Jacobi Type Continued Fractions for the Ordinary Generating Functions of Generalized Factorial Functions

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Abstract

The article studies a class of generalized factorial functions and symbolic product sequences through Jacobi type continued fractions (J-fractions) that formally enumerate the divergent ordinary generating functions of these sequences. The more general definitions of these J-fractions extend the known expansions of the continued fractions originally proved by Flajolet that generate the rising factorial function, or Pochhammer symbol, $(x)_n$, at any fixed non–zero indeterminate $x \in \mathbb{C}$. The rational convergents of these generalized J-fractions provide formal power series approximations to the ordinary generating functions that enumerate many specific classes of factorial–related integer sequences. These generalized rational convergent functions are expanded by particular cases of the special zeros of the confluent hypergeometric function and associated Laguerre polynomial sequences studied in the references.

The article also cites a number of specific identities, new integer congruence relations satisfied by generalized factorial-related product sequences, restatements of classical congruence properties concerning the primality of integer subsequences, among several other notable motivating examples as immediate applications of the new results. The convergent–based generating function techniques illustrated by the special case examples cited within the article are easily extended to enumerate the factorial-like product sequences arising in the context of many other specific applications. In this sense, the article serves as a semi–comprehensive, detailed survey reference that introduces applications to many established and otherwise well–known combinatorial identities, new cases of generating functions for factorial–function–related product sequences, and other examples of the generalized integer–valued multifactorial, or α–factorial, function sequences.

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1 Introduction

The focus of the new results established by this article is on new enumerating properties of the generalized symbolic product sequences, $p_n(\alpha, R)$ defined by (1.1), which are generated by the convergents to *Jacobi type continued fractions* (*J–fractions*) that represent formal power series expansions of the otherwise divergent ordinary generating functions (OGFs) for these sequences.

$$\begin{aligned} p_n(\alpha, R) &:= \prod_{0 \leq j < n} (R + \alpha j) + [n = 0]_\delta \qquad (1.1) \\ &= R(R+\alpha)(R+2\alpha) \times \cdots \times (R + (n-1)\alpha) + [n=0]_\delta \,. \end{aligned}$$

The related integer–valued cases of the multiple factorial function sequences of interest in the applications of this article are defined recursively for any fixed $\alpha \in \mathbb{Z}^+$ and $n \in \mathbb{N}$ by the following equation [22, §2]:

$$n!_{(\alpha)} = \begin{cases} n \cdot (n-\alpha)!_{(\alpha)}, & \text{if } n > 0; \\ 1, & \text{if } -\alpha < n \leq 0; \\ 0, & \text{otherwise.} \end{cases} \tag{1.2}$$

The particular new results studied within the article generalize the known series proved in the references [9; 10], including expansions of the series for generating functions enumerating the *rising* and *falling factorial* functions, $x^{\overline{n}} = (-1)^n (-x)^{\underline{n}} = p_n(1, x)$ and $x^{\underline{n}} = x!/(x-n)! = p_n(-1, x)$, and the *Pochhammer symbol*, $(x)_n$, expanded by the *Stirling numbers of the first kind*, $\left[{n \atop k}\right]$, as

$$\begin{aligned} (x)_n &= x(x+1)(x+2)\cdots(x+n-1)\,[n \geq 1]_\delta + [n = 0]_\delta \\ &= \sum_{k=1}^{n} \left[{n \atop k}\right] x^k\,[n \geq 1]_\delta + [n = 0]_\delta\,. \end{aligned}$$

The new forms of the J–fractions defined below also provide generalizations of several other known series expansions that formally enumerate the classical *single* and *double factorial* functions, $n!$ and $n!!$, and generalized forms of the Stirling number triangles.

We are especially interested in using the new results established in this article to formally enumerate the factorial–function–like product sequences, $p_n(\alpha, \beta n + \gamma)$, for some fixed parameters $\alpha, \beta, \gamma \in \mathbb{Q}$ when the symbolic indeterminate, R, depends linearly on n. The particular forms of the generalized product sequences of interest in the applications of this article are related to the *Gould polynomials*, $G_n(x; a, b) = \frac{x}{x-an} \cdot \left(\frac{x-an}{b}\right)_n$, in the forms of the following equation [22, §3.4.2] [13; 21]:

$$p_n(\alpha, \beta n + \gamma) = \frac{\alpha^n \cdot (\beta n + \gamma)}{\gamma} \times G_n(\gamma; -\beta, \alpha)\,. \tag{1.3}$$

The generalized product sequences in (1.1) also correspond to the definition of the *Pochhammer k–symbol*, $(x)_{n,k} = p_n(k, x)$, defined in the reference [7] for any fixed $k \neq 0$ and non–zero indeterminate, $x \in \mathbb{C}$ [*F.1].

Whereas the first results proved in the first articles [9; 10] are focused on establishing properties of divergent forms of the ordinary generating functions for a number of special sequence cases through more combinatorial interpretations of these continued fraction series, the emphasis in this article is more enumerative in flavor. The new identities involving the integer–valued cases of the multiple, α–factorial functions, $n!_{(\alpha)}$, defined in (1.2) obtained by this article extend the study of these sequences motivated by the distinct symbolic polynomial expansions of these functions originally considered in the reference [22]. This article extends a number of the examples considered as applications of the results from the 2010 article [22] briefly summarized in the next section.

[*F.1] *Note §1.1.* The generalized rising and falling factorial functions denote the products, $(x|\alpha)^{\overline{n}} = (x)_{n,\alpha}$ and $(x|\alpha)^{\underline{n}} = (x)_{n,-\alpha}$, defined in the reference, where the products, $(x)_n = (x|1)^{\overline{n}}$ and $x^{\underline{n}} = (x|1)^{\underline{n}}$, correspond to the particular special cases of these functions cited below [22, §2].

Note that Roman's *Umbral Calculus* reference employs the alternate, less standard notation of $\left(\frac{x}{a}\right)_n := \frac{x}{a}\left(\frac{x}{a}-1\right)\cdots\left(\frac{x}{a}-n+1\right)$ to denote the sequence *lower factorial polynomials*, and $x^{(n)}$ in place of the Pochhammer symbol to denote the *rising factorial polynomials*, which are connected by the following sum involving the *Lah numbers* [21, §4.1.2, §5; *cf.* §4.3.1]:

$$x^{\underline{n}} = \sum_{k=1}^{n} \binom{n-1}{k-1} \frac{n!}{k!} (x)_k\,.$$

1.1 Polynomial expansions of generalized α–factorial functions

For any fixed integer $\alpha \geq 1$ and $n, k \in \mathbb{N}$, the coefficients defined by the triangular recurrence relation in (1.4) provides one approach to enumerating the symbolic polynomial expansions of the generalized factorial function product sequences defined as special cases of (1.1) defined above †F.2 ‡F.3 .

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\alpha} = (\alpha n + 1 - 2\alpha) \begin{bmatrix} n-1 \\ k \end{bmatrix}_{\alpha} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{\alpha} + [n = k = 0]_{\delta} \tag{1.4}$$

The combinatorial interpretations of these coefficients motivated in the reference [22] leads to polynomial expansions in n of the multiple factorial function sequence variants in (1.2) that generalize the known formulas for the single and double factorial functions, $n!$ and $n!!$, involving the Stirling numbers of the first kind, $\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_1 = (-1)^{n-k} s(n,k)$, expanded in the forms of the following equations [13, §6] [19, §26.8] [A130534; A008275; A008277] §F.4 :

$$\begin{aligned} n! &= \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} (-1)^{n-m} n^m, \ \forall n \geq 1 \\ n!! &= \sum_{m=0}^{n} \begin{bmatrix} \lfloor \frac{n+1}{2} \rfloor \\ m \end{bmatrix} (-2)^{\lfloor \frac{n+1}{2} \rfloor - m} n^m, \ \forall n \geq 1 \\ n!_{(\alpha)} &= \sum_{m=0}^{n} \begin{bmatrix} \lceil n/\alpha \rceil \\ m \end{bmatrix} (-\alpha)^{\lceil \frac{n}{\alpha} \rceil - m} n^m, \ \forall n \geq 1, \alpha \in \mathbb{Z}^+. \end{aligned} \tag{1.5.a}$$

The polynomial expansions of the first two classical sequences in the previous equations are then generalized to the more general α–factorial function cases through the triangles defined as in (1.4) from the reference [22] through the next explicit finite sum formulas when $n \geq 1$.

$$n!_{(\alpha)} = \sum_{m=0}^{n} \begin{bmatrix} \lfloor \frac{n-1+\alpha}{\alpha} \rfloor + 1 \\ m+1 \end{bmatrix}_{\alpha} (-1)^{\lfloor \frac{n-1+\alpha}{\alpha} \rfloor - m} (n+1)^m, \ \forall n \geq 1, \alpha \in \mathbb{Z}^+ \tag{1.5.b}$$

The polynomial expansions in n of the generalized α–factorial functions, $(\alpha n - d)!_{(\alpha)}$, for fixed $\alpha \in \mathbb{Z}^+$ and $0 \leq d < \alpha$, are obtained similarly from (1.5.b) through the generalized coefficients in (1.4) as follows [22, *cf.* §2]:

$$(\alpha n - d)!_{(\alpha)} = (\alpha - d) \times \sum_{m=1}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_{\alpha} (-1)^{n-m} (\alpha n + 1 - d)^{m-1} \tag{1.5.c}$$

†F.2 *Note §1.2.* See Table A.1 starting on page 77 and Table A.2 starting on page 78.

‡F.3 *Note §1.3.* The symbolic product sequences formed by these coefficients defined by the reference [22, §3] are expanded by

$$p_n(\alpha, s \pm 1) = \sum_{m=0}^{n} \begin{bmatrix} n+1 \\ m+1 \end{bmatrix}_{\alpha} (\pm 1)^{n-m} s^m = (s \pm 1)(s \pm 1 + \alpha)(s \pm 1 + 2\alpha) \cdots (s \pm 1 + (n-1)\alpha).$$

§F.4 *Note §1.4.* The Stirling numbers of the first kind similarly provide non–polynomial exact finite sum formulas for the single and double factorial functions in the following forms for $n \geq 1$ where $(2n)!! = 2^n \times n!$ [13, §6.1] [4, §5.3]: (**TODO**)

$$\begin{aligned} n! &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \\ (2n-1)!! &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} 2^{n-k}. \end{aligned}$$

$$= \sum_{m=0}^{n} \begin{bmatrix} n+1 \\ m+1 \end{bmatrix}_{\alpha} (-1)^{n-m}(\alpha n+1-d)^m, \ \forall n \geq 1, \alpha \in \mathbb{Z}^+, 0 \leq d < \alpha.$$

A binomial–coefficient–themed phrasing of the products underlying the expansions of the more general factorial function sequences of this type (each formed by dividing through by a normalizing factor of $n!$) is suggested by the next expansions of these coefficients by the Pochhammer symbol [13, §5]:

$$\binom{\frac{s-1}{\alpha}}{n} = \frac{(-1)^n}{n!} \cdot \left(\frac{s-1}{\alpha}\right)_n = \frac{1}{\alpha^n \cdot n!} \prod_{j=0}^{n-1} (s-1-\alpha j)\,. \tag{1.6}$$

When the initially fixed indeterminate $s := s_n$ is considered modulo α in the form of $s_n := \alpha n + d$ for some fixed least integer residue, $0 \leq d < \alpha$, the prescribed setting of this offset d completely determines the numerical α–factorial function sequences of the forms in (1.5.c) generated by these products (see the examples cited below in Section 1.4.1 and the tables in the reference [22, *cf.* §6.1.2, Table 6.1]).

For any lower index $n \geq 1$, the binomial coefficient formulation of the multiple factorial function products in (1.6) provides the next several expansions by the exponential generating functions for the generalized coefficient triangles in (1.4), and their corresponding generalized Stirling polynomial analogs, $\sigma_k^{(\alpha)}(x)$, defined in the references [22, §5] [13, *cf.* §6, §7.4] ¶F.5 :

$$\binom{\frac{s-1}{\alpha}}{n} = \sum_{m=0}^{n} \begin{bmatrix} n+1 \\ n+1-m \end{bmatrix}_{\alpha} \frac{(-1)^m s^{n-m}}{\alpha^n n!} \tag{1.7.a}$$

$$= \sum_{m=0}^{n} \frac{(-1)^m \cdot (n+1)\sigma_m^{(\alpha)}(n+1)}{\alpha^m} \times \frac{(s/\alpha)^{n-m}}{(n-m)!}$$

$$\binom{\frac{s-1}{\alpha}}{n} = [z^n] \left(e^{(s-1+\alpha)z/\alpha} \left(\frac{-ze^{-z}}{e^{-z}-1} \right)^{n+1} \right) \tag{1.7.b}$$

$$= [z^n w^n] \left(-\frac{z \cdot e^{(s-1+\alpha)z/\alpha}}{1+wz-e^z} \right).$$

A more extensive treatment of the properties and generating function relations satisfied by the triangular coefficients defined by (1.4), including their similarities to the Stirling number triangles, Stirling polynomial sequences, and the generalized Bernoulli polynomials, among relations to several other notable special sequences, is provided in the references [22, *cf.* §5] [13; 15; 21]. The results in Section 6.4 and Section 6.5 of the article suggest further related applications of the generalized forms of these triangles enumerated in the references.

¶F.5 *Note §1.5.* The generalized forms of the *Stirling convolution polynomials*, $\sigma_n(x)$, and the α*–factorial polynomials*, $\sigma_n^{(\alpha)}(x)$, studied in the reference are defined for each $n \geq 0$ by the triangle in (1.4) as follows [22, §5.2]:

$$x \cdot \sigma_n^{(\alpha)}(x) := \begin{bmatrix} x \\ x-n \end{bmatrix}_{\alpha} \frac{(x-n-1)!}{(x-1)!} = [z^n] \left(e^{(1-\alpha)z} \left(\frac{\alpha z e^{\alpha z}}{e^{\alpha z}-1} \right)^x \right). \qquad \text{(Generalized Stirling Polynomials)}$$

Table A.2 (page 78) provides listings of the first several examples of these polynomials and the corresponding special case corresponding to the Stirling polynomials, $\sigma_n(x)$, when $\alpha := 1$.

1.2 Divergent ordinary generating functions approximated by the convergents to infinite Jacobi and Stieltjes type continued fractions

Another approach to enumerating the symbolic expansions of the generalized α–factorial function sequences outlined above is constructed as a new generalization of the continued fraction series representations of the ordinary generating function for the rising factorial function, or Pochhammer symbol, $(x)_n = \Gamma(x+n)/\Gamma(x)$, proved by Flajolet in the references [9; 10]. For any fixed non–zero indeterminate, $x \in \mathbb{C}$, the ordinary power series enumerating the rising factorial sequence is defined through the next infinite Jacobi–type J–fraction expansion [9, §2, p. 148]:

$$R_0(x,z) := \sum_{n\geq 0} (x)_n z^n = \cfrac{1}{1 - xz - \cfrac{1\cdot xz^2}{1-(x+2)z - \cfrac{2(x+1)z^2}{\cdots}}}. \tag{1.8}$$

Since we know symbolic polynomial expansions of the functions, $(x)_n$, through the Stirling numbers of the first kind, we notice that the terms in a convergent power series defined by (1.8) correspond to the coefficients of the following well–known two–variable "*double*", or "*super*", exponential generating functions (EGFs) for the Stirling number triangle when x is taken to be a fixed, formal parameter with respect to these series [13, §7.4] [19, §26.8(ii)] [9, *cf.* Prop. 9] ‖F.6 :

$$\sum_{n\geq 0} (x)_n \frac{z^n}{n!} = \frac{1}{(1-z)^x} \qquad \text{and} \qquad \sum_{m,n\geq 0} s(n,m) \frac{w^m z^n}{n!} = (1+z)^w.$$

When x depends linearly on n, the ordinary generating functions for the numerical factorial functions formed by $(x)_n$ do not converge for $z \neq 0$. However, the convergents of the continued fraction representations of these series still lead to partial, truncated series approximations enumerating these generalized product sequences, which in turn immediately satisfy a number of combinatorial properties, recurrence relations, and other established integer congruence properties implied by the rational convergents to the first continued fraction expansion given in (1.8).

Two particular divergent ordinary generating functions for the single factorial function sequences, $f_1(n) := n!$ and $f_2(n) := (n+1)!$, are cited in the references as examples of the Jacobi–type J–fraction results proved in Flajolet's articles [9; 10] [17, *cf.* §5.5]. The next pair of series expansions serve to illustrate the utility to enumerating each sequence formally with respect to z required by the results in this article [9, Thm. 3A; Thm. 3B] [A000142].

$$F_{1,\infty}(z) := \sum_{n\geq 0} n! \cdot z^n = \cfrac{1}{1 - z - \cfrac{1^2\cdot z^2}{1-3z-\cfrac{2^2z^2}{\cdots}}} \qquad \text{(Single Factorial J-Fractions)}$$

‖F.6 *Note §1.6.* For natural numbers $m \geq 1$ and fixed $\alpha \in \mathbb{Z}^+$, the coefficients defined by the generalized triangles in (1.4) are enumerated similarly by the generating function [22, *cf.* §3.3]

$$\sum_{m,n\geq 0} \begin{bmatrix} n+1 \\ m+1 \end{bmatrix}_\alpha \frac{w^m z^n}{n!} = (1-\alpha z)^{-(w+1)/\alpha}. \tag{1.9}$$

$$F_{2,\infty}(z) := \sum_{n \geq 0} (n+1)! \cdot z^n \qquad = \cfrac{1}{1 - 2z - \cfrac{1 \cdot 2z^2}{1 - 4z - \cfrac{2 \cdot 3z^2}{\cdots}}}$$

In each of these respective formal power series expansions, we immnediately see that for each finite $h \geq 1$, the h^{th} convergent functions, denoted $F_{i,h}(z)$ for $i = 1, 2$, satisfy $f_i(n) = [z^n]F_{i,h}(z)$ whenever $1 \leq n \leq 2h$. We also have that $[z^n]F_{i,h}(z) \equiv f_i(n) \pmod{p}$ for any $n \geq 0$ whenever p is a divisor of h [10] [17, *cf.* §5]. Similar expansions of other factorial–related continued fraction series are given in the references [9] [17, *cf.* §5.9].

For example, the next known *Stieltjes type* continued fractions (*S–fractions*), formally generating the double factorial function, $(2n-1)!!$, and the *Catalan numbers*, $C_n := \binom{2n}{n} \frac{1}{(n+1)}$, respectively, are expanded through the convergents of the following infinite continued fractions [9, Prop. 5; Thm. 2] [17, §5.5] [A000165; A000108]:

$$\sum_{n \geq 0} \underbrace{1 \cdot 3 \cdots (2n-1)}_{(2n-1)!!} \times z^{2n} = \cfrac{1}{1 - \cfrac{1 \cdot z^2}{1 - \cfrac{2 \cdot z^2}{1 - \cfrac{3 \cdot z^2}{\cdots}}}} \qquad \text{(Double Factorial S-Fractions)}$$

$$\sum_{n \geq 0} \underbrace{\frac{2^n (2n-1)!!}{(n+1)!}}_{C_n} \times z^{2n} = \cfrac{1}{1 - \cfrac{z^2}{1 - \cfrac{z^2}{1 - \cfrac{z^2}{\cdots}}}}.$$

For comparison, some related forms of regularized ordinary power series in z generating the single and double factorial function sequences from the previous examples are stated in terms of the *incomplete gamma function*, $\Gamma(a,z) = \int_z^\infty t^{a-1}e^{-t}dt$, as follows [19, §8; *cf.* §18.5–18.6] **F.7 :

$$\begin{aligned}
\sum_{n\geq 0} n!\cdot z^n &= -\frac{e^{-1/z}}{z}\times\Gamma\left(0,-\frac{1}{z}\right) \\
\sum_{n\geq 0} (n+1)!\cdot z^n &= -\frac{e^{-1/z}}{z^2}\times\Gamma\left(-1,-\frac{1}{z}\right) \\
\sum_{n\geq 1} (2n-1)!!\cdot z^n &= -\frac{e^{-1/2z}}{(-2z)^{3/2}}\times\Gamma\left(-\frac{1}{2},-\frac{1}{2z}\right).
\end{aligned} \tag{1.10}$$

The remarks given in Section 3.3 suggest similar approximations to the α–factorial functions generated by the generalized convergent functions defined in the next section, and their relations to the confluent hypergeometric functions and the associated Laguerre polynomial sequences [19, *cf.* §18.5(ii)] [21].

1.3 Generalized convergent functions for factorial–related product sequences

1.3.1 Definition of the generalized J–fractions and convergent function series expansions

We state the next definition as a generalization of the result for the rising factorial function due to Flajolet cited in (1.8) to form the analogous series enumerating the multiple, α–factorial product sequence cases defined by (1.1) and (1.2).

Definition 1.1 (Generalized J-Fraction Convergent Functions)**.** Suppose that the parameters $\alpha\in\mathbb{Z}^+$ and $R := R(n)$ are defined in the notation of the product–wise sequences from (1.1). For

**F.7 *Note §1.7.* Since $p_n(\alpha,R) = \alpha^n\,(R/\alpha)_n$, the exponential generating function for the generalized product sequences corresponds to the series [13, *cf.* §7, (7.55)] [15; 21]

$$\widehat{P}(\alpha,R;z) := \sum_{n=0}^\infty p_n(\alpha,R)\frac{z^n}{n!} = (1-\alpha z)^{-R/\alpha},$$

where for each fixed $\alpha\in\mathbb{Z}^+$ and $0\leq r<\alpha$, we have the identities, $(\alpha n - r)!_{(\alpha)} = p_n(\alpha,\alpha-r) = \alpha^n\left(1-\frac{r}{\alpha}\right)_n$. The form of this exponential generating function leads to the next forms of the regularized sums (see Remark 4.5) [19, *cf.* §8.6(i)].

$$\begin{aligned}
\widetilde{B}_{\alpha,-r}(z) &:= \sum_{n\geq 0}(\alpha n-r)!_{(\alpha)}z^n \\
&= \int_0^\infty \frac{e^{-t}}{(1-\alpha t z)^{1-r/\alpha}}dt = \frac{e^{-\frac{1}{\alpha z}}}{(-\alpha z)^{1-r/\alpha}}\times\Gamma\left(\frac{r}{\alpha},-\frac{1}{\alpha z}\right)
\end{aligned} \tag{Borel Regularized Sums}$$

The related form of a two–variable, diagonal–coefficient ordinary generating generating function for the product sequences, $p_n(\alpha,\beta n+\gamma)$, is obtained similarly by performing the integral transform

$$\begin{aligned}
\widetilde{P}_1(w,z) &:= \int_0^\infty e^{-t}\times\left(\sum_{n\geq 0}(1-\alpha z)^{-(\beta n+\gamma)/\alpha}(tw)^n\right)dt \\
&= -\frac{(1-\alpha z)^{(\beta-\gamma)/\alpha}}{w}\times\exp\left(-\frac{(1-\alpha z)^{\beta/\alpha}}{w}\right)\times\Gamma\left(0,-\frac{(1-\alpha z)^{\beta/\alpha}}{w}\right),
\end{aligned}$$

where $p_n(\alpha,\beta n+\gamma) = [z^n w^n]\widetilde{P}_1(w,z)$ for all $n\geq 0$.

$h \geq 0$ and $z \in \mathbb{C}$, let the component numerator and denominator convergent functions, denoted $\mathrm{FP}_h(\alpha, R; z)$ and $\mathrm{FQ}_h(\alpha, R; z)$, respectively, be defined by the next equations.

$$\mathrm{FP}_h(\alpha, R; z) := \begin{cases} (1-(R+2\alpha(h-1))z)\,\mathrm{FP}_{h-1}(\alpha,R;z)-\alpha(R+\alpha(h-2))(h-1)z^2\,\mathrm{FP}_{h-2}(\alpha,R;z), & \text{if } h \geq 2; \\ 1, & \text{if } h = 1; \\ 0, & \text{otherwise.} \end{cases} \tag{1.11}$$

$$\mathrm{FQ}_h(\alpha, R; z) := \begin{cases} (1-(R+2\alpha(h-1))z)\,\mathrm{FQ}_{h-1}(\alpha,R;z)-\alpha(R+\alpha(h-2))(h-1)z^2\,\mathrm{FQ}_{h-2}(\alpha,R;z), & \text{if } h \geq 2; \\ 1-Rz, & \text{if } h = 1; \\ 1, & \text{if } h = 0; \\ 0, & \text{otherwise.} \end{cases} \tag{1.12}$$

The corresponding convergent functions, $\mathrm{Conv}_h\,(\alpha, R; z)$, defined in the next equation provide the rational, formal power series approximations in z to the divergent ordinary generating functions of many factorial–related sequences formed as special cases of the symbolic products in (1.1).

$$\mathrm{Conv}_h\,(\alpha, R; z) = \cfrac{1}{1 - R\cdot z - \cfrac{\alpha R\cdot z^2}{1-(R+2\alpha)\cdot z - \cfrac{2\alpha(R+\alpha)\cdot z^2}{1-(R+4\alpha)\cdot z - \cfrac{3\alpha(R+2\alpha)\cdot z^2}{\cfrac{\cdots}{1-(R+2(h-1)\alpha)\cdot z}}}}}$$

$$\mathrm{Conv}_h\,(\alpha, R; z) := \frac{\mathrm{FP}_h(\alpha, R; z)}{\mathrm{FQ}_h(\alpha, R; z)} = \sum_{n=0}^{2h-1} p_n(\alpha, R) z^n + \sum_{n=2h}^{\infty} \widetilde{e}_{h,n}(\alpha, R) z^n \tag{1.13}$$

The first series coefficients on the right–hand–side of (1.13) enumerate the products, $p_n(\alpha, R)$, from (1.1), where the remaining forms of the power series coefficients, $\widetilde{e}_{h,n}(\alpha, R)$, correspond to "error terms" in the truncated formal series approximations to the exact sequence generating functions obtained from these convergent functions. Ⓓ

1.3.2 Properties of the generalized J–fraction convergent functions

A number of the immediate, noteworthy properties satisfied these generalized convergent functions are apparent from inspection of the first few special cases provided in Table A.3 (page 79) and in Table A.4 (page 80). The most important of these properties relevant to the new interpretations of the α–factorial function sequences proved in the next sections of the article are briefly summarized in the points stated below.

1. **Rationality of the convergent functions**:
   For any fixed $h \geq 1$, it is easy to show that the component convergent functions, $\mathrm{FP}_h(z)$ and $\mathrm{FQ}_h(z)$, defined by (1.11) and (1.12), respectively, are polynomials of finite degree in each of z, R, and α satisfying
   $$\deg_{z,R,\alpha}\{\mathrm{FP}_h(\alpha, R; z)\} = h - 1 \quad \text{and} \quad \deg_{z,R,\alpha}\{\mathrm{FQ}_h(\alpha, R; z)\} = h.$$
   For any $h, n \in \mathbb{Z}^+$, if $R := R(n)$ denotes some linear function of n, the product sequences, $p_n(\alpha, R)$, generated by the generalized convergent functions always correspond to polynomials in n (in R) of predictably finite degree with integer coefficients determined by the choice of $n \geq 1$ expanded by the results in Section 6.

2. **Exact representations by special functions**:
For all $h \geq 0$, and fixed non–zero parameters α and R, the power series in z generated by the generalized h^{th} convergents, $\mathrm{Conv}_h(\alpha, R; z)$, are characterized by the representations of the convergent denominator functions, $\mathrm{FQ}_h(\alpha, R; z)$, through the confluent hypergeometric functions, $U(a, b, w)$ and $M(a, b, w)$, and the associated Laguerre polynomial sequences, $L_n^{(\beta)}(x)$, as follows [19, §13; §18] [21]:

$$\underbrace{z^h \cdot \mathrm{FQ}_h\left(\alpha, R; z^{-1}\right)}_{\widetilde{\mathrm{FQ}}_h(\alpha, R; z)} = \alpha^h \times U\left(-h, \frac{R}{\alpha}, \frac{z}{\alpha}\right) \tag{1.14}$$

$$= (-\alpha)^h \, (R/\alpha)_h \times M\left(-h, \frac{R}{\alpha}, \frac{z}{\alpha}\right)$$

$$= (-\alpha)^h \cdot h! \times L_h^{(R/\alpha-1)}\left(\frac{z}{\alpha}\right).$$

The special function expansions of the reflected convergent denominator function sequences above lead to the statements of addition theorems, multiplication theorems, and several additional auxiliary recurrence relations for these functions proved in Section 5.1.

3. **Exact sequence formulas and congruence properties**:
If some ordering of the h zeros of (1.14) is fixed at each $h \geq 1$, we can define the next sequences forming special cases the zeros studied in the references [3; 11]. In particular, each of the following special zero sequence definitions provide factorizations over z of the denominator sequences, $\mathrm{FQ}_h(\alpha, R; z)$, parametrized by α and R:

$$\left(\ell_{h,j}(\alpha, R)\right)_{j=1}^{h} := \left\{z_j : \alpha^h \times U\left(-h, R/\alpha, \frac{z}{\alpha}\right) = 0,\ 1 \leq j \leq h\right\} \qquad \text{(Special Function Zeros)}$$

$$= \left\{z_j : \alpha^h \times L_h^{(R/\alpha-1)}\left(\frac{z}{\alpha}\right) = 0,\ 1 \leq j \leq h\right\}.$$

Let the sequences, $c_{h,j}(\alpha, R)$, denote a shorthand for the coefficients corresponding to an expansion of the generalized convergent functions, $\mathrm{Conv}_h(\alpha, R; z)$, by partial fractions in z [19, §1.2(iii)].

For $n \geq 1$ and any fixed integer $\alpha \neq 0$, these rational convergent functions provide the following formulas exactly generating the respective sequence cases in (1.1) and (1.2):

$$p_n(\alpha, R) = \sum_{j=1}^{n} c_{n,j}(\alpha, R) \times \ell_{n,j}(\alpha, R)^n \tag{1.15.a}$$

$$n!_{(\alpha)} = \sum_{j=1}^{n} c_{n,j}(-\alpha, n) \times \ell_{n,j}(-\alpha, n)^{\lfloor \frac{n-1}{\alpha} \rfloor}.$$

The corresponding congruences satisfied by each of these generalized sequence cases obtained from the h^{th} convergent function expansions in z are stated similarly modulo any prescribed integers $h \geq 2$ and fixed $\alpha \geq 1$ in the next forms.

$$p_n(\alpha, R) \equiv \sum_{j=1}^{h} c_{h,j}(\alpha, R) \times \ell_{h,j}(\alpha, R)^n \qquad \pmod{h} \tag{1.15.b}$$

$$n!_{(\alpha)} \equiv \sum_{j=1}^{h} c_{h,j}(-\alpha, n) \times \ell_{h,j}(-\alpha, n)^{\lfloor \frac{n-1}{\alpha} \rfloor} \qquad (\mathrm{mod}\ h, h\alpha, \cdots, h\alpha^h)$$

The notation $g_1(n) \equiv g_2(n) \pmod{n_1, n_2}$ in the previous equations denotes that the stated congruences hold modulo either base, n_1 or n_2. Section 1.4.2 and Section 4.1 provide several particular special case examples of the new congruence properties expanded by (1.15.b).

1.4 Examples of the new results

1.4.1 Examples: Factorial–related finite product sequences enumerated by the generalized convergent functions

There are a couple of noteworthy subtleties that arise in defining the specific numerical forms of the α–factorial function sequences defined by (1.2) and (1.5.b). First, since the generalized convergent functions generate the distinct symbolic products that characterize the forms of these expansions, we see that the following convergent–based enumerations of the multiple factorial sequence variants hold at each $\alpha, n \in \mathbb{Z}^+$, and some fixed choice of the prescribed offset, $0 \leq d < \alpha$:

$$\begin{aligned}
(\alpha n - d)!_{(\alpha)} &= \underbrace{(-\alpha)^n \cdot \left(\frac{d}{\alpha} - n\right)_n}_{p_n(-\alpha, \alpha n - d)} = [z^n]\,\mathrm{Conv}_n\left(-\alpha, \alpha n - d; z\right) \qquad (1.16.a) \\
&= \underbrace{\alpha^n \cdot \left(1 - \frac{d}{\alpha}\right)_n}_{p_n(\alpha, \alpha - d)} \qquad = [z^n]\,\mathrm{Conv}_n\left(\alpha, \alpha - d; z\right).
\end{aligned}$$

For example, some variants of the arithmetic progression sequences formed by the single factorial and double factorial functions, $n!$ and $n!!$, in Section 4.2.3 are generated by the particular shifted inputs to these functions highlighted by the special cases in the next equations [A000142; A001147; A000165]:

$$\begin{aligned}
(n!)_{n=1}^{\infty} &= \left((1)_n\right)_{n=1}^{\infty} && \xrightarrow{\texttt{A000142}} && (1, 2, 6, 24, 120, 720, 5040, \ldots) \\
((2n)!!)_{n=1}^{\infty} &= \left(2^n \cdot (1)_n\right)_{n=1}^{\infty} && \xrightarrow{\texttt{A001147}} && (2, 8, 48, 384, 3840, 46080, \ldots) \\
((2n-1)!!)_{n=1}^{\infty} &= \left(2^n \cdot (1/2)_n\right)_{n=1}^{\infty} && \xrightarrow{\texttt{A000165}} && (1, 3, 15, 105, 945, 10395, \ldots).
\end{aligned}$$

The next few special case variants of the α–factorial function sequences corresponding to $\alpha := 3, 4$, also expanded in Section 4.2.3, are given in the following sequence forms [A032031; A008554; A007559; A034176; A000407; A007696; A047053]:

$$\begin{aligned}
((3n)!!!)_{n=1}^{\infty} &= \left(3^n \cdot (1)_n\right)_{n=1}^{\infty} && \xrightarrow{\texttt{A032031}} && (3, 18, 162, 1944, 29160, \ldots) \\
((3n-1)!!!)_{n=1}^{\infty} &= \left(3^n \cdot (2/3)_n\right)_{n=1}^{\infty} && \xrightarrow{\texttt{A008554}} && (2, 10, 80, 880, 12320, 209440, \ldots) \\
((3n-2)!!!)_{n=1}^{\infty} &= \left(3^n \cdot (1/3)_n\right)_{n=1}^{\infty} && \xrightarrow{\texttt{A007559}} && (1, 4, 28, 280, 3640, 58240, \ldots) \\
\left((4n)!_{(4)}\right)_{n=0}^{\infty} &= \left(4^n \cdot (1)_n\right)_{n=0}^{\infty} && \xrightarrow{\texttt{A047053}} && (1, 4, 32, 384, 6144, 122880, \ldots) \\
\left((4n+1)!_{(4)}\right)_{n=0}^{\infty} &= \left(4^n \cdot (5/4)_n\right)_{n=0}^{\infty} && \xrightarrow{\texttt{A007696}} && (1, 5, 45, 585, 9945, 208845, \ldots)
\end{aligned}$$

$$\left(\tfrac{1}{2}\,(4n+2)!_{(4)}\right)_{n=0}^{\infty} = \left(4^n \cdot (3/2)_n\right)_{n=0}^{\infty} \quad \xrightarrow{\text{A000407}} \quad (1, 6, 60, 840, 15120, 332640, \ldots)$$

$$\left(\tfrac{1}{3}\,(4n+3)!_{(4)}\right)_{n=0}^{\infty} = \left(4^n \cdot (7/4)_n\right)_{n=0}^{\infty} \quad \xrightarrow{\text{A034176}} \quad (1, 7, 77, 1155, 21945, 504735, \ldots).$$

Likewise, given any $n \geq 1$ and fixed $\alpha \in \mathbb{Z}^+$, we can enumerate the somewhat less obvious full forms of the generalized α–factorial function sequences defined piecewise for the distinct residues, $n \in \{0, 1, \ldots, \alpha - 1\}$, modulo α by (1.2) and in (1.16.a). The multi–valued products defined by (1.1) for these functions are generated as follows:

$$\begin{aligned} n!_{(\alpha)} &= \left[z^{\lfloor (n+\alpha-1)/\alpha \rfloor}\right] \operatorname{Conv}_n(-\alpha, n; z) && \text{(1.16.b)} \\ &= [z^n] \left(\sum_{0 \leq d < \alpha} z^{-d} \cdot \operatorname{Conv}_n(\alpha, \alpha - d; z^\alpha) \right) \\ &= [z^{n+\alpha-1}] \left(\frac{1 - z^\alpha}{1 - z} \times \operatorname{Conv}_n(-\alpha, n; z^\alpha) \right). && \text{(1.16.c)} \end{aligned}$$

The complete sequences over the multi–valued symbolic products formed by the special cases of the double factorial function, the *triple factorial* function, $n!!!$, the *quadruple factorial* function, $n!!!! = n!_{(4)}$, the *quintuple factorial* (5–*factorial*) function, $n!_{(5)}$, and the 6–*factorial* function, $n!_{(6)}$, respectively, are generated by the convergent generating function approximations expanded in the next equations [A006882; A007661; A007662; A085157; A085158].

$$(n!!)_{n=1}^{\infty} = \left(\left[z^{\lfloor (n+1)/2 \rfloor}\right] \operatorname{Conv}_n(-2, n; z)\right)_{n=1}^{\infty} \quad \xrightarrow{\text{A006882}} \quad (1, 2, 3, 8, 15, 48, 105, 384, 945, 3840, \ldots)$$

$$(n!!!)_{n=1}^{\infty} = \left(\left[z^{\lfloor (n+2)/3 \rfloor}\right] \operatorname{Conv}_n(-3, n; z)\right)_{n=1}^{\infty} \quad \xrightarrow{\text{A007661}} \quad (1, 2, 3, 4, 10, 18, 28, 80, 162, 280, \ldots)$$

$$\left(n!_{(4)}\right)_{n=1}^{\infty} = \left(\left[z^{\lfloor (n+3)/4 \rfloor}\right] \operatorname{Conv}_n(-4, n; z)\right)_{n=1}^{\infty} \quad \xrightarrow{\text{A007662}} \quad (1, 2, 3, 4, 5, 12, 21, 32, 45, 120, 231, \ldots)$$

$$\left(n!_{(5)}\right)_{n=1}^{\infty} = \left(\left[z^{\lfloor (n+4)/5 \rfloor}\right] \operatorname{Conv}_n(-5, n; z)\right)_{n=1}^{\infty} \quad \xrightarrow{\text{A085157}} \quad (1, 2, 3, 4, 5, 6, 14, 24, 36, 50, 66, 168, \ldots)$$

$$\left(n!_{(6)}\right)_{n=1}^{\infty} = \left(\left[z^{\lfloor (n+5)/6 \rfloor}\right] \operatorname{Conv}_n(-6, n; z)\right)_{n=1}^{\infty} \quad \xrightarrow{\text{A085158}} \quad (1, 2, 3, 4, 5, 6, 7, 16, 27, 40, 55, 72, 91, \ldots)$$

Moreover, for each $n \in \mathbb{N}$ and prescribed constants $r, c \in \mathbb{Z}$ defined such that $c \mid n + r$, we also obtain rational convergent–based generating functions enumerating the modified multiple factorial function sequences given by

$$\left(\tfrac{n+r}{c}\right)! = [z^n] \operatorname{Conv}_h\left(-c, n + r; \frac{z}{c}\right) + \left[\frac{r}{c} = 0\right]_\delta [n = 0]_\delta, \; \forall\, h \geq \lfloor (n + r)/c \rfloor. \tag{1.17}$$

The identity corresponding to the special case of (1.17) when $c := 2$ leads to the convergent–based generating function expansions of the congruence for the primes of the form $p = 4k + 1$ [A002144] cited by (1.23) in the examples given below in Section 1.4.3. The rationality in z of the generalized convergent functions, $\operatorname{Conv}_h(\alpha, R; z)$, for all $h \geq 1$ also provides several of the new forms of generating function identities for many factorial–related product sequences and related expansions of the binomial coefficients that are easily established from the diagonal–coefficient, or Hadamard product, generating function results established in Section 4.2.

For example, for natural numbers $n \geq 1$, the next variants of the binomial–coefficient–related product sequences are enumerated by the following coefficient identities [A009120; A001448]:

$$\frac{(4n)!}{(2n)!} = \frac{4^{4n} \cancel{(1)_n} \cancel{\left(\tfrac{2}{4}\right)_n} \left(\tfrac{1}{4}\right)_n \left(\tfrac{3}{4}\right)_n}{2^{2n} (1)_n \left(\tfrac{1}{2}\right)_n} \tag{1.18}$$

$$\begin{aligned}
&= 4^n \times 4^n\,(1/4)_n \times 4^n\,(3/4)_n \\
&= [x_1^0 z^n]\left(\operatorname{Conv}_n\left(4, 3; \frac{4z}{x_1}\right)\operatorname{Conv}_n\left(4, 1; x_1\right)\right) \\
&= 4^n \times (4n-3)!_{(4)}\,(4n-1)!_{(4)} \\
&= [x_1^0 z^n]\left(\operatorname{Conv}_n\left(-4, 4n-3; \frac{4z}{x_1}\right)\operatorname{Conv}_n\left(-4, 4n-1; x_1\right)\right) \\
\binom{4n}{2n} &= [x_1^0 x_2^0 z^n]\left(\operatorname{Conv}_n\left(4, 3; \frac{4z}{x_2}\right)\operatorname{Conv}_n\left(4, 1; \frac{x_2}{x_1}\right) \times \underbrace{\cosh\left(\sqrt{x_1}\right)}_{\widehat{E}_2(z) = E_{2,1}(z)}\right) \\
&= [x_1^0 x_2^0 z^n]\left(\operatorname{Conv}_n\left(-4, 4n-3; \frac{4z}{x_2}\right)\operatorname{Conv}_n\left(-4, 4n-1; \frac{x_2}{x_1}\right) \times \underbrace{\cosh\left(\sqrt{x_1}\right)}_{\widehat{E}_2(z) = E_{2,1}(z)}\right).
\end{aligned}$$

The examples given in Section 4.2.2 cite related constructions of the hybrid rational convergent–based generating function products that generate the central binomial coefficients and several other notable cases of related sequences.

