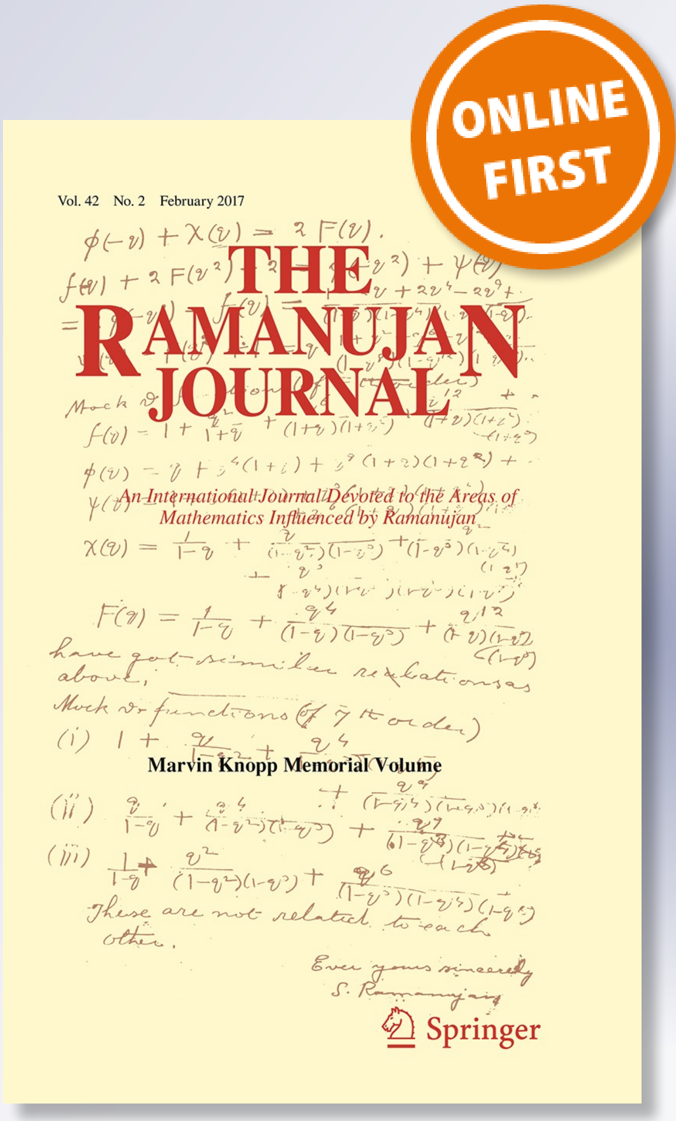
The Lambert series factorization theorem

Mircea Merca

The Ramanujan Journal
An International Journal Devoted to
the Areas of Mathematics Influenced by
Ramanujan

ISSN 1382-4090

Ramanujan J
DOI 10.1007/s11139-016-9856-3



Springer

Ramanujan J
DOI 10.1007/s11139-016-9856-3

The Lambert series factorization theorem

Mircea Merca[1]

Received: 27 December 2015 / Accepted: 15 September 2016

Abstract A factorization for partial sums of Lambert series is introduced in this paper. As corollaries, we derive some connections between partitions and divisors. These results can be easily used to discover and prove new combinatorial identities involving important functions from number theory: the Möbius function $\mu(n)$, Euler's totient $\varphi(n)$, Jordan's totient $J_k(n)$, Liouville's function $\lambda(n)$, the von Mangoldt function $\Lambda(n)$, and the divisor function $\sigma_x(n)$. The fascinating feature of these identities is their common nature.

Keywords Divisors · Lambert series · Partitions

Mathematics Subject Classification 11P81 · 11A25 · 11P84 · 05A17 · 05A19

1 Introduction

The Lambert series

$$\sum_{n=1}^{\infty} a_n \frac{q^n}{1-q^n}, \qquad |q| < 1,$$

where the a_n $(n = 1, 2, \ldots)$ are real or complex numbers is a natural generalization of the following formula related to the theory of numbers:

✉ Mircea Merca
mircea.merca@profinfo.edu.ro

1 Department of Mathematics, University of Craiova, Craiova, DJ 200585, Romania

Published online: 13 January 2017

$$\sum_{n=1}^{\infty} n^x \frac{q^n}{1-q^n} = \sum_{n=1}^{\infty} \sigma_x(n) q^n, \qquad |q| < 1.$$

In multiplicative number theory, the divisor function $\sigma_x(n)$, for a real or complex number x, is defined as the sum of the xth powers of the positive divisors of n,

$$\sigma_x(n) = \sum_{d|n} d^x,$$

where $d|n$ is shorthand for "d divides n." In general, assuming convergence, we have

$$\sum_{n=1}^{\infty} a_n \frac{q^n}{1 \pm q^n} = \sum_{n=1}^{\infty} b_{\pm}(n) q^n,$$

where

$$b_{\pm}(n) = \sum_{d|n} (\mp 1)^{1+n/d} a_d.$$

Lambert series have been the object of study in the classical works by Bromwich [4, pp. 102–103], Chrystal [5, pp. 345–346], Hardy and Wright [6, pp. 257–258], Knopp [7, pp. 448–452], MacMahon [8, pp. 26–32], Pólya and Szegő [15, pp. 125–129], and Titchmarsh [17, pp. 160–161]. These series frequently occur in Ramanujan's work on elliptic functions, theta functions, and mock theta functions [1].

