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# Factorization theorems for relatively prime divisor sums, GCD sums and generalized Ramanujan sums

Hamed Mousavi<sup>1</sup> · Maxie D. Schmidt<sup>1</sup>

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## Abstract

We build on and generalize recent work on so-termed factorization theorems for Lambert series generating functions. These factorization theorems allow us to express formal generating functions for special sums as invertible matrix transformations involving partition functions. In the Lambert series case, the generating functions at hand enumerate the divisor sum coefficients of  $q^n$  as  $\sum_{d|n} f(d)$  for some arithmetic function  $f$ . Our new factorization theorems provide analogs to these established expansions generating corresponding sums of the form  $\sum_{d:(d,n)=1} f(d)$  (type I sums) and the Anderson–Apostol sums  $\sum_{d|(m,n)} f(d)g(n/d)$  (type II sums) for any arithmetic functions  $f$  and  $g$ . Our treatment of the type II sums includes a matrix-based factorization method relating the partition function  $p(n)$  to arbitrary arithmetic functions  $f$ . We conclude the last section of the article by directly expanding new formulas for an arithmetic function  $g$  by the type II sums using discrete, and discrete time, Fourier transforms (DFT and DTFT) for functions over inputs of greatest common divisors.

**Keywords** Divisor sum · Totient function · Matrix factorization · Möbius inversion · Partition function

**Mathematics Subject Classification** 11N64 · 11A25 · 05A17

## 1 Notation and conventions

To make a common source of references to definitions of key functions and sequences defined within the article, we provide a comprehensive list of commonly used nota-

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tion and conventions. Where possible, we have included section references to local definitions of special sequences and functions where they are given in the text. We have organized the symbols list alphabetically by symbol. This listing of notation and conventions is a useful reference to accompany the article. It is provided in Appendix A.

## 2 Introduction

### 2.1 Motivation

The *average order* of an arithmetic function  $f$  is defined as the arithmetic mean of the summatory function,  $F(x) := \sum_{n \leq x} f(n)$ , as  $F(x)/x$ . We typically represent the average order of a function in the form of an asymptotic formula in the cases where the formula,  $F(x)/x$ , for the average order diverges as  $x \rightarrow \infty$ . For example, it is well known that the average order of  $\Omega(n)$ , which counts the number of prime factors of  $n$  (counting multiplicity), is  $\log \log n$ , and that the average order of Euler's totient function,  $\phi(n)$ , is given by  $\frac{6n}{\pi^2}$  [4]. We are motivated by considering the breakdown of the partial sums of an arithmetic function  $f(d)$  whose average order we would like to estimate into sums over the pairwise disjoint sets of component indices  $d \leq x$ :

$$\sum_{d \leq x} f(d) = \sum_{\substack{1 \leq d \leq x \\ (d,x)=1}} f(d) + \sum_{\substack{d|x \\ d>1}} f(d) + \sum_{\substack{1 < d \leq x \\ 1 < (d,x) < x}} f(d). \quad (1)$$

In particular, in evaluating the partial sums of an arithmetic function  $f(d)$  over all  $d \leq x$ , we wish to break the terms in these partial sums into three sets: those  $d$  relatively prime to  $x$ , the  $d$  dividing  $x$  (for  $d > 1$ ), and the somewhat less “round” set of indices  $d$  which are neither relatively prime to  $x$  nor proper divisors of  $x$ . Now if we let  $f$  denote any arithmetic function, we define the remainder terms in our average order expansions from (1) as follows:

$$\tilde{S}_f(x) = \sum_{d \leq x} f(d) - \sum_{\substack{1 \leq d \leq x \\ (d,x)=1}} f(d) - \sum_{\substack{d|x \\ d>1}} f(d). \quad (2)$$

For instance, when  $x = 24$  we have that

$$\tilde{S}_f(24) = f(9) + f(10) + f(14) + f(16) + f(18) + f(20) + f(21) + f(22).$$

We observe that the last divisor sum terms in (2) correspond to the coefficients of powers of  $q$  in the Lambert series generating function over  $f$  in the following form considered in the next subsection:

$$\sum_{d|x} f(d) = [q^x] \left( \sum_{n \geq 1} \frac{f(n)q^n}{1 - q^n} \right), \quad |q| < 1.$$

We can see that the average order sums on the left-hand side of (1) correspond to the hybrid of divisor and relatively prime divisor sums of the form

$$\sum_{d \leq x} f(d) = \sum_{m|x} \sum_{\substack{k=1 \\ (k, \frac{x}{m})=1}}^{\frac{x}{m}} f(km) = \sum_{m|x} \sum_{\substack{k=1 \\ (k,m)=1}}^m f\left(\frac{kx}{m}\right).$$

In this article, we study and prove new results relating both variants of the sums expanding the right-hand side of the previous equation to restricted partitions and special partition functions. Namely, a combination of the results we prove in Sect. 3 and the Lambert series factorization theorem results summarized in the next subsection allow us to write

$$\begin{aligned} \sum_{n \leq x} f(d) = \sum_{d|x+1} s_{x,d} \left[ \sum_{j=0}^d \sum_{k=1}^{j-1} \sum_{i=0}^j (-1)^{\left\lceil \frac{i}{2} \right\rceil} p(d-j) \chi_{1,k}(j-k-G_i) \right. \\ \left. \times [j-k-G_i \geq 1]_{\delta} \cdot f\left(\frac{(x+1)k}{d}\right) \right], \end{aligned}$$

where  $\chi_{1,k}(n)$  is the principal Dirichlet character modulo  $k$ ,  $[n=k]_{\delta} = \delta_{n,k}$  denotes Iverson's convention,  $G_j := \frac{1}{2} \lceil j/2 \rceil \lceil (3j+1)/2 \rceil$  denotes the sequence of interleaved, or generalized pentagonal numbers, the triangular sequence  $s_{n,k} := [q^n](q; q)_{\infty} \frac{q^k}{1-q^k}$  corresponds to the difference of restricted partition functions discussed in the next subsection, and  $p(n)$  is the classical partition function. The analysis of the asymptotic properties of these sums is a central topic in the study of the behavior of arithmetic functions, analytic number theory, and in applications such as algorithmic analysis. Our new results connect variants of such sums over multiplicative functions with the distinctly additive flavor of the theory of partitions.

## 2.2 Variations on recent work

There is a fairly complete and extensive set of expansions providing identities related to these Lambert series generating functions and their matrix factorizations in the form of so-termed ‘‘Lambert series factorization theorems’’ studied by Merca and Schmidt over 2017–2018 [8, 10, 14]. These results provide factorizations for a Lambert series generating function over the arbitrary arithmetic function  $f$  expanded in the form of

$$L_f(q) := \sum_{n \geq 1} \frac{f(n)q^n}{1-q^n} = \frac{1}{(q; q)_{\infty}} \sum_{n \geq 1} \left( \sum_{k=1}^n s_{n,k} f(k) \right) q^n, \quad (3)$$

where  $s_{n,k} = s_o(n, k) - s_e(n, k)$  is independent of  $f$  and is defined as the difference of the functions  $s_{o/e}(n, k)$  which, respectively, denote the number of  $k$ 's in all partitions of  $n$  into an odd (even) number of distinct parts. These so-termed factorization theorems, which effectively provide a matrix-based expansion of an ordinary generating function

for the divisor sums of the type enumerated by Lambert series expansions, connect the additive theory of partitions to the more multiplicative constructions of power series generating functions found in other branches on number theory. In these cases, it appears that it is most natural, in some sense, to expand these sums via the factorizations defined in (3) since the matrix entries (and their inverses) are also partition-related. It then leads us to the question of what other natural, or even canonical, analogous expansions can be formed for other more general variants of the above divisor sums.

More generally, we can form analogous matrix-based factorizations of the generating functions of the sequences of special sums in (4) provided that these transformations are invertible. That is, we can express generating functions for the sums  $s_n(f, \mathcal{A}) := \sum_{k \in \mathcal{A}_n} f(k)$  where we take  $\mathcal{A}_n \subseteq [1, n] \cap \mathbb{Z}$  for all  $n \geq 1$ :

$$\sum_{n \geq 1} \left( \sum_{\substack{k \in \mathcal{A}_n \\ \mathcal{A}_n \subseteq [1, n]}} f(k) \right) q^n = \frac{1}{(q; q)_\infty} \sum_{n \geq 1} \left( \sum_{k=1}^n v_{n,k}(\mathcal{A}) f(k) \right) q^n, \quad (4)$$

The sums in (5) below are referred to as *type I* and *type II sums*, respectively, in the next subsections. These sums are given by the special cases of (4) where when  $\mathcal{A}_{1,n} := \{d : 1 \leq d \leq n, (d, n) = 1\}$  and  $\mathcal{A}_{2,n} := \{d : 1 \leq d \leq n, d|(k, n)\}$  for some  $1 \leq k \leq n$ , respectively.

$$T_f(x) = \sum_{\substack{d=1 \\ (d,x)=1}}^x f(d), \quad (5)$$

$$L_{f,g,k}(x) = \sum_{d|(k,x)} f(d) g\left(\frac{x}{d}\right);$$

We define the following preliminary constructions for the factorizations of the Lambert-like series whose respective expansions involve the sums in (5). Notice that the invertible matrix coefficients,  $t_{n,k}$  and  $u_{n,k}(f, w)$ , in each expansion are defined such that the subequations in (6) are correct. We will identify precise formulas for these invertible matrices as primary results in the article:

$$T_f(x) = [q^x] \left( \frac{1}{(q; q)_\infty} \sum_{n \geq 2} \sum_{k=1}^n t_{n,k} f(k) \cdot q^n + f(1) \cdot q \right) \quad (6a)$$

$$g(x) = [q^x] \left( \frac{1}{(q; q)_\infty} \sum_{n \geq 2} \sum_{k=1}^n u_{n,k}(f, w) \left[ \sum_{m=1}^k L_{f,g,m}(k) w^m \right] \cdot q^n \right), \quad w \in \mathbb{C}. \quad (6b)$$

The sequences  $t_{n,k}$  and  $u_{n,k}(f, w)$  are lower triangular and invertible for suitable choices of the indeterminate parameter  $w$ . For a fixed  $N \geq 1$ , we can truncate these

sequences after  $N$  rows and form the  $N \times N$  matrices whose entries are  $t_{n,k}$  (respectively,  $u_{n,k}(f, w)$ ) for  $1 \leq n, k \leq N$ . The corresponding inverse matrices have terms denoted by  $t_{n,k}^{(-1)}$  (and  $u_{n,k}^{(-1)}(f, w)$ ), respectively). That is to say, for  $n \geq 2$ , these inverse matrices satisfy

$$f(n) = \sum_{k=1}^n t_{n,k}^{(-1)} \cdot [q^k] \left( (q; q)_{\infty} \times \sum_{n \geq 1} T_f(n) q^n \right), \quad (6c)$$

$$\sum_{m=1}^n L_{f,g,m}(k) w^m = \sum_{k=1}^n u_{n,k}^{(-1)}(f, w) \cdot [q^k] \left( (q; q)_{\infty} \times \sum_{n \geq 1} g(n) q^n \right), \quad w \in \mathbb{C}. \quad (6d)$$

Explicit representations for these inverse matrix sequences are proved in the article. We focus on the special expansions of each factorization type in Sects. 3 and 4, respectively, though we note that other related variants of these expansions are possible.