1.4.2 Examples: New congruences for the α–factorial functions, the Stirling numbers of the first kind, and the r–order harmonic number sequences

For any fixed $\alpha \in \mathbb{Z}^+$ and natural numbers $n \geq 1$, the generalized multiple, α–factorial functions, $n!_{(\alpha)}$, defined by (1.2) satisfy the following congruences modulo 2 (and 2α):

$$n!_{(\alpha)} \equiv \frac{n}{2}\left(\left(-\alpha + \sqrt{\alpha(\alpha-n)} + n\right)^{\left\lfloor \frac{n-1}{\alpha} \right\rfloor} + \left(-\alpha - \sqrt{\alpha(\alpha-n)} + n\right)^{\left\lfloor \frac{n-1}{\alpha} \right\rfloor}\right) \pmod{2, 2\alpha}. \tag{1.19}$$

Given that the definition of the single factorial function implies that $n! \equiv 0 \pmod 2$ whenever $n \geq 2$, the statement of (1.19) provides somewhat less obvious results for the generalized α–factorial function sequence cases when $\alpha \geq 2$. Table A.5 (page 84) provides specific listings of the result in (1.19) satisfied by the α–factorial functions, $n!_{(\alpha)}$, for $\alpha := 1, 2, 3, 4$. The corresponding, closely–related new forms of congruence properties satisfied by these functions expanded through analogous exact algebraic formulas modulo 3 (3α) and modulo 4 (4α) are also cited as special cases in the next examples.

To simplify notation, we first define the next shorthand for the respective (distinct) roots, $r_{p,i}^{(\alpha)}(n)$ for $1 \leq i \leq p$, corresponding to the special cases of the convergent denominator functions, $\mathrm{FQ}_p(\alpha, R; z)$, factorized over z for any fixed integers $n, \alpha \geq 1$ when $p := 3, 4, 5$ [19, §1.11(iii); *cf.* §4.43]:

$$\begin{aligned}
\left(r_{3,i}^{(\alpha)}(n)\right)_{i=1}^{3} &:= \left\{z_i : z_i^3 - 3z_i^2(2\alpha+n) + 3z_i(\alpha+n)(2\alpha+n) - n(\alpha+n)(2\alpha+n) = 0,\ 1 \leq i \leq 3\right\} \\
\left(r_{4,j}^{(\alpha)}(n)\right)_{j=1}^{4} &:= \left\{z_j : z_j^4 - 4z_j^3(3\alpha+n) + 6z_j^2(2\alpha+n)(3\alpha+n) - 4z_j(\alpha+n)(2\alpha+n)(3\alpha+n)\right. \\
&\qquad \left. + n(\alpha+n)(2\alpha+n)(3\alpha+n) = 0,\ 1 \leq j \leq 4\right\} \\
\left(r_{5,k}^{(\alpha)}(n)\right)_{k=1}^{5} &:= \left\{z_k : z_k^5 - 5(4\alpha+n)z_k^4 + 10(3\alpha+n)(4\alpha+n)z_k^3 - 10(2\alpha+n)(3\alpha+n)(4\alpha+n)z_k^2\right.
\end{aligned}$$

$$\begin{aligned}
&+5(\alpha+n)(2\alpha+n)(3\alpha+n)(4\alpha+n)z_k \\
&-n(\alpha+n)(2\alpha+n)(3\alpha+n)(4\alpha+n)=0,\ 1\leq k\leq 5\big\}.
\end{aligned} \tag{1.20}$$

Similarly, we define the following functions for any fixed $\alpha\in\mathbb{Z}^+$ and $n\geq 1$ to simplify the notation in stating next the congruences in (1.21) below:

$$\begin{aligned}
\widetilde{R}_3^{(\alpha)}(n) &:= \frac{\left(6\alpha^2+\alpha\left(6r_{3,1}^{(-\alpha)}(n)-4n\right)+\left(n-r_{3,1}^{(-\alpha)}(n)\right)^2\right) r_{3,1}^{(-\alpha)}(n)^{\left\lfloor\frac{n-1}{\alpha}\right\rfloor+1}}{\left(r_{3,1}^{(-\alpha)}(n)-r_{3,2}^{(-\alpha)}(n)\right)\left(r_{3,1}^{(-\alpha)}(n)-r_{3,3}^{(-\alpha)}(n)\right)} \\
&\quad+\frac{\left(6\alpha^2+\alpha\left(6r_{3,3}^{(-\alpha)}(n)-4n\right)+\left(n-r_{3,3}^{(-\alpha)}(n)\right)^2\right) r_{3,3}^{(-\alpha)}(n)^{\left\lfloor\frac{n-1}{\alpha}\right\rfloor+1}}{\left(r_{3,3}^{(-\alpha)}(n)-r_{3,1}^{(-\alpha)}(n)\right)\left(r_{3,3}^{(-\alpha)}(n)-r_{3,2}^{(-\alpha)}(n)\right)} \\
&\quad+\frac{\left(6\alpha^2+\alpha\left(6r_{3,2}^{(-\alpha)}(n)-4n\right)+\left(n-r_{3,2}^{(-\alpha)}(n)\right)^2\right) r_{3,2}^{(-\alpha)}(n)^{\left\lfloor\frac{n-1}{\alpha}\right\rfloor+1}}{\left(r_{3,2}^{(-\alpha)}(n)-r_{3,1}^{(-\alpha)}(n)\right)\left(r_{3,2}^{(-\alpha)}(n)-r_{3,3}^{(-\alpha)}(n)\right)} \\
C_{4,i}^{(\alpha)}(n) &:= 24\alpha^3-18\alpha^2\left(n-2\cdot r_{4,i}^{(-\alpha)}(n)\right)+\alpha\left(7n-12\cdot r_{4,i}^{(-\alpha)}(n)\right)\left(n-r_{4,i}^{(-\alpha)}(n)\right) \\
&\qquad-\left(n-r_{4,i}^{(-\alpha)}(n)\right)^3,\quad \text{for } 1\leq i\leq 4 \\
C_{5,k}^{(\alpha)}(n) &:= 120\alpha^4+2\alpha^2\left(23n^2-79n\cdot r_{5,k}^{(-\alpha)}(n)+60\cdot r_{5,k}^{(-\alpha)}(n)^2\right)+48\alpha^3(2n-5\cdot r_{5,k}^{(-\alpha)}(n)) \\
&\qquad+\alpha(11n-20r_{5,k}^{(-\alpha)}(n))(n-r_{5,k}^{(-\alpha)}(n))^2+(n-r_{5,k}^{(-\alpha)}(n))^4,\quad \text{for } 1\leq k\leq 5.
\end{aligned}$$

For fixed $\alpha\in\mathbb{Z}^+$ and $n\geq 0$, we obtain the following analogs to the first congruence result modulo 2 expanded by (1.19) for the α–factorial functions, $n_{(\alpha)}$, when $n\geq 1$:

$$\begin{aligned}
n!_{(\alpha)} &\equiv \widetilde{R}_3^{(\alpha)}(n) && \pmod{3, 3\alpha} \\
n!_{(\alpha)} &\equiv \underbrace{\sum_{1\leq i\leq 4}\frac{C_{4,i}^{(\alpha)}(n)}{\prod\limits_{j\neq i}\left(r_{4,i}^{(-\alpha)}(n)-r_{4,j}^{(-\alpha)}(n)\right)} r_{4,i}^{(-\alpha)}(n)^{\left\lfloor\frac{n+\alpha-1}{\alpha}\right\rfloor}}_{:=R_4^{(\alpha)}(n)} && \pmod{4, 4\alpha} \\
n!_{(\alpha)} &\equiv \underbrace{\sum_{1\leq k\leq 5}\frac{C_{5,k}^{(\alpha)}(n)}{\prod\limits_{j\neq k}\left(r_{5,k}^{(-\alpha)}(n)-r_{5,j}^{(-\alpha)}(n)\right)} r_{5,k}^{(-\alpha)}(n)^{\left\lfloor\frac{n+\alpha-1}{\alpha}\right\rfloor}}_{:=R_5^{(\alpha)}(n)} && \pmod{5, 5\alpha}.
\end{aligned} \tag{1.21}$$

Several particular concrete examples illustrating the results cited in (1.19) modulo 2 (and 2α), and in (1.21) modulo p (and $p\alpha$) for $p:=3,4,5$, corresponding to the first few cases of $\alpha\geq 1$ and $n\geq 1$ appear in Table A.5 (page 84).

Further computations of the congruences given in (1.21) modulo $p\alpha^i$ for $0\leq i\leq p$ are contained in the *Mathematica* summary notebook included as a supplementary file with the submission of this article (see Section 2.1 and the working reference [23]). The results in Section 4.1 provide statements of these new integer congruences for fixed $\alpha\neq 0$ modulo any integers $p\geq 2$. The analogous formulations of the new relations for the factorial–related product sequences modulo any

p and $p\alpha$ are easily established for the subsequent cases of integers $p \geq 6$ from the properties of the convergent functions, $\mathrm{Conv}_h(\alpha, R; z)$, cited in the particular listings in Table A.3 (page 79) and in Table A.4 (page 80) through the generalized convergent function properties proved in Section 5.

The results given in Section 4.1 also provide new congruences for the generalized Stirling number triangles in (1.4), as well as several new forms of rational generating functions that enumerate the scaled factorial–power veriants of the *r–order harmonic numbers*, $(n!)^r \times H_n^{(r)} := (n!)^r \times (1 + 2^{-r} + 3^{-r} + \cdots + n^{-r})$, modulo integers $p \geq 2$ [13, §6.3]. For example, the known congruences for the Stirling numbers of the first kind proved by the generating function techniques enumerated in the reference [26, §4.6] imply the next new congruence results satisfied by the binomial coefficients modulo 2.

$$\begin{aligned}
&\binom{\lfloor \frac{n}{2} \rfloor}{m - \lceil \frac{n}{2} \rceil} [1 \leq m \leq 6]_{\delta} \\
&\qquad \equiv \left\{\begin{array}{ll}
\frac{2^n}{4} & \text{if } m = 1; \\
\frac{3\cdot 2^n}{16}(n-1) & \text{if } m = 2; \\
\frac{2^n}{128}(9n-20)(n-1) & \text{if } m = 3; \\
\frac{2^n}{512}(3n-10)(3n-7)(n-1) & \text{if } m = 4; \\
\frac{2^n}{8192}(27n^3 - 279n^2 + 934n - 1008)(n-1) & \text{if } m = 5; \\
\frac{2^n}{163840}(9n^2 - 71n + 120)(3n-14)(3n-11)(n-1) & \text{if } m = 6; \\
0 & \text{otherwise.}
\end{array}\right\} \quad [n > m]_{\delta} \\
&\qquad\quad + [1 \leq m \leq 6]_{\delta} [n = m]_{\delta} \qquad\qquad \pmod{2}
\end{aligned}$$

1.4.3 Applications in Wilson's theorem, Clement's theorem, and other restatements of classical congruences and necessary conditions concerning primality

The first examples given in this section provide restatements of the necessary and sufficient integer–congruence–based conditions imposed in both statements of Wilson's theorem and Clement's theorem through the exact expansions of the factorial functions defined above. For odd integers $p \geq 3$, the congruences implicit to each of *Wilson's theorem* and *Clement's theorem* are enumerated as follows [20, §4.3] [5; 13; 14]:

$$\begin{aligned}
p \text{ prime} &\iff (p-1)! + 1 &&\equiv 0 \pmod{p} && \text{(Wilson)} \\
&\iff [z^{p-1}] \,\mathrm{Conv}_p(-1, p-1; z) + 1 &&\equiv 0 \pmod{p} \\
&\iff [z^{p-1}] \,\mathrm{Conv}_p(1, 1; z) + 1 &&\equiv 0 \pmod{p} \\
p, p+2 \text{ prime} &\iff 4\left((p-1)! + 1\right) + p &&\equiv 0 \pmod{p(p+2)} && \text{(Clement)} \\
&\iff 4[z^{p-1}] \,\mathrm{Conv}_{p(p+2)}(-1, p-1; z) + p + 4 &&\equiv 0 \pmod{p(p+2)} \\
&\iff 4[z^{p-1}] \,\mathrm{Conv}_{p(p+2)}(1, 1; z) + p + 4 &&\equiv 0 \pmod{p(p+2)}.
\end{aligned}$$

These particular congruences involving the expansions of the single factorial function are considered in the reference [22, §6.1.6] as an example of the first product–based symbolic factorial function expansions implicit to both Wilson's theorem and Clement's theorem. Section 6 of this article considers the particular cases of these two classically–phrased congruence statements as applications of the new polynomial expansions for the generalized product sequences, $p_n(\alpha, \beta n + \gamma)$, derived from the expansions of the convergent function sequences by finite difference equations. Related formulations of conditions concerning the primality of prime pairs, $(p, p+d)$, and then of other prime k–tuples, are similarly straightforward to obtain by elementary methods starting from the statement of Wilson's theorem [2; 14; 18].

For example, the new results proved in Section 6 are combined with the known congruences established in the reference [18, §3, §5] to obtain the cases of the next particular forms of alternate necessary and sufficient conditions for the twin primality of the odd positive integers $p_1 := 2n+1$ and $p_2 := 2n+3$ when $n \geq 1$ [A001359; A006512]:

$$\begin{aligned} & 2n+1, 2n+3 \text{ odd primes} && (1.22) \\ \Longleftrightarrow\ & 2\left(\sum_{i=0}^{n} \binom{(2n+1)(2n+3)}{i}^2 (-1)^i i!(n-i)!\right)^2 + (-1)^n(10n+7) \equiv 0 \pmod{(2n+1)(2n+3)} \\ \Longleftrightarrow\ & 4\left(\sum_{i=0}^{2n} \binom{(2n+1)(2n+3)}{i}^2 (-1)^i i!(2n-i)!\right) + 2n+5 \equiv 0 \pmod{(2n+1)(2n+3)}. \end{aligned}$$

The rationality in z of the convergent functions, $\mathrm{Conv}_h(\alpha, R; z)$, at each h leads to further alternate formulations of other well–known congruence statements concerning the divisibility of factorial functions. For example, we may characterize the primality of the odd integers, $p > 3$, of the form $p = 4k+1$ according to the next condition [14] [A002144]:

$$\left(\frac{p-1}{2}\right)!^2 \equiv -1 \pmod{p} \iff p \text{ is a prime of the form } 4k+1. \tag{1.23}$$

For an odd integer $p > 3$ to be both prime and satisfy $p \equiv 1 \pmod 4$, the congruence statement in (1.23) requires that the diagonals of the following rational two–variable convergent generating functions satisfy the following equivalent conditions where p_i is chosen so that $2^p p_i \mid p$ for each $i = 1, 2$:

$$\begin{aligned} \left(\tfrac{p-1}{2}\right)!^2 &= [z^{(p-1)/2}][x^0]\left(\mathrm{Conv}_{p_1}\left(-1, \frac{p-1}{2}; x\right)\mathrm{Conv}_{p_2}\left(-1, \frac{p-1}{2}; \frac{z}{x}\right)\right) && \equiv -1 \pmod{p} \\ \left(\tfrac{p-1}{2}\right)!^2 &= [z^{(p-1)/2}][x^0]\left(\mathrm{Conv}_{p_1}(-2, p-1; x)\,\mathrm{Conv}_{p_2}\left(-2, p-1; \frac{z}{4x}\right)\right) && \equiv -1 \pmod{p} \\ \left(\tfrac{p-1}{2}\right)!^2 &= [z^{(p-1)/2}][x^0]\left(\mathrm{Conv}_{p_1}\left(-1, \frac{p-1}{2}; x\right)\mathrm{Conv}_{p_2}\left(-2, p-1; \frac{z}{2x}\right)\right) && \equiv -1 \pmod{p}. \end{aligned}$$

Another related example is given by the next necessary condition following from *Wolstenholme's theorem*, which provides that $p^2 \mid (p-1)! \cdot H_{p-1}$ whenever $p > 3$ is prime, where $H_n = H_n^{(1)}$ is a *first–order harmonic number* [14, §VII.7.8] [A001008; A002805]. This requirement on the primality of the odd integers $p > 3$ is rewritten in this case as the following statements involving the generalized convergent functions and the corresponding exponential generating function for the Stirling numbers, $H_n^{(1)} = \left[{n+1 \atop 2}\right]\frac{1}{n!}$ [13, *cf.* §6, 7.4]:

$$\begin{aligned} p > 3 \text{ prime} \quad &\Longrightarrow\quad [z^{p-1}][x^0]\left(\mathrm{Conv}_{p^2}\left(-1, p-1; \frac{z}{x}\right)\frac{\mathrm{Log}(1-x)}{(1-x)}\right) \equiv 0 \pmod{p^2} && (1.24) \\ &\Longrightarrow\quad [x^0 z^{p-1}]\left(\mathrm{Conv}_{p^2}\left(1, 1; \frac{z}{x}\right)\frac{\mathrm{Log}(1-x)}{(1-x)}\right) \equiv 0 \pmod{p^2}. \end{aligned}$$

The results providing the new congruence properties for the α–factorial functions modulo the integers p, and $p\alpha^i$ for some $0 \leq i \leq p$, expanded in Section 6 also lead to alternate phrasings of the necessary and sufficient conditions on the primality of several notable subsequences of the odd positive integers $n \geq 3$.

For example, the sequence of *Wilson primes*, or the subsequence of odd integers $p \geq 5$ satisfying $n^2 \mid (n-1)!+1$, is characterized through each of the follwing additional divisibility requirements placed on the expansions of the single factorial function implicit to Wilson's theorem cited by the applications of the new results given below in Section 6.3.2 of the article [A007619; A007540]:

$$\underbrace{\sum_{i=0}^{n-1} \binom{n^2}{i}^2 (-1)^i i!(n-1-i)!}_{\equiv\ (n-1)!\pmod{n^2}} \equiv -1 \pmod{n^2}$$

$$\underbrace{\sum_{i=0}^{n-1} \binom{n^2}{i} (n^2-n)^{\underline{i}} \times (-1)^{n-1-i}(n-1)^{\underline{n-1-i}}}_{\equiv\ (n-1)!\pmod{n^2}} \equiv -1 \pmod{n^2}$$

$$\sum_{s=0}^{n-1}\sum_{i=0}^{s}\sum_{v=0}^{i}\left(\sum_{k=0}^{n-1}\sum_{m=0}^{k}\binom{n^2}{k}\binom{m}{s}\binom{i}{v}\binom{n^2+v}{v}\begin{bmatrix}k\\m\end{bmatrix}\begin{Bmatrix}s\\i\end{Bmatrix}(-1)^{i-v}(n-1)^{\underline{n-1-k}}(-n)^{m-s}i!\right) \equiv -1. \pmod{n^2}$$

The integer congruences satisfied by the double factorial function, $(2n-1)!!$, and the Pochhammer symbol cases, $2^n \times \left(\frac{1}{2}\right)_n$, expanded in Section 6.3.3 provide the next variants of the polynomial congruences for the *central binomial coefficients*, $\binom{2n}{n} = 2^n \times (2n-1)!!/n!$, reduced modulo integer multiples of the polynomial powers, n^p, for fixed integers $p \geq 2$ [A000984] (see Section 4.2.2):

$$\begin{aligned}\binom{2n}{n} &\equiv \sum_{i=0}^{n}\binom{n^p}{i}\binom{2n-2i}{n-i}(1/2-n^p)_i \times \frac{8^i \cdot (n-i)!}{n!} && \pmod{n^p} \\ &\equiv \sum_{i=0}^{n}\binom{n^p}{i}(-2)^i\,(1/2+n-n^p)_i\,(1/2-n)_{n-i} \times \frac{2^{2n}}{n!} && \pmod{n^p}.\end{aligned}$$

Additional specific expansions of related congruence properties characterizing the prime subsequences consisting of elements of the form $p := n^2+1$, the *cousin prime* pairs, the *sexy primes* and *sexy prime triplets*, the *factorial* (and *multifactorial*) *primes*, the *generalized Fermat numbers* and *Fermat prime* subsequences, the *Mersenne numbers* and *Mersenne primes*, and the *Sophie Germain primes*, among several of the other notable cases of famous prime subsequences suggested by Remark 6.10, are also cited as other particular applications of the new results given in Section 6.

2 Organization of the article, supplementary documents, notation, and conventions

2.1 Mathematica summary notebook information

The prepared summary notebook file, `multifact-cfracs-summary.nb`, attached to the submission of this manuscript contains working *Mathematica* code to verify the formulas, propositions, and other identities cited within the article [23]. Given the length of the article, the *Mathematica* summary notebook included with this submission is intended to help the reviewer to process the article more quickly, and to help the reader with verifying and modifying the examples presented as applications of the new results cited below. The summary notebook also contains numerical data to verify examples and congruences specifically referenced in several places by the applications given in the next subsections of the article.

2.2 A quick note on the page length of the article

The presentation of the multiple applications to special integer sequences in this manuscript is intended to give a more or less cohesive overview of the new convergent–based generating function and formal power series techniques, as enumerated through specific applications that highlight the new results proved in Section 3.1 and Section 5. The multitude of differing special case identities and integer congruence properties given in the context of the sequences considered below illustrate by example the more significant applications of these new results that are not readily partitioned into multiple separate articles related to the subject matter contained in the next few subsections.

2.3 Organization of the article contents by section

The content in the next several sections of the article is roughly organized according to the next topics summarized in the listings below. Several appropriate inline references related to the motivating examples and applications cited in these next few subsections of the article also suggest further topics for investigation and future research on the new results and factorial–related expansions of the integer congruence properties proved in this article. The final section after the bibliography section concluding the article (starting on page 77) is an appendix containing the listings of all of the tables referenced within the article.

- <u>Section 3</u>. **J–fraction expansions of the generalized convergent functions**.
  Section 3.1 provides a brief overview of the enumerative J–fraction properties established in Flajolet's articles that summarize the properties needed to prove the new results within this article. A short, direct proof of the convergent function representations for the more general product sequence expansions defined in (1.13) is given in Section 3.2. This proof follows as a straightforward adaptation of the known J-fraction expansions enumerating the Pochhammer symbol, $(x)_n$, and the two–variable generating function for the Stirling numbers of the first kind from the reference [9].

- <u>Section 4</u>. **Applications and motivating examples**.
  - **New congruences for the α–factorial functions, generalized cases of the symbolic product sequences, the Stirling number triangles, and the r–order harmonic numbers**:
    Section 4.1 treats the new forms of the integer congruence properties satisfied by special cases of the product sequences, $p_n(\alpha, R)$, and their expansions by the special function zeros from (1.14) in Section 1.3 studied in the references [3; 11].

    This subsection states several new congruence identities satisfied by the Stirling numbers of the first kind, $\left[{n \atop k}\right]$, modulo 2 and modulo 3 for comparison with some of the other known congruences for the triangle expanded in the references [26, §4.6] [6, *cf.* §5.8]. These results also imply the rational generating function expansions enumerating the scaled r–order harmonic number sequences, $(n!)^r \times H_n^{(r)}$, modulo fixed integers p for the first few integer cases of $p \geq 2$ and $r \geq 1$ expanded in Section 4.1.1.

    <u>**Sequence References**</u>:
    A001008, A002805, A007406, A007407, A007408, A007409, A008275, A087755, and A130534.
  - **Convergent–based generating function transformations and identities enumerating factorial–related sums and product sequences**:
    The examples cited in the subsections of Section 4.2 are intended to provide a number of

specific representative applications that arise in enumerating factorial–related sums, binomial power sequences, and other special products through the rational convergent function approximations, $\mathrm{Conv}_h(\alpha, R; z)$, provided by (1.13) through the proof in Section 3.2. These examples illustrate expansions corresponding to several well–known sequences and other related identities formally enumerated by special cases of the diagonal–coefficient, or Hadamard product, generating functions defined in (4.5) and (4.6) of the overview to Section 4.2.

A number of the examples given in this section provide specific applications of the formal Laplace–Borel–like generating function transformation technique outlined below in Remark 4.5. These formal series transformations are performed by diagonal coefficient extractions involving the rational convergent–function–based generating functions that enumerate the single and double factorial function terms implicit to the expansions of many factorial–related sums, products, and related special sequence identities.

Sequence References:
A000166, A000178, A000225, A000918, A000984, A001044, A002109, A003422, A005165, A008277, A008292, A009445, A010050, A024023, A024036, A024049, A027641, A027642, A033312, A061062, A066802, A100043, A100089, A100732, A104344, A166351, and A184877.

- **Section 5. Properties of the generalized convergent functions.**
Section 5.1 and Section 5.2 prove several additional properties and exact formulas satisfied by the respective convergent functions, $\mathrm{FQ}_h(\alpha, R; z)$ and $\mathrm{FP}_h(\alpha, R; z)$. The results for the convergent denominator functions, $\mathrm{FQ}_h(\alpha, R; z)$, stated by the propositions in Section 5.1 provide characterizations of these sequences by well–known special functions and orthogonal polynomial sequences, additional auxiliary recurrence relations, and analogs to the known addition and multiplication theorems for the confluent hypergeometric function, $U(-h, b, z)$. Besides the short direct proof given in Section 3.2, this section provides careful proofs of the convergent function properties needed to rigorously justify the results for the examples cited as particular applications elsewhere within the article.
- **Section 6. Applications of new expansions by finite difference equations.**
The last topics considered in Section 6 provide further new applications illustrated by example of the h–order finite difference equations suggested by the expansions of the rational h^{th} convergents, $\mathrm{Conv}_h(\alpha, R; z)$, when the parameter, R, denotes some initially fixed indeterminate parameter with respect to the generalized product sequences defined by (1.1).

 The corresponding finite–degree, rational polynomial expansions in n of the sequences, $p_n(\alpha, R)$, when R depends linearly on n offer a dual interpretation to the algebraic structure of the previous formulas for these functions given in terms of the special function zeros defined above. The resulting multiple sum expansions suggest many new applications to classical congruences and otherwise noteworthy special case identities concerning primality of subsequences of the odd positive integers and prime pairs.

 Sequence References:
 A000043, A000108, A000215, A000225, A000668, A000978, A001008, A001097, A001220, A002234, A002496, A002805, A002981, A002982, A005109, A005384, A006512, A007406, A007407, A007408, A007409, A007540, A007619, A019434, A022004, A022005, A023200, A023201, A023202, A023203, A046118, A046124, A046133, A077800, A078303, A080075, A088164, and A123176.
- **Section 7. Conclusions.**
- **Appendix A. Listings of tables referenced in the article.**
  - **Table A.1. The generalized α–factorial coefficient triangles.**

2.4 Notation and conventions

Most of the conventions in the article are consistent with the notation employed within the *Concrete Mathematics* reference, and the conventions defined in the introduction to the first article [22]. These conventions include the following particular non–standard notational variants: (i) the special notation for formal power series coefficient extraction, $f_n := [z^n] \left(\sum_k f_k z^k\right)$, (ii) the compact usage of Iverson's convention, $[i = j]_\delta \equiv \delta_{i,j}$, where $[n = k = 0]_\delta \equiv \delta_{n,0}\delta_{k,0}$, (iii) the alternate bracket notation for the Stirling number triangles, $\left[{n \atop k}\right] = (-1)^{n-k} s(n, k)$ and $\left\{{n \atop k}\right\} = S(n, k)$, and (iv) the notation for the first–order hamonic numbers, $H_n^{(1)} = H_n$, and the partial sums defining the more general cases of the r–order harmonic numbers, $H_n^{(r)} := 1 + 2^{-r} + 3^{-r} + \cdots + n^{-r}$, when $r, n \geq 1$ are integer–valued.

Within the article the notation $g_1(n) \equiv g_2(n) \pmod{N_1, N_2, \ldots, N_k}$ is also understood to mean that the congruence, $g_1(n) \equiv g_2(n) \pmod{N_j}$, holds modulo any of the bases, N_j, for $1 \leq j \leq k$. Other more standard notation for special functions cited within the article is consistent with the definitions employed within the *NIST Handbook of Mathematical Functions* reference. The respective symbols Ⓓ, Ⓔ, and Ⓡ are taken to denote the end markers following the conclusions for the definition, example, and remark environments expanded in manuscript text below.

3 The Jacobi type J–fractions for generalized factorial function sequences

3.1 Enumerative properties of Jacobi type J–fractions

To simplify the exposition in this article, we adopt the notation for the Jacobi type continued fractions, or J–fractions, in the references [9; 10] [19, *cf.* §3.10] [17, *cf.* §5.5]. Given some application–specific choices of the prescribed sequences, $\{a_k, b_k, c_k\}$, we consider the formal power series whose coefficients are generated by the rational convergents, $J^{[h]}(z) := J^{[h]}(\{a_k, b_k, c_k\}; z)$, of the infinite continued fractions, $J(z) := J^{[\infty]}(\{a_k, b_k, c_k\}; z)$, defined as follows:

$$J(z) = \cfrac{1}{1 - c_0 z - \cfrac{a_0 b_1 z^2}{1 - c_1 z - \cfrac{a_1 b_2 z^2}{\cdots}}}. \tag{3.1}$$

We briefly summarize the other enumerative properties from the references that are relevant in constructing the new factorial–function–related results given in the next subsections of this article [9; 10; 17] [13, *cf.* §6.7].

1. **Definitions of the h–order convergent series**:
   When $h \geq 1$, the h^{th} convergent functions, given by the equivalent notation of $J^{[h]}(z)$ and

$J^{[h]}(\{a_k, b_k, c_k\}; z)$ within this section, of the infinite continued fraction expansions of (3.1) are defined as the ratios, $J^{[h]}(z) := P_h(z)/Q_h(z)$.

The component functions corresponding to the convergent numerator and denominator sequences, $P_h(z)$ and $Q_h(z)$, each satisfy second–order finite difference equations (in h) of the respective forms defined by the next two equations.

$$P_h(z) = (1 - c_{h-1}z)P_{h-1}(z) - a_{h-2}b_{h-1}z^2 P_{h-2}(z) + [h = 1]_\delta$$
$$Q_h(z) = (1 - c_{h-1}z)Q_{h-1}(z) - a_{h-2}b_{h-1}z^2 Q_{h-2}(z) + (1 - c_0 z)\,[h = 1]_\delta + [h = 0]_\delta$$

2. **Rationality of truncated convergent function approximations**:
Let $p_n = p_n(\{a_k, b_k, c_k\}) := [z^n]J(z)$ denote the expected term corresponding to the coefficient of z^n in the formal power series expansion defined by the infinite J–fraction from (3.1). For all $n \geq 0$, we know that the h^{th} convergent functions have truncated power series expansions that satisfy
$$p_n(\{a_k, b_k, c_k\}) = [z^n]J^{[h]}(\{a_k, b_k, c_k\}; z),\ \forall n \leq h.$$
In particular, the series coefficients of the h^{th} convergents are always at least h–order accurate as formal power series expansions in z that exactly enumerate the expected sequence terms, $(p_n)_{n \geq 0}$.
The resulting "*eventually periodic*" nature suggested by the approximate sequences enumerated by the rational convergent functions in z is formalized in the congruence properties given below in (3.2) [10] [17, See §2, §5.7].

3. **Congruence properties modulo integer bases**:
Let $\lambda_k := a_{k-1}b_k$ and suppose that the corresponding bases, M_h, are formed by the products $M_h := \lambda_1\lambda_2\cdots\lambda_h$ for $h \geq 1$. Whenever $N_h \mid M_h$, and for any $n \geq 0$, we have that
$$p_n(\{a_k, b_k, c_k\}) \equiv [z^n]J^{[h]}(\{a_k, b_k, c_k\}; z) \pmod{N_h}, \tag{3.2}$$
which is also true of all partial sequence terms enumerated by the h^{th} convergent functions modulo any integer divisors of the M_h [17, *cf.* §5.7].

3.2 A short direct proof of the J–fraction representations for the generalized product sequence generating functions

We omit the details to a more combinatorially flavored proof that the J–fraction series defined by the convergent functions in (1.13) do, in fact, correctly enumerate the expected symbolic product sequences in (1.1). Instead, a short direct proof following from the J–fraction results given in Flajolet's first article is sketched below. Even further combinatorial interpretations of the sequences generated by these continued fraction series, their relations to the Stirling number triangles, and other properties tied to the coefficient triangles studied in depth by [22] based on the properties of these new J–fractions is suggested as a topic for later investigation.

The J–fraction results established in Flajolet's article prove two equivalent formulations of the expansions cited in the form of (1.8) defined in the introduction. In particular, the reference states one indirect form of this result generating the bivariate ordinary power series for the Stirling numbers of the first kind, and a second respective result directly providing the J–fraction expansions that generate the rising factorial function, or Pochhammer symbol, $(x)_n = n! \cdot \binom{x}{n}$, for any non–zero indeterminate, $x \in \mathbb{C}$ [9, Thm. 3B, Prop. 9].

The prescribed sequences in the J–fraction expansions defined by (3.1) in the previous section, corresponding to (i) the Pochhammer symbol, $(x)_n$, and (ii) the convergent functions enumerating the generalized products, $p_n(\alpha, R)$, or equivalently, the Pochhammer k–symbols, $(R)_{n,\alpha}$, over any fixed $\alpha \in \mathbb{Z}^+$ and indeterminate, R, are defined as follows:

$$\begin{aligned} \{\mathrm{as}_k(x), \mathrm{bs}_k(x), \mathrm{cs}_k(x)\} &\overset{\text{(i)}}{:=} \{x+k, k, x+2k\} && \text{(Pochhammer Symbol)} \\ \{\mathrm{af}_k(\alpha, R), \mathrm{bf}_k(\alpha, R), \mathrm{cf}_k(\alpha, R)\} &\overset{\text{(ii)}}{:=} \{R+k\alpha, k, \alpha\cdot(R+2k\alpha)\}. && \text{(Generalized Products)} \end{aligned}$$

We claim that these modified J–fractions, $R_0(R/\alpha, \alpha z)$, that generate the corresponding series expansions in (1.8) enumerate the analogous terms of the generalized symbolic product sequences, $p_n(\alpha, R)$, defined by (1.1).

Proof of the Claim. First, an appeal to the polynomial expansions of both the Pochhammer symbol, $(x)_n$, and then of the products, $p_n(\alpha, R)$, defined by (1.1) by the Stirling numbers of the first kind, yields the following sums:

$$\begin{aligned} p_n(\alpha, R) \times z^n &= \left(\prod_{j=0}^{n-1}(R+j\alpha) + [n=0]_\delta\right) \times z^n \\ &= \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \alpha^{n-k} R^k\right) \times z^n \\ &= \left(\alpha^n \cdot \left(\frac{R}{\alpha}\right)_n\right) \times z^n. \end{aligned} \tag{3.3}$$

Finally, after a parameter substitution of $x :\mapsto R/\alpha$ together with the change of variable $z :\mapsto \alpha z$ in the first results from Flajolet's article [9], we obtain identical forms of the convergent–based definitions for the generalized J–fraction definitions given in Definition 1.1. ⊡

3.3 Alternate exact expansions of the generalized convergent functions

The notational inconvenience introduced in the inner sums of (1.15.a) and (1.15.b), implicitly determined by the shorthand for the coefficients, $c_{h,j}(\alpha, R)$, each of which depend on the more difficult terms of the convergent numerator function sequences, $\mathrm{FP}_h(\alpha, R; z)$, is avoided in place of alternate recurrence relations for each finite convergent function, $\mathrm{Conv}_h(\alpha, R; z)$, involving paired products of the denominator polynomials, $\mathrm{FQ}_h(\alpha, R; z)$. These denominator function sequences are related to generalized forms of the Laguerre polynomial sequences as follows (see Section 5.1):

$$\mathrm{FQ}_h(\alpha, R; z) = (-\alpha z)^h \cdot h! \cdot L_h^{(R/\alpha-1)}\left((\alpha z)^{-1}\right). \tag{3.4}$$

The expansions provided by the formula in (3.4) suggest useful alternate formulations of the congruence results given below in Section 4.1 when the Laguerre polynomial, or corresponding confluent hypergeometric function, zeros are considered to be less complicated in form than the more involved sums expanded through the numerator functions, $\mathrm{FP}_h(\alpha, R; z)$ [3; 11].

Example 3.1 (Recurrence Relations for the Convergent Functions and Laguerre Polynomials)**.** The well–known cases of the enumerative properties satisfied by the expansions of the convergent function sequences given in the references immediately yield the following relations [9, §3] [19, §1.12(ii)]:

$$\mathrm{Conv}_h(\alpha, R; z) = \mathrm{Conv}_{h-k}(\alpha, R; z)$$

$$+\sum_{i=0}^{k-1}\frac{\alpha^{h-i-1}(h-i-1)!\cdot p_{h-i-1}(\alpha,R)\cdot z^{2(h-i-1)}}{\mathrm{FQ}_{h-i}(\alpha,R;z)\,\mathrm{FQ}_{h-i-1}(\alpha,R;z)},\ h>k\geq 1$$

$$\begin{aligned}\mathrm{Conv}_h(\alpha,R;z) &= \sum_{i=0}^{h-1}\frac{\alpha^{h-i-1}(h-i-1)!\cdot p_{h-i-1}(\alpha,R)\cdot z^{2(h-i-1)}}{\mathrm{FQ}_{h-i}(\alpha,R;z)\,\mathrm{FQ}_{h-i-1}(\alpha,R;z)} \\ &= \sum_{i=0}^{h-1}\binom{\frac{R}{\alpha}+i-1}{i}\times\frac{(-\alpha z)^{-1}}{(i+1)\cdot L_i^{(R/\alpha-1)}\left((\alpha z)^{-1}\right)L_{i+1}^{(R/\alpha-1)}\left((\alpha z)^{-1}\right)},\ h\geq 2.\end{aligned}\tag{3.5}$$

Topics aimed at finding new results obtained from other 'known, strictly continued–fraction–related properties of the convergent function sequences beyond the proofs given in Section 5 are suggested as a further avenue to approach the otherwise divergent ordinary generating functions for these more general cases of the integer–valued factorial–related product sequences (see also footnote **F.35 in Section 6 given on page 68). Ⓔ

Remark 3.2 (Related Convergent Function Generating Function Expansions)**.** For $h-i\geq 0$, we have a noteworthy Rodrigues–type formula satisfied by the Laguerre polynomial sequences in (3.4) stated by the reference as follows [19, §18.5(ii)]:

$$(-\alpha z)^{h-i}\cdot(h-i)!\times L_{h-i}^{(\beta)}\left(\frac{1}{\alpha z}\right)=\alpha^{h-i}\cdot z^{2h-2i+\beta+1}e^{1/\alpha z}\times\left\{\frac{e^{-1/\alpha z}}{z^{\beta+1}}\right\}^{(h-i)}.$$

The multiple derivatives implicit to this statement in the previous equation then have the additional expansions through the product rule analog provided by the formula of Halphen given in the form of [6, §3 Exercises, p. 161]

$$\left\{F\left(\frac{1}{z}\right)G(z)\right\}^{(n)}=\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{z^k}\cdot F^{(k)}\left(\frac{1}{z}\right)\left\{\frac{G(z)}{z^k}\right\}^{(n-k)},\qquad\text{(Halphen's Product Rule)}$$

for natural numbers $n\geq 0$, and where the notation for the functions, $F(z)$ and $G(z)$, employed in the previous equation corresponds to any prescribed choice of these functions that are each n times continuously differentiable at the point $z\neq 0$.

The next particular restatements of (3.5) then follow easily from the last two equations as

$$\begin{aligned}\mathrm{Conv}_h(\alpha,R;z) &= \mathrm{Conv}_{h-k}(\alpha,R;z) \\ &\quad+\sum_{i=0}^{k-1}\frac{(h-i-1)!}{(\alpha z)\cdot(R/\alpha)_{h-i}}\times\frac{1}{{}_1F_1\left(-(h-i);\frac{R}{\alpha};\frac{1}{\alpha z}\right){}_1F_1\left(-(h-i-1);\frac{R}{\alpha};\frac{1}{\alpha z}\right)},\ 1\leq k<h \\ \mathrm{Conv}_h(\alpha,R;z) &= \sum_{i=0}^{h-1}\frac{(h-i-1)!}{(\alpha z)\cdot(R/\alpha)_{h-i}}\times\frac{1}{{}_1F_1\left(-(h-i);\frac{R}{\alpha};\frac{1}{\alpha z}\right){}_1F_1\left(-(h-i-1);\frac{R}{\alpha};\frac{1}{\alpha z}\right)} \\ &= \sum_{i=1}^{h}\binom{\frac{R}{\alpha}+i-1}{i-1}^{-1}\times\frac{(-Rz)^{-1}}{{}_1F_1\left(1-i;\frac{R}{\alpha};\frac{1}{\alpha z}\right){}_1F_1\left(-i;\frac{R}{\alpha};\frac{1}{\alpha z}\right)},\end{aligned}\tag{3.6}$$

where the h^{th} convergents, $\mathrm{Conv}_h(\alpha,R;z)$, are rational in z for all $h\geq 1$. The functions ${}_1F_1(a;b;z)$, or $M(a,b,z)=\sum_{s\geq 0}\frac{(a)_s}{(b)_s s!}z^s$, in the previous equations denote *Kummer's confluent hypergeometric function* [19, §13.2] [13, §5.5]. Ⓡ

4 Applications and motivating examples

We require the next lemma to formally enumerate the generalized products and factorial function sequences already cited without proof in the examples from Section 1.4.