Recently, the author [11] proved a factorization for partial sums of Lambert series in the case $a_n = 1$,

$$\sum_{k=1}^{n} \frac{q^k}{1-q^k} = \frac{1}{(q;q)_n} \sum_{k=1}^{n} (-1)^{k+1} k q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}, \qquad |q| < 1, \tag{1.1}$$

and obtained the following result:

$$\sum_{n=1}^{\infty} \frac{q^n}{1-q^n} = \frac{1}{(q;q)_\infty} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n q^{\binom{n+1}{2}}}{(q;q)_n}, \qquad |q| < 1,$$

where

$$(a;q)_n = (1-a)(1-aq)(1-aq^2)\ldots(1-aq^{n-1})$$

is the q-shifted factorial, with $(a;q)_0 = 1$, and

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \dfrac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}, & \text{if } k \in \{0, 1, \ldots, n\}, \\ 0, & \text{otherwise} \end{cases}$$

is the q-binomial coefficient.

Shortly after that, the author [12] reconsidered the fact that the q-binomial coefficients are specializations of elementary symmetric functions and deduced the following result.

Theorem 1.1 *For* $|q| < 1$,

$$\sum_{n=1}^{\infty} \frac{q^n}{1 \pm q^n} = \frac{1}{(\mp q; q)_\infty} \sum_{n=1}^{\infty} (s_o(n) \pm s_e(n)) q^n,$$

where $s_o(n)$*, respectively,* $s_e(n)$ *denotes the number of parts in all partitions of* n *into odd, respectively even, number of distinct parts.*

A lot of new connections between partitions and divisors of a positive integer were derived by the author [12] as corollaries of this theorem. Among these, we particularly note a new formula for the number of divisors of a positive integer as a convolution

$$\tau(n) = \sum_{k=1}^{n} (s_o(k) - s_e(k)) p(n-k) \tag{1.2}$$

and the identity

$$\sum_{k=0}^{\infty} (-1)^{\lceil k/2 \rceil} \tau(n - G_k) = s_o(n) - s_e(n), \tag{1.3}$$

where $\tau(n) = \sigma_0(n)$ counts the number of divisors of the positive integer n, $p(n)$ is Euler's partition function, and

$$\{G_k\}_{k \geqslant 0} = \{0, 1, 2, 5, 7, 12, 15, 22, 26, 35, 40, 51, \ldots\}$$

is the sequence of the generalized pentagonal numbers, i.e.,

$$G_k = \frac{1}{2} \left\lceil \frac{k}{2} \right\rceil \left\lceil \frac{3k+1}{2} \right\rceil.$$

Recall that the divisors of numbers have been studied from the point of view of partitions of numbers for a long time [9]. It is well-known that the partition function $p(n)$ and the sum of divisors function $\sigma(n) = \sigma_1(n)$ satisfy a common recursive relation with only $p(0)$ different from $\sigma(0)$ [14]:

$$\sum_{k=0}^{\infty} (-1)^{\lceil k/2 \rceil} p(n - G_k) = \delta_{0,n}, \quad \text{with} \quad p(0) = 1,$$

and

$$\sum_{k=0}^{\infty} (-1)^{\lceil k/2 \rceil} \sigma(n - G_k) = 0, \tag{1.4}$$

with $\sigma(0)$ replaced by n. Moreover, there are two relations [14] that combine the functions $p(n)$ and $\sigma(n)$:

$$np(n) = \sum_{k=0}^{n} \sigma(k) p(n-k) \tag{1.5}$$

and

$$\sigma(n) = \sum_{k=0}^{\infty} (-1)^{\lceil k/2 \rceil - 1} G_k p(n - G_k). \tag{1.6}$$

In this paper, motivated by these results, we provide the following generalization of Theorem 1.1.

Theorem 1.2 *For $|q| < 1$,*

$$\sum_{n=1}^{\infty} a_n \frac{q^n}{1 \pm q^n} = \frac{1}{(\mp q; q)_\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{n} (s_o(n,k) \pm s_e(n,k)) a_k q^n,$$

where a_n $(n = 1, 2, \ldots)$ are real or complex numbers and $s_o(n,k)$, respectively, $s_e(n,k)$ denotes the number of k's in all partitions of n into odd, respectively even, number of distinct parts.

This result follows directly from the following lemma when $n \to \infty$.

Lemma 1.1 *Let n be a positive integer and let $a_1, \ldots, a_n$ be n real or complex numbers. For $|q| < 1$,*

$$\sum_{k=1}^{n} a_k \frac{q^k}{1 \pm q^k} = \frac{1}{(\mp q; q)_n} \sum_{k=1}^{n} \sum_{1 \leqslant i_1 < \ldots < i_k \leqslant n} (\pm 1)^{k-1} (a_{i_1} + \ldots + a_{i_k}) q^{i_1 + \ldots + i_k}.$$

The general nature of the numbers a_k allows for applications of Theorem 1.2 to many important functions from number theory: the Möbius function $\mu(n)$, Euler's totient $\varphi(n)$, Jordan's totient $J_k(n)$, Liouville's function $\lambda(n)$, the von Mangoldt function $\Lambda(n)$, and the divisor function $\sigma_x(n)$. The fascinating feature of these identities is their common nature.

