Jacobi Type Continued Fractions for the Ordinary Generating Functions of Generalized Factorial Functions

Maxie D. Schmidt
maxieds@gmail.com

Draft Last Updated: Wednesday 12th August, 2015 at 00:01.

Abstract

The article studies a class of generalized factorial functions through Jacobi type continued fractions (J-fractions) that formally enumerate the ordinary generating functions of these sequences. These continued fractions extend the known J-fraction expansions generating the rising factorial function, or Pochhammer symbol, $(x)_n$, at fixed indeterminates $x \in \mathbb{C}$. The factorial-related product sequences generated by the convergents to these generalized J-fractions defined by the article are related to special cases of the Gould polynomials and to the more general forms of the Pochhammer k-symbols, $(x)_{n,\alpha}$. The results established by the article provide new exact formulas and integer congruences modulo any $p \geq 2$ satisfied by these symbolic products, and the corresponding generalized α-factorial functions. These new results provide formulas expanded through the zeros of the confluent hypergeometric function and associated Laguerre polynomial sequences already studied in the references.

The article also provides a number of specific identities, classical congruence properties, and other motivating examples as immediate applications of the new results proved in the next subsections. The particular applications to factorial-related congruences in variants of Wilson's theorem extend the identities previously considered in the context of multiple (j-factorial) functions and their relations to generalized Stirling number triangles and Stirling polynomial sequences. The rational convergent–based generating function techniques illustrated by the special case examples cited within the article are easily extended to enumerate the factorial-like product sequences arising in the context of many other specific applications.

Keywords and Phrases: Continued fraction, J-fraction, S-fraction, congruence, factorial, multifactorial, multiple factorial, double factorial, superfactorial, rising factorial, Pochhammer symbol, Pochhammer k-symbol, Barnes G-function, hyperfactorial, triple factorial, quadruple factorial, quintuple factorial, Stirling number, Stirling number of the first kind, Wilson's theorem, Clement's theorem, Wolstenholme's theorem, confluent hypergeometric function, associated Laguerre polynomial; ordinary generating function, diagonal generating function, Hadamard product, divergent ordinary generating function;

Sequence Number References:
A000142; A000165; A000178; A000225; A000407; A000918; A000984; A001008; A001044; A001097; A001147; A002109; A002144; A002805; A003422; A005165; A006882; A007406; A007407; A007408; A007409; A007540; A007559; A007661; A007662; A007696; A008277; A008292; A008554; A024023; A024036; A024049; A027641; A027642; A032031; A033312; A034176; A047053; A061062; A085157; A085158; A094638; A104344; A157250; A184877.

Mathematics Subject Classifications:

1 Introduction

The focus of the new results established by this article is in enumerating properties of sequences generated by convergents to continued fraction representations for the ordinary generating function of the distinct polynomial expansions defined as

$$\begin{aligned} p_n(\alpha, R) &:= \prod_{j=0}^{n-1} (R + \alpha j) + [n = 0]_\delta \qquad (1.1) \\ &= R(R+\alpha)(R+2\alpha) \times \cdots \times (R + (n-1)\alpha), \ n \geq 1. \end{aligned}$$

Related integer sequences of interest in the applications of this article are defined recursively for any fixed $\alpha \in \mathbb{Z}^+$ and $n \in \mathbb{N}$ by the following equation:

$$n!_{(\alpha)} = \begin{cases} n \cdot (n-\alpha)!_{(\alpha)} & \text{if } n > 0 \\ 1 & \text{if } -\alpha < n \leq 0. \end{cases} \tag{1.2}$$

We are especially interested in using the new results established in this article to formally enumerate the sequences of factorial–function–like products, $p_n(\alpha, \beta n + \gamma)$, for some fixed parameters $\alpha, \beta, \gamma \in \mathbb{Q}$ when the symbolic indeterminate, R, depends linearly on n. These particular forms of the generalized product sequences of interest in the applications of this article are related to the *Gould polynomials*, $G_n(x; a, b) = \frac{x}{x-an} \cdot \left(\frac{x-an}{b}\right)_n$, in the form of [21, §3.4.2] [13; 20]

$$p_n\left(\alpha, \beta n + \gamma\right) = \frac{\alpha^n \cdot (\beta n + \gamma)}{\gamma} \times G_n\left(\gamma; \alpha, -\beta\right). \tag{1.3}$$

The generalized product sequences in (1.1) also correspond to the definition of the *Pochhammer k–symbol*, $p_n\left(\alpha, R\right) \equiv \left(R\right)_{n,\alpha}$, or equivalently, $p_n\left(k, x\right) \equiv \left(x\right)_{n,k}$, given in [8] for any fixed $k \neq 0$ and non–zero indeterminates, $x \in \mathbb{C}$. The new identities involving the generalized multiple factorial, or α–factorial, functions based on the symbolic products defined by (1.1) extend the study of these sequences motivated through the polynomial expansions originally considered in [21].

The particular new results studied within the article generalize known series defined through continued fraction representations of the combinatorial sequences proved in [9; 10], including several series enumerating the *rising factorial function*, $x^{\overline{n}} \equiv (-1)^n (-x)^{\underline{n}}$, or *Pochhammer symbol*, $(x)_n = x(x+1)(x+2)\cdots(x+n-1)$, and other related sequences involving the classical single and double factorial functions, $n!$ and $n!!$, and the Stirling numbers of the first kind. Whereas the first results proved in [9; 10] are focused on establishing properties of divergent forms of the ordinary generating functions for a number of special sequence cases through more combinatorial interpretations of these continued fraction series, the emphasis in this article is more enumerative in flavor. This article extends a number of the examples considered as applications of the results from the 2010 article summarized in the next section [21].

1.1 Polynomial Expansions of Generalized α–Factorial Functions

For any fixed integer $\alpha \geq 1$ and $n, k \in \mathbb{N}$, the coefficients defined by the triangular recurrence relation in (1.4) provides one approach to enumerating the symbolic expansions of the generalized factorial function products defined as special cases of (1.1).

$$\begin{bmatrix} n \\ k \end{bmatrix}_\alpha = (\alpha n + 1 - 2\alpha) \begin{bmatrix} n-1 \\ k \end{bmatrix}_\alpha + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_\alpha + [n = k = 0]_\delta \tag{1.4}$$

The combinatorial interpretations of these coefficients motivated in [21] leads to the polynomial expansions of the multiple factorial function sequence variants in (1.2) that generalize the forms of the single factorial function, $n!$, given in terms of the *Stirling numbers of the first kind*, $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] \equiv \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_1 \equiv (-1)^{n-k} s(n,k)$, through the next formula [13, §6] [19, §26.8] [22, A094638; A008277]:

$$n! = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} (-1)^{n-m} n^m, \ \forall n \geq 1.$$

The polynomial expansions of the generalized α–factorial functions, $(\alpha n-d)!_{(\alpha)}$, for fixed $\alpha \in \mathbb{Z}^+$ and $0 \leq d < \alpha$, are obtained similarly from (1.4) as follows [21, *cf.* §2][1]:

$$(\alpha n-d)!_{(\alpha)} = (\alpha-d) \times \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_{\alpha} (-1)^{n-m}(\alpha n+1-d)^{m-1}, \ \forall n \geq 1, \alpha \in \mathbb{Z}^+, 0 \leq d < \alpha. \tag{1.5}$$

A binomial–coefficient–themed phrasing of the products underlying the expansions of the more general factorial sequences of this type (each formed by dividing through by a normalizing factor of $k!$) is suggested by the alternate expansions of these coefficients through the Pochhammer symbol, $(x)_k$, provided below [13, §5]:

$$\binom{\frac{s-1}{\alpha}}{k} = \frac{(-1)^k}{k!} \cdot \left(\frac{s-1}{\alpha}\right)_k = \frac{1}{\alpha^k \cdot k!} \prod_{j=0}^{k-1} (s-1-\alpha j). \tag{1.6}$$

Whenever $\alpha \mid s-1$, we have similar formulas relating the symbolic expansions of these binomial coefficients result in the finite sums implicit to applying the binomial theorem to powers of $(a+b)^{(s-1)/\alpha}$ for fixed constants, a and b. Additionally, when the indeterminate $s \equiv s_k$ in the previous expansions is taken modulo α to be $s_k := \alpha k + d$ at some residue $0 \leq d < \alpha$, the prescribed setting of the single offset d completely determines the numerical α–factorial function sequences of the forms in (1.5) generated by these products.

For any lower index $k \geq 1$, the binomial coefficient formulation for the multiple factorial function cases provided by (1.6) leads to the next several expansions by the exponential generating functions for the generalized coefficient triangles in (1.4), and their corresponding Stirling polynomial analogs, $\sigma_n^{(\alpha)}(x)$, given as in the references [21, §5] [13, *cf.* §6, §7.4][2]:

$$\begin{aligned} \binom{\frac{s-1}{\alpha}}{k} &= \sum_{m=0}^{k} \begin{bmatrix} k+1 \\ k+1-m \end{bmatrix}_{\alpha} \frac{(-1)^m s^{k-m}}{\alpha^k k!} && (1.7) \\ &= \sum_{m=0}^{k} \frac{(-1)^m \cdot (k+1)\sigma_m^{(\alpha)}(k+1)}{\alpha^m} \times \frac{(s/\alpha)^{k-m}}{(k-m)!} \\ \binom{\frac{s-1}{\alpha}}{k} &= [z^k] \left\{ e^{(s-1+\alpha)z/\alpha} \left(\frac{-ze^{-z}}{e^{-z}-1}\right)^{k+1} \right\} && (1.8) \\ &= [z^k w^k] \left\{ -\frac{z \cdot e^{(s-1+\alpha)z/\alpha}}{1+wz-e^z} \right\}. \end{aligned}$$

A more extensive treatment of the properties and generating function relations satisfied by the triangular coefficients defined by (1.4) above, including their similarities to the Stirling number triangles, Stirling polynomial sequences, and the generalized Bernoulli polynomials, among relations to several other notable special function sequence cases, is provided in the references [21, *cf.* §5].

1.2 Divergent Ordinary Generating Functions Approximated by the Convergents to Infinite Continued Fractions

Another approach to enumerating the symbolic expansions of the α–factorial function sequences outlined above is constructed as a new generalization of the continued fraction series representations of the ordinary generating

[1] The symbolic polynomial expansions of the products, $p_n(-\alpha, s-1) \equiv (s-1)_{n,-\alpha}$, with respect to the indeterminate, $s \neq 0$, correspond to the first motivations behind constructing the generalized triangles defined in (1.4) originally considered in the reference [21] (2010). In particular, for all $n \geq 0$, $\alpha \in \mathbb{Z}^+$, and symbolic indeterminate parameters, $s \neq 0$, the first article studies the expansions of these products given in the forms of

$$\begin{aligned} (s-1)_{n,-\alpha} &= (s-1)(s-1-\alpha)(s-1-2\alpha)\cdots(s-1-(n-1)\alpha) \\ &= \sum_{m=0}^{n} \begin{bmatrix} n+1 \\ m+1 \end{bmatrix}_{\alpha} (-1)^{n-m} s^m, \end{aligned}$$

where the symbolic sums, $(x)_{n,k} = \sum_{m=0}^{n} \begin{bmatrix} n+1 \\ m+1 \end{bmatrix}_{k} (x-1)^m$, corresponds to the Pochhammer k–symbol defined as in the reference [8].

[2] The sequences of generalized *α–factorial polynomials*, $\sigma_n^{(\alpha)}(x)$, studied in the reference [21, §5] are defined as the coefficients of the Stirling–polynomial–related generating function, $x \cdot \sigma_n^{(\alpha)}(x) := [z^n] \left\{ e^{(1-\alpha)z} \left(\frac{\alpha z e^{\alpha z}}{e^{\alpha z}-1}\right)^x \right\}$. These polynomials are alternately defined through the triangles in (1.4) as $\sigma_n^{(\alpha)}(x) \equiv \begin{bmatrix} x \\ x-n \end{bmatrix}_{\alpha} (x-n-1)!/x!$ [21, §5.2].

function for the Pochhammer symbol, $(x)_n = \Gamma(x+n)/\Gamma(x)$, proved by Flajolet in [9; 10]. For any fixed indeterminate $x \in \mathbb{C}$, the ordinary power series enumerating this sequence is through the next infinite J–fraction [9, Prop. 9]:

$$R_0(x,z) := \sum_{n\geq 0} (x)_n z^n = \cfrac{1}{1 - xz - \cfrac{1\cdot xz^2}{1-(x+2)z - \cfrac{2(x+1)z^2}{\cdots}}}. \tag{1.9}$$

Since we know symbolic polynomial expansions of the functions, $(x)_n$, through the Stirling numbers of the first kind, we notice that the terms in a convergent power series defined by (1.9) correspond to the coefficients of the well–known "*double*", or "*super*", *generating functions* for the Stirling number triangle when x is taken to be a fixed, formal parameter with respect to these sequences [13, §7.4] [19, §26.8(ii)].

When x depends linearly on n, the ordinary generating functions for the numerical sequences formed by $(x)_n$ do not converge for $z \neq 0$. However, the convergents of the continued fraction representation of these series still lead to partial, truncated series approximations enumerating these generalized product sequences, which in turn immediately satisfy a number of established combinatorial properties, recurrence relations, and other integer congruence properties implied by the first continued fraction expansion in (1.9). Moreover, if we set aside convergence concerns in the limiting case on the right–hand–side of (1.9), for each finite $h \geq 2$, the h^{th} convergents to the infinite continued fraction may be treated as a rational function in z forming the ordinary power series generating the expected approximate sequence terms.

Two particular divergent ordinary generating functions for the single factorial function sequences, $f_1(n) := n!$ and $f_2(n) := (n+1)!$, are cited in [9; 10] and [17, *cf.* §5.5]. as examples of the *Jacobi–type continued fraction* (*J–fraction*) results proved in [9; 10]. The next pair of series serve to illustrate the utility required here to enumerating each sequence formally with respect to z [9, Thm. 3A; Thm. 3B]:

$$F_{1,\infty}(z) := \sum_{n\geq 0} n! \cdot z^n = \cfrac{1}{1 - z - \cfrac{1^2\cdot z^2}{1-3z - \cfrac{2^2 z^2}{\cdots}}} \tag{Single Factorial J-Fractions}$$

$$F_{2,\infty}(z) := \sum_{n\geq 0} (n+1)! \cdot z^n = \cfrac{1}{1 - 2z - \cfrac{1\cdot 2z^2}{1-4z - \cfrac{2\cdot 3z^2}{\cdots}}}.$$

In each of these respective series, we see that for each finite $h \geq 1$, the h^{th} convergent functions, denoted by $F_{i,h}(z)$ for $i = 1, 2$, satisfy $f_i(n) = [z^n]F_{i,h}(z)$ whenever $1 \leq n \leq h$. We also see that $[z^n]F_{i,h}(z) \equiv f_i(n) \pmod{p}$ for any $n \geq 0$ whenever p is a divisor of h [10] [17, *cf.* §5].

Similar examples of factorial–function–like series are given in the references [9, Thm. 2] [17, *cf.* §5.9]. For example, another known *Stieltjes–type continued fraction* (*S–fraction*), formally generating the double factorial function, $(2n-1)!! = 1 \cdot 3 \cdots (2n-1)$, is expanded through convergents of the infinite continued fraction

$$\sum_{n\geq 0} (2n-1)!! \cdot z^{2n} = \cfrac{1}{1 - \cfrac{1\cdot z^2}{1 - \cfrac{2\cdot z^2}{1 - \cfrac{3\cdot z^2}{\cdots}}}}. \tag{Double Factorial S-Fraction}$$

For comparison, some related forms of regularized ordinary power series in z generating the single and double factorial function sequences from the previous examples are stated in terms of the *incomplete gamma function*, $\Gamma(a,z) = \int_z^\infty t^{a-1}e^{-t}dt$, as follows [19, §8; *cf.* §18.5–18.6] [3]:

$$\sum_{n\geq 0} n! \cdot z^n = -\frac{e^{-1/z}}{z}\Gamma\left(0, -\frac{1}{z}\right)$$

[3] For $\alpha \in \mathbb{Z}^+$ and $0 \leq r < \alpha$, we have the identity, $(\alpha n - r)!_{(\alpha)} = p_\alpha(\alpha - r, \equiv)\, \alpha^n \left(1 - \frac{r}{\alpha}\right)_n$, where the exponential generating function for the generalized product sequences, $p_n(\alpha, R)$, corresponds to the series $(1-\alpha z)^{-R/\alpha}$ [13, *cf.* §7, (7,55)], which leads to the

$$\begin{aligned}\sum_{n\geq 0}(n+1)!\cdot z^n &= -\frac{e^{-1/z}}{z^2}\Gamma\left(-1,-\frac{1}{z}\right)\\ \sum_{n\geq 1}(2n-1)!!\cdot z^n &= -\frac{e^{-1/2z}}{(-2z)^{3/2}}\Gamma\left(-\frac{1}{2},-\frac{1}{2z}\right).\end{aligned} \tag{1.10}$$

The remarks given in Section 3.3 suggest similar approximations to the generalized α–factorial functions generated by the general convergent functions defined in the next section, and their relations to the confluent hypergeometric function and associated Laguerre polynomial sequences [19, *cf.* §18.5(ii)] [20].

1.3 Generalized Convergent Functions for Factorial–Related Sequences

We state the next definition to generalize Flajolet's original result cited in (1.9) to the analogous series satisfied by the expansions of the multiple factorial function sequences defined as special cases of the symbolic product sequences in (1.1).

Definition 1.1 (Generalized J-Fraction Convergent Functions)**.** Suppose that the parameters $\alpha \in \mathbb{Z}^+$ and $R \equiv R(n)$ are defined in the notation of the product–wise sequences from (1.1). For $h \geq 0$ and $z \in \mathbb{C}$, let the component numerator and denominator convergent functions, denoted by $\mathrm{FP}_h(\alpha, R; z)$ and $\mathrm{FQ}_h(\alpha, R; z)$, respectively, be defined by the following recurrence relations:

$$\mathrm{FP}_h(\alpha,R;z) := \begin{cases} (1-(R+2\alpha(h-1))z)\,\mathrm{FP}_{h-1}(\alpha,R;z)-\alpha(R+\alpha(h-2))(h-1)z^2\,\mathrm{FP}_{h-2}(\alpha,R;z) & h\geq 2\\ 1 & h=1\\ 0 & h<1;\end{cases} \tag{1.11}$$

$$\mathrm{FQ}_h(\alpha,R;z) := \begin{cases} (1-(R+2\alpha(h-1))z)\,\mathrm{FQ}_{h-1}(\alpha,R;z)-\alpha(R+\alpha(h-2))(h-1)z^2\,\mathrm{FQ}_{h-2}(\alpha,R;z) & h\geq 2\\ 1-Rz & h=1\\ 1 & h=0\\ 0 & h<0.\end{cases} \tag{1.12}$$

The series defined by (1.13) below gives a partial series for the h^{th} convergent approximating the infinite continued fraction such that for sufficiently large h the series coefficients correspond to the product form in (1.1).

$$\mathrm{Conv}_h(\alpha,R;z) = \cfrac{1}{1-R\cdot z-\cfrac{\alpha R\cdot z^2}{1-(R+2\alpha)\cdot z-\cfrac{2\alpha(R+\alpha)\cdot z^2}{1-(R+4\alpha)\cdot z-\cfrac{3\alpha(R+2\alpha)\cdot z^2}{\cfrac{\cdots}{1-(R+2(h-1)\alpha)\cdot z}}}}}$$

$$\mathrm{Conv}_h(\alpha,R;z) := \frac{\mathrm{FP}_h(\alpha,R;z)}{\mathrm{FQ}_h(\alpha,R;z)} = \sum_{n=0}^{2h-1} p_n(\alpha,R)z^n + \sum_{n=2h}^{\infty} \widetilde{e}_{h,n}(\alpha,R)z^n \tag{1.13}$$

The series coefficients on the right–hand–side of (1.13) enumerate the products, $p_n(\alpha, R)$, from (1.1), where the remaining forms of the power series coefficients, $\widetilde{e}_{h,n}(\alpha, R)$, correspond to "error terms" in the truncated series approximations to the exact sequence generating functions obtained from these convergent functions. ——— Ⓓ

A number of the immediate, noteworthy properties satisfied these convergent functions, $\mathrm{Conv}_h(\alpha, R; z)$, are apparent from inspection of the first several special cases provided in Table 1 (page 39). The most important of these properties in the new interpretations of the α–factorial function sequences proved in later sections of the article are briefly summarized in the next points given below.

next forms of these regularized sums given by

$$\int_0^\infty e^{-t}\left\{\sum_{n\geq 0}(\alpha n-r)!_{(\alpha)}\frac{(tz)^n}{n!}\right\}dt = \int_0^\infty \frac{e^{-t}}{(1-\alpha tz)^{1-r/\alpha}}dt = \frac{e^{-\frac{1}{\alpha z}}}{(-\alpha z)^{1-r/\alpha}}\times\Gamma\left(\frac{r}{\alpha},-\frac{1}{\alpha z}\right).$$

1. **Rationality of the Convergent Functions**:
For any fixed $h \geq 1$, it is easy to show that the convergent functions, $\mathrm{FP}_h(z)$ and $\mathrm{FQ}_h(z)$, defined by (1.11) and (1.12), respectively, are polynomials of finite degree in each of z, R, and α satisfying
$$\deg_{z,R,\alpha}\{\mathrm{FP}_h(\alpha, R; z)\} = h - 1 \quad \text{and} \quad \deg_{z,R,\alpha}\{\mathrm{FQ}_h(\alpha, R; z)\} = h.$$
For any $h, n \in \mathbb{Z}^+$, if $R \equiv R(n)$ corresponds to some linear function of n, the product sequences, $p_n(\alpha, R)$, generated by the generalized convergent functions always correspond to polynomials in n (in R) of predictably finite degree with integer coefficients determined by the choice of $n \geq 1$ in Section 5.3. The rationality of these convergent functions in z also provides the applications of the diagonal, or Hadamard product, generating functions given in Section 4.2 [7; 17; 24], as well as the applications of the h–order finite difference equation expansions given in Section 5.3.

2. **Exact Representations by Special Functions**:
For all $h \geq 0$ and fixed parameters $\alpha, R \neq 0$, the power series in z generated by the generalized forms of the h^{th} convergents, $\mathrm{Conv}_h(\alpha, R; z)$, are characterized by the representations of the convergent denominator functions, $\mathrm{FQ}_h(\alpha, R; z)$, by the confluent hypergeometric function, $U(a, b, z)$, and through the associated Laguerre polynomial sequences provided as follows [19, §13; §18]:
$$\begin{aligned} z^h \cdot \mathrm{FQ}_h\left(\alpha, R; z^{-1}\right) &= \alpha^h \times U\left(-h, R/\alpha, \frac{z}{\alpha}\right) \\ &= (-\alpha)^h \cdot h! \times L_h^{(R/\alpha-1)}\left(\frac{z}{\alpha}\right). \end{aligned} \tag{1.14}$$
These special function expansions of the convergent denominator functions lead to the statements of addition theorems, multiplication theorems, and several additional recurrence relations for these sequences proved in Section 5.1.

3. **Exact Sequence Formulas and Congruence Properties**:
If some ordering of the h zeros of (1.14) is fixed at each $h \geq 2$, we can define the next sequences formed as special cases the zeros studied by the references [4; 11]. In particular, each of the following sequences definitions provide factorizations over z of the denominator sequences, $\mathrm{FQ}_h(\alpha, R; z)$, parametrized by α and R:
$$\begin{aligned} \{\ell_{h,j}(\alpha, R)\}_{j=1}^h &:= \left\{z_j : \alpha^h \times U\left(-h, R/\alpha, \frac{z}{\alpha}\right) = 0,\ 1 \leq j \leq h\right\} \\ &= \left\{z_j : \alpha^h \times L_h^{(R/\alpha-1)}\left(\frac{z}{\alpha}\right) = 0,\ 1 \leq j \leq h\right\}. \end{aligned} \qquad \text{(Special Function Zero Sets)}$$
Let the sequences, $c_{h,j}(\alpha, R)$, denote a shorthand for the coefficients corresponding to an expansion of the convergent functions, $\mathrm{Conv}_h(\alpha, R; z)$, by partial fractions in z [19, §1.2(iii)].

