaborate on proof techniques that involve properties of strongly additive number theoretic functions
- Try to keep things in perspctive at a high level without bogging down the key takeway points with technical details

The plan of attack

1. Write the Mertens function via partial sums depending on classical ± 1-valued signs; and then **motivate why we should care** by arguing that the unsigned magnitudes of these summands are "nice" or somehow "special".
2. Conjecture and reason about heuristics that provide limiting distributions (in analog to the Erdős-Kac theorem) of certain unsigned sequences we will precisely identify in thew coming slides.
3. Hence, the high-level argument goes that the new exact formulas for $M(x)$ are worthwhile to study and consider more (*that they might be a revealing lense from which to view the classically hard partial sums*)

(Too much verbage – Let's get started on the tasks at hand!)

Definitions and notation

- Here, the function $\omega(n)$ (and $\Omega(n)$) counts the number of distinct prime factors of any n without (and with, respectively) multiplicity.
- The **Möbius function** is defined as

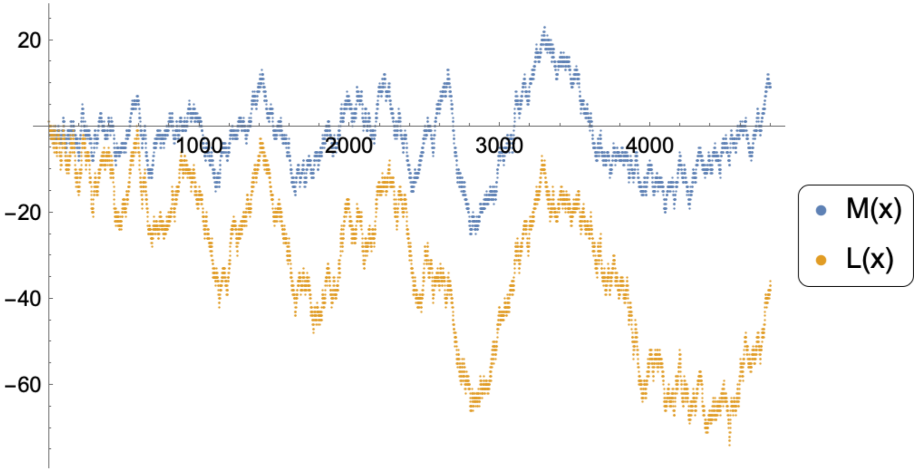
$$\mu(n) = \begin{cases} 1, & \text{if } n = 1; \\ (-1)^{\omega(n)}, & \text{if } \omega(n) = \Omega(n) \text{ (i.e., if this } n \geq 2 \text{ is squarefree);} \\ 0, & \text{otherwise.} \end{cases}$$

- The summatory function given by the **Mertens function** is defined as follows:

$$M(x) := \sum_{n \leq x} \mu(n), \text{ for } x \geq 1.$$

- Related functions include the **Liouville lambda function**, $\lambda(n) := (-1)^{\Omega(n)}$ for $n \geq 1$, and its partial sums $L(x) := \sum_{n \leq x} \lambda(n)$ for $x \geq 1$.

Visualizing the first oscillatory values



Definitions of auxiliary unsigned functions

- We fix the notation for the Dirichlet inverse function (inverse taken with respect to the operation of Dirichlet convolution, e.g., $(f * h)(n) = \sum_{d|n} f(d) h\left(\frac{n}{d}\right))$ as follows:

$$g(n) := (\omega + \mathbb{1})^{-1}(n), \text{ for } n \geq 1.$$

- We define the partial sums for $x \geq 1$ as

$$G(x) := \sum_{n \leq x} g(n) = \sum_{n \leq x} \lambda(n) |g(n)|.$$

- Where did the definition of $g(n)$ come from? Its partial sums are related to the classical prime counting function by

$$\chi_{\mathbb{P}}(n) + \delta_{n,1} = (\omega + \mathbb{1}) * \mu(n), n \geq 1,$$

by Möbius inversion since

$$\omega(n) = \sum_{p|n} 1 = \sum_{d|n} \chi_{\mathbb{P}}(d).$$

New explicit formulas for $M(x)$

Theorem (Schmidt, 2022)