**Theorem 2.1** (Exact Formulas for the Factorization Matrix Sequences) *The lower triangular sequence  $t_{n,k}$  is defined by the first expansion in (6a). The corresponding inverse matrix coefficients are denoted by  $t_{n,k}^{(-1)}$ . For integers  $n, k \geq 1$ , the two lower triangular factorization sequences defining the expansion of (6a) satisfy exact formulas given by*

$$t_{n,k} = \sum_{j=0}^n (-1)^{\lceil j/2 \rceil} \chi_{1,k}(n+1-G_j) [n-G_j \geq 1]_{\delta}, \quad (i)$$

$$t_{n,k}^{(-1)} = \sum_{d=1}^n p(d-k) \mu_{n,d}, \quad (ii)$$

where we define the sequence of interleaved pentagonal numbers  $G_j$  as in the introduction, and the sequence  $\mu_{n,k}$  as in Proposition 3.1.

The function  $\chi_{1,k}(n)$  defined in the glossary section starting in Appendix A refers to the principal Dirichlet character modulo  $k$  for some  $k \geq 1$ .

**Proposition 2.2** (Formulas for the inverse matrix sequences of  $u_{n,k}(f, w)$ ) *For all  $n \geq 1$  and  $1 \leq k \leq n$ , any fixed arithmetic function  $f$ , and  $w \in \mathbb{C}$ , we have that*

$$u_{n,k}^{(-1)}(f, w) = \sum_{m=1}^n \left( \sum_{d|(m,n)} f(d) p(n/d-k) \right) w^m.$$

Another formulation of the expression for the inverse sequence in the previous proposition is proved in Proposition 4.19 using constructions we define in the subsections leading up to that result.

## 2.3 A summary of applications of our new results

### 2.3.1 Forms of the type I sums

The identities and theorems we prove for the general sum case defined by (5) in Sect. 3 can be useful in constructing new undiscovered identities for well-known functions. These expansions which are phrased in terms of our new matrix factorization sequences are not well explored yet. So these expansions may yield additional information on the properties of the well-known function cases, which are examples of the type I sums. We give a few notable examples of summation identities which express classical functions and combinatorial objects in new ways below. These ways are to illustrate the style of an application of our new formulas. The concrete examples we cite below also serve to motivate the methods behind our new matrix factorizations for the special type I sums. We will set out to prove these identities in more generality in later sections of this article.

We obtain the following identities for Euler's totient function based on our new constructions:

$$\begin{aligned}\phi(n) &= \sum_{j=0}^n \sum_{k=1}^{j-1} \sum_{i=0}^j p(n-j)(-1)^{\lceil i/2 \rceil} \chi_{1,k}(j-k-G_i) [j-k-G_i \geq 1]_{\delta} \\ &\quad + [n=1]_{\delta} \\ \phi(n) &= \sum_{\substack{d=1 \\ (d,n)=1}}^n \left( \sum_{k=1}^{d+1} \sum_{i=1}^d \sum_{j=0}^k p(i+1-k)(-1)^{\lceil j/2 \rceil} \phi(k-G_j) \mu_{d,i} [k-G_j \geq 1]_{\delta} \right).\end{aligned}$$

To give another related example that applies to classical multiplicative functions, recall that we have a known representation for the Möbius function given as an exponential sum in terms of powers of the  $n$ th primitive roots of unity of the form [4, §16.6]

$$\mu(n) = \sum_{\substack{d=1 \\ (d,n)=1}}^n \exp\left(2\pi i \frac{d}{n}\right).$$

The *Mertens function*,  $M(x)$ , is defined as the summatory function over the Möbius function  $\mu(n)$  for all  $n \leq x$ . Using the definition of the Möbius function as one of our type I sums defined above, we have new expansions for the Mertens function given by (cf. Corollary 4.16)

$$\begin{aligned}M(x) &= \sum_{1 \leq k < j \leq n \leq x} \left( \sum_{i=0}^j p(n-j)(-1)^{\lceil i/2 \rceil} \chi_{1,k}(j-k-G_i) \right. \\ &\quad \times [j-k-G_i \geq 1]_{\delta} e^{2\pi i k/n} \Big).\end{aligned}$$

Finally, we can form another related polynomial sum of the type indicated above when we consider that the logarithm of the *cyclotomic polynomials* leads to the sums



$$\begin{aligned}\log \Phi_n(z) &= \sum_{\substack{1 \leq k \leq n \\ (k,n)=1}} \log \left( z - e^{2\pi i k/n} \right) \\ &= \sum_{1 \leq k < j \leq n} \left( \sum_{i=0}^j p(n-j)(-1)^{\lceil i/2 \rceil} \chi_{1,k}(j-k-G_i) \right. \\ &\quad \left. [j-k-G_i \geq 1]_{\delta} \log \left( z - e^{2\pi i k/n} \right) \right).\end{aligned}$$

### 2.3.2 Forms of the type II sums

The sums  $L_{f,g,k}(n)$  are sometimes referred to as *Anderson–Apostol sums* named after the authors who first defined them (cf. [2, §8.3] and [1]). Other variants and generalizations of these sums are studied in Refs. [5,6]. There are many number theoretic applications of the periodic sums factorized in this form. For example, the famous expansion of Ramanujan's sum  $c_q(n)$  is expressed as the following right-hand side divisor sum [3, §IX]:

$$c_q(n) = \sum_{\substack{d=1 \\ (d,n)=1}}^n e^{2\pi i dn/q} = \sum_{d|(q,n)} d \cdot \mu(q/d).$$

The applications of our new results to Ramanujan's sum include the expansions

$$\begin{aligned}c_n(x) &= [w^x] \left( \sum_{k=1}^n u_{n,k}^{(-1)}(\mu, w) \sum_{j \geq 0} (-1)^{\lceil j/2 \rceil} \mu(k - G_j) \right) \\ &= \sum_{k=1}^n \left( \sum_{d|(n,x)} d \cdot p(n/d - k) \right) \sum_{j \geq 0} (-1)^{\lceil j/2 \rceil} \mu(k - G_j),\end{aligned}$$

where the inverse matrices  $u_{n,k}^{(-1)}(\mu, w)$  are expanded according to Proposition 2.2. We then immediately have the following new results for the next special expansions of the generalized sum-of-divisors functions when  $\Re(s) > 0$ :

$$\begin{aligned}\sigma_s(n) &= n^s \zeta(s+1) \times \sum_{i=1}^{\infty} \sum_{k=1}^i \left( \sum_{d|(n,i)} d \cdot p(i/d - k) \right) \sum_{j \geq 0} \frac{(-1)^{\lceil j/2 \rceil} \mu(k - G_j)}{i^{s+1}} \\ &= n^s \zeta(s+1) \times \sum_{i=1}^n \sum_{k \geq 0} \frac{\mu \left( \frac{nk+i}{(n,i)} \right)}{(nk+i)^{s+1}}.\end{aligned}$$

Section 4.3 expands the left-hand side function  $g(x)$  in (6b) by considering a new indirect method involving the type II sums  $L_{f,g,k}(n)$ . The expansions we derive in that section employ results for discrete Fourier transforms of functions of the greatest

common divisor studied in [15, 18]. This method allows us to study the factorization forms in (6b) where we effectively bypass the complicated forms of the ordinary matrix coefficients  $u_{n,k}(f, w)$ . Results enumerating the ordinary matrices with coefficients given by  $u_{n,k}(f, w)$  are treated in Corollary 4.4 of Sect. 4.1. The discrete Fourier series methods we use to prove our theorems in these sections lead to the next key result proved in Theorem 4.14 which states that

$$\sum_{d|k} \sum_{r=0}^{k-1} d \cdot L_{f,g,r}(k) e\left(-\frac{rd}{k}\right) \mu(k/d) = \sum_{d|k} \phi(d) f(d) (k/d)^2 g(k/d),$$

where  $e(x) = \exp(2\pi i \cdot x)$  is standard notation for the complex exponential function.

## 2.4 Significance of our new results

Our new results provide generating function expansions for the type I and type II sums in the form of matrix-based factorization theorems. The matrix products involved in expressing the coefficients of these generating functions for arbitrary arithmetic functions  $f$  and  $g$  are closely related to the partition function  $p(n)$ . The known Lambert series factorization theorems proved in the references and which are summarized in the subsections on variants above demonstrate the flavor of the matrix-based expansions of these forms for ordinary divisor sums of the form  $\sum_{d|n} f(d)$ . Our extensions of these factorization theorem approached in the context of the new forms of the type I and type II sums similarly relate special arithmetic functions in number theory to partition functions and more additive branches of number theory. The last results proved in Sect. 4.3 are expanded in the spirit of these matrix factorization constructions using discrete Fourier transforms of functions (and sums of functions) evaluated at greatest common divisors. We pay special attention to illustrating our new results with many relevant examples and new identities expanding famous special number theoretic functions throughout the article.

## 3 Factorization theorems for sums of the first type

### 3.1 Inversion relations

We begin our exploration here by expanding an inversion formula which is analogous to Möbius inversion for ordinary divisor sums. We prove the following result which is the analog to the sequence inversion relation provided by the Möbius transform in the context of our sums over the integers relatively prime to  $n$  [12, cf. §2, §3].

**Proposition 3.1** (Inversion formula) *For all  $n \geq 2$ , there is a unique lower triangular sequence, denoted by  $\mu_{n,k}$ , which satisfies the lower triangular inversion relation (i.e., so that  $\mu_{n,d} = 0$  whenever  $n < d$ )*

1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	-1	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
-1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	-1	-1	1	0	0	0	0	0	0	0	0	0	0	0
1	0	-1	0	-1	0	1	0	0	0	0	0	0	0	0	0	0
-1	0	2	-1	0	0	-1	1	0	0	0	0	0	0	0	0	0
-1	0	0	0	1	0	-1	0	1	0	0	0	0	0	0	0	0
1	0	-1	1	0	-1	1	-1	-1	1	0	0	0	0	0	0	0
-1	0	1	0	0	0	-1	0	0	0	1	0	0	0	0	0	0
1	0	-1	0	0	0	1	0	0	-1	-1	1	0	0	0	0	0
3	0	-2	0	-2	0	2	0	-1	0	-1	0	1	0	0	0	0
-3	0	1	0	3	0	-1	-1	1	0	0	0	-1	1	0	0	0
-1	0	1	0	1	0	-1	0	0	0	0	0	-1	0	1	0	0
1	0	0	0	-2	0	0	1	0	0	1	-1	1	-1	-1	1	0
-3	0	2	0	2	0	-2	0	1	0	0	0	-1	0	0	0	1

$\mu_{n,k}$  for  $1 \leq n, k < 18$

Fig. 1 Inversion formula coefficient sequences

$$g(n) = \sum_{\substack{d=1 \\ (d,n)=1}}^n f(d) \iff f(n) = \sum_{d=1}^n g(d+1)\mu_{n,d}. \quad (7a)$$

Moreover, if we form the matrix  $(\mu_{i,j} [j \leq i]_{\delta})_{1 \leq i,j \leq n}$  for any  $n \geq 2$ , we have that the inverse sequence satisfies

$$\mu_{n,k}^{(-1)} = [(n+1, k) = 1]_{\delta} [k \leq n]_{\delta}. \quad (7b)$$

**Proof** Consider the  $(n-1) \times (n-1)$  matrix

$$([(i, j-1) = 1 \text{ and } j \leq i]_{\delta})_{1 \leq i,j < n}, \quad (8)$$

which effectively corresponds to the formula on the left-hand side of (7a) by applying the matrix to the vector of  $[f(1) f(2) \cdots f(n)]^T$  and extracting the  $(n+1)$ th column of the matrix formed by extracting the  $\{0, 1\}$ -valued coefficients of  $f(d)$ . Since  $\gcd(i, j-1) = 1$  for all  $i = j$  with  $i, j \geq 1$ , we see that the matrix (8) is lower triangular with ones on its diagonal. Thus the matrix is non-singular and its unique inverse, which we denote by  $(\mu_{i,j})_{1 \leq i,j < n}$ , leads to the sum on the right-hand side of the sum in (7a) when we shift  $n \mapsto n+1$ . The second equation stated in (7b) restates the form of the first matrix of  $\mu_{i,j}$  as on the right-hand side of (7a).  $\square$

**Remark 3.2** Figure 1 provides a listing of the relevant analogs to the role of the Möbius function in a Möbius inversion transform of the ordinary divisor sum over an arithmetic function. We do not know of a comparatively simple closed-form function for the sequence of  $\mu_{n,k}$  [16, cf. A096433]. However, we readily see by construction that the sequence and its inverse satisfy

$$\sum_{\substack{d=1 \\ (d,n)=1}}^n \mu_{d,k} = 0,$$

$$\sum_{\substack{d=1 \\ (d,n)=1}}^n \mu_{d,k}^{(-1)} = \phi(n),$$

where  $\phi(n)$  is Euler's totient function. The first columns of the  $\mu_{n,1}$  appear in the integer sequences database as the entry [16, A096433].