2 Proof of Lemma 1.1

We consider the following identity:

$$\sum_{k=1}^{n} \frac{a_k x_k}{1 \pm x_k} = S_n \prod_{k=1}^{n} \frac{1}{1 \pm x_k}, \tag{2.1}$$

where

$$S_n = \sum_{k=1}^{n} \sum_{1\leqslant i_1<\ldots<i_k\leqslant n} (\pm 1)^{k-1}(a_{i_1}+\ldots+a_{i_k})x_{i_1}\ldots x_{i_k}$$

and $a_1, a_2, \ldots, a_n, x_1, x_2, \ldots, x_n$ are independent variables such that $1 \pm x_k \neq 0$ for $k = 1, 2, \ldots, n$. We can write

$$\begin{aligned} S_{n+1} &= S_n \pm x_{n+1} \sum_{k=1}^{n} \sum_{1\leqslant i_1<\ldots<i_k\leqslant n} (\pm 1)^{k-1}(a_{i_1}+\ldots+a_{i_k}+a_{n+1})x_{i_1}\ldots x_{i_k} \\ &= S_n \pm x_{n+1}\left(S_n \pm a_{n+1} \sum_{k=1}^{n} \sum_{1\leqslant i_1<\ldots<i_k\leqslant n} (\pm 1)^{k} x_{i_1}\ldots x_{i_k}\right) \\ &= (1 \pm x_{n+1})S_n + a_{n+1}x_{n+1} \prod_{k=1}^{n}(1 \pm x_k). \end{aligned} \tag{2.2}$$

We are going to prove (2.1) by induction on n. For $n = 1$, we have

$$\frac{a_1x_1}{1 \pm x_1} = a_1x_1 \cdot \frac{1}{1 \pm x_1}$$

and the base case of induction is completed. We suppose that the relation

$$\sum_{k=1}^{n'} \frac{a_k x_k}{1 \pm x_k} = S_{n'} \prod_{k=1}^{n'} \frac{1}{1 \pm x_k}$$

is true for any integer n', $1 \leqslant n' \leqslant n$. Taking into account (2.2), we have

$$\begin{aligned} S_{n+1} \prod_{k=1}^{n+1} \frac{1}{1 \pm x_k} &= S_n \prod_{k=1}^{n} \frac{1}{1 \pm x_k} + \frac{a_{n+1}x_{n+1}}{1 \pm x_{n+1}} \\ &= \sum_{k=1}^{n} \frac{a_k x_k}{1 \pm x_k} + \frac{a_{n+1}x_{n+1}}{1 \pm x_{n+1}}. \end{aligned}$$

The proof of the identity (2.1) is completed. By (2.1), with x_k replaced by q^k, we arrive at our conclusion.

3 Connections between partitions and divisors

In this paper, $p_o(n)$, respectively, $p_e(n)$ denotes the number partitions of n into odd, respectively even, number of parts and $q_o(n)$, respectively, $q_e(n)$ denotes the number partitions of n into odd, respectively even, number of distinct parts. Clearly

$$p(n) = p_o(n) + p_e(n)$$

and

$$q(n) = q_o(n) + q_e(n),$$

where $q(n)$ is the classical notation for the number of partitions of n into distinct parts.

Recall [2] that the generating functions for $p_e(n) \pm p_o(n)$ and $q_e(n) \pm q_o(n)$ are given by

$$\frac{1}{(\pm q; q)_\infty} = \sum_{n=0}^{\infty} (p_e(n) \pm p_o(n))q^n, \qquad |q| < 1 \tag{3.1}$$

and

$$(\pm q; q)_\infty = \sum_{n=0}^{\infty} (q_e(n) \mp q_o(n))q^n, \qquad |q| < 1. \tag{3.2}$$

In addition, we remark Euler's Pentagonal Number Theorem

$$q_e(n) - q_o(n) = \begin{cases} (-1)^{\lceil k/2 \rceil}, & \text{if } n = G_k, k \in \mathbb{N}, \\ 0, & \text{otherwise}. \end{cases} \tag{3.3}$$

We denote by $s(n, k)$ the number of k's in all partitions of n into distinct parts. It is clear that

$$s(n, k) = s_o(k, d) + s_e(k, d).$$

Some connections between partitions and divisors can be easily derived from Theorem 1.1 taking into account these relations.

Corollary 3.1 *Let n be a positive integer and let $a_1, \ldots, a_n$ be n real or complex numbers. Then*

$$\sum_{d|k} (\pm 1)^{1+k/d} a_d = \sum_{j=1}^{k} A_\pm(j)(p_e(k-j) \pm p_o(k-j)),$$

where

$$A_\pm(n) = \sum_{k=1}^{n} (s_o(n, k) \mp s_e(n, k))a_k.$$

Proof By Theorem 1.2 and (3.1), we deduce that

$$\sum_{n=1}^{\infty} b_\pm(n)q^n = \left(\sum_{n=0}^{\infty} (p_e(n) \pm p_o(n))q^n \right) \left(\sum_{n=1}^{\infty} A_\pm(n)q^n \right), \qquad |q| < 1,$$

where

$$b_{\pm}(n) = \sum_{d|n} (\pm)^{1+n/d} a_d.$$

Taking into account the Cauchy product of two power series, the proof follows easily. $\square$