For all $x \geq 1$

(1a) $$M(x) = G(x) + \sum_{1 \leq k \leq x} |g(k)| \pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) \lambda(k),$$

(1b) $$M(x) = G(x) + \sum_{1 \leq k \leq \frac{x}{2}} \left(\pi\left(\left\lfloor \frac{x}{k} \right\rfloor\right) - \pi\left(\left\lfloor \frac{x}{k+1} \right\rfloor\right)\right) G(k),$$

(1c) $$M(x) = G(x) + \sum_{p \leq x} G\left(\left\lfloor \frac{x}{p} \right\rfloor\right).$$

Remarks and important takeaway features to follow ...

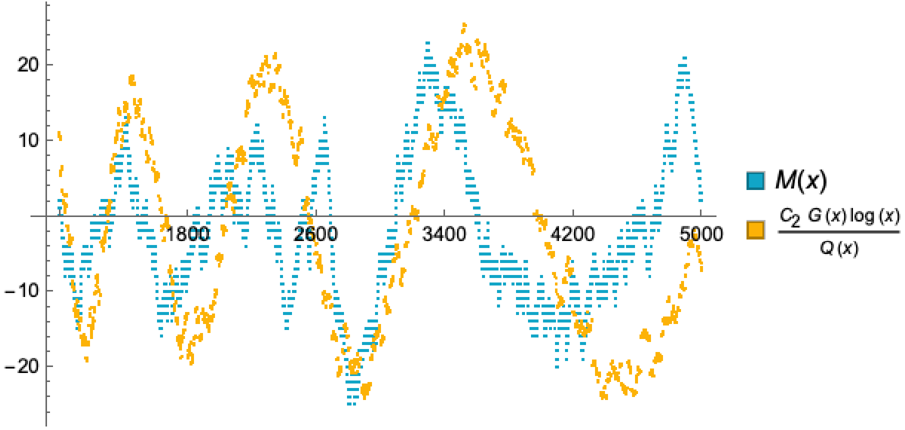
Remarks on the significance of the new formulas for $M(x)$ in terms of $G(x)$

- The summands are sign-weighted by $\lambda(n)$ with unsigned magnitudes that have "nicer" properties.
- For comparision, we have the less predictably signed expansion:

$$M(x) = \sum_{d \leq \sqrt{x}} \mu(d) L\left(\left\lfloor \frac{x}{d^2} \right\rfloor\right), \text{ for } x \geq 1. \tag{2}$$

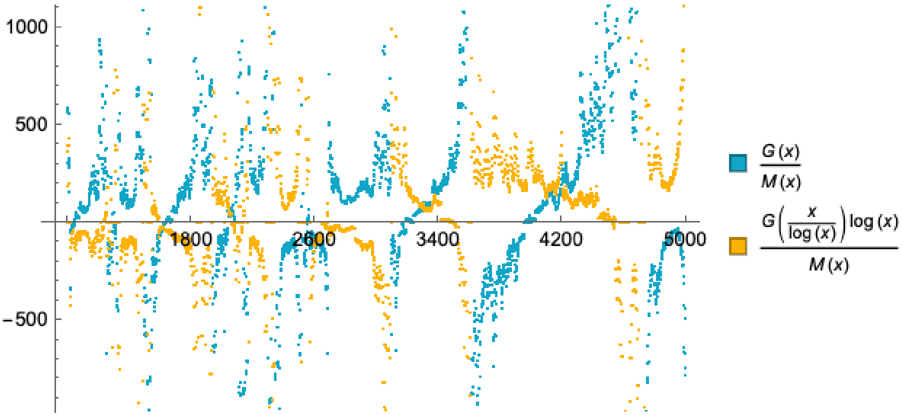
- Why are the unsigned summands in the previous theorem so much "nicer" than classical expansions of $M(x)$ like in (2)?
- We actually conjecture that there is limiting probability distribution that "characterizes" the spread of the unsigned values of $|g(n)|$ from $2 \leq n \leq x$ as $x \to \infty$.

