Probability Comprehensive Exam
Aug 24, 2016

Student Number:

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Suppose (X_n) is a sequence of random vectors such that for some sigma-algebra $\mathcal{F}$, one has X_n and $\mathcal{F}$ independent for all n. If $X_n \to X$ almost surely, show that X and $\mathcal{F}$ are independent.

Solution: Let $A \in \mathcal{F}$. If $\mathbb{P}(A) \in \{0,1\}$, then A is independent of all events, in particular of any event in the sigma-algebra generated by X. So we may assume that $\mathbb{P}(A) \in (0,1)$ and prove that A is independent of X.

For a random vector Z, let ϕ_Z be its characteristic function. Define the conditional measure $\mathbb{P}_A(\cdot) = \mathbb{P}(\cdot \mid A)$ and denote the corresponding characteristic function of a random variable Z by ϕ_Z^A. Then since $X_n \to X$ almost surely, the convergence also occurs in distribution relative to $\mathbb{P}$, and so $\phi_{X_n} \to \phi_X$ pointwise. By independence between X_n and A, one has

$$\phi_{X_n}^A(t) = \frac{\mathbb{E}e^{it\cdot X_n}\mathbf{1}_A}{\mathbb{P}(A)} = \phi_{X_n}(t),$$

so $\phi_{X_n}^A \to \phi_X$ pointwise as well. Since $X_n \to X$ almost surely, also for any t,

$$e^{it\cdot X_n}\mathbf{1}_A \to e^{it\cdot X}\mathbf{1}_A \text{ almost surely.}$$

By the bounded convergence theorem,

$$\phi_{X_n}^A(t) = \frac{\mathbb{E}e^{it\cdot X_n}\mathbf{1}_A}{\mathbb{P}(A)} \to \frac{\mathbb{E}e^{it\cdot X}\mathbf{1}_A}{\mathbb{P}(A)} = \phi_X^A(t).$$

This implies $\phi_X(t) = \phi_X^A(t)$, so since characteristic functions determine a distribution, the distribution of X under $\mathbb{P}$ is the same as that under $\mathbb{P}_A$. In other words, for $B \subset \mathbb{R}^n$ Borel,

$$\mathbb{P}(X \in B) = \mathbb{P}_A(X \in B) = \frac{\mathbb{P}(X \in B,\ A)}{\mathbb{P}(A)},$$

or X and A are independent. This implies X and $\mathcal{F}$ are independent.

2. Let X be a random variable with continuous density function f and $f(0) > 0$. Let Y be a random variable with

$$Y = \begin{cases} \frac{1}{X} & \text{if } X > 0 \\ 0 & \text{otherwise} \end{cases}.$$

and $Y_1, Y_2, \ldots$ be i.i.d. with distribution equal to that of Y. What is the value of the almost sure limit

$$\lim_n \frac{Y_1 + \cdots + Y_n}{n}?$$

Solution: First compute the probability for $y > 0$

$$\mathbb{P}(Y \geq y) = \mathbb{P}(X \in (0, 1/y]).$$

Choose $x_0 > 0$ small enough so that $f(x) \geq f(0)/2$ for $x \in [0, x_0]$. Then for $y > 1/x_0$, one has

$$\mathbb{P}(Y \geq y) = \int_0^{1/y} f(x) \, \mathrm{d}x \geq \frac{f(0)}{2y}.$$

Therefore

$$\begin{aligned} \mathbb{E}Y = \int_0^\infty \mathbb{P}(Y \geq y) \, \mathrm{d}y &\geq \int_{1/x_0}^\infty \mathbb{P}(Y \geq y) \, \mathrm{d}y \\ &\geq \int_{1/x_0}^\infty \frac{f(0)}{2y} \, \mathrm{d}y = \infty. \end{aligned}$$

Now since the Y_i's have infinite mean and are positive, we can show that $(Y_1 + \cdots + Y_n)/n \to \infty$ almost surely. To do so, pick any $M > 0$ and define

$$Y_i^{(M)} = \min\{Y_i, M\}.$$

By the strong law of large numbers,

$$\frac{Y_1^{(M)} + \cdots + Y_n^{(M)}}{n} \to \mathbb{E}Y_1^{(M)} \text{ almost surely.}$$

Therefore, for any $M > 0$, almost surely

$$\liminf_{n\to\infty} \frac{Y_1 + \cdots + Y_n}{n} \geq \lim_n \frac{Y_1^{(M)} + \cdots + Y_n^{(M)}}{n} = \mathbb{E}Y_1^{(M)}.$$

The event on which this inequality holds we denote by A_M. Then on $\cap_{M\in\mathbb{N}} A_M$ (which has probability one), we have

$$\liminf_{n\to\infty} \frac{Y_1 + \cdots + Y_n}{n} \geq \sup_{M\in\mathbb{N}} \mathbb{E}Y_1^{(M)} = \lim_{M\to\infty} \mathbb{E}Y_1^{(M)} = \infty,$$

where the last equation holds by the monotone convergence theorem.