For $n \geq 1$ and a fixed integer $\alpha \neq 0$, these convergent functions provide the following formulas exactly generating the each of the sequence cases defined in (1.1) and (1.2) above:
$$\begin{aligned} p_n(\alpha, R) &= \sum_{j=1}^n c_{n,j}(\alpha, R) \times \ell_{n,j}(\alpha, R)^n \\ n!_{(\alpha)} &= \sum_{j=1}^n c_{n,j}(-\alpha, n) \times \ell_{n,j}(-\alpha, n)^{\lfloor \frac{n-1}{\alpha} \rfloor + 1}. \end{aligned} \tag{1.15}$$
The corresponding congruences satisfied by these generalized sequence cases are stated similarly modulo any prescribed integers $p \geq 2$ in the forms of
$$\begin{aligned} p_n(\alpha, R) &\equiv \sum_{j=1}^p c_{p,j}(\alpha, R) \times \ell_{p,j}(\alpha, R)^n && \pmod{p, p\alpha} \\ n!_{(\alpha)} &\equiv \sum_{j=1}^p c_{p,j}(-\alpha, n) \times \ell_{p,j}(-\alpha, n)^{\lfloor \frac{n-1}{\alpha} \rfloor + 1} && \pmod{p, p\alpha}, \end{aligned} \tag{1.16}$$

where the notation $g_1(n) \equiv g_2(n) \pmod{n_1, n_2}$ is taken to mean that the congruence may be taken modulo either base, n_1 or n_2. Section 1.4.2 and Section 4.1 provide specific examples of the new congruence properties expanded in the forms of (1.16) for these sequences.

1.4 Examples of the New Results

1.4.1 Factorial–Related Sequences Enumerated by the Generalized Convergent Functions

There are a couple of noteworthy subtleties that arise in defining the specific numerical forms of the α–factorial function sequences suggested in (1.5). Since the convergent functions only enumerate the distinct symbolic products that characterize the forms of these expansions, we see that the following convergent–based enumerations of the multiple factorial sequence variants hold at each $n \in \mathbb{Z}^+$, and for any prescribed offset, $0 \leq d < \alpha$:

$$\begin{aligned} (\alpha n - d)!_{(\alpha)} &= [z^n] \operatorname{Conv}_n(-\alpha, \alpha n - d; z) = (-\alpha)^n \cdot \left(\frac{d}{\alpha} - n\right)_n \\ &= [z^n] \operatorname{Conv}_n(\alpha, \alpha - d; z) \qquad = \alpha^n \cdot \left(1 - \frac{d}{\alpha}\right)_n . \end{aligned} \tag{1.17}$$

For example, some of the sequence variants formed by the single factorial and double factorial functions, $n!$ and $n!!$, respectively, are generated by the particular shifted arithmetic progressions highlighted by the special cases given as in the next few equations [22, A000142; A001147, A000165]:

$$\begin{aligned} \{n!\}_{n=1}^{\infty} = \{(1)_n\}_{n=1}^{\infty} &\xrightarrow{\text{A000142}} \{1, 2, 6, 24, 120, 720, 5040, \ldots\} \\ \{(2n)!!\}_{n=1}^{\infty} = \{2^n \cdot (1)_n\}_{n=1}^{\infty} &\xrightarrow{\text{A001147}} \{2, 8, 48, 384, 3840, 46080, \ldots\} \\ \{(2n-1)!!\}_{n=1}^{\infty} = \{2^n \cdot (1/2)_n\}_{n=1}^{\infty} &\xrightarrow{\text{A000165}} \{1, 3, 15, 105, 945, 10395, \ldots\}. \end{aligned}$$

Additional related cases of the generalized multiple factorial function sequences are generated similarly from (1.17) through the distinct symbolic products defined by (1.1). The next several special cases corresponding to $\alpha := 3, 4$ are given by [22, A032031, A008554, A007559; A034176, A000407, A007696, A047053]

$$\begin{aligned} \{(3n)!!!\}_{n=1}^{\infty} = \{3^n \cdot (1)_n\}_{n=1}^{\infty} &\xrightarrow{\text{A032031}} \{3, 18, 162, 1944, 29160, \ldots\} \\ \{(3n-1)!!!\}_{n=1}^{\infty} = \{3^n \cdot (2/3)_n\}_{n=1}^{\infty} &\xrightarrow{\text{A008554}} \{2, 10, 80, 880, 12320, 209440, \ldots\} \\ \{(3n-2)!!!\}_{n=1}^{\infty} = \{3^n \cdot (1/3)_n\}_{n=1}^{\infty} &\xrightarrow{\text{A007559}} \{1, 4, 28, 280, 3640, 58240, \ldots\} \\ \left\{(4n)!_{(4)}\right\}_{n=0}^{\infty} = \{4^n \cdot (1)_n\}_{n=0}^{\infty} &\xrightarrow{\text{A047053}} \{1, 4, 32, 384, 6144, 122880, \ldots\} \\ \left\{(4n+1)!_{(4)}\right\}_{n=0}^{\infty} = \{4^n \cdot (5/4)_n\}_{n=0}^{\infty} &\xrightarrow{\text{A007696}} \{1, 5, 45, 585, 9945, 208845, \ldots\} \\ \left\{\tfrac{1}{2}(4n+2)!_{(4)}\right\}_{n=0}^{\infty} = \{4^n \cdot (3/2)_n\}_{n=0}^{\infty} &\xrightarrow{\text{A000407}} \{1, 6, 60, 840, 15120, 332640, \ldots\} \\ \left\{\tfrac{1}{3}(4n+3)!_{(4)}\right\}_{n=0}^{\infty} = \{4^n \cdot (7/4)_n\}_{n=0}^{\infty} &\xrightarrow{\text{A034176}} \{1, 7, 77, 1155, 21945, 504735, \ldots\}. \end{aligned}$$

Likewise, given any $n \geq 1$ and fixed $\alpha \in \mathbb{Z}^+$, we can generate the less obvious forms of the full α–factorial function sequences defined case–wise for $n \in \{0, 1, \ldots, \alpha - 1\} \pmod{\alpha}$ by (1.2) through the multi–valued products given by (1.1) as follows:

$$\begin{aligned} n!_{(\alpha)} &= \left[z^{\lfloor (n+\alpha-1)/\alpha \rfloor}\right] \operatorname{Conv}_n(-\alpha, n; z) \\ &= [z^n] \left\{ \sum_{0 \leq d < \alpha} z^{-d} \cdot \operatorname{Conv}_n(\alpha, \alpha - d; z^{\alpha}) \right\}. \end{aligned} \tag{1.18}$$

For example, the complete sequence forms of the multi-valued products formed by the special cases of the double factorial, *triple factorial*, $n!!!$, *quadruple factorial*, $n!!!!$ or $n!_{(4)}$, *quintuple factorial*, or 5–*factorial*, $n!_{(5)}$, and the 6–*factorial*, $n!_{(6)}$, functions, respectively, are generated as in the following equations [22, A006882; A007661; A007662;

A085157; A085158]:

$$\begin{aligned}
\{n!!\}_{n=1}^{\infty} &= \left\{\left[z^{\lfloor (n+1)/2\rfloor}\right] \mathrm{Conv}_n\left(-2, n; z\right)\right\}_{n=1}^{\infty} & \xrightarrow{\text{A006882}} \quad & \{1, 2, 3, 8, 15, 48, 105, 384, 945, 3840, \ldots\} \\
\{n!!!\}_{n=1}^{\infty} &= \left\{\left[z^{\lfloor (n+2)/3\rfloor}\right] \mathrm{Conv}_n\left(-3, n; z\right)\right\}_{n=1}^{\infty} & \xrightarrow{\text{A007661}} \quad & \{1, 2, 3, 4, 10, 18, 28, 80, 162, 280, \ldots\} \\
\left\{n!_{(4)}\right\}_{n=1}^{\infty} &= \left\{\left[z^{\lfloor (n+3)/4\rfloor}\right] \mathrm{Conv}_n\left(-4, n; z\right)\right\}_{n=1}^{\infty} & \xrightarrow{\text{A007662}} \quad & \{1, 2, 3, 4, 5, 12, 21, 32, 45, 120, 231, \ldots\} \\
\left\{n!_{(5)}\right\}_{n=1}^{\infty} &= \left\{\left[z^{\lfloor (n+4)/5\rfloor}\right] \mathrm{Conv}_n\left(-5, n; z\right)\right\}_{n=1}^{\infty} & \xrightarrow{\text{A085157}} \quad & \{1, 2, 3, 4, 5, 6, 14, 24, 36, 50, 66, 168, \ldots\} \\
\left\{n!_{(6)}\right\}_{n=1}^{\infty} &= \left\{\left[z^{\lfloor (n+5)/6\rfloor}\right] \mathrm{Conv}_n\left(-6, n; z\right)\right\}_{n=1}^{\infty} & \xrightarrow{\text{A085158}} \quad & \{1, 2, 3, 4, 5, 6, 7, 16, 27, 40, 55, 72, 91, \ldots\}.
\end{aligned}$$

For each $n \in \mathbb{N}$ and prescribed constants $r, c \in \mathbb{Z}$ defined such that $c \mid n + r$, we also obtain convergent functions generating the modified multiple factorial function sequences given by

$$\left(\frac{n+r}{c}\right)! = [z^n] \, \mathrm{Conv}_h\left(-c, n + r; \frac{z}{c}\right) + \left[\frac{r}{c} = 0\right]_{\delta} [n = 0]_{\delta}, \ \forall \, h \geq \lfloor (n + r)/c \rfloor.$$

The first particular form of the single factorial function is considered in [21, §6.16] as an example of the product–based symbolic factorial function expansions in Wilson's theorem, which is also considered separately as an application of the new results given below.

1.4.2 New Congruences for the α–Factorial Functions

For any fixed $\alpha \in \mathbb{Z}^+$ and natural numbers $n \geq 1$, the generalized multiple, α–factorial functions, $n!_{(\alpha)}$, defined by (1.2) satisfy the following congruence relations modulo 2 and 2α:

$$n!_{(\alpha)} \equiv \frac{1}{2} n \left(\left(-\alpha + \sqrt{\alpha(\alpha - n)} + n\right)^{\left\lfloor \frac{n-1}{\alpha} \right\rfloor} + \left(-\alpha - \sqrt{\alpha(\alpha - n)} + n\right)^{\left\lfloor \frac{n-1}{\alpha} \right\rfloor}\right) \pmod{2, 2\alpha}. \tag{1.19}$$

Given that the definition of the single factorial function implies that $n! \equiv 0 \pmod{2}$ whenever $n \geq 2$, the statement of (1.19) provides somewhat less obvious results for the generalized sequence cases when $\alpha \geq 2$. The corresponding, closely–related new forms of congruence properties satisfied expansions of these functions through analogous exact algebraic formulas modulo 3 (3α) and modulo 4 (4α) are also cited as special cases in the next examples.

To simplify notation, we define shorthand for the respective (distinct) polynomial zeros, denoted by $r_{p,i}^{(\alpha)}(n)$ where $1 \leq i \leq p$ for the special cases of $p := 3, 4, 5$ below, factorized over z for any fixed integers $n, \alpha \geq 1$ as follows [19, §1.11(iii); *cf.* §4.43]:

$$\begin{aligned}
\left\{r_{3,i}^{(\alpha)}(n)\right\}_{i=1}^{3} &:= \left\{z_i : z_i^3 - 3z_i^2(2\alpha + n) + 3z_i(\alpha + n)(2\alpha + n) - n(\alpha + n)(2\alpha + n) = 0, \ 1 \leq i \leq 3\right\} \qquad (1.20) \\
\left\{r_{4,j}^{(\alpha)}(n)\right\}_{j=1}^{4} &:= \big\{z_j : z_j^4 - 4z_j^3(3\alpha + n) + 6z_j^2(2\alpha + n)(3\alpha + n) \\
&\qquad - 4z_j(\alpha + n)(2\alpha + n)(3\alpha + n) + n(\alpha + n)(2\alpha + n)(3\alpha + n) = 0, \ 1 \leq j \leq 4\big\} \\
\left\{r_{5,k}^{(\alpha)}(n)\right\}_{k=1}^{5} &:= \big\{z_k : z_k^5 - 5(4\alpha + n)z_k^4 + 10(3\alpha + n)(4\alpha + n)z_k^3 - 10(2\alpha + n)(3\alpha + n)(4\alpha + n)z_k^2 \\
&\qquad + 5(\alpha + n)(2\alpha + n)(3\alpha + n)(4\alpha + n)z_k \\
&\qquad - n(\alpha + n)(2\alpha + n)(3\alpha + n)(4\alpha + n) = 0, \ 1 \leq k \leq 5\big\}.
\end{aligned}$$

Similarly, we define the following functions for any fixed $\alpha \in \mathbb{Z}^+$ and $n \geq 1$ to simplify notation in the next congruence properties stated as in (1.21) below:

$$\begin{aligned}
R_3^{(\alpha)}(n) &:= \frac{\left[6\alpha^2 + \alpha\left(6r_{3,1}^{(-\alpha)}(n) - 4n\right) + \left(n - r_{3,1}^{(-\alpha)}(n)\right)^2\right] r_{3,1}^{(-\alpha)}(n)^{\left\lfloor \frac{n-1}{\alpha} \right\rfloor + 1}}{\left(r_{3,1}^{(-\alpha)}(n) - r_{3,2}^{(-\alpha)}(n)\right)\left(r_{3,1}^{(-\alpha)}(n) - r_{3,3}^{(-\alpha)}(n)\right)} \\
&\quad + \frac{\left[6\alpha^2 + \alpha\left(6r_{3,3}^{(-\alpha)}(n) - 4n\right) + \left(n - r_{3,3}^{(-\alpha)}(n)\right)^2\right] r_{3,3}^{(-\alpha)}(n)^{\left\lfloor \frac{n-1}{\alpha} \right\rfloor + 1}}{\left(r_{3,3}^{(-\alpha)}(n) - r_{3,1}^{(-\alpha)}(n)\right)\left(r_{3,3}^{(-\alpha)}(n) - r_{3,2}^{(-\alpha)}(n)\right)}
\end{aligned}$$

$$+\frac{\left[6\alpha^2+\alpha\left(6r_{3,2}^{(-\alpha)}(n)-4n\right)+\left(n-r_{3,2}^{(-\alpha)}(n)\right)^2\right]r_{3,2}^{(-\alpha)}(n)^{\left\lfloor\frac{n-1}{\alpha}\right\rfloor+1}}{\left(r_{3,2}^{(-\alpha)}(n)-r_{3,1}^{(-\alpha)}(n)\right)\left(r_{3,2}^{(-\alpha)}(n)-r_{3,3}^{(-\alpha)}(n)\right)}$$

$$\begin{aligned}C_{4,i}^{(\alpha)}(n) &:= 24\alpha^3-18\alpha^2\left(n-2\cdot r_{4,i}^{(-\alpha)}(n)\right)+\alpha\left(7n-12\cdot r_{4,i}^{(-\alpha)}(n)\right)\left(n-r_{4,i}^{(-\alpha)}(n)\right)\\ &\qquad -\left(n-r_{4,i}^{(-\alpha)}(n)\right)^3, \quad \text{for } 1\le i\le 4\\ C_{5,k}^{(\alpha)}(n) &:= 120\alpha^4+2\alpha^2\left(23n^2-79n\cdot r_{5,k}^{(-\alpha)}(n)+60\cdot r_{5,k}^{(-\alpha)}(n)^2\right)+48\alpha^3(2n-5\cdot r_{5,k}^{(-\alpha)}(n))\\ &\qquad +\alpha(11n-20r_{5,k}^{(-\alpha)}(n))(n-r_{5,k}^{(-\alpha)}(n))^2+(n-r_{5,k}^{(-\alpha)}(n))^4, \quad \text{for } 1\le k\le 5.\end{aligned}$$

Then for fixed $\alpha\in\mathbb{Z}^+$ and $n\ge 0$, we obtain the following analogs to the first congruence result in (1.19) satisfied by the α–factorial functions, $n_{(\alpha)}$, when $n\ge 1$:

$$\begin{aligned}n!_{(\alpha)} &\equiv R_3^{(\alpha)}(n) && \pmod{3, 3\alpha} \qquad (1.21)\\ n!_{(\alpha)} &\equiv \sum_{1\le i\le 4}\frac{C_{4,i}^{(\alpha)}(n)}{\prod_{j\ne i}\left(r_{4,i}^{(-\alpha)}(n)-r_{4,j}^{(-\alpha)}(n)\right)}r_{4,i}^{(-\alpha)}(n)^{\left\lfloor\frac{n+\alpha-1}{\alpha}\right\rfloor} && \pmod{4, 4\alpha}\\ n!_{(\alpha)} &\equiv \sum_{1\le k\le 5}\frac{C_{5,k}^{(\alpha)}(n)}{\prod_{j\ne k}\left(r_{5,k}^{(-\alpha)}(n)-r_{5,j}^{(-\alpha)}(n)\right)}r_{5,k}^{(-\alpha)}(n)^{\left\lfloor\frac{n+\alpha-1}{\alpha}\right\rfloor} && \pmod{5, 5\alpha}.\end{aligned}$$

Several particular concrete examples illustrating the result cited in (1.21) modulo 3 (and 3α) for special cases of $\alpha\ge 1$ and $n\ge 1$ appear in Table 2 (page 40). The results in Section 4.1 provide statements of these new integer congruences for fixed $\alpha\ne 0$ modulo any integers $p\ge 2$. These new relations for the factorial–related product sequences modulo any p and $p\alpha$ are easily established from the properties of the convergent functions, $\mathrm{Conv}_h(\alpha, R; z)$, cited as in Table 1 (page 39), and proved for the more general cases of $p>6$ in Section 5.

1.4.3 Classical Congruence Statements and Necessary Conditions Concerning Primality

Wilson's theorem provides a classical necessary and sufficient characterization of primality formed by the integer residues of the single factorial functions modulo natural numbers $n\ge 2$. The examples given in this section provide the restatements of the necessary and sufficient congruence conditions imposed in both classical statements of Wilson's theorem and Clement's theorem concerning twin prime pairs, through these exact finite–sum expansions. For integers $p\ge 2$, the congruences implicit to each theorem are enumerated as follows [6; 13; 14]:

$$\begin{aligned} p \text{ prime} &\iff (p-1)!+1 && \equiv 0 \pmod{p} && \text{(Wilson)}\\ &\iff [z^{p-1}]\,\mathrm{Conv}_p(-1, p-1; z)+1 && \equiv 0 \pmod{p}\\ &\iff [z^{p-1}]\,\mathrm{Conv}_p(1, 1; z)+1 && \equiv 0 \pmod{p}\\ p, p+2 \text{ prime} &\iff 4\left[(p-1)!+1\right]+p && \equiv 0 \pmod{p(p+2)} && \text{(Clement)}\\ &\iff 4[z^{p-1}]\,\mathrm{Conv}_{p(p+2)}(-1, p-1; z)+p+4 && \equiv 0 \pmod{p(p+2)}\\ &\iff 4[z^{p-1}]\,\mathrm{Conv}_{p(p+2)}(1, 1; z)+p+4 && \equiv 0 \pmod{p(p+2)}.\end{aligned}$$

Related formulations of conditions concerning the primality of prime pairs, $(p, p+d)$, and then of other prime k–tuples and constellations, are similarly straightforward to obtain by elementary methods starting from the statement of Wilson's theorem [3; 14; 18].

The rationality in z of the convergent functions, $\mathrm{Conv}_h(\alpha, R; z)$, at each h also leads to alternate formulations of other well–known congruence statements concerning the divisibility of factorial functions. For example, we may characterize the *quad*, or *cousin primes*, according to the next condition for odd integers $p>3$ [22, A002144] [14]:

$$\left[\left(\tfrac{p-1}{2}\right)!\right]^2\equiv -1 \pmod{p} \iff p \text{ is a prime of the form } 4k+1. \qquad (1.22)$$

For an odd integer $p > 3$ to be both prime and satisfy $p \equiv 1 \bmod 4$, the congruence statement given in (1.22) requires that the diagonals of the following two–variable convergent generating function products satisfy the following equivalent conditions where p_i is taken such that $2^p p_i \mid p$ for each $i = 1, 2$:

$$\begin{aligned}
\left[\left(\tfrac{p-1}{2}\right)!\right]^2 &= [z^{(p-1)/2}][x^0]\left\{\mathrm{Conv}_{p_1}\left(-1, \frac{p-1}{2}; x\right)\mathrm{Conv}_{p_2}\left(-1, \frac{p-1}{2}; \frac{z}{x}\right)\right\} && \equiv -1 \pmod{p} \\
\left[\left(\tfrac{p-1}{2}\right)!\right]^2 &= [z^{(p-1)/2}][x^0]\left\{\mathrm{Conv}_{p_1}\left(-2, p-1; \frac{x}{2}\right)\mathrm{Conv}_{p_2}\left(-2, p-1; \frac{z}{2x}\right)\right\} && \equiv -1 \pmod{p} \\
\left[\left(\tfrac{p-1}{2}\right)!\right]^2 &= [z^{(p-1)/2}][x^0]\left\{\mathrm{Conv}_{p_1}\left(-1, \frac{p-1}{2}; x\right)\mathrm{Conv}_{p_2}\left(-2, p-1; \frac{z}{2x}\right)\right\} && \equiv -1 \pmod{p}.
\end{aligned}$$

Another example is given as the next necessary condition following from *Wolstenholme's theorem*, which provides that $p^2 \mid (p-1)! \cdot H_{p-1}$ whenever $p > 3$ is prime, where H_n is a *first–order harmonic number* [22, A001008; A002805]. This requirement for the primality of odd integers $p > 3$ is rewritten in this case as the following statement involving the generalized convergent functions and the corresponding generating function for the Stirling numbers, $H_n = \left[{n+1 \atop 2}\right]\frac{1}{n!}$ [13]:

$$\begin{aligned}
p > 3 \text{ prime} \qquad &\implies \quad [z^{p-1}][x^0]\left\{\mathrm{Conv}_{p^2}\left(-1, p-1; \frac{z}{x}\right)\frac{\mathrm{Log}(1-x)}{(1-x)}\right\} && \equiv 0 \pmod{p^2} \qquad (1.23) \\
&\implies \quad [x^0 z^{p-1}]\left\{\mathrm{Conv}_{p^2}\left(1, 1; \frac{z}{x}\right)\frac{\mathrm{Log}(1-x)}{(1-x)}\right\} && \equiv 0 \pmod{p^2}.
\end{aligned}$$

Further examples of prime–related congruence identities and variants of other factorial function sequences generated by diagonal, or Hadamard product, generating functions are provided as applications in Section 4.2 and Section 5.3.

2 Organization of the Article and Notation

2.1 Organization of the Article

The content in the next several sections of the article is roughly organized into the following topics summarized below. The included summary notebook file, `summary-*.nb`, contains working *Mathematica* code to verify the formulas and identities cited in the article.

- **<u>Section 3</u>**. *The J-Fraction Expansions of the Generalized Convergent Functions*:
  Section 3.1 provides a brief overview of the enumerative J–fraction properties established in Flajolet's articles that summarize the properties needed to prove the new results within this article. A short, direct proof of the convergent function representations for the more general product sequence expansions defined in (1.13) is given in Section 3.2. This proof follows as a straightforward adaptation of the known J-fraction expansions enumerating the Pochhammer symbol, $(x)_n$, and the two–variable generating function for the Stirling numbers of the first kind in the reference [9]. ⊗
- **<u>Section 4</u>**. *Applications and Motivating Examples*:
  Section 4.1 treats the new forms of the integer congruence properties satisfied by special cases of the product sequences, $p_n(\alpha, R)$, and their expansions by the special function zeros of (1.14) from Section 1.3 studied in the references [4; 11]. The examples cited in the subsections of Section 4.2 are intended to provide a number of specific representative applications that arise in enumerating factorial–related sum, power, and product sequences by the rational convergent function approximations, $\mathrm{Conv}_h(\alpha, R; z)$, provided by (1.13) and the proofs in Section 3.2.

  These examples illustrate the expansions of corresponding to several well–known sequences and other related identities formally enumerated by special cases of the diagonal, or Hadamard product, generating generating functions defined in (4.3) and (4.4) of the overview to this section. The motivating special case sequences and identities enumerated through the new forms provided by these rational convergent functions for the generalized product sequences include the applications to the following factorial–related sequences: arithmetic progressions of the single factorial function, generalized superfactorial products and their relations to the Barnes G–function, $G(z+2)$, at rational–valued z, sums of over powers of the natural numbers, and the p^{th} power sequences, $2^p - 1$ and $(s+1)^p - 1$, for $s > 0$ when $p \geq 1$. ⊡

- **Section 5**. *Properties of the Generalized Convergent Functions*:
  Section 5.1 and Section 5.2 prove several additional properties and exact formulas satisfied by the respective convergent function sequences, $\mathrm{FQ}_h(\alpha, R; z)$ and $\mathrm{FP}_h(\alpha, R; z)$. The results for the convergent denominator functions, $\mathrm{FQ}_h(\alpha, R; z)$, stated by propositions in Section 5.1 provide characterizations of these sequences by well–known special functions and orthogonal polynomials, auxiliary recurrence relations, and analogs to the known addition and multiplication theorems for the confluent hypergeometric function, $U(-h, b, z)$.

