Announcements:

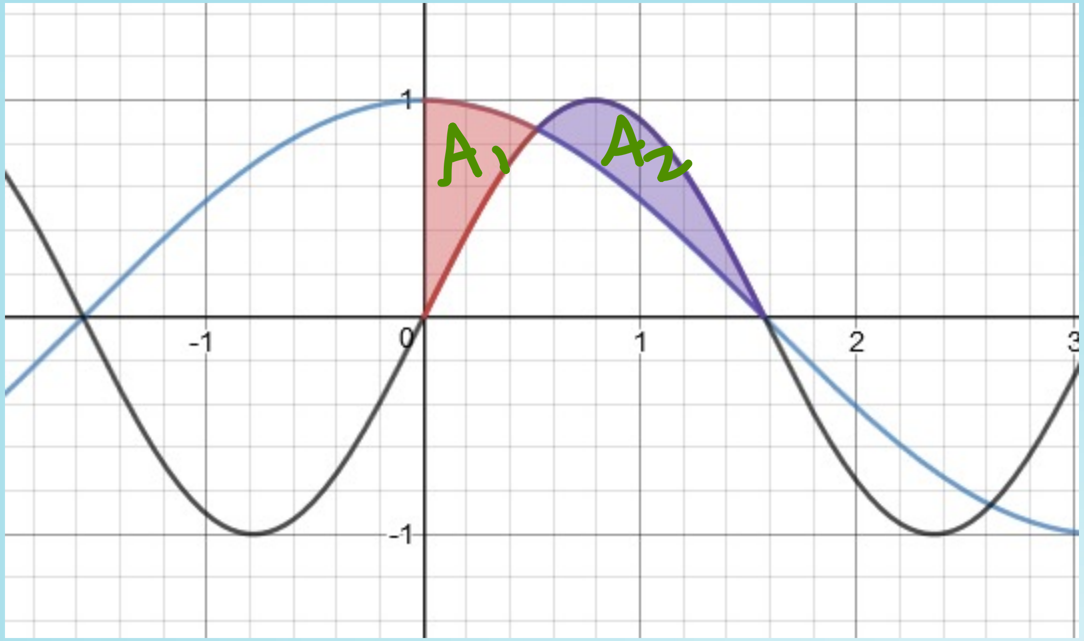
- office hours today from 1-2PM
- complete the TurningPoint poll on Canvas
- tentative quiz 1 stats:
  - → avg: 62%
    - stddev: 2.19
  - → lowest quiz score gets dropped

Example 4:

Find the area of the region bounded by the curves

$y_1 = \cos x$ and $y_2 = \sin(2x)$ on $\left[0, \frac{\pi}{2}\right]$.

→ Start this today, finish on Friday



asked to find $A = A_1 + A_2$

y_2 (in gray)

y_1 (in blue)

Subints: $[0, \pi/6]$ and $[\pi/6, \pi/2]$

to compute A_1 ... A_2

→ $y_1 \geq y_2$, $y_2 \geq y_1$

step 1: find int. points of $y_1 = y_2$ for $x \in [0, \pi/2]$:

$y_1 = \cos x$ and $y_2 = \sin(2x)$ on $\left[0, \dfrac{\pi}{2}\right]$.

first recall that $\sin(2x) = 2\cos(x)\cdot\sin(x)$

$y_1 = y_2 \longleftrightarrow \cos(x) = 2\cos(x)\cdot\sin(x)$

$\rightarrow \frac{1}{2} = \sin(x) \longleftrightarrow \sin^{-1}(1/2) = x$

$= \pi/6$

Subints: $[0, \pi/6]$ and $[\pi/6, \pi/2]$

to compute A_1 $\qquad$ A_2

$\rightarrow$ $y_1 \geq y_2$, $y_2 \geq y_1$

$y_1 = \cos(x)$

$y_2 = \sin(2x)$

Step3: setup integrals to compute $A = A_1 + A_2$:

$$A = \int_0^{\pi/6} (y_1(x) - y_2(x))\,dx + \int_{\pi/6}^{\pi/2} (y_2(x) - y_1(x))\,dx$$