### 3.2 Exact formulas for the factorization matrices

The next result is key to proving the exact formulas for the matrix sequences,  $t_{n,k}$  and  $t_{n,k}^{(-1)}$ , and their expansions by the partition functions defined in the introduction. We prove the following result first as a lemma which we will use in the proof of Theorem 2.1 given below. The first several rows of the matrix sequence  $t_{n,k}$  and its inverse implicit to the factorization theorem in (6) are tabulated in Fig. 2 for intuition on the formulas we prove in the next proposition and following theorem.

**Lemma 3.3** (A convolution identity for relatively prime integers) *For all natural numbers  $n \geq 2$  and  $k \geq 1$  with  $k \leq n$ , we have the following expression for the principal Dirichlet character modulo  $k$ :*

$$\sum_{j=1}^n t_{j,k} p(n-j) = \chi_{1,k}(n).$$

Equivalently, we have that

$$t_{n,k} = \sum_{i=0}^n (-1)^{[i/2]} \chi_{1,k}(n - G_i) [n - G_i \geq k + 1]_{\delta}$$

$$= \sum_{b=\pm 1} \left[ \sum_{i=0}^{\left\lfloor \frac{\sqrt{24(n-k-1)+1}-b}{6} \right\rfloor} (-1)^{[i/2]} \chi_{1,k} \left( n - \frac{i(3i-b)}{2} \right) \right]. \quad (9)$$

**Proof** We begin by noticing that the right-hand side expression in the statement of the lemma is equal to  $\mu_{n,k}^{(-1)}$  by the construction of the sequence in Proposition 3.1. Next, we see that the factorization in (6a) is equivalent to the expansion

1														
0	1													
-1	-1	1												
-1	0	0	1											
-1	-1	-2	-1	1										
0	0	0	0	0	1									
0	0	0	-1	-1	-1	1								
1	0	-1	0	-1	-1	0	1							
1	1	1	0	-2	0	-1	-1	1						
1	0	1	0	1	1	-1	0	0	1					
1	1	0	1	1	0	-1	-1	-2	-1	1				
1	0	1	0	1	0	0	0	0	0	0	1			
0	1	1	1	1	0	-1	0	0	-1	-1	-1	1		
0	-1	0	0	-1	-1	2	0	-1	-1	-1	-1	0	1	

(i)  $t_{n,k}$

1																				
0	1																			
1	1	1																		
1	0	0	1																	
4	3	2	1	1																
0	0	0	0	0	1															
5	3	2	2	1	1	1														
4	4	3	1	1	1	0	1													
15	11	8	5	4	2	1	1	1												
-1	-1	-1	1	0	0	1	0	0	1											
32	24	18	12	9	6	4	3	2	1	1										
-6	-4	-3	-1	-1	0	0	0	0	0	0	0	1								
24	17	13	12	8	7	6	3	2	2	1	1	1								

$$(ii) \ t_{n,k}^{(-1)}$$

**Fig. 2** The factorization matrices,  $t_{n,k}$  and  $t_{n,k}^{(-1)}$ , for  $1 \leq n, k < 14$

$$\sum_{d=1}^{n-1} f(d) \mu_{n,d}^{(-1)} = \sum_{j=1}^n \sum_{k=1}^j p(n-j) t_{j,k} \cdot f(k). \quad (10)$$

Since  $\mu_{n,k}^{(-1)} = [(n+1, k) = 1]_\delta$ , we may take the coefficients of  $f(k)$  on each side of (10) for each  $1 \leq k < n$  to establish the result we have claimed in this lemma. The equivalent statement of the result follows by a generating function argument applied to the product that generates the left-hand side Cauchy product in (9).

**Proof** (Theorem 2.1: Proof of (i)) It is plain to see by the considerations in our construction of the factorization theorem that both matrix sequences are lower triangular. Thus, we need only consider the cases where  $n \leq k$ . By a convolution of generating functions, the identity in Lemma 3.3 shows that

$$t_{n,k} = \sum_{j=k}^n [q^{n-j}](q; q)_{\infty} \cdot [(j+1, k) = 1]_{\delta}.$$

Then shifting the index of summation in the previous equation implies (i).

**Proof** (Theorem 2.1: Proof of (ii)) To prove (ii), we consider the factorization theorem when  $f(n) := t_{n,r}^{(-1)}$  for some fixed  $r \geq 1$ . We then expand (6a) as

$$\begin{aligned} \sum_{\substack{d=1 \\ (d,n)=1}}^n t_{d,r}^{(-1)} &= [q^n] \frac{1}{(q; q)_{\infty}} \sum_{n \geq 1} \sum_{k=1}^{n-1} t_{n,k} \cdot t_{k,r}^{(-1)} \cdot q^n \\ &= \sum_{j=1}^n p(n-j) \times \sum_{k=1}^{j-1} t_{j,k} t_{k,r}^{(-1)} \\ &= \sum_{j=1}^n p(n-j) [r = j-1]_{\delta} \\ &= p(n-1-r). \end{aligned}$$

Hence we may perform the inversion by Proposition 3.1 to the left-hand side sum in the previous equations to obtain our stated result.

**Remark 3.4** (Relations to the Lambert series factorization theorems) We notice that by inclusion-exclusion applied to the right-hand side of (6a), we may write our matrices  $t_{n,k}$  in terms of the triangular sequence expanded as differences of restricted partitions in (3). For example, when  $k := 12$  we see that

$$\sum_{n \geq 12} [(n, 12) = 1]_{\delta} q^n = \frac{q^{12}}{1-q} - \frac{q^{12}}{1-q^2} - \frac{q^{12}}{1-q^3} + \frac{q^{12}}{1-q^6}.$$

In general, when  $k > 1$  we can expand

$$\sum_{n \geq k} [(n, k) = 1]_{\delta} q^n = \sum_{d|k} \frac{q^k \mu(d)}{1-q^d}.$$

Thus we can relate the triangles  $t_{n,k}$  in this article to the  $s_{n,k} = [q^n](q; q)_\infty q^k / (1 - q^k)$  for  $n \geq k \geq 1$  employed in the expansions from the references as follows:

$$t_{n,k} = \begin{cases} s_{n,k}, & k = 1 \\ \sum_{d|k} s_{n+1-k+d,d} \cdot \mu(d), & k > 1. \end{cases}$$

### 3.3 Completing the proofs of the main applications

We remark that as in the Lambert series factorization results from Ref. [8], we have three primary types of expansion identities that we will consider for any fixed choice of the arithmetic function  $f$  in the forms of

$$\sum_{\substack{d=1 \\ (d,n)=1}}^n f(d) = \sum_{j=1}^n \sum_{k=1}^{j-1} p(n-j)t_{j-1,k} f(k) + f(1)[n=1]_\delta, \quad (11a)$$

$$\sum_{k=1}^{n-1} t_{n-1,k} f(k) = \sum_{j=1}^n \sum_{\substack{d=1 \\ (d,j)=1}}^j [q^{n-j}](q; q)_\infty \cdot f(d) - [q^{n-1}](q; q)_\infty \cdot f(1), \quad (11b)$$

and the corresponding inverted formula providing that

$$f(n) = \sum_{k=1}^n t_{n,k}^{(-1)} \left( \sum_{\substack{j \geq 0 \\ k+1-G_j > 0}} (-1)^{\left\lceil \frac{j}{2} \right\rceil} T_f(k+1-G_j) - [q^k](q; q)_\infty \cdot f(1) \right). \quad (11c)$$

Now the applications cited in the introduction follow immediately and require no further proof other than to cite these results for the respective special cases of  $f$ . We provide other similar corollaries and examples of these factorization theorem results below.

**Example 3.5** (Sum-of-divisors functions) For any  $\alpha \in \mathbb{C}$ , the expansion identity given in (11c) also implies the following new formula for the generalized sum-of-divisors functions,  $\sigma_\alpha = \sum_{d|n} d^\alpha$ :

$$\sigma_\alpha(n) = \sum_{d|n} \sum_{k=1}^d t_{d,k}^{(-1)} \left( \sum_{\substack{j \geq 0 \\ k+1-G_j > 0}} (-1)^{\left\lceil \frac{j}{2} \right\rceil} \phi_\alpha(k+1-G_j) - [q^k](q; q)_\infty \right).$$

In particular, when  $\alpha := 0$  we obtain the next identity for the divisor function  $d(n) \equiv \sigma_0(n)$  expanded in terms of Euler's totient function,  $\phi(n)$ .

$$d(n) = \sum_{d|n} \sum_{k=1}^d t_{d,k}^{(-1)} \left( \sum_{\substack{j \geq 0 \\ k+1-G_j > 0}} (-1)^{\lceil \frac{j}{2} \rceil} \phi(k+1-G_j) - [q^k](q; q)_{\infty} \right).$$

**Remark 3.6** There are also numerous noteworthy applications of the expansions of the type I sums from this section in the context of exact (and asymptotic) expansions of named partition functions. For instance, Rademacher's exact series formula for the partition function  $p(n)$  involves Dedekind sums implicitly expanded through sums of this type. Similarly, an asymptotic approximation for the named special function  $q(n) = [q^n](-q; q)_{\infty}$  [16, A000009] which counts the number of partitions of  $n$  into distinct parts involves an infinite series over modified Bessel functions and nested Kloosterman sums [11, §26.10(vi)]. We have not attempted to study the usefulness of our new finite sums in these contexts in deconstructing asymptotic properties in these more famous examples of partition formulas. A detailed treatment is nonetheless suggested as an exercise to readers which may unravel some undiscovered combinatorial twists to the expansions of such sums.