Lemma 4.1 (*Sequences Generated by the Generalized Convergent Functions*).
For fixed integers $\alpha \neq 0$, $0 \leq d < \alpha$, and each $n \geq 1$, the generalized α–factorial sequences defined in (1.2) satisfy the following expansions by the generalized products in (1.1):

$$\begin{aligned}(\alpha n - d)!_{(\alpha)} &= p_n(-\alpha, \alpha n - d) && (4.1.a)\\ &= p_n(\alpha, \alpha - d) && (4.1.b)\\ n!_{(\alpha)} &= p_{\lfloor (n+\alpha-1)/\alpha \rfloor}(-\alpha, n). && (4.1.c)\end{aligned}$$

Proof. The related cases of each of these identities cited in the equations above correspond proving to the equivalent expansions of the product–wise representations for the α–factorial functions given in each of the next equations:

$$\begin{aligned}(\alpha n - d)!_{(\alpha)} &= \prod_{j=0}^{n-1} (\alpha n - d - j\alpha) && (a)\\ &= \prod_{j=0}^{n-1} (\alpha - d + j\alpha) && (b)\\ n!_{(\alpha)} &= \prod_{j=0}^{\lfloor (n+\alpha-1)/\alpha \rfloor - 1} (n - i\alpha). && (c)\end{aligned}$$

The first product in (a) is easily obtained from (1.2) by induction on n, which then implies the second result in (b). Similarly, an inductive argument applied to the definition provided by (1.2) proves the last product representation given in (c). ⊡

Corollaries. The proof of the lemma provides immediate corollaries to the special cases of the α–factorial functions, $(\alpha n - d)!_{(\alpha)}$, expanded by the results from (1.16.a). We explicitly state the following particular special cases of the lemma corresponding to $d := 0$ in (4.2.a) below, and then to $d := 1$ in (4.2.b) below, respectively, for later use in Section 4.2.3 and Section 6 of the article below:

$$\begin{aligned}(\alpha n)!_{(\alpha)} &= \alpha^n \cdot (1)_n = [z^n] \operatorname{Conv}_{n+n_0}(-\alpha, \alpha n; z)\,, \ \forall n_0 \geq 0 && (4.2.a)\\ &= \alpha^n \cdot n! \quad = [z^n] \operatorname{Conv}_{n+n_0}(-1, n; \alpha z)\,, \ \forall n_0 \geq 0\\ (\alpha n - 1)!_{(\alpha)} &= p_n(-\alpha, \alpha n - 1) = (-\alpha)^n \left(\frac{1}{\alpha} - n\right)_n && (4.2.b)\\ &= p_n(\alpha, \alpha - 1) \qquad = \alpha^n \left(1 - \frac{1}{\alpha}\right)_n .\end{aligned}$$

The first two results are employed by the convergent–based formulations to the applications given in Section 4.2.3. The last pair of results given in (4.2.b) are employed to phrase the generalized expansions for the identities cited in Example 6.4 of Section 6.1. ⊡

Lemma 4.1 provides proofs of the convergent–function–based generating function identities enumerating the α–factorial sequences given in (1.16.a) and (1.16.b) of the introduction. The last convergent–based generating function identity that enumerates the α–factorial functions, $n!_{(\alpha)}$, when $n > \alpha$ expanded in the form of (1.16.c) from Section 1.4 follows from the product function expansions provided in (4.1.c) of the lemma by applying a result proved in exercises of the reference [13, §7, Ex. 36; p. 569].

4.1 New congruences for α–factorial functions, the generalized Stirling number triangles, and the Pochhammer k–symbols

The particular cases of the J–fraction representations enumerating the product sequences defined by (1.1) always yield a factor of $h := N_h \mid M_h$ in the statement of (3.2) (see Remark 4.4). One consequence of this property implicit to each of the generalized factorial–like sequences observed so far, is that it is straightforward to formulate new congruence relations for these sequences modulo any fixed integers $p \geq 2$. The particular results stated in this section follow immediately from the congruences properties modulo integer divisors of the M_h summarized by Section 3.1 considered as in the references [9; 10] [17, *cf.* §5.7].

4.1.1 Examples: Congruences for the Stirling numbers of the first kind

The special case of (1.19) from the examples given in the introduction when $\alpha := 1$ corresponding to the single factorial function, $n!$, agrees with the known congruence for the Stirling numbers of the first kind derived in the reference [26, §4.6] [6, *cf.* §5.8]. In particular, for all $n \geq 1$ we can prove that

$$n! \equiv \sum_{m=1}^{n} \binom{\lfloor n/2 \rfloor}{m - \lceil n/2 \rceil} (-1)^{n-m} n^m + [n = 0]_\delta \pmod{2}.$$

For integers $h \geq 3$ and $n \geq m \geq 1$, the (unsigned) Stirling numbers of the first kind, $\left[{n \atop m}\right]$, are generated by the polynomial expansions of the rising factorial function, or Pochhammer symbol, $x^{\overline{n}} = (x)_n$, as follows [13, §7.4; §6] [A130534; A008275] *F.8 :

$$\begin{aligned} \left[{n \atop m}\right] &= [R^m] R(R+1)\cdots(R+n-1) \\ &= [z^n][R^m] \operatorname{Conv}_h(1, R; z), \quad \text{for } 1 \leq m \leq n \leq 2h. \end{aligned}$$

Analogous formulations of new congruence results for the α–factorial triangles defined by (1.4), and the corresponding forms of the generalized harmonic number sequences employed in stating the results in Section 6.4 of the article below, are expanded by noting that for all $n, m \geq 1$, and integers $p \geq 2$, we have the following expansions [22]:

$$\begin{aligned} \left[{n \atop m}\right]_\alpha &= [s^{m-1}](s+1)(s+1+\alpha)\cdots(s+1+(n-2)\alpha) \\ &= [s^{m-1}] p_{n-1}(\alpha, s+1) \end{aligned}$$

*F.8 *Note §4.1.* The approach to the rational convergent functions in z by the finite difference equations outlined in the applications of Section 6 provides related expansions of the series coefficients of the h^{th} convergents, $\operatorname{Conv}_h(\alpha, R; z)$, which are also rational in R for all $h \geq 1$ when $\alpha \neq 0$ is fixed. The corresponding properties of these coefficients of the generalized convergent functions with respect to formal power series expansions in R as functions of α and z are not explored in depth within this article.

$$\begin{bmatrix} n \\ m \end{bmatrix}_{\alpha} \equiv [z^{n-1}][R^{m-1}] \operatorname{Conv}_p(\alpha, R+1; z) \pmod{p}.$$

The coefficients of R^m series expansions of the convergent functions, $\operatorname{Conv}_h(1, R; z)$, in the formal variable R are rational functions of z with denominators given by m^{th} powers of the reflected polynomials defined in the next equation. For a fixed $h \geq 3$, let the zeros, $\omega_{h,i}$, be defined as follows:

$$(\omega_{h,i})_{i=1}^{h-1} := \left\{ \omega_j : \sum_{i=0}^{h-1} \binom{h-1}{i} \frac{h!}{(i+1)!} (-\omega_j)^i = 0,\ 1 \leq j < h \right\}. \tag{4.3}$$

The forms of both exact formulas and congruences for the Stirling numbers of the first kind modulo any prescribed integers $h \geq 3$ are then expanded as

$$\begin{aligned} \begin{bmatrix} n \\ m \end{bmatrix} &= \left(\sum_{i=0}^{h-1} p_{h,i}^{[m]}(n) \times \omega_{h,i}^n \right) [n > m]_\delta + [n = m]_\delta, & m \leq n \leq 2h-1 \\ \begin{bmatrix} n \\ m \end{bmatrix} &\equiv \left(\sum_{i=0}^{h-1} p_{h,i}^{[m]}(n) \times \omega_{h,i}^n \right) [n > m]_\delta + [n = m]_\delta \pmod{h},\ \forall n \geq m, \end{aligned}$$

where the functions, $p_{h,i}^{[m]}(n)$, denote fixed polynomials of degree m in n for each h, m, and i [13, §7.2]. For example, when $h := 2, 3$, the respective reflected roots defined by the previous equations in (4.3) are given exactly by

$$\{\omega_{2,1}\} := \{2\} \qquad \text{and} \qquad (\omega_{3,i})_{i=1}^2 := \left\{ 3 - \sqrt{3}, 3 + \sqrt{3} \right\}.$$

For comparison with the known result for the Stirling numbers of the first kind modulo 2 expanded as in the result from the reference stated above, several particular cases of these congruences for the Stirling numbers, $\begin{bmatrix} n \\ m \end{bmatrix}$, modulo 2 are guven by [A087755]

$$\begin{aligned} \begin{bmatrix} n \\ 1 \end{bmatrix} &\equiv \frac{2^n}{4} [n \geq 2]_\delta + [n = 1]_\delta && \pmod{2} \\ \begin{bmatrix} n \\ 2 \end{bmatrix} &\equiv \frac{3 \cdot 2^n}{16} (n-1) [n \geq 3]_\delta + [n = 2]_\delta && \pmod{2} \\ \begin{bmatrix} n \\ 3 \end{bmatrix} &\equiv 2^{n-7}(9n-20)(n-1) [n \geq 4]_\delta + [n = 3]_\delta && \pmod{2} \\ \begin{bmatrix} n \\ 4 \end{bmatrix} &\equiv 2^{n-9}(3n-10)(3n-7)(n-1) [n \geq 5]_\delta + [n = 4]_\delta && \pmod{2} \\ \begin{bmatrix} n \\ 5 \end{bmatrix} &\equiv 2^{n-13}(27n^3 - 279n^2 + 934n - 1008)(n-1) [n \geq 6]_\delta + [n = 5]_\delta && \pmod{2} \\ \begin{bmatrix} n \\ 6 \end{bmatrix} &\equiv \frac{2^{n-15}}{5}(9n^2 - 71n + 120)(3n-14)(3n-11)(n-1) [n \geq 7]_\delta + [n = 6]_\delta && \pmod{2}, \end{aligned}$$

where $\binom{\lfloor n/2 \rfloor}{m - \lceil n/2 \rceil} \equiv \begin{bmatrix} n \\ m \end{bmatrix} \pmod{2}$ for all $m \geq 1$ [26, §4.6], and then in the next few particular forms in the following special case congruences satsfied these coefficients modulo 3:

$$\begin{bmatrix} n \\ 1 \end{bmatrix} \equiv \sum_{j=\pm 1} \tfrac{1}{36} \left(9 - 5j\sqrt{3}\right) \times \left(3 + j\sqrt{3}\right)^n [n \geq 2]_\delta + [n = 1]_\delta \pmod{3}$$

$$\begin{bmatrix} n \\ 2 \end{bmatrix} \equiv \sum_{j=\pm 1} \tfrac{1}{216} \left((44n-41)-(25n-24)\cdot j\sqrt{3}\right) \times \left(3+j\sqrt{3}\right)^n [n \geq 3]_\delta + [n=2]_\delta \pmod{3}$$

$$\begin{aligned} \begin{bmatrix} n \\ 3 \end{bmatrix} \equiv \sum_{j=\pm 1} \tfrac{1}{15552} & \left((1299n^2-3837n+2412)-(745n^2-2217n+1418)\cdot j\sqrt{3}\right) \times \\ & \times \left(3+j\sqrt{3}\right)^n [n \geq 4]_\delta + [n=3]_\delta \pmod{3} \end{aligned}$$

$$\begin{aligned} \begin{bmatrix} n \\ 4 \end{bmatrix} \equiv \sum_{j=\pm 1} \tfrac{1}{179936} & \left((6409n^3-383778n^2+70901n-37092)\right. \\ & \left. -(3690n^3-22374n^2+41088n-21708)\cdot j\sqrt{3}\right) \times \\ & \times \left(3+j\sqrt{3}\right)^n [n \geq 5]_\delta + [n=4]_\delta . \pmod{3} \end{aligned}$$

Additional congruences for the Stirling numbers of the first kind modulo 4 and modulo 5 are straightforward to expand by related formulas with exact algebraic expressions for the roots of the third–degree and fourth–degree equations defined as in (4.3) above. The expansions of the integer–order harmonic number sequences cited in the reference [22, §4.3.3] also yield congruences for the terms, $(n!)^r \times H_n^{(r)}$, provided by the noted identities for these functions involving the Stirling numbers of the first kind modulo any fixed integers $p \geq 2$.

The next several particular cases of the congruences satisfied by the *first–order harmonic numbers*, $H_n^{(1)}$, are stated exactly in terms the rational generating functions in z that lead to generalized forms of the congruences in the last equations modulo the integers $p := 2, 3$ for these subsequent cases of the integers $p \geq 4$ [13, §6.3] [A001008; A002805].

$$\begin{aligned} n! \times H_n^{(1)} &= \begin{bmatrix} n+1 \\ 2 \end{bmatrix} \\ &\equiv [z^{n+1}] \left(\tfrac{36z^2-48z+325}{576} + \tfrac{17040z^2+1782z+6467}{576(24z^3-36z^2+12z-1)} + \tfrac{78828z^2-33987z+3071}{288(24z^3-36z^2+12z-1)^2}\right) && \pmod{4} \\ &\equiv [z^n] \left(\tfrac{3z-4}{48} + \tfrac{1300z^2+890z+947}{96(24z^3-36z^2+12z-1)} + \tfrac{24568z^2-10576z+955}{96(24z^3-36z^2+12z-1)^2}\right) && \pmod{4} \\ &\equiv [z^{n-1}] \left(\tfrac{1}{16} + \tfrac{-96z^2+794z+397}{48(24z^3-36z^2+12z-1)} + \tfrac{5730z^2-2453z+221}{24(24z^3-36z^2+12z-1)^2}\right) && \pmod{4} \\ &\equiv [z^n] \left(\tfrac{12z-29}{300} + \tfrac{80130z^3+54450z^2+79113z+108164}{900(120z^4-240z^3+120z^2-20z+1)}\right. \\ &\qquad \left. + \tfrac{17470170z^3-11428050z^2+2081551z-108077}{900(120z^4-240z^3+120z^2-20z+1)^2}\right) && \pmod{5} \\ &\equiv [z^n] \left(\tfrac{10z-37}{360} + \tfrac{1419408z^4+903312z^3+1797924z^2+2950002z+4780681}{2160(720z^5-1800z^4+1200z^3-300z^2+30z-1)}\right. \\ &\qquad \left. + \tfrac{5581246248z^4-4906594848z^3+1347715644z^2-140481648z+4780903}{2160(720z^5-1800z^4+1200z^3-300z^2+30z-1)^2}\right) && \pmod{6} \end{aligned}$$

The *second–order* and *third–order harmonic numbers*, $H_n^{(2)}$ and $H_n^{(3)}$, respectively, are expanded exactly through the following formulas involving the Stirling numbers of the first kind modulo any fixed integers $p \geq 2$, and where the Stirling number sequences, $\left[\begin{smallmatrix} n \\ m \end{smallmatrix}\right] \pmod{p}$ at each fixed $m := 1, 2, 3, 4$, are generated by the predictably rational functions of z enumerated through the identities stated above [22, §4.3.3] [A007406; A007407; A007408; A007409]:

$$(n!)^2 \times H_n^{(2)} = (n!)^2 \times \sum_{k=1}^{n} \frac{1}{k^2}$$

$$\equiv \begin{bmatrix} n+1 \\ 2 \end{bmatrix}^2 - 2\begin{bmatrix} n+1 \\ 1 \end{bmatrix}\begin{bmatrix} n+1 \\ 3 \end{bmatrix} \pmod{p}$$

$$(n!)^3 \times H_n^{(3)} = (n!)^3 \times \sum_{k=1}^{n} \frac{1}{k^3}$$

$$\equiv \begin{bmatrix} n+1 \\ 2 \end{bmatrix}^3 - 3\begin{bmatrix} n+1 \\ 1 \end{bmatrix}\begin{bmatrix} n+1 \\ 2 \end{bmatrix}\begin{bmatrix} n+1 \\ 3 \end{bmatrix} + 3\begin{bmatrix} n+1 \\ 1 \end{bmatrix}^2\begin{bmatrix} n+1 \\ 4 \end{bmatrix}. \pmod{p}$$

The Hadamard product, or diagonal–coefficient, generating function constructions formulated in the examples introduced by Section 4.2 below give expansions of rational convergent–function–based generating functions in z that generate these corresponding r–order sequence cases modulo any fixed integers $p \geq 2$.

The proof of Theorem 2.4 given in the reference [17, §2] suggests the direct method for obtaining the next rational generating functions for these sequences documented in the source code from the summary notebook reference [23], each of which generate series coefficients for these particular harmonic number sequence variants (modulo p) that always satisfy some finite–degree linear difference equation with constant coefficients over n when α and R are fixed parameters.

$$\begin{aligned}
(n!)^2 \times H_n^{(2)} &\equiv [z^n]\left(\frac{z\left(1-3z+9z^2-8z^3\right)}{(1-4z)^2}\right) && \pmod{2} \\
&= [z^n]\left(\frac{5z}{16} - \frac{z^2}{2} + \frac{11}{64(1-4z)^2} - \frac{11}{64(1-4z)}\right) \\
&= [z^n]\left(z + 5z^2 + 33z^3 + 176z^4 + 880z^5 + 4224z^6 + 19712z^7 + \cdots\right) \\
(n!)^2 \times H_n^{(2)} &\equiv [z^n]\left(\frac{z\left(1-61z+1339z^2-13106z^3+62284z^4-144264z^5+151776z^6-124416z^7+41472z^8\right)}{(1-6z)^3(1-24z+36z^2)^2}\right) && \pmod{3} \\
&= [z^n]\Bigg(-\frac{13}{324} + \frac{14z}{81} - \frac{4z^2}{27} + \frac{25}{1944(-1+6z)^3} + \frac{115}{1944(-1+6z)^2} + \frac{5}{162(-1+6z)} \\
&\qquad + \frac{-787+17624z}{216(1-24z+36z^2)^2} + \frac{2377+3754z}{648(1-24z+36z^2)}\Bigg) \\
&= [z^n]\left(z + 5z^2 + 49z^3 + 820z^4 + 18232z^5 + 437616z^6 + 10619568z^7 + \cdots\right) \\
(n!)^3 \times H_n^{(3)} &\equiv [z^n]\left(\frac{z\left(1-7z+49z^2-144z^3+192z^4\right)}{(1-8z)^2}\right) && \pmod{2} \\
&= [z^n]\left(\frac{11z}{32} - \frac{3z^2}{2} + 3z^3 + \frac{21}{256(1-8z)^2} - \frac{21}{256(1-8z)}\right) \\
&= [z^n]\left(z + 9z^2 + 129z^3 + 1344z^4 + 13440z^5 + 129024z^6 + 1204224z^7 + \cdots\right) \\
(n!)^3 \times H_n^{(3)} &\equiv [z^n]\Bigg(-\frac{143}{5832} + \frac{625z}{2916} - \frac{4z^2}{9} + \frac{4z^3}{3} + \frac{115(-6719+711956z)}{93312(1-108z+216z^2)^2} && \pmod{3} \\
&\qquad + \frac{774079+1459082z}{93312(1-108z+216z^2)} - \frac{125(-11+312z)}{11664(1-36z+216z^2)^4} \\
&\qquad - \frac{10(1+306z)}{729(1-36z+216z^2)^3} + \frac{-20677+269268z}{93312(1-36z+216z^2)^2} + \frac{11851+89478z}{93312(1-36z+216z^2)}\Bigg) \\
&= [z^n]\left(z + 9z^2 + 251z^3 + 16280z^4 + 1586800z^5 + 171547200z^6 + \cdots\right)
\end{aligned}$$

The next cases of the rational generating functions enumerating the terms of these two sequences modulo 4 and 5 lead to less compact formulas expanded in partial fractions over z, roughly approximated in form by the generating function expansions from the previous formulas. The factored denominators, denoted $\operatorname{Denom}_{r, \bmod p}[\![z]\!]$ immediately below, of the rational generating functions over the respective second–order and third–order cases of the r–order sequences modulo $p := 4, 5$

are provided in the following equations:

$$\begin{aligned}
\mathrm{Denom}_{2,\bmod 4}[\![z]\!] &= \left(-1+72z-720z^2+576z^3\right)^2\left(-1+36z-288z^2+576z^3\right)^3 \\
\mathrm{Denom}_{2,\bmod 5}[\![z]\!] &= \left(1-160z+5040z^2-28800z^3+14400z^4\right)^2 \times \\
&\quad\times\left(1-120z+4680z^2-76800z^3+561600z^4-1728000z^5+1728000z^6\right)^3 \\
\mathrm{Denom}_{3,\bmod 4}[\![z]\!] &= (1-24z)^4\left(-1+504z-17280z^2+13824z^3\right)^2 \times \\
&\quad\times\left(-1+144z-5184z^2+13824z^3\right)^4\left(-1+216z-3456z^2+13824z^3\right)^4 \\
\mathrm{Denom}_{3,\bmod 5}[\![z]\!] &= \left(1-1520z+273600z^2-4320000z^3+1728000z^4\right)^2 \times \\
&\quad\times\left(1-240z+14400z^2-288000z^3+1728000z^4\right)^4 \times \\
&\quad\times\big(1-1680z+1051200z^2-319776000z^3+51914304000z^4 \\
&\qquad -4764026880000z^5+251795865600000z^6-7537618944000000z^7 \\
&\qquad +121956544512000000z^8-998751928320000000z^9 \\
&\qquad +4084826112000000000z^{10}-7739670528000000000z^{11} \\
&\qquad +5159780352000000000z^{12}\big)^4.
\end{aligned}$$

The summary notebook reference contains further complete expansions of the rational generating functions enumerating these r–order sequence cases for $r := 1, 2, 3, 4$ modulo the next few prescribed cases of the integers $p \geq 6$ [23] [22, *cf.* §4.3.3].

4.1.2 Generalized expansions of the new integer congruences for the α–factorial functions and the symbolic product sequences

Example 4.2 (The Special Cases Modulo 2, 3, and 4)**.** The first congruences for the α–factorial functions, $n!_{(\alpha)}$, modulo the prescribed integer bases, 2 and 2α, cited in (1.19) from the introduction result by applying Lemma 4.1 to the series for the generalized convergent function, $\mathrm{Conv}_2(\alpha, R; z)$, expanded by following equations:

$$\begin{aligned}
p_n(\alpha, R) &\equiv [z^n]\left(\frac{1-z(2\alpha+R)}{R(\alpha+R)z^2-2(\alpha+R)z+1}\right) && \pmod{2, 2\alpha} \\
&= \sum_{b=\pm1}\frac{\left(\sqrt{\alpha(\alpha+R)}-b\cdot\alpha\right)\left(\alpha+b\cdot\sqrt{\alpha(\alpha+R)}+R\right)^n}{2\sqrt{\alpha(\alpha+R)}} && \pmod{2, 2\alpha}.
\end{aligned}$$

The next congruences for the α–factorial function sequences modulo 3 (3α) and modulo 4 (4α) cited as particulars example in Section 1.4 are established similarly by applying the previous lemma to the series coefficients of the next cases of the convergent functions, $\mathrm{Conv}_p(\alpha, R; z)$, for $p := 3, 4$ and where $\alpha :\mapsto -\alpha$ and $R :\mapsto n$.

$$p_n(\alpha, R) \equiv [z^n]\left(\frac{1-2(3\alpha+R)z+z^2\left(R^2+4\alpha R+6\alpha^2\right)}{1-3(2\alpha+R)z+3(\alpha+R)(2\alpha+R)z^2-R(\alpha+R)(2\alpha+R)z^3}\right) \pmod{3, 3\alpha}$$

$$p_n(\alpha, R) \equiv [z^n]\left(\frac{1-3(R+4\alpha)z+z^2\left(3R^2+19R\alpha+36\alpha^2\right)-(R+4\alpha)\left(R^2+3R\alpha+6\alpha^2\right)z^3}{1-4(R+3\alpha)z+6(R+2\alpha)(R+3\alpha)z^2-4(R+\alpha)(R+2\alpha)(R+3\alpha)z^3+R(R+\alpha)(R+2\alpha)(R+3\alpha)z^4}\right) \pmod{4, 4\alpha}$$

The particular cases of the new congruence properties satisfied modulo 3 (3α) and 4 (4α) cited in (1.21) from Section 1.4 of the introduction also phrase results that are expanded through exact algebraic formulas involving the reciprocal zeros of the convergent denominator functions, $\mathrm{FQ}_3(-\alpha, n; z)$

and $\mathrm{FQ}_4(-\alpha, n; z)$, given in Table A.3 (page 79). [19, *cf.* §1.11(iii), §4.43]. The congruences cited in the example cases from the introduction then correspond to the respective special cases of the reflected numerator polynomials provided in Table A.4 (page 80). Ⓔ

Definition 4.3. More generally, let the p–order roots, $\ell_{p,i}^{(\alpha)}(R)$, and the corresponding reflected partial fraction coefficients, $C_{p,i}^{(\alpha)}(R)$, be defined for $p \geq 2$ and each $1 \leq i \leq p$ as follows:

$$\begin{aligned}\left(\ell_{p,i}^{(\alpha)}(R)\right)_{i=1}^{p} &:= \left\{z_i : z_i^h \cdot \mathrm{FQ}_h\left(\alpha, R; z_i^{-1}\right) = 0,\ 1 \leq i \leq p\right\} \\ C_{p,i}^{(\alpha)}(R) &:= \widetilde{\mathrm{FP}}_h\left(\alpha, R; \ell_{p,i}^{(\alpha)}(R)\right),\ 1 \leq i \leq p.\end{aligned}$$

The roots, $\ell_{p,i}^{(\alpha)}(R)$, defined as above in terms of the reflected denominator polynomials, denoted by $\widetilde{\mathrm{FQ}}_h(\alpha, R; z) := z^h \cdot \mathrm{FQ}_h\left(\alpha, R; z^{-1}\right)$, correspond to special zeros of the confluent hypergeometric function, $U(-h, b, w)$, and the associated Laguerre polynomials, $L_p^{(\beta)}(w)$, defined as in Section 1.3 [19, §18.2(vi), §18.16] [3; 11] †F.9 . The listings given in Table A.4 (page 80) provide the first few simplified cases of the reflected numerator polynomial sequences, denoted by $\widetilde{\mathrm{FP}}_h(\alpha, R; z) := z^{h-1}\,\mathrm{FP}_h(\alpha, R; 1/z)$, which lead to the explicit formulations of the congruences modulo p (and $p\alpha$) at each of the particular cases of $p := 4, 5$ given in (1.21), and then for the next few small special cases for subsequent cases of the integers $p \geq 6$. Ⓓ

The notation in the previous definition is then employed in the next statements generalizing the exact formula expansions and congruence properties cited in (1.15.a) and (1.15.b) of Section 1.3, and the formulas given in the particular special case results from (1.21) of Section 1.4.2 when the fixed parameters $\alpha, R \neq 0$ are integer–valued (see Remark 4.4 below):

$$\begin{aligned}
p_n(\alpha, R) &= \sum_{1 \leq i \leq h} \frac{C_{h,i}^{(\alpha)}(R)}{\prod_{j \neq i}\left(\ell_{h,i}^{(\alpha)}(R) - \ell_{h,j}^{(\alpha)}(R)\right)} \times \left(\ell_{h,i}^{(\alpha)}(R)\right)^{n+1}, && \forall h \geq n \geq 1 \qquad (4.4) \\
p_n(\alpha, R) &\equiv \sum_{1 \leq i \leq p} \frac{C_{p,i}^{(\alpha)}(R)}{\prod_{j \neq i}\left(\ell_{p,i}^{(\alpha)}(R) - \ell_{p,j}^{(\alpha)}(R)\right)} \times \left(\ell_{p,i}^{(\alpha)}(R)\right)^{n+1} && \left(\mathrm{mod}\ p, p\alpha, p\alpha^2, \ldots, p\alpha^p\right) \\
n!_{(\alpha)} &= \sum_{1 \leq i \leq h} \frac{C_{h,i}^{(-\alpha)}(n)}{\prod_{j \neq i}\left(\ell_{h,i}^{(-\alpha)}(n) - \ell_{h,j}^{(-\alpha)}(n)\right)} \times \left(\ell_{h,i}^{(-\alpha)}(n)\right)^{\left\lfloor \frac{n-1}{\alpha} \right\rfloor + 1}, && \forall h \geq n \geq 1 \\
n!_{(\alpha)} &\equiv \underbrace{\sum_{1 \leq i \leq p} \frac{C_{p,i}^{(-\alpha)}(n)}{\prod_{j \neq i}\left(\ell_{p,i}^{(-\alpha)}(n) - \ell_{p,j}^{(-\alpha)}(n)\right)} \times \left(\ell_{p,i}^{(-\alpha)}(n)\right)^{\left\lfloor \frac{n-1}{\alpha} \right\rfloor + 1}}_{:= R_p^{(\alpha)}(n)} && \left(\mathrm{mod}\ p, p\alpha, p\alpha^2, \ldots, p\alpha^p\right).
\end{aligned}$$

The first pair of expansions given in (4.4) for the generalized product sequences, $p_n(\alpha, R)$, provide exact formulas and the corresponding new congruence properties for the Pochhammer symbol and Pochhammer k–symbol, in the respective special cases where $(x)_n :\mapsto \alpha^{-n} p_n(\alpha, \alpha x)$ and $(x)_{n,\alpha} :\mapsto p_n(\alpha, x)$ in the equations above.

†F.9 *Note §4.2.* The characterizations of the convergent denominator functions, $\mathrm{FQ}_h(\alpha, R; z)$, by the confluent hypergeometric function and Laguerre polynomial sequences suggests possible approaches to these factorial–function–related sequence formulas by Turan–type inequalities for the special function sequences in (1.14) [19, *cf.* §18.14(ii), §18.16(iv)].

Remark 4.4 (Congruences for Rational–Valued Parameters)**.** The J–fraction parameters, $\lambda_h = \lambda_h(\alpha, R)$ and $M_h = M_h(\alpha, R)$, defined as in the summary of the enumerative properties from Section 3.1, corresponding to the expansions of the generalized convergents defined by the proof in Section 3.2 satisfy

$$\begin{aligned}
\lambda_k(\alpha, R) &:= \mathrm{as}_{k-1}(\alpha, R) \cdot \mathrm{bs}_k(\alpha, R) \\
&= \alpha(R + (k-1)\alpha) \cdot k \\
M_h(\alpha, R) &:= \lambda_1(\alpha, R) \cdot \lambda_2(\alpha, R) \times \cdots \times \lambda_h(\alpha, R) \\
&= \alpha^h \cdot h! \times p_h(\alpha, R) \\
&= \alpha^h \cdot h! \times (R)_{h,\alpha},
\end{aligned}$$

so that for integer divisors, $N_h(\alpha, R) \mid M_h(\alpha, R)$, we have that

$$p_n(\alpha, R) \equiv [z^n] \operatorname{Conv}_h(\alpha, R; z) \pmod{N_h(\alpha, R)}.$$

So far we have restricted ourselves to examples of the particular product sequence cases, $p_n(\alpha, R)$, where $\alpha \neq 0$ is integer–valued, i.e., so that $p, p\alpha^i \mid M_p(\alpha, R)$ for $1 \leq i \leq p$ whenever $p \geq 2$ is a fixed natural number. Identities arising in some related applications that involve finding results analogous to the explicit new congruence properties stated so far for other factorial–related sequence cases, specifically when the choice of $\alpha \neq 0$ is strictly rational–valued, are intentionally not treated in the examples above. ®

4.2 Applications of rational diagonal–coefficient generating functions and Hadamard product sequences

4.2.1 Overview: Notation and generalized definitions

We define the next extended notation for the *Hadamard product* generating functions, $(F_1 \odot F_2)(z)$ and $(F_1 \odot \cdots \odot F_k)(z)$, at some fixed, formal $z \in \mathbb{C}$ [6; 17] [25, §6.1]. Phrased in slightly different wording, we define (4.5) as an alternate notation for the *diagonal generating functions* that enumerate the corresponding product sequences generated by the diagonal coefficients of the multiple–variable product series in k formal variables treated as in the reference [25, §6.3].

$$F_1 \odot F_2 \odot \cdots \odot F_k := \sum_{n \geq 0} f_{1,n} f_{2,n} \cdots f_{k,n} \times z^n \quad \text{where} \quad F_i(z) := \sum_{n \geq 0} f_{i,n} z^n \text{ for } 1 \leq i \leq k \tag{4.5}$$

When $F_i(z)$ is a rational function of z for each $1 \leq i \leq k$, we have particularly nice expansions of the coefficient extraction formulas of the rational diagonal generating functions from [25] [17, §2.4]. In particular, when $F_i(z)$ is rational in z at each respective i, these rational generating functions are expanded through the next few useful formulas:

$$\begin{aligned}
F_1 \odot F_2 &= [x_1^0]\left(F_2\left(\frac{z}{x_1}\right) \cdot F_1(x_1)\right) \qquad (4.6) \\
F_1 \odot F_2 \odot F_3 &= [x_2^0 x_1^0]\left(F_3\left(\frac{z}{x_2}\right) \cdot F_2\left(\frac{x_2}{x_1}\right) \cdot F_1(x_1)\right) \\
F_1 \odot F_2 \odot \cdots \odot F_k &= [x_{k-1}^0 \cdots x_2^0 x_1^0]\left(F_k\left(\frac{z}{x_{k-1}}\right) \cdot F_{k-1}\left(\frac{x_{k-1}}{x_{k-2}}\right) \times \cdots \times F_2\left(\frac{x_2}{x_1}\right) \cdot F_1(x_1)\right).
\end{aligned}$$

Analytic formulas for the Hadamard products, $F_1 \odot F_2 = F_1(z) \odot F_2(z)$, when the component sequence generating functions are well enough behaved in some neighborhood of $z_0 = 0$ are given in the references [6; 8] [25, *cf.* §6.3] ‡F.10 .

We regard the rational convergents approximating the otherwise divergent ordinary generating functions for the generalized factorial function sequences strictly as formal power series in z whenever possible in this article. The remaining examples in this section illustrate this more formal approach taken with the generating functions enumerating the factorial–related product sequences considered here. The next several subsections aim to provide several concrete applications and some notable special cases illustrating the utility of this approach to the more general formal sequence products enumerated through the rational convergent functions, especially when combined with other generating function techniques discussed elsewhere and in the references [6; 13; 16; 17; 25; 26].

4.2.2 Examples: Constructing hybrid rational generating function approximations from the generalized convergent functions

When one of the generating functions of an individual sequence from the Hadamard product representations in (4.6) is not rational in z, we still proceed, however slightly more carefully, to formally enumerate the terms of these sequences that arise in applications. For example, the *central binomial coefficients* are enumerated by the next convergent–based generating functions whenever $n \geq 1$ [13, *cf.* §5.3] [A000984] §F.11 .

$$\begin{aligned} \binom{2n}{n} &= \frac{2^{2n}}{n!} \times (1/2)_n \quad = [z^n][x^0]\left(e^{2x}\operatorname{Conv}_n\left(2, 1; \frac{z}{x}\right)\right) && \text{(Central Binomial Coefficients)} \\ &= \frac{2^n}{n!} \times (2n-1)!! = [z^n][x^1]\left(e^{2x}\operatorname{Conv}_n\left(-2, 2n-1; \frac{z}{x}\right)\right). \end{aligned}$$

Since the reciprocal factorial squared terms, $(n!)^{-2}$, are generated by the power series for the *modified Bessel function of the first kind*, $I_0(2\sqrt{z}) = \sum_{n\geq 0} z^n/(n!)^2$, these central binomial coefficients are also enumerated as the diagonal coefficients of the following convergent function products [19, §10.25(ii)] [13, §5.5]:

$$\begin{aligned} \binom{2n}{n} &= \frac{2^{2n}\,(1)_n\left(\frac{1}{2}\right)_n}{(n!)^2} && = [x_1^0 x_2^0 z^n]\left(\operatorname{Conv}_n\left(2, 2; \frac{z}{x_2}\right)\operatorname{Conv}_n\left(2, 1; \frac{x_2}{x_1}\right) I_0\left(2\sqrt{x_1}\right)\right) \\ &= \frac{(2n)!!(2n-1)!!}{(n!)^2} && = [x_1^0 x_2^0 z^n]\left(\operatorname{Conv}_n\left(-2, 2n; \frac{z}{x_2}\right)\operatorname{Conv}_n\left(-2, 2n-1; \frac{x_2}{x_1}\right) I_0\left(2\sqrt{x_1}\right)\right). \end{aligned}$$

The next binomial coefficient product sequence is enumerated through a similar construction of the convergent–based generating function identities expanded in the previous equations:

$$\binom{3n}{n}\binom{2n}{n} = \frac{3^{3n}\left(\frac{1}{3}\right)_n\left(\frac{2}{3}\right)_n}{(n!)^2}$$

‡F.10 *Note §4.3.* Compare with the next known formula when both sequence generating functions, $F_1(z)$ and $F_2(z)$, are absolutely convergent for some $|z| \leq r < 1$:

$$\left(F_1 \odot F_2\right)(z^2) = \frac{1}{2\pi}\int_0^{2\pi} F_1\left(ze^{\imath t}\right) F_2\left(ze^{-\imath t}\right) dt. \qquad \text{(Hadamard Product Integral Formula)}$$

§F.11 *Note §4.4.* Section 6.3.3 provides further examples of congruences for the double factorial function, $(2n-1)!!$, related to these coefficients modulo integer powers of n and n^p.

$$\begin{aligned}
&= [x_1^0 x_2^0 z^n] \left(\mathrm{Conv}_n\left(3, 2; \frac{3z}{x_2}\right) \mathrm{Conv}_n\left(3, 1; \frac{x_2}{x_1}\right) I_0\left(2\sqrt{x_1}\right)\right) \\
&= \frac{3^n}{(n!)^2} \times (3n-1)!_{(3)}\,(3n-2)!_{(3)} \\
&= [x_1^0 x_2^0 z^n] \left(\mathrm{Conv}_n\left(-3, 3n-1; \frac{3z}{x_2}\right) \mathrm{Conv}_n\left(-3, 3n-2; \frac{x_2}{x_1}\right) I_0\left(2\sqrt{x_1}\right)\right).
\end{aligned}$$

The next few identities for the convergent generating function products over the binomial coefficient variants cited in (1.18) from the introduction are generated as the diagonal coefficients of the corresponding products of the convergent functions convolved with arithmetic progressions extracted from the exponential series in the form of the following equation, where $\omega_a := \exp(2\pi\imath/a)$ denotes the primitive a^{th} root of unity for integers $a \geq 2$ [16, §1.2.9] [6, Ex. 1.26, p. 84] ¶F.12 :

$$\widehat{E}_a(z) := \sum_{n\geq 0} \frac{z^n}{(an)!} = \frac{1}{a}\left(e^{z^{1/a}} + e^{\omega_a \cdot z^{1/a}} + e^{\omega_a^2 \cdot z^{1/a}} + \cdots + e^{\omega_a^{a-1}\cdot z^{1/a}}\right), \quad a > 1. \tag{4.7}$$

The next particular special cases of these diagonal–coefficient generating functions corresponding to the binomial coefficient sequence variants from (1.18) of Section 1.4.1 are then given through the following coefficient extraction identities provided by (4.6) [A166351; A066802]:

$$\begin{aligned}
\frac{(6n)!}{(3n)!} &= \frac{6^{6n} \cancel{(1)_n} \cancel{\left(\frac{2}{6}\right)_n} \cancel{\left(\frac{3}{6}\right)_n} \left(\frac{1}{6}\right)_n \left(\frac{3}{6}\right)_n \left(\frac{5}{6}\right)_n}{3^{3n}\,(1)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n} \\
&= 24^n \times 6^n\,(1/6)_n \times 2^n\,(1/2)_n \times 6^n\,(5/6)_n \\
&= [x_2^0 x_1^0 z^n] \left(\mathrm{Conv}_n\left(6, 5; \frac{24z}{x_2}\right) \mathrm{Conv}_n\left(2, 1; \frac{x_2}{x_1}\right) \mathrm{Conv}_n(6, 1; x_1)\right) \\
&= 8^n \times (6n-5)!_{(6)}\,(6n-3)!_{(6)}\,(6n-1)!_{(6)} \\
&= [x_2^0 x_1^0 z^n] \left(\mathrm{Conv}_n\left(-6, 6n-5; \frac{8z}{x_2}\right) \mathrm{Conv}_n\left(-6, 6n-3; \frac{x_2}{x_1}\right) \mathrm{Conv}_n(-6, 6n-1; x_1)\right) \\
\binom{6n}{3n} &= [x_3^0 x_2^0 x_1^0 z^n] \Biggl(\mathrm{Conv}_n\left(6, 5; \frac{8z}{x_3}\right) \mathrm{Conv}_n\left(6, 3; \frac{x_3}{x_2}\right) \mathrm{Conv}_n\left(6, 1; \frac{x_2}{x_1}\right) \times \\
&\qquad \times \underbrace{\frac{1}{3}\left(e^{x_1^{1/3}} + 2e^{-\frac{x_1^{1/3}}{2}} \cos\left(\frac{\sqrt{3}\cdot x_1^{1/3}}{2}\right)\right)}_{\widehat{E}_3(z)}\Biggr) \\
&= [x_3^0 x_2^0 x_1^0 z^n] \Biggl(\mathrm{Conv}_n\left(-6, 6n-5; \frac{8z}{x_3}\right) \mathrm{Conv}_n\left(-6, 6n-3; \frac{x_3}{x_2}\right) \times
\end{aligned}$$

¶F.12 *Note §4.5.* The modified generating functions, $\widehat{E}_a(z) = E_{a,1}(z)$, correspond to special cases of the *Mittag–Leffler function*, $E_{a,b}(z)$, defined as in the references [19, §10.46], which then denote the power series expansions of arithmetic progressions over the coefficients of the ordinary generating function for the exponential series sequences, $f_n := 1/n!$ and $f_{an} = 1/(an)!$. For $a := 2, 3, 4$, the particular cases of these exponential series generating functions are given by

$$\widehat{E}_2(z) = \cosh\left(\sqrt{z}\right), \widehat{E}_3(z) = \frac{1}{3}\left(e^{z^{1/3}} + 2e^{-\frac{z^{1/3}}{2}} \cos\left(\frac{\sqrt{3}\cdot z^{1/3}}{2}\right)\right), \text{ and } \widehat{E}_4(z) = \frac{1}{2}\left(\cos\left(z^{1/4}\right) + \cosh\left(z^{1/4}\right)\right),$$

where the powers of the a^{th} roots of unity in these special cases satisfy $\omega_2 = -1$, $\omega_3 = \frac{\imath}{2}\left(\imath + \sqrt{3}\right)$, $\omega_3^2 = -\frac{\imath}{2}\left(-\imath + \sqrt{3}\right)$, and $\left(\omega_4^m\right)_{1\leq m\leq 4} = (\imath, -1, -\imath, 1)$.