3. Let $X_1, X_2, \ldots$ be i.i.d. with

$$\mathbb{P}(X_1 = 1) = 1/2 = \mathbb{P}(X_1 = -1).$$

Let C be the set of factorials:

$$C = \{k! : k \in \mathbb{N}\}.$$

Show that

$$\lim_n \mathbb{P}(X_1 + \cdots + X_n \in C) = 0.$$

Hint. You may want to start by covering part of $[0, \infty)$ by small intervals. Let $\epsilon > 0$ and $I > 0$, consider intervals $I_1, \ldots, I_I$, where $I_i = [(i-1)\epsilon, i\epsilon)$, and show that for large n, at most two of the sets $I_i \cap (C/\sqrt{n})$ are nonempty.

Solution: Fix $\delta > 0, \epsilon > 0$ and consider the sets

$$I_1 = [0, \epsilon),\ \ I_2 = [\epsilon, 2\epsilon),$$

and generally $I_i = [(i-1)\epsilon, i\epsilon)$. Write $C/\sqrt{n} = \{k!/\sqrt{n} : k \in \mathbb{N}\}$ and for $I > 0$ fixed, estimate

$$\limsup_n \mathbb{P}(S_n \in C) \leq \limsup_n \Bigg[\sum_{i=1}^{I} \mathbb{P}(S_n/\sqrt{n} \in (C/\sqrt{n}) \cap I_i) \\ + \mathbb{P}(S_n/\sqrt{n} \geq I\epsilon) \Bigg].$$

We claim that for large n, at most one of the sets $(C/\sqrt{n}) \cap I_i$ is nonempty, for $i = 2, \ldots, I$. Indeed, assuming at least one is nonempty, let $i_0 \geq 2$ be the minimal index i such that $(C/\sqrt{n}) \cap I_i$ is nonempty. Then there is $\ell \in \mathbb{N}$ such that

$$\ell!/\sqrt{n} \in [(i_0 - 1)\epsilon, i_0\epsilon).$$

Note that since $\ell! \geq \epsilon\sqrt{n}$, we must have for n large

$$\ell \geq I.$$

For such large n,

$$(\ell+1)!/\sqrt{n} = (\ell+1) \cdot \frac{\ell!}{\sqrt{n}} \geq \ell\epsilon \geq I\epsilon,$$

so all of the sets $(C/\sqrt{n}) \cap I_i$ for $i = i_0 + 1, \ldots, I$ are empty.

Putting this together, we get

$$\limsup_n \mathbb{P}(S_n \in C) \\ \leq 3 \limsup_n \max \left\{ \mathbb{P}(S_n/\sqrt{n} \geq I\epsilon), \max_{i \in [1,I]} \mathbb{P}(S_n/\sqrt{n} \in I_i) \right\}.$$

By the CLT, if X is a standard normal random variable, this converges to

$$3 \limsup_n \max \left\{ \mathbb{P}(X \geq I\epsilon), \max_{i \in [1,I]} \mathbb{P}(|X| \in I_i) \right\}.$$

Since the distribution of X has no atoms, we may choose ϵ so small and I so large that this is at most δ.

4. Let (X, Y) be a normal vector in $\mathbb{R}^2$ with mean zero and covariance matrix Σ, where

$$\Sigma = \begin{pmatrix} 5 & 1 \\ 1 & 10 \end{pmatrix}.$$

Find $\mathbb{E}X^2Y^2$.

Solution: If (X_1, X_2) is a standard Gaussian vector, then if A is an arbitrary 2×2 real matrix,

$$A \cdot \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} =: \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix} \cdot \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

is an arbitrary mean zero Gaussian vector (Y_1, Y_2). The covariance matrix Σ is given by

$$\Sigma_{i,j} = \mathbb{E}Y_iY_j = \sum_{k,\ell=1}^{2} \mathbb{E}A_{k,i}X_kA_{\ell,j}X_\ell = \sum_{k=1}^{2} A_{k,i}A_{k,j} = (A^TA)_{i,j}.$$

Solving in our case for A in terms of the covariance matrix Σ, we find that

$$X = X_1 + 2X_2,\ Y = 3X_1 - X_2,$$

where (X, Y) is in the statement of the problem, and (X_1, X_2) is a standard normal vector.