  The last topics considered in Section 5.3 give additional notable applications of the h–order finite difference equations formed by the rational h^{th} convergents, $\mathrm{Conv}_h(\alpha, R; z)$, when the parameter, R, denotes some initially fixed indeterminate parameter with respect to the symbolic product sequences defined by (1.1). The finite–degree, rational polynomial expansions of the sequences, $p_n(\alpha, R)$, when R depends linearly on n offer a dual interpretation to the algebraic structure of the previous formulas given in terms of the special function zeros above. The resulting multiple sum expansions suggest many new applications in classical and otherwise noteworthy special case identities. ———— ⊛

2.2 Notation and Conventions

Most of the notational conventions within the article are consistent with those employed in the *Concrete Mathematics* reference. The particular conventions cited in the introduction to [21] are employed within this article. These particular conventions include the special notation for sequences, formal power series coefficient extraction, $f_n := [z^n]\left\{\sum_k f_k z^k\right\}$, Iverson's convention, $[i = j]_\delta \equiv \delta_{i,j}$, where $[n = k = 0]_\delta \equiv \delta_{n,0}\delta_{k,0}$, and the alternate bracket notation for the Stirling number triangles, $\left[{n \atop k}\right] \equiv (-1)^{n-k} s(n, k)$ and $\left\{{n \atop k}\right\} \equiv S(n, k)$. Within the article the notation $g_1(n) \equiv g_2(n) \pmod{N_1, N_2, \ldots, N_k}$ is also understood to mean that the congruence, $g_1(n) \equiv g_2(n) \pmod{N_j}$, holds modulo any of the bases, N_j, for $1 \leq j \leq k$. Other standard notation for special functions employed within the article is consistent with the definitions in the *NIST Handbook of Mathematical Functions*.

2.3 Listings of Main Results, Examples, and Referenced Tables

The next listings provide citations to the relevant applications, examples, and main results established in the article. The page references to the table citations referenced within the next several sections of the article are also provided in the listings given immediately below.

Examples and Results in the Article

Tables Referenced in the Article

Contents

3 Jacobi Type J-Fractions for Generalized Factorial Function Sequences

3.1 Enumerative Properties of Jacobi Type J–Fractions

To simplify the exposition in this article, we adopt the notation for the *Jacobi–type* continued fractions (*J–fractions*) in [9] [19, *cf.* §3.10]. Given some application–specific choices of the prescribed sequences, $\{a_k, b_k, c_k\}$, we consider the formal power series whose coefficients are generated by the rational convergents, $J^{[h]}(z) := J^{[h]}(\{a_k, b_k, c_k\}; z)$, of the infinite continued fractions, $J(z) := J^{[\infty]}(\{a_k, b_k, c_k\}; z)$, defined through the next equation.

$$J(z) = \cfrac{1}{1 - c_0 z - \cfrac{a_0 b_1 z^2}{1 - c_1 z - \cfrac{a_1 b_2 z^2}{\cdots}}}. \tag{3.1}$$

We briefly summarize the other enumerative properties from the references that are relevant in constructing the new factorial–function–related results in this article [9; 10; 17].

1. **Definitions of the h–Order Convergent Series**:
When $h \geq 1$, the h^{th} convergent functions, given in the equivalent forms of $J^{[h]}(z) := J^{[h]}(\{a_k, b_k, c_k\}; z)$ in this section, to the infinite expansion of (3.1) are defined as the ratios, $J^{[h]}(z) := \mathrm{FP}_h(z)/\,\mathrm{FQ}_h(z)$, also denoted by $\mathrm{Conv}_h(\alpha, R; z)$ within this article. The component functions corresponding to the convergent numerator and denominator sequences each satisfy second–order finite difference equations (in h) of the respective forms defined as in (3.2) and (3.3).
$$P_h(z) = (1 - c_{h-1}z)P_{h-1}(z) - a_{h-2}b_{h-1}z^2 P_{h-2}(z) + [h = 1]_\delta \tag{3.2}$$
$$Q_h(z) = (1 - c_{h-1}z)Q_{h-1}(z) - a_{h-2}b_{h-1}z^2 Q_{h-2}(z) + (1 - c_0 z)\,[h = 1]_\delta + [h = 0]_\delta \tag{3.3}$$

2. **Rationality of Truncated Convergent Function Approximations**:
Let $p_n \equiv p_n(\{a_k, b_k, c_k\}) := [z^n]J(z)$ denote the expected term corresponding to the coefficient of z^n in the formal power series expansion of the infinite J–fraction from (3.1). Then for all $n \geq 0$, we know that the h^{th} convergent functions have truncated power series expansions that satisfy
$$p_n(\{a_k, b_k, c_k\}) = [z^n]J^{[h]}(\{a_k, b_k, c_k\}; z), \quad \text{for all } h \geq n \geq 0.$$
In particular, the series coefficients of the h^{th} convergents are always at least h–order accurate as power series expansions in z. Since the h^{th} convergent functions, $J^{[h]}(z)$, are rational functions of z for each finite $h \in \mathbb{N}$, these truncated sequences forms lead to additional representations through the h–order difference equations specified by the sequences from (3.3) [13, §7.2].

3. **Congruence Properties Modulo Integer Bases**:
The resulting "*eventually periodic*" nature [10] [17, See §2; §5.7] suggested by the approximate sequences enumerated by the rational convergent functions is formalized in the congruence properties given in (3.4). Let $\lambda_k := a_{k-1}b_k$ and suppose that the corresponding bases, M_h, are formed by the products $M_h := \lambda_1\lambda_2\cdots\lambda_h$ for each $h \geq 1$. Then whenever $N_h \mid M_h$, and for any $n \geq 0$, we have that
$$p_n(\{a_k, b_k, c_k\}) \equiv [z^n]J^{[h]}(\{a_k, b_k, c_k\}; z) \pmod{N_h}, \tag{3.4}$$
which is also true of all partial sequence terms enumerated by the h^{th} convergent functions modulo any integer divisors of the M_h.

3.2 Proof of the J–Fraction Representations for the Generalized Product Sequence Generating Functions

We omit the details to a more combinatorially flavored proof that the J–fraction series defined through the convergent functions in (1.13) do, in fact, correctly enumerate the expected symbolic product sequences from (1.1). Instead, a short direct proof following from the J–fractions in Flajolet's first article is sketched below. Even further combinatorial interpretations of the sequences generated by these continued fraction series, their relations to the Stirling number triangles, and other properties tied to the coefficient triangles studied in depth by [21] based on the properties of these new J–fractions is suggested as a topic for later investigation.

The prescribed sequences in the J–fraction expansions defined by (3.1) in the previous section, corresponding to (i) the generalized convergent forms where $\{a_k, b_k, c_k\} := \{\mathrm{af}_k(\alpha, R), \mathrm{bf}_k(\alpha, R), \mathrm{cf}_k(\alpha, R)\}$, and (ii) the Pochhammer symbol, $(x)_n$, for any fixed indeterminate $x \in \mathbb{C}$, where $\{a_k, b_k, c_k\} := \{\mathrm{as}_k(x), \mathrm{bs}_k(x), \mathrm{cs}_k(x)\}$, are defined as follows:

$$\{\mathrm{af}_k(\alpha, R), \mathrm{bf}_k(\alpha, R), \mathrm{cf}_k(\alpha, R)\} := \{R + k\alpha, k, \alpha \cdot (R + 2k\alpha)\} \qquad \text{(Generalized Products)}$$
$$\{\mathrm{as}_k(x), \mathrm{bs}_k(x), \mathrm{cs}_k(x)\} := \{x + k, k, x + 2k\}. \qquad \text{(Pochhammer Symbol)}$$

We claim that the modified J–fractions, $R_0(R/\alpha, \alpha z)$, that generate the series expansions in (1.9) enumerate the analogous terms of the generalized symbolic product sequences defined by (1.1).

Proof of the Claim. The J–fraction results established in Flajolet's article prove two equivalent formulations of the expansions cited in the form of (1.9) above. In particular, the reference states one indirect form of the result generating the bivariate ordinary power series for the Stirling numbers of the first kind, and a second respective result directly providing the J–fraction expansions that generate the Pochhammer symbol, or rising factorial function, $(r)_k \equiv k! \cdot \binom{r}{k}$, for some indeterminate $r \in \mathbb{C}$ [9, Thm. 3B, Prop. 9].

After an appeal to the polynomial expansions of both the Pochhammer symbols, $(x)_n$, and then of the products, $p_n(\alpha, R)$, defined as in (1.1) by the Stirling numbers of the first kind, we see that

$$\begin{aligned} p_n(\alpha, R) \times z^n &= \left\{\prod_{j=0}^{n-1}(R + j\alpha) + [n = 0]_\delta\right\} \times z^n \\ &= \left\{\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \alpha^{n-k} R^k\right\} \times z^n \\ &= \left\{\alpha^n \cdot \left(\frac{R}{\alpha}\right)_n\right\} \times z^n \end{aligned} \tag{3.5}$$

Finally, after the parameter substitution of $x :\mapsto R/\alpha$ together with the change of variable $z :\mapsto \alpha z$ in the results from Flajolet's article [9], we obtain identical forms of the convergent–based definitions of the generalized J–fractions definitions given in Definition 1.1. ⊡

3.3 Alternate Exact Expansions of the Generalized Convergent Functions

The notational inconvenience introduced in the inner sums of (1.15) and (1.16), each of which depend implicitly on the more difficult terms of the numerator polynomials, $\mathrm{FP}_h(\alpha, R; z)$, is avoided in place of alternate recurrence relations for each $\mathrm{Conv}_h(\alpha, R; z)$ involving paired products of the denominator polynomials, $\mathrm{FQ}_h(\alpha, R; z)$, which are related to generalized forms of the Laguerre polynomial sequences as follows (see Section 5.1):

$$\mathrm{FQ}_h(\alpha, R; z) = (-\alpha z)^h \cdot h! \cdot L_h^{(R/\alpha - 1)}\left((\alpha z)^{-1}\right). \tag{3.6}$$

Topics aimed at finding new results obtained from known, strictly continued–fraction–related properties of the convergent function sequences beyond the proofs given in Section 5 are suggested as a further avenue to work on enumerating the otherwise divergent generating functions of more general integer–valued factorial–like sequences.

The expansions provided by the formula in (3.6) suggest useful alternate formulations of the congruence results stated in Section 4.1 when the Laguerre polynomial, or alternately confluent hypergeometric function, zeros [4; 11] are considered to be less complicated in form than the more involved sums expanded through the numerator functions, $\mathrm{FP}_h(\alpha, R; z)$. For example, the enumerative properties of the convergent function sequences given in the references provide the following known relations [9, §3] [19, §1.12(ii)]:

$$\begin{aligned} \mathrm{Conv}_h(\alpha, R; z) &= \mathrm{Conv}_{h-k}(\alpha, R; z) + \sum_{i=0}^{k-1} \frac{\alpha^{h-i-1}(h-i-1)! \cdot p_{h-i-1}(\alpha, R) \cdot z^{2(h-i-1)}}{\mathrm{FQ}_{h-i}(\alpha, R; z)\, \mathrm{FQ}_{h-i-1}(\alpha, R; z)}, \ \forall h > k \geq 1 \\ \mathrm{Conv}_h(\alpha, R; z) &= \sum_{i=0}^{h-1} \frac{\alpha^{h-i-1}(h-i-1)! \cdot p_{h-i-1}(\alpha, R) \cdot z^{2(h-i-1)}}{\mathrm{FQ}_{h-i}(\alpha, R; z)\, \mathrm{FQ}_{h-i-1}(\alpha, R; z)}, \ \forall h \geq 2. \end{aligned} \tag{3.7}$$

The expansions of the convergent denominator functions, $\mathrm{FQ}_h(\alpha, R; z)$, through the confluent hypergeometric function provided by Proposition 5.1 lead to a number of the new recurrence relations proved by Corollary 5.2, as well as the analogous forms of the addition and multiplication theorems phrased by Proposition 5.3, respectively, in Section 5.1.

Remark 3.1 (Related Convergent Generating Function Expansions)**.** For $h-i \geq 0$, we have a noteworthy identity satisfied by the Laguerre polynomial sequences cited as follows [19, §18.5(ii)]:

$$(-\alpha z)^{h-i} \cdot (h-i)! \times L_{h-i}^{(\beta)}\left(\frac{1}{\alpha z}\right) = \alpha^{h-i} \cdot z^{2h-2i+\beta+1} e^{1/\alpha z} \times \left\{\frac{e^{-1/\alpha z}}{z^{\beta+1}}\right\}^{(h-i)}.$$

The multiple derivatives implicit to the statement of the previous equation then have the additional expansions through the product rule analog provided by the formula of Halphen stated in the form of [7, §3 Exercises]

$$\left\{F\left(\frac{1}{z}\right) G(z)\right\}^{(n)} = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{z^k} \cdot F^{(k)}\left(\frac{1}{z}\right) \left\{\frac{G(z)}{z^k}\right\}^{(n-k)}, \qquad \text{(Halphen's Product Rule)}$$

for natural numbers $n \geq 0$, and where the notation for the functions, $F(z)$ and $G(z)$, employed in the previous equation corresponds to any prescribed choice of these functions that are each n times continuously differentiable at the point $z \neq 0$. The particular restatements of (3.7) given by

$$\begin{aligned}
\mathrm{Conv}_h(\alpha, R; z) &= \mathrm{Conv}_{h-k}(\alpha, R; z) \\
&\quad + \sum_{i=0}^{k-1} \frac{(h-i-1)!}{(\alpha z) \cdot (R/\alpha)_{h-i}} \times \frac{1}{{}_1F_1\left(-(h-i); \frac{R}{\alpha}; \frac{1}{\alpha z}\right) {}_1F_1\left(-(h-i-1); \frac{R}{\alpha}; \frac{1}{\alpha z}\right)}, \quad 1 \leq k < h \\
\mathrm{Conv}_h(\alpha, R; z) &= \sum_{i=0}^{h-1} \frac{(h-i-1)!}{(\alpha z) \cdot (R/\alpha)_{h-i}} \times \frac{1}{{}_1F_1\left(-(h-i); \frac{R}{\alpha}; \frac{1}{\alpha z}\right) {}_1F_1\left(-(h-i-1); \frac{R}{\alpha}; \frac{1}{\alpha z}\right)},
\end{aligned} \tag{3.8}$$

are then easily obtained from the previous two formulas, where ${}_1F_1(a; b; z)$, or $M(a, b, z) = \sum_{s \geq 0} \frac{(a)_s}{(b)_s s!} z^s$, denotes *Kummer's confluent hypergeometric function* [19, §13.2]. ——— ®

4 Applications and Motivating Examples

4.1 New Congruence Properties for Generalized Factorial Functions

The particular cases of the J–fraction representations enumerating the product sequences defined by (1.1) always yield a factor of $h := N_h \mid M_h$ in the statement of (3.4). One consequence of this property implicit to each of the generalized factorial–like sequences cited so far, is that it is straightforward to formulate new congruence relations for these sequences modulo any fixed integers $p \geq 2$. The particular results stated in this section follow immediately from the congruences properties modulo integer divisors of the M_h summarized by Section 3.1 for the more general J–fraction cases enumerating the combinatorial sequences already considered by [9; 10].

The generalized product sequences, $p_n(\alpha, R)$, defined by (1.1) correspond to the J–fraction parameters, λ_h and M_h, formed by the generalized sequence cases from the proof in Section 3.2. The new congruence properties corresponding to the particular variants of the α–factorial functions, $n!_{(\alpha)}$ and $(\alpha n - d)!_{(\alpha)}$, are then constructed as special cases of the more well–known enumerative properties satisfied by the generalized convergent functions, $\mathrm{Conv}_h(\alpha, R; z)$. We require the next lemma to formally enumerate the generalized products and factorial function sequences already cited without proof in the examples from Section 1.4.

Lemma 4.1 (*Sequences Generated by the Generalized Convergent Functions*).
For fixed integers $\alpha \neq 0$, $0 \leq d < \alpha$, and each $n \geq 1$, the generalized α–factorial sequences defined in (1.2) satisfy the following product–wise expansions given by (1.1):

$$\begin{aligned}
(\alpha n - d)!_{(\alpha)} &= p_n(-\alpha, \alpha n - d) && \text{(i)} \\
&= p_n(\alpha, \alpha - d) && \text{(ii)} \\
n!_{(\alpha)} &= p_{\lfloor (n+\alpha-1)/\alpha \rfloor}(-\alpha, n). && \text{(iii)}
\end{aligned}$$

Proof. The related identities cited above correspond to the equivalent expansions of the product–wise representations for the α–factorial functions given in each of the next equations:

$$(\alpha n-d)!_{(\alpha)} = \prod_{j=0}^{n-1} (\alpha n-d-j\alpha) \tag{i.2}$$

$$= \prod_{j=0}^{n-1} (\alpha-d+j\alpha) \tag{ii.2}$$

$$n!_{(\alpha)} = \prod_{j=0}^{\lfloor (n+\alpha-1)/\alpha\rfloor -1} (n-i\alpha). \tag{iii.2}$$

The first product in (i.2) is easily obtained from (1.2) by induction on n, which then implies the second result in (ii.2). Similarly, an inductive argument applied to the definition provided by (1.2) proves the last product representation given in (iii.2). ⊡

Corollaries. The proof of the lemma provides immediate corollaries to the special cases of the α–factorial functions, $(\alpha n-d)!_{(\alpha)}$, expanded by the results from (1.17). We explicitly state the following particular special cases of the lemma corresponding to $d := 0$ in (4.1), and then $d := 1$ in (4.2), respectively, for later use in Section 4.2.1 and Section 5.3 of the article below:

$$\begin{aligned}(\alpha n)!_{(\alpha)} &= \alpha^n \cdot (1)_n = [z^n] \operatorname{Conv}_{n+n_0}(-\alpha, \alpha n; z), \ \forall n_0 \geq 0 \\ &= \alpha^n \cdot n! \quad = [z^n] \operatorname{Conv}_{n+n_0}(-1, n; \alpha z), \ \forall n_0 \geq 0\end{aligned} \tag{4.1}$$

$$\begin{aligned}(\alpha n-1)!_{(\alpha)} &= p_n(-\alpha, \alpha n-1) = (-\alpha)^n \left(\frac{1}{\alpha}-n\right)_n \\ &= p_n(\alpha, \alpha-1) \qquad = \alpha^n \left(1-\frac{1}{\alpha}\right)_n.\end{aligned} \tag{4.2}$$

The previous results follow as corollaries of the lemma concerning the functions, $(\alpha n-d)!_{(\alpha)}$, when $d := 0, 1$, and where the rightmost statements in the equations provided by (4.2) correspond to the respective forms of the product sequences, $p_n(\alpha, R)$, given by (3.5). The first two results are employed by the convergent–based formulations to the applications given in Section 4.2.1. The last pair of results given in (4.2) are employed in phrasing the generalized expansions for the identities cited in Example 5.5 of the article below. ⊡

Example 4.2 (The Special Cases Modulo 2, 3, and 4)**.** The first congruences for the α–factorial functions, $n!_{(\alpha)}$, modulo the prescribed integer bases, 2 and 2α, cited in (1.19) result by applying Lemma 4.1 to the series for the generalized convergent function, $\operatorname{Conv}_2(\alpha, R; z)$, expanded by following equations:

$$\begin{aligned}p_n(\alpha, R) &\equiv [z^n] \left\{\frac{1-z(2\alpha+R)}{R(\alpha+R)z^2-2(\alpha+R)z+1}\right\} && \pmod{2, 2\alpha} \\ &\equiv \sum_{b=\pm 1} \frac{\left(\sqrt{\alpha(\alpha+R)}-b\cdot\alpha\right)\left(\alpha+b\cdot\sqrt{\alpha(\alpha+R)}+R\right)^n}{2\sqrt{\alpha(\alpha+R)}} && \pmod{2, 2\alpha}.\end{aligned}$$

The special case of (1.19) when $\alpha = 1$ corresponding to the single factorial function, $n!$, agrees with the known congruence for the Stirling numbers of the first kind derived in [25, §4.6]. In particular, for $n \geq 0$ we obtain that

$$n! \equiv \sum_{m=1}^{n} \binom{\lfloor n/2\rfloor}{m-\lceil n/2\rceil} (-1)^{n-m} n^m + [n=0]_\delta \pmod{2}.$$

The next congruence satisfied by the α–factorial function sequences modulo 3 (3α) cited as a particular example in Section 1.4 is established similarly by applying the previous lemma to the series coefficients of the next convergent function, $\operatorname{Conv}_3(\alpha, R; z)$, when $\alpha :\mapsto -\alpha$ and $R :\mapsto n$:

$$p_n(\alpha, R) \equiv [z^n] \left\{\frac{-z^2\left(R^2+4\alpha R+6\alpha^2\right)+2(3\alpha+R)z-1}{R(\alpha+R)(2\alpha+R)z^3-3(\alpha+R)(2\alpha+R)z^2+3(2\alpha+R)z-1}\right\} \pmod{3, 3\alpha}.$$

The particular cases of the new congruence properties satisfied modulo 3 (3α) and 4 (4α) cited in (1.21) from Section 1.4 of the introduction phrase results that are also expanded through exact algebraic formulas involving the reciprocal roots corresponding to the denominators of the convergent functions, $\mathrm{Conv}_3(-\alpha, n; z)$ and $\mathrm{Conv}_4(-\alpha, n; z)$, provided as in Table 1 (page 39) [19, *cf.* §1.11(iii), §4.43]. ——— Ⓔ

More generally, let the p–order zeros, $\ell_{p,i}^{(\alpha)}(R)$, and the reflected partial fraction coefficients, $C_{p,i}^{(\alpha)}(R)$, be defined for $p \geq 2$, and each $1 \leq i \leq p$, as follows:

$$\left\{\ell_{p,i}^{(\alpha)}(R)\right\}_{i=1}^{p} := \left\{z_i : z^h \cdot \mathrm{FQ}_h\left(\alpha, R; z^{-1}\right) = 0\right\}_{i=1}^{p} \quad \text{and} \quad C_{p,i}^{(\alpha)}(R) := \widetilde{\mathrm{FP}}_h\left(\alpha, R; \ell_{p,i}^{(\alpha)}(R)\right),\ 1 \leq i \leq p.$$

The listings given in Table 3 (page 40) provide the first few simplified cases of the reflected numerator polynomial sequences, denoted by $\widetilde{\mathrm{FP}}_h(\alpha, R; z) := z^{h-1}\,\mathrm{FP}_h(\alpha, R; 1/z)$, which lead to the explicit formulations of the congruences modulo p (and modulo $p\alpha$) at each of the particular cases of $p := 4, 5$ given in (1.21), and then for the next few small special cases of subsequent integers $p \geq 6$.