Example For $n \leqslant 7$, we have the following partitions into distinct parts:

$$\begin{array}{lllll}
1\,, & & & & \\
2\,, & & & & \\
3\,, & 2+1\,, & & & \\
4\,, & 3+1\,, & & & \\
5\,, & 4+1\,, & 3+2\,, & & \\
6\,, & 5+1\,, & 4+2\,, & 3+2+1\,, & \\
7\,, & 6+1\,, & 5+2\,, & 4+3\,, & 4+2+1.
\end{array}$$

We see that

$$[s_o(n,k)]_{7\times 7} = \begin{bmatrix}
1 & & & & & & \\
0 & 1 & & & & & \\
0 & 0 & 1 & & & & \\
0 & 0 & 0 & 1 & & & \\
0 & 0 & 0 & 0 & 1 & & \\
1 & 1 & 1 & 0 & 0 & 1 & \\
1 & 1 & 0 & 1 & 0 & 0 & 1
\end{bmatrix}$$

and

$$[s_e(n,k)]_{7\times 7} = \begin{bmatrix}
0 & & & & & & \\
0 & 0 & & & & & \\
1 & 1 & 0 & & & & \\
1 & 0 & 1 & 0 & & & \\
1 & 1 & 1 & 1 & 0 & & \\
1 & 1 & 0 & 1 & 1 & 0 & \\
1 & 1 & 1 & 1 & 1 & 1 & 0
\end{bmatrix}.$$

So we have

$$\begin{aligned}
a_1 + a_7 = p(6)\cdot a_1 + p(5)\cdot a_2 + p(4)\cdot(-a_1 - a_2 + a_3) \\
+p(3)\cdot(-a_1 - a_3 + a_4) + p(2)\cdot(-a_1 - a_2 - a_3 - a_4 + a_5) \\
+p(1)\cdot(a_3 - a_4 - a_5 + a_6) + p(0)\cdot(-a_3 - a_5 - a_6 + a_7).
\end{aligned}$$

The following result is a natural consequence of Corollary 3.1.

Corollary 3.2 *Let d and n be two positive integers such that $d \leqslant n$. Then*

$$\sum_{k=d}^{n} (s_o(k,d) \mp s_e(k,d))(p_e(n-k) \pm p_o(n-k)) = \delta_{0,n \bmod d},$$

where $\delta_{i,j}$ is the Kronecker delta.

More explicitly, this result can be written as

$$\sum_{k=d}^{n} (s_o(k,d) - s_e(k,d))p(n-k) = \delta_{0,n \bmod d}$$

and

$$\sum_{k=d}^{n} s(k,d)(p_e(n-k) - p_o(n-k)) = \delta_{0,n \bmod d}.$$

Another consequence of Theorem 1.2 is given by

Corollary 3.3 *Let n be a positive integer and let $a_1, \ldots, a_n$ be n real or complex numbers. Then*

$$\sum_{k=1}^{n} b_{\pm}(k)(q_e(n-k) \pm q_o(n-k)) = \sum_{j=1}^{n} (s_o(n,j) \pm s_e(n,j))a_j,$$

where

$$b_{\pm}(n) = \sum_{d|n} (\mp 1)^{1+n/d} a_d.$$

Proof By Theorem 1.2 and (3.2), we obtain

$$\left(\sum_{n=1}^{\infty} b_{\pm}(n)q^n\right)\left(\sum_{n=0}^{\infty} (q_e(n) \pm q_o(n))q^n\right)$$
$$= \sum_{n=0}^{\infty}\sum_{j=1}^{n} (s_o(n,j) \pm s_e(n,j))a_j q^n, \quad |q| < 1.$$

Applying Cauchy multiplication, we arrive at our conclusion. □

On the other hand, by Corollary 3.3, we deduce

Corollary 3.4 *Let d, k, and n be three positive integers such that $d \leqslant n$ and $G_{k-1} < n \leqslant G_k$. Then*

$$\sum_{j=0}^{k-1} (-1)^{\lceil j/2 \rceil} \delta_{0,(n-G_j) \bmod d} = s_o(n,d) - s_e(n,d).$$

Corollary 3.5 *Let k and n be two positive integers. Then*

$$\sum_{j=1}^{\lfloor n/k \rfloor} (-1)^{j-1} q(n - j \cdot k) = s(n,k).$$

4 On the number of k's in partitions of n into distinct parts

We have seen that the Lambert series can be expressed using the numbers $s_o(n,k)$ and $s_e(n,k)$. Then some identities involving $s_o(n,k)$ and $s_e(n,k)$ have been deduced, see for example Corollaries 3.1–3.5. By the relation (3.3) and Corollary 3.4, we deduce that $s_o(n,1) - s_e(n,1)$ is the coefficient of q^{n-1} in $(q^2;q)_\infty$, i.e.,

$$\begin{aligned}
(q^2;q)_\infty &= \frac{1}{1-q}(q;q)_\infty \\
&= \frac{1}{1-q}\sum_{n=0}^{\infty}(q_e(n) - q_o(n))q^n \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}(q_e(k) - q_o(k))q^n.
\end{aligned}$$

In other words, we have

$$s_o(n,1) - s_e(n,1) = \sum_{k=0}^{n-1}\left(q_e(k) - q_o(k)\right).$$

Taking into account Euler's Pentagonal Number Theorem (3.3), it is not difficult to show that

$$s_o(n,1) - s_e(n,1) = \begin{cases} (-1)^k, & \text{if } \frac{k(3k+1)}{2} \leqslant n-1 \leqslant \frac{k(3k+1)}{2} + 2k,\ k \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

So it is natural to ask about the numbers $s_o(n,k)$ and $s_e(n,k)$. First, we see that these numbers satisfy the relations

$$s_o(n,k) = q_e(n-k) - s_e(n-k,k) \tag{4.1}$$

and

$$s_e(n,k) = q_o(n-k) - s_o(n-k,k). \tag{4.2}$$

The following identities are easily obtained by an alternating use of (4.1) and (4.2).