**Example 3.7** (Menon's identity and related arithmetical sums) We can use our new results proved in this section to expand new identities for known closed-forms of special arithmetic sums. For example, *Menon's identity* [17] states that

$$\phi(n)d(n) = \sum_{\substack{1 \leq k \leq n \\ (k,n)=1}} \gcd(k-1, n),$$

where  $\phi(n)$  is Euler's totient function and  $d(n) = \sigma_0(n)$  is the divisor function. We can then expand the right-hand side of Menon's identity as follows:

$$\begin{aligned} \phi(n)d(n) &= \sum_{j=0}^n \sum_{k=1}^{j-1} \sum_{i=0}^j p(n-j)(-1)^{\lceil i/2 \rceil} \chi_{1,k}(j-k-G_i) \\ &\quad \times [j-k-G_i \geq 1]_{\delta} \gcd(k-1, n). \end{aligned}$$

As another application, we show a closely related identity considered by Tóth [17]. Tóth's identity states that (cf. [7]) for an arithmetic function  $f$  we have

$$\sum_{\substack{1 \leq k \leq n \\ (k,n)=1}} f(\gcd(k-1, n)) = \phi(n) \cdot \sum_{d|n} \frac{(\mu * f)(d)}{\phi(d)}.$$

We can use our new formulas to write a gcd-related recurrence relation for  $f$  in two steps. First, we observe that the right-hand side divisor sum in the previous equation is expanded by



$$\begin{aligned} \sum_{d|n} \frac{(\mu * f)(d)}{\phi(d)} &= \frac{1}{\phi(n)} \cdot \sum_{j=0}^n \sum_{k=1}^{j-1} \sum_{i=0}^j p(n-j)(-1)^{\lceil i/2 \rceil} \chi_{1,k}(j-k-G_i) \\ &\quad \times [j-k-G_i \geq 1]_{\delta} f(\gcd(k-1, n)) \\ &\quad + f(1) [n=1]_{\delta}. \end{aligned}$$

The notation  $(f * g)(n) := \sum_{d|n} f(d)g(n/d)$  denotes the Dirichlet convolution of two arithmetic functions,  $f$  and  $g$ .

Next, by Möbius inversion and noting that the Dirichlet inverse of  $\mu(n)$  is  $\mu * 1 = \varepsilon$ , where  $\varepsilon(n) = \delta_{n,1}$  is the multiplicative identity with respect to Dirichlet convolution, we can express  $f(n)$  as follows:

$$\begin{aligned} f(n) &= \sum_{d|n} \sum_{r|d} \sum_{j=0}^r \sum_{k=1}^{j-1} \sum_{i=0}^j \left[ p(r-j)(-1)^{\lceil i/2 \rceil} \chi_{1,k}(j-k-G_i) [j-k-G_i \geq 1]_{\delta} \right. \\ &\quad \left. \times f(\gcd(k-1, r)) \frac{\phi(d)}{\phi(r)} \mu\left(\frac{d}{r}\right) \right] + f(1) \cdot \sum_{d|n} \phi(d) \mu(d). \end{aligned}$$

## 4 Factorization theorems for sums of the second type

### 4.1 Formulas for the inverse matrices

It happens that in the case of the series expansions we defined in (6b) of the introduction, the corresponding terms of the inverse matrices  $u_{n,k}^{(-1)}(f, w)$  satisfy considerably simpler formulas than the ordinary matrix entries themselves. We first prove a partition-related explicit formula for these inverse matrices as Proposition 2.2 and then discuss several applications of this result.

**Proof of Proposition 2.2** Let  $1 \leq r \leq n$  and for some suitably chosen arithmetic function  $g$  define

$$u_{n,r}^{(-1)}(f, w) := \sum_{m=1}^n L_{f,g,m}(n) w^m. \quad (\text{i})$$

By directly expanding the series on the right-hand side of (6b), we obtain that

$$\begin{aligned} g(n) &= \sum_{j=0}^n \left( \sum_{k=1}^j u_{j,k}(f, w) \cdot u_{k,r}^{(-1)}(f, w) \right) p(n-j) \\ &= \sum_{j=0}^n p(n-j) [j=r]_{\delta} = p(n-r). \end{aligned}$$

Hence the choice of the function  $g$  which satisfies (i) above is given by  $g(n) := p(n - r)$ . The claimed expansion of the inverse matrices then follows.

**Proposition 4.1** *We in fact prove the following generalized identity for arbitrary arithmetic functions  $f, g$ :*

$$L_{f,g,m}(n) = \sum_{k=1}^n \sum_{d|(m,n)} f(d) p\left(\frac{n}{d} - k\right) \times \sum_{\substack{j \geq 0 \\ k > G_j}} (-1)^{\left\lceil \frac{j}{2} \right\rceil} g(k - G_j). \quad (12)$$

**Proof** Since the coefficients on the left-hand side of the next equation correspond to a right-hand side matrix product as

$$[q^n](q; q)_{\infty} \sum_{m \geq 1} g(m) q^m = \sum_{k=1}^n u_{n,k}(f, w) \sum_{m=1}^k L_{f,g,m}(k) w^m,$$

we can invert the matrix product on the right to obtain that

$$\begin{aligned} \sum_{m=1}^k L_{f,g,m}(k) w^m &= \sum_{k=1}^n \left( \sum_{m=1}^n \sum_{d|(n,m)} f(d) p\left(\frac{n}{d} - k\right) \cdot w^m \right) \\ &\quad \times [q^k](q; q)_{\infty} \sum_{m \geq 1} g(m), \end{aligned}$$

so that by comparing coefficients of  $w^m$  for  $1 \leq m \leq n$ , we obtain (12).

**Corollary 4.2** (A new formula for Ramanujan sums) *For any natural numbers  $x, m \geq 1$ , we have that*

$$c_x(m) = \sum_{k=1}^x \sum_{d|(m,x)} d \cdot p\left(\frac{x}{d} - k\right) \times \sum_{\substack{j \geq 0 \\ k > G_j}} (-1)^{\left\lceil \frac{j}{2} \right\rceil} \mu(k - G_j).$$

**Proof** The Ramanujan sums correspond to the special case of Proposition 4.1 where  $f(n) := n$  is the identity function and  $g(n) := \mu(n)$  is the Möbius function.

**Remark 4.3** We define the following shorthand notation:

$$\widehat{L}_{f,g}(n; w) := \sum_{m=1}^n L_{f,g,m}(n) w^m.$$

In this notation we have that  $u_{n,k}^{(-1)}(f, w) = \widehat{L}_{f,g}(n; w)$  when  $g(n) := p(n - k)$ . Moreover, if we denote by  $T_n(x)$  the polynomial  $T_n(x) := 1 + x + x^2 + \cdots + x^{n-1} = \frac{x^n - 1}{x - 1}$ , then we have expansions of these sums as convolved ordinary divisor sums “twisted” by polynomial terms of the form

$$\begin{aligned}\widehat{L}_{f,g}(n; w) &= \sum_{d|n} w^d f(d) T_{n/d}(w^d) g\left(\frac{n}{d}\right) \\ &= (w^n - 1) \times \sum_{d|n} \frac{w^d}{w^d - 1} f(d) g\left(\frac{n}{d}\right).\end{aligned}\quad (13)$$

The Dirichlet inverse of these divisor sums is also not difficult to express, though we will not give its formula here. These sums lead to a first formula for the more challenging expressions for the ordinary matrix entries  $u_{n,k}(f, w)$  given by the next corollary.

**Corollary 4.4** (A formula for the ordinary matrix entries) *To distinguish notation, let  $\widehat{P}_{f,k}(n; w) := \widehat{L}_{f(n),p(n-k)}(n; w)$ , which is an immediate shorthand for the matrix inverse terms  $u_{n,k}^{(-1)}(f, w)$  that we will precisely enumerate below. For  $n \geq 1$  and  $1 \leq k < n$ , we have the following formula:*

$$\begin{aligned}u_{n,k}(f, w) &= -\frac{(1-w)^2}{w^2 \cdot (1-w^n)(1-w^k) \cdot f(1)^2} \\ &\times \left( \widehat{P}_{f,k}(n; w) + \sum_{m=1}^{n-k-1} \left( \frac{w-1}{wf(1)} \right)^m \right. \\ &\times \left[ \sum_{k \leq i_1 < \dots < i_m < n} \frac{\widehat{P}_{f,k}(i_1; w) \widehat{P}_{f,i_1}(i_2; w) \widehat{P}_{f,i_2}(i_3; w) \cdots \widehat{P}_{f,i_{m-1}}(i_m; w) \widehat{P}_{f,i_m}(n; w)}{(1-w^{i_1})(1-w^{i_2}) \cdots (1-w^{i_m})} \right] \Bigg)\end{aligned}$$

When  $k = n$ , we have that

$$u_{n,n}(f, w) = \frac{1-w}{w(1-w^n) \cdot f(1)}.$$

**Proof** This follows inductively from the inversion relation between the coefficients of a matrix and its inverse. For any invertible lower triangular  $n \times n$  matrix  $(a_{i,j})_{1 \leq i,j \leq n}$ , we can express a non-recursive formula for the inverse matrix entries as follows:

$$\begin{aligned}a_{n,k}^{(-1)} &= \frac{1}{a_{n,n}} \left( -\frac{a_{n,k}}{a_{k,k}} \right. \\ &+ \sum_{m=1}^{n-k-1} (-1)^{m+1} \left[ \sum_{k \leq i_1 < \dots < i_m < n} \frac{a_{i_1,k} a_{i_2,i_1} a_{i_3,i_2} \cdots a_{i_m,i_{m-1}} a_{n,i_m}}{a_{k,k} a_{i_1,i_1} a_{i_2,i_2} \cdots a_{i_m,i_m}} \right] \Bigg) [k < n]_{\delta} \\ &+ \frac{[k = n]_{\delta}}{a_{n,n}}.\end{aligned}\quad (14)$$

The proof of our result is then just an application of the formula in (14) when  $a_{n,k} := u_{n,k}^{-1}(f, w)$ . While the identity in (14) is not immediately obvious from the known inversion formulas between inverse matrices in the form of

**Table 1** The simplified matrix entries  $\widehat{u}_{n,k}(f, w)$  for  $1 \leq n, k \leq 6$  where  $\widehat{f}(n) = \frac{w^n}{w^n - 1} \cdot f(n)$  for arithmetic functions  $f$  such that  $f(1) \neq 0$

$\frac{1}{\widehat{f}(1)}$	0	0	0	0	0
$-\frac{\widehat{f}(2)}{\widehat{f}(1)^2} - \frac{1}{\widehat{f}(1)}$	$\frac{1}{\widehat{f}(1)}$	0	0	0	0
$\frac{\widehat{f}(2)}{\widehat{f}(1)^2} - \frac{\widehat{f}(3)}{\widehat{f}(1)^2} - \frac{1}{\widehat{f}(1)}$	$-\frac{1}{\widehat{f}(1)}$	$\frac{1}{\widehat{f}(1)}$	0	0	0
$\frac{\widehat{f}(2)^2}{\widehat{f}(1)^3} + \frac{\widehat{f}(2)}{\widehat{f}(1)^2} + \frac{\widehat{f}(3)}{\widehat{f}(1)^2} - \frac{\widehat{f}(4)}{\widehat{f}(1)^2}$	$-\frac{\widehat{f}(2)}{\widehat{f}(1)^2} - \frac{1}{\widehat{f}(1)}$	$-\frac{1}{\widehat{f}(1)}$	$\frac{1}{\widehat{f}(1)}$	0	0
$-\frac{\widehat{f}(2)^2}{\widehat{f}(1)^3} + \frac{\widehat{f}(3)}{\widehat{f}(1)^2} + \frac{\widehat{f}(4)}{\widehat{f}(1)^2} - \frac{\widehat{f}(5)}{\widehat{f}(1)^2}$	$\frac{\widehat{f}(2)}{\widehat{f}(1)^2}$	$-\frac{1}{\widehat{f}(1)}$	$-\frac{1}{\widehat{f}(1)}$	$\frac{1}{\widehat{f}(1)}$	0
$-\frac{\widehat{f}(2)^2}{\widehat{f}(1)^3} + \frac{2\widehat{f}(3)\widehat{f}(2)}{\widehat{f}(1)^3} + \frac{\widehat{f}(4)}{\widehat{f}(1)^2} + \frac{\widehat{f}(5)}{\widehat{f}(1)^2} - \frac{\widehat{f}(6)}{\widehat{f}(1)^2} + \frac{1}{\widehat{f}(1)}$	$\frac{\widehat{f}(2)}{\widehat{f}(1)^2} - \frac{\widehat{f}(3)}{\widehat{f}(1)^2}$	$-\frac{\widehat{f}(2)}{\widehat{f}(1)^2}$	$-\frac{1}{\widehat{f}(1)}$	$-\frac{1}{\widehat{f}(1)}$	$\frac{1}{\widehat{f}(1)}$

$$a_{n,k}^{(-1)} = \frac{[n=k]_\delta}{a_{n,n}} - \frac{1}{a_{n,n}} \sum_{j=1}^{n-k-1} a_{n,j} a_{j,k}^{(-1)},$$

the result is easily obtained by induction on  $n$  so we do not prove it here.