The notation in the previous equation is then employed in the next statements generalizing the exact formula expansions and congruence properties cited in (1.15) and (1.16) of Section 1.3, and given in the particular special case results from (1.21) of Section 1.4.2:

$$\begin{aligned}
p_n(\alpha, R) &= \sum_{1 \leq i \leq h} \frac{C_{h,i}^{(\alpha)}(R)}{\prod\limits_{j \neq i}\left(\ell_{h,i}^{(\alpha)}(R) - \ell_{h,j}^{(\alpha)}(R)\right)} \times \left\{\ell_{h,i}^{(\alpha)}(R)\right\}^{n+1}, \quad \text{for any } h \geq n \geq 1 \\
p_n(\alpha, R) &\equiv \sum_{1 \leq i \leq p} \frac{C_{p,i}^{(\alpha)}(R)}{\prod\limits_{j \neq i}\left(\ell_{p,i}^{(\alpha)}(R) - \ell_{p,j}^{(\alpha)}(R)\right)} \times \left\{\ell_{p,i}^{(\alpha)}(R)\right\}^{n+1} && \pmod{p, p\alpha} \\
n!_{(\alpha)} &= \sum_{1 \leq i \leq h} \frac{C_{h,i}^{(-\alpha)}(n)}{\prod\limits_{j \neq i}\left(\ell_{h,i}^{(-\alpha)}(n) - \ell_{h,j}^{(-\alpha)}(n)\right)} \times \left\{\ell_{h,i}^{(-\alpha)}(n)\right\}^{\left\lfloor \frac{n-1}{\alpha} \right\rfloor + 1}, \quad \text{for any } h \geq n \geq 1 \\
n!_{(\alpha)} &\equiv \sum_{1 \leq i \leq p} \frac{C_{p,i}^{(-\alpha)}(n)}{\prod\limits_{j \neq i}\left(\ell_{p,i}^{(-\alpha)}(n) - \ell_{p,j}^{(-\alpha)}(n)\right)} \times \left\{\ell_{p,i}^{(-\alpha)}(n)\right\}^{\left\lfloor \frac{n-1}{\alpha} \right\rfloor + 1} && \pmod{p, p\alpha}.
\end{aligned}$$

The special zeros, $\ell_{p,i}^{(\alpha)}(R)$, again defined as above in terms of the reflected denominator polynomials, correspond to special cases of the special zeros of the confluent hypergeometric function, $U(-h, b, w)$, and the associated Laguerre polynomials, $L_p^{(\beta)}(w)$, defined as in Section 1.3 [19, §18.2(vi), §18.16] [4; 11][4].

Remark 4.3 (Congruences for Rational–Valued Parameters)**.** The J–fraction parameters, $\lambda_h \equiv \lambda_h(\alpha, R)$ and $M_h \equiv M_h(\alpha, R)$, defined as in the summary from Section 3.1, that correspond to the expansions of the generalized convergents defined in the proof from Section 3.2 satisfy

$$\begin{aligned}
\lambda_k(\alpha, R) &:= \mathrm{as}_{k-1}(\alpha, R) \cdot \mathrm{bs}_k(\alpha, R) \\
&= \alpha(R + (k-1)\alpha) \cdot k \\
M_h(\alpha, R) &:= \lambda_1(\alpha, R) \cdot \lambda_2(\alpha, R) \times \cdots \times \lambda_h(\alpha, R) \\
&= \alpha^h \cdot h! \times p_h(\alpha, R) \\
&= \alpha^h \cdot h! \times (R)_{h,\alpha},
\end{aligned}$$

so that for integer divisors, $N_h(\alpha, R) \mid M_h(\alpha, R)$, we have that

$$p_n(\alpha, R) \equiv [z^n]\,\mathrm{Conv}_h(\alpha, R; z) \pmod{N_h(\alpha, R)}.$$

So far we have restricted ourselves above to the particular cases of the product sequences, $p_n(\alpha, R)$, where $\alpha \neq 0$ is integer–valued, i.e., so that $p \mid M_p(\alpha, R)$ whenever $p \geq 2$ is a fixed natural number. Identities arising in some

[4] The characterizations of the convergent denominator functions, $\mathrm{Conv}_h(\alpha, R; z)$, by the confluent hypergeometric function and Laguerre polynomial sequences also suggests possible approaches to these factorial–function–related formulas by Turan–type inequalities for the special function sequences [19, *cf.* §18.14(ii), §18.16(iv)].

related applications that involve finding results analogous to the explicit new congruence properties stated so far for other factorial–related sequence cases, specifically when the choice of $\alpha \neq 0$ is strictly rational–valued, are intentionally not treated in the examples cited above. ——— ®

4.2 Diagonal Generating Functions and Hadamard Product Sequences

We define the next notation for the *Hadamard product* generating functions at fixed $z \in \mathbb{C}$ [7; 17]. Phrased in slightly different wording, we define (4.3) as an alternate notation for the "*diagonal*" *generating functions* that enumerate the corresponding product sequences generated as the diagonal coefficients of the series in k formal variables treated as in the reference [24, §6.3].

$$F_1 \odot F_2 \odot \cdots \odot F_k := \sum_{n \geq 0} f_{1,n} f_{2,n} \cdots f_{k,n} \times z^n \quad \text{where} \quad F_i(z) := \sum_{n \geq 0} f_{i,n} z^n \text{ for } 1 \leq i \leq k \tag{4.3}$$

When $F_i(z)$ is a rational function of z for each $1 \leq i \leq k$, we have particularly nice expansions of the coefficient extraction formulas of the diagonal generating functions from [24]. In particular, provided that $F_i(z)$ is rational in z at each respective i, these generating functions are expanded through the next few useful formulas:

$$\begin{aligned} F_1 \odot F_2 &= [x_1^0]\left\{F_2\left(\frac{z}{x_1}\right) \cdot F_1(x_1)\right\} \qquad (4.4) \\ F_1 \odot F_2 \odot F_3 &= [x_2^0 x_1^0]\left\{F_3\left(\frac{z}{x_2}\right) \cdot F_2\left(\frac{x_2}{x_1}\right) \cdot F_1(x_1)\right\} \\ F_1 \odot F_2 \odot \cdots \odot F_k &= [x_{k-1}^0 \cdots x_2^0 x_1^0]\left\{F_k\left(\frac{z}{x_{k-1}}\right) \cdot F_{k-1}\left(\frac{x_{k-1}}{x_{k-2}}\right) \times \cdots \times F_2\left(\frac{x_2}{x_1}\right) \cdot F_1(x_1)\right\}. \end{aligned}$$

Analytic formulas for the Hadamard products, $F_1 \odot F_2 \equiv F_1(z) \odot F_2(z)$, when the component sequence generating functions are well enough behaved in some neighborhood of $z_0 := 0$ are given in [7]. We regard the rational convergents approximating the otherwise divergent ordinary generating functions for the generalized factorial function sequences strictly as formal power series in z whenever possible in this article.

Remark 4.4 (Hybrid Generating Functions)**.** We also notice that when one of the generating functions of an individual sequence from the Hadamard product representations in (4.3) is not rational in z, we still proceed, however slightly more carefully, to formally enumerate the terms of these sequences that arise in applications. For example, the *central binomial coefficients* [22, A000984] are enumerated by the following convergent generating functions whenever $n \geq 1$ [13, *cf.* §5.3]:

$$\begin{aligned} \binom{2n}{n} &= \frac{2^n}{n!} \times (2n-1)!! = [z^n][x^1]\left\{e^{2x}\,\mathrm{Conv}_n\left(-2, 2n-1; \frac{z}{x}\right)\right\} \qquad \text{(Central Binomial Coefficients)} \\ &= \frac{2^{2n}}{n!} \times (1/2)_n \quad = [z^n][x^0]\left\{e^{2x}\,\mathrm{Conv}_n\left(2, 1; \frac{z}{x}\right)\right\}. \end{aligned}$$

The necessary condition for primality from the example given in (1.23) of the introduction is constructed by employing a similar technique with the Stirling number generating functions given by the following equations [13, §7.4]:

$$\sum_{n \geq 0} \begin{bmatrix} n+1 \\ p+1 \end{bmatrix} \frac{z^n}{n!} = \frac{\partial}{\partial z}\left\{\frac{\mathrm{Log}\left(\frac{1}{1-z}\right)^{p+1}}{(p+1)!}\right\} = \frac{(-1)^p}{p!} \cdot \frac{\mathrm{Log}(1-z)^p}{(1-z)}.$$

The remaining examples in this section illustrate the more formal approach taken with the generating functions enumerating the sequences considered here. Example 4.6 provides one particular application to enumerating the more challenging cases of these known interesting sums and combinatorial identities involving the double factorial function, $(2n-1)!!$, through the convergent–based generating function methods outlined in the next sections. ®

The next several subsections aim to provide several concrete applications and some notable special cases illustrating the utility of this approach to the more general product sequences provided by the rational convergent functions, especially when combined with other generating function techniques discussed elsewhere and in the references [7; 13; 17; 25].

4.2.1 Example: Expanding Arithmetic Progressions of the Single Factorial Function

One application suggested by the diagonal generating functions of the corresponding products of rational convergents formed by these sequences cases provides a–fold reductions of the h–order series approximations otherwise required to exactly enumerate arithmetic progressions of the single factorial function according to the next result:

$$(an+r)! = [z^{an+r}]\,\mathrm{Conv}_h\left(-1, an+r; z\right), \ \forall n \geq 1, a \in \mathbb{Z}^+, 0 \leq r < a, \forall h \geq an+r. \tag{4.5}$$

The statement of *Gauss's multiplication formula* provides decompositions of the single factorial functions, $(an+r)!$, into a finite product over a of the integer–valued multiple factorial sequences defined by (1.2) at $n \geq 1$ and whenever $a \geq 2$ and $0 \leq r < a$ are fixed natural numbers [19, §5.5(iii)]. For example, it is easy enough to see that[5]

$$\begin{aligned}(an)! &= a^{an} \cdot (an)!_{(a)} \cdot (an-1)!_{(a)} \times \cdots \times (an-a+1)!_{(a)} \\ &= a^n \cdot n! \times a^{(a-1)n} \cdot \left(\frac{1}{a}\right)_n \cdot \left(\frac{2}{a}\right)_n \times \cdots \times \left(\frac{a-1}{a}\right)_n, \forall a, n \in \mathbb{Z}^+.\end{aligned}$$

The arithmetic progression sequences formed in the particular special cases of $a := 2, 3$ expanded in the next examples cited below demonstrate the utility to these convergent–based formal generating function approximations.

In the particular cases where $a := 2$ (with $r := 0, 1$), we obtain the corresponding alternate expansions of (4.5) enumerated by the diagonal coefficients of the next Hadamard product generating functions given for all $n \geq 1$ as follows [23, *cf.* §2]:

$$\begin{aligned}(2n)! &= 2^n n! \times (2n-1)!! \\ &= [z^n][x^0]\left\{\mathrm{Conv}_n\left(-1, n; \frac{2z}{x}\right)\mathrm{Conv}_n\left(-2, 2n-1; x\right)\right\} \\ &= 2^n n! \times 2^n \left(1/2\right)_n \\ &= [x^0 z^n]\left\{\mathrm{Conv}_n\left(1, 1; \frac{2z}{x}\right)\mathrm{Conv}_n\left(2, 1; x\right)\right\} \\ (2n+1)! &= 2^n n! \times (2n+1)!! \\ &= [z^n][x^0]\left\{\mathrm{Conv}_n\left(-1, n; \frac{2z}{x}\right)\mathrm{Conv}_n\left(-2, 2n+1; x\right)\right\} \\ &= 2^n n! \times 2^n \left(3/2\right)_n \\ &= [x^1 z^n]\left\{\mathrm{Conv}_n\left(1, 1; \frac{2z}{x}\right)\mathrm{Conv}_n\left(2, 1; x\right)\right\} \\ &= [x^0 z^n]\left\{\mathrm{Conv}_n\left(1, 1; \frac{2z}{x}\right)\mathrm{Conv}_n\left(2, 3; x\right)\right\}.\end{aligned}$$

When $a := 3$ we obtain the next several alternate expansions generating the triple factorial products for the arithmetic progression sequences in (4.5) stated in the following equations for all $n \geq 2$ [23, §2]:

$$\begin{aligned}(3n)! &= (3n)!!! \times (3n-1)!!! \times (3n-2)!!! \\ &= [z^n][x_2^0 x_1^0]\left\{\mathrm{Conv}_n\left(-1, n; \frac{3z}{x_2}\right)\mathrm{Conv}_n\left(-3, 3n-1; \frac{x_2}{x_1}\right)\mathrm{Conv}_n\left(-3, 3n-2; x_1\right)\right\} \\ &= 3^n n! \times 3^n \left(2/3\right)_n \times 3^n \left(1/3\right)_n\end{aligned}$$

[5] The following variations of *Gauss's multiplication formula* satisfied by the *gamma function*, $(an)! \equiv \Gamma(an+1)$, given in [19, §5.5(ii)] also form the following multiplication formula analogs corresponding to the expansions of the Pochhammer symbols, $(a)_n \equiv \Gamma(a+n)/\Gamma(a)$ [26]:

$$\begin{aligned}(a)_{mn+r} &= (a)_r \times n^{mn} \times \prod_{j=0}^{n-1}\left(\frac{a+j+r}{n}\right)_m, \ n \in \mathbb{N} \\ (an+b)_m &= n^m \times \prod_{j=0}^{n-1}\left(a + \frac{b+j}{n}\right)_{m/n}, \ n \in \mathbb{Z}^+.\end{aligned}$$

$$\begin{aligned}
&= [x_1^0 x_2^0 z^n]\left\{\operatorname{Conv}_n\left(1,1;\frac{3z}{x_2}\right)\operatorname{Conv}_n\left(3,1;\frac{x_2}{x_1}\right)\operatorname{Conv}_n\left(3,2;x_1\right)\right\} \\
(3n+1)! &= (3n)!!! \times (3n-1)!!! \times (3(n+1)-2)!!! \\
&= [z^n][x_2^0 x_1^{-1}]\left\{\operatorname{Conv}_n\left(-1,n;\frac{3z}{x_2}\right)\operatorname{Conv}_n\left(-3,3n-1;\frac{x_2}{x_1}\right)\operatorname{Conv}_n\left(-3,3n+1;x_1\right)\right\} \\
&= 3^n n! \times 3^n\,(2/3)_n \times 3^n\,(4/3)_n \\
&= [x_1^0 x_2^0 z^n]\left\{\operatorname{Conv}_n\left(1,1;\frac{3z}{x_2}\right)\operatorname{Conv}_n\left(3,4;\frac{x_2}{x_1}\right)\operatorname{Conv}_n\left(3,2;x_1\right)\right\} \\
(3n+2)! &= (3n)!!! \times (3(n+1)-1)!!! \times (3(n+1)-2)!!! \\
&= [z^n][x_2^{-1} x_1^0]\left\{\operatorname{Conv}_n\left(-1,n;\frac{3z}{x_2}\right)\operatorname{Conv}_n\left(-3,3n+2;\frac{x_2}{x_1}\right)\operatorname{Conv}_n\left(-3,3n+1;x_1\right)\right\} \\
&= 2 \times 3^n n! \times 3^n\,(5/3)_n \times 3^n\,(4/3)_n \\
&= [x_1^1 x_2^0 z^n]\left\{\operatorname{Conv}_n\left(1,1;\frac{3z}{x_2}\right)\operatorname{Conv}_n\left(3,4;\frac{x_2}{x_1}\right)\operatorname{Conv}_n\left(3,2;x_1\right)\right\}.
\end{aligned}$$

The additional forms of the diagonal generating functions corresponding to the special cases where $(a,r) := (4,2)$ and $(a,r) := (5,3)$, respectively. involving the *quadruple* and *quintuple factorial* functions are also cited in the next equations to further illustrate the procedure outlined above.

$$\begin{aligned}
(4n+2)! &= (4n)!!!! \times (4n-1)!!!! \times (4(n+1)-2)!!!! \times (4(n+1)-3)!!!! \\
&= [z^n][x_3^0 x_2^{-1} x_1^0]\Bigg\{\operatorname{Conv}_n\left(-1,n;\frac{4z}{x_3}\right)\operatorname{Conv}_n\left(-4,4n-1;\frac{x_3}{x_2}\right) \times \\
&\qquad \times \operatorname{Conv}_n\left(-4,4n+2;\frac{x_2}{x_1}\right)\operatorname{Conv}_n\left(-4,4n+1;x_1\right)\Bigg\} \\
&= [x_1^0 x_2^1 x_3^0 z^n]\left\{\operatorname{Conv}_n\left(1,1;\frac{4z}{x_3}\right)\operatorname{Conv}_n\left(4,3;\frac{x_3}{x_2}\right)\operatorname{Conv}_n\left(4,2;\frac{x_2}{x_1}\right)\operatorname{Conv}_n\left(4,1;x_1\right)\right\},\ n \geq 2 \\
(5n+3)! &= (5n)!_{(5)} \times (5n-1)!_{(5)} \times (5(n+1)-2)!_{(5)} \times (5(n+1)-3)!_{(5)} \times (5(n+1)-4)!_{(5)} \\
&= [z^n][x_4^0 x_3^{-1} x_2^0 x_1^0]\Bigg\{\operatorname{Conv}_n\left(-1,n;\frac{5z}{x_4}\right)\operatorname{Conv}_n\left(-5,5n-1;\frac{x_4}{x_3}\right) \times \\
&\qquad \times \operatorname{Conv}_n\left(-5,5n+3;\frac{x_3}{x_2}\right)\operatorname{Conv}_n\left(-5,5n+2;\frac{x_2}{x_1}\right) \times \\
&\qquad \times \operatorname{Conv}_n\left(-5,5n+1;x_1\right)\Bigg\} \\
&= [x_1^0 x_2^0 x_3^1 x_4^0 z^n]\Bigg\{\operatorname{Conv}_n\left(1,1;\frac{5z}{x_4}\right)\operatorname{Conv}_n\left(5,4;\frac{x_4}{x_3}\right)\operatorname{Conv}_n\left(5,3;\frac{x_3}{x_2}\right) \times \\
&\qquad \times \operatorname{Conv}_n\left(5,2;\frac{x_2}{x_1}\right)\operatorname{Conv}_n\left(5,1;x_1\right)\Bigg\},\ n \geq 2.
\end{aligned}$$

The truncated power series approximations formulated by the last several examples are compared to the known forms of related results for extracting arithmetic progressions from any formal, ordinary generating function of an arbitrary sequence through the primitive a^{th} roots of unity, $\omega_a := \exp(2\pi\imath/a)$, considered in the references [13; 16; 25].

4.2.2 Example: The Superfactorial and the Barnes G–Functions

The *superfactorial function*, $S_1(n)$, also denoted $S_{1,0}(n)$ below, is defined as the factorial product [22, A000178]

$$S_1(n) := \prod_{k=1}^{n} k! + [n=0]_{\delta} \qquad \xrightarrow{\text{A000178}} \qquad \{1,1,2,12,288,34560,24883200,\ldots\}. \qquad \text{(Superfactorial Function)}$$

These superfactorial functions are given in terms of the *Barnes G–function*, $G(z)$, for $z \in \mathbb{Z}^+$ through the relation $S_1(n) \equiv G(n+2)$. The Barnes G–function, $G(z)$, corresponds to a so–termed "*double gamma function*" satisfying a functional equation of the form

$$G(n+2) = \Gamma(n+1)G(n+1) + [n=1]_\delta \qquad \text{(Barnes G-Function)}$$

for natural numbers $n \geq 1$ [19, §5.17]. We can similarly expand the superfactorial functions, $S_1(n)$, by unfolding the factorial products recursively according to the formulas in the next equation.

$$S_1(n) = n! \cdot (n-1)! \times \cdots \times (n-k+1)! \cdot S_1(n-k),\ 0 \leq k < n$$

The product sequences involving the single factorial functions formed in the last equations then suggests yet another application of the diagonal generating functions involving the rational convergents enumerating the sequences, $(n-k)!$, when $n-k \geq 1$.

In particular, these Hadamard–product–like sequence forms involving the single factorial functions are then generated as the diagonal coefficients

$$\begin{aligned} S_1(n) = \{[z^n]\operatorname{Conv}_{n+1}(-1,n;z)\} & \times \{[z^n]z \cdot \operatorname{Conv}_n(-1,n-1;z)\} \times \\ & \times \left\{[z^n]z^2 \cdot \operatorname{Conv}_{n-1}(-1,n-2;z)\right\} \times \cdots \times \{[z^n]z^n \cdot \operatorname{Conv}_2(-1,1;z)\} \\ S_1(n) = \{[z^n]\operatorname{Conv}_n(1,1;z)\} \quad & \times \{[z^n]z \cdot \operatorname{Conv}_{n-1}(1,1;z)\} \times \\ & \times \left\{[z^n]z^2 \cdot \operatorname{Conv}_{n-2}(1,1;z)\right\} \times \cdots \times \{[z^n]z^n \cdot \operatorname{Conv}_1(1,1;z)\}, \end{aligned}$$

Stated more precisely, the superfactorial sequence is generated by the following finite, rational products of the generalized convergent functions for any $n \geq 2$:

$$\begin{aligned} S_1(n) &= [x_1^{-1}x_2^{-1}\cdots x_{n-1}^{-1}x_n^n]\left\{\prod_{i=0}^{n-2}\operatorname{Conv}_n\left(-1,n-i;\frac{x_{n-i}}{x_{n-i-1}}\right) \times \operatorname{Conv}_n(-1,1;x_1)\right\} \\ S_1(n) &= [x_1^{-1}x_2^{-1}\cdots x_{n-1}^{-1}x_n^n]\left\{\prod_{i=0}^{n-2}\operatorname{Conv}_n\left(1,1;\frac{x_{n-i}}{x_{n-i-1}}\right) \times \operatorname{Conv}_n(1,1;x_1)\right\}. \end{aligned} \tag{4.6}$$

Example 4.5 (Enumerating Identities Involving Sums of Factorials)**.** The coefficients in the previous equations are also compared to the constructions of convergent–based generating functions involved in enumerating several analogous summation–based identities involving the single factorial function [22, A003422, A061062; A005165; A033312; A104344, A001044] [7, *cf.* §3; Ex. 3.30 p. 168][6]:

$$\begin{aligned} \operatorname{sf}_1(n) &:= \sum_{k=0}^{n-1} k! && \xrightarrow[\text{A061062}]{\text{A003422}} \quad \{0,1,2,4,10,34,154,874,\ldots\} \\ &= [z^n]\left\{\frac{z}{(1-z)} \cdot \operatorname{Conv}_n(1,1;z)\right\} && \text{(Left Factorials)} \\ \operatorname{af}(n) &:= \sum_{k=1}^{n}(-1)^{n-k} \cdot k! + [n=0]_\delta && \xrightarrow{\text{A005165}} \quad \{1,1,5,19,101,619,4421,\ldots\} \\ &= n! - \operatorname{af}(n-1) + [n=0]_\delta && \text{(Alternating Factorials)} \\ &= [z^n]\left\{\frac{1}{(1-z)} \cdot \left(\operatorname{Conv}_n(1,1;z) - 1\right)\right\} \\ \operatorname{sf}_2(n) &:= \sum_{k=1}^{n} k \cdot k! && (4.7) \end{aligned}$$

[6] A generalization of the second identity given in (4.7) due to Gould is cited in [7, p. 168] as follows:

$$\sum_{k=0}^{n}\binom{x}{k}^p\left(\frac{k!}{x^{k+1}}\right)^p\{(x-k)^p - x^p\} = \binom{x}{n+1}^p\left(\frac{(n+1)!}{x^{n+1}}\right)^p - 1.$$

$$= (n+1)! - 1 \qquad \xrightarrow{\text{A033312}} \qquad \{1, 5, 23, 119, 719, 5039, \ldots\}$$

$$= [x^0 z^n] \left\{ \frac{1}{(1-z)} \frac{x}{(1-x)^2} \operatorname{Conv}_n \left(1, 1; \frac{z}{x}\right) \right\}$$

$$\operatorname{sf}_3(n) := \sum_{k=1}^{n} (k!)^2 \qquad \xrightarrow[\text{A001044}]{\text{A104344}} \qquad \{1, 5, 41, 617, 15017, 533417, \ldots\}$$

$$= [x^0 z^n] \left\{ \frac{1}{(1-z)} \left(\operatorname{Conv}_n (1, 1; x) \operatorname{Conv}_n \left(1, 1; \frac{z}{x}\right) - 1 \right) \right\}.$$

Additionally, we can generalize the last sum, denoted $\operatorname{sf}_3(n)$ in (4.7), to form the following variants of sums over the squares of the α–factorial functions, $n!!$ and $n!!!$, enumerated by the generating function identities from the right–hand–side of (1.18) [22, *cf.* A184877]:

$$\operatorname{sf}_{3,2}(n) := \sum_{k=0}^{n} (k!!)^2 \qquad \xrightarrow[\text{A184877}]{} \qquad \{1, 2, 6, 15, 79, 304, 2608, 13633, 161089, \ldots\}$$

$$= [x^0 z^n] \left\{ \frac{1}{(1-z)} \left(\operatorname{Conv}_n (2, 2; x) \operatorname{Conv}_n \left(2, 2; \frac{z}{x}\right) + z \cdot \operatorname{Conv}_n (2, 1; x) \operatorname{Conv}_n \left(2, 1; \frac{z}{x}\right) \right) \right\}$$

$$\operatorname{sf}_{3,3}(n) := \sum_{k=0}^{n} (k!!!)^2 \qquad \longrightarrow \qquad \{1, 2, 6, 15, 31, 131, 455, 1239, 7639, 33883, \ldots\}$$

$$= [x^0 z^n] \Bigg\{ \frac{1}{(1-z)} \Bigg(\operatorname{Conv}_n (3, 3; x) \operatorname{Conv}_n \left(3, 3; \frac{z}{x}\right) + z^{-1} \cdot \operatorname{Conv}_n (3, 2; x) \operatorname{Conv}_n \left(3, 2; \frac{z}{x}\right)$$

$$+ z^{-2} \cdot \operatorname{Conv}_n (3, 1; x) \operatorname{Conv}_n \left(1, 1; \frac{z}{x}\right) - \frac{1}{z} - \frac{1}{z^2} \Bigg) \Bigg\}.$$