Some comparisons to $G(x)$



(Here, $Q(x) := \sum_{n \leq x} \mu^2(n)$ and we take the absolute constant as $C_2 := \frac{\pi^2}{6}$.)

Some comparisons to $G(x)$ (cont'd)



Note that the plot axis values are SMALL – But still heuristically interesting to explore, nonetheless ...

Properties of the unsigned sequences

- For all $n \geq 1$, $\operatorname{sgn}(g(n)) = \lambda(n)$
- An exact expression is given by:

$$\lambda(n)g(n) = \sum_{d|n} \mu^2\left(\frac{n}{d}\right) C_\Omega(d), n \geq 1,$$

where

$$C_\Omega(n) = \begin{cases} 1, & \text{if } n = 1; \\ (\Omega(n))! \times \prod_{p^\alpha || n} \frac{1}{\alpha!}, & \text{if } n \geq 2. \end{cases}$$

- For all squarefree integers $n \geq 1$, we have that

$$|g(n)| = \sum_{m=0}^{\omega(n)} \binom{\omega(n)}{m} \times m!$$

Notice the nicer values of these integer sequences

n	n	$g(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}}{\lvert g(n)\rvert}$	$G(x)$
2	2^1	-2	1.000	-1
3	3^1	-2	1.000	-3
4	2^2	2	1.500	-1
5	5^1	-2	1.000	-3
6	$2^1 3^1$	5	1.000	2
7	7^1	-2	1.000	0
8	2^3	-2	2.000	-2
9	3^2	2	1.500	0
10	$2^1 5^1$	5	1.000	5
11	11^1	-2	1.000	3
12	$2^2 3^1$	-7	1.286	-4
13	13^1	-2	1.000	-6
14	$2^1 7^1$	5	1.000	-1
15	$3^1 5^1$	5	1.000	4
16	2^4	2	2.500	6
17	17^1	-2	1.000	4
18	$2^1 3^2$	-7	1.286	-3
19	19^1	-2	1.000	-5
20	$2^2 5^1$	-7	1.286	-12
21	$3^1 7^1$	5	1.000	-7
22	$2^1 11^1$	5	1.000	-2
23	23^1	-2	1.000	-4
24	$2^3 3^1$	9	1.556	5
25	5^2	2	1.500	7
26	$2^1 13^1$	5	1.000	12
27	3^3	-2	2.000	10

n	n	$g(n)$	$\frac{\sum_{d\mid n} C_{\Omega(d)}}{\lvert g(n)\rvert}$	$G(x)$
28	$2^2 7^1$	-7	1.286	3
29	29^1	-2	1.000	1
30	$2^1 3^1 5^1$	-16	1.000	-15
31	31^1	-2	1.000	-17
32	2^5	-2	3.000	-19
33	$3^1 11^1$	5	1.000	-14
34	$2^1 17^1$	5	1.000	-9
35	$5^1 7^1$	5	1.000	-4
36	$2^2 3^2$	14	1.357	10
37	37^1	-2	1.000	8
38	$2^1 19^1$	5	1.000	13
39	$3^1 13^1$	5	1.000	18
40	$2^3 5^1$	9	1.556	27
41	41^1	-2	1.000	25
42	$2^1 3^1 7^1$	-16	1.000	9
43	43^1	-2	1.000	7
44	$2^2 11^1$	-7	1.286	0
45	$3^2 5^1$	-7	1.286	-7
46	$2^1 23^1$	5	1.000	-2
47	47^1	-2	1.000	-4
48	$2^4 3^1$	-11	1.818	-15
49	7^2	2	1.500	-13
50	$2^1 5^2$	-7	1.286	-20
51	$3^1 17^1$	5	1.000	-15
52	$2^2 13^1$	-7	1.286	-22
53	53^1	-2	1.000	-24

Properties of the unsigned sequences (cont'd)

- Erdős-Kac (EK) theorem (standard **normal tending** CLT-type statement):

$$\frac{1}{x} \times \#\left\{2 \leq n \leq x : \frac{\omega(n) - \log\log x}{\sqrt{\log\log x}} \leq z\right\} = \Phi(z) + o(1).$$