Now we can compute the expected value as

$$\begin{aligned}
&\mathbb{E}(X_1 + 2X_2)^2(3X_1 - X_2)^2 \\
&= \mathbb{E}((X_1^2 + 4X_1X_2 + 4X_2^2)(9X_1^2 - 6X_1X_2 + X_2^2)) \\
&= \mathbb{E}(9X_1^4 + 30X_1^3X_2 + 13X_1^2X_2^2 - 20X_1X_2^3 + 4X_2^4).
\end{aligned}$$

Using the i.i.d. assumption with symmetry, we obtain

$$13\mathbb{E}X_1^4 + 13(\mathbb{E}X_1^2)^2 = 13 \cdot 3 + 13 = 52.$$

5. Let ξ_1 and ξ_2 be independent random variables with characteristic functions $\varphi_1(u) = \frac{1-iu}{1+u^2}$ and $\varphi_2(u) = \frac{1+iu}{1+u^2}$ respectively. Find the probability that $\xi_1 + \xi_2$ takes values in $(3, +\infty)$.

Solution: Since ξ_1 and ξ_2 are independent, the characteristic function of $\xi_1 + \xi_2$ equals the product of their characteristic functions:

$$\varphi(u) = \varphi_1(u)\varphi_2(u) = \frac{1}{1+u^2}.$$

Therefore, $\xi_1 + \xi_2$ has bilateral distribution with density

$$f(x) = \frac{1}{2} e^{-|x|},$$

and

$$\mathbb{P}(\xi_1 + \xi_2 > 3) = \frac{1}{2} \int_3^\infty e^{-x} dx = \frac{e^{-3}}{2}.$$

Answer: $\frac{e^{-3}}{2}$.

6. Let $\{A_n\}$ be an infinite collection of independent events. Suppose that $\mathbb{P}(A_n) < 1$ for every $n \geq 1$. Show that $\mathbb{P}(A_n \, i.o.) = 1$ if and only if $\mathbb{P}(\cup A_n) = 1$.

Solution: For any $N \geq 1$,

$$\begin{aligned} \mathbb{P}(A_1^c \cap \cdots \cap A_N^c)\mathbb{P}(\cup_{n>N} A_n) &= \mathbb{P}(A_1^c \cap ... \cap A_N^c \cap (\cup_{n>N} A_n)) \\ &= \mathbb{P}(A_1^c \cap ... \cap A_N^c \cap (\cup_{n\geq 1} A_n)). \end{aligned} \tag{1}$$

If $\mathbb{P}(\cup_{n\geq 1} A_n) = 1$, then (1) equals

$$\mathbb{P}(A_1^c \cap \cdots \cap A_N^c).$$

Since $\mathbb{P}(A_n) < 1$ for all n,

$$\mathbb{P}(A_1^c \cap \cdots \cap A_N^c) = \prod_{i=1}^N \mathbb{P}(A_i^c) > 0,$$

so we may divide both sides of

$$\mathbb{P}(A_1^c \cap \cdots \cap A_N^c)\mathbb{P}(\cup_{n>N} A_n) = \mathbb{P}(A_1^c \cap \cdots \cap A_N^c)$$

by $\mathbb{P}(A_1^c \cap \cdots \cap A_N^c)$, and we get

$$\mathbb{P}(\cup_{n>N} A_n) = 1.$$

This is true for all $N \geq 1$, hence

$$P(A_n i.o.) = P(\cap_N \cup_{n\geq N} A_n) = 1.$$

Conversely, if $P(A_n \, i.o.) = 1$, then since

$$\{A_n \text{ i.o.}\} = \cap_N \cup_{n\geq N} A_n \subset \cup_n A_n$$

we also have $\mathbb{P}(\cup_n A_n) = 1$.

7. Let X be a random variable taking values on the interval $[1,2]$. Find sharp lower and upper estimates on the quantity $\mathbb{E}X \cdot \mathbb{E}\frac{1}{X}$. Provide an example of a random variable for which the lower estimate is attained. Provide an example of a random variable for which the upper estimate is attained.

   Hint. For the upper bound, justify and use the inequality
$$ab \leq \frac{1}{2}\left(\frac{a}{2}+b\right)^2.$$

Solution: To obtain the lower bound, we use Jensen's inequality:
$$\mathbb{E}X \cdot \mathbb{E}\frac{1}{X} \geq \mathbb{E}X \cdot \frac{1}{\mathbb{E}X} = 1.$$
The lower bound is attained when $X = c$ almost surely for some $c \in [1,2]$.