The second factorial sum variant, denoted $\operatorname{sf}_2(n)$, in (4.7) is enumerated by an alternate approach by the more interesting summations given in [7, §3, p. 168]. In particular, we have another set of identities for the sums, $\operatorname{sf}_2(n)$, given by the following equations:

$$(n+1)! - 1 = (n+1)! \times \sum_{k=0}^{n} \frac{k}{(k+1)!}$$

$$= [z^n x^0] \left\{ \left(\frac{1}{x \cdot (1-x)} - \frac{e^x}{x} \right) \times \operatorname{Conv}_{n+2} \left(-1, n+1; \frac{z}{x}\right) \right\}$$

$$= [x^0 z^{n+1}] \left\{ \left(\frac{1}{(1-x)} - e^x \right) \times \operatorname{Conv}_{n+2} \left(1, 1; \frac{z}{x}\right) \right\}.$$

A related challenge is posed by the phrasings of several finite sums involving the double factorial function stated in the references [5; 12]. ———— Ⓔ

Example 4.6 (Enumerating More Challenging Sums)**.** Since we know that $(2k-1)!! \equiv [z^k] \operatorname{Conv}_n (2, 1; z)$ for all $0 \leq k < n$, we can generate modified product sequences through the following forms of the convergent function series:

$$\frac{(k+1)}{k!} \cdot (2k-1)!! = [x^0][z^k] \left\{ \operatorname{Conv}_k \left(-1, 2k-1; \frac{z}{x}\right) \cdot (x+1) e^x \right\}$$

$$= [x^0][z^k] \left\{ \operatorname{Conv}_k \left(2, 1; \frac{z}{x}\right) \cdot (x+1) e^x \right\}.$$

The following forms of n^{th}–order approximating exponential generating functions for the following known identity are easily obtained from the last equation:

$$(2n-1)!! = \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!} \cdot k \cdot (2k-3)!!$$

$$= (n-1)! \times [x_2^n][x_1^0] \left\{ \frac{x_2}{(1-x_2)} \times \mathrm{Conv}_n\left(2, 1; \frac{x_2}{x_1}\right) \times (x_1+1)e^{x_1} \right\}$$
$$= [x_1^0 x_2^0 x_3^{n-1}] \left\{ \mathrm{Conv}_n\left(1, 1; \frac{x_3}{x_2}\right) \mathrm{Conv}_n\left(2, 1; \frac{x_2}{x_1}\right) \times \frac{(x_1+1)}{(1-x_2)} \cdot e^{x_1} \right\}.$$

Notice that since we restrict our focus to enumerations of these factorial–function–related sequences through formal power series generating functions, we are somewhat limited here by the rationality requirement imposed in the second Hadamard product expansions from (4.4) in forming the corresponding approximate ordinary generating functions for these sums given in the last equation. ——— Ⓔ

Remark 4.7 (Generalized Superfactorial Sequences)**.** Let the more general superfactorial functions, $S_{\alpha,d}(n)$, formed as analogous products of the multiple, α–factorial functions given by the integer–valued sequences in (1.2) correspond to the expansions defined by the next equation.

$$S_{\alpha,d}(n) := \prod_{j=1}^{n} (\alpha j - d)!_{(\alpha)}, \ \alpha \in \mathbb{Z}^+, 0 \leq d < \alpha, n \geq 1 \tag{4.8}$$

Observe that whenever $n \geq 1$ and for any fixed $\alpha \in \mathbb{Z}^+$, we immediately obtain the following identity corresponding to a so–termed "ordinary" superfactorial functions, $S_1(n) \equiv S_{1,0}(n)$, defined as above from the corollary of Lemma 4.1 cited in (4.1):

$$S_1(n) = \alpha^{-\binom{n+1}{2}} \prod_{j=1}^{n} (\alpha j)!_{(\alpha)}, \ \forall \alpha \in \mathbb{Z}^+.$$

For the more general cases of $d > 0$, the generalized superfactorial function products defined by (4.8) are enumerated in similar fashion to the outline of the previous examples above by extending the convergent–based generating function constructions given as in (4.6) at each $n \geq 2$.

The special case sequences formed by the double factorial products, $S_{2,1}(n)$, and the quadruple factorial products, $S_{4,2}(n)$, are simplified using *Mathematica* to obtain the next closed–form expression given by

$$S_{2,1}(n) := \prod_{j=1}^{n} (2j-1)!! = \frac{A^{3/2}}{2^{1/24} e^{1/8} \pi^{1/4}} \cdot \frac{2^{n(n+1)/2}}{\pi^{n/2}} \times G\left(n + \frac{3}{2}\right)$$
$$S_{4,2}(n) := \prod_{j=1}^{n} (4j-2)!!!! = \frac{A^{3/2}}{2^{1/24} e^{1/8} \pi^{1/4}} \cdot \frac{4^{n(n+1)/2}}{\pi^{n/2}} \times G\left(n + \frac{3}{2}\right),$$

where $A \approx 1.2824271$ denotes *Glaisher's constant* [19, §5.17], and where the particular constant multiples in the previous equation correspond to the special case values, $\Gamma(1/2) \equiv \sqrt{\pi}$ and $G(3/2) \equiv A^{-3/2} 2^{1/24} e^{1/8} \pi^{1/4}$. Since the sequences generated by (4.8) are also expanded as the products

$$S_{\alpha,d}(n) = \prod_{j=1}^{n} \left\{ \alpha^j \cdot \left(1 - \frac{d}{\alpha}\right)_j \right\} = \prod_{j=1}^{n} \left\{ \frac{\alpha^j \cdot \Gamma\left(j + 1 - \frac{d}{\alpha}\right)}{\Gamma\left(1 - \frac{d}{\alpha}\right)} \right\},$$

further computation with *Mathematica* also yields the next several special cases:

$$S_{3,1}(n) = \prod_{j=1}^{n} (3j-1)!!! = 3^{n(n-1)/2} \left\{ \frac{2 \cdot G\left(\frac{5}{3}\right)}{G\left(\frac{8}{3}\right)} \right\}^n \times \frac{G\left(n + \frac{5}{3}\right)}{G\left(\frac{5}{3}\right)}$$
$$S_{4,1}(n) = \prod_{j=1}^{n} (4j-1)!!!! = 4^{n(n-1)/2} \left\{ \frac{3 \cdot G\left(\frac{7}{4}\right)}{G\left(\frac{11}{4}\right)} \right\}^n \times \frac{G\left(n + \frac{7}{4}\right)}{G\left(\frac{7}{4}\right)}$$
$$S_{5,1}(n) = \prod_{j=1}^{n} (5j-1)!_{(5)} = 5^{n(n-1)/2} \left\{ \frac{4 \cdot G\left(\frac{9}{5}\right)}{G\left(\frac{14}{5}\right)} \right\}^n \times \frac{G\left(n + \frac{9}{5}\right)}{G\left(\frac{9}{5}\right)}$$
$$S_{5,2}(n) = \prod_{j=1}^{n} (5j-2)!_{(5)} = 5^{n(n-1)/2} \left\{ \frac{3 \cdot G\left(\frac{8}{5}\right)}{G\left(\frac{13}{5}\right)} \right\}^n \times \frac{G\left(n + \frac{8}{5}\right)}{G\left(\frac{8}{5}\right)}.$$

We are then led to conjecture inductively, without proof in this example, that these sequences are generated by the next equation involving the Barnes G–function over the rational–valued inputs prescribed by the formula

$$S_{\alpha,d}(n) = \frac{\alpha^{n(n-1)/2} \cdot (\alpha-d)^n}{\Gamma\left(2-\frac{d}{\alpha}\right)^n} \times \frac{G\left(n+2-\frac{d}{\alpha}\right)}{G\left(2-\frac{d}{\alpha}\right)}.$$

Considering modified selections of the initial conditions satisfied by the product sequences, $p_n(\alpha, R)$, suggests further avenues to enumerating other particular forms of the Barnes G–function formed by the generalized integer–parameter product cases cited above through the rational generating function methods already outlined in this section [1; 2; 8], which then suggest additional identities for the Barnes G-functions, $G(z+2)$, over rational–valued $z > 0$ involving the special function zeros already defined by Section 1.3 [4; 11].

The generalized superfactorial sequences defined by (4.8) are also related to the *hyperfactorial function*, $H_1(n) := \prod_{1 \leq j \leq n} j^j$, and the corresponding relations to the factorial power sequences given by $H_1(n) \cdot G(n+1) = \{n!\}^n$ for natural numbers $n \geq 1$ [22, A002109]. Statements of congruence properties and other relations connecting these sequences are also considered in the references [3]. ———— ®

4.2.3 Example: Sums of Powers of Natural Numbers and the Binomial Coefficients

As a starting point for expansions of integral powers by the binomial coefficients, let $p \geq 2$ be fixed, and suppose that $m \in \mathbb{Z}^+$. The convergent–based series over the integer powers, m^p, are generated by appealing to the binomial theorem forming the next sums:

$$m^p - 1 = (p-1)! \cdot \left(p \times \sum_{k=0}^{p-1} \frac{(m-1)^{p-k}}{k!(p-k)!}\right) \tag{4.9}$$

$$m^p - 1 = [z^{p-1}][x^0]\left\{\operatorname{Conv}_p\left(-1, p-1; \frac{z}{x}\right) \times (me^{mx} - e^x)\right\}. \tag{4.10}$$

Next, we consider the following generating functions formed as sums over $0 \leq m \leq n$ in (4.9):

$$\widetilde{B}(w,x) := \sum_{n \geq 0}\left[\sum_{m=0}^{n}(me^{mx} - e^x)\right]w^n = -\frac{w^2e^{3x} - 2we^{2x} + (w^2 - w + 1)e^x}{(1-w)^2(we^x - 1)^2}.$$

We then obtain the following results exactly enumerating these finite sum sequences over p^{th} integral powers of the natural numbers $m \leq n$:

$$\sum_{m=0}^{n} m^p = (n+1) + [z^{p-1}w^n][x^0]\left\{\operatorname{Conv}_p\left(-1, p-1; \frac{z}{x}\right)\widetilde{B}(w,x)\right\}. \tag{4.11}$$

A somewhat related result for variations of more general cases of the power sums given above are enumerated similarly for $p \geq 1$ and some fixed indeterminate u according to the generating functions given by

$$\sum_{m=0}^{n} m^p u^m = \frac{u^{n+1}-1}{u-1} - [z^{p-1}w^n][x^0]\left\{\operatorname{Conv}_p\left(-1, p-1; \frac{z}{x}\right)\frac{u^2w^2e^{3x} - 2uwe^{2x} + (u^2w^2 - uw + 1)e^x}{(1-w)(1-uw)(uwe^x - 1)^2}\right\}. \tag{4.12}$$

The two–variable generating functions involved in enumerating the sequences in each of (4.11) and (4.12) are related to the generating functions for the *Bernoulli* and *Euler polynomials*, $B_n(x)$ and $E_n(x)$, given in [19, §24.2]. For $u := \pm 1$, the sums on the left–hand–sides of the previous two equations also correspond to special cases of the following known identities involving these polynomial sequences [19, §24.4(iii)]:

$$\sum_{m=0}^{m}(am+b)^p = \frac{a^p}{p+1}\left[B_{p+1}\left(n+1+\frac{b}{a}\right) - B_{p+1}\left(\frac{b}{a}\right)\right] \qquad \text{(Sums of Powers Formulas)}$$

$$\sum_{m=0}^{m}(-1)^m(am+b)^p = \frac{a^p}{2}\left[(-1)^n \cdot E_p\left(n+1+\frac{b}{a}\right) + E_p\left(\frac{b}{a}\right)\right].$$

The results in the previous equations are compared to the forms of other well–known sequence generating functions involving the *Bernoulli numbers*, B_n, the *first–order Eulerian numbers*, $\left\langle {n \atop m} \right\rangle$, and the *Stirling numbers of the second kind*, $\left\{ {n \atop k} \right\}$, in the next few established identities for these sequences given in [22, A027641, A027642; A008292; A008277] [13, *cf.* §6] given in Remark 4.8.

Remark 4.8 (Comparisons to Other Known Sequence Generating Functions)**.** The sequences enumerated by (4.11) are first compared with the following known identities that exactly generate these finite sums over $n \geq 0$ and $p \geq 1$ [19, §24.4(iii), §24.2] [13, §6, §7.4]:

$$\begin{aligned}
\sum_{m=0}^{n} m^p &= \frac{B_{p+1}(n+1) - B_{p+1}(0)}{p+1} \\
&= \sum_{s=0}^{p} \binom{p+1}{s} \frac{B_s \cdot (n+1)^{p+1-s}}{(p+1)} \\
\sum_{m=0}^{n} m^p &= [z^n] \left\{ \sum_{i \geq 0} \left\langle {p \atop i} \right\rangle \frac{z^{i+1}}{(1-z)^{p+2}} \right\} \\
&= p! \cdot [w^n z^p] \left\{ \frac{w \cdot e^z}{(1-w)(1-we^z)} \right\}.
\end{aligned}$$

Similarly, the generalized sums generated by (4.12) are related to more well–known identities expanded as follows [19, §26.8] [13, §7.4]:

$$\begin{aligned}
\sum_{m=0}^{n} m^p u^m &= \sum_{j=0}^{p} \left\{ {p \atop j} \right\} u^j \times \frac{\partial^{(j)}}{\partial u^{(j)}} \left\{ \frac{1}{1-u} - \frac{u^{n+1}}{1-u} \right\} \\
&= [w^n] \left\{ \sum_{j=0}^{p} \left\{ {p \atop j} \right\} \frac{(uw)^j \cdot j!}{(1-w)(1-uw)^{j+1}} \right\} \\
&= p! \cdot [w^n z^p] \left\{ \frac{uw \cdot e^z}{(1-w)(1-uwe^z)} \right\}.
\end{aligned}$$

The role of the parameter p corresponding to the forms of the special sequence triangles in the identities given above is phrased through the implicit dependence of the convergent functions on this $p \geq 1$ in each of (4.10), (4.11), and (4.12). ——— ®

A second motivating example highlighting the procedure outlined above expands the sequences, $2^p - 1$ for $p \geq 1$, formed by an extension of the first result given in (4.9) when $m := 2$. The finite sums for integer powers provided by the binomial theorem in these cases correspond to removing, or selectively peeling off, the r uppermost–indexed terms from the first sum in the for the subsequent cases of $p \geq r \leq 1$ given by

$$m^p - 1 = \sum_{i=0}^{r} \binom{p}{p+1-i} (m-1)^i + (p-r-1)! \cdot \left(p(p-1) \cdots (p-r) \times \sum_{k=0}^{p-r-1} \frac{(m-1)^{k+1}}{(k+1)!(p-1-k)!} \right)$$

The generating function identities phrased by (4.9) are then modified slightly according to the previous equation for the next several special cases of $r \geq 1$. These identities are employed to obtain the following analogous generating function expansions generalizing the result from (4.10) above [22, A000225, A000918]:

$$\begin{aligned}
2^p - 1 &= [z^{p-2}][x^0] \left\{ \frac{1}{(1-z)} + \left(4e^{2x} - 2e^x\right) \times \mathrm{Conv}_p\left(-1, p-2; \frac{z}{x}\right) \right\}, \ p \geq 2 \\
&= [z^{p-3}][x^0] \left\{ \frac{4-3z}{(1-z)^2} + \left(8e^{2x} - e^x \cdot (x+5)\right) \times \mathrm{Conv}_p\left(-1, p-3; \frac{z}{x}\right) \right\}, \ p \geq 3 \\
&= [z^{p-4}][x^0] \left\{ \frac{11-17z+7z^2}{(1-z)^3} + \left(16e^{2x} - \frac{e^x}{2} \cdot (x^2+10x+24)\right) \times \mathrm{Conv}_p\left(-1, p-4; \frac{z}{x}\right) \right\}, \ p \geq 4.
\end{aligned}$$

The next few special cases of these generating functions for the p^{th} powers given above are expanded in the next more general convergent function identities for $p > m \geq 1$:

$$\begin{aligned}
2^p - 1 &= [z^{p-m-1}x^0] \left\{ \frac{\widetilde{\ell}_{m,2}(z)}{(1-z)^m} + \left(2^{m+1} \cdot e^{2x} - \frac{e^x}{(m-1)!} \cdot \widetilde{p}_{m,2}(x)\right) \times \operatorname{Conv}_p\left(-1, p-m-1; \frac{z}{x}\right) \right\} \\
&= [x^0 z^{p-m-1}] \left\{ \frac{\widetilde{\ell}_{m,2}(z)}{(1-z)^m} + \left(2^{m+1} \cdot e^{2x} - \frac{e^x}{(m-1)!} \cdot \widetilde{p}_{m,2}(x)\right) \times \operatorname{Conv}_p\left(1, 1; \frac{z}{x}\right) \right\}, \ p > m \geq 1.
\end{aligned}$$

The listings provided in Table 4 (page 41) cite several particular special cases of the polynomials, $\ell_{m,2}(z)$ and $p_{m,2}(x)$, that provide the generalizations of the first cases expanded in the previous equations given above. The constructions of these new identities, including those formed by the binomial coefficient sums for the powers, $2^p - 2$, are motivated in the context of divisibility modulo p by [14, §8].

Further cases of the more general p^{th} power sequences, $(s+1)^p - 1$, for any fixed scalars, $s > 0$, are also enumerated similarly according to the next formulas:

$$\begin{aligned}
(s+1)^p - 1 = [z^{p-m-1}x^0] &\left\{ \frac{s^2 \ell_{m,s+1}(z)}{(1-sz)^m} + \left(-e^x + (s+1)^{m+1} \cdot e^{(s+1)x} - \frac{s^2 e^{sx}}{(m-1)!} \cdot p_{m,s+1}(sx)\right) \times \right. \\
&\left. \times \operatorname{Conv}_p\left(-1, p-m-1; \frac{z}{x}\right) \right\}, \ p > m \geq 1 \\
= [x^0 z^{p-m-1}] &\left\{ \frac{s^2 \ell_{m,s+1}(z)}{(1-sz)^m} + \left(-e^x + (s+1)^{m+1} \cdot e^{(s+1)x} - \frac{s^2 e^{sx}}{(m-1)!} \cdot p_{m,s+1}(sx)\right) \times \right. \\
&\left. \times \operatorname{Conv}_p\left(1, 1; \frac{z}{x}\right) \right\}, \ p > m \geq 1.
\end{aligned}$$

The second listings provided in Table 4 (page 41) cite several additional special cases corresponding to the polynomial sequences, $\ell_{m,s+1}(z)$ and $p_{m,s+1}(x)$, required to generate these more general p^{th} power sequences when $p > m \geq 1$ [22, *cf.* A000225; A024023; A024036; A024049].

5 Properties of the Generalized Convergent Functions

The respective convergent function component sequences, $\mathrm{FP}_h(\alpha, R; z)$ and $\mathrm{FQ}_h(\alpha, R; z)$, satisfy essentially the same second–order difference equation over h, with the exception of the prescribed initial conditions defining each respective function in (1.11) and (1.12). It happens that the denominator polynomial sequences, $\mathrm{FQ}_h(\alpha, R; z)$, are easily related to particular special cases of the confluent hypergeometric function, $U(-h, b, z)$, and the associated Laguerre polynomials, $L_h^{(\beta)}(z)$. The numerator convergent sequences have less obvious expansions through special functions, or otherwise more well–known polynomial sequences[7]. Since the reciprocal zeros of the denominator functions, $\mathrm{FQ}_h(\alpha, R; z)$, determine the series expansions for the convergent functions up to parametrized constant multiples, we prefer to focus on the comparatively simple factored expressions of the denominator sequences expanded by the zeros of the confluent hypergeometric function and generalized Laguerre polynomial sequences studied in the references [19, *cf.* §18.6(iv); §13.9(ii)] [4; 11].

The identification of the convergent denominator functions as special cases of the confluent hypergeometric function yields additional identities providing analogous addition and multiplication theorems for these functions with respect to z, as well as a number of further, new recurrence relations derived from established relations, such as those provided by *Kummer's transformations*. These properties form a superset of extended results beyond the immediate, more combinatorial, known relations for the J–fractions summarized in Section 3.1 [9; 10; 19]. The latter characterization of the generalized convergent functions by the Laguerre polynomials also provides factorizations taken over the zeros of the classical orthogonal polynomial sequence forms studied in the references [4; 11], for example, as employed to state the results provided by (1.15) and (1.16) in Section 1.3.

[7] A point concerning the relative simplicity for the expressions of the denominator convergent polynomials compared to the numerator convergent sequences is also mentioned in §3.1 of Flajolet's article [9].

5.1 The Convergent Denominator Functions

In contrast with the convergent numerator functions, $\mathrm{FP}_h(\alpha, R; z)$, discussed next in Section 5.2, the corresponding denominator functions, $\mathrm{FQ}_h(\alpha, R; z)$, are readily expressed through well–known special functions. The first several special cases given in Table 1 (page 39) suggest the next identity, which is proved following Proposition 5.1 below.

$$\mathrm{FQ}_h(\alpha, R; z) = \sum_{k=0}^{h} \binom{h}{k} (-1)^k \left[\prod_{j=0}^{k-1} (R + (h-1-j)\alpha)\right] z^k \tag{5.1}$$

The convergent denominator functions are expanded by the *confluent hypergeometric function*, $U(-h, b, w)$, or equivalently by the *associated Laguerre polynomials*, $L_h^{(b-1)}(w)$, when $b :\mapsto R/\alpha$ and $w :\mapsto (\alpha z)^{-1}$ through the relations proved in the next proposition [19; 20].