## 4.2 Formulas for simplified variants of the ordinary matrices

In Corollary 4.4 we proved an exact, however somewhat implicit and unsatisfying, expansion of the ordinary matrix entries  $u_{n,k}(f, w)$  by sums of weighted products of the inverse matrices  $u_{n,k}^{(-1)}(f, w)$  expressed in closed form through Proposition 2.2. We will now develop the machinery needed to more precisely express the ordinary forms of these matrices for general cases of the indeterminate indexing parameter  $w \in \mathbb{C}$ .

**Remark 4.5** (Simplifications of the matrix terms) Using the formula for the coefficients of  $u_{n,k}(f, w)$  in (6b) expanded by (13), we can simplify the form of the matrix entries we seek closed-form expressions for in the next calculations. In particular, we make the following definitions for  $1 \leq k \leq n$ :

$$\begin{aligned}\widehat{f}(n) &:= \frac{w^n}{w^n - 1} f(n) \\ \widehat{u}_{n,k}(f, w) &:= (w^k - 1)u_{n,k}(f, w).\end{aligned}$$

Then an equivalent formulation of finding the exact formulas for  $u_{n,k}(f, w)$  is to find exact expressions expanding the triangular sequence of  $\widehat{u}_{n,k}(f, w)$  satisfying

$$\sum_{\substack{j \geq 0 \\ n-G_j > 0}} (-1)^{\left\lceil \frac{j}{2} \right\rceil} g(n - G_j) = \sum_{k=1}^n \widehat{u}_{n,k}(f, w) \sum_{d|k} \widehat{f}(d) g\left(\frac{n}{d}\right).$$

**Table 2** The multiple convolution function  $D_f(n)$  for  $2 \leq n \leq 16$  where  $\widehat{f}(n) := \frac{w^n}{w^n - 1} \cdot f(n)$  for an arbitrary arithmetic function  $f$  such that  $f(1) \neq 0$

$n$	$D_f(n)$	$n$	$D_f(n)$	$n$	$D_f(n)$
2	$-\frac{\widehat{f}(2)}{\widehat{f}(1)^2}$	7	$-\frac{\widehat{f}(7)}{\widehat{f}(1)^2}$	12	$\frac{2\widehat{f}(3)\widehat{f}(4)+2\widehat{f}(2)\widehat{f}(6)-\widehat{f}(1)\widehat{f}(12)}{\widehat{f}(1)^3}$ $-\frac{3\widehat{f}(2)^2\widehat{f}(3)}{\widehat{f}(1)^4}$
3	$-\frac{\widehat{f}(3)}{\widehat{f}(1)^2}$	8	$\frac{2\widehat{f}(2)\widehat{f}(4)-\widehat{f}(1)\widehat{f}(8)}{\widehat{f}(1)^3} - \frac{\widehat{f}(2)^3}{\widehat{f}(1)^4}$	13	$-\frac{\widehat{f}(13)}{\widehat{f}(1)^2}$
4	$\frac{\widehat{f}(2)^2-\widehat{f}(1)\widehat{f}(4)}{\widehat{f}(1)^3}$	9	$\frac{\widehat{f}(3)^2-\widehat{f}(1)\widehat{f}(9)}{\widehat{f}(1)^3}$	14	$\frac{2\widehat{f}(2)\widehat{f}(7)-\widehat{f}(1)\widehat{f}(14)}{\widehat{f}(1)^3}$
5	$-\frac{\widehat{f}(5)}{\widehat{f}(1)^2}$	10	$\frac{2\widehat{f}(2)\widehat{f}(5)-\widehat{f}(1)\widehat{f}(10)}{\widehat{f}(1)^3}$	15	$\frac{2\widehat{f}(3)\widehat{f}(5)-\widehat{f}(1)\widehat{f}(15)}{\widehat{f}(1)^3}$
6	$\frac{2\widehat{f}(2)\widehat{f}(3)-\widehat{f}(1)\widehat{f}(6)}{\widehat{f}(1)^3}$	11	$-\frac{\widehat{f}(11)}{\widehat{f}(1)^2}$	16	$\frac{\widehat{f}(2)^4}{\widehat{f}(1)^5} - \frac{3\widehat{f}(4)\widehat{f}(2)^2}{\widehat{f}(1)^4}$ $+\frac{\widehat{f}(4)^2+2\widehat{f}(2)\widehat{f}(8)}{\widehat{f}(1)^3} - \frac{\widehat{f}(16)}{\widehat{f}(1)^2}$

We will obtain precisely such formulas in the next few results. Table 1 provides the first few rows of our simplified matrix entries.

**Definition 4.6** (Special multiple convolutions) For  $n, j \geq 1$ , we define the following nested  $j$ -convolutions of the function  $\widehat{f}(n)$  [9]:

$$\text{ds}_j(f; n) = \begin{cases} (-1)^{\delta_{n,1}} \widehat{f}(n) & \text{if } j = 1 \\ \sum_{\substack{d|n \\ d>1}} \widehat{f}(d) \text{ds}_{j-1}\left(f; \frac{n}{d}\right) & \text{if } j \geq 2. \end{cases}$$

Then we define our primary multiple convolution function of interest as

$$D_f(n) := \sum_{j=1}^n \frac{\text{ds}_{2j}(f; n)}{\widehat{f}(1)^{2j+1}}.$$

For example, the first few cases of  $D_f(n)$  for  $2 \leq n \leq 16$  are computed in Table 2. The examples in the table should clarify precisely what multiple convolutions we are defining by the function  $D_f(n)$ . Namely, a signed sum of all possible ordinary  $k$  Dirichlet convolutions of  $\widehat{f}$  with itself evaluated at  $n$ .

**Lemma 4.7** We claim that for all  $n \geq 1$

$$(D_f * \widehat{f})(n) \equiv \sum_{d|n} f(d) D_f(n/d) = -\frac{\widehat{f}(n)}{\widehat{f}(1)} + \varepsilon(n),$$

where  $\varepsilon(n) \equiv \delta_{n,1}$  is the multiplicative identity function with respect to Dirichlet convolution.

**Proof** We note that the statement of the lemma is equivalent to showing that

$$\left(D_f + \frac{\varepsilon}{\widehat{f}(1)}\right)(n) = \widehat{f}^{-1}(n). \quad (15)$$

A general recursive formula for the inverse of  $\widehat{f}(n)$  is given by [2]

$$\widehat{f}^{-1}(n) = \left(-\frac{1}{\widehat{f}(1)} \sum_{\substack{d|n \\ d>1}} \widehat{f}(d) \widehat{f}^{-1}(n/d)\right) [n > 1]_\delta + \frac{1}{\widehat{f}(1)} [n = 1]_\delta.$$

This definition is almost how we defined  $\text{ds}_j(f; n)$  above. Let's see how to modify this recurrence relation to obtain the formula for  $D_f(n)$ . We can recursively substitute in the formula for  $\widehat{f}^{-1}(n)$  until we hit the point where successive substitutions only leave the base case of  $\widehat{f}^{-1}(1) = 1/\widehat{f}(1)$ . This occurs after  $\Omega(n)$  substitutions where  $\Omega(n)$  denotes the number of prime factors of  $n$  counting multiplicity. We can write the nested formula for  $\text{ds}_j(f; n)$  as

$$\text{ds}_j(f; n) = \widehat{f}_\pm * \underbrace{(\widehat{f} - \widehat{f}(1)\varepsilon) * \cdots * (\widehat{f} - \widehat{f}(1)\varepsilon)}_{j-1 \text{ factors}}(n),$$

where we define  $\widehat{f}_\pm(n) := \widehat{f}(n) [n > 1]_\delta - \widehat{f}(1) [n = 1]_\delta$ . Next, define the nested  $k$ -convolutions  $C_k(n)$  recursively by

$$C_k(n) = \begin{cases} \widehat{f}(n) - \widehat{f}(1)\varepsilon(n) & \text{if } k = 1 \\ \sum_{d|n} (\widehat{f}(d) - \widehat{f}(1)\varepsilon(d)) C_{k-1}(n/d) & \text{if } k \geq 2. \end{cases}$$

Then we can express the inverse of  $\widehat{f}(n)$  using this definition as follows:

$$\widehat{f}^{-1}(n) = \sum_{d|n} \widehat{f}(d) \left[ \sum_{j=1}^{\Omega(n)} \frac{C_{2k}(n/d)}{\widehat{f}(1)^{\Omega(n)+1}} - \frac{\varepsilon(n/d)}{\widehat{f}(1)^2} \right].$$

Then based on the initial conditions for  $k = 1$  (or  $j = 1$ ) in the definitions of  $C_k(n)$  (or  $\text{ds}_j(f; n)$ ), we see that the function in (15) is in fact the inverse of  $\widehat{f}(n)$ .

**Proposition 4.8** *For all  $n \geq 1$  and  $1 \leq k \leq n$ , we have that*

$$\sum_{i=0}^{n-1} p(i) \widehat{u}_{n-i,k}(f, w) = D_f\left(\frac{n}{k}\right) [n \equiv 0 \pmod{k}]_\delta + \frac{1}{\widehat{f}(1)} [n = k]_\delta.$$

**Proof** We notice that Lemma 4.7 implies that

$$\varepsilon(n) = \left( \left( D_f + \frac{\varepsilon}{\widehat{f}(1)} \right) * \widehat{f} \right) (n),$$

where  $\varepsilon(n)$  is the multiplicative identity for Dirichlet convolutions. The last equation implies that

$$g(n) = \left( \left( D_f + \frac{\varepsilon}{\widehat{f}(1)} \right) * \widehat{f} * g \right) (n). \quad (\text{i})$$

Additionally, we know by the expansion of (6b) and that  $\widehat{u}_{n,n}(f, w) = 1/\widehat{f}(1)$  that we also have the expansion

$$g(n) = \sum_{k \geq 1} \left[ \sum_{j=0}^{n-1} p(j) \widehat{u}_{n-j,k} \right] \sum_{d|k} \widehat{f}(d) g(k/d). \quad (\text{ii})$$

So we can equate (i) and (ii) to see that

$$\sum_{j=0}^{n-1} p(j) \widehat{u}_{n-j,k} = D_f \left( \frac{n}{k} \right) [k|n]_{\delta} + \frac{[n=k]_{\delta}}{\widehat{f}(1)}.$$

This establishes our claim.