$$= \int_0^{\pi/6} (\cos x - \sin(2x))\,dx - \int_{\pi/6}^{\pi/2} (\cos x - \sin(2x))\,dx$$

$$= \left(+\sin x + \frac{1}{2}\cos(2x) \right) \Big|_0^{\pi/6}$$

$$- \left(\sin(x) + \frac{1}{2}\cos(2x) \right) \Big|_{\pi/6}^{\pi/2}$$

$$\left(\begin{array}{ll} \cos(0) = 1 & \sin(0) = 0 \\ \cos(\pi/3) = 1/2 & \sin(\pi/6) = 1/2 \\ \cos(\pi) = -1 & \sin(\pi/2) = 1 \end{array} \right)$$

$$= \left(+\sin(\pi/6) + \frac{1}{2}\cos(\pi/3) \right) - \left(\sin(0) + \frac{1}{2}\cos(0) \right)$$

$$- \left(\sin(\pi/2) + \frac{1}{2}\cos(\pi) \right) + \left(\sin(\pi/6) + \frac{1}{2}\cos(\pi/3) \right)$$

$$= \left(\frac{1}{2} + \frac{1}{4}\right) - \left(0 + \frac{1}{2}\right)$$

$$- \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} + 1/4\right) = \boxed{1/2}$$

Section 8.2: Integration by parts

Math 1552 lecture slides adapted from the course materials
By Klara Grodzinsky (GA Tech, *School of Mathematics*, Summer 2021)

Review Question: Evaluate the integral.

$$\int x\left(\frac{1}{3}\right)^{x^2} dx = I$$

$$\boxed{(A)\ -\frac{1}{2\ln 3}\left(\frac{1}{3}\right)^{x^2} + C}$$

$$(B)\ \frac{1}{2(x^2+1)}\left(\frac{1}{3}\right)^{x^2+1} + C$$

$$(C)\ -\frac{1}{\ln 3}\left(\frac{1}{3}\right)^{x^2} + C$$

$$(D)\ \frac{\ln 3}{2}\cdot\left(\frac{1}{3}\right)^{x^2} + C$$

Evaluate with an u-sub:

$$u = x^2,\ du = 2x\,dx$$

$$\int a^{bx}\,dx$$

$$e^{\ln(3^{-u})}$$

$$I = \frac{1}{2}\int 3^{-u}\,du$$

$$= \frac{1}{2}\int e^{-u\cdot\ln(3)}\,du$$

$$= -\frac{1}{2\ln(3)}\left(\frac{1}{3}\right)^{u} + C$$

$$= -\frac{1}{2\ln(3)}\left(\frac{1}{3}\right)^{x^2} + C$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C \quad (a = -\ln(3))$$

Learning Goals

- Identify which functions can be solved using the method of integration by parts (IBP)
- Understand how to choose the values of "u" and "dv"
- Evaluate integrals using integration by parts

Formula for Integration by Parts

Integration by parts comes from the product rule for differentiation.

$$\boxed{\int u \cdot dv = uv - \int v \cdot du}$$

(know this formula, or memorize it)

Differentiate u to obtain du.

Find v by taking an antiderivative of dv.

$$(fg)' = f'g + fg' \implies$$

$$f(x)g(x) = \int f'(x)g(x)dx + \int f(x)g'(x)dx$$

Rules to Apply Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

- The original integral CANNOT be evaluated by a normal *u*-substitution alone.
- Begin by rewriting the original function as the product of two pieces, *u* and *dv*.
- We must be able to integrate *dv*! e.g., must be able to eval. v from our choice of dv!
- The new integral should be easier than the original problem. If not, try a different choice for *u* and *dv*.