Proposition 5.1 (*Exact Representations by Special Functions*).
The convergent denominator functions, $\mathrm{FQ}_h(\alpha, R; z)$, are expanded in terms of the confluent hypergeometric function and the associated Laguerre polynomials through the following results:

$$\begin{aligned} \mathrm{FQ}_h(\alpha, R; z) &= (\alpha z)^h \times U\left(-h, R/\alpha, (\alpha z)^{-1}\right) && (5.2) \\ &= (-\alpha z)^h \cdot h! \times L_h^{(R/\alpha - 1)}\left((\alpha z)^{-1}\right). && (5.3) \end{aligned}$$

Proof. We proceed to prove the first identity in (5.2) by induction. It easy to verify by computation (see Table 1) that the left–hand–side and right–hand–sides of (5.2) coincide when $h = 0$ and $h = 1$. For $h \geq 2$, we apply the recurrence relation from (1.11) to write the right–hand–side of (5.2) as

$$\begin{aligned} \mathrm{FQ}_h(\alpha, R; z) = (1 - (R + 2\alpha(h-1))z) U\left(-h+1, R/\alpha, (\alpha z)^{-1}\right) (\alpha z)^{h-1} \\ - \alpha(R + \alpha(h-2))(h-1) z^2 U\left(-h+2, R/\alpha, (\alpha z)^{-1}\right) (\alpha z)^{h-2}. \end{aligned} \tag{5.4}$$

The proof is completed using the known recurrence relation for the confluent hypergeometric function given by [19, §13.3(i)]

$$U(-h, b, u) = (u - b - 2(h-1)) U(-h+1, b, u) - (h-1)(b+h-2) U(-h+2, b, u). \tag{5.5}$$

In particular, we can rewrite (5.4) as

$$\begin{aligned} \mathrm{FQ}_h(\alpha, R; z) = (\alpha z)^h \Biggl[\left((\alpha z)^{-1} - \left(\frac{R}{\alpha} + 2(h-1)\right)\right) U\left(-h+1, R/\alpha, (\alpha z)^{-1}\right) \\ - \left(\frac{R}{\alpha} + h - 2\right)(h-1) U\left(-h+2, R/\alpha, (\alpha z)^{-1}\right) \Biggr], \end{aligned} \tag{5.6}$$

which implies (5.2) in the special case of (5.5) where $(b, u) := (R/\alpha, (\alpha z)^{-1})$. The second characterization of $\mathrm{FQ}_h(\alpha, R; z)$ by the Laguerre polynomials in (5.3) follows from the first result whenever $h \geq 0$ [19, §18.11]. ⊡

Proof of Equation (5.1)*.* The first identity for the denominator functions, $\mathrm{FQ}_h(\alpha, R; z)$, conjectured in (5.1) follows from the first statement of the previous proposition. We cite the particular expansions of $U(-n, b, z)$ when $n \geq 0$ is integer–valued involving the Pochhammer symbol, $(x)_n$, stated as follows [19, §13.2(i)]:

$$\begin{aligned} U(-n, b, z) &= \sum_{k=0}^{n} \binom{n}{k} (b+k)_{n-k} (-1)^n (-z)^k \\ &= \sum_{k=0}^{n} \binom{n}{k} (b+n-k)_k (-1)^k z^{n-k} \end{aligned} \tag{5.7}$$

Then (5.7) implies that (5.2) can be expanded as

$$\mathrm{FQ}_h(\alpha, R; z) = (\alpha z)^h U\left(-h, R/\alpha, (\alpha z)^{-1}\right)$$

$$\begin{aligned}
&= (\alpha z)^h \sum_{k=0}^{h} \binom{h}{k} (-1)^k \left(\frac{R}{\alpha}+h-k\right)_k (\alpha z)^{k-h} \\
&= \sum_{k=0}^{h} \binom{h}{k} \left(\frac{R}{\alpha}+h-k\right)_k (-\alpha z)^k \\
&= \sum_{k=0}^{h} \binom{h}{k} \left[\prod_{j=0}^{k-1} (R+(h-1-j)\alpha)\right] (-z)^k,
\end{aligned}$$

which completes the proof. ⊡

The coefficients of z from (5.1) also yield the next identities involving the product sequences from (1.1) that are employed in formulating several of the new results given in Section 5.3. In particular, we obtain the alternate restatements of these coefficients provided by the following equations:

$$\begin{aligned}
[z^k]\, \mathrm{FQ}_p(\alpha, R; z) &= \binom{p}{k} (-1)^k p_k(-\alpha, R+(p-1)\alpha) \cdot [0 \leq k \leq p]_\delta \\
&= \binom{p}{k} p_k(\alpha, -R-(p-1)\alpha) \cdot [0 \leq k \leq p]_\delta .
\end{aligned} \tag{5.8}$$

Corollary 5.2 (*Auxiliary Recurrence Relations*).
For $h \geq 0$ and any integers $s > -h$, the convergent denominator functions, $\mathrm{FQ}_h(\alpha, R; z)$, satisfy the reflection identity given by

$$\mathrm{FQ}_h(\alpha, \alpha s; z) = \mathrm{FQ}_{h+s-1}(\alpha; \alpha(2-s); z). \tag{5.9}$$

Additionally, for $h \geq 2$ these functions satisfy recurrence relations of the following forms:

$$\begin{aligned}
(R+\alpha h-\alpha)z\, \mathrm{FP}_h(\alpha, R-\alpha; z) + (\alpha z - Rz - 1)\, \mathrm{FQ}_h(\alpha, R; z) + \mathrm{FQ}_h(\alpha, R+\alpha; z) &= 0 \\
\mathrm{FQ}_h(\alpha, R; z) + \alpha h z\, \mathrm{FQ}_{h-1}(\alpha, R; z) - \mathrm{FQ}_h(\alpha, R-\alpha; z) &= 0 \\
(R+\alpha h)z\, \mathrm{FQ}_h(\alpha, R; z) + \mathrm{FQ}_{h+1}(\alpha, R; z) - \mathrm{FQ}_h(\alpha, R+\alpha; z) &= 0 \\
(1-\alpha h z)\, \mathrm{FQ}_h(\alpha, R; z) - \mathrm{FQ}_h(\alpha, R+\alpha; z) - \alpha h(R+\alpha h-\alpha)z^2\, \mathrm{FQ}_{h-1}(\alpha, R; z) &= 0 \\
(1-\alpha h z-\alpha z)\, \mathrm{FQ}_h(\alpha, R; z) - \mathrm{FQ}_{h+1}(\alpha, R; z) + (\alpha - \alpha h - R)z\, \mathrm{FQ}_h(\alpha, R-\alpha; z) &= 0 \\
-\alpha(h-1)z\, \mathrm{FQ}_{h-2}(\alpha, R+2\alpha; z) + (1-Rz)\, \mathrm{FQ}_{h-1}(\alpha, R+\alpha; z) - \mathrm{FQ}_h(\alpha, R; z) &= 0
\end{aligned} \tag{5.10}$$

Proof. The first equation results from *Kummer's transformation* for the confluent hypergeometric function, $U(a, b, z)$, given by [19, §13.2(vii)]

$$U(a, b, z) = z^{1-b} U(a-b+1, 2-b, z).$$

In particular, when $R := \alpha s$ and $h+s-1 \geq 0$ Proposition 5.1 implies that

$$\begin{aligned}
\mathrm{FQ}_h(\alpha, R; z) &= (\alpha z)^{h+R/\alpha-1}\, U\left(-(h+\frac{R}{\alpha}-1), \frac{2\alpha-R}{\alpha}, (\alpha z)^{-1}\right) \\
&= \mathrm{FQ}_{h+R/\alpha-1}(\alpha, 2\alpha-R; z).
\end{aligned}$$

The recurrence relations stated in (5.10) follow similarly consequences of the first proposition by applying the known results for the confluent hypergeometric functions cited in the reference [19, §13.3(i)]. ⊡

Proposition 5.3 (*Addition and Multiplication Theorems*).
Let $z, w \in \mathbb{C}$ with $z \neq w$ and suppose that $z \neq 0$. For a fixed $\alpha \in \mathbb{Z}^+$ and $h \geq 0$, the following finite sums provide two addition theorem analogs satisfied by the sequences of convergent denominator functions:

$$\begin{aligned}
\mathrm{FQ}_h(\alpha, R; z-w) &= \sum_{n=0}^{h} \frac{(-h)_n (-w)^n (z-w)^{h-n}}{z^h \cdot n!} \mathrm{FQ}_{h-n}(\alpha, R+\alpha n; z) \\
\mathrm{FQ}_h(\alpha, R; z-w) &= \sum_{n=0}^{h} \frac{(-h)_n \left(1-h-\frac{R}{\alpha}\right)_n (\alpha w)^n}{n!} \mathrm{FQ}_{h-n}(\alpha, R; z).
\end{aligned} \tag{5.11}$$

The corresponding multiplication theorems for the denominator functions are stated similarly for $h \geq 0$ in the form of the following equations:

$$\mathrm{FQ}_h(\alpha, R; zw) = \sum_{n=0}^{h} \frac{(-h)_n (w-1)^n w^{h-n}}{n!} \mathrm{FQ}_{h-n}(\alpha, R+\alpha n; z) \tag{5.12}$$

$$\mathrm{FQ}_h(\alpha, R; zw) = \sum_{n=0}^{h} \frac{(-h)_n \left(1-h-\frac{R}{\alpha}\right)_n (1-w)^n (\alpha z)^n}{n!} \mathrm{FQ}_{h-n}(\alpha, R; z).$$

Proof of the Addition Theorems. The sums stated in (5.11) follow from special cases of established addition theorems for the confluent hypergeometric function, $U(a, b, x+y)$, cited in [19, §13.13(ii)]. The particular addition theorems required in the proof are provided as follows:

$$U(a, b, x+y) = \sum_{n=0}^{\infty} \frac{(a)_n (-y)^n}{n!} U(a+n, b+n, x), \ |y| < |x| \tag{5.13}$$

$$U(a, b, x+y) = \left(\frac{x}{x+y}\right)^a \sum_{n=0}^{\infty} \frac{(a)_n (1+a-b)_n y^n}{n!(x+y)^n} U(a+n, b, x), \ \Re[y/x] > -\frac{1}{2}.$$

First, observe that in the special case inputs to $U(a, b, z)$ resulting from the application of Proposition 5.1 involving the functions

$$\mathrm{FQ}_h(\alpha, R; z) = U\left(-h, R/\alpha, (\alpha z)^{-1}\right),$$

in the infinite sums of (5.13) lead to to finite sum identities corresponding to the inputs, h, to $\mathrm{FQ}_h(\alpha, R; z)$ where $h \geq 0$. More precisely, the definition of the convergent denominator sequences provided by (1.12) requires that $\mathrm{FP}_h(\alpha, R; z) \equiv 0$ whenever $h < 0$.

To apply the cited results for $U(a, b, x+y)$ in these cases, let $z \neq w$, assume that both $\alpha, z \neq 0$, and suppose the parameters, x and y, in (5.11) are defined so that

$$x := (\alpha z)^{-1}, \ y := \frac{1}{\alpha}\left((z-w)^{-1} - z^{-1}\right), \ x+y = (\alpha(z-w))^{-1} \tag{5.14}$$

Since each of the sums in (5.11) involve only finitely–many terms, we ignore treatment of the convergence conditions given on the right–hand–side equations in (5.13) to justify these two restatements of the addition theorem analogs provided above. ⊡

Proof of the Multiplication Theorems. The second pair of identities stated in (5.12) are formed by the multiplication theorems for $U(a, b, z)$ noted as in [19, §13.13(iii)]. The proof is derived similarly from the first parameter definitions of x and y given in the addition theorem proof, with an additional adjustment employed in these cases corresponding to the change of variable $\widehat{y} \mapsto (y-1)x$, which is selected so that $x + \widehat{y} \equiv xy$ in the above proof. The analog to (5.14) that results in these two cases then yields the parameters, $x := (\alpha z)^{-1}$ and $y := (w^{-1}-1) \cdot (\alpha z)^{-1}$, in the first identities for the confluent hypergeometric function, $U(a, b, x+y)$, given by (5.13). ⊡

5.2 The Convergent Numerator Functions

The most direct expansion of the convergent numerator functions, $\mathrm{FP}_h(\alpha, R; z)$, is obtained from the *erasing operator*, defined as in Flajolet's first article, which performs the formal truncation operation given by the next equation.

$$\mathrm{E}_m \left[\!\left[\sum_i p_i z^i\right]\!\right] := \sum_i p_i \cdot z^i \cdot [i \leq m]_\delta \tag{Erasing Operator}$$

The numerator polynomials are then given in this notation through the following equations [9, *cf.* §3]:

$$\mathrm{FP}_h(\alpha, R; z) = \mathrm{E}_{h-1}\left[\!\left[\mathrm{FQ}_h(\alpha, R; z) \cdot \mathrm{Conv}_h(\alpha, R; z)\right]\!\right]$$

$$= \sum_{k=0}^{h-1} \left\{ \sum_{i=0}^{k} \binom{h}{i} (-1)^i p_{k-i}(\alpha, R) \cdot p_i\left(-\alpha, R + (h-1)\alpha\right) \right\} \times (\alpha z)^k. \tag{5.15}$$

The coefficients of z^k expanded by the last equation are rewritten slightly in terms of the Pochhammer symbols, $(x)_n$, to arrive at an equivalent formula given in (5.16).

$$[z^k]\,\mathrm{FP}_h(\alpha, R; z) = \left\{ \sum_{i=0}^{k} \binom{h}{i} (R/\alpha)_{k-i} \,(1 - h - R/\alpha)_i \right\} \times \alpha^k, \ 0 \leq k < h \tag{5.16}$$

These sums are remarkably similar in form to the next binomial–type convolution formula, or *Vandermonde identity*, stated as follows [7; 26][8]:

$$\begin{aligned} (x+y)_k &= \sum_{i=0}^{k} \binom{k}{i} (x)_i \,(y)_{k-i} \qquad \text{(Vandermonde Convolution)} \\ &= \sum_{i=0}^{k} \binom{k}{i} (-1)^k \,(x+i)_{k-i} \,(-y)_i \,. \end{aligned}$$

A separate treatment of other properties implicit to these more complicated expansions is briefly explored through the definitions of the three additional forms of auxiliary coefficient sequences, denoted in respective order by $C_{h,n}(\alpha, R) := [z^n]\,\mathrm{FP}_h(\alpha, R; z)$, $R_{h,k}(\alpha; z) := [R^k]\,\mathrm{FP}_h(\alpha, R; z)$, and $T_h^{(\alpha)}(n, k) := [z^n R^k]\,\mathrm{FP}_h(\alpha, R; z)$, in the next remark.

Remark 5.4 (Alternate Forms of the Convergent Numerator Sequences)**.** The next results summarize recurrence relations satisfied by three particular variations of the numerator sequences considered, respectively, as polynomials in z and R. For $h \geq 2$, fixed $\alpha \in \mathbb{Z}^+$, and $n, k \geq 0$, we consider the following auxiliary numerator coefficient subsequences:

$$\begin{aligned} C_{h,n}(\alpha, R) &:= [z^n]\,\mathrm{FP}_h(\alpha, R; z)\,, \quad \text{for } \ 0 \leq n \leq h-1 \\ &= C_{h-1,n}(\alpha, R) - (R + 2\alpha(h-1))C_{h-1,n-1}(\alpha, R) - \alpha(R + \alpha(h-2))(h-1)C_{h-2,n-2}(\alpha, R) \\ R_{h,k}(\alpha; z) &:= [R^k]\,\mathrm{FP}_h(\alpha, R; z)\,, \quad \text{for } \ 0 \leq k \leq h-1 \\ &= (1 - 2\alpha(h-1)z)R_{h-1,k}(\alpha; z) - \alpha^2(h-1)(h-2)z^2 R_{h-2,k}(\alpha; z) - zR_{h-1,k-1}(\alpha; z) \\ &\quad - \alpha(h-1)z^2 R_{h-2,k-1}(\alpha; z) \\ T_h^{(\alpha)}(n, k) &:= [z^n R^k]\,\mathrm{FP}_h(\alpha, R; z)\,, \quad \text{for } \ 0 \leq n, k \leq h-1 \\ &= T_{h-1}^{(\alpha)}(n, k) - T_{h-1}^{(\alpha)}(n-1, k-1) - 2\alpha(h-1)T_{h-1}^{(\alpha)}(n-1, k) \\ &\quad - \alpha(h-1)T_{h-2}^{(\alpha)}(n-2, k-1) - \alpha^2(h-1)(h-2)T_{h-2}^{(\alpha)}(n-2, k) \\ &\quad + \left([z^n R^0]\,\mathrm{FP}_h(z)\right)[h \geq 1]_\delta \,[n \geq 0]_\delta \,[k = 0]_\delta \,. \end{aligned}$$

Each of the difference equations cited in the previous equations are derived from (1.11) by a straightforward application of the coefficient extraction method first motivated in [21]. Table 5 (page 42) and Table 6 (page 43) list the first few special cases of the first two auxiliary forms of these component polynomial subsequences.

We also state, without proof, a number of multiple, alternating sums involving the Stirling number triangles that generate these auxiliary polynomial subsequences for reference in the next several equations. In particular, for $h \geq 1$ and $0 \leq n < h$, the sequences, $C_{h,n}(\alpha, R)$, are expanded by the following sums:

$$C_{h,n}(\alpha, R) = \sum_{\substack{0 \leq m \leq k \leq n \\ 0 \leq s \leq n}} \left\{ \binom{h}{k} \binom{m}{s} \begin{bmatrix} k \\ m \end{bmatrix} (-1)^m \alpha^n \left(\frac{R}{\alpha}\right)_{n-k} \left(\frac{R}{\alpha} - 1\right)^{m-s} \right\} \times h^s \tag{5.17}$$

[8] The product–wise *connection formulas* for the Pochhammer symbol given by

$$(x)_n \,(x)_m = \sum_{k=0}^{\min(m,n)} \binom{m}{k} \binom{n}{k} k! \cdot (x)_{m+n-k} \,, n \neq m,$$

provides additional expansions of (5.16) involving terms of the Pochhammer symbols, $(R/\alpha)_j$ and $(h)_j$.

$$= \sum_{\substack{0\leq m\leq k\leq n\\ 0\leq t\leq s\leq n}} \left\{\binom{h}{k}\binom{m}{t}\begin{bmatrix}k\\m\end{bmatrix}\begin{bmatrix}n-k\\s-t\end{bmatrix}(-1)^m\alpha^{n-s}(h-1)^{m-t}\right\} \times R^s$$

$$= \sum_{\substack{0\leq m\leq k\leq n\\ 0\leq i\leq s\leq n}} \binom{h}{k}\binom{h}{i}\binom{m}{s}\begin{bmatrix}k\\m\end{bmatrix}\begin{Bmatrix}s\\i\end{Bmatrix}(-1)^m\alpha^n\left(\frac{R}{\alpha}\right)_{n-k}\left(\frac{R}{\alpha}-1\right)^{m-s} \times i!$$

$$= \sum_{\substack{0\leq m\leq k\leq n\\ 0\leq v\leq i\leq s\leq n}} \binom{h}{k}\binom{m}{s}\binom{i}{v}\binom{h+v}{v}\begin{bmatrix}k\\m\end{bmatrix}\begin{Bmatrix}s\\i\end{Bmatrix}(-1)^{m+i-v}\alpha^{n+s-m}\left(\frac{R}{\alpha}\right)_{n-k}\left(\frac{R}{\alpha}-1\right)^{m-s} \times i!.$$

Similarly, for all $h \geq 1$ and $0 \leq k < h$, the sequences, $R_{h,k}(\alpha, R)$, are expanded as follows:

$$\begin{aligned} R_{h,k}(\alpha; z) &= \sum_{\substack{0\leq m\leq i\leq n<h\\ 0\leq t\leq k}} \left\{\binom{h}{i}\binom{m}{t}\begin{bmatrix}i\\m\end{bmatrix}\begin{bmatrix}n-i\\k-t\end{bmatrix}(-1)^m\alpha^{n-k}(h-1)^{m-t}\right\} \times z^n \\ &= \sum_{\substack{0\leq m\leq i\leq n<h\\ 0\leq t\leq k\\ 0\leq p\leq m-t}} \left\{\binom{h}{i}\binom{m}{t}\binom{h-1}{p}\begin{bmatrix}i\\m\end{bmatrix}\begin{bmatrix}n-i\\k-t\end{bmatrix}\begin{Bmatrix}m-t\\p\end{Bmatrix}(-1)^m\alpha^{n-k}\times p!\right\} \times z^n. \end{aligned} \tag{5.18}$$

Since the exact special function expansions of the denominator sequences, $\mathrm{FQ}_h(\alpha, R; z)$, proved in Section 5.1 characterize the series coefficients of $\mathrm{Conv}_h(\alpha, R; z)$, a more careful treatment of the properties of these subsequences is omitted from this section for brevity. ——— ®

5.3 New Identities Resulting From Expansions by Finite Difference Equations

The rationality of the convergent functions, $\mathrm{Conv}_h(\alpha, R; z)$, in z for all h provides new forms of h–order finite difference equations with respect to n satisfied by the product sequences, $p_n(\alpha, R)$, when α and R correspond to fixed parameters independent of the sequence indices n. The next multiple summation identities obtained from these difference equations are satisfied by both exact expansions of the functions, $p_n(\alpha, R)$, and by the generalized α–factorial functions modulo any prescribed integers $p \geq 2$. The rationality of the h^{th} convergent functions immediately suggests the results for both forms of the congruence properties given in (5.19) modulo integers $p \geq 2$, and for the exact expansions of the generalized products, $p_n(\alpha, R)$, given below in (5.20) below [7; 13; 17; 25].

For fixed $\alpha \neq 0$ and integers $p, n-s \geq 1$, we employ (5.8) from the properties of the convergent denominator functions, $\mathrm{FQ}_h(\alpha, R; z)$, already noted in Section 5.1 above to obtain the following results where the coefficients, $C_{p,n-s} := [z^{n-s}]\,\mathrm{FP}_p(\alpha, R; z)$, correspond to the respective formulas given in (5.16) and (5.17) of Section 5.2:

$$p_{n-s}(\alpha, R) \equiv \begin{Bmatrix} \sum\limits_{k=0}^{p-1}\binom{p}{k+1}(-1)^k p_{k+1}(-\alpha, R+(n-1)\alpha)p_{n-s-1-k}(\alpha, R) & n-s\geq p \\ [z^{n-s}]\,\mathrm{FP}_p(\alpha, R; z) & n-s<p. \end{Bmatrix} \pmod{p} \tag{5.19}$$

$$\begin{aligned} p_{n-s}(\alpha, R) &= \sum_{k=0}^{n-s-1}\binom{n-s}{k+1}(-1)^k p_{k+1}(-\alpha, R+(n-s-1)\alpha)p_{n-s-1-k}(\alpha, R) \\ &= \sum_{k=0}^{n-s-1}\binom{n-s}{k}(-1)^{n-1-s-k} p_k(\alpha, R)p_{n-s-k}(-\alpha, R+(n-s-1)\alpha). \end{aligned} \tag{5.20}$$

When the initially indeterminate parameter, R, assumes an implicit dependence on the sequence index, n, the results phrased by the previous equations, somewhat counter intuitively, do not imply difference equations satisfied between the generalized product sequences, either exactly, or modulo the prescribed choices of $p \geq 2$. The new formulas connecting the generalized product sequences, $p_n(\alpha, \beta n + \gamma)$, resulting from (5.19) and (5.20) in these cases are, however, reminiscent of the relations satisfied between the generalized Stirling polynomial sequences studied in the references [13; 15; 21].

The product sequence forms given in terms of the *Pochhammer symbol* and *Pochhammer k–symbols*, expanded as $p_n(\alpha, R) \equiv \alpha^n (R/\alpha)_n$ and $p_n(k, x) \equiv (x)_{n,k}$, respectively, yield comparisons between various known generalized forms of Vandermonde's convolution identity, stated as in Section 5.2 from above, with the new formulas cited in (5.19) and (5.20) [7; 8; 13; 20]. The identity from (1.3) relating the generalized product sequences, $p_{n-s}(-\alpha, \beta n + \gamma)$, to the Gould polynomials, $G_n(x; a, b)$, combined with the congruence properties in (5.19) provides the next relations for this Sheffer sequence modulo any prescribed integers $p \geq 2$ whenever $n - s > p$[9]:

$$\binom{\frac{\beta n+\gamma}{\alpha}}{n-s}\alpha^{n-s}(n-s)! \equiv -\sum_{k=1}^{p}\binom{p}{k}\binom{\frac{(\alpha-\beta)n-\gamma-\alpha}{\alpha}}{k}\binom{\frac{\beta n+\gamma}{\alpha}}{n-s-k}\times \times \alpha^{n-s}k!(n-s-k)! \pmod{p}$$

$$\binom{\frac{\beta n+\gamma}{\alpha}+n-s-1}{n-s}\alpha^{n-s}(n-s)! \equiv -\sum_{k=1}^{p}\binom{p}{k}\binom{\frac{(\alpha+\beta)n+\gamma-\alpha}{\alpha}}{k}\binom{\frac{\beta n+\gamma}{\alpha}+n-s-k-1}{n-s-k}\times \times \alpha^{n-s}(-1)^{k+1}\cdot k!(n-s-k)! \pmod{p}$$

The first formula given in (5.20) for these sequences also yields the exact sums over the corresponding variants of the binomial coefficients expanded as follows:

$$\binom{\frac{\beta n+\gamma}{\alpha}}{n-s} = -\sum_{k=1}^{n-s}\binom{\frac{(\alpha-\beta)n-\gamma-(s+1)\alpha}{\alpha}}{k}\binom{\frac{\beta n+\gamma}{\alpha}}{n-s-k}.$$

Example 5.5 (Finite Sums Involving the α–Factorial Functions)**.** The double factorial function, $(2n-1)!!$, satisfies a number of known expansions through the finite sum identities summarized in [5; 12]. The particular "round number" identity in the form of (5.21) is remarkably similar to the statement of the first sum in (5.20) satisfied by the more general product cases.

$$(2n-1)!! = \sum_{k=0}^{n-1}\binom{n}{k+1}(2k-1)!!(2n-2k-3)!!. \tag{5.21}$$

If we assume that $\alpha \geq 2$ is integer–valued, and proceed to expand these α–factorial functions according to the expansions from (4.2) and (5.20) above, we see readily that[10] [11]

$$(\alpha n-1)!_{(\alpha)} = \sum_{k=0}^{n-1}\binom{n-1}{k+1}(-1)^k \times \left(\frac{1}{\alpha}\right)_{-(k+1)}\left(\frac{1}{\alpha}-n\right)_{k+1} \times (\alpha k+\alpha-1)!_{(\alpha)}(\alpha n-\alpha k-\alpha-1)!_{(\alpha)}\times$$

$$(\alpha n-1)!_{(\alpha)} = \sum_{k=0}^{n-1}\binom{n-1}{k+1}(-1)^k \times \binom{\frac{1}{\alpha}+k-n}{k+1}\binom{\frac{1}{\alpha}-1}{k+1}^{-1} \times (\alpha k+\alpha-1)!_{(\alpha)}(\alpha n-\alpha k-\alpha-1)!_{(\alpha)}.$$

The construction of further analogs for generalized variants of the finite summations and more well–known combinatorial identities satisfied by the double factorial function cases when $\alpha := 2$ is suggested as another topic for future investigation. The next few particularly interesting special cases corresponding to the triple and quadruple factorial functions, $n!!!$ and $n!!!!$, respectively, are one starting point for approaching the generalized forms of the identities summarized in the references [5; 12]. ──── Ⓔ

[9] Since $(x)_k = \binom{x+k-1}{k}\cdot k! = (-1)^k\binom{-x}{k}\cdot k!$.