$$\times \operatorname{Conv}_n\left(-6, 6n-1; \frac{x_2}{x_1}\right) \times \underbrace{\frac{1}{3}\left(e^{x_1^{1/3}} + 2e^{-\frac{x_1^{1/3}}{2}} \cos\left(\frac{\sqrt{3} \cdot x_1^{1/3}}{2}\right)\right)}_{\widehat{E}_3(z)} \Bigg).$$

Similarly, the following related sequence cases forming particular expansions of these binomial coefficient variants are generated by

$$\begin{aligned}
\binom{8n}{4n} &= \frac{2^{16n}}{(4n)!} \times \left(\tfrac{1}{8}\right)_n \left(\tfrac{3}{8}\right)_n \left(\tfrac{5}{8}\right)_n \left(\tfrac{7}{8}\right)_n \\
&= [x_1^0 x_2^0 x_3^0 x_4^0 z^n]\Bigg(\operatorname{Conv}_n\left(8, 7; \frac{16z}{x_4}\right) \operatorname{Conv}_n\left(8, 5; \frac{x_4}{x_3}\right) \operatorname{Conv}_n\left(8, 3; \frac{x_3}{x_2}\right) \times \\
&\qquad \times \operatorname{Conv}_n\left(8, 1; \frac{x_2}{x_1}\right) \times \frac{1}{2}\left(\cos\left(x_1^{1/4}\right) + \cosh\left(x_1^{1/4}\right)\right)\Bigg) \\
&= \frac{2^{4n}}{(4n)!} \times (8n-7)!_{(8)} (8n-5)!_{(8)} (8n-3)!_{(8)} (8n-3)!_{(8)} \\
&= [x_1^0 x_2^0 x_3^0 x_4^0 z^n]\Bigg(\operatorname{Conv}_n\left(-8, 8n-7; \frac{16z}{x_4}\right) \operatorname{Conv}_n\left(-8, 8n-5; \frac{x_4}{x_3}\right) \times \\
&\qquad \times \operatorname{Conv}_n\left(-8, 8n-3; \frac{x_3}{x_2}\right) \operatorname{Conv}_n\left(-8, 8n-1; \frac{x_2}{x_1}\right) \times \widehat{E}_4(x_1)\Bigg).
\end{aligned}$$

Remark 4.5 (Laplace–Borel Transformations of Formal Power Series)**.** Another pair of convergent–based generating function identities enumerating the sequence of *subfactorials*, $(!n)_{n\geq 1}$, or *derangements*, $(n¡)_{n\geq 1}$, are expanded for $n \geq 1$ as follows [13, §5.3] [2, §3] [19, *cf.* §8.4] [A000166] (see Remark 4.8 and the related examples cited in Section 4.2.5):

$$\begin{aligned}
!n &:= n! \times \sum_{i=0}^{n} \frac{(-1)^i}{i!} && \text{(Subfactorial Function)} \\
&= [z^n x^0]\left(\frac{e^{-x}}{(1-x)} \times \operatorname{Conv}_n\left(-1, n; \frac{z}{x}\right)\right) \\
&= [x^0 z^n]\left(\frac{e^{-x}}{(1-x)} \times \operatorname{Conv}_n\left(1, 1; \frac{z}{x}\right)\right).
\end{aligned}$$

The sequence of subfactorials are enumerated through the previous equations as the diagonals of generating function products where the rational convergent functions, $\operatorname{Conv}_n(\alpha, R; z)$, generate the sequence multiplier of $n!$ corresponding to the (*formal*) *Laplace–Borel transform*, $\mathcal{L}(f(t); z)$, defined as the integral transform in the next equation [9, *cf.* §2.2] [8, §B.4] [13, p. 566], applied termwise to the power series given by the exponential generating function, $f(x) := e^{-x}/(1-x)$, for this sequence.

$$\mathcal{L}(f(t); z) := \int_0^\infty e^{-t} f(tz) dt, \qquad \text{(Laplace-Borel Transform Integral Formula)}$$

The necessary condition for primality from the example given in (1.24) of the introduction is constructed by employing a similar technique with the Stirling number generating functions given by the following equation when $p := 1$ [13, §7.4]:

$$\sum_{n\geq 0} \begin{bmatrix} n+1 \\ p+1 \end{bmatrix} \frac{z^n}{n!} = \frac{\partial}{\partial z} \left[\frac{\operatorname{Log}\left(\frac{1}{1-z}\right)^{p+1}}{(p+1)!} \right] = \frac{(-1)^p}{p!} \cdot \frac{\operatorname{Log}(1-z)^p}{(1-z)}.$$

The applications cited in Section 4.2.5 and Section 4.2.6 in this article below employ this generating function particular technique to enumerate the factorial function multipliers provided by these rational convergent functions in several particular cases of sequences involving finite sums over factorial functions, sums of powers sequences, and new forms of approximate generating functions for the binomial coefficients and sequences of binomials. Ⓡ

4.2.3 Example: Expanding arithmetic progressions of the single factorial function

One application suggested by the results in the previous section provides a–fold reductions of the h–order series approximations otherwise required to exactly enumerate arithmetic progressions of the single factorial function according to the next result.

$$(an+r)! = [z^{an+r}] \operatorname{Conv}_h(-1, an+r; z), \ \forall n \geq 1, a \in \mathbb{Z}^+, 0 \leq r < a, \forall h \geq an+r \tag{4.8}$$

The statement of *Gauss's multiplication formula* for the gamma function yields the following decompositions of the single factorial functions, $(an+r)!$, into a finite product over a of the integer–valued multiple factorial sequences defined by (1.2) for $n \geq 1$ and whenever $a \geq 2$ and $0 \leq r < a$ are fixed natural numbers [19, §5.5(iii)] [24, §2] [27] ‖F.13 :

$$\begin{aligned}(an+r)! &= (an+r)!_{(a)} \cdot (an+r-1)!_{(a)} \times \cdots \times (an+r-a+2)!_{(a)} \, (an+r-a+1)!_{(a)} \\ &= r! \cdot a^{an} \left(\frac{1+r}{a}\right)_n \left(\frac{2+r}{a}\right)_n \times \cdots \times \left(\frac{a-1+r}{a}\right)_n \left(\frac{a+r}{a}\right)_n \\ &= r! \cdot a^{an} \left(1+\frac{r}{a}\right)_n \left(1+\frac{r-1}{a}\right)_n \times \cdots \times \left(1+\frac{r-a+1}{a}\right)_n, \ \forall a, n \in \mathbb{Z}^+, r \geq 0.\end{aligned}$$

The arithmetic progression sequences of the single factorial function formed in the particular special cases when $a := 2, 3$ are expanded in the examples cited below illustrate the utility to these convergent–based formal generating function approximations.

In the particular cases where $a := 2$ (with $r := 0, 1$), we obtain the following forms of the corresponding alternate expansions of (4.8) enumerated by the diagonal coefficients of the next convergent function product generating functions for all $n \geq 1$ [24, *cf.* §2] [`A010050`; `A009445`]:

$$(2n)! = 2^n n! \times (2n-1)!! \qquad \text{(Double Factorial Function Expansions)}$$

‖F.13 *Note §4.6.* These identities correspond to the expansions of the single factorial function given by

$$n! = n!! \cdot (n-1)!! = n!!! \cdot (n-1)!!! \cdot (n-2)!!! = \prod_{i=0}^{a-1} (a-i)!_{(a)}, \ a \in \mathbb{Z}^+,$$

and the *multiplication formula* for the Pochhammer symbol expanded in the form of [27]

$$(x)_{an+r} = (x)_r \times a^{an} \times \prod_{j=0}^{a-1} \left(\frac{x+j+r}{a}\right)_n, \qquad \text{(Pochhammer Symbol Multiplication Formula)}$$

for integers $a \geq 1$, $r \geq 0$, and where $(an+r)! = (1)_{an+r}$ by Lemma 4.1.

$$\begin{aligned}
&= [z^n][x^0]\left(\operatorname{Conv}_n\left(-1, n; \frac{2z}{x}\right)\operatorname{Conv}_n\left(-2, 2n-1; x\right)\right) \\
&= 2^n n! \times 2^n \left(1/2\right)_n \\
&= [x^0 z^n]\left(\operatorname{Conv}_n\left(1, 1; \frac{2z}{x}\right)\operatorname{Conv}_n\left(2, 1; x\right)\right) \\
(2n+1)! &= 2^n n! \times (2n+1)!! \\
&= [z^n][x^0]\left(\operatorname{Conv}_n\left(-1, n; \frac{2z}{x}\right)\operatorname{Conv}_n\left(-2, 2n+1; x\right)\right) \\
&= 2^n n! \times 2^n \left(3/2\right)_n \\
&= [x^1 z^n]\left(\operatorname{Conv}_n\left(1, 1; \frac{2z}{x}\right)\operatorname{Conv}_n\left(2, 1; x\right)\right) \\
&= [x^0 z^n]\left(\operatorname{Conv}_n\left(1, 1; \frac{2z}{x}\right)\operatorname{Conv}_n\left(2, 3; x\right)\right).
\end{aligned}$$

When $a := 3$ we obtain the next several alternate expansions generating the triple factorial products for the arithmetic progression sequences in (4.8) stated in the following equations for $n \geq 2$ by extending the constructions of the identities for the expansions of the double factorial products in the previous equations [A100732; A100089; A100043] [24, §2]:

$$\begin{aligned}
(3n)! &= (3n)!!! \times (3n-1)!!! \times (3n-2)!!! \qquad \text{(Triple Factorial Function Expansions)} \\
&= [z^n][x_2^0 x_1^0]\left(\operatorname{Conv}_n\left(-1, n; \frac{3z}{x_2}\right)\operatorname{Conv}_n\left(-3, 3n-1; \frac{x_2}{x_1}\right)\operatorname{Conv}_n\left(-3, 3n-2; x_1\right)\right) \\
&= 3^n n! \times 3^n \left(2/3\right)_n \times 3^n \left(1/3\right)_n \\
&= [x_1^0 x_2^0 z^n]\left(\operatorname{Conv}_n\left(1, 1; \frac{3z}{x_2}\right)\operatorname{Conv}_n\left(3, 1; \frac{x_2}{x_1}\right)\operatorname{Conv}_n\left(3, 2; x_1\right)\right) \\
(3n+1)! &= (3n)!!! \times (3n-1)!!! \times (3(n+1)-2)!!! \\
&= [z^n][x_2^0 x_1^{-1}]\left(\operatorname{Conv}_n\left(-1, n; \frac{3z}{x_2}\right)\operatorname{Conv}_n\left(-3, 3n-1; \frac{x_2}{x_1}\right)\operatorname{Conv}_n\left(-3, 3n+1; x_1\right)\right) \\
&= 3^n n! \times 3^n \left(2/3\right)_n \times 3^n \left(4/3\right)_n \\
&= [x_1^0 x_2^0 z^n]\left(\operatorname{Conv}_n\left(1, 1; \frac{3z}{x_2}\right)\operatorname{Conv}_n\left(3, 4; \frac{x_2}{x_1}\right)\operatorname{Conv}_n\left(3, 2; x_1\right)\right) \\
(3n+2)! &= (3n)!!! \times (3(n+1)-1)!!! \times (3(n+1)-2)!!! \\
&= [z^n][x_2^{-1} x_1^0]\left(\operatorname{Conv}_n\left(-1, n; \frac{3z}{x_2}\right)\operatorname{Conv}_n\left(-3, 3n+2; \frac{x_2}{x_1}\right)\operatorname{Conv}_n\left(-3, 3n+1; x_1\right)\right) \\
&= 2 \times 3^n n! \times 3^n \left(5/3\right)_n \times 3^n \left(4/3\right)_n \\
&= [x_1^1 x_2^0 z^n]\left(\operatorname{Conv}_n\left(1, 1; \frac{3z}{x_2}\right)\operatorname{Conv}_n\left(3, 4; \frac{x_2}{x_1}\right)\operatorname{Conv}_n\left(3, 2; x_1\right)\right).
\end{aligned}$$

The additional forms of the diagonal–coefficient generating functions corresponding to the special cases of the sequences in (4.8) where $(a, r) := (4, 2)$ and $(a, r) := (5, 3)$, respectively involving the *quadruple* and *quintuple factorial* functions are also cited in the next equations to further illustrate the procedure outlined by the previous two example cases.

$$(4n+2)! = (4n)!!!! \times (4n-1)!!!! \times (4(n+1)-2)!!!! \times (4(n+1)-3)!!!!$$

$$
\begin{aligned}
&= [z^n][x_3^0 x_2^{-1} x_1^0]\Bigg(\operatorname{Conv}_n\left(-1, n; \frac{4z}{x_3}\right) \operatorname{Conv}_n\left(-4, 4n-1; \frac{x_3}{x_2}\right) \times \\
&\qquad \times \operatorname{Conv}_n\left(-4, 4n+2; \frac{x_2}{x_1}\right) \operatorname{Conv}_n\left(-4, 4n+1; x_1\right)\Bigg) \\
&= [x_1^0 x_2^1 x_3^0 z^n]\Bigg(\operatorname{Conv}_n\left(1, 1; \frac{4z}{x_3}\right) \operatorname{Conv}_n\left(4, 3; \frac{x_3}{x_2}\right) \operatorname{Conv}_n\left(4, 2; \frac{x_2}{x_1}\right) \times \\
&\qquad \times \operatorname{Conv}_n\left(4, 1; x_1\right)\Bigg), \quad n \geq 2
\end{aligned}
$$

$$
\begin{aligned}
(5n+3)! &= (5n)!_{(5)} \times (5n-1)!_{(5)} \times (5(n+1)-2)!_{(5)} \times (5(n+1)-3)!_{(5)} \times (5(n+1)-4)!_{(5)} \\
&= [z^n][x_4^0 x_3^{-1} x_2^0 x_1^0]\Bigg(\operatorname{Conv}_n\left(-1, n; \frac{5z}{x_4}\right) \operatorname{Conv}_n\left(-5, 5n-1; \frac{x_4}{x_3}\right) \times \\
&\qquad \times \operatorname{Conv}_n\left(-5, 5n+3; \frac{x_3}{x_2}\right) \operatorname{Conv}_n\left(-5, 5n+2; \frac{x_2}{x_1}\right) \times \\
&\qquad \times \operatorname{Conv}_n\left(-5, 5n+1; x_1\right)\Bigg) \\
&= [x_1^0 x_2^0 x_3^1 x_4^0 z^n]\Bigg(\operatorname{Conv}_n\left(1, 1; \frac{5z}{x_4}\right) \operatorname{Conv}_n\left(5, 4; \frac{x_4}{x_3}\right) \operatorname{Conv}_n\left(5, 3; \frac{x_3}{x_2}\right) \times \\
&\qquad \times \operatorname{Conv}_n\left(5, 2; \frac{x_2}{x_1}\right) \operatorname{Conv}_n\left(5, 1; x_1\right)\Bigg), \quad n \geq 2.
\end{aligned}
$$

The truncated power series approximations generating the single factorial functions formulated in the last few examples expanded in this section are compared to the known results for extracting arithmetic progressions from any formal, ordinary power series generating function of an arbitrary sequence through the primitive a^{th} roots of unity, $\omega_a := \exp(2\pi\imath/a)$, stated in the references [16, §1.2.9] [6, Ex. 1.26, p. 84] [13; 26], and in the special cases of the exponential series generating functions defined in (4.7) of the previous section.

4.2.4 Example: The superfactorial and the Barnes G–functions

The *superfactorial function*, $S_1(n)$, also denoted $S_{1,0}(n)$ below, is defined as the factorial product [A000178]

$$S_1(n) := \prod_{k \leq n} k! \quad \xrightarrow{\text{A000178}} \quad \{1, 1, 2, 12, 288, 34560, 24883200, \ldots\}. \qquad \text{(Superfactorial Function)}$$

These superfactorial functions are given in terms of the *Barnes G–function*, $G(z)$, for $z \in \mathbb{Z}^+$ through the relation $S_1(n) = G(n+2)$. The Barnes G–function, $G(z)$, corresponds to a so–termed "*double gamma function*" satisfying a functional equation of the following form for natural numbers $n \geq 1$ [19, §5.17] [1]:

$$G(n+2) = \Gamma(n+1)G(n+1) + [n=1]_\delta \qquad \text{(Barnes G-Function)}$$

We can similarly expand the superfactorial functions, $S_1(n)$, by unfolding the factorial products in the previous definition recursively according to the formulas given in the next equation.

$$S_1(n) = n! \cdot (n-1)! \times \cdots \times (n-k+1)! \cdot S_1(n-k), \ 0 \leq k < n$$

The product sequences over the single factorial functions formed by the last equations then lead to another application of the diagonal–coefficient product generating functions involving the rational convergent functions that enumerate the functions, $(n-k)!$, when $n-k \geq 1$.

In particular, these diagonal coefficient, Hadamard–product–like sequences involving the single factorial function are generated as the coefficients

$$\begin{aligned} S_1(n) = ([z^n] \operatorname{Conv}_n(-1, n; z)) & \times ([z^n] z \cdot \operatorname{Conv}_n(-1, n-1; z)) \times \\ & \times \left([z^n] z^2 \cdot \operatorname{Conv}_n(-1, n-2; z)\right) \times \cdots \times \\ & \times ([z^n] z^n \cdot \operatorname{Conv}_n(-1, 1; z)) \\ S_1(n) = ([z^n] \operatorname{Conv}_n(1, 1; z)) \qquad & \times ([z^n] z \cdot \operatorname{Conv}_n(1, 1; z)) \times \\ & \times \left([z^n] z^2 \cdot \operatorname{Conv}_n(1, 1; z)\right) \times \cdots \times ([z^n] z^n \cdot \operatorname{Conv}_n(1, 1; z)) . \end{aligned}$$

Stated more precisely, the superfactorial sequence is generated by the following finite, rational products of the generalized convergent functions for any $n \geq 2$:

$$\begin{aligned} S_1(n) &= [x_1^{-1} x_2^{-1} \cdots x_{n-1}^{-1} x_n^n] \left(\prod_{i=0}^{n-2} \operatorname{Conv}_n\left(-1, n-i; \frac{x_{n-i}}{x_{n-i-1}}\right) \times \operatorname{Conv}_n(-1, 1; x_1) \right) \\ S_1(n) &= [x_1^{-1} x_2^{-1} \cdots x_{n-1}^{-1} x_n^n] \left(\prod_{i=0}^{n-2} \operatorname{Conv}_n\left(1, 1; \frac{x_{n-i}}{x_{n-i-1}}\right) \times \operatorname{Conv}_n(1, 1; x_1) \right). \end{aligned} \tag{4.9}$$

Remark 4.6 (Generating Generalized Superfactorial Sequences)**.** Let the more general superfactorial functions, $S_{\alpha,d}(n)$, forming the analogous products of the integer–valued multiple, α–factorial function cases from (1.2) correspond to the expansions defined by the next equation.

$$S_{\alpha,d}(n) := \prod_{j=1}^{n} (\alpha j - d)!_{(\alpha)}, \ \alpha \in \mathbb{Z}^+, 0 \leq d < \alpha, n \geq 1 \tag{4.10}$$

Observe that the corollary of Lemma 4.1 cited in (4.2.a) implies that whenever $n \geq 1$, and for any fixed $\alpha \in \mathbb{Z}^+$, we immediately obtain the following identity corresponding to the so–termed "ordinary" case of these superfactorial functions, $S_1(n) = S_{1,0}(n)$, in the notation for these sequences defined above.

$$S_1(n) = \alpha^{-\binom{n+1}{2}} \prod_{j=1}^{n} (\alpha j)!_{(\alpha)} = \alpha^{-\binom{n+1}{2}} S_{\alpha,0}(n), \ \forall \alpha \in \mathbb{Z}^+, \ n \geq 1.$$

For other cases of the parameter $d > 0$, the generalized superfactorial function products defined by (4.10) are enumerated in a similar fashion to the previous constructions of the convergent–based generating function identities expanded by (4.9).

The special case sequences formed by the double factorial products, $S_{2,1}(n)$, and the quadruple factorial products, $S_{4,2}(n)$, are simplified by *Mathematica* to obtain the next closed–form expressions given by

$$S_{2,1}(n) := \prod_{j=1}^{n} (2j-1)!! = \frac{A^{3/2}}{2^{1/24} e^{1/8} \pi^{1/4}} \cdot \frac{2^{n(n+1)/2}}{\pi^{n/2}} \times G\left(n + \frac{3}{2}\right)$$

$$S_{4,2}(n) := \prod_{j=1}^{n} (4j-2)!!!! = \frac{A^{3/2}}{2^{1/24} e^{1/8} \pi^{1/4}} \cdot \frac{4^{n(n+1)/2}}{\pi^{n/2}} \times G\left(n+\frac{3}{2}\right),$$

where $A \approx 1.2824271$ denotes *Glaisher's constant* [19, §5.17], and where the particular constant multiples in the previous equation correspond to the special case values, $\Gamma(1/2) = \sqrt{\pi}$ and $G(3/2) = A^{-3/2} 2^{1/24} e^{1/8} \pi^{1/4}$ [1].

Since the sequences defined by (4.10) are also expanded as the products

$$S_{\alpha,d}(n) = \prod_{j=1}^{n} \left(\alpha^j \cdot \left(1-\frac{d}{\alpha}\right)_j\right) = \prod_{j=1}^{n} \left(\frac{\alpha^j \cdot \Gamma\left(j+1-\frac{d}{\alpha}\right)}{\Gamma\left(1-\frac{d}{\alpha}\right)}\right),$$

further computations with *Mathematica* yield the next few representative special cases of these generalized superfactorial functions when $\alpha := 3, 4, 5$ [1, *cf.* §2]:

$$\begin{aligned}
S_{3,1}(n) &= \prod_{j=1}^{n} (3j-1)!!! = 3^{n(n-1)/2} \left(\frac{2 \cdot G\left(\frac{5}{3}\right)}{G\left(\frac{8}{3}\right)}\right)^n \times \frac{G\left(n+\frac{5}{3}\right)}{G\left(\frac{5}{3}\right)} \\
S_{4,1}(n) &= \prod_{j=1}^{n} (4j-1)!!!! = 4^{n(n-1)/2} \left(\frac{3 \cdot G\left(\frac{7}{4}\right)}{G\left(\frac{11}{4}\right)}\right)^n \times \frac{G\left(n+\frac{7}{4}\right)}{G\left(\frac{7}{4}\right)} \\
S_{5,1}(n) &= \prod_{j=1}^{n} (5j-1)!_{(5)} = 5^{n(n-1)/2} \left(\frac{4 \cdot G\left(\frac{9}{5}\right)}{G\left(\frac{14}{5}\right)}\right)^n \times \frac{G\left(n+\frac{9}{5}\right)}{G\left(\frac{9}{5}\right)} \\
S_{5,2}(n) &= \prod_{j=1}^{n} (5j-2)!_{(5)} = 5^{n(n-1)/2} \left(\frac{3 \cdot G\left(\frac{8}{5}\right)}{G\left(\frac{13}{5}\right)}\right)^n \times \frac{G\left(n+\frac{8}{5}\right)}{G\left(\frac{8}{5}\right)}.
\end{aligned}$$

We are then led to conjecture inductively, without proof given in this example, that these sequences satisfy the form of the next equation involving the Barnes G–function over the rational–valued inputs prescribed according to the formula

$$S_{\alpha,d}(n) = \frac{\alpha^{n(n-1)/2} \cdot (\alpha-d)^n}{\Gamma\left(2-\frac{d}{\alpha}\right)^n} \times \frac{G\left(n+2-\frac{d}{\alpha}\right)}{G\left(2-\frac{d}{\alpha}\right)}.$$

Considering modified selections of the initial conditions satisfied by the product sequences, $p_n(\alpha, R) = (R)_{n,\alpha}$, suggests further avenues to enumerating other particular forms of the Barnes G–function formed by these generalized integer–parameter product sequence cases cited in the examples above. These functions are generated by extending the constructions of the rational generating function methods outlined in this section [1; 7], which then suggest additional identities for the Barnes G-functions, $G(z+2)$, over rational–valued $z > 0$ involving the special function zeros already defined by Section 1.3 and in Section 4.1 [3; 11]. Ⓡ

The generalized superfactorial sequences defined by (4.10) in the previous remark are also related to the *hyperfactorial function*, $H_1(n) := \prod_{j \leq n} j^j$, and the corresponding relations to the factorial function power sequences in the form of $H_1(n) \times G(n+1) = (n!)^n$ for $n \geq 1$ [A002109]. Statements of congruence properties and other relations connecting these sequences are also considered in the references [1; 2; 7].

4.2.5 Example: Enumerating sequences involving sums of factorial functions

The coefficients of the convergent–based generating function constructions for the factorial product sequences given in the previous section are compared to the next several identities expanding the corresponding sequences of finite sums involving factorial functions [A003422; A061062; A005165; A033312; A104344; A001044] [6, *cf.* §3; Ex. 3.30 p. 168] **F.14 *F.15 :

$$\begin{aligned}
\operatorname{af}(n) &:= \sum_{k=1}^{n} (-1)^{n-k} \cdot k! = [z^n] \left(\frac{1}{(1+z)} \cdot (\operatorname{Conv}_n(1,1;z) - 1) \right) && \text{(Alternating Factorials)} \\
L!n &:= \sum_{k=0}^{n-1} k! = [z^n] \left(\frac{z}{(1-z)} \cdot \operatorname{Conv}_n(1,1;z) \right) && \text{(Left Factorials)} \\
\operatorname{sf}_2(n) &:= \sum_{k=1}^{n} k \cdot k! = (n+1)! - 1 && (4.11) \\
&= [x^0 z^n] \left(\frac{1}{(1-z)} \frac{x}{(1-x)^2} \operatorname{Conv}_n\left(1,1;\frac{z}{x}\right) \right) \\
\operatorname{sf}_3(n) &:= \sum_{k=1}^{n} (k!)^2 && \text{(Sums of Factorial Squares)} \\
&= [x^0 z^n] \left(\frac{1}{(1-z)} \times \left(\operatorname{Conv}_n(1,1;x) \operatorname{Conv}_n\left(1,1;\frac{z}{x}\right) - 1 \right) \right) \\
\operatorname{sf}_4(n) &:= \sum_{k=0}^{n} (k!)^3 && \text{(Sums of Factorial Cubes)} \\
&= [x_1^0 x_2^0 z^n] \left(\frac{1}{(1-z)} \times \operatorname{Conv}_n\left(1,1;\frac{z}{x_2}\right) \operatorname{Conv}_n\left(1,1;\frac{x_2}{x_1}\right) \operatorname{Conv}_n(1,1;x_1) + 1 \right).
\end{aligned}$$

The expansion of the second to last sum, denoted $\operatorname{sf}_3(n)$ in (4.11), is generalized to form the following variants of sums over the squares of the α–factorial functions, $n!!$ and $n!!!$, through the generating function identities given in (1.16.b) of the introduction [A184877]:

$$\begin{aligned}
\operatorname{sf}_{3,2}(n) &:= \sum_{k=0}^{n} (k!!)^2 \\
&= [x^0 z^n] \Bigg(\frac{1}{(1-z)} \times \Bigg(\operatorname{Conv}_n(2,2;x) \operatorname{Conv}_n\left(2,2;\frac{z^2}{x}\right) \\
&\qquad + z \cdot \operatorname{Conv}_n(2,3;x) \operatorname{Conv}_n\left(2,3;\frac{z^2}{x}\right) \Bigg) \Bigg) \\
&= [x^0 z^n] \Bigg(\frac{1}{(1-z)} \times \Bigg(\operatorname{Conv}_n(2,2;x) \operatorname{Conv}_n\left(2,2;\frac{z^2}{x}\right) \\
&\qquad + z^{-1} \cdot \operatorname{Conv}_n(2,1;x) \operatorname{Conv}_n\left(2,1;\frac{z^2}{x}\right) - 1 \Bigg) \Bigg)
\end{aligned}$$

**F.14 *Note §4.7.* See also the MathWorld site for definitions of several other factorial–related finite sums and series.

*F.15 *Note §4.8.* A generalization of the second identity given in (4.11) due to Gould is stated in the reference as [6, p. 168]

$$\sum_{k=0}^{n} \binom{x}{k}^p \left(\frac{k!}{x^{k+1}}\right)^p ((x-k)^p - x^p) = \binom{x}{n+1}^p \left(\frac{(n+1)!}{x^{n+1}}\right)^p - 1.$$

$$\begin{aligned}\mathrm{sf}_{3,3}(n) &:= \sum_{k=0}^{n} (k!!!)^2 \\ &= [x^0 z^n]\Bigg(\frac{1}{(1-z)} \times \Bigg(\mathrm{Conv}_n\left(3,3;x\right)\mathrm{Conv}_n\left(3,3;\frac{z^3}{x}\right) \\ &\qquad\qquad + z^{-1}\cdot \mathrm{Conv}_n\left(3,2;x\right)\mathrm{Conv}_n\left(3,2;\frac{z^3}{x}\right) \\ &\qquad\qquad + z^{-2}\cdot \mathrm{Conv}_n\left(3,1;x\right)\mathrm{Conv}_n\left(3,1;\frac{z^3}{x}\right) - 1 - \frac{1}{z}\Bigg)\Bigg).\end{aligned}$$

The next form of the cube–factorial–power sequences, $\mathrm{sf}_4(n)$, defined in (4.11) corresponding to the next sums taken over powers of the double factorial function are similarly generated by †F.16

$$\begin{aligned}\mathrm{sf}_{4,2}(n) &:= \sum_{k=0}^{n} (k!!)^3 \\ &= [x_1^0 x_2^0 z^n]\Bigg(\frac{1}{(1-z)} \times \Bigg(\mathrm{Conv}_n\left(2,2;\frac{z^2}{x_2}\right)\mathrm{Conv}_n\left(2,2;\frac{x_2}{x_1}\right)\mathrm{Conv}_n\left(2,2;x_1\right) \\ &\qquad + z^{-1}\cdot \mathrm{Conv}_n\left(2,1;\frac{z^2}{x_2}\right)\mathrm{Conv}_n\left(2,1;\frac{x_2}{x_1}\right)\mathrm{Conv}_n\left(2,1;x_1\right) - \frac{1}{z}\Bigg) + 2\Bigg).\end{aligned}$$

The second variant of the factorial sums, $\mathrm{sf}_2(n)$, in (4.11) is enumerated through an alternate approach provided by the more interesting summation identities cited in the reference [6, §3, p. 168]. In particular, we have another pair of identities generating these sums expanded as

$$\begin{aligned}(n+1)! - 1 &= (n+1)! \times \sum_{k=0}^{n} \frac{k}{(k+1)!} \\ &= [z^n x^0]\left(\left(\frac{1}{x\cdot(1-x)} - \frac{e^x}{x}\right) \times \mathrm{Conv}_{n+2}\left(-1, n+1; \frac{z}{x}\right)\right) \\ &= [x^0 z^{n+1}]\left(\left(\frac{1}{(1-x)} - e^x\right) \times \mathrm{Conv}_{n+1}\left(1,1;\frac{z}{x}\right)\right).\end{aligned}$$

The convergent–based generating function identities enumerating the sequences stated next in Example 4.7 and Remark 4.8 below provide additional examples of the termwise formal Laplace–Borel–like transform provided by coefficient extractions involving these rational convergent functions outlined by Remark 4.5.

Example 4.7 (Enumerating Sums Involving Double Factorials)**.** Since we know that $(2k-1)!! = [z^k]\,\mathrm{Conv}_n\left(2,1;z\right)$ for all $0 \leq k < n$, the terms of the next modified product sequences are generated through the following related forms obtained from the formal series expansions of the convergent generating functions:

$$\frac{(k+1)}{k!}\cdot (2k-1)!! = [x^0][z^k]\left(\mathrm{Conv}_k\left(-1, 2k-1; \frac{z}{x}\right)\cdot (x+1)e^x\right)$$

†F.16 *Note §4.9.* The sums over factorial function power sequences defined by the functions $\mathrm{sf}_{3,2}(n)$, $\mathrm{sf}_{3,3}(n)$, and $\mathrm{sf}_{4,2}(n)$ do not appear to have corresponding entries in the current OEIS database.

$$= [x^0][z^k] \left(\mathrm{Conv}_k\left(2, 1; \frac{z}{x}\right) \cdot (x+1)e^x\right).$$

The convergent–based expansions of the next "round number" identity generating the double factorial function given cited in the reference are then easily obtained from the previous equations in the following forms [4, §4.3]:

$$\begin{aligned}
(2n-1)!! &= \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!} \cdot k \cdot (2k-3)!! \\
&= (n-1)! \times [x_2^n][x_1^0] \left(\frac{x_2}{(1-x_2)} \times \mathrm{Conv}_n\left(2, 1; \frac{x_2}{x_1}\right) \times (x_1+1)e^{x_1}\right) \\
&= [x_1^0 x_2^0 x_3^{n-1}] \left(\mathrm{Conv}_n\left(1, 1; \frac{x_3}{x_2}\right) \mathrm{Conv}_n\left(2, 1; \frac{x_2}{x_1}\right) \times \frac{(x_1+1)}{(1-x_2)} \cdot e^{x_1}\right).
\end{aligned}$$

Related challenges are posed in the statements of several other finite sum identities involving the double factorial function cited in the references [4; 12]. Ⓔ

Remark 4.8 (Other Identities Generating the Subfactorial Function)**.** The first convergent–based generating function expansions approximating the formal ordinary power series over the subfactorial sequence given in Section 4.2.2 are expanded as [A000166]

$$!n = n! \times \sum_{i=0}^{n} \frac{(-1)^i}{i!} = [x_1^0 z^n] \left(\frac{e^{-x_1}}{(1-x_1)} \times \mathrm{Conv}_n\left(1, 1; \frac{z}{x_1}\right)\right).$$

These constructions of convergent–based formal power series for the ordinary generating functions of the subfactorial function, $!n$, outlined in the previous section are extended to enumerate a few other special case identities for this sequence related to the sums from Example 4.7.

For example, the next pair of alternate, factorial–function–like auxiliary recurrence relations defining the subfactorial function when $n \geq 2$ lead to additional identities involving the first convergent–based generating function expansions that approximate the subfactorial sequence terms at each n [13, §5.3] [6, §4.2]. In particular, the following identities exactly enumerating the subfactorial function for all $n \geq 0$ correspond to the respective expansions of the previous equation involving the first–order partial derivatives of the convergent functions, $\mathrm{Conv}_n(\alpha, R; t)$, with respect to t [13, *cf.* §7.2] [26, *cf.* §2.2]:

$$\begin{aligned}
!n &= (n-1)\left(!(n-1) + !(n-2)\right) \qquad \text{(Factorial-Related Recurrences)} \\
&= (n-1) \times !(n-1) + (n-2) \times !(n-2) + !(n-2) \\
&= [x_1^0 z^n] \left(\frac{(z^2+z^3) \cdot e^{-x_1}}{x_1 \cdot (1-x_1)} \times \mathrm{Conv}_n^{(1)}\left(1, 1; \frac{z}{x_1}\right) + \frac{z^2 \cdot e^{-x_1}}{(1-x_1)} \times \mathrm{Conv}_n\left(1, 1; \frac{z}{x_1}\right) + 1\right) \\
!n &= n \times !(n-1) + (-1)^n \\
&= (n-1) \times !(n-1) + !(n-1) + (-1)^n \\
&= [x_1^0 z^n] \left(\frac{z^2 \cdot e^{-x_1}}{x_1 \cdot (1-x_1)} \times \mathrm{Conv}_n^{(1)}\left(1, 1; \frac{z}{x_1}\right) + \frac{z \cdot e^{-x_1}}{(1-x_1)} \times \mathrm{Conv}_n\left(1, 1; \frac{z}{x_1}\right) + \frac{1}{(1+z)}\right).
\end{aligned}$$

The next sums provide another summation–based recursive formula for the subfactorial function derived from the known exponential generating function, $\widehat{D}_{n_i}(z) = e^{-z} \cdot (1-z)^{-1}$, for this sequence [13, §5.4] [6, §4.2].

$$!n = n! - \sum_{i=1}^{n} \binom{n}{i} !(n-i) \qquad \text{(Basic Subfactorial Recurrence)}$$

$$
\begin{aligned}
&= n! \times \left(1 - \sum_{i=1}^{n} \frac{1}{i!} \cdot \frac{!(n-i)}{(n-i)!}\right) \\
&= n! \times \left(1 - [x_1^0 x_2^0 x_3^n] \left((e^{x_3} - 1) \operatorname{Conv}_n\left(1, 1; \frac{x_3}{x_2 x_1}\right) \frac{e^{x_2 - x_1}}{(1 - x_1)}\right)\right) \\
&= [x_x^0 x_2^0 x_3^0 z^n] \left(\operatorname{Conv}_n\left(1, 1; \frac{z}{x_3}\right) \left(\frac{1}{(1 - x_3)} - \operatorname{Conv}_n\left(1, 1; \frac{x_3}{x_2 x_1}\right) \frac{e^{x_2 - x_1} \cdot (e^{x_3} - 1)}{(1 - x_1)}\right)\right).
\end{aligned}
$$

The rational convergent–based expansions that generate the last equation immediately above then correspond to the effect of performing a termwise Laplace–Borel transformation approximating the complete integral transform, $\mathcal{L}(\widehat{D}_{n_\mathrm{i}}(t); z)$, defined by Remark 4.5, which is related to the regularized sums involving the incomplete gamma function given in the examples from Section 1.2 of the introduction. ®

4.2.6 Example: Generating sums of powers of natural numbers and the binomial coefficients

As a starting point for the next generating function identities that provide expansions of the sums of powers sequences defined by (4.14) in this section below, let $p \geq 2$ be fixed, and suppose that $m \in \mathbb{Z}^+$. The convergent–based generating function series over the integer powers, m^p, are generated through an appeal to the binomial theorem to form the next sums:

$$
m^p - 1 = (p-1)! \cdot \left(p \times \sum_{k=0}^{p-1} \frac{(m-1)^{p-k}}{k!(p-k)!}\right) \tag{4.12.a}
$$

$$
m^p - 1 = [z^{p-1}][x^0] \left(\operatorname{Conv}_p\left(-1, p-1; \frac{z}{x}\right) \times (me^{mx} - e^x)\right). \tag{4.12.b}
$$

Next, consider the generating function expansions enumerating the finite sums of the p^{th} power sequences in (4.12.a) summed over $0 \leq m \leq n$ as follows [13, *cf.* §7.6]:

$$
\begin{aligned}
\widetilde{B}_u(w, x) &:= \sum_{n \geq 0} \left(\sum_{m=0}^{n} (me^{mx} - e^x) u^m\right) w^n \qquad (4.13) \\
&= \sum_{n \geq 0} \left(\left(\frac{1}{1-u} + \frac{ne^{nx}}{ue^x - 1} - \frac{e^{nx}}{(ue^x - 1)^2}\right) e^x u^{n+1} + \left(\frac{u}{(ue^x - 1)^2} - \frac{1}{1-u}\right) e^x\right) w^n \\
&= -\frac{u^2 w^2 e^{3x} - 2uwe^{2x} + (u^2 w^2 - uw + 1)e^x}{(1-w)(1-uw)(uwe^x - 1)^2} \\
&= \frac{e^x}{(1-w)(1-uw)} - \frac{1}{(1-w)(e^x uw - 1)^2} + \frac{1}{(1-w)(e^x uw - 1)} \\
\widetilde{B}_{a,b,u}(w, x) &:= \sum_{n \geq 0} \left(\sum_{m=0}^{n} ((am+b)e^{(am+b)x} - e^x) u^m\right) w^n \\
&= \frac{e^x - be^{bx} + \left((b-a)e^{(a+b)x} - 2e^{(a+1)x} + be^{bx}\right) uw + \left((a-b)e^{(a+b)x} + e^{(2a+1)x}\right) u^2 w^2}{(1-w)(1-uw)(uwe^{ax} - 1)^2} \\
&= \frac{be^{bx} + (a-b)e^{(a+b)x} uw}{(1-w)(uwe^{ax} - 1)^2} - \frac{e^x - 2e^{(a+1)x} uw + e^{(2a+1)x} u^2 w^2}{(1-w)(1-uw)(uwe^{ax} - 1)^2}.
\end{aligned}
$$

We then obtain the next cases of the following convergent–based generating function identities exactly enumerating the corresponding forms of the sums of powers sequences obtained from (4.12.a) [13, §6.5, §7.6]:

$$\begin{aligned} S_p(n) &:= \sum_{m=0}^{n-1} m^p && \text{(Sums of Powers Sequences)} \\ &= n + [w^{n-1}][z^{p-1}x^0]\left(\mathrm{Conv}_p\left(-1, p-1; \frac{z}{x}\right) \widetilde{B}_1(w, x)\right) && \text{(4.14.a)} \\ &= n + [w^{n-1}][x^0 z^{p-1}]\left(\mathrm{Conv}_p\left(1, 1; \frac{z}{x}\right) \widetilde{B}_1(w, x)\right). \end{aligned}$$

A somewhat related set of results for variations of more general cases of the power sums expanded above is expanded similarly for $p \geq 1$, fixed scalars $a, b \neq 0$, and any non–zero indeterminate u according to the next convergent function identities given by

$$\begin{aligned} S_p(u, n) &:= \sum_{m=0}^{n-1} m^p u^m && \text{(Generalized Sums of Powers Sequences)} \\ &= \frac{u^n - 1}{u - 1} + [w^{n-1}][z^{p-1}x^0]\left(\mathrm{Conv}_p\left(-1, p-1; \frac{z}{x}\right) \widetilde{B}_u(w, x)\right) \\ &= \frac{u^n - 1}{u - 1} + [w^{n-1}][x^0 z^{p-1}]\left(\mathrm{Conv}_p\left(1, 1; \frac{z}{x}\right) \widetilde{B}_u(w, x)\right) && \text{(4.14.b)} \\ S_p(a, b; u, n) &:= \sum_{m=0}^{n-1} (am+b)^p u^m \\ &= \frac{u^n - 1}{u - 1} + [w^{n-1}][z^{p-1}x^0]\left(\mathrm{Conv}_p\left(-1, p-1; \frac{z}{x}\right) \widetilde{B}_{a,b,u}(w, x)\right) \\ &= \frac{u^n - 1}{u - 1} + [w^{n-1}][x^0 z^{p-1}]\left(\mathrm{Conv}_p\left(1, 1; \frac{z}{x}\right) \widetilde{B}_{a,b,u}(w, x)\right). \end{aligned}$$

The bivariate, two–variable generating functions, $\widetilde{B}_u(w, x)$ and $\widetilde{B}_{a,b,u}(w, x)$, involved in enumerating the respective sequences in each of (4.14.a) and (4.14.b) are related to the generating functions for the *Bernoulli* and *Euler polynomials*, $B_n(x)$ and $E_n(x)$, defined in the references [19, §24.2] [21, §4.2.2, §4.2.3]. For $u := \pm 1$, the sums defined by the left–hand–sides of the previous two equations also correspond to special cases of the following known identities involving these polynomial sequences [19, §24.4(iii)]:

$$\begin{aligned} \sum_{m=0}^{n} (am+b)^p &= \frac{a^p}{p+1}\left(B_{p+1}\left(n+1+\frac{b}{a}\right) - B_{p+1}\left(\frac{b}{a}\right)\right) && \text{(Sums of Powers Formulas)} \\ \sum_{m=0}^{n} (-1)^m (am+b)^p &= \frac{a^p}{2}\left((-1)^n \cdot E_p\left(n+1+\frac{b}{a}\right) + E_p\left(\frac{b}{a}\right)\right). \end{aligned}$$

The results in the previous equations are also compared to the forms of other well–known sequence generating functions involving the *Bernoulli numbers*, B_n, the *first–order Eulerian numbers*, $\left\langle {n \atop m} \right\rangle$, and the *Stirling numbers of the second kind*, $\left\{ n \atop k \right\}$, in the next few cases of the established identities for these sequences expanded in Remark 4.9 [13, *cf.* §6] [A027641; A027642; A008292; A008277].