Corollary 4.1 *The numbers $s_o(n,k)$ and $s_e(n,k)$ can be expressed in terms of the partition functions q_o and q_e:*

$$s_o(n,k) = \sum_{j=1}^{\left\lfloor \frac{n+k}{2k} \right\rfloor} q_e(n-(2j-1)\cdot k) - \sum_{j=1}^{\left\lfloor \frac{n}{2k} \right\rfloor} q_o(n-2j\cdot k)$$

and

$$s_e(n,k) = \sum_{j=1}^{\left\lfloor \frac{n+k}{2k} \right\rfloor} q_o(n-(2j-1)\cdot k) - \sum_{j=1}^{\left\lfloor \frac{n}{2k} \right\rfloor} q_e(n-2j\cdot k),$$

where $\lfloor x \rfloor$ *is the largest integer not greater than* x.

A new expansion for $s_o(n,k) - s_e(n,k)$ can be easily derived from this corollary:

$$s_o(n,k) - s_e(n,k) = \sum_{j=1}^{\lfloor n/k \rfloor} (q_e(n-j\cdot k) - q_o(n-j\cdot k)).$$

Is it possible to deduce Corollary 3.4 from this identity? Anyway, we can write

Corollary 4.2 *Let* d, k, *and* n *be three positive integers such that* $d \leqslant n$ *and* $G_{k-1} < n \leqslant G_k$. *Then*

$$\sum_{j=0}^{k-1} (-1)^{\lceil j/2 \rceil} \delta_{0,(n-G_j) \bmod d} = \sum_{j=1}^{\lfloor n/d \rfloor} (q_e(n-j\cdot d) - q_o(n-j\cdot d)).$$

In addition, it is clear that Corollary 3.5 can be easily obtained from Corollary 4.1.

Corollary 4.3 *For* $k > 0$, *the generating functions for the numbers* $s_o(n,k)$ *and* $s_e(n,k)$ *are given by*

$$\sum_{n=0}^{\infty} s_o(n,k) q^n = \frac{1}{2} \cdot \frac{q^k}{1+q^k} \cdot (-q;q)_\infty + \frac{1}{2} \cdot \frac{q^k}{1-q^k} \cdot (q;q)_\infty$$

and

$$\sum_{n=0}^{\infty} s_e(n,k) q^n = \frac{1}{2} \cdot \frac{q^k}{1+q^k} \cdot (-q;q)_\infty - \frac{1}{2} \cdot \frac{q^k}{1-q^k} \cdot (q;q)_\infty.$$

Proof By the relations (4.1) and (4.2), for $k > 0$, we deduce that

$$\sum_{n=0}^{\infty} (s_o(n,k) - s_e(n,k)) q^n = \frac{q^k}{1-q^k} \cdot (q;q)_\infty$$

and

$$\sum_{n=0}^{\infty}(s_o(n,k)+s_e(n,k))q^n = \frac{q^k}{1+q^k}\cdot(-q;q)_\infty.$$

Here we invoked the relation (3.2). □

5 Some applications

5.1 Positive divisor functions $\tau(n)$ and $\sigma(n)$

Considering the relation [16, A035116]

$$\sum_{d|n}\tau(d^2)=\tau^2(n),$$

Corollaries 3.1 and 3.3 can be written as follows:

Corollary 5.1 *For $n>0$,*

$$\tau^2(n)=\sum_{k=1}^{n}\sum_{j=1}^{k}(s_o(k,j)-s_e(k,j))\tau(j^2)p(n-k).$$

Corollary 5.2 *Let k and n be two positive integers such that $G_{k-1}<n\leqslant G_k$. Then*

$$\sum_{j=0}^{k-1}(-1)^{\lceil j/2\rceil}\tau^2(n-G_j)=\sum_{j=1}^{n}(s_o(n,j)-s_e(n,j))\tau(j^2).$$

New convolutions for the number of divisors function have been recently published by Ballantine and Merca [3], as applications of Theorem 1.2.

By Theorem 1.2, with a_k replaced by k, we get

$$\begin{aligned}\sum_{n=1}^{\infty}\sigma(n)q^n &= \frac{1}{(q;q)_\infty}\sum_{n=1}^{\infty}\sum_{0<i_1<\cdots<i_n}(-1)^{n-1}(i_1+\ldots+i_n)q^{i_1+\ldots+i_n}\\ &= \frac{1}{(q;q)_\infty}\sum_{n=1}^{\infty}(q_o(n)-q_e(n))\,nq^n.\end{aligned}$$

Taking into account (3.3), we obtain a known factorization for the generating function of $\sigma(n)$.