**Corollary 4.9** (An exact formula for the ordinary matrix entries) *For all  $n \geq 1$  and  $1 \leq k \leq n$ , we have that*

$$\begin{aligned} \widehat{u}_{n,k}(f, w) = \sum_{\substack{j \geq 0 \\ n-G_j > 0}} (-1)^{\left\lceil \frac{j}{2} \right\rceil} \left( D_f \left( \frac{n-G_j}{k} \right) [n-G_j \equiv 0 \pmod{k}]_{\delta} \right. \\ \left. + \frac{1}{\widehat{f}(1)} [n-G_j = k]_{\delta} \right). \end{aligned}$$

**Proof** This is an immediate consequence of Proposition 4.8 by noting that the generating function for  $p(n)$  is  $(q; q)_{\infty}^{-1}$  and that

$$(q; q)_{\infty} = \sum_{j \geq 0} (-1)^{\left\lceil \frac{j}{2} \right\rceil} q^{G_j}.$$

□

### 4.3 The general matrices expressed through discrete Fourier transforms

The proof of the result given in Theorem 4.14 builds on several key ideas for discrete Fourier transforms of the greatest common divisor function  $(k, n) \equiv \gcd(k, n)$  developed in [18]. We adopt the common convention that the function  $e(x)$  denotes the

exponential function  $e(x) := e^{2\pi i x}$ . Throughout the remainder of this section we take  $k \geq 1$  to be fixed and consider divisor sums of the following form which are periodic with respect to  $k$ :

$$L_{f,g,k}(n) := \sum_{d|(n,k)} f(d)g\left(\frac{n}{d}\right).$$

In [18] these sums are called  $k$ -convolutions of  $f$  and  $g$ . We will first need to discuss some terminology related to discrete Fourier transforms.

A *discrete Fourier transform* (DFT) maps a finite sequence of complex numbers  $\{f[n]\}_{n=0}^{N-1}$  onto their associated Fourier coefficients  $\{F[k]\}_{k=0}^{N-1}$  defined according to the following reversion formulas relating these sequences:

$$\begin{aligned} F[k] &= \sum_{n=0}^{N-1} f[n]e\left(-\frac{kn}{N}\right) \\ f[k] &= \frac{1}{N} \sum_{n=0}^{N-1} F[k]e\left(\frac{kn}{N}\right). \end{aligned}$$

The discrete Fourier transform of functions of the greatest common divisor, which we will employ repeatedly to prove Theorem 4.14, is summarized by the formula in the next lemma [15, 18].

**Lemma 4.10** (Typical relations between periodic divisor sums and Fourier series) *If we take any two arithmetic functions  $f$  and  $g$ , we can express periodic divisor sums modulo any  $k \geq 1$  of the form*

$$s_k(f, g; n) := \sum_{d|(n,k)} f(d)g(k/d) = \sum_{m=1}^k a_k(f, g; m) \cdot e^{2\pi i \cdot mn/k}, \quad (16a)$$

where the discrete Fourier coefficients in the right-hand side equation are given by

$$a_k(f, g; m) = \sum_{d|(m,k)} g(d)f(k/d) \cdot \frac{d}{k}. \quad (16b)$$

**Proof** For a proof of these relations consult Refs. [2, §8.3] and [11, cf. §27.10]. These relations are also related to the gcd-transformations proved in [15, 18].

**Notation 4.11** The function  $c_m(a)$  defined by

$$c_m(a) := \sum_{\substack{k=1 \\ (k,m)=1}}^m e\left(\frac{ka}{m}\right),$$



is Ramanujan's sum. Ramanujan's sum is expanded as in the divisor sums in Corollary 4.2 of the last subsection. In the next lemma, the convolution operator  $*$  on two arithmetic functions  $f, g$  corresponds to the Dirichlet convolution,  $f * g$ .

**Lemma 4.12** (DFT of functions of the greatest common divisor) *Let  $h$  be any arithmetic function. For natural numbers  $m \geq 1$ , the discrete Fourier transform (DFT) of  $h$  is defined by the following function:*

$$\widehat{h}[a](m) := \sum_{k=1}^m h(\gcd(k, m)) e\left(\frac{ka}{m}\right).$$

*This variant of the DFT of  $h(\gcd(n, k))$  (with  $k$  a free parameter) satisfies  $\widehat{h}[a] = h * c_-(a)$  where the function  $\widehat{h}[a]$  is summed explicitly for  $n \geq 1$  as the Dirichlet convolution*

$$\widehat{h}[a](n) = (h * c_-(a))(n) = \sum_{d|n} h(n/d) c_d(a).$$

**Definition 4.13** (Notation and special exponential sums) In what follows, we denote the  $\ell$ th Fourier coefficient with respect to  $k$  of the function  $L_{f,g,k}(n)$  by  $a_{k,\ell}$  which is well defined since  $L_{f,g,k}(n) = L_{f,g,k}(n+k)$  is periodic with period  $k$ . We then have an expansion of this function in the form of

$$L_{f,g,k}(n) = \sum_{\ell=0}^{k-1} a_{k,\ell} \cdot e\left(\frac{\ell n}{k}\right),$$

where we can compute these coefficients directly from  $L_{f,g,k}(n)$  according to the formula

$$a_{k,\ell} = \sum_{n=0}^{k-1} L_{f,g,k}(n) e\left(-\frac{\ell n}{k}\right).$$

We also notice that these Fourier coefficients are given explicitly in terms of  $f$  and  $g$  by the formulas cited in (16) above.

**Theorem 4.14** *For all arithmetic functions  $f, g$  and natural numbers  $k \geq 1$ , we have that*

$$\sum_{d|k} \sum_{r=0}^{k-1} d \cdot L_{f,g,r}(k) e\left(-\frac{rd}{k}\right) \mu(k/d) = \sum_{d|k} \phi(d) f(d) (k/d)^2 g(k/d), \quad (17)$$

where  $\phi(n)$  is Euler's totient function.

**Proof** We notice that the left-hand side of (17) is a divisor sum of the form

$$\sum_{d|k} \sum_{r=0}^{k-1} d \cdot L_{f,g,r}(k) e\left(-\frac{rd}{k}\right) \mu(k/d) = \sum_{d|k} d \cdot a_{k,d} \cdot \mu(k/d),$$

where the Fourier coefficients in this expansion are given by (16) [2, §8.3] and [11, §27.10]. In particular, we have that

$$a_{k,d} = k \cdot \sum_{r|(k,d)} g(r) f(k/r) \frac{r}{k}.$$

The left-hand side of (17) then becomes (cf. (19))

$$\begin{aligned} \sum_{d|k} d \cdot a_{k,d} \mu(k/d) &= \sum_{d|k} \sum_{r|d} r g(r) f\left(\frac{k}{r}\right) \cdot d \mu\left(\frac{k}{d}\right) \\ &= \sum_{d=1}^k [d|k]_{\delta} d \cdot \mu\left(\frac{k}{d}\right) \times \sum_{r|d} r g(r) f\left(\frac{k}{r}\right) \\ &= \sum_{r|k} r g(r) f\left(\frac{k}{r}\right) \times \sum_{d=1}^{k/r} dr \cdot \mu\left(\frac{k}{dr}\right) [d|k]_{\delta} \\ &= \sum_{r|k} r^2 g(r) \cdot f\left(\frac{k}{r}\right) \phi\left(\frac{k}{r}\right). \end{aligned}$$

We notice that while the exponential sums in the original statement of the claim are desirable in expanding applications, this direct expansion is difficult to manipulate algebraically. Therefore, we have effectively swapped out the exponential sum for the known divisor sum formula for the Fourier coefficients implicit in the statement of (17) in order to prove our key result.

**Corollary 4.15** (An exact formula for  $g(n)$ ) For any  $n \geq 1$  and arithmetic functions  $f, g$  we have the formula

$$g(n) = \sum_{d|n} \sum_{j|d} \sum_{r=0}^{d-1} \frac{j \cdot L_{f,g,r}(d)}{d^2} e\left(-\frac{rj}{d}\right) \mu(d/j) y_f(n/d),$$

where  $y_f(n) = (\phi f \text{Id}_{-2})^{-1}(n)$  is the Dirichlet inverse of  $f(n)\phi(n)/n^2$  and  $\text{Id}_k(n) := n^k$  for  $n \geq 1$ .

**Proof** We first divide both sides of the result in Theorem 4.14 by  $k^2$ . Then we apply a Dirichlet convolution of the left-hand side of the formula in Theorem 4.14 with  $y_f(n)$  defined as above to obtain the exact expansion for  $g(n)$ .

**Corollary 4.16** (The Mertens function) *For all  $x \geq 1$ , the Mertens function defined in the introduction is expanded by Ramanujan's sum as*

$$M(x) = \sum_{d=1}^x \sum_{n=1}^{\lfloor \frac{x}{d} \rfloor} \sum_{r=0}^{d-1} \left( \sum_{j|d} \frac{j}{d} \cdot e\left(-\frac{rj}{d}\right) \mu\left(\frac{d}{j}\right) \right) \frac{c_d(r)}{d} y(n), \quad (18)$$

where  $y(n) = (\phi \text{Id}_{-1})^{-1}(n)$  is the Dirichlet inverse of  $\phi(n)/n$ .

**Proof** We begin by citing Theorem 4.14 in the special case corresponding to  $L_{f,g,k}(n)$  a Ramanujan sum for  $f(n) = n$  and  $g(n) = \mu(n)$ . Then we sum over the left-hand side  $g(n)$  in Corollary 4.15 to obtain the initial summation identity for  $M(x)$  given by

$$M(x) = \sum_{n \leq x} \sum_{d|n} \sum_{j|d} \sum_{r=0}^{d-1} \frac{j}{d^2} c_d(r) e\left(-\frac{rj}{d}\right) \mu\left(\frac{d}{j}\right) y\left(\frac{n}{d}\right). \quad (i)$$

We can then apply the identity that for any arithmetic functions  $h, u, v$  we can interchange nested divisor sums as<sup>1</sup>

$$\sum_{k=1}^n \sum_{d|k} h(d) u(k/d) v(k) = \sum_{d=1}^n h(d) \left( \sum_{k=1}^{\lfloor \frac{n}{d} \rfloor} u(k) v(dk) \right). \quad (19)$$

Application of this identity to (i) leads to the first form for  $M(x)$  stated in (18).