Remark 4.9 (Comparisons to Other Known Sequence Generating Functions)**.** The sequences enumerated by (4.14.a) are first compared to the following known expansions that exactly generate these finite sums over $n \geq 0$ and $p \geq 1$ [19, §24.4(iii), §24.2] [13, §6, §7.4] ‡F.17 :

$$\begin{aligned}\sum_{m=0}^{n} m^p &= \frac{B_{p+1}(n+1) - B_{p+1}(0)}{p+1} \\ &= \sum_{s=0}^{p} \binom{p+1}{s} \frac{B_s \cdot (n+1)^{p+1-s}}{(p+1)} \\ \sum_{m=0}^{n} m^p &= [z^n] \left(\sum_{j=0}^{p} \left\{ p \atop j \right\} \frac{z^j \cdot j!}{(1-z)^{j+2}} \right) \\ &= [z^n] \left(\sum_{i \geq 0} \left\langle p \atop i \right\rangle \frac{z^{i+1}}{(1-z)^{p+2}} \right) \\ &= p! \cdot [w^n z^p] \left(\frac{w \cdot e^z}{(1-w)(1-we^z)} \right).\end{aligned}$$

Similarly, the generalized forms of the sums generated by (4.14.b) are related to the more well–known combinatorial sequence identities expanded as follows [19, §26.8] [13, §7.4]:

$$\begin{aligned}\sum_{m=0}^{n} m^p u^m &= \sum_{j=0}^{p} \left\{ p \atop j \right\} u^j \times \frac{\partial^{(j)}}{\partial u^{(j)}} \left(\frac{1}{1-u} - \frac{u^{n+1}}{1-u} \right) \\ &= [w^n] \left(\sum_{j=0}^{p} \left\{ p \atop j \right\} \frac{(uw)^j \cdot j!}{(1-w)(1-uw)^{j+1}} \right) \\ &= p! \cdot [w^n z^p] \left(\frac{uw \cdot e^z}{(1-w)(1-uwe^z)} \right).\end{aligned}$$

As in the examples of termwise applications of the formal Laplace–Borel transforms noted above, the role of the parameter p corresponding to the forms of the special sequence triangles in the identities given above is phrased through the implicit dependence of the convergent functions on the fixed $p \geq 1$ in each of (4.12.b), (4.14.a), and (4.14.b). Ⓡ

A second motivating example highlighting the procedure outlined in the examples above expands the binomial power sequences, $2^p - 1$ for $p \geq 1$, through an extension of the first result given in (4.12.a) when $m := 2$. The finite sums for the integer powers provided by the binomial theorem in

‡F.17 *Note §4.10.* Two bivariate "super" generating function for the *first–order Eulerian numbers*, $\left\langle n \atop m \right\rangle$, are given by the following equations where $\left\langle n \atop m \right\rangle = \left\langle n \atop n-1-m \right\rangle$ for all $n \geq 1$ and $0 \leq m < n$ by the row–wise symmetry in the triangle [13, §7.4, §6.2] [19, §26.14(ii)]:

$$\sum_{m,n \geq 0} \left\langle n \atop m \right\rangle \frac{w^m z^n}{n!} = \frac{1-w}{e^{(w-1)z} - w} \qquad \text{(First-Order Eulerian Number EGFs)}$$

$$\sum_{m,n \geq 0} \left\langle m+n+1 \atop m \right\rangle \frac{w^m z^n}{(m+n+1)!} = \frac{e^w - e^z}{we^z - ze^w}.$$

these cases correspond to removing, or selectively peeling off, the r uppermost–indexed terms from the first sum for subsequent choices of the $p \geq r \geq 1$ in the following forms [20, *cf.* §2.2, §2.4]:

$$\begin{aligned} m^p - 1 &= \sum_{i=0}^{r} \binom{p}{p+1-i} (m-1)^i \\ & + (p-r-1)! \cdot \left(p(p-1)\cdots(p-r) \times \sum_{k=0}^{p-r-1} \frac{(m-1)^{k+1}}{(k+1)!(p-1-k)!}\right). \end{aligned}$$

The generating function identities phrased in terms of (4.12.a) in the previous examples are then modified slightly according to this equation for the next few special cases of $r \geq 1$.

For example, these sums are employed to obtain the next analogous forms of the convergent–based generating function expansions generalizing the result in (4.12.b) above [`A000225`; `A000918`].

$$\begin{aligned} 2^p - 1 &= [z^{p-2}][x^0]\left(\frac{1}{(1-z)} + \left(4e^{2x} - 2e^x\right) \times \operatorname{Conv}_p\left(-1, p-2; \frac{z}{x}\right)\right), \ p \geq 2 \\ &= [z^{p-3}][x^0]\left(\frac{4-3z}{(1-z)^2} + \left(8e^{2x} - e^x \cdot (x+5)\right) \times \operatorname{Conv}_p\left(-1, p-3; \frac{z}{x}\right)\right), \ p \geq 3 \\ &= [z^{p-4}][x^0]\left(\frac{11-17z+7z^2}{(1-z)^3}\right. \\ & \left. + \left(16e^{2x} - \frac{e^x}{2} \cdot (x^2+10x+24)\right) \times \operatorname{Conv}_p\left(-1, p-4; \frac{z}{x}\right)\right), \ p \geq 4 \end{aligned}$$

The special cases of these generating functions for the p^{th} powers defined above are also expanded in the next more general forms of these convergent function identities for $p > m \geq 1$.

$$\begin{aligned} 2^p - 1 &= [z^{p-m-1}x^0]\left(\frac{\widetilde{\ell}_{m,2}(z)}{(1-z)^m} + \left(2^{m+1} \cdot e^{2x} - \frac{e^x}{(m-1)!} \cdot \widetilde{p}_{m,2}(x)\right) \times \operatorname{Conv}_p\left(-1, p-m-1; \frac{z}{x}\right)\right) \\ &= [x^0 z^{p-m-1}]\left(\frac{\widetilde{\ell}_{m,2}(z)}{(1-z)^m} + \left(2^{m+1} \cdot e^{2x} - \frac{e^x}{(m-1)!} \cdot \widetilde{p}_{m,2}(x)\right) \times \operatorname{Conv}_p\left(1, 1; \frac{z}{x}\right)\right), \ p > m \end{aligned}$$

The listings provided in Table A.6 (page 85) cite the particular special cases of the polynomials, $\ell_{m,2}(z)$ and $p_{m,2}(x)$, that provide the generalizations of the first cases expanded in the previous equations. The constructions of these new identities, including the variations for the sequences formed by the binomial coefficient sums for the powers, $2^p - 2$, are motivated in the context of divisibility modulo p by the reference [14, §VIII].

Further cases of the more general p^{th} power sequences of the form $(s+1)^p - 1$ for any fixed $s > 0$ are enumerated similarly through the next formulas.

$$\begin{aligned} (s+1)^p - 1 = [z^{p-m-1}x^0]&\left(\frac{s^2 \ell_{m,s+1}(z)}{(1-sz)^m} + \left(-e^x + (s+1)^{m+1} \cdot e^{(s+1)x} - \frac{s^2 e^{sx}}{(m-1)!} \cdot p_{m,s+1}(sx)\right) \times \right. \\ &\left. \times \operatorname{Conv}_p\left(-1, p-m-1; \frac{z}{x}\right)\right), \ p > m \geq 1 \end{aligned}$$

$$= [x^0 z^{p-m-1}]\left(\frac{s^2 \ell_{m,s+1}(z)}{(1-sz)^m} + \left(-e^x + (s+1)^{m+1} \cdot e^{(s+1)x} - \frac{s^2 e^{sx}}{(m-1)!} \cdot p_{m,s+1}(sx)\right) \times \right.$$
$$\left. \times \operatorname{Conv}_p\left(1, 1; \frac{z}{x}\right)\right), \ p > m \geq 1$$

The second set of listings provided in Table A.6 (page 85) expand several additional special cases corresponding to the polynomial sequences, $\ell_{m,s+1}(z)$ and $p_{m,s+1}(x)$, required to generate the more general cases of these particular p^{th} power sequences when $p > m \geq 1$ [A000225; A024023; A024036; A024049]. Related expansions of the sequences of binomials of the forms $a^n \pm 1$ and $a^n \pm b^n$ are considered in the references [20, *cf.* §2.2, §2.4].

5 Properties of the generalized convergent functions

The respective convergent function component sequences, $\mathrm{FP}_h(\alpha, R; z)$ and $\mathrm{FQ}_h(\alpha, R; z)$, satisfy essentially the same second–order difference equation over h, with the exception of the prescribed initial conditions defining each of the respective function cases in (1.11) and (1.12). It happens that the denominator polynomial sequences, $\mathrm{FQ}_h(\alpha, R; z)$, are easily related to particular special cases of the confluent hypergeometric function, $U(-h, b, z)$, and the associated Laguerre polynomials, $L_h^{(\beta)}(z)$ [19, *cf.* §18.6(iv); §13.9(ii)] [3; 11]. The numerator convergent sequences have less obvious expansions through special functions, or otherwise more well–known polynomial sequences [§F.18].

The identification of the convergent denominator functions as special cases of the confluent hypergeometric function yields additional identities providing analogous addition and multiplication theorems for these functions with respect to the parameter z, as well as a number of further, new recurrence relations derived from established relations stated in the references, such as those provided by *Kummer's transformations*. These properties form a superset of extended results beyond the immediate, more combinatorial, known relations for the J–fractions summarized in Section 3.1 and in Section 3.3 [9; 10; 17; 19]. The latter characterization of the generalized convergent functions by the Laguerre polynomials provides the factorizations over the zeros of the classical orthogonal polynomial sequences studied in the references [3; 11], required to state the results provided by (1.15.a) and (1.15.b) from Section 1.3, and more generally by (4.4) in Section 4.1.

Since the reciprocal zeros of the denominator functions, $\mathrm{FQ}_h(\alpha, R; z)$, determine the series expansions for the convergent functions up to parametrized constant multiples, we first focus on the comparatively simple factored expressions for the series coefficients of the denominator sequences in the results proved in Section 5.1. The expansions of the convergent numerator functions stated in Section 5.2, and of the auxiliary subsequences defined by Section 5.2.1, provide further identities for the particular congruences satisfied by many of the prime–related identities and notable prime number subsequences cited as the applications of the new results expanded in Section 6.

5.1 The convergent denominator functions

In contrast to the convergent numerator functions, $\mathrm{FP}_h(\alpha, R; z)$, discussed next in Section 5.2, the corresponding denominator functions, $\mathrm{FQ}_h(\alpha, R; z)$, are readily expressed through well–known special functions. The first several special cases given in Table A.3 (page 79) suggest the next

[§F.18] *Note §5.1.* A point concerning the relative simplicity for the expressions of the denominator convergent polynomials compared to the numerator convergent sequences is also mentioned in §3.1 of Flajolet's article [9].

identity, which is proved following Proposition 5.1 below.

$$\mathrm{FQ}_h(\alpha, R; z) = \sum_{k=0}^{h} \binom{h}{k} (-1)^k \left(\prod_{j=0}^{k-1} (R + (h-1-j)\alpha)\right) z^k \tag{5.1}$$

The convergent denominator functions are expanded by the *confluent hypergeometric function*, $U(-h, b, w)$, or equivalently by the *associated Laguerre polynomials*, $L_h^{(b-1)}(w)$, when $b :\mapsto R/\alpha$ and $w :\mapsto (\alpha z)^{-1}$ through the relations proved by the next proposition [19; 21].

Proposition 5.1 (*Exact Representations by Special Functions*).
The convergent denominator functions, $\mathrm{FQ}_h(\alpha, R; z)$, are expanded in terms of the confluent hypergeometric function and the associated Laguerre polynomials through the following results:

$$\begin{aligned} \mathrm{FQ}_h(\alpha, R; z) &= (\alpha z)^h \times U\left(-h, R/\alpha, (\alpha z)^{-1}\right) && (5.2) \\ &= (-\alpha z)^h \cdot h! \times L_h^{(R/\alpha - 1)}\left((\alpha z)^{-1}\right). && (5.3) \end{aligned}$$

Proof. We proceed to prove the first identity in (5.2) by induction. It easy to verify by computation (see Table A.3) that the left–hand–side and right–hand–sides of (5.2) coincide when $h = 0$ and $h = 1$. For $h \geq 2$, we apply the recurrence relation from (1.11) to write the right–hand–side of (5.2) as

$$\begin{aligned} \mathrm{FQ}_h(\alpha, R; z) = (1 - (R + 2\alpha(h-1))z) U\left(-h+1, R/\alpha, (\alpha z)^{-1}\right)(\alpha z)^{h-1} \\ - \alpha(R + \alpha(h-2))(h-1)z^2 U\left(-h+2, R/\alpha, (\alpha z)^{-1}\right)(\alpha z)^{h-2}. \end{aligned} \tag{5.4}$$

The proof is completed using the known recurrence relation for the confluent hypergeometric function stated in reference as [19, §13.3(i)]

$$U(-h, b, u) = (u - b - 2(h-1)) U(-h+1, b, u) - (h-1)(b+h-2) U(-h+2, b, u). \tag{5.5}$$

In particular, we can rewrite (5.4) as

$$\begin{aligned} \mathrm{FQ}_h(\alpha, R; z) = (\alpha z)^h \Bigg[\left((\alpha z)^{-1} - \left(\frac{R}{\alpha} + 2(h-1)\right)\right) U\left(-h+1, R/\alpha, (\alpha z)^{-1}\right) \\ - \left(\frac{R}{\alpha} + h - 2\right)(h-1) U\left(-h+2, R/\alpha, (\alpha z)^{-1}\right) \Bigg], \end{aligned} \tag{5.6}$$

which implies (5.2) in the special case of (5.5) where $(b, u) := (R/\alpha, (\alpha z)^{-1})$. The second characterization of $\mathrm{FQ}_h(\alpha, R; z)$ by the Laguerre polynomials stated in (5.3) follows from the first result whenever $h \geq 0$ [19, §18.11]. ⊡

Proof of Equation (5.1). The first identity for the denominator functions, $\mathrm{FQ}_h(\alpha, R; z)$, conjectured from the special case table listings by (5.1) follows from the first statement of the previous proposition. We cite the particular expansions of $U(-n, b, z)$ when $n \geq 0$ is integer–valued involving the Pochhammer symbol, $(x)_n$, stated as follows [19, §13.2(i)]:

$$\begin{aligned} U(-n, b, z) &= \sum_{k=0}^{n} \binom{n}{k} (b+k)_{n-k} (-1)^n (-z)^k \\ &= \sum_{k=0}^{n} \binom{n}{k} (b+n-k)_k (-1)^k z^{n-k} && (5.7) \end{aligned}$$

The second sum for the confluent hypergeometric function given in (5.7) then implies that the right–hand–side of (5.2) can be expanded as follows:

$$\begin{aligned}
\mathrm{FQ}_h(\alpha, R; z) &= (\alpha z)^h\, U\left(-h, R/\alpha, (\alpha z)^{-1}\right) \\
&= (\alpha z)^h \sum_{k=0}^{h} \binom{h}{k} (-1)^k \left(\frac{R}{\alpha} + h - k\right)_k (\alpha z)^{k-h} \\
&= \sum_{k=0}^{h} \binom{h}{k} \left(\frac{R}{\alpha} + h - k\right)_k (-\alpha z)^k \\
&= \sum_{k=0}^{h} \binom{h}{k} \underbrace{\left((-1)^k \times \prod_{j=0}^{k-1} (R + (h-1-j)\alpha)\right)}_{(\pm 1)^k p_k\left(\mp\alpha, \pm R \pm (h-1)\alpha\right)} z^k.
\end{aligned}$$

The last line of previous equations provides the required expansion completing a proof of the first identity cited in (5.1). ⊡

Corollaries. The coefficients of z from (5.1) also yield the next identities involving the product sequences from (1.1) that are employed in formulating several of the new results given in Section 6. In particular, we obtain the alternate restatements of these coefficients provided by the following equations:

$$\begin{aligned}
[z^k]\, \mathrm{FQ}_p(\alpha, R; z) &= \binom{p}{k} (-1)^k p_k(-\alpha, R + (p-1)\alpha) \cdot [0 \leq k \leq p]_\delta \qquad (5.8) \\
&= \binom{p}{k} \alpha^k \left(1 - p - R/\alpha\right)_k \cdot [0 \leq k \leq p]_\delta \\
[z^k]\, \mathrm{FQ}_p(\alpha, R; z) &= \binom{p}{k} p_k(\alpha, -R - (p-1)\alpha) \cdot [0 \leq k \leq p]_\delta \\
&= \binom{p}{k} (-\alpha)^k \left(R/\alpha + p - 1\right)^{\underline{k}} \cdot [0 \leq k \leq p]_\delta\,.
\end{aligned}$$

⊡

Corollary 5.2 (*Auxiliary Recurrence Relations*).
For $h \geq 0$ and any integers $s > -h$, the convergent denominator functions, $\mathrm{FQ}_h(\alpha, R; z)$, satisfy the reflection identity given by

$$\mathrm{FQ}_h(\alpha, \alpha s; z) = \mathrm{FQ}_{h+s-1}(\alpha; \alpha(2-s); z). \qquad (5.9)$$

Additionally, for $h \geq 0$ these functions satisfy recurrence relations of the following forms:

$$\begin{aligned}
(R + (h-1)\alpha) z\, \mathrm{FP}_h(\alpha, R - \alpha; z) + ((\alpha - R)z - 1)\, \mathrm{FQ}_h(\alpha, R; z) + \mathrm{FQ}_h(\alpha, R + \alpha; z) &= 0 \qquad (5.10) \\
\mathrm{FQ}_h(\alpha, R; z) + \alpha h z\, \mathrm{FQ}_{h-1}(\alpha, R; z) - \mathrm{FQ}_h(\alpha, R - \alpha; z) &= 0 \\
(R + \alpha h) z\, \mathrm{FQ}_h(\alpha, R; z) + \mathrm{FQ}_{h+1}(\alpha, R; z) - \mathrm{FQ}_h(\alpha, R + \alpha; z) &= 0 \\
(1 - \alpha h z)\, \mathrm{FQ}_h(\alpha, R; z) - \mathrm{FQ}_h(\alpha, R + \alpha; z) - \alpha h (R + (h-1)\alpha) z^2\, \mathrm{FQ}_{h-1}(\alpha, R; z) &= 0 \\
(1 - (h+1)\alpha z)\, \mathrm{FQ}_h(\alpha, R; z) - \mathrm{FQ}_{h+1}(\alpha, R; z) - (R + (h-1)\alpha) z\, \mathrm{FQ}_h(\alpha, R - \alpha; z) &= 0 \\
\alpha (h-1) z\, \mathrm{FQ}_{h-2}(\alpha, R + 2\alpha; z) - (1 - Rz)\, \mathrm{FQ}_{h-1}(\alpha, R + \alpha; z) + \mathrm{FQ}_h(\alpha, R; z) &= [h = 0]_\delta
\end{aligned}$$

Proof. The first equation results from *Kummer's transformation* for the confluent hypergeometric function, $U(a,b,z)$, given by [19, §13.2(vii)]

$$U(a,b,z) = z^{1-b}U(a-b+1, 2-b, z).$$

In particular, when $R := \alpha s$ and $h+s-1 \geq 0$ Proposition 5.1 implies that

$$\begin{aligned}\mathrm{FQ}_h(\alpha, R; z) &= (\alpha z)^{h+R/\alpha-1}\, U\left(-(h+\frac{R}{\alpha}-1), \frac{2\alpha-R}{\alpha}, (\alpha z)^{-1}\right) \\ &= \mathrm{FQ}_{h+R/\alpha-1}(\alpha, 2\alpha-R; z).\end{aligned}$$

The recurrence relations stated in (5.10) follow similarly consequences of the first proposition by applying the known results for the confluent hypergeometric functions cited in the reference [19, §13.3(i)]. ⊡

Proposition 5.3 (*Addition and Multiplication Theorems*).
Let $z, w \in \mathbb{C}$ with $z \neq w$ and suppose that $z \neq 0$. For a fixed $\alpha \in \mathbb{Z}^+$ and $h \geq 0$, the following finite sums provide two addition theorem analogs satisfied by the sequences of convergent denominator functions:

$$\begin{aligned}\mathrm{FQ}_h(\alpha, R; z-w) &= \sum_{n=0}^{h} \frac{(-h)_n(-w)^n(z-w)^{h-n}}{z^h \cdot n!}\, \mathrm{FQ}_{h-n}(\alpha, R+\alpha n; z) \\ \mathrm{FQ}_h(\alpha, R; z-w) &= \sum_{n=0}^{h} \frac{(-h)_n \left(1-h-\frac{R}{\alpha}\right)_n (\alpha w)^n}{n!}\, \mathrm{FQ}_{h-n}(\alpha, R; z).\end{aligned} \tag{5.11}$$

The corresponding multiplication theorems for the denominator functions are stated similarly for $h \geq 0$ in the form of the following equations:

$$\begin{aligned}\mathrm{FQ}_h(\alpha, R; zw) &= \sum_{n=0}^{h} \frac{(-h)_n(w-1)^n w^{h-n}}{n!}\, \mathrm{FQ}_{h-n}(\alpha, R+\alpha n; z) \\ \mathrm{FQ}_h(\alpha, R; zw) &= \sum_{n=0}^{h} \frac{(-h)_n \left(1-h-\frac{R}{\alpha}\right)_n (1-w)^n(\alpha z)^n}{n!}\, \mathrm{FQ}_{h-n}(\alpha, R; z).\end{aligned} \tag{5.12}$$

Proof of the Addition Theorems. The sums stated in (5.11) follow from special cases of established addition theorems for the confluent hypergeometric function, $U(a,b,x+y)$, cited in [19, §13.13(ii)]. The particular addition theorems required in the proof are provided as follows:

$$\begin{aligned}U(a,b,x+y) &= \sum_{n=0}^{\infty} \frac{(a)_n(-y)^n}{n!} U(a+n, b+n, x), \ |y| < |x| \\ U(a,b,x+y) &= \left(\frac{x}{x+y}\right)^a \sum_{n=0}^{\infty} \frac{(a)_n(1+a-b)_n y^n}{n!(x+y)^n} U(a+n, b, x), \ \Re[y/x] > -\frac{1}{2}.\end{aligned} \tag{5.13}$$

First, observe that in the special case inputs to $U(a,b,z)$ resulting from the application of Proposition 5.1 involving the functions

$$\mathrm{FQ}_h(\alpha, R; z) = U\left(-h, R/\alpha, (\alpha z)^{-1}\right),$$

in the infinite sums of (5.13) lead to to finite sum identities corresponding to the inputs, h, to $\mathrm{FQ}_h(\alpha, R; z)$ where $h \geq 0$. More precisely, the definition of the convergent denominator sequences provided by (1.12) requires that $\mathrm{FP}_h(\alpha, R; z) = 0$ whenever $h < 0$.

To apply the cited results for $U(a, b, x+y)$ in these cases, let $z \neq w$, assume that both $\alpha, z \neq 0$, and suppose the parameters corresponding to x and y in (5.11) are defined so that

$$x := (\alpha z)^{-1},\ y := \frac{1}{\alpha}\left((z-w)^{-1} - z^{-1}\right),\ x + y = (\alpha(z-w))^{-1} \tag{5.14}$$

Since each of the sums in (5.11) involve only finitely–many terms, we ignore treatment of the convergence conditions given on the right–hand–sides of the equations in (5.13) to justify these two restatements of the addition theorem analogs provided above. ⊡

Proof of the Multiplication Theorems. The second pair of identities stated in (5.12) are formed by the multiplication theorems for $U(a, b, z)$ noted as in [19, §13.13(iii)]. The proof is derived similarly from the first parameter definitions of x and y given in the addition theorem proof, with an additional adjustment employed in these cases corresponding to the change of variable $\widehat{y} \mapsto (y-1)x$, which is selected so that $x + \widehat{y} :\mapsto xy$ in the above proof. The analog to (5.14) that results in these two cases then yields the parameters, $x := (\alpha z)^{-1}$ and $y := (w^{-1} - 1) \cdot (\alpha z)^{-1}$, in the first identities for the confluent hypergeometric function, $U(a, b, x+y)$, given by (5.13). ⊡

5.2 The convergent numerator functions

The most direct expansion of the convergent numerator functions, $\mathrm{FP}_h(\alpha, R; z)$, is obtained from the *erasing operator*, defined as in Flajolet's first article, which performs the formal power series truncation operation defined by the next equation [9, §3].

$$\mathrm{E}_m\left[\!\!\left[\sum_i g_i z^i\right]\!\!\right] := \sum_i g_i z^i \cdot [i \leq m]_\delta \tag{Erasing Operator}$$

The numerator polynomials are then given through this notation by the expansions in the following equations:

$$\begin{aligned}\mathrm{FP}_h(\alpha, R; z) &= \mathrm{E}_{h-1}\left[\!\!\left[\mathrm{FQ}_h(\alpha, R; z) \cdot \mathrm{Conv}_h(\alpha, R; z)\right]\!\!\right] \\ &= \sum_{k=0}^{h-1} \underbrace{\left(\sum_{i=0}^{k} [z^i]\, \mathrm{FQ}_h(\alpha, R; z) \times p_{k-i}(\alpha, R)\right)}_{C_{h,k}(\alpha, R) := [z^k]\, \mathrm{FP}_h(\alpha, R; z)} \times z^k.\end{aligned}$$

The coefficients of z^k expanded in the last equation are rewritten slightly in terms of (5.8) and the Pochhammer symbol representations of the product sequences, $p_n(\alpha, R)$, to arrive at the pair of next formulas expanded as follows ¶F.19 :

$$C_{h,k}(\alpha, R) = \sum_{i=0}^{k} \binom{h}{i} (-1)^i p_i(-\alpha, R + (h-1)\alpha)\, p_{k-i}(\alpha, R), \qquad h > k \geq 0 \tag{5.15.a}$$

¶F.19 *Note §5.2.* These coefficients are expanded through the auxiliary subsequences, $C_{h,k}(\alpha, R)$, defined by (5.16) in the next subsection and the corresponding finite, multiple summation identities satisfied by these sequences expanded in (5.17).

$$= \sum_{i=0}^{k} \binom{h}{i} (1-h-R/\alpha)_i \, (R/\alpha)_{k-i} \times \alpha^k, \qquad h > k \geq 0. \tag{5.15.b}$$

These sums are remarkably similar in form to the next binomial–type convolution formula, or *Vandermonde identity*, stated as follows [6; 27] ||F.20 :

$$\begin{aligned}(x+y)_k &= \sum_{i=0}^{k} \binom{k}{i} (x)_i \, (y)_{k-i} && \text{(Vandermonde Convolution)} \\ &= \sum_{i=0}^{k} \binom{k}{i} (-1)^k \, (x+i)_{k-i} \, (-y)_i \,. \end{aligned}$$

A separate treatment of other properties implicit to the more complicated expansions of these convergent function subsequences is briefly explored through the definitions of the three additional forms of auxiliary coefficient sequences, denoted in respective order by $C_{h,n}(\alpha, R) := [z^n] \, \mathrm{FP}_h(\alpha, R; z)$, $R_{h,k}(\alpha; z) := [R^k] \, \mathrm{FP}_h(\alpha, R; z)$, and $T_h^{(\alpha)}(n, k) := [z^n R^k] \, \mathrm{FP}_h(\alpha, R; z)$, considered in the next subsection.

Remark 5.4 (Reflected Convergent Numerator Function Sequences)**.** The special cases of the reflected numerator polynomials given in Table A.4 (page 80) also suggest a consideration of the numerator convergent functions factored with respect to powers of $\pm(z-R)$ by expanding these sequences with respect to another formal auxiliary variable, w, when $R :\mapsto z \mp w$. The tables contained in the attached summary notebook provide working *Mathematica* code to expand and factor these modified forms of the reflected numerator polynomial sequences employed in stating the generalized congruence results for the α–factorial functions, $n!_{(\alpha)}$, from the examples cited in Section 1.4.2, and more generally by the results proved in Section 4.1, satisfied by the α–factorial functions and generalized product sequence expansions modulo integers $p \geq 2$. Ⓡ

5.2.1 Alternate forms of the convergent numerator function subsequences

The next results summarize three semi–triangular recurrence relations satisfied by the particular variations of the numerator function subsequences considered, respectively, as polynomials with respect to z and R. For $h \geq 2$, fixed $\alpha \in \mathbb{Z}^+$, and $n, k \geq 0$, we consider the following forms of these auxiliary numerator coefficient subsequences:

$$\begin{aligned} C_{h,n}(\alpha, R) &:= [z^n] \, \mathrm{FP}_h(\alpha, R; z), \quad \text{for } \ 0 \leq n \leq h-1 \\ &= C_{h-1,n}(\alpha, R) - (R + 2\alpha(h-1)) C_{h-1,n-1}(\alpha, R) \\ &\quad - \alpha(R + \alpha(h-2))(h-1) C_{h-2,n-2}(\alpha, R) \\ R_{h,k}(\alpha; z) &:= [R^k] \, \mathrm{FP}_h(\alpha, R; z), \quad \text{for } \ 0 \leq k \leq h-1 \\ &= (1 - 2\alpha(h-1)z) R_{h-1,k}(\alpha; z) - \alpha^2 (h-1)(h-2) z^2 R_{h-2,k}(\alpha; z) - z R_{h-1,k-1}(\alpha; z) \\ &\quad - \alpha(h-1) z^2 R_{h-2,k-1}(\alpha; z) \end{aligned} \tag{5.16}$$

||F.20 *Note §5.3.* The next product–wise *connection formulas* for the Pochhammer symbol also provides additional expansions of (5.15.b) involving terms of the Pochhammer symbols, $(R/\alpha)_{j_1}$ and $(h)_{j_2}$ [6, Ex. 1.23, p. 83]:

$$(x)_n \, (x)_m = \sum_{k=0}^{\min(m,n)} \binom{m}{k} \binom{n}{k} k! \cdot (x)_{m+n-k} \,, n \neq m.$$

$$\begin{aligned}
T_h^{(\alpha)}(n,k) &:= [z^n R^k] \operatorname{FP}_h(\alpha, R; z), \quad \text{for} \quad 0 \leq n, k \leq h-1 \\
&= T_{h-1}^{(\alpha)}(n,k) - T_{h-1}^{(\alpha)}(n-1,k-1) - 2\alpha(h-1)T_{h-1}^{(\alpha)}(n-1,k) \\
&\quad - \alpha(h-1)T_{h-2}^{(\alpha)}(n-2,k-1) - \alpha^2(h-1)(h-2)T_{h-2}^{(\alpha)}(n-2,k) \\
&\quad + \left([z^n R^0] \operatorname{FP}_h(z)\right) [h \geq 1]_\delta \, [n \geq 0]_\delta \, [k=0]_\delta .
\end{aligned}$$

Each of the recurrence relations for the triangles cited in the previous equations are derived from (1.11) by a straightforward application of the coefficient extraction method first motivated in [22]. Table A.7 (page 87) and Table A.8 (page 88) list the first few special cases of the first two auxiliary forms of these component polynomial subsequences. Special cases of the third component coefficient subsequence are then obtained from the results in these tables.

We also state, without proof, a number of multiple, alternating sums involving the Stirling number triangles that generate these auxiliary subsequences for reference in the next several equations. In particular, for $h \geq 1$ and $0 \leq n < h$, the sequences, $C_{h,n}(\alpha, R)$, are expanded by the following sums:

$$\begin{aligned}
C_{h,n}(\alpha,R) &= \sum_{\substack{0\leq m\leq k\leq n \\ 0 \leq s \leq n}} \left(\binom{h}{k}\binom{m}{s}\begin{bmatrix}k\\m\end{bmatrix}(-1)^m \alpha^n \left(\frac{R}{\alpha}\right)_{n-k}\left(\frac{R}{\alpha}-1\right)^{m-s}\right) \times h^s && \text{(5.17.a)} \\
&= \sum_{\substack{0\leq m\leq k\leq n \\ 0 \leq t \leq s \leq n}} \left(\binom{h}{k}\binom{m}{t}\begin{bmatrix}k\\m\end{bmatrix}\begin{bmatrix}n-k\\s-t\end{bmatrix}(-1)^m \alpha^{n-s}(h-1)^{m-t}\right) \times R^s && \text{(5.17.b)} \\
&= \sum_{\substack{0\leq m\leq k\leq n \\ 0 \leq i \leq s \leq n}} \binom{h}{k}\binom{h}{i}\binom{m}{s}\begin{bmatrix}k\\m\end{bmatrix}\begin{Bmatrix}s\\i\end{Bmatrix}(-1)^m \alpha^n \left(\frac{R}{\alpha}\right)_{n-k}\left(\frac{R}{\alpha}-1\right)^{m-s} \times i! && \text{(5.17.c)} \\
&= \sum_{\substack{0\leq m\leq k\leq n \\ 0 \leq v \leq i \leq s \leq n}} \binom{h}{k}\binom{m}{s}\binom{i}{v}\binom{h+v}{v}\begin{bmatrix}k\\m\end{bmatrix}\begin{Bmatrix}s\\i\end{Bmatrix}(-1)^{m+i-v}\alpha^{n+s-m} \times && \text{(5.17.d)} \\
&\qquad \times \left(\frac{R}{\alpha}\right)_{n-k}\left(\frac{R}{\alpha}-1\right)^{m-s} \times i!.
\end{aligned}$$

Since the powers of R in the second identity are expanded by the Stirling numbers of the second kind as [13, §6.1]

$$R^p = \alpha^p \times \sum_{i=0}^{p} \begin{Bmatrix}p\\i\end{Bmatrix}(-1)^{p-i}\left(\frac{R}{\alpha}\right)_i,$$

for all natural numbers $p \geq 0$, the multiple sum identity in (5.17.b) also implies the next finite multiple sum expansion for these auxiliary coefficient subsequences.

$$C_{h,n}(\alpha,R) = \sum_{i=0}^{n} \underbrace{\left(\sum_{\substack{0\leq m\leq k\leq n \\ 0 \leq t \leq s \leq n}} \binom{h}{k}\binom{m}{t}\begin{bmatrix}k\\m\end{bmatrix}\begin{bmatrix}n-k\\s-t\end{bmatrix}\begin{Bmatrix}s\\i\end{Bmatrix}(-1)^{m+s-i}(h-1)^{m-t}\right)}_{\text{polynomial function of } h \text{ only } := \frac{(-1)^n m_{n,h}}{n!} \times \binom{n}{i} p_{n,i}(h)} \times \alpha^n \left(\frac{R}{\alpha}\right)_i \tag{5.17.e}$$

Similarly, for all $h \geq 1$ and $0 \leq k < h$, the sequences, $R_{h,k}(\alpha, R)$, are expanded as follows:

$$\begin{aligned} R_{h,k}(\alpha; z) &= \sum_{\substack{0 \leq m \leq i \leq n < h \\ 0 \leq t \leq k}} \left(\binom{h}{i}\binom{m}{t}\begin{bmatrix} i \\ m \end{bmatrix}\begin{bmatrix} n-i \\ k-t \end{bmatrix}(-1)^m \alpha^{n-k}(h-1)^{m-t}\right) \times z^n \\ &= \sum_{\substack{0 \leq m \leq i \leq n < h \\ 0 \leq t \leq k \\ 0 \leq p \leq m-t}} \left(\binom{h}{i}\binom{m}{t}\binom{h-1}{p}\begin{bmatrix} i \\ m \end{bmatrix}\begin{bmatrix} n-i \\ k-t \end{bmatrix}\left\{\begin{matrix} m-t \\ p \end{matrix}\right\}(-1)^m \alpha^{n-k} \times p!\right) \times z^n. \end{aligned} \tag{5.18}$$

A more careful immediate treatment of the properties satisfied by these subsequences is omitted from this section for brevity. A number of the new congruence results cited in the next section do, at any rate, have alternate expansions given by the more involved termwise structure implicit to these finite multiple sums modulo some application–specific prescribed functions of h.

6 Applications of new identities resulting from expansions by finite difference equations

6.1 Exact formulas and finite sum representations for the generalized product sequences modulo p

The rationality of the convergent functions, $\operatorname{Conv}_h(\alpha, R; z)$, in z for all h suggests new forms of h–order finite difference equations with respect to n satisfied by the product sequences, $p_n(\alpha, R)$, when α and R correspond to fixed parameters independent of the sequence indices n. In particular, the rationality of the h^{th} convergent functions immediately implies the next results stated in Proposition 6.1 below, which provides both forms of the congruence properties given in (6.1.b) modulo integers $p \geq 2$, and for the exact expansions of the generalized products, $p_n(\alpha, R)$, given below in (6.2) below [17, §2.3] [13, §7.2] [6; 26].