Corollary 5.3 *For $|q|<1$,*

$$\sum_{n=1}^{\infty}\sigma(n)q^n=\frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^{\lceil n/2\rceil-1}G_nq^{G_n}.$$

The relations (1.4) and (1.6) are immediate consequences of this corollary. More details about these relations can be seen in [14] and references therein.

Because

$$\sum_{d|n}(-1)^{1+n/d}d = \sigma_{odd}(n),$$

where $\sigma_{odd}(n)$ denotes the sum of odd divisors of n [16, A000593], the other case $a_k = k$ of Theorem 1.2 can be written as

Corollary 5.4 *For* $|q| < 1$,

$$\sum_{n=1}^{\infty}\sigma_{odd}(n)q^n = \frac{1}{(-q;q)_\infty}\sum_{n=0}^{\infty}nq(n)q^n.$$

Thus, we deduce the following two identities.

Corollary 5.5 *For* $n > 0$,

$$\sigma_{odd}(n) = \sum_{k=1}^{n}kq(k)(p_e(n-k) - p_o(n-k)).$$

Corollary 5.6 *For* $n > 0$,

$$nq(n) = \sum_{k=1}^{n}\sigma_{odd}(k)q(n-k).$$

We remark that the last identity is known [16, A000009] and is similar to the well-known convolution (1.5) involving the partition function $p(n)$ and sum of divisors function $\sigma(n)$.

5.2 Möbius function

The classical Möbius function $\mu(n)$ is an important multiplicative function in number theory and combinatorics. This function is defined for all positive integers n and has its values in $\{-1, 0, 1\}$ depending on the factorization of n into prime factors:

$$\mu(n) = \begin{cases} 0, & \text{if } n \text{ has a squared prime factor,} \\ (-1)^k, & \text{if } n \text{ is a product of } k \text{ distinct primes.} \end{cases}$$

The sum over all positive divisors of n of the Möbius function is zero except when $n = 1$ [6, Theorem 263], i.e.,

$$\sum_{d|n}\mu(d) = \delta_{n,1}.$$

On the other hand, since

$$\sum_{n=1}^{\infty} a_n \frac{q^n}{1+q^n} = \sum_{n=1}^{\infty} a_n \frac{q^n}{1-q^n} - 2\sum_{n=1}^{\infty} a_n \frac{q^{2n}}{1-q^{2n}}, \qquad |q| < 1,$$

we deduce that

$$\sum_{d|n} (-1)^{1+n/d} \mu(d) = \begin{cases} 1, & \text{if } n = 1, \\ -2, & \text{if } n = 2, \\ 0, & \text{otherwise.} \end{cases}$$

So by Corollaries 3.1 and 3.3, we derive the following results.

Corollary 5.7 *Let n be a positive integer. Then*

1. $\displaystyle\sum_{k=1}^{n}\sum_{j=1}^{k}(s_o(k,j) - s_e(k,j))\mu(j)p(n-k) = 0, \qquad n > 1.$
2. $\displaystyle\sum_{k=1}^{n}\sum_{j=1}^{k}s(k,j)\mu(j)(p_e(n-k) - p_o(n-k)) = 0, \qquad n > 2.$

Corollary 5.8 *Let n be a positive integer. Then*

1. $\displaystyle\sum_{k=1}^{1+G_n}(s_o(1+G_n,k) - s_e(1+G_n,k))\mu(k) = (-1)^{\lceil n/2 \rceil}.$
2. $\displaystyle\sum_{k=1}^{n}s(n,k)\mu(k) = q(n-1) - 2q(n-2).$

Recall that a natural number d is a *unitary divisor* of a number n if d is a divisor of n and if d and n/d are coprime. The sum over all positive divisors of n of the absolute value of the Möbius function is equal to the number of unitary divisors of n [6, Theorem 264],

$$\sum_{d|n} |\mu(d)| = 2^{\omega(n)},$$

where $\omega(n)$ is an additive function defined as the number of distinct primes dividing n. By Corollary 3.1, we get the following identity.

Corollary 5.9 *For $n > 0$,*

$$\sum_{k=1}^{n}\sum_{j=1}^{k}(s_o(k,j) - s_e(k,j))\,|\mu(j)|\,p(n-k) = 2^{\omega(n)}.$$

According to Corollary 3.3, the Möbus function and generalized pentagonal numbers can be combined as follows:

Corollary 5.10 *Let k and n be two positive integers such that $G_{k-1} < n \leqslant G_k$. Then*

$$\sum_{j=0}^{k-1} (-1)^{\lceil j/2 \rceil} 2^{w(n-G_j)} = \sum_{j=1}^{n} (s_o(n,j) - s_e(n,j)) \, |\mu(j)| \, .$$

Moreover, considering the relation [10, Exercise 1.52]

$$\sum_{d|n} 2^{\omega(d)} = \tau(n^2),$$

Corollaries 3.1 and 3.3 can be written as follows:

Corollary 5.11 *For $n > 0$,*

$$\tau(n^2) = \sum_{k=1}^{n} \sum_{j=1}^{k} (s_o(k,j) - s_e(k,j)) 2^{\omega(j)} p(n-k).$$

Corollary 5.12 *Let k and n be two positive integers such that $G_{k-1} < n \leqslant G_k$. Then*

$$\sum_{j=0}^{k-1} (-1)^{\lceil j/2 \rceil} \tau((n-G_j)^2) = \sum_{j=1}^{n} (s_o(n,j) - s_e(n,j)) 2^{\omega(j)}.$$

5.3 Jordan's totient function

Euler's totient or phi function, $\varphi(n)$, is a multiplicative function that counts the totatives of n, that is the positive integers less than or equal to n that are relatively prime to n. According to Euler's classical formula [6, Theorem 63],

$$\sum_{d|n} \varphi(d) = n$$

and

$$\sum_{d|n} (-1)^{1+n/d} \varphi(d) = n \cdot \delta_{1, n \bmod 2}.$$

By Corollary 3.1, we obtain the following identities.