[10] For the non–integral cases of $z, w \in \mathbb{C}$ satisfying $w, z-w \neq -1, -2, -3, \ldots$, the generalized forms of the binomial coefficients are defined by $\binom{z}{w} := \Gamma(z+1)/\left[\Gamma(w+1)\Gamma(z-w+1)\right]$ [19, §5] [13, *cf.* §5].

[11] We also note the simplifications [26]: $\left(\frac{1}{\alpha}\right)_{-(k+1)} = \frac{(-\alpha)^{k+1}}{(\alpha(k+1)-1)!_{(\alpha)}}$.

We are primarily concerned with cases of the α–factorial functions formed the products, $p_n(\alpha, R_n)$, when the parameter $R_n := \beta n + \gamma$ depends linearly on n. Strictly speaking, once we evaluate the indeterminate, R, as a function of n in the sequences, $p_n(\alpha, R)$, the generating functions over the convergent sequences no longer correspond to predictably rational functions of z. We may, however, still prefer to work with these sequences formulated as finite–degree polynomials in n through a few useful forms of the multiple summation identities expanded below. The resulting forms of the generalized factorial–function–like sequences offer a dual interpretation to the corresponding exact formulas for these sequence cases stated in terms of the special function zeros already cited above to provide the sequence expansions given in Section 1.3.

Remark 5.6 (Generalized Polynomial Expansions by Finite Sum Identities)**.** The following particular cases of finite, multiple sums for the generalized factorial functions, $p_{n-s}(\alpha, \beta n + \gamma)$, are provided where $n - s \geq 1$ and $\alpha, \beta, \gamma \in \mathbb{Q}$ are taken to be fixed parameters:

$$\begin{aligned} p_{n-s}(\alpha, \beta n + \gamma) = \sum_{\substack{0 \leq m \leq k < n-s \\ 0 \leq r \leq p \leq n-s}} \sum_{t=0}^{n-s-k} \binom{m}{r} \binom{n-s}{k} \binom{t}{p-r} \begin{bmatrix} k \\ m \end{bmatrix} \begin{bmatrix} n-s-k \\ t \end{bmatrix} \times & \\ \times (-\alpha - \beta)^{p-r} \alpha^{n-s-m-t} \beta^r \gamma^{m-r} (\alpha(s+1) - \gamma)^{t+r-p} \times & \\ \times n^p & \end{aligned} \tag{5.22}$$

$$\begin{aligned} p_{n-s}(\alpha, \beta n + \gamma) = \sum_{\substack{0 \leq r \leq p \leq u \leq 3n \\ 0 \leq m \leq k, i < n-s}} \sum_{t=0}^{n-s-k} \binom{m}{r} \binom{i}{p-u} \binom{t}{p-r} \begin{bmatrix} k \\ i \end{bmatrix} \begin{bmatrix} k \\ m \end{bmatrix} \begin{bmatrix} n-k-s \\ t \end{bmatrix} \times & \\ \times \frac{(-1)^{u-r+t-k}}{k!} \alpha^{n-s-m-t} \beta^r (\alpha + \beta)^{p-r} \gamma^{m-r} \times & \\ \times s^{p-u+i} (\alpha(s+1) - \gamma)^{t+r-p} \times n^u. & \end{aligned}$$

The forms of these expansions for the generalized factorial function sequence variants in the equations above are provided here without citing the details to a somewhat tedious, and unnecessary, proof derived from the well–known polynomial expansions of the products, $p_n(\alpha, R) \equiv \alpha^n \left(R/\alpha\right)_n$ by the Stirling number triangles.

More concretely, for $n, k \geq 0$ and fixed $\alpha, \beta, \gamma, \rho, n_0 \in \mathbb{Q}$, the following particular expansions suffice to show enough of the detail needed to more carefully prove each of the multiple sum identities cited in (5.22) starting from the first statements provided in (5.20):

$$\begin{aligned} p_k(\alpha, \beta n + \gamma + \rho) &= \alpha^k \cdot \left(\frac{\beta n + \gamma + \rho}{\alpha}\right)_k \\ &= \sum_{m=0}^{k} \begin{bmatrix} m \\ k \end{bmatrix} \alpha^{k-m} \left(\beta n + \gamma + \rho\right)^m \\ &= \sum_{p=0}^{k} \left\{ \sum_{m=p}^{k} \begin{bmatrix} k \\ m \end{bmatrix} \binom{m}{p} \alpha^{k-m} \beta^p \left(\gamma + \rho + \beta n_0\right)^{m-p} \right\} \times (n - n_0)^p. \end{aligned}$$

Simplified triple sum expansions of interest in the next example correspond to a straightforward simplification of the more general multiple finite quintuple 5–sums and sextuple 6–sum summations that exactly enumerate the functions, $p_{n-s}(\alpha, \beta n + \gamma)$, when $(s, \alpha, \beta, \gamma) := (1, -1, 1, 0)$. ——— Ⓡ

One immediate consequence of Remark 5.6 phrases the form of triple sums that exactly enumerate the single factorial functions, $(n-s)!$, modulo any prescribed $p \geq 2$. In particular, these results lead to the following finite, triple sum expansions of the single factorial function[12]:

$$(n-1)! = \sum_{p=0}^{n} \left[\sum_{0 \leq t \leq k < n} \binom{n}{n-1-k} \begin{bmatrix} n-1-k \\ p \end{bmatrix} \begin{bmatrix} k \\ k-t \end{bmatrix} (-1)^{n-1-p} \right] \times (n-1)^p \quad \text{(Single Factorial Triple Sums)}$$

[12] The third exact triple sum identity given in (5.23) is further expanded through the formula of Riordan cited in [7, p. 173] as follows:

$$\sum_{0 \leq k < n} \binom{n-1}{k} (k+1)! \cdot n^{n-1-k} = n^n.$$

$$= \sum_{p=0}^{n} \left[\sum_{\substack{0 \leq k < n \\ 0 \leq t \leq n-1-k}} \binom{n}{k} \begin{bmatrix} k \\ p \end{bmatrix} \begin{bmatrix} n-1-k \\ n-1-k-t \end{bmatrix} (-1)^{n-1-p} \right] \times (n-1)^p$$

$$= \sum_{p=0}^{n} \left[\sum_{0 \leq t \leq k < n} \binom{n}{n-1-k} \begin{bmatrix} n-1-k \\ n-p \end{bmatrix} \begin{bmatrix} k \\ k-t \end{bmatrix} (-1)^{p+1} \right] \times (n-1)^{n-p}. \tag{5.23}$$

For comparison, the next several equations provide related forms of finite, triple sum identities for the double factorial function, $(2n-1)!!$. The following expansions are obtained from (4.2) applied to the known double sum identities involving the Stirling numbers of the first kind documented in the references [5, §5.3]:

$$(2n-1)!! = \sum_{1 \leq j \leq k \leq n} \begin{bmatrix} k-1 \\ j-1 \end{bmatrix} 2^{n-j} (-1)^{n-k} (1-n)_{n-k} \qquad \text{(Double Factorial Multiple Sum Identities)}$$

$$= \sum_{\substack{1 \leq j \leq k \leq n \\ 0 \leq m \leq n-k}} \begin{bmatrix} k-1 \\ j-1 \end{bmatrix} \begin{bmatrix} n-k+1 \\ m+1 \end{bmatrix} 2^{n-j} (-1)^{n-k-m} n^m$$

$$(2n-1)!! = \sum_{1 \leq j \leq k \leq n} \binom{2n-k-1}{k-1} \begin{bmatrix} k \\ j \end{bmatrix} (2n-2k-1)!!$$

$$= \sum_{\substack{1 \leq j \leq k \leq n \\ 0 \leq m \leq n-k}} \binom{2n-k-1}{k-1} \begin{bmatrix} k \\ j \end{bmatrix} \begin{bmatrix} n-k \\ m \end{bmatrix} 2^{n-k-m}$$

$$= \sum_{\substack{1 \leq j \leq k \leq n \\ 0 \leq m \leq n-k}} \binom{2n-k-1}{k-1} \begin{bmatrix} k \\ j \end{bmatrix} \begin{bmatrix} n-k+1 \\ m+1 \end{bmatrix}_2 (-1)^{n-k-m} (2n-2k)^m .$$

A couple of the characteristic examples of the polynomial expansions in n by the Stirling numbers of the first kind in (5.23) are given next to illustrate the notable special cases of Wilson theorem and Clement's theorem given in Example 5.7. The next rephrasings of the classical primality conditions introduced in Section 1.4.3 provide one possible application to determining properties required modulo some as yet unspecified odd prime, $n \geq 3$.

Example 5.7 (Applications to Variants of Wilson's Theorem)**.** We first define the following variants of the first triple sum given in (5.23), denoted by $F_{\omega,n}(x_{\mathrm{p}}, x_{\mathrm{t}}, x_{\mathrm{k}})$, for application–dependent, prescribed functions $N_{\omega,p}(n)$ and $M_\omega(n)$, where the formal variables $\{x_{\mathrm{p}}, x_{\mathrm{t}}, x_{\mathrm{k}}\}$, index the terms corresponding to each individual sum over the respective variables, p, t, and k[13]:

$$F_{\omega,n}(x_{\mathrm{p}}, x_{\mathrm{t}}, x_{\mathrm{k}}) := \sum_{\substack{0 \leq t \leq k < n \\ 0 \leq p \leq n}} \binom{n}{n-1-k} \begin{bmatrix} n-1-k \\ p \end{bmatrix} \begin{bmatrix} k \\ k-t \end{bmatrix} \times \tag{5.24}$$

$$\times (-1)^{n-1-p} \times \{x_{\mathrm{p}}^p x_{\mathrm{t}}^t x_{\mathrm{k}}^k\} \times N_{\omega,p}(n) \qquad \pmod{M_\omega(n)}.$$

The next specialized forms of the parameters implicit to the previous congruence are defined as follows to form another restatement of Wilson's theorem corresponding to the result given immediately below:

$$(\omega, N_{\omega,p}(n), M_\omega(n)) :\mapsto (\mathrm{WT}, (-1)^p, n)\,. \qquad \text{(Wilson)}$$

Then we see that

$$n \geq 2 \text{ prime} \iff F_{\mathrm{WT},n}(1,1,1) \equiv -1 \pmod{M_{\mathrm{WT}}(n)}. \qquad \text{(Wilson's Theorem)}$$

Numerical computations with *Mathematica's* `PolynomialMod` function suggest several nice properties satisfied by the trivariate polynomial sequences, $F_{\omega,n}(x_{\mathrm{p}}, x_{\mathrm{t}}, x_{\mathrm{k}})$, defined by (5.24) when n is prime, particularly as formed in the cases taken over the following polynomial configurations of the three formal variables, x_{p}, x_{t}, and x_{k}:

$$(x_{\mathrm{p}}, x_{\mathrm{t}}, x_{\mathrm{k}}) \in \{(x,1,1), (1,x,1), (1,1,x)\}\,.$$

[13] Notice that when $N_{\omega,p}(n) := (n-1)^p$, the function $F_{\omega,n}(1,1,1)$ exactly generates the single factorial function, $(n-1)!$.

The divisibility of the Stirling numbers of the first kind is tied to well–known expansions of the triangle involving the generalized *r–order harmonic numbers*, $H_n^{(r)}$, for integer–order $r \geq 1$ [13, §6] [7, *cf.* §5.7] [14, *cf.* §7–8]. For example, the applications cited in the reference [21, §4.3] provide statements of the following established special case identities for these coefficients [22, A001008, A002805; A007406, A007407; A007408, A007409]:

$$\begin{bmatrix} n+1 \\ 2 \end{bmatrix} = n! \cdot H_n, \begin{bmatrix} n+1 \\ 3 \end{bmatrix} = \frac{n!}{2}\left(H_n^2 - H_n^{(2)}\right), \quad \text{and} \quad \begin{bmatrix} n+1 \\ 4 \end{bmatrix} = \frac{n!}{6}\left(H_n^3 - 3H_nH_n^{(2)} + 2H_n^{(3)}\right).$$

A couple of related approaches to congruence–based primality conditions on prime pairs through the triple sum expansions phrased above are also pointed out in the last examples of these results provided below. In particular, it is not difficult to prove that the following congruence holds for integers $p \geq 0$, $n \geq 1$, and any fixed $k \geq 1$[14]:

$$(n-1)^p \equiv \frac{(-1)^p}{k}\left[k + (1-(k+1)^p)\cdot n\right] \pmod{n(n+k)}.$$

The special case of the congruence relation formulated in Clement's theorem concerning a characterization of the *twin primes* [22, A001097] is of particular interest in continuing the discussion from Section 1.4. For $k := 2$, the parameters in (5.24) employed to state the next result are then formed as follows:

$$(\omega, N_{\omega,p}(n), M_\omega(n)) :\mapsto \left(\mathrm{CT}, \frac{(-1)^p}{2}\left(2 + (1-3^p)\cdot n\right), n(n+2)\right). \qquad \text{(Clement)}$$

Then we have an alternate formulation of Clement's theorem provided as in following equation:

$$n, n+2 \text{ prime} \iff 4 \cdot F_{\mathrm{CT},n}(1,1,1) + 4 + n \equiv 0 \pmod{M_{\mathrm{CT}}(n)}. \qquad \text{(Clement's Theorem)}$$

It is similarly straightforward to obtain related congruences satisfied by the p^{th} powers, $(n-1)^p$, modulo $(n-k_1)(n+k_2)$ at some prescribed choices of the integers k_1 and k_2. For example, the next equation states another particular congruence relation following from an appeal to the binomial theorem[15]:

$$(n-1)^p \equiv \frac{(k_1-1)^p(n+k_2) - (k_2+1)^p(n-k_1)}{k_1+k_2} \pmod{(n-k_1)(n+k_2)}.$$

There are numerous other examples of prime–related congruences that are also easily expanded by extending the procedure in the classical cases given above. The references include analogous factorial–related results phrasing integer congruence requirements on prime tuples and prime constellations [3; 14; 18]. Ⓔ

Remark 5.8 (Other Related Congruences)**.** The first result for the product sequences, $p_{n-s}(\alpha, R)$, modulo p when $p > n-s$ given in (5.19) yields additional congruences similar in form to the results from the last example. For example, when $n, n-s, d \in \mathbb{Z}^+$ are chosen so that $n+d > n-s$, the coefficient identity for the sequences, $p_{n-s}(1,1) \equiv [z^{n-s}]\,\mathrm{FP}_{n+d}(1,1;z) \pmod{n+d}$, stated in (5.16) of Section 5.2 provides that

$$(n-s)! \equiv \sum_{i=0}^{n-s}\binom{n+d}{i}\left(-(n+d)\right)_i (n-s-1)! \qquad \pmod{n+d}.$$

[14] i.e., since a naïve expansion via the binomial theorem shows that

$$(n-1)!^p = (-1)^p + \sum_{s=1}^{p}\binom{p}{s}(-1)^{p-s}\cdot n\cdot(-k)^{s-1} + \sum_{s=1}^{p}\sum_{r=1}^{s-1}\binom{p}{s}\binom{s-1}{r} n\cdot(n+k)\times(-1)^{p-s}(-k)^{s-1-r}(n+k)^{r-1}.$$

[15] Repeated applications of the binomial theorem lead to the exact expansions

$$\begin{aligned}(n-1)^p &= (k_1-1)^p + \sum_{s=1}^{p}\binom{p}{s}(n-k_1)\cdot(k_1-1)^{p-s}\left[-(k_1+k_2)\right]^{s-1} \\ &\quad + \sum_{s=1}^{p}\sum_{r=1}^{s-1}\binom{p}{s}\binom{s-1}{r}(n-k_1)\cdot(n+k_2)\times(k_1-1)^{p-s}\left[-(k_1+k_2)\right]^{s-1-r}(n+k_2)^{r-1}.\end{aligned}$$

For natural numbers $n-s, k_1, k_2$ such that $(n-k_1)(n+k_2) > n-s \geq 1$, another application of the identity cited above yields the following congruences for the single factorial function, $(n-s)!$:

$$(n-s)! \equiv \sum_{i=0}^{n-s} \frac{(-1)^i}{i!} \times \left\{(-(n-k_1)(n+k_2))_i\right\}^2 \cdot (n-s-1)! \qquad (\mathrm{mod}\ (n-k_1)(n+k_2)).$$

The congruence satisfied by the sequence of *Wilson primes*, denoting the primes p where $W(p) := [(p-1)!+1]/p \in \mathbb{Z}$, corresponds to the additional requirement [22, A007540; A157250]

$$\sum_{i=0}^{n-1} \frac{(-1)^i}{i!} \times \left\{\left(-n^2\right)_i\right\}^2 \times (n-1)! + 1 \equiv 0 \pmod{n^2}. \qquad \text{(Wilson Primes)}$$

Similarly, the necessary condition for the primality of an odd integer, $p := 2n+1$, proved in [14, §8.9; Thm. 133] requires that[16]

$$\sum_{i=0}^{n} \binom{p^2}{i} \left\{2^i \left(1/2-p^2\right)_i (2n-2i-1)!! - (-8)^n \left(-p^2\right)_i (n-i)!\right\} \equiv 0 \pmod{p^2}.$$

The treatment of the modular congruence identities involved in these few example cases is by no means exhaustive, but serves to demonstrate the utility of the approach in rephrasing several non–trivial prime number results with many notable applications. ———— ®

The sums over the Stirling numbers in the examples given above also satisfy further expansions by the Stirling polynomial sequences, and by the generalized Bernoulli polynomials in the immediate forms given in (5.23). The properties of the Stirling polynomial sequences cited in [13; 15; 20], and the related identities corresponding to their generalized forms in [21], suggest another possible avenue towards the simplifying similar forms of the factorial–function–related congruences cited so far as applications in the article.

6 Conclusions

We have defined several new forms of ordinary power series approximations to the normally divergent ordinary generating functions of generalized multiple, or α–factorial, function sequences. The generalized forms of these convergent functions provide partial truncated approximations to the sequences formally enumerated by these divergent power series. The exponential generating functions for the special case product sequences, $p_n(\alpha, s-1)$, are studied in the reference [21, §5]. The exponential generating functions that enumerate the cases corresponding to the more general factorial–like sequences, $p_n(\alpha, \beta n+\gamma)$, are less obvious in form. Along the way, we have also suggested a number of new, alternate approaches to enumerating the factorial function sequences that arise in applications, including classical identities involving the single and double factorial functions, and in the forms of several other noteworthy special cases.

The key ingredient to the short proof given in Section 3 employs known characterizations of the Pochhammer symbols, $(x)_n$, by generalized Stirling number triangles as polynomial expansions in the indeterminate, x, each with predictably small finite–integral–degree at any fixed n. The more combinatorial proof in the spirit of Flajolet's articles suggested by the discussions in Section 3.2 may lead to further interesting interpretations of the α–factorial functions, $(s-1)!_{(\alpha)}$, which motivate the investigations of the coefficient-wise symbolic polynomial expansions of the functions first considered by [21]. A separate proof of these new continued fractions formulated in terms of the generalized α–factorial function coefficients defined by (1.4), and by their strikingly Stirling–number–like combinatorial properties motivated in the introduction, is notably missing from this article.

The rationality of these convergent functions for all h suggests new insight to generating numeric sequences of interest, including several specific new congruence properties, derivations of finite difference equations that hold for these exact sequences modulo any integer p, and perhaps more interestingly, exact expansions of the

[16] For $p := 2n+1$ an odd prime, we employ the second form of the congruence from the reference given as in the following equations:

$$\begin{aligned}(2n)! &\equiv (-1)^n 2^{4n} \left\{n!\right\}^2 && \pmod{p^2} \\ (2n-1)!! &\equiv (-1)^n 2^{3n} \cdot n! && \pmod{p^2}.\end{aligned}$$

classical single and double factorial functions by the special zeros of the generalized Laguerre polynomials and confluent hypergeometric functions. The techniques behind the specific identities given here are easily generalized and extended to further specific applications. The particular examples cited within this article are intended as suggestions at new starting points to tackling the expansions that arise in many other practical situations, both implicitly and explicitly involving the generalized cases of factorial–function–like product sequences, $p_n(\alpha, R)$.

7 Acknowledgments

The original work on the article is an extension of the forms of the generalized factorial function variants considered in my article published in the *Journal of Integer Sequences* (2010). The research on the continued fraction representations for the generalized factorial functions considered in this article began as the topic for my final project in the *Introduction to Mathematical Research* course at the University of Illinois at Urbana–Champaign around the time of the first publication. Thanks especially to Professor Bruce Reznick at the University of Illinois at Urbana–Champaign, and also to Jimmy McLaughlin, for their helpful input on revising previous drafts of the article.

References

[1] V. Adamchik, On the Barnes function, in *Proceedings of the 2001 International Symposium on Symbolic and Algebraic Computation (July 22-25 2001, London, Canada)*. New York: Academic Press, 2001a, pp. 15–20.

[2] V. S. Adamchik, Contributions to the theory of the Barnes function, **abs/0308086v1** (2003).

[3] C. Aebi and G. Cairns, Generalizations of Wilson's theorem for double–, hyper–, sub–, and superfactorials, *The American Mathematical Monthly* **122** (2015), 433–443.

[4] S. Ahmed, Properties of the zeros of confluent hypergeometric functions, *Journal of Approximation Theory* (1980), 335–347.

[5] D. Callan, A combinatorial survey of combinatorial identities for the double factorial, **abs/0906.1317** (2009).

[6] P. A. Clement, Congruences for sets of primes, *Amer. Math. Monthly* **56** (1949), 23–25.

[7] L. Comtet, *Advanced Combinatorics: The Art of Finite and Infinite Expressions*, D. Reidel Publishing Company, 1974.

[8] R. Diaz and E. Pariguan, On hypergeometric functions and k–Pochhammer symbol, **abs/0405596v2** (2005).

[9] P. Flajolet, Combinatorial aspects of continued fractions, *Discrete Mathematics* **32** (1980), 125–161.

[10] P. Flajolet, On congruences and continued fractions for some classical combinatorial quantities, *Discrete Mathematics* **41** (1982), 145–153.

[11] W. Gautschi and C. Giordano, Luigi Gatteschi's work on asymptotics of special functions and their zeros, *Numer Algor* (2008).

[12] H. Gould and J. Quaintance, Double fun with double factorials*Mathematics Magazine* **85** , 177–192.

[13] R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics: A Foundation for Computer Science*, Addison-Wesley, 1994.

[14] G. H. Hardy and E. M. Wright, eds., *An Introduction to the Theory of Numbers*, Oxford University Press, 2008 (Sixth Edition).

[15] D. E. Knuth, Convolution polynomials, *The Mathematica J.* **2** (1992), 67–78.

[16] D. E. Knuth, *The Art of Computer Programming: Fundamental Algorithms*, Vol. 1, Addison-Wesley, 1997.

[17] S. K. Lando, *Lectures on Generating Functions*, American Mathematical Society, 2002.

[18] C. Lin and L. Zhipeng, On Wilson's theorem and Polignac conjecture, *Math. Medley* **6** (2005).

[19] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, eds., *NIST Handbook of Mathematical Functions*, Cambridge University Press, 2010.

[20] S. Roman, *The Umbral Calculus*, Dover, 1984.

[21] M. D. Schmidt, Generalized j–factorial functions, polynomials, and applications, *J. Integer Seq.* **13** (2010).

[22] N. J. A. Sloane, The Online Encyclopedia of Integer Sequences,

http://www.research.att.com/~njas/sequences/.

[23] J. Spanier and K. B. Oldham, *An Atlas of Functions*, Taylor & Francis / Hemisphere, 1987.