Proposition 6.1 (*Exact Formulas and Finite Sum Representations Modulo h*).
Given any fixed integer $\alpha \neq 0$, any prescribed $h \geq 2$, and natural numbers $n, n-s \geq 0$, and $0 \leq r \leq h$, the following finite sum identities for the generalized product sequences, $p_n(\alpha, R)$ and $p_{n-s}(\alpha, R)$, defined by (1.1) modulo h are provided in terms of the coefficients, $C_{p,n-s}(\alpha, R) := [z^{n-s}] \operatorname{FP}_p(\alpha, R; z)$, from (5.15.b) and (5.17) of Section 5.2:

$$p_n(\alpha, R) \equiv \sum_{k=1}^{n} \binom{h}{k} (-1)^k p_k(-\alpha, R+(h-1)\alpha) p_{n-k}(\alpha, R) + C_{h,n}(\alpha, R) \pmod{h} \tag{6.1.a}$$

$$\begin{aligned} p_n(\alpha, R) &\equiv \sum_{k=0}^{n} \binom{h}{k} (-\alpha)^k p_k(-\alpha, R+(h-1)\alpha) p_{n-k}(\alpha, R) \pmod{h} \qquad (6.1.b) \\ &\equiv \sum_{k=0}^{n} \binom{h}{k} \alpha^{n+k} (1-h-R/\alpha)_k (R/\alpha)_{n-k} \pmod{h} \\ &\equiv \sum_{k=0}^{n} \binom{h}{k} \alpha^{n+(r+1)k} (1-h-R/\alpha)_k (R/\alpha)_{n-k} \pmod{h\alpha^r}. \end{aligned}$$

For any fixed $h_0 \geq 0$, the following finite sums provide the additional exact formula expansions of the generalized product sequences, $p_{n+1-s}(\alpha, R)$, in (1.1):

$$\begin{aligned} p_{n+1-s}(\alpha, R) &= \sum_{k=0}^{n-s} \tbinom{n+1-s}{k+1} (-1)^{k+1} p_{k+1}(-\alpha, R+(n-s)\alpha) p_{n-s-k}(\alpha, R) \qquad (6.2) \\ &= \sum_{k=0}^{n-s} \tbinom{n+1-s+h_0}{k+1} (-1)^{k+1} p_{k+1}(-\alpha, R+(n-s+h_0)\alpha) p_{n-s-k}(\alpha, R) \\ &\qquad + C_{n+1-s+h_0, n+1-s}(\alpha, R). \end{aligned}$$

Proof. If we let the shorthand, $p_{n,h}(\alpha, R) := [z^n] \operatorname{Conv}_h(\alpha, R; z)$, denote the series coefficients of the h^{th} convergent function, we notice that these terms satisfy the following congruence properties:

$$\begin{aligned} p_{n,h}(\alpha, R) &= p_n(\alpha, R), && \forall h \geq n \\ p_{n,h}(\alpha, R) &\equiv p_n(\alpha, R) \pmod{h}, && \forall n \geq 0. \end{aligned}$$

For fixed $\alpha \neq 0$ and integers $h \geq 2$ and $n, n-s \geq 0$, we employ the identities given in (5.8) from the properties of the convergent denominator functions, $\operatorname{FQ}_h(\alpha, R; z)$, already noted in Section 5.1

to expand the series in the next equation.

$$\underbrace{\left(\sum_{n\geq 0} p_{n,h}(\alpha, R)\, z^n\right)}_{\operatorname{Conv}_h(\alpha,R;z)} \times \underbrace{\left(1-\sum_{i=1}^{h}\binom{h}{i}(-1)^{i+1}p_i\left(-\alpha, R+(h-1)\alpha\right)z^i\right)}_{\operatorname{FQ}_h(\alpha,R;z)} = \underbrace{\sum_{n=0}^{h-1} C_{h,n}(\alpha,R)z^n}_{\operatorname{FP}_h(\alpha,R;z)}$$

Since the auxiliary terms, $C_{h,n}(\alpha, R)$, are zero–valued whenever $n \geq h$, the series in the previous equation gives a h–order finite difference equation for the approximate products corresponding to the convergent function coefficients, $p_{n,h}(\alpha, R) \equiv p_n(\alpha, R) \pmod{h}$, of the following forms:

$$p_n(\alpha,R) \equiv \begin{cases} \sum\limits_{k=1}^{h}\binom{h}{k}(-1)^{k+1}p_k(-\alpha,R+(h-1)\alpha)[p_{n-k}(\alpha,R) \pmod{h}], & \text{if } 1 < h \leq n; \\ \sum\limits_{k=1}^{n}\binom{h}{k}(-1)^{k+1}p_k(-\alpha,R+(h-1)\alpha)p_{n-k}(\alpha,R)+C_{h,n}(\alpha,R), & \text{if } 0 \leq n < h. \end{cases} \pmod{h}.$$

The previous equations immediately imply the first congruence stated in (6.1.a). Since $p_n(\alpha, R) = [z^n]\operatorname{Conv}_{n+n_0}(\alpha, R; z)$ for all $0 \leq n \leq n+n_0$, we similarly obtain the pair of exact formulas expanded in (6.2). ⊡

Remark 6.2 (Congruences for the Single Factorial Function)**.** The second cases of the generalized factorial function congruences in (6.1.b) are of particular utility in expanding several of the non–trivial results given in Section 6.3 below when $h-(n-s) \geq 1$. In particular, since $n! = p_n(-1, n)$ and $n! = p_n(1, 1)$ for all $n \geq 1$, whenever $h \geq 2$ is fixed (or when h corresponds to some fixed function with an implicit dependence on the index n), the single factorial function, $(n-s)!$, satisfies the following congruences:

$$\begin{aligned}
(n-s)! &\equiv C_{h,n-s}(1,1) && \pmod{h} \\
&= \sum_{i=0}^{n-s}\binom{h}{i}(-h)_i\,(n-s-i)! \\
&= \sum_{i=0}^{n-s}\binom{h}{i}^2(-1)^i i!(n-s-i)! \\
&= \sum_{i=0}^{n-s}\binom{h}{i}\binom{i-h-1}{i} i!(n-s-i)! \\
(n-s)! &\equiv C_{h,n-s}(-1,n-s) && \pmod{h} \\
&= \sum_{i=0}^{n-s}\binom{h}{i}(n+1-s-h)_i \times (-1)^{n-s-i}\left(-(n-s)\right)_{n-s-i} \\
&= \sum_{i=0}^{n-s}\binom{h}{i}\binom{n-s}{i}\binom{h-n+s-1}{i}(-1)^i i! \times (n-s-i)!. && (6.3)
\end{aligned}$$

The right–hand–side terms, $C_{h,n-s}(\alpha, R)$, in the previous two equations correspond to the auxiliary convergent function sequences defined in Section 5.2, and the corresponding multiple sum expansions stated in (5.15) and in (5.17) highlighted by the listings given in Table A.7 starting on page 86.

The results stated above in (6.3) are provided as lemmas needed to state many of the congruence results for the prime–related sequence cases given as examples in Section 6.3.2 and Section 6.3.4. The results related to the double factorial functions and the central binomial coefficients expanded through the congruences in Section 6.3.3 also employ the second cases of (6.1.b) stated in Proposition 6.1, which we do not prove explicitly above. Ⓡ

6.1.1 Examples

When the initially indeterminate parameter, R, assumes an implicit dependence on the sequence index, n, the results phrased by the previous equations, somewhat counterintuitively, do not immediately imply difference equations satisfied between the generalized product sequences, either exactly, or modulo the prescribed choices of $p \geq 2$. The new formulas connecting the generalized product sequences, $p_n(\alpha, \beta n + \gamma)$, resulting from (6.1.b) and (6.2) in these cases are, however, reminiscent of the relations satisfied between the generalized Stirling polynomial sequences studied by the references [13; 15; 22].

The product sequence forms expanded in terms of the Pochhammer symbol, $p_n(\alpha, R) = \alpha^n (R/\alpha)_n$, and the Pochhammer k–symbol, $p_n(\alpha, x) = (x)_{n,\alpha}$, yield comparisons between various known generalized forms of Vandermonde's convolution identity, stated as in Section 5.2 above, with the new formulas stated in the previous equations [6; 7; 13; 15; 21] **F.21 .

Example 6.3 (Binomial–Coefficient–Related Congruences and Convolution Identities)**.** The particular identity noted in (1.3) of the introduction relating the generalized product sequences, $p_{n-s}(-\alpha, \beta n + \gamma)$, to the Gould polynomials, $G_n(x; a, b)$, combined with the congruence properties in (6.1.b) provides the next binomial–coefficient–related expansions of this Sheffer sequence modulo any prescribed integers p, and $p\alpha^r$, for $p \geq 2$ and some $0 \leq r \leq p$ *F.22 :

$$\begin{aligned}
\binom{\frac{\beta n+\gamma}{\alpha}}{n-s}(-\alpha)^{n-s}(n-s)! &\equiv \sum_{k=0}^{n-s}\binom{p}{k}\binom{\frac{\beta n+\gamma}{\alpha}+p-1}{k}\binom{\frac{\beta n+\gamma}{\alpha}}{n-s-k}\times(-1)^{rk}(-\alpha)^{n-s+(r+1)k}\times && \pmod{p\alpha^r}\\
&\qquad\times k!(n-s-k)!\\
&\equiv \sum_{k=0}^{n-s}\binom{p}{k}\binom{\frac{\beta n+\gamma}{\alpha}-p+1}{k}\binom{\frac{\beta n+\gamma}{\alpha}}{n-s-k}\times(-1)^{k}(-\alpha)^{n-s+(r+1)k}\times && \pmod{p\alpha^r}\\
&\qquad\times k!(n-s-k)!\\
&\equiv \sum_{k=0}^{n-s}\binom{p}{k}\binom{-\frac{\beta n+\gamma}{\alpha}+p-1}{k}\binom{\frac{\beta n+\gamma}{\alpha}}{n-s-k}\times(-\alpha)^{n-s+(r+1)k}\times && \pmod{p\alpha^r}\\
&\qquad\times k!(n-s-k)!\\
\binom{\frac{\beta n+\gamma}{\alpha}+n-s-1}{n-s}\alpha^{n-s}(n-s)! &\equiv \sum_{k=0}^{n-s}\binom{p}{k}\binom{\frac{\beta n+\gamma}{\alpha}+p-1}{k}\binom{\frac{\beta n+\gamma}{\alpha}+n-s-k-1}{n-s-k}\times\alpha^{n-s+(r+1)k}\times && \pmod{p\alpha^r}\\
&\qquad\times k!(n-s-k)!.
\end{aligned}$$

The previous equation is also compared to the known generalization of Vandermonde's identity for the coefficients of the *generalized binomial series*, $\binom{x+tn}{n}\frac{x}{x+tn} = [z^n]\mathcal{B}_t(z)^x = x(x+tn-1)\cdots(x+tn-n+1)/n!$, expanded by the convolution formulas given in the references [15] [13, §5; Table 169, Table 202]. Ⓔ

Example 6.4 (Finite Sums Involving the α–Factorial Functions)**.** The double factorial function, $(2n-1)!!$, satisfies a number of known expansions through the finite sum identities summarized as in [4; 12]. The particular combinatorial identity for the double factorial function expanded in the form of (6.4) below is remarkably similar to the statement of the first sum in (6.2) satisfied by the

**F.21 *Note §6.1.* For $s \neq 0$, the polynomials, $f_n(x) := (x)_{n,-s}$, also form a convolution family with corresponding exponential generating function given by $F(z)^x = (1-sz)^{-x/s}$ [15].

*F.22 *Note §6.2.* Since $(x)_k = \binom{x+k-1}{k}\cdot k!$ and $(x)_k = (-1)^k\binom{-x}{k}\cdot k!$ for fixed x and $k \geq 1$ where $(-x)_k = (-1)^k x^{\underline{k}} = (-1)^k(x-k+1)_k$ [15; 27].

more general product cases [4, §4.1].

$$(2n-1)!! = \sum_{k=0}^{n-1} \binom{n}{k+1} (2k-1)!!(2n-2k-3)!!. \tag{6.4}$$

If we assume that $\alpha \geq 2$ is integer–valued, and proceed to expand these cases of the α–factorial functions according to the expansions from (4.2.b) and (6.2) above, we see readily that †F.23 ‡F.24

$$\begin{aligned}
(\alpha n-1)!_{(\alpha)} &= \sum_{k=0}^{n-1} \binom{n-1}{k+1} (-1)^k \times \left(\frac{1}{\alpha}\right)_{-(k+1)} \left(\frac{1}{\alpha}-n\right)_{k+1} \\
&\qquad \times (\alpha k+\alpha-1)!_{(\alpha)} (\alpha n-\alpha k-\alpha-1)!_{(\alpha)} \times \\
(\alpha n-1)!_{(\alpha)} &= \sum_{k=0}^{n-1} \binom{n-1}{k+1} (-1)^k \times \binom{\frac{1}{\alpha}+k-n}{k+1} \binom{\frac{1}{\alpha}-1}{k+1}^{-1} \\
&\qquad \times (\alpha k+\alpha-1)!_{(\alpha)} (\alpha n-\alpha k-\alpha-1)!_{(\alpha)}.
\end{aligned}$$

The first sum above combined with the expansions of the Pochhammer symbols, $(\pm x)_n$, given in footnotes *F.22 and ‡F.24, and the form of Vandermonde's convolution stated in Section 5.2 also leads to the following pair of double sum identities for the α–factorial functions when $\alpha, n \geq 2$ are integer–valued:

$$\begin{aligned}
(\alpha n-1)!_{(\alpha)} &= \sum_{k=0}^{n-1} \sum_{i=0}^{k+1} \binom{n-1}{k+1} \binom{k+1}{i} (-1)^k \alpha^{k+1-i} (\alpha i-1)!_{(\alpha)} (\alpha(n-1-k)-1)!_{(\alpha)} (n-1-k)_{k+1-i} \\
&= \sum_{k=0}^{n-1} \sum_{i=0}^{k+1} \binom{n-1}{k+1} \binom{k+1}{i} \binom{n-1-i}{k+1-i} (-1)^k \alpha^{k+1-i} (\alpha i-1)!_{(\alpha)} (\alpha(n-1-k)-1)!_{(\alpha)} (k+1-i)!.
\end{aligned}$$

The construction of further analogs for generalized variants of the finite summations and more well–known combinatorial identities satisfied by the double factorial function cases when $\alpha := 2$ is suggested as a topic for future investigation. The next few particularly interesting special cases corresponding to the triple and quadruple factorial functions, $n!!!$ and $n!!!!$, respectively, are one starting point for approaching the generalized forms of the identities summarized in the references [4; 12] [24, §2]. Ⓔ

6.2 Multiple summation identities and finite–degree polynomial expansions of the generalized product sequences in n

We are primarily concerned with cases of the α–factorial functions formed the products, $p_n(\alpha, R_n)$, when the parameter $R_n := \beta n + \gamma$ depends linearly on n. Strictly speaking, once we evaluate the indeterminate, R, as a function of n in the sequences, $p_n(\alpha, R)$, the generating functions over the convergent sequences no longer correspond to predictably rational functions of z.

We may, however, still prefer to work with these sequences formulated as finite–degree polynomials in n through a few useful forms of the next multiple sums expanded below related to the

†F.23 *Note §6.3.* For the non–integral cases of $z, w \in \mathbb{C}$ satisfying $w, z-w \neq -1, -2, -3, \ldots$, the generalized forms of the binomial coefficients are defined by $\binom{z}{w} := \Gamma(z+1) / (\Gamma(w+1)\Gamma(z-w+1))$ [19, §5] [13, *cf.* §5].

‡F.24 *Note §6.4.* We note the simplification $\left(\frac{1}{\alpha}\right)_{-(k+1)} = \frac{(-\alpha)^{k+1}}{(\alpha(k+1)-1)!_{(\alpha)}}$ where the expansions of the α–factorial functions, $(\alpha n-1)!_{(\alpha)}$, by the Pochhammer symbol correspond to the results given in Lemma 4.1 and in the corollaries from (4.2.b) of Section 4.1 (see the Pochhammer symbol identities cited in the reference [27]).

identities given in Section 5.2.1. The resulting forms of these generalized factorial–function–like sequence cases offer a dual interpretation to the corresponding exact formulas for these sequence cases stated in terms of the special function zeros already cited above to provide the sequence expansions given in Section 1.3 and Section 4.1.

Corollary 6.5 (*Generalized Polynomial Expansions by Finite Sum Identities*).
The following particular cases of finite, multiple sums for the generalized factorial function cases expanded below are provided where $n, s, n-s \geq 1$ and $\alpha, \beta, \gamma \in \mathbb{Q}$ are taken to be fixed parameters:

$$\begin{aligned} p_{n-s}(\alpha, \beta n+\gamma) = \sum_{\substack{0 \leq m \leq k < n-s \\ 0 \leq r \leq p \leq n-s}} \sum_{t=0}^{n-s-k} & \binom{m}{r}\binom{n-s}{k}\binom{t}{p-r}\left[\begin{matrix}k \\ m\end{matrix}\right]\left[\begin{matrix}n-s-k \\ t\end{matrix}\right] \times \\ & \times (-1)^{p-r-1} \alpha^{n-s-m-t} \beta^r \gamma^{m-r} (\alpha+\beta)^{p-r} \times \\ & \times (\alpha(s+1)-\gamma)^{t-(p-r)} \times n^p \\ & + [0 \leq n \leq s]_{\delta} \end{aligned} \tag{6.5}$$

$$\begin{aligned} p_{n-s}(\alpha, \beta n+\gamma) = \sum_{\substack{0 \leq r \leq p \leq u \leq 3n \\ 0 \leq m, i \leq k < n-s}} \sum_{t=0}^{n-s-k} & \binom{m}{r}\binom{i}{u-p}\binom{t}{p-r}\left[\begin{matrix}k \\ m\end{matrix}\right]\left[\begin{matrix}k \\ i\end{matrix}\right]\left[\begin{matrix}n-s-k \\ t\end{matrix}\right] \times \\ & \times \frac{(-1)^{u-r+k+1}}{k!} \alpha^{n-s-m-t} \beta^r \gamma^{m-r} (\alpha+\beta)^{p-r} \times \\ & \times (\alpha(s+1)-\gamma)^{t-(p-r)} \times s^{p-u+i} n^u \\ & + [0 \leq n \leq s]_{\delta}. \end{aligned}$$

Proof Outline. The forms of these expansions for the generalized factorial function sequence variants in the equations above are provided here without citing the details to a somewhat tedious, and unnecessary, proof derived from the well–known polynomial expansions of the products, $p_n(\alpha, R) = \alpha^n (R/\alpha)_n$ by the Stirling number triangles.

More concretely, for $n, k \geq 0$ and fixed $\alpha, \beta, \gamma, \rho, n_0 \in \mathbb{Q}$, the following particular expansions suffice to show enough of the detail needed to more carefully prove each of the multiple sum identities cited in (6.5) starting from the first statements provided in (6.2):

$$\begin{aligned} p_k(\alpha, \beta n+\gamma+\rho) &= \alpha^k \cdot \left(\frac{\beta n+\gamma+\rho}{\alpha}\right)_k \\ &= \sum_{m=0}^{k} \left[\begin{matrix}m \\ k\end{matrix}\right] \alpha^{k-m} (\beta n+\gamma+\rho)^m \\ &= \sum_{p=0}^{k} \left(\sum_{m=p}^{k} \left[\begin{matrix}k \\ m\end{matrix}\right]\binom{m}{p} \alpha^{k-m} \beta^p (\gamma+\rho+\beta n_0)^{m-p}\right) \times (n-n_0)^p. \end{aligned}$$

The simplified triple sum expansions of interest in the Example 6.7 below correspond to a straightforward simplification of the more general multiple finite quintuple 5–sums and 6–sum identities that exactly enumerate the functions, $p_{n-s}(\alpha, \beta n+\gamma)$, when $(s, \alpha, \beta, \gamma) := (1, -1, 1, 0)$. ⊡

One immediate consequence of Corollary 6.5 phrases the form of the next multiple sums that exactly generate the single factorial functions, $(n-s)!$, modulo any prescribed $p \geq 2$. In particular, these results lead to the following finite, triple sum expansions of the single factorial function cases implicit to the statements of both Wilson's theorem and Clement's theorem considered as examples in the next subsection [14, *cf.* §VII] §F.25 :

$$\begin{aligned}
(n-1)! &= \sum_{p=0}^{n} \left(\sum_{0 \leq t \leq k < n} \binom{n}{n-1-k} \begin{bmatrix} n-1-k \\ p \end{bmatrix} \begin{bmatrix} k \\ k-t \end{bmatrix} (-1)^{n-1-p} \right) \times (n-1)^p \qquad (6.6) \\
&= \sum_{p=0}^{n} \left(\sum_{\substack{0 \leq k < n \\ 0 \leq t \leq n-1-k}} \binom{n}{k} \begin{bmatrix} k \\ p \end{bmatrix} \begin{bmatrix} n-1-k \\ n-1-k-t \end{bmatrix} (-1)^{n-1-p} \right) \times (n-1)^p \\
&= \sum_{p=0}^{n} \left(\sum_{0 \leq t \leq k < n} \binom{n}{n-1-k} \begin{bmatrix} n-1-k \\ n-p \end{bmatrix} \begin{bmatrix} k \\ k-t \end{bmatrix} (-1)^{p+1} \right) \times (n-1)^{n-p}.
\end{aligned}$$

For comparison, the next several equations provide related forms of finite, triple sum identities for the double factorial function, $(2n-1)!!$. The following expansions are obtained from the lemma in (4.2.b) applied to the known double sum identities involving the Stirling numbers of the first kind documented in the reference [4, §6]:

$$\begin{aligned}
(2n-1)!! &= \sum_{1 \leq j \leq k \leq n} \begin{bmatrix} k-1 \\ j-1 \end{bmatrix} 2^{n-j} (-1)^{n-k} (1-n)_{n-k} \qquad \text{(Double Factorial Triple Sums)} \\
&= \sum_{\substack{1 \leq j \leq k \leq n \\ 0 \leq m \leq n-k}} \begin{bmatrix} k-1 \\ j-1 \end{bmatrix} \begin{bmatrix} n-k+1 \\ m+1 \end{bmatrix} 2^{n-j} (-1)^{n-k-m} n^m \\
(2n-1)!! &= \sum_{1 \leq j \leq k \leq n} \binom{2n-k-1}{k-1} \begin{bmatrix} k \\ j \end{bmatrix} (2n-2k-1)!! \\
&= \sum_{\substack{1 \leq j \leq k \leq n \\ 0 \leq m \leq n-k}} \binom{2n-k-1}{k-1} \begin{bmatrix} k \\ j \end{bmatrix} \begin{bmatrix} n-k \\ m \end{bmatrix} 2^{n-k-m} \\
&= \sum_{\substack{1 \leq j \leq k \leq n \\ 0 \leq m \leq n-k}} \binom{2n-k-1}{k-1} \begin{bmatrix} k \\ j \end{bmatrix} \begin{bmatrix} n-k+1 \\ m+1 \end{bmatrix}_2 (-1)^{n-k-m} (2n-2k)^m .
\end{aligned}$$

A couple of the characteristic examples of these polynomial expansions in n by the Stirling numbers of the first kind in (6.6) are given next to illustrate the notable special cases of Wilson theorem and Clement's theorem modulo some as yet unspecified odd prime, $n \geq 3$, considered by Example 6.7.

§F.25 *Note §6.5.* The third exact triple sum identity given in (6.6) is further expanded through the formula of Riordan cited in [6, p. 173] as follows:

$$\sum_{0 \leq k < n} \binom{n-1}{k} (k+1)! \cdot n^{n-1-k} = n^n.$$

6.3 Expansions of the classical congruence cases and several other new results for prime–related subsequences

6.3.1 Applications to variants of Wilson's theorem and Clement's theorem

Definition 6.6. We first define the following variants of the first triple sum given in (6.6), denoted by $F_{\omega,n}(x_{\rm p}, x_{\rm t}, x_{\rm k})$, for application–dependent, prescribed functions $N_{\omega,p}(n)$ and $M_\omega(n)$, where the formal variables $\{x_{\rm p}, x_{\rm t}, x_{\rm k}\}$, index the terms corresponding to each individual sum over the respective variables, p, t, and k:

$$\begin{aligned} F_{\omega,n}(x_{\rm p}, x_{\rm t}, x_{\rm k}) := \sum_{\substack{0 \leq t \leq k < n \\ 0 \leq p \leq n}} \binom{n}{n-1-k} \begin{bmatrix} n-1-k \\ p \end{bmatrix} \begin{bmatrix} k \\ k-t \end{bmatrix} \times & \qquad (6.7) \\ \times (-1)^{n-1-p} \times N_{\omega,p}(n) \times \{x_{\rm p}^p x_{\rm t}^t x_{\rm k}^k\} \qquad & \pmod{M_\omega(n)}. \end{aligned}$$

Notice that when $N_{\omega,p}(n) := (n-1)^p$, the function $F_{\omega,n}(1,1,1)$ exactly generates the single factorial function, $(n-1)!$. Section 6.3.4 contains related constructions for the prime triplet congruence functions, $T_{d_1,d_2,d_3}(x_1, x_2, x_3; p)$, defined by Example 6.11 below. Alternate expansions of the triple sums in (6.6), and the corresponding forms of the parametrized congruences formulated by (6.7), given in terms of the Stirling polynomials, $\sigma_n(x)$, are stated by the results in Section 6.5. Ⓓ

Example 6.7 (Variants of Wilson's Theorem and Clement's Theorem)**.** The next specialized forms of the parameters implicit to the congruence in (6.7) of the previous definition are chosen as follows to form another restatement of Wilson's theorem given immediately below:

$$(\omega, N_{\omega,p}(n), M_\omega(n)) :\mapsto (\mathrm{WT}, (-1)^p, n)\,. \qquad \text{(Wilson Parameters)}$$

Then we see that

$$n \geq 2 \text{ prime} \iff F_{\mathrm{WT},n}(1,1,1) \equiv -1 \pmod{M_{\mathrm{WT}}(n)}. \qquad \text{(Wilson's Theorem)}$$

Numerical computations with *Mathematica*'s `PolynomialMod` function suggest several nice properties satisfied by the trivariate polynomial sequences, $F_{\mathrm{WT},n}(x_{\rm p}, x_{\rm t}, x_{\rm k})$, defined by (6.7) when n is prime, particularly as formed in the cases taken over the following polynomial configurations of the three formal variables, $x_{\rm p}$, $x_{\rm t}$, and $x_{\rm k}$ [¶F.26] :

$$(x_{\rm p}, x_{\rm t}, x_{\rm k}) \in \{(x,1,1), (1,x,1), (1,1,x)\}\,.$$

The special case of the congruence formulated in Clement's theorem concerning a classical characterization of the *twin primes* [A001097] is also of particular interest in continuing the discussion from Section 1.4.

[¶F.26] *Note §6.6.* In particular, these computations suggest the following properties satisfied by these sums for integers $n \geq 2$ where the coefficients of the functions, $F_{\mathrm{WT},n}(x_{\rm p}, x_{\rm t}, x_{\rm k})$, are computed termwise with respect to the formal variables, $\{x_{\rm p}, x_{\rm t}, x_{\rm k}\}$, modulo each $M_{\mathrm{WT}}(n) := n$:

1. $F_{\mathrm{WT},n}(x_{\rm p},1,1) \equiv n-1 \pmod{n}$ when n is prime where $\deg_{x_{\rm p}}\{F_{\mathrm{WT},n}(x_{\rm p},1,1) \pmod{n}\} > 0$ when n is composite;
2. $F_{\mathrm{WT},n}(1,x_{\rm k},1) \equiv (n-1)\cdot x_{\rm k}^{n-1}\,[n \text{ prime}]_\delta \pmod{n}$; and
3. $F_{\mathrm{WT},n}(1,1,x_{\rm t}) \equiv \sum_{i=0}^{n-2} x_{\rm t}^i \pmod{n}$ when n is prime where $\deg_{x_{\rm t}}\{F_{\mathrm{WT},n}(1,1,x_{\rm t}) \pmod{n}\} < n-2$ when n is composite.

The computations in the attached summary notebook file, `multifact-cfracs-summary.nb`, provide several specific examples of the properties suggested by these configurations of the special congruence polynomials for these cases.

It is not difficult to prove that the following congruence holds for integers $p \geq 0$, $n \geq 1$, and any fixed $k \geq 1$ ‖F.27 :

$$(n-1)^p \equiv \frac{(-1)^p}{k}\left(k + (1-(k+1)^p)\cdot n\right) \pmod{n(n+k)}.$$

When $k := 2$, the parameters in (6.7) are then employed to state the next result are then formed as follows:

$$(\omega, N_{\omega,p}(n), M_{\omega}(n)) :\mapsto \left(\mathrm{CT}, \frac{(-1)^p}{2}\left(2 + (1-3^p)\cdot n\right), n(n+2)\right). \qquad \text{(Clement Parameters)}$$

The corresponding expansion of this alternate formulation of Clement's theorem stated as in Section 1.4.3 of the introduction then results in the following equation [20, §4.3] **F.28 :

$$n, n+2 \text{ prime} \iff 4\cdot F_{\mathrm{CT},n}(1,1,1) + 4 + n \equiv 0 \pmod{M_{\mathrm{CT}}(n)}. \qquad \text{(Clement's Theorem)}$$

It is similarly straightforward to obtain related congruences satisfied by the p^{th} powers, $(n-1)^p$, modulo $(n-k_1)(n+k_2)$ at some prescribed choices of the integers k_1 and k_2. For example, the next equation states another particular congruence relation following from an appeal to the binomial theorem *F.29 :

$$(n-1)^p \equiv \frac{(k_1-1)^p(n+k_2) - (k_2+1)^p(n-k_1)}{k_1+k_2} \pmod{(n-k_1)(n+k_2)}.$$

There are numerous other examples of prime–related congruences that are also easily adapted by extending the procedure for the classical cases given above. The references prove analogous factorial–related results phrasing elementary integer congruence requirements on prime pairs, prime triplets, and the more general forms of prime k–tuples suggested as applications of these new congruence results in Example 6.11 [2; 14; 18]. Ⓔ

‖F.27 *Note §6.7.* i.e., since a naïve expansion via the binomial theorem shows that

$$\begin{aligned}(n-1)^p = (-1)^p &+ \sum_{s=1}^{p}\binom{p}{s}(-1)^{p-s}\cdot n\cdot(-k)^{s-1} \\ &+ \sum_{s=1}^{p}\sum_{r=1}^{s-1}\binom{p}{s}\binom{s-1}{r} n\cdot(n+k)\times(-1)^{p-s}(-k)^{s-1-r}(n+k)^{r-1}.\end{aligned}$$

**F.28 *Note §6.8.* Further special cases of the properties noted in footnote ¶F.26 for the expansions of Wilson's theorem defined above are also tabulated for these polynomial variants corresponding to Clement's theorem in the attached summary notebook. One other noteworthy property satisfied by the sums, $F_{\mathrm{CT},n}(x_{\mathrm{p}}, x_{\mathrm{t}}, x_{\mathrm{k}})$, modulo each prescribed $M_{\mathrm{CT}}(n) := n(n+2)$ for the first several cases of the integers $n \geq 3$, suggests that whenever n is prime and $n+2$ is composite we have that

$$F_{\mathrm{CT},n}(1,1,x_{\mathrm{k}}) \equiv n + 4 + (n^2-4)x_{\mathrm{k}}^{n-1} \pmod{n(n+2)},$$

where $\deg_{x_{\mathrm{k}}}\{F_{\mathrm{CT},n}(1,1,x_{\mathrm{k}}) \pmod{n(n+2)}\} > 0$ when n is prime.

*F.29 *Note §6.9.* Repeated appeals to the binomial theorem lead to the exact expansions

$$\begin{aligned}(n-1)^p = (k_1-1)^p &+ \sum_{s=1}^{p}\binom{p}{s}(n-k_1)\cdot(k_1-1)^{p-s}\left(-(k_1+k_2)\right)^{s-1} \\ &+ \sum_{s=1}^{p}\sum_{r=1}^{s-1}\binom{p}{s}\binom{s-1}{r}(n-k_1)\cdot(n+k_2)\times(k_1-1)^{p-s}\left(-(k_1+k_2)\right)^{s-1-r}(n+k_2)^{r-1}.\end{aligned}$$

The divisibility of the Stirling numbers of the first kind is tied to well–known expansions of the triangle involving the generalized *r–order harmonic numbers*, $H_n^{(r)} := \sum_{k=1}^{n} k^{-r}$, for integer–order $r \geq 1$ [13, §6] [6, *cf.* §5.7] [14, *cf.* §VII–VIII]. The applications cited in the references provide statements of the following established special case identities for these coefficients [22, §4.3] [13, §6.3] [A001008; A002805; A007406; A007407; A007408; A007409]:

$$\begin{aligned} \begin{bmatrix} n+1 \\ 2 \end{bmatrix} &= n! \cdot H_n && \text{(Harmonic Number Expansions of the Stirling Numbers)} \\ \begin{bmatrix} n+1 \\ 3 \end{bmatrix} &= \frac{n!}{2}\left(H_n^2 - H_n^{(2)}\right) \\ \begin{bmatrix} n+1 \\ 4 \end{bmatrix} &= \frac{n!}{6}\left(H_n^3 - 3H_nH_n^{(2)} + 2H_n^{(3)}\right) \\ \begin{bmatrix} n+1 \\ 5 \end{bmatrix} &= \frac{n!}{24}\left(H_n^4 - 6H_n^2H_n^{(2)} + 3\left(H_n^{(2)}\right)^2 + 8H_nH_n^{(3)} - 6H_n^{(4)}\right). \end{aligned} \tag{6.8}$$

The reference [13, p. 554, Ex. 6.51] gives a related precise statement of the necessary condition on the primality of odd integers $p > 3$ implied by Wolstenholme's theorem from the example cited in Section 1.4.3 of the introduction in the following form:

$$p > 3 \text{ prime} \implies \qquad \text{(Stirling Number Variant of Wolstenholme's Theorem)}$$
$$p^2 \mid \begin{bmatrix} p \\ 2 \end{bmatrix}, \; p^2 \mid p\begin{bmatrix} p \\ 3 \end{bmatrix} - p^2\begin{bmatrix} p \\ 4 \end{bmatrix} + \cdots + p^{p-2}\begin{bmatrix} p \\ p \end{bmatrix}.$$

A couple of related approaches to congruence–based primality conditions for prime pairs formulated through the triple sum expansions phrased in Example 6.7 above are provided in the applications to the next examples of the prime–related subsequences highlighted in Section 6.3.2 and in Section 6.3.4 below.

6.3.2 Examples: Expansions of several other prime–related congruences and subsequence identities

The first result for the generalized symbolic product sequences, $p_{n-s}(\alpha, R)$, modulo p when $p > n-s$ given in (6.1.b) yields additional congruences similar in form to the results in the last example. If $n, n-s, d, an+r \in \mathbb{Z}^+$ are selected so that $n+d, an+r > n-s$, the coefficient identities for the sequences, $p_{n-s}(1,1) = [z^{n-s}]\,\mathrm{FP}_{n+d}(1,1;z) \pmod{n+d, an+r}$, stated in (5.15.b) and (5.17) of Section 5.2 provide that (see the further symbolic computations with these sums outlined in Remark 6.8)

$$\begin{aligned} (n-s)! &\equiv \sum_{i=0}^{n-s} \binom{n+d}{i}^2 (-1)^i i! \times (n-s-i)! && \pmod{n+d} \\ (n-s)! &\equiv \sum_{i=0}^{n-s} \binom{an+r}{i} \left(-(an+r)\right)_i (n-s-i)! && \pmod{an+r}. \end{aligned} \tag{6.9}$$

The previous identities lead to additional examples phrasing congruences equivalent to the primality condition in Wilson's theorem involving products of the single factorial functions, $n!$ and $(n+1)!$, modulo some odd integer $p := 2n+1$ of unspecified primality to be determined by an application of these results.

For example, we can see that for $n \geq 1$, an odd integer $p := 2n+1$ is prime if and only if †F.30

$$\begin{aligned} 2^{1-n} \cdot n! \cdot (n+1)! &\equiv (-1)^{\binom{n+2}{2}} && \pmod{2n+1} \qquad (6.10) \\ (n!)^2 &\equiv (-1)^{n+1} && \pmod{2n+1}. \end{aligned}$$

The first congruence in (6.10) yields the following additional form of the necessary and sufficient condition on the primality of the odd integers, $p := 2n+1$, resulting from Wilson's theorem:

$$\frac{1}{2^{n-1}} \times \left(\prod_{s:=0,1} \sum_{i=0}^{n+s} \binom{2n+1}{i}^2 (-1)^i i!(n+s-i)! \right) \equiv (-1)^{(n+1)(n+2)/2} \pmod{2n+1}. \qquad (6.11)$$

If we further seek to determine new properties of the odd primes of the form $p := n^2+1 \geq 5$, obtained starting from adaptations of the new forms given by these sums, the second consequence of Wilson's theorem provided in (6.10) above leads to an analogous requirement expanded in the form of the next equations [20] [A002496].

$$\begin{aligned} n^2+1 \text{ prime} \quad &\iff \left(\sum_{i=0}^{n^2/2} \binom{n^2+1}{i} \left(-(n^2+1)\right)_i \left(\tfrac{1}{2}(n^2-2i)\right)! \right)^2 && \equiv (-1)^{n^2/2+1} && \pmod{n^2+1} \\ &\iff \left(\sum_{i=0}^{n^2/2} \binom{n^2+1}{i} \binom{i-n^2-2}{i} i! \left(\tfrac{1}{2}(n^2-2i)\right)! \right)^2 && \equiv (-1)^{n^2/2+1} && \pmod{n^2+1} \end{aligned}$$

For comparison, the first classical statement of Wilson's theorem stated as in the introduction is paired with the next expansions of the fourth and fifth multiple sums stated in (5.17) to show that an odd integer $p \geq 5$ of the form $p := n^2+1$ for some even $n \geq 2$ is prime if and only if ‡F.31

$$\begin{aligned} \sum_{\substack{0 \leq m \leq k \leq n^2 \\ 0 \leq v \leq i \leq s \leq n^2}} \underbrace{\binom{n^2+1}{k}\binom{m}{s}\binom{i}{v}\binom{n^2+1+v}{v}\left[{k \atop m}\right]\left\{{s \atop i}\right\}(-1)^{m+i-v} i! \left(-n^2\right)_{n^2-k} (n^2+1)^{m-s}}_{(C_{h,k}(\alpha,R):\ h:\mapsto n^2+1,\ k:\mapsto n^2,\ \alpha:\mapsto -1,\ R:\mapsto n^2)} &\equiv -1 \pmod{n^2+1} \\ \sum_{\substack{0 \leq i \leq n^2 \\ 0 \leq m \leq k \leq n^2 \\ 0 \leq t \leq s \leq n^2}} \underbrace{\binom{n^2+1}{k}\binom{m}{t}\left[{k \atop m}\right]\left[{n^2-k \atop s-t}\right]\left\{{s \atop i}\right\}(-1)^{m+s-i} n^{2m-2t} \times i!}_{(C_{h,n}(\alpha,R):\ h:\mapsto n^2+1,\ n:\mapsto n^2,\ \alpha:\mapsto 1,\ R:\mapsto 1)} &\equiv -1 \pmod{n^2+1}. \end{aligned}$$

Next, for natural numbers $n-s, k_1, k_2 \geq 0$ such that $(n-k_1)(n+k_2) > n-s \geq 1$, a related application to the identities cited in Example 6.7 above yields the following congruences for the single factorial function, $(n-s)!$:

$$\begin{aligned} (n-s)! &\equiv \sum_{i=0}^{n-s} \binom{(n-k_1)(n+k_2)}{i}^2 (-1)^i i!(n-s-i)! && \pmod{(n-k_1)(n+k_2)} \\ &\equiv \sum_{i=0}^{n-s} \binom{(n-k_1)(n+k_2)}{i} \binom{i-(n-k_1)(n+k_2)-1}{i} i!(n-s-i)! && \pmod{(n-k_1)(n+k_2)}. \end{aligned}$$

Congruences characterizing the sequence of *Wilson primes*, or odd primes n where the *Wilson quotient*, $W(n)/n := [(n-1)!+1]/n^2 \in \mathbb{Z}$, correspond to the imposing the following additional

†F.30 *Note §6.10.* The first equation restates a result proved by Szántó in 2005 given on the *MathWorld* site.

‡F.31 *Note §6.11.* These congruences are straightforward to adapt to form related results characterizing prime subsequences of the form $p := an^2+bn+c$ for some fixed constants $a, b, c \in \mathbb{Z}$ satisfying the constraints given in the reference at natural numbers $n \geq 1$ [14, §II.2.8].

equivalent requirements on the divisibility of the single factorial function modulo n in Wilson's theorem [20, §5.4] [14, *cf.* §VI.6.6] [A007619; A007540]:

$$\underbrace{\sum_{i=0}^{n-1}\binom{n^2}{i}^2(-1)^i i!(n-1-i)!}_{C_{n^2,n-1}(1,1)\ \equiv\ (n-1)!\pmod{n^2}} \equiv -1 \pmod{n^2} \qquad \text{(Wilson Primes)}$$

$$\underbrace{\sum_{i=0}^{n-1}\binom{n^2}{i}\binom{i-n^2-1}{i} i!(n-1-i)!}_{(n-1)!\pmod{n^2}} \equiv -1 \pmod{n^2}$$

$$\underbrace{\sum_{i=0}^{n-1}\binom{n^2}{i}(n^2-n)^{\underline{i}}\times(-1)^{n-1-i}(n-1)^{\underline{n-1-i}}}_{C_{n^2,n-1}(-1,n-1)\ \equiv\ (n-1)!\pmod{n^2}} \equiv -1 \pmod{n^2}.$$

The congruence in the previous equation is verified numerically in the summary notebook to hold for the first few hundred primes, p_n, only when $p_n \in \{5, 13, 563\}$. The third and fourth multiple sum expansions of the coefficients, $C_{n^2,n-1}(-1, n-1)$, given in (5.17) similarly provide that an odd integer $n > 3$ is a Wilson prime if and only if either of the following pair of congruences holds modulo integers n^2:

$$\sum_{s=0}^{n-1}\sum_{i=0}^{s}\left(\sum_{k=0}^{n-1}\sum_{m=0}^{k}\binom{n^2}{k}\binom{n^2}{i}\binom{m}{s}\left[{k \atop m}\right]\left\{{s \atop i}\right\}(-1)^{n-1-k}(1-n)_{n-1-k}(-n)^{m-s}i!\right) \equiv -1 \pmod{n^2}$$

$$\sum_{s=0}^{n-1}\sum_{i=0}^{s}\sum_{v=0}^{i}\left(\sum_{k=0}^{n-1}\sum_{m=0}^{k}\binom{n^2}{k}\binom{m}{s}\binom{i}{v}\binom{n^2+v}{v}\left[{k \atop m}\right]\left\{{s \atop i}\right\}(-1)^{i-v}(n-1)^{\underline{n-1-k}}(-n)^{m-s}i!\right) \equiv -1 \pmod{n^2}.$$

The constructions of these new results are also combined with the known congruences established in the reference [18, §3, §5] to obtain the alternate necessary and sufficient conditions for the *twin prime* pairs [A077800; A006512] given in (1.22) of the introduction, as well as in the analogous expansions of the congruence statements corresponding to characterizations of the *cousin prime* and *sexy prime* pairs stated as follows [A023200; A023201]:

$$\begin{aligned}
&2n+1, 2n+5 \text{ odd primes} && \text{(Cousin Prime Pairs)}\\
&\iff 36\left(\sum_{i=0}^{n}\binom{(2n+1)(2n+5)}{i}^2(-1)^i i!(n-i)!\right)^2 \\
&\qquad + (-1)^n(29-14n) \equiv 0 \pmod{(2n+1)(2n+5)} \\
&\iff 96\left(\sum_{i=0}^{2n}\binom{(2n+1)(2n+5)}{i}^2(-1)^i i!(2n-i)!\right) \\
&\qquad + 46n+119 \equiv 0 \pmod{(2n+1)(2n+5)} \\
&2n+1, 2n+7 \text{ odd primes} && \text{(Sexy Prime Pairs)}\\
&\iff 1350\left(\sum_{i=0}^{n}\binom{(2n+1)(2n+7)}{i}^2(-1)^i i!(n-i)!\right)^2 \\
&\qquad + (-1)^n(578n+1639) \equiv 0 \pmod{(2n+1)(2n+7)} \\
&\iff 4320\left(\sum_{i=0}^{2n}\binom{(2n+1)(2n+7)}{i}^2(-1)^i i!(2n-i)!\right)
\end{aligned}$$

$$+1438n+5039 \qquad \equiv 0 \pmod{(2n+1)(2n+7)}. \tag{6.12}$$

The treatment of the modular congruence identities involved in these few notable example cases in Example 6.7, and in the last several examples from the remarks above, is by no means exhaustive, but serves to demonstrate the utility of this approach in formulating several new forms of non–trivial prime number results with many notable applications.