**Corollary 4.17** (Euler's Totient function) *For any  $n \geq 1$  we have*

$$\phi(n) = n \cdot \sum_{d|n} \sum_{j|d} \sum_{r=0}^{d-1} \frac{j}{d^2} c_d(r) e\left(-\frac{rj}{d}\right) \mu\left(\frac{d}{j}\right).$$

Additionally, we have the following expansion of the average order sums for  $\phi(n)$  given by

$$\sum_{2 \leq n \leq x} \phi(n) = \sum_{d=1}^x \sum_{r=0}^{d-1} \frac{c_d(r)}{2d} \left\lfloor \frac{x}{d} \right\rfloor \left( \left\lfloor \frac{x}{d} \right\rfloor - 1 \right) \times \sum_{j|d} j e\left(-\frac{rj}{d}\right) \mu\left(\frac{d}{j}\right).$$

<sup>1</sup> We also have a related identity which allows us to interchange the order of summation in the Anderson–Apostol sums of the following form for any natural numbers  $x \geq 1$  and arithmetic functions  $f, g, h : \mathbb{N} \rightarrow \mathbb{C}$ :

$$\sum_{d=1}^x f(d) \sum_{r|(d,x)} g(r) h\left(\frac{d}{r}\right) = \sum_{r|x} g(r) \sum_{d=1}^{x/r} h(d) f(\gcd(x, r)d).$$

**Proof** We consider the formula in Theorem 4.14 with  $f(n) = n$  and  $g(n) = \mu(n)$ . Since the Dirichlet inverse of the Möbius function is  $\mu * 1 = \varepsilon$ , we obtain our result by convolution and multiplication by the factor of  $n$ . The average order identity follows from the first expansion by applying (19).

#### 4.4 An approach via polynomials and orthogonality relations

In Corollary 4.9 of Sect. 4.2 we proved an exact formula for the modified ordinary matrix entries  $\widehat{u}_{n,k}(f, w)$  defined by the simplifications of the original  $u_{n,k}(f, w)$  from (6b) in Remark 4.5 (cf. Table 1). We proved the exact formula for  $\widehat{u}_{n,k}(f, w)$  in the previous subsection using a more combinatorial argument involving the multiple convolutions of the function  $D_f(n)$  constructed recursively in Definition 4.6 (see Table 2). In this section we define a sequence of related polynomials  $P_j(w; t)$  whose coefficients are the corresponding simplified forms of the inverse matrices. We prove in Proposition 4.18 below that

$$\sum_{k=1}^n \widehat{u}_{n,k}(f, w) \cdot P_k(w, t) = t^n.$$

where the  $P_k(w, t)$  sequence is defined by (21). We then find and prove the form of a weight function  $\omega(t)$  which provides us with the orthogonality condition

$$\int_{|t|=1, t \in \mathbb{C}} \omega(t) P_i(w; t) P_j(w; t) dt =: \hat{c}_i(w) [i = j]_{\delta}, \quad (20a)$$

where we define the right-hand side coefficients by

$$\hat{c}_n(w) := \int_{|t|=1, t \in \mathbb{C}} \omega(t) (P_n(w; t))^2 dt. \quad (20b)$$

This construction, which we develop and make rigorous below, provides us with another method by which we may exactly extract the form of the simplified matrices  $\widehat{u}_{n,k}(f, w)$ . Namely, we have that for all  $n \geq 1$  and  $1 \leq k \leq n$  the operation

$$\widehat{u}_{n,k}(f, w) = \frac{1}{\hat{c}_k(w)} \int_{|t|=1, t \in \mathbb{C}} \omega(t) t^n P_k(w, t) dt, \quad (20c)$$

yields an exact formula for our matrix entries of interest here. We now develop the requisite machinery to prove that this construction holds.

**Proposition 4.18** (A partition-related polynomial sum) *Let an arithmetic function  $f$  be fixed and for an indeterminate  $w \in \mathbb{C}$  let  $\widehat{f}(n)$  denote*

$$\widehat{f}(n) := \frac{w^n}{w^n - 1} f(n).$$

For natural numbers  $j \geq 1$  and any indeterminate  $w$ , let the polynomials

$$P_j(w; t) := \sum_{i=1}^j \left( \sum_{d|j} \widehat{f}(d) p\left(\frac{j}{d} - i\right) \right) t^i. \quad (21)$$

Then for all  $n \geq 1$  we have that

$$\sum_{k=1}^n \widehat{u}_{n,k}(f, w) \cdot P_k(w, t) = t^n.$$

**Proof** The claim is equivalent to proving that for each  $n \geq 1$ , we have that

$$(w^n - 1) \cdot P_n(w; t) = \sum_{k=1}^n u_{n,k}^{(-1)}(f, w) \cdot t^k. \quad (22)$$

Notice that equation (22) also implies that

$$\begin{aligned} P_n(w; t) &= \sum_{k=1}^n \widehat{u}_{n,k}^{(-1)}(f, w) \cdot t^k = \sum_{k=1}^n \left( \sum_{d|n} \widehat{f}(d) p\left(\frac{n}{d} - k\right) \right) t^k \\ &= \sum_{d|n} \widehat{f}(d) \times \sum_{i=0}^{\frac{n}{d}-1} p(i) \cdot t^{\frac{n}{d}-i}. \end{aligned} \quad (23)$$

Now finally, for each  $1 \leq k \leq n$ , we can expand the coefficients of the left-hand side as

$$\begin{aligned} [t^k](w^n - 1)P_n(w; t) &= \sum_{d|n} \frac{(w^n - 1)w^d}{w^d - 1} f(d) p(n/d - k) \\ &= \sum_{d|n} f(d) p(n/d - k) \left( \sum_{i=1}^{n/d} w^{id} \right) \\ &= \sum_{m=1}^n \left( \sum_{d|m} f(d) p(n/d - k) [d|n]_{\delta} \right) w^m \quad m = id \\ &= \sum_{m=1}^n \left( \sum_{d|(m,n)} f(d) p(n/d - k) \right) w^m. \end{aligned}$$

Hence by the formula for the inverse matrices given in Proposition 2.2, we have proved our claim.

**Proposition 4.19** (Another matrix formula) *For  $n \geq 1$  and  $1 \leq k \leq n$ , we have the following formula for the simplified matrix entries:*

$$\widehat{u}_{n,k}(f, w) = \sum_{\substack{j \geq 0 \\ n - G_j > 0 \\ k | n - G_j}} (-1)^{\left\lceil \frac{j}{2} \right\rceil} \cdot \widehat{f}^{-1} \left( \frac{n - G_j}{k} \right).$$

**Proof** According to the last expansion in (23), we have that

$$\sum_{i=0}^{n-1} p(i) t^{n-i} = \left( \widehat{f}^{-1} * P_-(w; t) \right) (n),$$

or equivalently that

$$t^n = \sum_{\substack{j \geq 0 \\ n - G_j > 0}} (-1)^{\left\lceil \frac{j}{2} \right\rceil} \cdot \left( \widehat{f}^{-1} * P_-(w; t) \right) (n - G_j). \quad (24)$$

Then by substituting equation (24) into (20c) we have our result.

**Theorem 4.20** *Suppose that the form of the sequence  $\{\widehat{c}_k(\omega)\}_{k \geq 1}$  is given. Let  $D_f(n) := \text{DTFT}[f](n)$  denote the discrete time Fourier transform of  $f$  at  $n$ . Then writing  $t := e^{iu}$  for  $0 \leq u \leq 2\pi$ , we have the following exact expression for the weight function  $\omega(t)$  which varies only depending on the prescribed sequence of  $\widehat{c}_k(\omega)$ :*

$$\begin{aligned} \omega \left( e^{\frac{iu}{2}} \right) &= 2 \cdot D_f \left( \sum_{G_r < i} (-1)^{\left\lceil \frac{r}{2} \right\rceil} \right. \\ &\quad \times \left. \sum_{G_l < i - G_r} (-1)^{\left\lceil \frac{l}{2} \right\rceil} \left( (\widehat{c}_- * \widehat{f}^{-1}) * \widehat{f}^{-1} \right) (i - G_l - G_r) \right) (u). \end{aligned}$$

**Proof** Using (20b)

$$\widehat{c}_i(\omega) = \sum_{d|i} \widehat{f}(d) \times \sum_{r=0}^{\frac{i}{d}-1} p(r) \times \sum_{c|i} \widehat{f}(c) \times \sum_{l=0}^{\frac{i}{c}-1} p(l) \times \int_{|t|=1} \omega(t) t^{\frac{i}{d} + \frac{i}{c} - r - l} dt.$$

For  $t := e^{iu}$  and  $0 \leq u \leq 2\pi$  defined above, let  $h(u) = \omega(e^{iu}) = \omega(t)$ . By a direct appeal to Möebius inversion we see that

$$((\widehat{c}_- * \widehat{f}^{-1}) * \widehat{f}^{-1})(i) = \sum_{r=0}^{i-1} \sum_{l=0}^{i-1} p(r) p(l) D_h^{-1}(2i - l - r).$$

Then we can obtain that

$$\sum_{r=0}^{i-1} p(r) D_h^{-1}(2i-r) = \sum_{G_l < i} (-1)^{\left\lceil \frac{l}{2} \right\rceil} \left( (\hat{c}_- * \hat{f}^{-1}) * \hat{f}^{-1} \right) (i - G_l),$$

and

$$D_h^{-1}(2i) = \sum_{G_r < i} (-1)^{\left\lceil \frac{r}{2} \right\rceil} \sum_{G_l < i - G_r} (-1)^{\left\lceil \frac{l}{2} \right\rceil} \left( (\hat{c}_- * \hat{f}^{-1}) * \hat{f}^{-1} \right) (i - G_l - G_r).$$

Thus by taking DTFT of both sides we arrive at the formula

$$\begin{aligned} \frac{1}{2} h\left(\frac{u}{2}\right) &= D_h \left( \sum_{G_r < i} (-1)^{\left\lceil \frac{r}{2} \right\rceil} \right. \\ &\quad \times \left. \sum_{G_l < i - G_r} (-1)^{\left\lceil \frac{l}{2} \right\rceil} \left( (\hat{c}_- * \hat{f}^{-1}) * \hat{f}^{-1} \right) (i - G_l - G_r) \right) (u) \end{aligned}$$

Since  $h(u) = \omega(e^{tu}) = \omega(t)$  this proves our key formula.

## 5 Conclusions

We have proved several new expansions of the type I and type II sums defined by (5) for any prescribed arithmetic functions  $f$  and  $g$ . Our new results proved in the article include treatments of the expansions of these two sum types by both matrix-based factorization theorems and analogous identities formulated through discrete Fourier transforms of special function sums. The type I sums implicitly define many special number theoretic functions and sequences by exponential sum variants of this type. Perhaps the most notable canonical example of this sum type is given by Euler's totient function which counts the number of integers relatively prime to a natural number  $n$ . The Möbius function also has a representation in the form of a type I sum.

The type II sums form an alternate flavor of the ordinary divisor sums enumerated by Lambert series generating functions and the Dirichlet convolutions of two arithmetic functions  $f$  and  $g$ . These sums are sometimes referred to as Anderson–Apostol sums, or  $k$ -convolutions in the references. The prototypical example of sums of this type is given by the Ramanujan sums  $c_q(n)$  which form expansions of many other special number theoretic functions by composition and infinite series. Our results provide new and useful expansions that characterize common and important classes of sums that arise in applications. Our results are unique in that we are able to relate partition functions to the expansions of these general classes of sums in both cases.