Corollary 5.13 *For $n > 0$,*

1. $\displaystyle\sum_{k=1}^{n} \sum_{j=1}^{k} (s_o(k,j) - s_e(k,j)) \varphi(j) p(n-k) = n.$

2. $\sum_{k=1}^{n}\sum_{j=1}^{k} s(k,j)\varphi(j)(p_e(n-k)-p_o(n-k)) = n\cdot\delta_{1,n \bmod 2}$.

The following identity is a special case of Corollary 3.3.

Corollary 5.14 *For $n > 0$,*

$$\sum_{k=1}^{n} s(n,k)\varphi(k) = \sum_{k=1}^{\lceil n/2\rceil} (2k-1)q(n-2k+1).$$

In number theory, Jordan's totient function of a positive integer n, $J_t(n)$, is the number of t-tuples of positive integers all less than or equal to n that form a coprime $(t+1)$-tuple together with n. This is a generalization of Euler's totient function, which is J_1. Considering the identity [13, eq. 27.6.8, p. 641]

$$\sum_{d|n} J_t(d) = n^t,$$

by Corollaries 3.1 and 3.3, we obtain the following results.

Corollary 5.15 *For $n > 0$,*

$$\sum_{k=1}^{n}\sum_{j=1}^{k}(s_o(k,j)-s_e(k,j))J_t(j)p(n-k) = n^t, \qquad n,t > 0.$$

Corollary 5.16 *Let k, n, and t be positive integers such that $G_{k-1} < n \leqslant G_k$. Then*

$$\sum_{j=0}^{k-1}(-1)^{\lceil j/2\rceil}(n-G_j)^t = \sum_{j=1}^{n}(s_o(n,j)-s_e(n,j))J_t(j).$$

5.4 Liouville's function

For a positive integer n, the Liouville function $\lambda(n)$ is a completely multiplicative function defined as follows:

$$\lambda(n) = (-1)^{\Omega(n)},$$

where $\Omega(n)$ is the number of not necessarily distinct prime factors of n, with $\Omega(1) = 0$. We remark that $\Omega(n)$ is a completely additive function. Taking into account the relation [13, eq. 27.7.6, p. 641]

$$\sum_{n=1}^{\infty}\lambda(n)\frac{q^n}{1-q^n} = \sum_{n=1}^{\infty} q^{n^2}, \qquad |q| < 1,$$

we can write

$$\sum_{n=1}^{\infty}\left(\sum_{d|n}(-1)^{1+n/d}\lambda(d)\right)q^n = \sum_{n=1}^{\infty}\lambda(n)\frac{q^n}{1+q^n}$$
$$= \sum_{n=1}^{\infty}\lambda(n)\frac{q^n}{1-q^n} - 2\sum_{n=1}^{\infty}\lambda(n)\frac{q^{2n}}{1-q^{2n}}$$
$$= \sum_{n=1}^{\infty}q^{n^2} - 2\sum_{n=1}^{\infty}q^{2n^2}, \qquad |q|<1.$$

It is clear that

$$\sum_{d|n}(-1)^{1+n/d}\lambda(d) = \begin{cases} 1, & \text{if } n \text{ is a square,}\\ -2, & \text{if } n \text{ is twice a square,}\\ 0, & \text{otherwise.}\end{cases}$$

So by Corollary 3.1, we deduce the following identities.

Corollary 5.17 *For $n>0$,*

1. $\displaystyle\sum_{k=1}^{n^2}\sum_{j=1}^{k}(s_o(k,j)-s_e(k,j))\lambda(j)p(n^2-k)=1.$
2. $\displaystyle\sum_{k=1}^{n^2}\sum_{j=1}^{k}s(k,j)\lambda(j)(p_e(n^2-k)-p_o(n^2-k))=1.$
3. $\displaystyle\sum_{k=1}^{2n^2}\sum_{j=1}^{k}s(k,j)\lambda(j)(p_e(2n^2-k)-p_o(2n^2-k))=-2.$

The following identities are special cases of Corollary 3.3.

Corollary 5.18 *For $n>0$,*

$$\sum_{k=1}^{n}s(n,k)\lambda(k)=\sum_{k=1}^{\infty}\left(q(n-k^2)-2q(n-2k^2)\right).$$

Corollary 5.19 *Let k and n be two positive integers such that $G_{k-1}<n\leqslant G_k$. Then*

$$\sum_{j=0}^{k-1}(-1)^{\lceil j/2\rceil}\tau(n-G_j) \bmod 2=\sum_{j=1}^{n}(s_o(n,j)-s_e(n,j))\lambda(j).$$

Proof We use the relation [13, eq. 27.6.1, p. 641]

$$\sum_{d|n} \lambda(d) = \tau(n) \bmod 2.$$

□

5.5 von Mangoldt function

The von Mangoldt function, conventionally written as $\Lambda(n)$, is an arithmetic function defined as

$$\Lambda(n) = \begin{cases} \ln(p), & \text{if } n = p^k \text{ for some prime } p \text{ and positive integer } k, \\ 0, & \text{otherwise.} \end{cases}$$

This function is an example of an important arithmetic function that is neither multiplicative nor additive. It satisfies the identity [6, Theorem 296]

$$\sum_{d|n} \Lambda(d) = \ln(n).$$

This is proved by the fundamental theorem of arithmetic, since the terms that are not powers of primes are equal to 0. By Corollary 3.1, we derive the following identity.