Remark 6.8 (Symbolic Computations with Mathematica's Sigma Package)**.** The working summary notebook, `multifact-cfracs-summary.nb`, attached to the article includes computations with *Mathematica*'s `Sigma` package that yield additional forms of the identities expanded in (6.9), (6.10), and (6.11), for the single factorial function, $(n-s)!$, when $s:=0$. For example, alternate forms of the identity for the first sum in (6.9) are expanded as follows [§F.32] : (**TODO**)

$$\begin{aligned}
n! &\equiv \sum_{i=0}^{n} \binom{n+d}{i} \left(-(n+d)\right)_i (n-i)! && \pmod{n+d} \\
&= \sum_{i=0}^{n} \binom{n+d}{i}^2 (-1)^i i!(n-i)! \\
&= \sum_{i=0}^{n} \binom{n+d}{i} \binom{i-n-d-1}{i} i!(n-i)! \\
&= (-1)^n (2d)_n \times \left(1+d^2 \times \sum_{i=1}^{n} \binom{i+d}{i} \frac{(-1)^i \left(-(i+d)\right)_i}{(i+d)^2 (2d)_i}\right).
\end{aligned}$$

Further computation with *Mathematica*'s `Sigma` package similarly yields the following alternate form of the second sums in (6.9) implicit to the congruence identities stated in (6.10) and (6.11) above:

$$\begin{aligned}
n! &\equiv \sum_{i=0}^{n} \binom{2n+1}{i}^2 (-1)^i i!(n-i)! \qquad \pmod{2n+1} \qquad (6.13)\\
&= \frac{(-1)^n (3n+1)!}{8\,(n!)^2} \times \left(8 - \sum_{i=1}^{n} \binom{2i+1}{i}^2 \frac{(i!)^3}{2\cdot(3i+1)!} \left(11 - \frac{20}{i} - \frac{8}{(2i+1)} + \frac{1}{(2i+1)^2}\right)\right) \\
&= \frac{(-1)^n (3n+1)!}{8\,(n!)^2} \times \left(8 - \sum_{i=1}^{n} \binom{2i+1}{i}^2 \frac{(i!)^3}{(3i)!} \left(\frac{10}{i} + \frac{5}{(2i+1)} - \frac{1}{(2i+1)^2} - \frac{32}{(3i+1)}\right)\right).
\end{aligned}$$

The second (non–square) sums implicit to the congruences providing characterizations of the *twin prime* pairs given in (1.22) of the introduction, and of the *cousin* and *sexy primes* expanded in (6.12) of the previous remark, are easily generalized to form related results for other prime pairs [A023202; A023203; A046133].

For positive integers $d \geq 1$, these expansions lead to more general congruence–based characterizations of the odd prime pairs, $(2n+1, 2n+1+2d)$, in the form of the following equation for some

[§F.32] *Note §6.12.* The Pochhammer symbol, $(2x)_j$, is factored according to known duplication identities for these functions over cases of the even and odd parity of the input integers $j \geq 0$ [19, *cf.* §5.5(iii)]. For $i \leq n$, the ratio of the Pochhammer symbols in the second sum also satisfies the following identity employed to simplify the next ratio terms in (6.15) of this remark immediately below [27]:

$$\frac{(x)_n}{(x)_i} = (x+i)_{n-i}.$$

$a_d, b_d, c_d \in \mathbb{Z}$ and where the parameter $h := (2n+1)(2n+1+2d) > 2n$ implicit to these sums depends quadratically on n [18, *cf.* §3, §5]:

$$2n+1, 2n+1+2d \text{ prime} \quad \Longleftrightarrow \quad a_d \times \underbrace{\sum_{i=0}^{2n} \binom{h}{i}^2 (-1)^i i!(2n-i)!}_{:= S_n(h)} + b_d n + c_d \equiv 0 \pmod{h}.$$

Let the shorthand for the functions, $T_{h,i}$, $H_{h,i}$, $T_i(n)$, $H_i(n)$, be defined as in the next equations:

$$\begin{aligned} T_{h,i} &:= \prod_{j=0}^{i} \left(\frac{(h-2j)^2(h+1-2j)^2}{2\cdot(2h+1-2j)(h-j)}\right) = 4^i \times \frac{\left(\frac{1-h}{2}\right)_i^2 \left(1-\frac{h}{2}\right)_i^2}{\left(\frac{1}{2}-h\right)_i (1-h)_i} \qquad (6.14) \\ T_i(n) &:= \underbrace{4^i \times \frac{\left(\frac{1-(2n+1)(2n+3)}{2}\right)_i^2 \cdot \left(1-\frac{(2n+1)(2n+3)}{2}\right)_i^2}{\left(\frac{1}{2}-(2n+1)(2n+3)\right)_i (1-(2n+1)(2n+3))_i}}_{T_{(2n+1)(2n+3),i}} \\ H_{h,i} &:= \frac{h(h+1)(2h-1)}{(h-1)(h-i)} + \frac{2(h+1)^2(2h+1)}{h(2h+1-2i)} + \frac{2(h+1)}{h(h-1)(h+1-2i)} \\ &= \frac{(h+1)(2h+1-4i)(h-2i)}{(2h+1-2i)(h+1-2i)(h-i)} \\ H_i(n) &:= \underbrace{\frac{2(n+1)^2(4n^2+8n+3-2i)(8n^2+16n+7-4i)}{(2n^2+4n+2-i)(4n^2+8n+3-i)(8n^2+16n+7-2i)}}_{H_{(2n+1)(2n+3),i}}. \end{aligned}$$

Computations with the Sigma package then lead to the following necessary and sufficient condition obtained from the second congruence statement in (1.22) of Section 1.4.3:

$$\begin{aligned} &2n+1, 2n+3 \text{ prime} \quad \Longleftrightarrow \\ &\qquad 4 \times \sum_{i=0}^{n} \binom{(2n+1)(2n+3)}{2i}^2 \frac{T_n(n)H_i(n)}{T_i(n)} \times (2i)! + 2n+5 \equiv 0 \pmod{(2n+1)(2n+3)}. \end{aligned}$$

The ratios of the products on the right–hand–side of (6.14) defining the functions, $T_{h,i}$, also form the next alternate, simplified terms involving the Pochhammer symbol:

$$\frac{T_{h,n}}{T_{h,i}} = \frac{(1-h)_{2n}^2}{(1-2h)_{2n}} \times \frac{(1-2h)_{2i}}{(1-h)_{2i}^2} = \frac{(1-h+2i)_{2n-2i}^2}{(1-2h+2i)_{2n-2i}},\ n \geq i. \qquad (6.15)$$

The forms of the generalized sums, $S_n(h)$, obtained as in the special case identity above using the Sigma software package routines are then expanded as ¶F.33

$$S_n(h) = \sum_{i=0}^{n} \binom{h}{2n-2i}^2 \frac{(2n-2i)!}{h(h-1)} \times \frac{(1-h+2n-2i)_{2i}^2}{(1-2h+2n-2i)_{2i}} \times \left(\frac{h^2(h+1)(2h-1)}{(h-n+i)} + \frac{2(h+1)^2(2h+1)(h-1)}{(2h+1-2n+2i)} + \frac{2(h+1)}{(h+1-2n+2i)}\right)$$

¶F.33 *Note §6.13.* Note that the binomial coefficient identity, $\binom{\binom{k}{2}}{2} = 3\binom{k+1}{4}$, given in the exercises section of the reference [13, p. 535, Ex. 5.67], suggests simplifications of the sums, $S_n(h)$, when h denotes some fixed, implicit application–dependent quadratic function of n obtained by first expanding the inner terms, $\binom{h}{i}$, as a (finite) linear combination of binomial coefficient terms whose upper index corresponds to a linear function of n. The Sigma package is able to obtain alternate forms of these pre–processed finite sums defining the functions, $S_n(\beta n+\gamma)$, for scalar–valued β, γ that generalize the last two expansions provided above in (6.13).

$$= \sum_{i=0}^{n} \binom{h}{2n-2i}^2 (2n-2i)! \times \frac{(1-h+2n-2i)_{2i}^2}{(1-2h+2n-2i)_{2i}} \times \left(\frac{(h+1)(2h+1-4(n-i))(h-2(n-i))}{(2h+1-2(n-i))(h+1-2(n-i))(h-n+i)}\right)$$

The sums in the previous several equations provide approaches to these congruences for the prime–related subsequences cited as applications in the examples above through expansions by generalized Stirling number triangles, Stirling and Bernoulli polynomials, and the corresponding sequences of generalized r–order harmonic numbers noted in the identities given in Section 6.4. In particular, the expansions of the multiple factorial functions underlying the various forms of these congruences suggest analogs to known Wolstenholme–prime–like identities and related necessary conditions involving generalized harmonic number sequences for the primality of odd integer pairs and other cases of the prime–related subsequences. ®

6.3.3 Expansions of congruences involving the double factorial function

The coefficient expansion given by the first identity in (6.1.b) provides the alternate forms of congruences for the double factorial functions, $(2n-1)!! = p_n(-2, 2n-1)$ and $2^n (1/2)_n = p_n(2, 1)$, stated in the next equations modulo $2^s \cdot h$ for fixed integers $h \geq 2$ and any $0 \leq s \leq h$ [‖F.34] :

$$\begin{aligned}
(2n-1)!! &\equiv \sum_{i=0}^{n} \binom{h}{i} 2^{n+(s+1)i} (1/2-h)_i (1/2)_{n-i} && \pmod{2^s h} \\
&\equiv \sum_{i=0}^{n} \binom{h}{i} \binom{2n-2i}{n-i} \frac{2^{n+(s+1)i}}{4^{n-i}} \times (1/2-h)_i (n-i)! && \pmod{2^s h} \\
&\equiv \sum_{i=0}^{n} \binom{h}{i} (-2)^{n+(s+1)i} (1/2+n-h)_i (1/2-n)_{n-i} && \pmod{2^s h}.
\end{aligned}$$

The next congruences satisfied by the central binomial coefficients modulo polynomial powers of n^p with integer coefficients also provide additional examples of some of the double–factorial–related phrasings of the expansions of (6.1.b) following from the noted identity given in (5.15.b) of Section 5.2:

$$\begin{aligned}
\binom{2n}{n} &= \frac{2^n}{n!} \times (2n-1)!! && (6.16) \\
&\equiv \sum_{i=0}^{n} \binom{n^p}{i} 2^i (1/2-n^p)_i (1/2)_{n-i} \times \frac{2^{2n}}{n!} && \pmod{n^p} \\
&\equiv \sum_{i=0}^{n} \binom{n^p}{i} \binom{2n-2i}{n-i} (1/2-n^p)_i \times \frac{8^i \cdot (n-i)!}{n!} && \pmod{n^p} \\
&\equiv \sum_{i=0}^{n} \binom{n^p}{i} (-2)^i (1/2+n-n^p)_i (1/2-n)_{n-i} \times \frac{2^{2n}}{n!} && \pmod{n^p}.
\end{aligned}$$

[‖F.34] *Note §6.14.* For natural numbers $n \geq 0$, the central binomial coefficients satisfy an expansion by the following identity given in the reference [13, §5.3]:

$$(1/2)_n = \binom{-1/2}{n} \times (-1)^n n! = \binom{2n}{n} \times \frac{n!}{4^n}.$$

The special cases of the congruences in (6.16) corresponding to $p := 3$ and $p := 4$, respectively, are related to the necessary condition for the primality of odd integers $n > 3$ in *Wolstenholme's theorem* and to the sequence of *Wolstenholme primes* [20, §2.2] [14, *cf.* §VII] [A088164] **F.35 .

As another example of the applications of these new results expanded through the double factorial function, notice that the following identity gives the form of another exact, finite sum expansion of the single factorial function over convolved products of double factorial functions *F.36 :

$$\begin{aligned}(n-1)! = (2n-3)!! &+ \sum_{k=1}^{n-2}\sum_{j=k}^{n-1} (-1)^{j+1} (-j)_k (-(2n-k-j-2))_{j-k} (2n-2j-3)!! \\ &+ \sum_{k=1}^{n-2}\sum_{j=k+1}^{n-1} (-1)^{j} (-j)_{k+1} (-(2n-k-j-3))_{j-k-1} (2n-2j-3)!!\end{aligned}$$

A modified approach involving the congruence techniques outlined in either the first cases cited in Example 6.7, or as suggested by the previous few example cases given in Section 6.3.2, provides even further applications to adapting the results for new variants of the established, or otherwise well–known, special case congruence–based identities suggested in the previous topics.

6.3.4 Applications of Wilson's theorem to other prime subsequences

The constructions of the congruences in the examples above also lead to necessary and sufficient conditions on the primality for sequences of composite functions based on Wilson's theorem considered by the references. Specifically, the result in Wilson's theorem provides additional congruences for prime subsequences formed by functions of p when p is prime, denoted $p \in \mathbb{P}$ in the results below. The congruences cited next in Example 6.9 and Example 6.11 provide several specific examples of these new prime–related results in the context of the factorial prime sequences, the (generalized) Fermat number and prime subsequences, the Mersenne numbers and sequence of Mersenne primes, Sophie Germain primes, and notable special cases of prime k–tuples, including the sexy prime triplets.

Example 6.9 (Applications of Wilson's Theorem to Special Case Prime Subsequences)**.** The sequences of *factorial primes* of the form $p := n! \pm 1$ for some $n \geq 1$ satisfy congruences of the following form modulo $n! \pm 1$ given by the expansions of (5.15.b) and (5.17) from Section 5.2 [20, *cf.* §2.2] [A002981; A002982]:

$n! + 1$ prime $\iff$ (Factorial Prime Congruences)

$$\underbrace{\sum_{i=0}^{n!} \binom{n!+1}{i}^2 (-1)^i i!(n!-i)!}_{(n!)! \equiv C_{n!+1,n!}(1,1) \pmod{n!+1}} \equiv -1 \pmod{n!+1}$$

**F.35 *Note §6.15.* The rational convergents to the S–fraction series for the ordinary generating function of the *Catalan numbers*, $C_n = \binom{2n}{n}\frac{1}{(n+1)}$, defined by Section 1.2 and in the references suggests alternate continued–fraction–based approaches to the congruences satisfied by these coefficients modulo the integer powers n^p utilizing the methods employed to stated the particular examples above [17, §5.5] [9, Prop. 5] [13, *cf.* §5.3] [A000108].

*F.36 *Note §6.16.* This identity is straightforward to prove starting from the first non–round sum given in §5.1 of the reference combined with second identity for the component summation terms in §6.3 of the same article [4].

$$n!-1 \text{ prime} \iff \underbrace{\sum_{i=0}^{n!-2}\binom{n!-1}{i}^2(-1)^i i!(n!-2-i)!}_{(n!-2)!\equiv C_{n!-1,n!-2}(1,1)\pmod{n!-1}} \equiv -1 \pmod{n!-1}.$$

The *Fermat numbers*, F_n, generating the subsequence of *Fermat primes* of the form $p:=2^m+1$ where $m=2^n$ for some $n\geq 0$ similarly satisfy the next congruences expanded through the identities for the single and double factorial functions expanded above [20, §2.6] [14, §II.2.5] [A000215; A019434]:

$$\begin{aligned} F_n:=2^{2^n}+1 \text{ prime} &\iff 2^{2^n}+1 \mid \left(2^{2^n}\right)!+1 && \text{(Fermat Prime Congruences)} \\ &\iff 2^{2^n}+1 \mid 2^{2^{2^n-1}}\sum_{i=0}^{2^{2^n}}\binom{2^{2^n}+1}{i}^2(-1)^i i!\left(2^{2^n}-i\right)!+1 \\ &\iff 2^{2^n}+1 \mid 2^{2^{2^n-1}}\left(2^{2^n-1}\right)!\left(2^{2^n}-1\right)!!+1 \\ &\iff 2^{2^n}+1 \mid 2^{\frac{3}{4}\cdot 2^{2^n}}\left(2^{2^n-2}\right)!\left(2^{2^n-1}-1\right)!!\left(2^{2^n}-1\right)!!+1 \\ &\iff 2^{2^n}+1 \mid 2^{\frac{7}{8}\cdot 2^{2^n}}\left(2^{2^n-3}\right)!\left(2^{2^n-2}-1\right)!!\left(2^{2^n-1}-1\right)!!\left(2^{2^n}-1\right)!!+1. \end{aligned}$$

For integers $h\geq 2$ and $r\geq 1$ such that $2^r \mid h$, the expansions of the congruences in the previous several equations correspond to forming the products of the single and double factorial functions modulo $h+1$ from the previous examples to require that

$$2^{\left(1-2^{-r}\right)\cdot h}\times\left(\frac{h}{2^r}\right)!\left(\frac{h}{2^{r-1}}-1\right)!!\left(\frac{h}{2^{r-2}}-1\right)!!\times\cdots\times\left(\frac{h}{2}-1\right)!!\,(h-1)!!\equiv -1 \pmod{h+1}.$$

The *generalized Fermat numbers*, $F_n(\alpha):=\alpha^{2^n}+1$, and the corresponding *generalized Fermat prime* subsequences when $\alpha:=2,4,6$ suggest generalizations of the approach to the results in the previous equations through the procedure to the multiple, α–factorial function expansions outlined in Section 6.4 that generalized the procedure to expanding the congruences above for the Fermat primes when $\alpha:=2$ [A078303].

The *Mersenne primes* correspond to prime pairs of the form (p, M_p) for p prime and where $M_n:=2^n-1$ is a *Mersenne number* for some $n\geq 1$ [20, §2.7] [14, §II.2.5; §VI.6.15] [13, *cf.* §4.3, §4.8] [A000225; A000668; A000043]. The requirements in Wilson's theorem for the primality of both p and M_p provide elementary proofs of the following equivalent necessary and sufficient conditions for the primality of the prime pairs of these forms, which are then expanded by the results in (5.15.b) and (5.17) from Section 5.2 through the second cases of the more general product function congruences stated in (6.1.b) above:

$$\begin{aligned} (p,2^p-1)\in\mathbb{P}^2 \iff & && \text{(Mersenne Numbers and Prime Congruences)} \\ & p(2^p-1) \mid (p-1)!(2^p-2)!+(p-1)!+(2^p-2)!+1 \\ \iff & p(2^p-1) \mid \big(C_{p(2^p-1),p-1}(-1,p-1)C_{p(2^p-1),2^p-2}(-1,2^p-2) \\ & \qquad + C_{p(2^p-1),p-1}(-1,p-1)+C_{p(2^p-1),2^p-2}(-1,2^p-2)+1\big) \\ \iff & p(2^p-1) \mid \big(C_{p(2^p-1),p-1}(1,1)C_{p(2^p-1),2^p-2}(1,1) \\ & \qquad + C_{p(2^p-1),p-1}(1,1)+C_{p(2^p-1),2^p-2}(1,1)+1\big) \\ \iff & p(2^p-1) \mid 2^{2^{p-1}-1}(p-1)!(2^{p-1}-1)!(2^p-3)!!+(p-1)!+(2^p-2)!+1. \end{aligned}$$

Wilson's theorem similarly implies the next related congruence–based characterizations of the *Sophie Germain primes* corresponding to the prime pairs of the form $(p, 2p+1)$ expanded through the results given above when $p, p-1, 2p < p(2p+1)$ [20, §5.2] [A005384]:

$$\begin{aligned} & (p, 2p+1) \in \mathbb{P}^2 && \text{(Sophie Germain Prime Congruences)} \\ & \iff (p-1)!(2p)! + (p-1)! + (2p)! && \equiv -1 \pmod{p(2p+1)} \\ & \iff 2^p p!(p-1)!(2p-1)!! + (p-1)! + (2p)! && \equiv -1 \pmod{p(2p+1)}. \end{aligned}$$

The expansions of the generalized forms of the Sophie Germain primes noted in the reference [20, §5.2] also provide applications of the multiple, α–factorial function identities providing further expansions of the arithmetic progressions of the factorial functions from Section 4.2.3 cited in Section 6.4 of the article below. Ⓔ

Remark 6.10. The integer congruences obtained from Wilson's theorem for the particular special sequence cases noted in Example 6.9 are easily generalized to give constructions over the forms other prime subsequences including the following: primes of the form $p := 2^u 3^v + 1$ for some $u, v \in \mathbb{N}$ [A005109], primes of the form $p := n2^n \pm 1$ [A002234; A080075], the primes tuples of the form $\left(p, \frac{1}{3}(2^p+1)\right) \in \mathbb{P}^2$ [A000978; A123176], and for the generalized cases of the multifactorial prime sequences tabulated in the reference [20, Table 6, §2.2].

Notice that most of the factorial function expansions involved in the results formulated by the previous few examples do not immediately imply corresponding congruences obtained from (6.1.b) satisfied by the *Wieferich prime* sequence [20, §5.3] [A001220], or for the variations of the sequences of binomials enumerated by the rational convergent–function–based generating function identities over the binomial coefficient sums constructed in Section 4.2.6 modulo prime powers p^m for $m \geq 2$. Ⓡ

Example 6.11 (Some Related Results for Prime Triplets and Prime k–Tuples)**.** For natural numbers $p \geq 2$, fixed $k \geq 2$, and some prescribed integer offsets $d_k > \cdots > d_2 > d_1 \geq 0$, let the function $M_{k,p}$ denote the shorthand for the products, $M_{k,p} := (p+d_1)(p+d_2) \times \cdots \times (p+d_k)$, employed in the statements of the next results [†F.37] .

$$\begin{aligned} & (p+d_1, p+d_2, \cdots, p+d_k) \in \mathbb{P}^k && \text{(Wilson's Theorem for Prime Tuples)} \\ & \iff \sum_{j=1}^{k} \left(\sum_{1 \leq i_1 < i_2 < \cdots < i_j \leq k} (p+d_{i_1}-1)! \times \cdots \times \left(p+d_{i_j}-1\right)! \right) && \equiv -1 \pmod{M_{k,p}} \\ & \iff \sum_{j=1}^{k} \left(\sum_{1 \leq i_1 < i_2 < \cdots < i_j \leq k} \prod_{m=1}^{j} C_{M_{k,p}, p+d_{i_m}-1}(1,1) \right) && \equiv -1 \pmod{M_{k,p}} \\ & \iff \prod_{j=1}^{k} \underbrace{\left(1 + \sum_{i=0}^{p+d_j-1} \binom{(p+d_1)\cdots(p+d_k)}{i}^2 (-1)^i i!(p+d_j-1-i)! \right)}_{1+C_{M_{k,p},p+d_j-1}(1,1)} && \equiv \ 0 \pmod{M_{k,p}} \end{aligned}$$

The special case of the previous equation when $k := 3$ and $d_1 := 0$ yields the next particular results providing the Wilson's theorem congruence expansions for primes triplets, or 3–tuples, expanded as follows:

$$(p, p+d_2, p+d_3) \in \mathbb{P}^3 \iff \qquad \text{(Wilson's Theorem for Prime Triplets)}$$

[†F.37] *Note §6.17.* These results are obtained easily from Wilson's theorem by induction on k. Note that related identities providing constructions of similar necessary and sufficient conditions on integer tuples with elements chosen as composite functions of some prime p, for example, as stated through the previous few results given in Example 6.9 above, are also straightforward to obtain by induction starting from Wilson's theorem.

$$(1+(p-1)!)(1+(p+d_2-1)!)(1+(p+d_3-1)!) \equiv 0 \pmod{p(p+d_2)(p+d_3)}.$$

For example, the *sexy prime triplets* corresponding to integer triples of the form $(p, p+6, p+12)$ for prime $p \geq 7$, and where it is assumed that $p+18 \notin \mathbb{P}$, correspond to the following condition obtained from the repeated applications of Wilson's theorem phrased above [A046118; A046124]:

$$(p, p+6, p+12) \in \mathbb{P}^3 \iff \qquad \text{(Partial Congruence for the Sexy Prime Triplets)}$$

$$\prod_{d \in \{-1,5,11\}} \left(1 + \sum_{i=0}^{p+d} \binom{p(p+6)(p+12)}{i}^2 (-1)^i i!(p+d-i)!\right) \equiv 0 \pmod{p(p+6)(p+12)}$$

The working summary notebook contains computations related to noteworthy cases of these triples with respect to the formal variables, $\{x_1, x_2, x_3\}$, and the generalized prime triple congruences, $T_{0,d_2,d_3}(x_1, x_2, x_3; p)$ and $T_{0,d_2,d_3}(x, x, x; p)$, defined as in the parametrized congruences from Example 6.7 in Section 6.3.1 by the next equation, each reduced modulo $p(p+d_2)(p+d_3)$ for the first several hundred specific cases of the integers $p \geq 2$.

$$T_{d_1,d_2,d_3}(x_1, x_2, x_3; p) := \prod_{1 \leq j \leq 3} \left(1 + x_j \times \underbrace{\sum_{i=0}^{p+d_j-1} \binom{(p+d_1)(p+d_2)(p+d_3)}{i}^2 (-1)^i i!(p+d_j-1-i)!}_{C_{(p+d_1)(p+d_2)(p+d_3),p+d_j-1}(1,1)}\right)$$

The other notable special case triples of interest in the references include applications to the prime 3–tuples of the forms $(p+d_1, p+d_2, p+d_3)$ for $(d_1, d_2, d_3) \in \{(0,2,6), (0,4,6)\}$ [14, *cf.* §I.4] [20, §4.4] [A022004; A022005]. Ⓔ

6.4 More general expansions of the new congruence results by multiple factorial functions and generalized harmonic number sequences

Additional identities formed as cases of other known prime–related congruences involving both the single and double factorial functions suggest even further applications of the properties of the α–factorial functions phrased for the more general sequences cases by (6.1.b) from above. If the form the odd primes $p := an + r$ is known in the previous congruences cited above in Example 6.7, Section 6.3.2, and Section 6.3.3, the expansions by the arithmetic arithmetic progressions of the single factorial function given in Section 4.2 suggest applications of the generalized Stirling number triangles in (1.4) to interpreting the new forms of these congruences (see Table A.1 starting on page 77 and Table A.2 starting on page 78).

The identities cited in Section 4.2.3 provide decompositions of the functions, $(an+r-1)!$, into $a > 1$ distinct factors of the multiple, α–factorial functions, $(an-d)!_{(a)}$, defined by (1.2) through the expansions of these functions by the generalized products, $p_n(\alpha, R)$, proved by Lemma 4.1. These expansions by the α–factorial functions then lead to related formulations of the previous congruence identities discussed above as products of finite sums with respect to n modulo the functions of $p := an + r$ implicit to the second cases of (6.1.b) from the previous results given in this section.

For example, generalizations of the congruences for the *Sophie Germain primes* given in Example 6.9 of the previous section provide the following analogous results for prime pairs of the form $(p, 2kp+1)$ when $k := 2, 3$ [20, §5.2] [A005384]:

$$(p, 4p+1) \in \mathbb{P}^2 \iff (p-1)!(4p)! + (p-1)! + (4p)! \equiv -1 \pmod{p(4p+1)}$$

$$\begin{aligned}
&\iff p(4p+1) \mid (p-1)! \left(1 + (4p)!_{(4)} (4p-1)!_{(4)} (4p-2)!_{(4)} (4p-3)!_{(4)}\right) \\
&\qquad\qquad + (4p)! + 1 \\
&\iff p(4p+1) \mid (p-1)! \left(1 + 4^{4p} (1)_p \left(\tfrac{1}{4}\right)_p \left(\tfrac{1}{2}\right)_p \left(\tfrac{3}{4}\right)_p\right) + (4p)! + 1 \\
(p, 6p+1) \in \mathbb{P}^2 &\iff (p-1)!(6p)! + (p-1)! + (6p)! \equiv -1 \pmod{p(6p+1)} \\
&\iff p(6p+1) \mid C_{p(6p+1),p-1}(-1, p-1) \left(1 + \prod_{i=0}^{5} C_{p(6p+1),p}(-6, 6p-i)\right) \\
&\qquad\qquad + C_{p(6p+1),6p}(-1, 6p) + 1 \\
&\iff p(6p+1) \mid C_{p(6p+1),p-1}(-1, p-1) \left(1 + \prod_{i=1}^{6} 6^p \times C_{p(6p+1),p}(6, i)\right) \\
&\qquad\qquad + C_{p(6p+1),6p}(-1, 6p) + 1.
\end{aligned}$$

The noted relations of the divisibility of the Stirling numbers of the first kind to the r–order harmonic number sequences expanded by the special cases from (6.8) are also generalized to the α–factorial function coefficient coefficient cases through the following forms of the exponential generating functions given in (1.9) of Section 1.1 [22, *cf.* §3.3]:

$$\sum_{n \geq 0} \begin{bmatrix} n+1 \\ m+1 \end{bmatrix}_\alpha \frac{z^n}{n!} = \frac{(1-\alpha z)^{-1/\alpha}}{m! \cdot \alpha^m} \times \operatorname{Log}\left(\frac{1}{1-\alpha z}\right)^m.$$

The next special cases of the coefficients generated by the previous equation when $m := 1, 2$ are then expanded as in the following equations:

$$\begin{aligned}
\begin{bmatrix} n+1 \\ 2 \end{bmatrix}_\alpha \frac{1}{n!} &= \alpha^{n-1} \times \sum_{k=0}^{n} \binom{1-\frac{1}{\alpha}}{k} (-1)^k H_{n-k} \\
&= \frac{\alpha^n \cdot \left(\frac{1}{\alpha}\right)_n}{n!} \times \sum_{1 \leq k \leq n} \frac{1}{(\alpha k + 1 - \alpha)} \\
\begin{bmatrix} n+1 \\ 3 \end{bmatrix}_\alpha \frac{1}{n!} &= \frac{\alpha^{n-2}}{2} \times \sum_{k=0}^{n} \binom{1-\frac{1}{\alpha}}{k} (-1)^k \left(H_{n-k}^2 - H_{n-k}^{(2)}\right) \\
&= \frac{\alpha^n \cdot \left(\frac{1}{\alpha}\right)_n}{2 \cdot n!} \times \left(\left(\sum_{1 \leq k \leq n} \frac{1}{(\alpha k + 1 - \alpha)}\right)^2 - \sum_{1 \leq k \leq n} \frac{1}{(\alpha k + 1 - \alpha)^2}\right).
\end{aligned}$$

Identities providing expansions of the generalized α–factorial triangles in (1.4) at other specific cases of the lower indices $m \geq 3$ that involve the slightly generalized cases of the r–order harmonic number sequences, $H_{\alpha,n}^{(r)}$ where $H_n^{(r)} = H_{1,n}^{(r)}$, defined by the next equation are expanded by related constructions.

$$H_{\alpha,n}^{(r)} := \sum_{k=1}^{n} \frac{1}{(\alpha k + 1 - \alpha)^r}, \ n \geq 1, \alpha, r \in \mathbb{Z}^+ \qquad \text{(Generalized Harmonic Numbers)}$$

When $\alpha := 2$, we have a relation between the sequences, $H_{2n}^{(r)}$, and the r–order harmonic numbers of the form $H_{2,n}^{(r)} = H_{2n}^{(r)} - 2^{-r} H_n^{(r)}$, which provides the next particular coefficient expansions given by

$$\begin{bmatrix} n+1 \\ 2 \end{bmatrix}_2 = 2^n \cdot (1/2)_n \times H_{2,n}^{(1)}$$

$$
\begin{aligned}
&= \frac{(2n-1)!!}{2} \times (2H_{2n} - H_n) \\
\begin{bmatrix} n+1 \\ 3 \end{bmatrix}_2 &= 2^{n-1} \cdot (1/2)_n \times \left(\left(H_{2,n}^{(1)} \right)^2 - H_{2,n}^{(2)} \right) \\
&= \frac{(2n-1)!!}{8} \times \left((2H_{2n} - H_n)^2 - \left(4H_{2n}^{(2)} - H_n^{(2)} \right) \right) \\
\begin{bmatrix} n+1 \\ 4 \end{bmatrix}_2 &= \frac{2^n \cdot (1/2)_n}{6} \times \left(\left(H_{2,n}^{(1)} \right)^3 - 3H_{2,n}^{(1)} H_{2,n}^{(2)} + 2H_{2,n}^{(3)} \right) \\
&= \frac{(2n-1)!!}{48} \times \left((2H_{2n} - H_n)^3 - 3\left(2H_{2n} - H_n\right) \left(4H_{2n}^{(2)} - H_n^{(2)} \right) + 2\left(8H_{2n}^{(3)} - H_n^{(3)} \right) \right).
\end{aligned}
$$

The expansions of the prime–related congruences involving the double factorial function cited in Section 6.3.3 above suggest additional applications to finding Wolstenholme–prime–like congruences and necessary conditions involving these harmonic number sequences related to other more general forms of congruences for prime pairs and subsequences (see also Remark 6.8).

6.5 Sums involving Stirling polynomials and generalized Bernoulli numbers

The sums over the Stirling numbers defined by the applications cited above also satisfy further expansions by the Stirling polynomial sequences, and by the Nörlund polynomials, or generalized Bernoulli numbers, in the immediate forms given by (6.6). For $n, x \in \mathbb{N}$, let the modified Stirling polynomials, $\sigma_n^*(x)$, corresponding to the first cases of the polynomial sequences in Table A.2 (page 78) be defined by the next equation:

$$
\sigma_n^*(x) := \begin{cases} \frac{x! \cdot (x-n)}{(x-n)!} \cdot \sigma_n(x), & \text{if } x > 0; \\ 1, & \text{if } x = 0. \end{cases} \qquad \text{(Modified Stirling Polynomials)}
$$

The last form of the triple sums given in (6.6) is then expanded by the forms in the following equations where the respective terms with respect to each individual component sum each correspond to rational polynomials in n prescribed by the generalized Stirling polynomials and Bernoulli number sequences noted above [13, *cf.* §6.2] [22]:

$$
\begin{aligned}
(n-1)! &= \sum_{0 \leq t \leq k < p \leq n} \binom{n}{n-1-k} (-1)^{p+1} \sigma_t^*(k) \sigma_{p-1-k}^*(n-1-k)(n-1)^{n-p} \\
&= \sum_{\substack{0 \leq t \leq k < p \leq n \\ 0 \leq r \leq n-p}} \binom{n}{n-1-k} \binom{n-p}{r} \times (-1)^{p+1} \sigma_t^*(k) \sigma_{p-1-k}^*(n-1-k) \times \\
&\qquad \times (n-\ell_0)^r (\ell_0 - 1)^{n-p-r}, \ \ell_0 \neq 1.
\end{aligned}
$$

The properties of the Stirling polynomial sequences cited in [13; 15; 21], and the related identities corresponding to their generalized forms in [22], then suggest another possible avenue towards simplifying similar forms of the factorial–function–related congruences provided as as applications of the new results established by this article.

7 Conclusions

We have defined several new forms of ordinary power series approximations to the normally divergent ordinary generating functions of generalized multiple, or α–factorial, function sequences. The generalized forms of these convergent functions provide partial truncated approximations to the sequences formally enumerated by these divergent power series. The exponential generating functions for the special case product sequences, $p_n(\alpha, s-1)$, are studied in the reference [22, §5]. The exponential generating functions that enumerate the cases corresponding to the more general factorial–like sequences, $p_n(\alpha, \beta n+\gamma)$, are less obvious in form. We have also suggested a number of new, alternate approaches to enumerating the factorial function sequences that arise in applications, including classical identities involving the single and double factorial functions, and in the forms of several other noteworthy special cases.

The key ingredient to the short proof given in Section 3 employs known characterizations of the Pochhammer symbols, $(x)_n$, by generalized Stirling number triangles as polynomial expansions in the indeterminate, x, each with predictably small finite–integral–degree at any fixed n. The more combinatorial proof in the spirit of Flajolet's articles suggested by the discussions in Section 3.2 may lead to further interesting interpretations of the α–factorial functions, $(s-1)!_{(\alpha)}$, which motivate the investigations of the coefficient-wise symbolic polynomial expansions of the functions first considered in [22]. A separate proof of the expansions of these new continued fractions formulated in terms of the generalized α–factorial function coefficients defined by (1.4), and by their strikingly Stirling–number–like combinatorial properties motivated in the introduction, is notably missing from this article.

The rationality of these convergent functions for all h suggests new insight to generating the numeric sequences of interest, including several specific new congruence properties, derivations of finite difference equations that hold for these exact sequences modulo any integers p, and perhaps more interestingly, exact expansions of the classical single and double factorial functions by the special zeros of the generalized Laguerre polynomials and confluent hypergeometric functions. The techniques behind the specific identities given here are easily generalized and extended to further specific applications. The particular examples cited within this article are intended as suggestions at new starting points to tackling the expansions that arise in many other practical situations, both implicitly and explicitly involving the generalized variants of the factorial–function–like product sequences, $p_n(\alpha, R)$.