## Appendix A: notation and conventions in the article

Symbol	Definition
$a_k(f, g; n)$	Discrete Fourier coefficients of the periodic divisor sums $s_k(f, g; n)$ defined on Sect. 4.3 and as symbol $s_k(f; g; n)$ in this glossary. The precise definition of these sums is given by $a_k(f, g; n) = \sum_{d (k,n)} g(d) f(n/d) \frac{d}{k}$
$a_{k,\ell}$	Sequence of coefficients that are defined explicitly on Sect. 4.3 in the discrete Fourier series expansion of the type II sums $L_{f,g,k}(x)$ . These coefficients are implicitly defined by Definition 4.13 by the sums $L_{f,g,k}(n) = \sum_{\ell=0}^{k-1} a_{k,\ell} \cdot e(\ell n/k)$ , where $e(x)$ is the shorthand for the complex exponential terms in the exponential sums we define in the article
$[x]$	The ceiling function $[x] := x + 1 - \{x\}$ where $0 \leq \{x\} < 1$ denotes the fractional part of $x \in \mathbb{R}$
$\chi_{1,k}(n)$	The principal Dirichlet character modulo $k$ , i.e., the indicator function of the natural numbers which are relatively prime for $n, k \geq 1$ , $\chi_{1,k}(n) = [(n, k) = 1]_\delta$
$C_k(n)$	Sequence of nested $k$ -convolutions of an arithmetic function $f$ with itself defined on Sect. 4.2. The precise definition of this sequence is given by $C_k(n) = \begin{cases} \hat{f}(n) - \hat{f}(1)\varepsilon(n), & \text{if } k = 1; \\ \sum_{d n} (\hat{f}(d) - \hat{f}(1)\varepsilon(d)) C_{k-1}(n/d), & \text{if } k \geq 2, \end{cases}$ where the symbol $\hat{f}(n)$ is defined in glossary entry $\hat{f}(n)$
$[q^n]F(q)$	The coefficient of $q^n$ in the power series expansion of $F(q)$ about zero
$c_q(n)$	Ramanujan's sum, $c_q(n) := \sum_{d (q,n)} d\mu(q/d)$
$D_f(n)$	Function related to the Dirichlet inverse of a function $f$ defined on Sect. 4.2. More precisely, this function is defined by the sum $D_f(n) := \sum_{j=1}^n \frac{ds_{2j}(f;n)}{\hat{f}(1)^{2j+1}}$ , where this definition involves the glossary symbols $ds_j(f; n)$ and $\hat{f}(n)$ . Lemma 4.7 relates this function to the Dirichlet inverse of the function $\hat{f}(n)$
$d(n)$	The ordinary divisor function, $d(n) := \sum_{d n} 1$
$ds_j(f; n)$	Summands in the formula for the Dirichlet inverse of an arithmetic function defined on Sect. 4.2. The precise definition of this function is given by $ds_j(f; n) = \begin{cases} (-1)^{\delta_{n,1}} \hat{f}(n), & \text{if } j = 1; \\ \sum_{d n} \hat{f}(d) ds_{j-1}(f; \frac{n}{d}), & \text{if } j \geq 2, \end{cases}$ where the fixed function $\hat{f}$ is defined by glossary symbol $\hat{f}(n)$
$\text{DFT}[f](k)$	The discrete Fourier transform (DFT) of $f$ at $k$ . We use this transformation in Sect. 4.3 of the article.
$\text{DTFT}[f](k)$	The discrete time Fourier transform (DTFT) of $f$ at $k$ , also denoted by $F[k]$ .
$\varepsilon(n)$	The multiplicative identity with respect to Dirichlet convolution, $\varepsilon(n) = \delta_{n,1}$
$e(x)$	The complex exponential function, $e(x) := \exp(2\pi i \cdot x)$
$f * C_{-}(m)$	This notation indicates that the index over which we perform the Dirichlet convolution is given by the dash parameter, $(f * C_{-}(m))(n) := \sum_{d n} f(d) C_{n/d}(m)$
$f * C_k(-)$	This notation indicates that the index over which we perform the Dirichlet convolution is given by the dash parameter, $(f * C_k(-))(n) := \sum_{d n} f(d) C_k(n/d)$
$*; f * g$	The Dirichlet convolution of $f$ and $g$ , $f * g(n) := \sum_{d n} f(d)g(n/d)$ , for $n \geq 1$ . This symbol for the discrete convolution of two arithmetic functions is the only notion of convolution of functions we employ within the article
$\hat{f}(n)$	A shorthand notation for scaled arithmetic function terms $\hat{f}(n) := w^n / (w^n - 1) f(n)$ for some indeterminate $w$ . The notation is defined



$f^{-1}(n)$	The Dirichlet inverse of $f$ with respect to convolution defined recursively by $f^{-1}(n) = -\frac{1}{f(1)} \sum_{\substack{d n \\ d>1}} f(d)f^{-1}(n/d)$ provided that $f(1) \neq 0$
$F[k]$	Discrete Fourier transform coefficients defined
$\lfloor x \rfloor$	The floor function $\lfloor x \rfloor := x - \{x\}$ where $0 \leq \{x\} < 1$ denotes the fractional part of $x \in \mathbb{R}$
$f_{\pm}(n)$	For any arithmetic function $f$ , we define $f_{\pm}(n) = f(n)[n > 1]_{\delta} - f(1)[n = 1]_{\delta}$ , i.e., the function that has identical values as $f$ for all $n \geq 2$ , and whose initial value is $f_{\pm}(1) := -f(1)$ when $n = 1$
$G_j$	Denotes the interleaved (or generalized) sequence of pentagonal numbers defined explicitly by the formula $G_j := \frac{1}{2} \left\lceil \frac{j}{2} \right\rceil \left\lceil \frac{3j+1}{2} \right\rceil$ . The sequence begins as $\{G_j\}_{j \geq 0} = \{0, 1, 2, 5, 7, 12, 15, 22, 26, 35, 40, 51, \dots\}$ .
$\text{Id}_k(n)$	The power-scaled identity function, $\text{Id}_k(n) := n^k$ for $n \geq 1$
$[n = k]_{\delta}$	Synonym for $\delta_{n,k}$ which is one if and only if $n = k$ , and zero otherwise
$[\text{cond}]_{\delta}$	For a boolean-valued cond, $[\text{cond}]_{\delta}$ evaluates to one precisely when cond is true, and zero otherwise.
$L_{f,g,k}(x)$	The type II Anderson–Apostol sum over the arithmetic functions $f, g$ , $L_{f,g,k}(x) := \sum_{d (k,x)} f(d)g(x/d)$
$\text{gcd}(m, n); (m, n)$	The greatest common divisor of $m$ and $n$ . Both notations for the GCD are used interchangeably within the article
$\mu(n)$	The Möbius function
$\mu_{n,k}$	Matrix sequence defined on Sect. 3.1. The invertible sequence is an analog to the role of the Möbius function in Möbius inversion. In this case these inversion coefficients are defined such that $g(n) = \sum_{\substack{d=1 \\ (d,n)=1}}^n f(d) \iff f(n) = \sum_{d=1}^n g(d+1)\mu_{n,d}$ . See Proposition 3.1 and Sect. 3 for the relation of this sequence (and its inverse) to the factorizations of type I sums
$\mu_{n,k}^{(-1)}$	Inverse matrix sequence of $\mu_{n,k}$ defined
$M(x)$	The Mertens function which is the summatory function over $\mu(n)$ , $M(x) := \sum_{n \leq x} \mu(n)$
$\omega(t)$	Orthogonal polynomial orthogonality weight function defined on Sect. 4.4. This function is related to glossary symbol $P_k(w, t)$ .
$\phi_k(n)$	Generalized totient function, $\phi_k(n) := \sum_{\substack{1 \leq d \leq n \\ (d,n)=1}} d^k$
$\phi(n)$	Euler's classical totient function, $\phi(n) := \sum_{\substack{1 \leq d \leq n \\ (d,n)=1}} 1$
$\Phi_n(z)$	The $n$ th cyclotomic polynomial in $z$ defined by $\Phi_n(z) := \prod_{\substack{1 \leq k \leq n \\ (k,n)=1}} (z - e^{2\pi i k/n})$ .
$P_k(w, t)$	Type of orthogonal polynomial function defined on Sect. 4.4. It satisfies that $\sum_{k=1}^n \hat{u}_{n,k}(f, w) P_k(w, t) = t^n$
$p(n)$	The partition function generated by $p(n) = [q^n] \prod_{n \geq 1} (1 - q^n)^{-1}$
$(q; q)_{\infty}$	The infinite $q$ -Pochhammer symbol defined as the product $(q; q)_{\infty} := \prod_{n \geq 1} (1 - q^n)$ for $ q  < 1$
$\sigma_{\alpha}(n)$	The generalized sum-of-divisors function, $\sigma_{\alpha}(n) := \sum_{d n} d^{\alpha}$ , for any $n \geq 1$ and $\alpha \in \mathbb{C}$
$s_k(f, g; n)$	Shorthand for the periodic (modulo $k$ ) divisor sums defined on Sect. 4.3 and expanded by the functions listed in $a_k(f, g; n)$ of this glossary. The precise expansion and corresponding finite Fourier series expansion of this function is given by $s_k(f, g; n) = \sum_{d (n,k)} f(d)g(k/d) = \sum_{m=1}^k a_k(f, g; m)e^{2\pi i \cdot mn/k}$ .
$s_{n,k}$	Matrix coefficients in Lambert series type factorizations defined on Sect. 2.2. These coefficients are defined precisely as the coefficients of the generating function $[q^n](q; q)_{\infty} q^k / (1 - q^k)$ for $k \geq 1$ where $(q; q)_{\infty}$ is the infinite $q$ -Pochhammer symbol

$T_f(x)$	The type I sum over an arithmetic function $f$ , $T_f(n) := \sum_{\substack{d \leq x \\ (d,x)=1}} f(d)$
$t_{n,k}$	Matrix sequence involved in the generating function expansions of the type I sums defined on Sect. 2.2 as $T_f(x) = [q^x] \left( \frac{1}{(q;q)_\infty} \sum_{n \geq 2} \sum_{k=1}^n t_{n,k} f(k) \cdot q^n + f(1) \cdot q \right)$
$t_{n,k}^{(-1)}$	Inverse matrix of the sequence $t_{n,k}$ that is defined
$\hat{u}_{n,k}(f, w)$	Matrix coefficients defined in terms of an indeterminate parameter $w$ as $\hat{u}_{n,k}(f, w) := (w^k - 1) \cdot u_{n,k}(f, w).$
$u_{n,k}(f, w)$	Matrix sequence defined on Sect. 2.2 in the expansion of the generating functions for the type II sums as $g(x) =$ $[q^x] \left( \frac{1}{(q;q)_\infty} \sum_{n \geq 2} \sum_{k=1}^n u_{n,k}(f, w) \left[ \sum_{m=1}^k L_{f,g,m}(k) w^m \right] \cdot q^n \right), \quad w \in \mathbb{C}.$
$u_{n,k}^{(-1)}(f, w)$	Inverse matrix terms of the sequence $u_{n,k}(f, w)$ defined
$y_f(n)$	The function $y_f(n)$ denotes the Dirichlet inverse of the function $h(n) := f(n)\phi(n)/n^2$ where $\phi(n)$ is Euler's totient function and $f$ is any invertible arithmetic function such that $f(1) \neq 0$ . This function is used to express the result in Corollary 4.15. A special case, denoted by $y(n)$ , corresponding to the case where $f(n) \equiv n$ is employed in stating Corollary 4.16 in Sect. 4.3
$\zeta(s)$	The Riemann zeta function, defined by $\zeta(s) := \sum_{n \geq 1} n^{-s}$ when $\Re(s) > 1$ , and by analytic continuation to the entire complex plane with the exception of a simple pole at $s = 1$

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