[24] R. P. Stanley, *Enumerative Combinatorics*, Vol. 2, Cambridge, 1999.

[25] H. S. Wilf, *Generatingfunctionology*, Academic Press, 1994.

[26] Wolfram Functions Site, Introduction to the factorials and binomials, http://funtions.wolfram.com/GammaBetaErf/Pochhammer/introductions/FactorialBinomials/ (2015).

Appendix: List of Tables

h	$\mathrm{FP}_h(\alpha, R; z)$
0	0
1	1
2	$1-(2\alpha+R)z$
3	$1-(6\alpha+2R)z+(6\alpha^2+4\alpha R+R^2)z^2$
4	$1-(12\alpha+3R)z+(36\alpha^2+19\alpha R+3R^2)z^2-(24\alpha^3+18\alpha^2R+7\alpha R^2+R^3)z^3$
5	$1-(20\alpha+4R)z+(120\alpha^2+51\alpha R+6R^2)z^2-(240\alpha^3+158\alpha^2R+42\alpha R^2+4R^3)z^3$ $+(120\alpha^4+96\alpha^3R+46\alpha^2R^2+11\alpha R^3+R^4)z^4$
6	$1-(30\alpha+5R)z+(300\alpha^2+106\alpha R+10R^2)z^2$ $-(1200\alpha^3+668\alpha^2R+138\alpha R^2+10R^3)z^3$ $-(1800\alpha^4+1356\alpha^3R+469\alpha^2R^2+78\alpha R^3+5R^4)z^4$ $+(720\alpha^5+600\alpha^4R+326\alpha^3R^2+101\alpha^2R^3+16\alpha R^4+R^5)z^5$

Table 1.1: The Convergent Numerator Functions, $\mathrm{FP}_h(\alpha, R; z)$

h	$\mathrm{FQ}_h(\alpha, R; z)$
0	1
1	$1-Rz$
2	$1-2(\alpha+R)z+R(\alpha+R)z^2$
3	$1-3(2\alpha+R)z+3(\alpha+R)(2\alpha+R)z^2-R(\alpha+R)(2\alpha+R)z^3$
4	$1-4(3\alpha+R)z+6(2\alpha+R)(3\alpha+R)z^2-4(\alpha+R)(2\alpha+R)(3\alpha+R)z^3$ $+R(\alpha+R)(2\alpha+R)(3\alpha+R)z^4$
5	$1-5(4\alpha+R)z+10(3\alpha+R)(4\alpha+R)z^2-10(2\alpha+R)(3\alpha+R)(4\alpha+R)z^3$ $+5(\alpha+R)(2\alpha+R)(3\alpha+R)(4\alpha+R)z^4-R(\alpha+R)(2\alpha+R)(3\alpha+R)(4\alpha+R)z^5$
6	$1-6(5\alpha+R)z+15(4\alpha+R)(5\alpha+R)z^2-20(3\alpha+R)(4\alpha+R)(5\alpha+R)z^3$ $+15(2\alpha+R)(3\alpha+R)(4\alpha+R)(5\alpha+R)z^4$ $-6(\alpha+R)(2\alpha+R)(3\alpha+R)(4\alpha+R)(5\alpha+R)z^5$ $+R(\alpha+R)(2\alpha+R)(3\alpha+R)(4\alpha+R)(5\alpha+R)z^6$

Table 1.2: The Convergent Denominator Functions, $\mathrm{FQ}_h(\alpha, R; z)$

Table 1: The Convergent Numerator and Denominator Function Sequences

n	$n!$	$R_3^{(1)}(n)$	(3)	$n!_{(2)}$	$R_3^{(2)}(n)$	(3)	(6)	$n!_{(3)}$	$R_3^{(3)}(n)$	(3)	(9)
1	1	13	1	1	25	1	1	1	37	1	1
2	2	134	2	2	26	2	2	2	38	2	2
3	6	1680	0	3	471	0	3	3	39	0	3
4	24	25896	0	8	536	2	2	4	1012	1	4
5	120	477708	0	15	11115	0	3	10	1108	1	1
6	720	10283904	0	48	13440	0	0	18	1206	0	0
7	5040	253190772	0	105	328977	0	3	28	35074	1	1
8	40320	7016155200	0	384	414336	0	0	80	40040	2	8
9	362880	216058418928	0	945	11787633	0	3	162	45360	0	0
10	3628800	7317516146688	0	3840	15286656	0	0	280	1535968	1	1

n	$n!_{(4)}$	$R_3^{(4)}(n)$	(3)	(12)	$n!_{(5)}$	$R_3^{(5)}(n)$	(3)	(15)	$n!_{(6)}$	$R_3^{(6)}(n)$	(3)	(18)
1	1	49	1	1	1	61	1	1	1	73	1	1
2	2	50	2	2	2	62	2	2	2	74	2	2
3	3	51	0	3	3	63	0	3	3	75	0	3
4	4	52	1	4	4	64	1	4	4	76	1	4
5	5	1757	2	5	5	65	2	5	5	77	2	5
6	12	1884	0	0	6	2706	0	6	6	78	0	6
7	21	2013	0	9	14	2864	2	14	7	3859	1	7
8	32	2144	2	8	24	3024	0	9	16	4048	1	16
9	45	80325	0	9	36	3186	0	6	27	4239	0	9
10	120	88920	0	0	50	3350	2	5	40	4432	1	4
11	231	97983	0	3	66	153636	0	6	55	4627	1	1
12	384	107520	0	0	168	166848	0	3	72	4824	0	0
13	585	4658073	0	9	312	180642	0	12	91	261775	1	1
14	1680	5263632	0	0	504	195024	0	9	224	280592	2	8
15	3465	5919993	0	9	750	210000	0	0	405	300105	0	9
16	6144	6629376	0	0	1056	11090976	0	6	640	320320	1	10
17	9945	329036025	0	9	2856	12244716	0	6	935	341243	2	17
18	30240	377204256	0	0	5616	13476456	0	6	1296	362880	0	0

Table 2: The α–Factorial Functions, $n!_{(\alpha)}$, Modulo 3 and Modulo 3α

h	$z^{h-1} \cdot \mathrm{FP}_h\left(\alpha, R; z^{-1}\right)$
1	1
2	$-(2\alpha+R)+z$
3	$6\alpha^2+\alpha(4R-6z)+(R-z)^2$
4	$-24\alpha^3-18\alpha^2(R-2z)-\alpha(7R-12z)(R-z)-(R-z)^3$
5	$120\alpha^4+2\alpha^2\left(23R^2-79Rz+60z^2\right)+48\alpha^3(2R-5z)+\alpha(11R-20z)(R-z)^2+(R-z)^4$
6	$-720\alpha^5-2\alpha^3\left(163R^2-678Rz+600z^2\right)-\alpha^2\left(101R^2-368Rz+300z^2\right)(R-z)$ $-600\alpha^4(R-3z)-2\alpha(8R-15z)(R-z)^3-(R-z)^5$
7	$5040\alpha^6+36\alpha^4\left(71R^2-347Rz+350z^2\right)+\alpha^2\left(197R^2-740Rz+630z^2\right)(R-z)^2$ $+\alpha^3\left(932R^3-5102R^2z+8322Rz^2-4200z^3\right)+2160\alpha^5(2R-7z)$ $+2\alpha(11R-21z)(R-z)^4+(R-z)^6$
8	$-40320\alpha^7-36\alpha^5\left(617R^2-3466Rz+3920z^2\right)-\alpha^2\left(351R^2-1342Rz+1176z^2\right)(R-z)^3$ $+\alpha^4\left(-9080R^3+57286R^2z-105144Rz^2+58800z^3\right)$ $-\alpha^3\left(2311R^3-13040R^2z+22210Rz^2-11760z^3\right)(R-z)-35280\alpha^6(R-4z)$ $-\alpha(29R-56z)(R-z)^5-(R-z)^7$

Table 3: The Reflected Numerator Polynomials, $\widetilde{\mathrm{FP}}_h(\alpha, R; z) := z^{h-1} \cdot \mathrm{FP}_h\left(\alpha, R; z^{-1}\right)$

m	$\widetilde{\ell}_{m,2}(z)$	$\widetilde{p}_{m,2}(x)$
1	1	2
2	$4-3z$	$x+5$
3	$11-17z+7z^2$	$x^2+10x+24$
4	$26-62z+52z^2-15z^3$	$x^3+18x^2+96x+192$
5	$57-186z+238z^2-139z^3+31z^4$	$x^4+28x^3+264x^2+1008x+1392$
6	$120-501z+868z^2-769z^3+346z^4-63z^5$	$x^5+40x^4+580x^3+3840x^2+11880x+14520$

Table 4.1: Generating the p^{th} Power Sequences, 2^p-1

m	$\ell_{m,2}(z)$	$p_{m,2}(x)$
1	1	1
2	$4-3z$	$x+4$
3	$11-17z+7z^2$	$x^2+10x+22$
4	$26-62z+52z^2-15z^3$	$x^3+18x^2+96x+156$
5	$57-186z+238z^2-139z^3+31z^4$	$x^4+28x^3+264x^2+1008x+1368$
m	$\ell_{m,3}(z)$	$p_{m,3}(x)$
1	1	1
2	$5-8z$	$x+5$
3	$18-60z+52z^2$	$x^2+12x+36$
4	$58-300z+532z^2-320z^3$	$x^3+21x^2+144x+348$
5	$179-1268z+3436z^2-4192z^3+1936z^4$	$x^4+32x^3+372x^2+1968x+4296$
m	$\ell_{m,4}(z)$	$p_{m,4}(x)$
1	1	1
2	$6-15z$	$x+6$
3	$27-141z+189z^2$	$x^2+14x+54$
4	$112-906z+2484z^2-2295z^3$	$x^3+24x^2+204x+672$
5	$453-4998z+20898z^2-39123z^3+27621z^4$	$x^4+36x^3+504x^2+3504x+10872$
m	$\ell_{m,5}(z)$	$p_{m,5}(x)$
1	1	1
2	$7-24z$	$x+7$
3	$38-272z+496z^2$	$x^2+16x+76$
4	$194-2144z+7984z^2-9984z^3$	$x^3+27x^2+276x+1164$
5	$975-14640z+82960z^2-209920z^3+199936z^4$	$x^4+40x^3+660x^2+5760x+23400$

Table 4.2: Generating the p^{th} Power Sequences, 2^p-1, 3^p-1, 4^p-1, and 5^p-1

m	$\ell_{m,s+1}(z)$	$p_{m,s+1}(x)$
1	1	1
2	$3+s(1-2z)-s^2z$	$3+s(1+x)$
3	$6+s^4z^2-4s(-1+2z)+s^3z(-2+3z)$ $+s^2(1-7z+3z^2)$	$12+8s(1+x)+s^2(2+2x+x^2)$
4	$10-s^6z^3-10s(-1+2z)-s^5z^2(-3+4z)$ $+5s^2(1-5z+3z^2)-s^4z(3-13z+6z^2)$ $+s^3(1-14z+21z^2-4z^3)$	$60+60s(1+x)+15s^2(2+2x+x^2)$ $+s^3(6+6x+3x^2+x^3)$
5	$15+s^8z^4-20s(-1+2z)+s^7z^3(-4+5z)$ $+5s^2(3-13z+9z^2)+s^6z^2(6-21z+10z^2)$ $-3s^3(-2+18z-27z^2+8z^3)$ $+s^5z(-4+33z-44z^2+10z^3)$ $+s^4(1-23z+73z^2-46z^3+5z^4)$	$360+480s(1+x)+180s^2(2+2x+x^2)$ $+24s^3(6+6x+3x^2+x^3)$ $+s^4(24+24x+12x^2+4x^3+x^4)$

Table 4.3: Generating the p^{th} Power Sequences, $(s+1)^p-1$

Table 4: Convergent–Based Generating Function Identities for p^{th} Power Sequences

n	m_h	$(-1)^n n! \cdot m_h^{-1} \cdot C_{h,n}(\alpha, R)$
0	1	1
1	1	$-(h-1)(R+h\alpha)$
2	$(h-2)$	$\alpha\left(2h^2-3h-1\right)R+\alpha^2(h-1)^2h+(h-1)R^2$
3	$(h-3)$	$3\alpha(h-2)\left(h^2-2h-1\right)R^2+\alpha^2(h-2)\left(3h^3-9h^2+2h-2\right)R$
		$\quad+\alpha^3(h-2)^2(h-1)^2h+(h-2)(h-1)R^3$
4	$(h-3)(h-4)$	$\alpha^2\left(6h^4-36h^3+53h^2-9h+22\right)R^2+2\alpha^3\left(2h^5-15h^4+36h^3-36h^2+19h+6\right)R$
		$\quad+\alpha^4(h-3)(h-2)^2(h-1)^2h+(h-2)(h-1)R^4+2\alpha(h-3)(h-2)(2h+1)R^3$
5	$(h-3)(h-4)(h-5)$	$5\alpha(h-2)\left(h^2-3h-2\right)R^4+5\alpha^2\left(2h^4-14h^3+23h^2-h+14\right)R^3$
		$\quad+5\alpha^3\left(2h^5-18h^4+49h^3-49h^2+40h+20\right)R^2$
		$\quad+\alpha^4\left(5h^6-55h^5+215h^4-395h^3+374h^2-72h+48\right)R$
		$\quad+\alpha^5(h-4)(h-3)(h-2)^2(h-1)^2h+(h-2)(h-1)R^5$
6	$(h-4)(h-5)(h-6)$	$3\alpha(h-3)(h-2)\left(2h^2-7h-5\right)R^5+5\alpha^2(h-3)\left(3h^4-24h^3+44h^2+3h+34\right)R^4$
		$\quad+5\alpha^3\left(4h^6-54h^5+256h^4-519h^3+520h^2-357h-270\right)R^3$
		$\quad+\alpha^4\left(15h^7-240h^6+1455h^5-4335h^4+7114h^3-6129h^2+764h-1644\right)R^2$
		$\quad+\alpha^5\left(6h^8-111h^7+826h^6-3246h^5+7378h^4-9603h^3+6478h^2-2448h-720\right)R$
		$\quad+\alpha^6(h-5)(h-4)(h-3)^2(h-2)^2(h-1)^2h+(h-3)(h-2)(h-1)R^6$

n	m_h	$(-1)^n n! \cdot m_h^{-1} \cdot C_{h,n}(\alpha, R)$
2	$(h-2)$	$\alpha^2h^3+2\alpha h^2(R-\alpha)+h\left(\alpha^2+R^2-3\alpha R\right)-R(\alpha+R)$
3	$(h-3)$	$\alpha^3h^5+3\alpha^2h^4(R-2\alpha)+\alpha h^3\left(13\alpha^2+3R^2-15\alpha R\right)$
		$\quad+h^2\left(-12\alpha^3+R^3-12\alpha R^2+20\alpha^2R\right)+h\left(4\alpha^3-3R^3+9\alpha R^2-6\alpha^2R\right)$
		$\quad+2R(\alpha+R)(2\alpha+R)$
4	$(h-3)(h-4)$	$\alpha^4h^6+\alpha^3h^5(4R-9\alpha)+\alpha^2h^4\left(31\alpha^2+6R^2-30\alpha R\right)$
		$\quad+\alpha h^3\left(-51\alpha^3+4R^3-36\alpha R^2+72\alpha^2R\right)$
		$\quad+h^2\left(40\alpha^4+R^4-18\alpha R^3+53\alpha^2R^2-72\alpha^3R\right)$
		$\quad+h\left(-12\alpha^4-3R^4+14\alpha R^3-9\alpha^2R^2+38\alpha^3R\right)+2R(\alpha+R)(2\alpha+R)(3\alpha+R)$
5	$(h-3)(h-4)(h-5)$	$\alpha^5h^7+\alpha^4h^6(5R-13\alpha)+\alpha^3h^5\left(67\alpha^2+10R^2-55\alpha R\right)$
		$\quad+5\alpha^2h^4\left(-35\alpha^3+2R^3-18\alpha R^2+43\alpha^2R\right)$
		$\quad+\alpha h^3\left(244\alpha^4+5R^4-70\alpha R^3+245\alpha^2R^2-395\alpha^3R\right)$
		$\quad+h^2\left(-172\alpha^5+R^5-25\alpha R^4+115\alpha^2R^3-245\alpha^3R^2+374\alpha^4R\right)$
		$\quad+h\left(48\alpha^5-3R^5+20\alpha R^4-5\alpha^2R^3+200\alpha^3R^2-72\alpha^4R\right)$
		$\quad+2R(\alpha+R)(2\alpha+R)(3\alpha+R)(4\alpha+R)$

Table 5.1: (a) Alternate Factored Forms of the Auxiliary Numerator Sequences, $C_{h,n}(\alpha, R) := [z^n]\,\mathrm{FP}_h(\alpha, R; z)$

n	$n! \cdot C_{h,n}(\alpha, R)$
2	$\alpha^2h^4+2\alpha h^3(R-2\alpha)+h^2\left(5\alpha^2+R^2-7\alpha R\right)-h(3R-2\alpha)(R-\alpha)+2R(\alpha+R)$
3	$-\alpha^3h^6-3\alpha^2h^5(R-3\alpha)-\alpha h^4\left(31\alpha^2+3R^2-24\alpha R\right)+h^3\left(51\alpha^3-R^3+21\alpha R^2-65\alpha^2R\right)$
	$\quad+h^2\left(-40\alpha^3+6R^3-45\alpha R^2+66\alpha^2R\right)-h(R-\alpha)\left(12\alpha^2+11R^2-10\alpha R\right)+6R(\alpha+R)(2\alpha+R)$
4	$\alpha^4h^8+4\alpha^3h^7(R-4\alpha)+2\alpha^2h^6\left(53\alpha^2+3R^2-29\alpha R\right)+2\alpha h^5\left(-188\alpha^3+2R^3-39\alpha R^2+165\alpha^2R\right)$
	$\quad+h^4\left(769\alpha^4+R^4-46\alpha R^3+377\alpha^2R^2-936\alpha^3R\right)-2h^3\left(452\alpha^4+5R^4-94\alpha R^3+406\alpha^2R^2-703\alpha^3R\right)$
	$\quad+h^2\left(564\alpha^4+35R^4-302\alpha R^3+721\alpha^2R^2-1118\alpha^3R\right)-2h(R-\alpha)\left(-72\alpha^3+25R^3-17\alpha R^2+114\alpha^2R\right)$
	$\quad+24R(\alpha+R)(2\alpha+R)(3\alpha+R)$
5	$-\alpha^5h^{10}-5\alpha^4h^9(R-5\alpha)-5\alpha^3h^8\left(54\alpha^2+2R^2-23\alpha R\right)-10\alpha^2h^7\left(-165\alpha^3+R^3-21\alpha R^2+111\alpha^2R\right)$
	$\quad-\alpha h^6\left(6273\alpha^4+5R^4-190\alpha R^3+1795\alpha^2R^2-5860\alpha^3R\right)$
	$\quad+h^5\left(15345\alpha^5-R^5+85\alpha R^4-1425\alpha^2R^3+8015\alpha^3R^2-18519\alpha^4R\right)$
	$\quad+5h^4\left(-4816\alpha^5+3R^5-111\alpha R^4+1055\alpha^2R^3-4011\alpha^3R^2+7205\alpha^4R\right)$
	$\quad-5h^3\left(-4660\alpha^5+17R^5-339\alpha R^4+1947\alpha^2R^3-5703\alpha^3R^2+8438\alpha^4R\right)$
	$\quad+h^2\left(-12576\alpha^5+225R^5-2200\alpha R^4+7975\alpha^2R^3-22900\alpha^3R^2+26400\alpha^4R\right)$
	$\quad-2h(R-\alpha)\left(1440\alpha^4+137R^4+7\alpha R^3+1802\alpha^2R^2-1848\alpha^3R\right)+120R(\alpha+R)(2\alpha+R)(3\alpha+R)(4\alpha+R)$

Table 5.2: (b) The Auxiliary Numerator Subsequences, $C_{h,n}(\alpha, R) := [z^n]\,\mathrm{FP}_h(\alpha, R; z)$

Table 5: The Convergent Numerator Function Subsequences, $C_{h,n}(\alpha, R)$

k	$(-1)^{h-k} z^{-(h-k)} \cdot R_{h,h-k}(\alpha; z)$
1	1
2	$-\frac{1}{2}\alpha\left(h^2-h+2\right)z+h-1$
3	$\frac{1}{2}(h-2)(h-1)-\frac{1}{2}\alpha(h-2)\left(h^2+3\right)z+\frac{1}{24}\alpha^2\left(3h^4-10h^3+21h^2-14h+24\right)z^2$
4	$-\frac{1}{4}\alpha(h-3)(h-2)\left(h^2+h+4\right)z+\frac{1}{24}\alpha^2(h-3)\left(3h^4-4h^3+19h^2-2h+56\right)z^2$ $-\frac{1}{48}\alpha^3\left(h^6-7h^5+23h^4-37h^3+48h^2-28h+48\right)z^3$ $+\frac{1}{6}(h-3)(h-2)(h-1)$
5	$-\frac{1}{12}\alpha(h-4)(h-3)(h-2)\left(h^2+2h+5\right)z+\frac{1}{48}\alpha^2(h-4)(h-3)\left(3h^4+2h^3+23h^2+16h+100\right)z^2$ $-\frac{1}{48}\alpha^3(h-4)\left(h^6-4h^5+14h^4-16h^3+61h^2-12h+180\right)z^3$ $+\frac{\alpha^4}{5760}\left(15h^8-180h^7+950h^6-2688h^5+4775h^4-5340h^3+5780h^2-3312h+5760\right)z^4$ $+\frac{1}{24}(h-4)(h-3)(h-2)(h-1)$

Table 6.1: (a) Alternate Factored Forms of the Auxiliary Numerator Sequences, $R_{h,k}(\alpha; z) := [R^k]\,\mathrm{FP}_h(\alpha, R; z)$

k	$k!(-1)^{h-k} z^{-(h-k)} \cdot R_{h,h-k}(\alpha; z)$
1	1
2	$-\alpha h^2 z+h(\alpha z+2)-2(\alpha z+1)$
3	$\frac{3}{4}\alpha^2h^4z^2-\frac{1}{2}\alpha h^3z(5\alpha z+6)+\frac{3}{4}h^2\left(7\alpha^2z^2+8\alpha z+4\right)$ $+\frac{1}{2}h\left(-7\alpha^2z^2-18\alpha z-18\right)+6\left(\alpha^2z^2+3\alpha z+1\right)$
4	$-\frac{1}{2}\alpha^3h^6z^3+\frac{1}{2}\alpha^2h^5z^2(7\alpha z+6)-\frac{1}{2}\alpha h^4z\left(23\alpha^2z^2+26\alpha z+12\right)$ $+\frac{1}{2}h^3\left(37\alpha^3z^3+62\alpha^2z^2+48\alpha z+8\right)$ $+h^2\left(-24\alpha^3z^3-59\alpha^2z^2-30\alpha z-24\right)$ $+2h\left(7\alpha^3z^3+31\alpha^2z^2+42\alpha z+22\right)-24\left(\alpha^3z^3+7\alpha^2z^2+6\alpha z+1\right)$
5	$\frac{5}{16}\alpha^4h^8z^4-\frac{5}{4}\alpha^3h^7z^3(3\alpha z+2)+\frac{5}{24}\alpha^2h^6z^2\left(95\alpha^2z^2+96\alpha z+36\right)$ $-\frac{1}{2}\alpha h^5z\left(112\alpha^3z^3+150\alpha^2z^2+95\alpha z+20\right)$ $+\frac{5}{48}h^4\left(955\alpha^4z^4+1728\alpha^3z^3+1080\alpha^2z^2+672\alpha z+48\right)$ $-\frac{5}{4}h^3\left(89\alpha^4z^4+250\alpha^3z^3+242\alpha^2z^2+104\alpha z+40\right)$ $+\frac{5}{12}h^2\left(289\alpha^4z^4+1536\alpha^3z^3+1584\alpha^2z^2+408\alpha z+420\right)$ $+h\left(-69\alpha^4z^4-570\alpha^3z^3-1270\alpha^2z^2-820\alpha z-250\right)$ $+120\left(\alpha^4z^4+15\alpha^3z^3+25\alpha^2z^2+10\alpha z+1\right)$

Table 6.2: (b) Alternate Factored Forms of the Auxiliary Numerator Sequences, $R_{h,k}(\alpha; z) := [R^k]\,\mathrm{FP}_h(\alpha, R; z)$

Table 6: The Convergent Numerator Function Subsequences, $R_{h,k}(\alpha; z)$