When to use Integration by Parts

Use integration by parts to evaluate the integrals of:

- Inverse functions
- Logarithmic functions
- Functions that are combinations of more than one type of function (i.e., polynomials, trigonometric, exponential, logarithmic functions)
- **Note:** We can combine IBP with the methods we have learned so far (e.g., start with a u-sub and then apply IBP after simplifying)
- After practice, you should be able to spot IBP type integrals quickly

Hints about IBP techniques

- **DO NOT** use tables, or tabular integration methods, you have seen before in this class!
- Start with a blank slate of parameters you need to find organized like the following:

$$\left\{ \begin{array}{cc} u = & dv = \\ du = & v = \end{array} \right\}$$

- Be prepared to apply IBP more than once, e.g., to evaluate $\boxed{\int x^2 e^x \, dx}$
- If nothing else works, you can always take $dv = 1 \cdot dx$
- We will see many examples in the next slides

$V = X$

for example:

$$\int \underbrace{\tan^{-1}(x)}_{u} \overbrace{dx}^{dv}$$

Order in which to choose *u*

Choose *u* according to the *ILATE* rule:

u

I – Inverse Functions $\sin^{-1}(x), \cos^{-1}(x), \tan^{-1}(x)$

L – Logarithmic Functions $\ln(x), \log(x), \log_b(x)$ for $b > 0$

A – Algebraic Expressions (polynomials, rational functions, etc.) $1, x, x^2$

T – Trigonometric Functions $\sin(x), \cos(x), \tan(x)$

E – Exponential Functions $e^x, e^{-2x}, 3^x$

Tip: In the event of a "tie" in the *ILATE* rule, pick u to be the simplest of the two functions.

useful

Example 1 (inverse functions):

$\int u\,dv = uv - \int v\,du$

Evaluate the integral $\int \sin^{-1}(x)dx. = I$

→ apply IBP

ILATE

$u = \sin^{-1}(x)$ $\quad dv = dx$

$du = \dfrac{dx}{\sqrt{1-x^2}}$ $\quad v = x$

$$I = x \cdot \sin^{-1}(x) - \boxed{\int \frac{x\,dx}{\sqrt{1-x^2}}}\ I_2$$

- evaluate I_2 by a u-sub

$u = 1 - x^2$, $du = -2x\,dx$

$$I_2 = -\frac{1}{2}\int \frac{du}{\sqrt{u}} = -\frac{1}{2}\cdot 2\sqrt{u} + C_1$$

$$= -(1-x^2)^{1/2} + C_1$$

so, in total:

$$I = x\cdot \sin^{-1}(x) + (1-x^2)^{1/2} + C$$

What should we choose for the value of *u* in the integral

ILATE rule

$\xrightarrow{\text{trig}}$

$\downarrow$

alg(x)

Hint:

$\sin(2x) = 2\sin(x)\cos(x)$

$$\int x\sin(x)\cos(x)dx\,?$$

A. x

B. sin(x)

C. cos(x)

D. sin(x)cos(x)

(if we tried, a u-sub first)

$u = \sin(x)$, $du = \cos x\,dx$,

$x = \sin^{-1}(u)$

$\rightarrow I = \int \sin^{-1}(u)\cdot u\,du$

Example 2: Evaluate the integral: $\int x\sin(x)\cos(x)dx. = I$, apply IBP with $u=x$

know this identity

Hint:

$\sin(2x) = 2\boxed{\sin(x)\cos(x)}$

$\int u\,dv = uv - \int v\,du$

$= \frac{1}{2}\int x \cdot \sin(2x)dx$

$u = x$ $\quad dv = \sin(2x)dx$

$du = dx$ $\quad v = -\frac{\cos(2x)}{2}$

$2I = -\frac{x\cos(2x)}{2} + \frac{1}{2}\int \cos(2x)dx$

$$2I = -\frac{x\cos(2x)}{2} + \frac{1}{4}\sin(2x) + C$$

Example 3: Evaluate the integral: $\int (\ln x)^2 dx = I$, apply the IBP method

$\int u\,dv = uv - \int v\,du$

ILATE

$\hookrightarrow (\ln x)^2 = u$

$u = (\ln x)^2 \qquad dv = dx$

$du = \dfrac{2\ln x\,dx}{x} \qquad v = x$

$$I = x(\ln x)^2 - 2\underbrace{\boxed{\int \ln x \cdot dx}}_{I_2}$$

To evaluate I_2, apply IBP again!