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A Appendix: Tables Referenced in the Article

n \ k	0	1	2	3	4	5	6
0	1						
1	0	1					
2	0	1	1				
3	0	3	4	1			
4	0	15	23	9	1		
5	0	105	176	86	16	1	
6	0	945	1689	950	230	25	1

Table A.1.1. The double factorial function triangle, $\left[{n \atop k}\right]_2$

n \ k	0	1	2	3	4	5	6
0	1						
1	0	1					
2	0	1	1				
3	0	4	5	1			
4	0	28	39	12	1		
5	0	280	418	159	22	1	
6	0	3640	5714	2485	445	35	1

Table A.1.2. The triple factorial function triangle, $\left[{n \atop k}\right]_3$

n \ k	0	1	2	3	4	5	6
0	1						
1	0	1					
2	0	1	1				
3	0	5	6	1			
4	0	45	59	15	1		
5	0	585	812	254	28	1	
6	0	9945	14389	5130	730	45	1

Table A.1.3. The quadruple factorial function triangle, $\left[{n \atop k}\right]_4$

Table A.1: Special cases of the α–factorial coefficient triangles

n	$\sigma_n(x)$
0	$\frac{1}{x}$
1	$\frac{1}{2}$
2	$\frac{1}{24}(3x-1)$
3	$\frac{1}{48}x(x-1)$
4	$\frac{1}{5760}\left(15x^3-30x^2+5x+2\right)$
5	$\frac{1}{11520}x(x-1)\left(3x^2-7x-2\right)$
6	$\frac{1}{2903040}\left(63x^5-315x^4+315x^3+91x^2-42x-16\right)$
7	$\frac{1}{5806080}x(x-1)\left(9x^4-54x^3+51x^2+58x+16\right)$
8	$\frac{1}{1393459200}\left(135x^7-1260x^6+3150x^5-840x^4-2345x^3-540x^2+404x+144\right)$

Table A.2.1. The Stirling polynomials, $\sigma_n(x) := \left[\begin{smallmatrix} x \\ x-n \end{smallmatrix}\right] \frac{(x-n-1)!}{x!}$

n	$(2n)! \times x\sigma_n^{(\alpha)}(x)$
0	1
1	$\alpha x - 2(\alpha-1)$
2	$3\alpha^2x^2 - \alpha(13\alpha-12)x + 12(\alpha-1)^2$
3	$15\alpha^3x^3 - 15\alpha^2(7\alpha-6)x^2 + 30\alpha(7\alpha-6)(\alpha-1)x - 120(\alpha-1)^3$
4	$105\alpha^4x^4 - 210\alpha^3(5\alpha-4)x^3 + 35\alpha^2\left(97\alpha^2-168\alpha+72\right)x^2$
	$-14\alpha\left(299\alpha^3-840\alpha^2+780\alpha-240\right)x + 1680(\alpha-1)^4$
5	$945\alpha^5x^5 - 3150\alpha^4(4\alpha-3)x^4 + 1575\alpha^3\left(37\alpha^2-60\alpha+24\right)x^3$
	$-630\alpha^2\left(184\alpha^3-485\alpha^2+420\alpha-120\right)x^2$
	$+1260\alpha\left(79\alpha^3-220\alpha^2+200\alpha-60\right)(\alpha-1)x - 30240(\alpha-1)^5$
6	$10395\alpha^6x^6 - 10395\alpha^5(17\alpha-12)x^5 + 51975\alpha^4(3\alpha-2)(7\alpha-6)x^4$
	$-1155\alpha^3\left(2687\alpha^3-6660\alpha^2+5400\alpha-1440\right)x^3$
	$+6930\alpha^2\left(617\alpha^4-2208\alpha^3+2910\alpha^2-1680\alpha+360\right)x^2$
	$-1320\alpha\left(2081\alpha^5-9954\alpha^4+18837\alpha^3-17640\alpha^2+8190\alpha-1512\right)x$
	$+665280(\alpha-1)^6$

n	$x\sigma_n^{(\alpha)}(x)$
1	$1+\frac{1}{2}\alpha(x-2)$
2	$\frac{1}{2}+\frac{1}{2}\alpha(x-2)+\frac{1}{24}\alpha^2(x-3)(3x-4)$
3	$\frac{1}{6}+\frac{1}{4}\alpha(x-2)+\frac{1}{24}\alpha^2(x-3)(3x-4)+\frac{1}{48}\alpha^3(x-4)(x-2)(x-1)$
4	$\frac{1}{24}+\frac{1}{12}\alpha(x-2)+\frac{1}{48}\alpha^2(x-3)(3x-4)+\frac{1}{48}\alpha^3(x-4)(x-2)(x-1)$
	$+\frac{1}{5760}\alpha^4(x-5)\left(15x^3-75x^2+110x-48\right)$
5	$\frac{1}{120}+\frac{1}{48}\alpha(x-2)+\frac{1}{144}\alpha^2(x-3)(3x-4)+\frac{1}{96}\alpha^3(x-4)(x-2)(x-1)$
	$+\frac{1}{5760}\alpha^4(x-5)\left(15x^3-75x^2+110x-48\right)$
	$+\frac{1}{11520}\alpha^5(x-6)(x-2)(x-1)\left(3x^2-13x+8\right)$
6	$\frac{1}{720}+\frac{1}{240}\alpha(x-2)+\frac{1}{576}\alpha^2(x-3)(3x-4)+\frac{1}{288}\alpha^3(x-4)(x-2)(x-1)$
	$+\frac{1}{11520}\alpha^4(x-5)\left(15x^3-75x^2+110x-48\right)$
	$+\frac{1}{11520}\alpha^5(x-6)(x-2)(x-1)\left(3x^2-13x+8\right)$
	$+\frac{1}{2903040}\left(63x^6-1071x^5+6615x^4-18809x^3+25914x^2-16648x+4032\right)$

Table A.2.2. Factored forms of the generalized α–factorial polynomials, $\sigma_n^{(\alpha)}(x)$

Table A.2: The generalized Stirling and α–factorial polynomial sequences

h	$\mathrm{FP}_h(\alpha, R; z)$
1	1
2	$1-(2\alpha+R)z$
3	$1-(6\alpha+2R)z+(6\alpha^2+4\alpha R+R^2)z^2$
4	$1-(12\alpha+3R)z+(36\alpha^2+19\alpha R+3R^2)z^2-(24\alpha^3+18\alpha^2 R+7\alpha R^2+R^3)z^3$
5	$1-(20\alpha+4R)z+(120\alpha^2+51\alpha R+6R^2)z^2-(240\alpha^3+158\alpha^2 R+42\alpha R^2+4R^3)z^3$ $+(120\alpha^4+96\alpha^3 R+46\alpha^2 R^2+11\alpha R^3+R^4)z^4$
6	$1-(30\alpha+5R)z+(300\alpha^2+106\alpha R+10R^2)z^2$ $-(1200\alpha^3+668\alpha^2 R+138\alpha R^2+10R^3)z^3$ $-(1800\alpha^4+1356\alpha^3 R+469\alpha^2 R^2+78\alpha R^3+5R^4)z^4$ $+(720\alpha^5+600\alpha^4 R+326\alpha^3 R^2+101\alpha^2 R^3+16\alpha R^4+R^5)z^5$

Table A.3.1. The convergent numerator functions, $\mathrm{FP}_h(\alpha, R; z)$

h	$\mathrm{FQ}_h(\alpha, R; z)$
0	1
1	$1-Rz$
2	$1-2(\alpha+R)z+R(\alpha+R)z^2$
3	$1-3(2\alpha+R)z+3(\alpha+R)(2\alpha+R)z^2-R(\alpha+R)(2\alpha+R)z^3$
4	$1-4(3\alpha+R)z+6(2\alpha+R)(3\alpha+R)z^2-4(\alpha+R)(2\alpha+R)(3\alpha+R)z^3$ $+R(\alpha+R)(2\alpha+R)(3\alpha+R)z^4$
5	$1-5(4\alpha+R)z+10(3\alpha+R)(4\alpha+R)z^2-10(2\alpha+R)(3\alpha+R)(4\alpha+R)z^3$ $+5(\alpha+R)(2\alpha+R)(3\alpha+R)(4\alpha+R)z^4-R(\alpha+R)(2\alpha+R)(3\alpha+R)(4\alpha+R)z^5$
6	$1-6(5\alpha+R)z+15(4\alpha+R)(5\alpha+R)z^2-20(3\alpha+R)(4\alpha+R)(5\alpha+R)z^3$ $+15(2\alpha+R)(3\alpha+R)(4\alpha+R)(5\alpha+R)z^4$ $-6(\alpha+R)(2\alpha+R)(3\alpha+R)(4\alpha+R)(5\alpha+R)z^5$ $+R(\alpha+R)(2\alpha+R)(3\alpha+R)(4\alpha+R)(5\alpha+R)z^6$

Table A.3.2. The convergent denominator functions, $\mathrm{FQ}_h(\alpha, R; z)$

h	$\mathrm{FP}_h(1,1;z)$	$\mathrm{FQ}_h(1,1;z)$
1	1	$1-z$
2	$1-3z$	$1-4z+2z^2$
3	$1-8z+11z^2$	$1-9z+18z^2-6z^3$
4	$1-15z+58z^2-50z^3$	$1-16z+72z^2-96z^3+24z^4$
5	$1-24z+177z^2-444z^3+274z^4$	$1-25z+200z^2-600z^3+600z^4-120z^5$
6	$1-35z+416z^2-2016z^3$ $+3708z^4-1764z^5$	$1-36z+450z^2-2400z^3+5400z^4$ $-4320z^5+720z^6$

Table A.3.3. The convergent generating functions, $\mathrm{Conv}_h(1,1;z) := \mathrm{FP}_h(1,1;z)/\mathrm{FQ}_h(1,1;z)$, enumerating the single factorial function, $n!$, for all $0 \leq n \leq h$ and $h \geq 1$

h	$\mathrm{FP}_h(1,1;z)$	$\mathrm{FQ}_h(1,1;z)$
1	1	$1-z$
2	$1-5z$	$1-6z+3z^2$
3	$1-14z+33z^2$	$1-15z+45z^2-15z^3$
4	$1-27z+185z^2-279z^3$	$1-28z+210z^2-420z^3+105z^4$
5	$1-44z+588z^2-2640z^3+2895z^4$	$1-45z+630z^2-3150z^3+4725z^4-945z^5$
6	$1-65z+1422z^2-12558z^3$ $+41685z^4-35685z^5$	$1-66z+1485z^2-13860z^3+51975z^4$ $-62370z^5+10395z^6$

Table A.3.4. The convergent generating functions, $\mathrm{Conv}_h(2,1;z) := \mathrm{FP}_h(2,1;z)/\mathrm{FQ}_h(2,1;z)$, enumerating the double factorial function, $(2n-1)!! = 2^n \times \left(\frac{1}{2}\right)_n$, for all $0 \leq n \leq h$ and $h \geq 1$

Table A.3: The generalized convergent numerator and denominator function sequences

h	$z^{h-1} \cdot \mathrm{FP}_h\left(\alpha, R; z^{-1}\right)$
1	1
2	$-(2\alpha+R)+z$
3	$6\alpha^2+\alpha(4R-6z)+(R-z)^2$
4	$-24\alpha^3-18\alpha^2(R-2z)-\alpha(7R-12z)(R-z)-(R-z)^3$
5	$120\alpha^4+2\alpha^2\left(23R^2-79Rz+60z^2\right)+48\alpha^3(2R-5z)+\alpha(11R-20z)(R-z)^2+(R-z)^4$
6	$-720\alpha^5-2\alpha^3\left(163R^2-678Rz+600z^2\right)-\alpha^2\left(101R^2-368Rz+300z^2\right)(R-z)$ $-600\alpha^4(R-3z)-2\alpha(8R-15z)(R-z)^3-(R-z)^5$
7	$5040\alpha^6+36\alpha^4\left(71R^2-347Rz+350z^2\right)+\alpha^2\left(197R^2-740Rz+630z^2\right)(R-z)^2$ $+\alpha^3\left(932R^3-5102R^2z+8322Rz^2-4200z^3\right)+2160\alpha^5(2R-7z)$ $+2\alpha(11R-21z)(R-z)^4+(R-z)^6$
8	$-40320\alpha^7-36\alpha^5\left(617R^2-3466Rz+3920z^2\right)-\alpha^2\left(351R^2-1342Rz+1176z^2\right)(R-z)^3$ $+\alpha^4\left(-9080R^3+57286R^2z-105144Rz^2+58800z^3\right)$ $-\alpha^3\left(2311R^3-13040R^2z+22210Rz^2-11760z^3\right)(R-z)-35280\alpha^6(R-4z)$ $-\alpha(29R-56z)(R-z)^5-(R-z)^7$

Table A.4.1. The reflected numerator polynomials, $\widetilde{\mathrm{FP}}_h(\alpha, R; z) := z^{h-1} \cdot \mathrm{FP}_h\left(\alpha, R; z^{-1}\right)$

h	$z^{h-1} \cdot \mathrm{FP}_h\left(\alpha, z-w; z^{-1}\right)$
2	$w-2\alpha$
3	$6\alpha^2+w^2-4\alpha w-2\alpha z$
4	$-24\alpha^3+w^3-7\alpha w^2+w\left(18\alpha^2-5\alpha z\right)+18\alpha^2 z$
5	$120\alpha^4+w^4-11\alpha w^3+w^2\left(46\alpha^2-9\alpha z\right)+w\left(66\alpha^2z-96\alpha^3\right)+8\alpha^2z^2-144\alpha^3z$
6	$-720\alpha^5+w^5-16\alpha w^4+w^3\left(101\alpha^2-14\alpha z\right)+w^2\left(166\alpha^2z-326\alpha^3\right)$ $+w\left(600\alpha^4+33\alpha^2z^2-704\alpha^3z\right)-170\alpha^3z^2+1200\alpha^4z$
7	$5040\alpha^6+w^6-22\alpha w^5+w^4\left(197\alpha^2-20\alpha z\right)+w^3\left(346\alpha^2z-932\alpha^3\right)$ $+w^2\left(2556\alpha^4+87\alpha^2z^2-2306\alpha^3z\right)$ $+w\left(-4320\alpha^5-914\alpha^3z^2+7380\alpha^4z\right)-48\alpha^3z^3+2664\alpha^4z^2-10800\alpha^5z$
8	$-40320\alpha^7+w^7-29\alpha w^6+w^5\left(351\alpha^2-27\alpha z\right)+w^4\left(640\alpha^2z-2311\alpha^3\right)$ $+w^3\left(9080\alpha^4+185\alpha^2z^2-6107\alpha^3z\right)+w^2\left(-22212\alpha^5-3063\alpha^3z^2+30046\alpha^4z\right)$ $+w\left(35280\alpha^6-279\alpha^3z^3+17812\alpha^4z^2-80352\alpha^5z\right)$ $+1862\alpha^4z^3-38556\alpha^5z^2+105840\alpha^6z$
h	$z^{h-1} \cdot \mathrm{FP}_h\left(-\alpha, z-w; z^{-1}\right)$
3	$6\alpha^2+w^2+\alpha(4w+2z)$
4	$24\alpha^3+w^3+\alpha\left(7w^2+5wz\right)+\alpha^2(18w+18z)$
5	$120\alpha^4+w^4+\alpha^2\left(46w^2+66wz+8z^2\right)+\alpha\left(11w^3+9w^2z\right)+\alpha^3(96w+144z)$
6	$720\alpha^5+w^5+\alpha^3\left(326w^2+704wz+170z^2\right)+\alpha\left(16w^4+14w^3z\right)$ $+\alpha^2\left(101w^3+166w^2z+33wz^2\right)+\alpha^4(600w+1200z)$
7	$5040\alpha^6+w^6+\alpha^4\left(2556w^2+7380wz+2664z^2\right)+\alpha\left(22w^5+20w^4z\right)$ $+\alpha^3\left(932w^3+2306w^2z+914wz^2+48z^3\right)+\alpha^2\left(197w^4+346w^3z+87w^2z^2\right)$ $+\alpha^5(4320w+10800z)$
8	$40320\alpha^7+w^7+\alpha^5\left(22212w^2+80352wz+38556z^2\right)+\alpha\left(29w^6+27w^5z\right)$ $+\alpha^4\left(9080w^3+30046w^2z+17812wz^2+1862z^3\right)+\alpha^2\left(351w^5+640w^4z+185w^3z^2\right)$ $+\alpha^3\left(2311w^4+6107w^3z+3063w^2z^2+279wz^3\right)+\alpha^6(35280w+105840z)$

Table A.4.2. Modified forms of the reflected numerator polynomials, $\widetilde{\mathrm{FP}}_h(\pm\alpha, z-w; z)$

Table A.4: The reflected convergent numerator function sequences

n	$n!_{(1)}$	$\widetilde{R}_2^{(1)}(n)$	(2)	(2)	$\widetilde{R}_3^{(1)}(n)$	(3)	(3)	$\widetilde{R}_4^{(1)}(n)$	(4)	(4)	$\widetilde{R}_5^{(1)}(n)$	(5)	(5)
0	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	2	0	0	2	2	2	2	2	2	2	2	2
3	6	6	0	0	6	0	0	6	2	2	6	1	1
4	24	0	0	0	24	0	0	24	0	0	24	4	4
5	120	-560	0	0	120	0	0	120	0	0	120	0	0
6	720	-15000	0	0	1440	0	0	720	0	0	720	0	0
7	5040	-355320	0	0	44100	0	0	5040	0	0	5040	0	0
8	40320	-8605184	0	0	1568448	0	0	0	0	0	40320	0	0
9	362880	-220557312	0	0	54676944	0	0	-3193344	0	0	362880	0	0
10	3628800	-6037169760	0	0	1896099840	0	0	-206599680	0	0	7257600	0	0
11	39916800	-176606100000	0	0	66812223060	0	0	-10648281600	0	0	512265600	0	0
12	479001600	-5507542216704	0	0	2422878480000	0	0	-509993003520	0	0	39734323200	0	0

Table A.5.1. Congruences for the single factorial function, $n! = n!_{(1)}$, modulo h for $h := 2, 3, 4, 5$.

n	$n!_{(2)}$	$\widetilde{R}_2^{(2)}(n)$	(2)	(4)	$\widetilde{R}_3^{(2)}(n)$	(3)	(6)	$\widetilde{R}_4^{(2)}(n)$	(4)	(8)	$\widetilde{R}_5^{(2)}(n)$	(5)	(10)
0	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	2	0	2	2	2	2	2	2	2	2	2	2
3	3	3	1	3	3	0	3	3	3	3	3	3	3
4	8	8	0	0	8	2	2	8	0	0	8	3	8
5	15	15	1	3	15	0	3	15	3	7	15	0	5
6	48	48	0	0	48	0	0	48	0	0	48	3	8
7	105	-175	1	1	105	0	3	105	1	1	105	0	5
8	384	0	0	0	384	0	0	384	0	0	384	4	4
9	945	-13671	1	1	945	0	3	945	1	1	945	0	5
10	3840	-17920	0	0	3840	0	0	3840	0	0	3840	0	0
11	10395	-633501	1	3	43659	0	3	10395	3	3	10395	0	5
12	46080	-960000	0	0	92160	0	0	46080	0	0	46080	0	0
13	135135	-28498041	1	3	3532815	0	3	135135	3	7	135135	0	5
14	645120	-45480960	0	0	5644800	0	0	645120	0	0	645120	0	0
15	2027025	-1343937855	1	1	257161905	0	3	-5386095	1	1	2027025	0	5
16	10321920	-2202927104	0	0	401522688	0	0	0	0	0	10321920	0	0
17	34459425	-67747539375	1	1	17642360385	0	3	-1211768415	1	1	34459425	0	5
18	185794560	-112925343744	0	0	27994595328	0	0	-1634992128	0	0	185794560	0	0
19	654729075	-3664567145437	1	3	1200706189875	0	3	-141536175885	3	3	3315215475	0	5
20	3715891200	-6182061834240	0	0	1941606236160	0	0	-211558072320	0	0	7431782400	0	0
21	13749310575	-212363430514977	1	3	83236453970607	0	3	-14054409745425	3	7	679112772975	0	5

Table A.5.2. Congruences for the double factorial function, $n!! = n!_{(2)}$, modulo h (and $2h$) for $h := 2, 3, 4, 5$. Supplementary listings containing computational data for the congruences, $n!! \equiv R_h^{(2)}(n) \pmod{2^i h}$, for $2 \leq i \leq h \leq 5$ are tabulated in the summary notebook reference.

n	$n!_{(3)}$	$\widetilde{R}_2^{(3)}(n)$	(2)	(6)	$\widetilde{R}_3^{(3)}(n)$	(3)	(9)	$\widetilde{R}_4^{(3)}(n)$	(4)	(12)	$\widetilde{R}_5^{(3)}(n)$	(5)	(15)
0	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	2	0	2	2	2	2	2	2	2	2	2	2
3	3	3	1	3	3	0	3	3	3	3	3	3	3
4	4	4	0	4	4	1	4	4	0	4	4	4	4
5	10	10	0	4	10	1	1	10	2	10	10	0	10
6	18	18	0	0	18	0	0	18	2	6	18	3	3
7	28	28	0	4	28	1	1	28	0	4	28	3	13
8	80	80	0	2	80	2	8	80	0	8	80	0	5
9	162	162	0	0	162	0	0	162	2	6	162	2	12
10	280	-980	0	4	280	1	1	280	0	4	280	0	10
11	880	-704	0	4	880	1	7	880	0	4	880	0	10
12	1944	0	0	0	1944	0	0	1944	0	0	1944	4	9
13	3640	-92300	0	4	3640	1	4	3640	0	4	3640	0	10
14	12320	-115192	0	2	12320	2	8	12320	0	8	12320	0	5
15	29160	-136080	0	0	29160	0	0	29160	0	0	29160	0	0
16	58240	-6186752	0	4	395200	1	1	58240	0	4	58240	0	10
17	209440	-8349992	0	4	633556	1	1	209440	0	4	209440	0	10
18	524880	-10935000	0	0	1049760	0	0	524880	0	0	524880	0	0
19	1106560	-411766784	0	4	51684256	1	1	1106560	0	4	1106560	0	10
20	4188800	-572266240	0	2	70505120	2	2	4188800	0	8	4188800	0	5
21	11022480	-777084840	0	0	96446700	0	0	11022480	0	0	11022480	0	0
22	24344320	-28922921456	0	4	5645314048	1	4	-144674816	0	4	24344320	0	10
23	96342400	-40807520000	0	4	7668245080	1	1	-116486720	0	4	96342400	0	10
24	264539520	-56458612224	0	0	10290587328	0	0	0	0	0	264539520	0	0
25	608608000	-2177450514800	0	4	577086766300	1	1	-41321139200	0	4	608608000	0	10
26	2504902400	-3101148709984	0	2	793943072000	2	8	-52040160640	0	8	2504902400	0	5
27	7142567040	-4341229572096	0	0	1076206288752	0	0	-62854589952	0	0	7142567040	0	0
28	17041024000	-176120000000000	0	4	58548072721600	1	1	-7074936915200	0	4	153556480000	0	10

Table A.5.3. Congruences for the triple factorial function, $n!!! = n!_{(3)}$, modulo h (and $3h$) for $h := 2, 3, 4, 5$. Supplementary listings containing computational data for the congruences, $n!!! \equiv R_h^{(3)}(n) \pmod{3^i h}$, for $2 \leq i \leq h \leq 5$ are tabulated in the summary notebook reference.

n	$n!_{(4)}$	$\widetilde{R}_2^{(4)}(n)$	(2)	(8)	$\widetilde{R}_3^{(4)}(n)$	(3)	(12)	$\widetilde{R}_4^{(4)}(n)$	(4)	(16)	$\widetilde{R}_5^{(4)}(n)$	(5)	(20)
0	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	2	0	2	2	2	2	2	2	2	2	2	2
3	3	3	1	3	3	0	3	3	3	3	3	3	3
4	4	4	0	4	4	1	4	4	0	4	4	4	4
5	5	5	1	5	5	2	5	5	1	5	5	0	5
6	12	12	0	4	12	0	0	12	0	12	12	2	12
7	21	21	1	5	21	0	9	21	1	5	21	1	1
8	32	32	0	0	32	2	8	32	0	0	32	2	12
9	45	45	1	5	45	0	9	45	1	13	45	0	5
10	120	120	0	0	120	0	0	120	0	8	120	0	0
11	231	231	1	7	231	0	3	231	3	7	231	1	11
12	384	384	0	0	384	0	0	384	0	0	384	4	4
13	585	-3159	1	1	585	0	9	585	1	9	585	0	5
14	1680	-2800	0	0	1680	0	0	1680	0	0	1680	0	0
15	3465	-1815	1	1	3465	0	9	3465	1	9	3465	0	5
16	6144	0	0	0	6144	0	0	6144	0	0	6144	4	4
17	9945	-364871	1	1	9945	0	9	9945	1	9	9945	0	5
18	30240	-437472	0	0	30240	0	0	30240	0	0	30240	0	0
19	65835	-508725	1	3	65835	0	3	65835	3	11	65835	0	15
20	122880	-573440	0	0	122880	0	0	122880	0	0	122880	0	0
21	208845	-32086803	1	5	1990989	0	9	208845	1	13	208845	0	5
22	665280	-40544064	0	0	2794176	0	0	665280	0	0	665280	0	0
23	1514205	-50324483	1	5	4031325	0	9	1514205	1	13	1514205	0	5
24	2949120	-61440000	0	0	5898240	0	0	2949120	0	0	2949120	0	0
25	5221125	-2829930075	1	5	358222725	0	9	5221125	1	5	5221125	0	5
26	17297280	-3647749248	0	0	452200320	0	0	17297280	0	0	17297280	0	0
27	40883535	-4637561553	1	7	570989007	0	3	40883535	3	15	40883535	0	15
28	82575360	-5821562880	0	0	722534400	0	0	82575360	0	0	82575360	0	0
29	151412625	-264205859375	1	1	52114215825	0	9	-1438808175	1	1	151412625	0	5
30	518918400	-344048090880	0	0	65833447680	0	0	-1378840320	0	0	518918400	0	0
31	1267389585	-442855631151	1	1	82524474513	0	9	-979895151	1	1	1267389585	0	5
32	2642411520	-563949338624	0	0	102789808128	0	0	0	0	0	2642411520	0	0
33	4996616625	-26469713463567	1	1	7078405640625	0	9	-516689348175	1	1	4996616625	0	5
34	17643225600	-34686740160000	0	0	9032888517120	0	0	-620425428480	0	0	17643225600	0	0

Table A.5.4. Congruences for the quadruple factorial (4–factorial) function, $n!!!! = n!_{(4)}$, modulo h (and $4h$) for $h := 2, 3, 4, 5$. Supplementary listings containing computational data for the congruences, $n!!!! \equiv R_h^{(4)}(n) \pmod{4^i h}$, for $2 \leq i \leq h \leq 5$ are tabulated in the summary notebook reference.

Table A.5. The α–factorial functions modulo h (and $h\alpha$) for $h := 2, 3, 4, 5$ defined by the special case expansions from Section 1.4.2 of the introduction and in Section 4.1 where $\widetilde{R}_p^{(\alpha)}(n) := \left[z^{\lfloor (n+\alpha-1)/\alpha \rfloor}\right] \operatorname{Conv}_p(-\alpha, n; z)$.

□

m	$\widetilde{\ell}_{m,2}(z)$	$\widetilde{p}_{m,2}(x)$
1	1	2
2	$4-3z$	$x+5$
3	$11-17z+7z^2$	$x^2+10x+24$
4	$26-62z+52z^2-15z^3$	$x^3+18x^2+96x+192$
5	$57-186z+238z^2-139z^3+31z^4$	$x^4+28x^3+264x^2+1008x+1392$
6	$120-501z+868z^2-769z^3+346z^4-63z^5$	$x^5+40x^4+580x^3+3840x^2+11880x+14520$

Table A.6.1. Generating the p^{th} power sequences, 2^p-1

m	$\ell_{m,2}(z)$	$p_{m,2}(x)$
1	1	1
2	$4-3z$	$x+4$
3	$11-17z+7z^2$	$x^2+10x+22$
4	$26-62z+52z^2-15z^3$	$x^3+18x^2+96x+156$
5	$57-186z+238z^2-139z^3+31z^4$	$x^4+28x^3+264x^2+1008x+1368$
m	$\ell_{m,3}(z)$	$p_{m,3}(x)$
1	1	1
2	$5-8z$	$x+5$
3	$18-60z+52z^2$	$x^2+12x+36$
4	$58-300z+532z^2-320z^3$	$x^3+21x^2+144x+348$
5	$179-1268z+3436z^2-4192z^3+1936z^4$	$x^4+32x^3+372x^2+1968x+4296$
m	$\ell_{m,4}(z)$	$p_{m,4}(x)$
1	1	1
2	$6-15z$	$x+6$
3	$27-141z+189z^2$	$x^2+14x+54$
4	$112-906z+2484z^2-2295z^3$	$x^3+24x^2+204x+672$
5	$453-4998z+20898z^2-39123z^3+27621z^4$	$x^4+36x^3+504x^2+3504x+10872$
m	$\ell_{m,5}(z)$	$p_{m,5}(x)$
1	1	1
2	$7-24z$	$x+7$
3	$38-272z+496z^2$	$x^2+16x+76$
4	$194-2144z+7984z^2-9984z^3$	$x^3+27x^2+276x+1164$
5	$975-14640z+82960z^2-209920z^3+199936z^4$	$x^4+40x^3+660x^2+5760x+23400$

Table A.6.2. Generating the p^{th} power sequences of binomials, 2^p-1, 3^p-1, 4^p-1, and 5^p-1

m	$\ell_{m,s+1}(z)$	$p_{m,s+1}(x)$
1	1	1
2	$3+s(1-2z)-s^2z$	$3+s(1+x)$
3	$6+s^4z^2-4s(-1+2z)+s^3z(-2+3z)$ $+s^2(1-7z+3z^2)$	$12+8s(1+x)+s^2(2+2x+x^2)$
4	$10-s^6z^3-10s(-1+2z)-s^5z^2(-3+4z)$ $+5s^2(1-5z+3z^2)-s^4z(3-13z+6z^2)$ $+s^3(1-14z+21z^2-4z^3)$	$60+60s(1+x)+15s^2(2+2x+x^2)$ $+s^3(6+6x+3x^2+x^3)$
5	$15+s^8z^4-20s(-1+2z)+s^7z^3(-4+5z)$ $+5s^2(3-13z+9z^2)+s^6z^2(6-21z+10z^2)$ $-3s^3(-2+18z-27z^2+8z^3)$ $+s^5z(-4+33z-44z^2+10z^3)$ $+s^4(1-23z+73z^2-46z^3+5z^4)$	$360+480s(1+x)+180s^2(2+2x+x^2)$ $+24s^3(6+6x+3x^2+x^3)$ $+s^4(24+24x+12x^2+4x^3+x^4)$

Table A.6.3. Generating the p^{th} power sequences, $(s+1)^p-1$

Table A.6: Convergent–based generating function identities for the binomial p^{th} power sequences enumerated by the examples in Section 4.2.6

n	$p_{n,0}(h)$	$p_{n,1}(h)$	$p_{n,2}(h)$
0	1	0	0
1	h	1	0
2	$h(h-1)^2$	$h(h-2)$	$h-1$
3	$h(h-1)^2(h-2)$	$h(h-1)(h-3)$	$h(h-3)$
4	$h(h-1)^2(h-2)^2(h-3)$	$h(h-1)(h-2)^2(h-4)$	$h(h-1)(h-3)(h-4)$
5	$h(h-1)^2(h-2)^2(h-3)(h-4)$	$h(h-1)(h-2)^2(h-3)(h-5)$	$h(h-1)(h-2)(h-4)(h-5)$
6	$h(h-1)^2(h-2)^2(h-3)^2(h-4)(h-5)$	$h(h-1)(h-2)^2(h-3)^2(h-4)(h-6)$	$h(h-1)(h-2)(h-3)^2(h-5)(h-6)$
7	$h(h-1)^2(h-2)^2(h-3)^2(h-4)(h-5)(h-6)$	$h(h-1)(h-2)^2(h-3)^2(h-4)(h-5)(h-7)$	$h(h-1)(h-2)(h-3)^2(h-4)(h-6)(h-7)$

n	$p_{n,3}(h)$	$p_{n,4}(h)$	$p_{n,5}(h)$
0	0	0	0
1	0	0	0
2	0	0	0
3	$h-1$	0	0
4	$h(h-2)(h-4)$	$(h-1)(h-2)$	0
5	$h(h-1)(h-2)(h-4)(h-5)$	$h(h-2)(h-5)$	$(h-1)(h-2)$
6	$h(h-1)(h-2)(h-4)(h-5)(h-6)$	$h(h-1)(h-3)(h-5)(h-6)$	$h(h-2)(h-3)(h-6)$
7	$h(h-1)(h-2)(h-3)(h-5)(h-6)(h-7)$	$h(h-1)(h-2)(h-5)(h-6)(h-7)$	$h(h-1)(h-3)(h-6)(h-7)$

n	$p_{n,6}(h)$	$p_{n,7}(h)$	$m_{n,h}$
0	0	0	1
1	0	0	$h-1$
2	0	0	$h-2$
3	0	0	$(h-2)(h-3)$
4	0	0	$(h-3)(h-4)$
5	0	0	$(h-3)(h-4)(h-5)$
6	$(h-1)(h-2)(h-3)$	0	$(h-4)(h-5)(h-6)$
7	$h(h-2)(h-3)(h-7)$	$(h-1)(h-2)(h-3)$	$(h-4)(h-5)(h-6)(h-7)$

Table A.7.1. The auxiliary numerator subsequences, $C_{h,n}(\alpha, R) :\mapsto \frac{(-\alpha)^n m_{n,h}}{n!} \times \sum_{i=0}^{n} \binom{n}{i} p_{n,i}(h)\,(R/\alpha)_i$, expanded by the finite–degree polynomial sequence terms defined by the Stirling number sums in (5.17.e) of Section 5.2.1.

n	m_h	$(-1)^n n! \cdot m_h^{-1} \cdot C_{h,n}(\alpha, R)$
0	1	1
1	1	$-(h-1)(R+h\alpha)$
2	$(h-2)$	$\alpha\left(2h^2-3h-1\right)R+\alpha^2(h-1)^2h+(h-1)R^2$
3	$(h-3)$	$3\alpha(h-2)\left(h^2-2h-1\right)R^2+\alpha^2(h-2)\left(3h^3-9h^2+2h-2\right)R$
		$+\alpha^3(h-2)^2(h-1)^2h+(h-2)(h-1)R^3$
4	$(h-3)(h-4)$	$\alpha^2\left(6h^4-36h^3+53h^2-9h+22\right)R^2+2\alpha^3\left(2h^5-15h^4+36h^3-36h^2+19h+6\right)R$
		$+\alpha^4(h-3)(h-2)^2(h-1)^2h+(h-2)(h-1)R^4+2\alpha(h-3)(h-2)(2h+1)R^3$
5	$(h-3)(h-4)(h-5)$	$5\alpha(h-2)\left(h^2-3h-2\right)R^4+5\alpha^2\left(2h^4-14h^3+23h^2-h+14\right)R^3$
		$+5\alpha^3\left(2h^5-18h^4+49h^3-49h^2+40h+20\right)R^2$
		$+\alpha^4\left(5h^6-55h^5+215h^4-395h^3+374h^2-72h+48\right)R$
		$+\alpha^5(h-4)(h-3)(h-2)^2(h-1)^2h+(h-2)(h-1)R^5$
6	$(h-4)(h-5)(h-6)$	$3\alpha(h-3)(h-2)\left(2h^2-7h-5\right)R^5+5\alpha^2(h-3)\left(3h^4-24h^3+44h^2+3h+34\right)R^4$
		$+5\alpha^3\left(4h^6-54h^5+256h^4-519h^3+520h^2-357h-270\right)R^3$
		$+\alpha^4\left(15h^7-240h^6+1455h^5-4335h^4+7114h^3-6129h^2+764h-1644\right)R^2$
		$+\alpha^5\left(6h^8-111h^7+826h^6-3246h^5+7378h^4-9603h^3+6478h^2-2448h-720\right)R$
		$+\alpha^6(h-5)(h-4)(h-3)^2(h-2)^2(h-1)^2h+(h-3)(h-2)(h-1)R^6$

n	m_h	$(-1)^n n! \cdot m_h^{-1} \cdot C_{h,n}(\alpha, R)$
0	1	1
1	1	$\alpha h^2+(R-\alpha)h-R$
2	$(h-2)$	$\alpha^2h^3+2\alpha h^2(R-\alpha)+h\left(\alpha^2+R^2-3\alpha R\right)-R(\alpha+R)$
3	$(h-3)$	$\alpha^3h^5+3\alpha^2h^4(R-2\alpha)+\alpha h^3\left(13\alpha^2+3R^2-15\alpha R\right)$
		$+h^2\left(-12\alpha^3+R^3-12\alpha R^2+20\alpha^2R\right)+h\left(4\alpha^3-3R^3+9\alpha R^2-6\alpha^2R\right)$
		$+2R(\alpha+R)(2\alpha+R)$
4	$(h-3)(h-4)$	$\alpha^4h^6+\alpha^3h^5(4R-9\alpha)+\alpha^2h^4\left(31\alpha^2+6R^2-30\alpha R\right)$
		$+\alpha h^3\left(-51\alpha^3+4R^3-36\alpha R^2+72\alpha^2R\right)$
		$+h^2\left(40\alpha^4+R^4-18\alpha R^3+53\alpha^2R^2-72\alpha^3R\right)$
		$+h\left(-12\alpha^4-3R^4+14\alpha R^3-9\alpha^2R^2+38\alpha^3R\right)+2R(\alpha+R)(2\alpha+R)(3\alpha+R)$
5	$(h-3)(h-4)(h-5)$	$\alpha^5h^7+\alpha^4h^6(5R-13\alpha)+\alpha^3h^5\left(67\alpha^2+10R^2-55\alpha R\right)$
		$+5\alpha^2h^4\left(-35\alpha^3+2R^3-18\alpha R^2+43\alpha^2R\right)$
		$+\alpha h^3\left(244\alpha^4+5R^4-70\alpha R^3+245\alpha^2R^2-395\alpha^3R\right)$
		$+h^2\left(-172\alpha^5+R^5-25\alpha R^4+115\alpha^2R^3-245\alpha^3R^2+374\alpha^4R\right)$
		$+h\left(48\alpha^5-3R^5+20\alpha R^4-5\alpha^2R^3+200\alpha^3R^2-72\alpha^4R\right)$
		$+2R(\alpha+R)(2\alpha+R)(3\alpha+R)(4\alpha+R)$

Table A.7.2. Alternate factored forms of the convergent numerator function subsequences, $C_{h,n}(\alpha, R) := [z^n]\,\mathrm{FP}_h(\alpha, R; z)$, gathered with respect to powers of R (or R/α scaled by α^n) and h.

n	$n! \cdot C_{h,n}(\alpha, R)$
2	$\alpha^2h^4+2\alpha h^3(R-2\alpha)+h^2\left(5\alpha^2+R^2-7\alpha R\right)-h(3R-2\alpha)(R-\alpha)+2R(\alpha+R)$
3	$-\alpha^3h^6-3\alpha^2h^5(R-3\alpha)-\alpha h^4\left(31\alpha^2+3R^2-24\alpha R\right)+h^3\left(51\alpha^3-R^3+21\alpha R^2-65\alpha^2R\right)$
	$+h^2\left(-40\alpha^3+6R^3-45\alpha R^2+66\alpha^2R\right)-h(R-\alpha)\left(12\alpha^2+11R^2-10\alpha R\right)+6R(\alpha+R)(2\alpha+R)$
4	$\alpha^4h^8+4\alpha^3h^7(R-4\alpha)+2\alpha^2h^6\left(53\alpha^2+3R^2-29\alpha R\right)+2\alpha h^5\left(-188\alpha^3+2R^3-39\alpha R^2+165\alpha^2R\right)$
	$+h^4\left(769\alpha^4+R^4-46\alpha R^3+377\alpha^2R^2-936\alpha^3R\right)-2h^3\left(452\alpha^4+5R^4-94\alpha R^3+406\alpha^2R^2-703\alpha^3R\right)$
	$+h^2\left(564\alpha^4+35R^4-302\alpha R^3+721\alpha^2R^2-1118\alpha^3R\right)-2h(R-\alpha)\left(-72\alpha^3+25R^3-17\alpha R^2+114\alpha^2R\right)$
	$+24R(\alpha+R)(2\alpha+R)(3\alpha+R)$
5	$-\alpha^5h^{10}-5\alpha^4h^9(R-5\alpha)-5\alpha^3h^8\left(54\alpha^2+2R^2-23\alpha R\right)-10\alpha^2h^7\left(-165\alpha^3+R^3-21\alpha R^2+111\alpha^2R\right)$
	$-\alpha h^6\left(6273\alpha^4+5R^4-190\alpha R^3+1795\alpha^2R^2-5860\alpha^3R\right)$
	$+h^5\left(15345\alpha^5-R^5+85\alpha R^4-1425\alpha^2R^3+8015\alpha^3R^2-18519\alpha^4R\right)$
	$+5h^4\left(-4816\alpha^5+3R^5-111\alpha R^4+1055\alpha^2R^3-4011\alpha^3R^2+7205\alpha^4R\right)$
	$-5h^3\left(-4660\alpha^5+17R^5-339\alpha R^4+1947\alpha^2R^3-5703\alpha^3R^2+8438\alpha^4R\right)$
	$+h^2\left(-12576\alpha^5+225R^5-2200\alpha R^4+7975\alpha^2R^3-22900\alpha^3R^2+26400\alpha^4R\right)$
	$-2h(R-\alpha)\left(1440\alpha^4+137R^4+7\alpha R^3+1802\alpha^2R^2-1848\alpha^3R\right)+120R(\alpha+R)(2\alpha+R)(3\alpha+R)(4\alpha+R)$

Table A.7: The auxiliary convergent numerator function subsequences, $C_{h,n}(\alpha, R) := [z^n]\,\mathrm{FP}_h(\alpha, R; z)$, defined in Section 5.2.

k	$(-1)^{h-k} z^{-(h-k)} \cdot R_{h,h-k}(\alpha; z)$
1	1
2	$-\frac{1}{2}\alpha\left(h^2-h+2\right)z+h-1$
3	$\frac{1}{2}(h-2)(h-1)-\frac{1}{2}\alpha(h-2)\left(h^2+3\right)z+\frac{1}{24}\alpha^2\left(3h^4-10h^3+21h^2-14h+24\right)z^2$
4	$-\frac{1}{4}\alpha(h-3)(h-2)\left(h^2+h+4\right)z+\frac{1}{24}\alpha^2(h-3)\left(3h^4-4h^3+19h^2-2h+56\right)z^2$ $-\frac{1}{48}\alpha^3\left(h^6-7h^5+23h^4-37h^3+48h^2-28h+48\right)z^3$ $+\frac{1}{6}(h-3)(h-2)(h-1)$
5	$-\frac{1}{12}\alpha(h-4)(h-3)(h-2)\left(h^2+2h+5\right)z+\frac{1}{48}\alpha^2(h-4)(h-3)\left(3h^4+2h^3+23h^2+16h+100\right)z^2$ $-\frac{1}{48}\alpha^3(h-4)\left(h^6-4h^5+14h^4-16h^3+61h^2-12h+180\right)z^3$ $+\frac{\alpha^4}{5760}\left(15h^8-180h^7+950h^6-2688h^5+4775h^4-5340h^3+5780h^2-3312h+5760\right)z^4$ $+\frac{1}{24}(h-4)(h-3)(h-2)(h-1)$

k	$k!(-1)^{h-k} z^{-(h-k)} \cdot R_{h,h-k}(\alpha; z)$
1	1
2	$-\alpha h^2 z+h(\alpha z+2)-2(\alpha z+1)$
3	$\frac{3}{4}\alpha^2 h^4 z^2-\frac{1}{2}\alpha h^3 z(5\alpha z+6)+\frac{3}{4}h^2\left(7\alpha^2 z^2+8\alpha z+4\right)$ $+\frac{1}{2}h\left(-7\alpha^2 z^2-18\alpha z-18\right)+6\left(\alpha^2 z^2+3\alpha z+1\right)$
4	$-\frac{1}{2}\alpha^3 h^6 z^3+\frac{1}{2}\alpha^2 h^5 z^2(7\alpha z+6)-\frac{1}{2}\alpha h^4 z\left(23\alpha^2 z^2+26\alpha z+12\right)$ $+\frac{1}{2}h^3\left(37\alpha^3 z^3+62\alpha^2 z^2+48\alpha z+8\right)$ $+h^2\left(-24\alpha^3 z^3-59\alpha^2 z^2-30\alpha z-24\right)$ $+2h\left(7\alpha^3 z^3+31\alpha^2 z^2+42\alpha z+22\right)-24\left(\alpha^3 z^3+7\alpha^2 z^2+6\alpha z+1\right)$
5	$\frac{5}{16}\alpha^4 h^8 z^4-\frac{5}{4}\alpha^3 h^7 z^3(3\alpha z+2)+\frac{5}{24}\alpha^2 h^6 z^2\left(95\alpha^2 z^2+96\alpha z+36\right)$ $-\frac{1}{2}\alpha h^5 z\left(112\alpha^3 z^3+150\alpha^2 z^2+95\alpha z+20\right)$ $+\frac{5}{48}h^4\left(955\alpha^4 z^4+1728\alpha^3 z^3+1080\alpha^2 z^2+672\alpha z+48\right)$ $-\frac{5}{4}h^3\left(89\alpha^4 z^4+250\alpha^3 z^3+242\alpha^2 z^2+104\alpha z+40\right)$ $+\frac{5}{12}h^2\left(289\alpha^4 z^4+1536\alpha^3 z^3+1584\alpha^2 z^2+408\alpha z+420\right)$ $+h\left(-69\alpha^4 z^4-570\alpha^3 z^3-1270\alpha^2 z^2-820\alpha z-250\right)$ $+120\left(\alpha^4 z^4+15\alpha^3 z^3+25\alpha^2 z^2+10\alpha z+1\right)$

Table A.8: The auxiliary convergent numerator function subsequences, $R_{h,k}(\alpha; z) := [R^k]\,\mathrm{FP}_h(\alpha, R; z)$, defined by Section 5.2.1.