Let $f(x)$ be the density of X. Let μ be the distribution of X. As for the upper bound, we apply the inequality $ab \leq \frac{1}{2}\left(\frac{a}{2}+b\right)^2$ with
$$a = \mathbb{E}X \text{ and } b = \mathbb{E}\frac{1}{X},$$
giving
$$\begin{aligned}
\mathbb{E}X \cdot \mathbb{E}\frac{1}{X} &= \int_1^2 x \,\mathrm{d}\mu(x) \cdot \int_1^2 \frac{1}{x}\,\mathrm{d}\mu(x) \\
&= 2\int_1^2 \frac{x}{2}\,\mathrm{d}\mu(x) \cdot \int_1^2 \frac{1}{x}\,\mathrm{d}\mu(x) \\
&\leq 2 \cdot \frac{1}{4}\left(\int_1^2 \frac{x}{2} f(x)dx + \int_1^2 \frac{1}{x} f(x)dx\right)^2 \\
&= \frac{1}{2}\left(\int_1^2 (\frac{x}{2}+\frac{1}{x}) f(x)dx\right)^2.
\end{aligned}$$
Here we used the inequality
$$ab \leq \frac{1}{4}(a+b)^2.$$
$$\mathbb{E}X \cdot \mathbb{E}\frac{1}{X} \leq \frac{1}{2}\left(\frac{1}{2}\mathbb{E}X + \mathbb{E}\frac{1}{X}\right)^2 = \frac{1}{2}\left(\mathbb{E}\left(\frac{X}{2}+\frac{1}{X}\right)\right)^2.$$
Observe that the maximum of $\frac{x}{2}+\frac{1}{x}$ over $[1,2]$ is attained when $x = 2$ (or when $x = 1$), where the function takes value $\frac{3}{2}$. Therefore,
$$\mathbb{E}X \cdot \mathbb{E}\frac{1}{X} \leq \frac{1}{2}\left(\frac{3}{2}\right)^2 = \frac{9}{8}.$$

For an example attaining this value, let the random variable X_0 take values 1 and 2 with probability $\frac{1}{2}$ each. Then

$$\mathbb{E}X_0 \cdot \mathbb{E}\frac{1}{X_0} = \frac{3}{2} \cdot \frac{3}{4} = \frac{9}{8}.$$

8. Show that for a sequence of random variables X_n, one has $X_n \to X$ in probability if and only if
$$\mathbb{E}\left[e^{\min\{2,|X_n-X|\}} - 1\right] \to 0,$$
as $n \to \infty$.

Solution: First suppose that $X_n \to X$ in probability. Then also $|X_n - X| \to 0$ in probability and therefore in distribution. By Portmanteau's theorem, for any bounded continuous $f : \mathbb{R} \to \mathbb{R}$, one has $\mathbb{E}f(|X_n - X|) \to f(0)$. Applying this to the function $f(x) = e^{\min\{2,x\}} - 1$, we obtain the convergence in the problem.

and let $\epsilon > 0$. Chose n_0 large enough such that for every $n \geq n_0$,

$$\mathbb{P}\left(|X_n - X| \geq \min\left\{\frac{\epsilon}{2}, 1\right\}\right) < e^{-2}\frac{\epsilon}{2}.$$

But then, for all $n \geq n_0$,

$$\begin{aligned}
&\mathbb{E}e^{\min\{2,|X-X_n|\}} \\
&= \int_{|X_n-X|\geq\min\{\frac{\epsilon}{2},1\}} e^{\min\{2,|X-X_n|\}} \, \mathrm{d}\mathbb{P} + \int_{|X_n-X|<\min\{\frac{\epsilon}{2},1\}} e^{\min\{2,|X-X_n|\}} \, \mathrm{d}\mathbb{P} \\
&\leq e^2\mathbb{P}\left(|X_n - X| \geq \min\left\{\frac{\epsilon}{2}, 1\right\}\right) + \min\left\{\frac{\epsilon}{2}, 1\right\} \\
&< \epsilon.
\end{aligned}$$

Conversely, suppose that $\mathbb{E}\left[e^{\min\{2,|X_n-X|\}} - 1\right] \to 0$, take $\epsilon \in (0, 2)$, and estimate by the Chebychev (Markov) inequality

$$\begin{aligned}
\mathbb{P}\left(|X_n - X| \geq \epsilon\right) &= \mathbb{P}(\min\{2, |X_n - X|\} > \epsilon) \\
&= \mathbb{P}\left(e^{\min\{2,|X_n-X|\}} - 1 > e^\epsilon - 1\right) \\
&\leq \frac{1}{e^\epsilon - 1}\mathbb{E}\left[e^{\min\{2,|X-X_n|\}} - 1\right].
\end{aligned}$$

Here we have used that $e^{\min\{2,|X_n-X|\}} - 1 \geq 0$ almost surely.

By the condition of the problem,

$$\lim_{n\to\infty} \frac{1}{e^{\epsilon}-1}\mathbb{E}\left[e^{\min(2,|X-X_n|)}-1\right]=0,$$

and hence

$$\mathbb{P}(|X_n - X| > \epsilon) \to 0$$

as $n \to \infty$. This means that $X_n \to X$ in probability.