- In analog, we expect that there is a limiting probability distribution with CDF denoted by $\Phi_\Omega(z)$ to express the spread of $C_\Omega(n)$ over $n \leq x$ in the following form:

$$\frac{1}{x} \times \#\left\{3 \leq n \leq x : \frac{C_\Omega(n) - \mu_\Omega(x)}{\sigma_\Omega(x)} \leq z\right\} = \Phi(z) + o(1),$$

  for $z \in \mathbb{R}$ as $x \to \infty$.
- The functions $\mu_\Omega(x)$ and $\sigma_\Omega(x)$ serve the role of mean and variance analogs from the EK CLT-type theorem, but for now, we only require (conjecture as in the preprint) that they are monotone and unbounded functions of x.

Properties of the unsigned sequences (cont'd)

- More generally, let

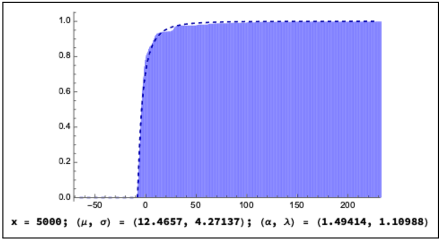
$$\mathcal{D}_{\Omega}\left(\mu_{\Omega}, \sigma_{\Omega}; x, z\right) :=$$
$$\frac{1}{x} \times \#\left\{3 \leq n \leq x : \frac{|g(n)| - \frac{1}{n} \times \sum_{k \leq n} |g(k)| - \frac{6}{\pi^2}\mu_{\Omega}(x)}{\sigma_{\Omega}(x)} \leq z\right\}.$$

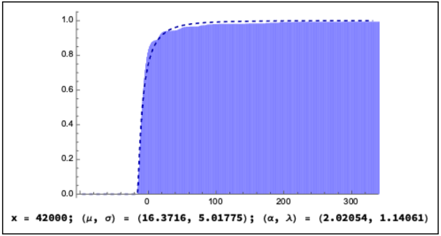
- **Conjecture.** There are explicit unbounded monotone increasing functions, $\mu_{\Omega}(x)$ and $\sigma_{\Omega}(x)$, and a limiting probability measure on $\mathbb{R}$ with associated CDF Φ_{Ω} so that for any $y \in (-\infty, \infty)$

$$\mathcal{D}_{\Omega}\left(\mu_{\Omega}, \sigma_{\Omega}; x, y\right) = \Phi_{\Omega}\left(\frac{\pi^2 y}{6}\right) + o(1), \text{ as } x \to \infty.$$

- If we find a limiting probability measure such that the conjectured result on the distribution of the $C_{\Omega}(n)$ holds (for some specific μ_{Ω}, σ_{Ω}), then the result for $|g(n)|$ on the last line follows easily.

Numerical evidence for the conjecture in a special case





- The plots show the empirical histogram distribution of $\mathcal{D}_{\Omega}\left((\log x)\sqrt{\log\log x}, \sqrt{(\log x)(\log\log x)}; x, z\right)$ at the specified values of $x := 5000, 42000$.
- The dashed lines provide an approximate fit by the CDF of a shifted log-normal distribution with mean α and standard deviation λ.
- Similar features appear even for these log-logarithmically small x.

Roadmap for the rest of the talk

1. Give the average order (i.e., first moment) asymptotics for the two unsigned sequences – **We can prove these carefully!**
2. Sketch the adaptations of well-known methods used in the nitty-gritty technical “guts” of the article.
3. Spend some time discussing why (under which heuristic models) we should expect the conjectured distributions to hold (keeping things very informal and intentionally hand-wavy for now).