Corollary 5.20 *For $n > 0$,*

$$\sum_{k=1}^{n} \sum_{j=1}^{k} (s_o(k, j) - s_e(k, j)) \Lambda(j) p(n-k) = \ln(n).$$

Replacing a_j by the von Mangoldt function $\Lambda(j)$ in Corollary 3.3, we get the following result.

Corollary 5.21 *Let k and n be two positive integers such that $G_{k-1} < n \leqslant G_k$. Then*

$$\sum_{j=0}^{k-1} (-1)^{\lceil j/2 \rceil} \ln(n - G_j) = \sum_{j=1}^{n} (s_o(n, j) - s_e(n, j)) \Lambda(j).$$

6 Concluding remarks

A factorization for the partial sum of Lambert series has been introduced in this paper. A generalization of the well-known connections between partitions and the sum of divisors function $\sigma(n)$ has been obtained as a consequence of this fact. This allows us to derive some identities involving partitions and few important functions from number theory: the Möbius function $\mu(n)$, Euler's totient $\varphi(n)$, Jordan's totient $J_k(n)$, Liouville's function $\lambda(n)$, the von Mangoldt function $\Lambda(n)$ and the divisor function $\sigma_x(n)$.

Moreover, Theorem 1.2 can be used to discover and prove numerous identities. For instance, replacing a_k by $a_k q^k$ in this theorem, we derive the following identity:

$$\sum_{n=1}^{\infty} a_n \frac{q^{2n}}{1 \pm q^n} = \frac{1}{(\mp q; q)_\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{n} (s_o(n,k) \pm s_e(n,k)) a_k q^{n+k}, \qquad |q| < 1,$$

which can be rewritten as

Corollary 6.1 *For $|q| < 1$,*

$$\sum_{n=1}^{\infty} a_n \frac{q^{2n}}{1 \pm q^n} = \frac{1}{(\mp q; q)_\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\lfloor n/2 \rfloor} (s_o(n-k,k) \pm s_e(n-k,k)) a_k q^n,$$

where $a_1, \ldots, a_n$ are real or complex numbers.

On the other hand, it is not difficult to prove that the series

$$\sum_{n=1}^{\infty} a_n \frac{q^{2n}}{1 \pm q^n}$$

is the generating function of

$$\sum_{\substack{d|n \\ d<n}} (\mp 1)^{n/d} a_d.$$

Thus, the applications presented in the previous section of the paper can be repeated in a slightly different context.

Acknowledgements The author likes to thank the referees for their helpful comments. The author also likes to mention his special thanks to Dr. Oana Merca for the careful reading of the manuscript and helpful remarks.

References

1. Andrews, G.E., Berndt, B.C.: Ramanujan's Lost Notebook, Part 1. Springer, New York (2005)
2. Andrews, G.E., Eriksson, K.: Integer Partitions. Cambridge University Press, Cambridge (2004)
3. Ballantine, C., Merca, M.: New convolutions for the number of divisors. J. Number Theory **170**, 17–34 (2017)
4. Bromwich, T.J.: An Introduction to the Theory of Infinite Series, 2nd edn. Macmillan, New York (1926)
5. Chrystal, G.: Algebra, vol. 2. Chelsea, New York (1952)
6. Hardy, G.H., Wright, E.M.: An Introduction to the Theory of Numbers, 5th edn. Clarendon Press, Oxford (1979)
7. Knopp, K.: Theory and Application of Infinite Series. Dover, New York (1990)
8. MacMahon, P.A.: Combinatory Analysis. Chelsea, New York (1960)

9. MacMahon, P.A.: Divisors of numbers and their continuations in the theory of partitions. Proc. Lond. Math. Soc. **s2–19**(1), 75–113 (1921)
10. McCarthy, P.J.: Introduction to Arithmetical Functions. Springer, New York (1986)
11. Merca, M.: A new look on the generating function for the number of divisors. J. Number Theory **149**, 57–69 (2015)
12. Merca, M.: Combinatorial interpretations of a recent convolution for the number of divisors of a positive integer. J. Number Theory **160**, 60–75 (2016)
13. Olver, F.W.J., Lozier, D.W., Boisvert, R.F., Clark, C.W.: NIST Handbook of Mathematical Functions. Cambridge University Press, Cambridge (2010)
14. Osler, T.J., Hassen, A., Chandrupatla, T.R.: Surprising connections between partitions and divisors. Coll. Math. J. **38**(4), 278–287 (2007)
15. Pólya, G., Szegő, G.: Problems and Theorems in Analysis II. Springer, Berlin (1976)
16. Sloane, N.J.A.: The on-line encyclopedia of integer sequences. http://oeis.org (2015)
17. Titchmarsh, E.C.: The Theory of Functions, 2nd edn. Oxford University Press, London (1939)