Average order asymptotics

- It takes a lot of technical machinery and analytic methods to get these first moments of the unsigned sequences!
- There is an absolute constant $B_0 > 0$ so that

$$\frac{1}{n} \times \sum_{k \leq n} C_{\Omega}(k) = B_0 \sqrt{\log\log n} \left(1 + O\left(\frac{1}{\log\log n}\right)\right).$$

- The average order of $|g(n)|$ is given by

$$\frac{1}{n} \times \sum_{k \leq n} |g(k)| = \frac{6B_0}{\pi^2} \cdot (\log n) \sqrt{\log\log n} \left(1 + O\left(\frac{1}{\log\log n}\right)\right).$$

Proofs of the first moment formulas are technical

- An extended application of the **Selberg-Delange** method shows uniformly for $1 \leq k \leq \frac{3}{2} \log\log x$, there is an absolute constant $A_0 > 0$ such that

$$(3) \qquad \sum_{\substack{n \leq x \\ \Omega(n)=k}} C_{\Omega}(n) \sim \frac{A_0 \sqrt{2\pi} x}{\log x} \times \widehat{G}\left(\frac{k-1}{\log\log x}\right) \frac{(\log\log x)^{k-\frac{1}{2}}}{(k-1)!}.$$

- We have defined the leading coefficients by

$$\widehat{G}(z) := \frac{1}{\Gamma(1+z)(1+P(2)z)} \times \prod_p \left(1 - \frac{1}{p^2}\right)^z, \text{ for } |z| < 2.$$

- The asymptotic tail of the partial sums are bounded by an adaptation of **Rankin's method** (note that $\frac{1}{2} - \frac{3}{2} \log\left(\frac{3}{2}\right) \approx -0.108198 < 0$):

$$\sum_{\substack{n \leq x \\ \Omega(n) \geq r \log\log x}} C_{\Omega}(n) \ll_r x (\log x)^{r-1-r\log r} \sqrt{\log\log x}, \text{ for } 1 \leq r < 2.$$

Intuition and a start to proving the conjecture

- An intuition for why there is a distribution underneath $C_{\Omega}(n)$ and $|g(n)|$ is found in a special property of certain arithmetic functions.
- Let the function $\mathcal{E}[n] \vdash (\alpha_1, \alpha_2, \ldots, \alpha_r)$ denote the unordered partition of exponents for which $n = p_1^{\alpha_1} \times \cdots \times p_r^{\alpha_r}$ is the factorization of n into powers of distinct primes.
- An arithmetic function h is said to be **factorization symmetric** if

$$\mathcal{E}[n_1] = \mathcal{E}[n_2] \implies h(n_1) = h(n_2), \text{ for all } n_1, n_2 \geq 2.$$

Intuition and a start to proving the conjecture (cont'd)

- We notice that $C_\Omega(n)$ and $g(n)$ are both factorization symmetric, as are the strongly additive functions $\omega(n)$ and $\Omega(n)$, which each have CLT analogs to the normal tending **Erdős-Kac theorem**.
- But so are the functions $\mu(n)$ and $\lambda(n)$, which are classically difficult to deal with and sum over $n \leq x$.
- Recall the exact formula given by

$$C_\Omega(n) = \begin{cases} 1, & \text{if } n = 1; \\ (\Omega(n))! \times \prod_{p^\alpha || n} \frac{1}{\alpha!}, & \text{if } n \geq 2. \end{cases} \tag{4}$$

- The formula in (4) corresponds to a **multinomial coefficient**, which is suggestive of the type of limiting distribution we can expect under some (reasonable) probabilistic models.

Conclusions – Taking a step back – What we've done

- We defined $g(n) := (\omega + \mathbb{1})^{-1}(n)$ as the shifted Dirichlet inverse of the strongly additive function, $\omega(n)$.
- We precisely connected $C_{\Omega}(n)$ to $g(n)$ and used it to prove formulas for the average orders of the unsigned sequences.
- We have conjectured limiting distributions exist underneath $C_{\Omega}(n)$ and $|g(n)|$ for $n \leq x$ as $x \to \infty$.
- We connected the Mertens function $M(x)$ with the partial sums $G(x) := \sum_{n \leq x} \lambda(n)|g(n)|$ via exact formulas for all $x \geq 1$.

Conclusions

The End

Questions?

Comments?

Feedback?

Thank you for attending!