SOME INITIAL PROPERTIES OF FACTORIZATIONS FOR RELATIVELY PRIME DIVISOR SUMS

MAXIE D. SCHMIDT
GEORGIA INSTITUTE OF TECHNOLOGY
SCHOOL OF MATHEMATICS
117 SKILES BUILDING
686 CHERRY STREET NW
ATLANTA, GA 30332
USA
MAXIEDS@GMAIL.COM
MSCHMIDT34@GATECH.EDU

Abstract.

1. Introduction

1.1. **Factorization theorems.** We construct analogs to the Lambert series factorization theorems proved in [**?**] for sums over relatively prime divisors of the form

$$\sum_{d:(d,n)=1} f(d) = [q^n] \frac{1}{C_*(q)} \sum_{n \geq 1} \sum_{k=1}^{n} s_{n,k}^{*} f(k) \cdot q^n, \tag{1}$$

where $f(n)$ denotes an arbitrary arithmetic function, typically with $f(1) = 1$, though not by necessity in our initial construction, and where the generating function $C_*(q)$ is combinatorially motivated in our results. In particular, we define the generating function for the number of partitions of n into pairwise relatively prime parts, denoted here by $\mathrm{rpp}(n)$ for $n \geq 0$, [4, A051424] as

$$P_\Psi(q) := \sum_{n \geq 0} \mathrm{rpp}(n) q^n = 1+q+2q^2+3q^3+4q^4+6q^5+7q^6+10q^7+12q^8+15q^9+\cdots.$$

We then define ancillary sequence, $\mathrm{rpp}_2(n)$, for non-negative integers $n \geq 0$ to be the sequence whose reciprocal generating function is $P_\Psi(q)$ defined as above. The first few terms of the second partition sequence are expanded as

$$\{\mathrm{rpp}_2(n)\}_{n \geq 0} = \{1, -1, -1, 0, 1, 0, 1, -1, 0, 0, 1, -3, 2, 0, 3, -1, -2, -10, 8, 5, \ldots\}.$$

We use each of these partition funtion definitions to identify the characteristic expansions of our primary special case of the factorization in (1) defined for an arbitrary arithmetic function f by

$$\sum_{d:(d,n)=1} f(d) = [q^n] \frac{1}{P_\Psi(q)} \sum_{n \geq 1} \sum_{k=1}^{n-1} \widetilde{s}_{n,k} f(k) \cdot q^n. \tag{2}$$

Since the matrix for the sequence $\widetilde{s}_{n,k}$ is not invertible if the index $n = 1$ is included, we consider the corresponding inverse sequence, $s_{n,k}^{(-1)}$, to invert the matrices

---

Date: Monday 7$^{\text{th}}$ August, 2017.

$(s_{i+1,j})_{1 \leq i,j<n}$ for integers $n \geq 2$ and then perform an adjustment later in our formulas that employ these sequences. Namely, we re-write (2) in the form of[1]

$$\sum_{d:(d,n)=1} f(d) = [q^n] \left(\frac{1}{P_\Psi(q)} \sum_{n \geq 2} \sum_{k=1}^{n-2} s_{n,k} f(k) \cdot q^n + f(1) \cdot q \right). \tag{3}$$

The explicit expansion in (3) is the key form of our modified factorization theorem result that we study within this article.

1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
−1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
−1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	−1	−1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
−1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	−1	−1	1	0	0	0	0	0	0	0	0	0	0	0
1	0	−1	0	−1	0	1	0	0	0	0	0	0	0	0	0	0
−1	0	2	−1	0	0	−1	1	0	0	0	0	0	0	0	0	0
−1	0	0	0	1	0	−1	0	1	0	0	0	0	0	0	0	0
1	0	−1	1	0	−1	1	−1	−1	1	0	0	0	0	0	0	0
−1	0	1	0	0	0	−1	0	0	0	1	0	0	0	0	0	0
1	0	−1	0	0	0	1	0	0	−1	−1	1	0	0	0	0	0
3	0	−2	0	−2	0	2	0	−1	0	−1	0	1	0	0	0	0
−3	0	1	0	3	0	−1	−1	1	0	0	0	−1	1	0	0	0
−1	0	1	0	1	0	−1	0	0	0	0	0	−1	0	1	0	0
1	0	0	0	−2	0	0	1	0	0	1	−1	1	−1	−1	1	0
−3	0	2	0	2	0	−2	0	1	0	0	0	−1	0	0	0	1

$\mu_{n,k}$ *for* $1 \leq n, k < 18$

Figure 1.1. *Inversion formula coefficient sequences*

1.2. **Applications and examples of our new results.**

2. Proofs of the main results

2.1. **Inversion relations.** We begin our exploratory analysis here by expanding an inversion formula which is analogous to Möbius inversion for ordinary divisor sums. We prove the following result which is the analog to the sequence inversion relation provided by the Möbius transform in the context of our sums over the integers relatively prime to n [3, *cf.* §2, §3].

[1] <u>*Notation*</u>: *Iverson's convention* compactly specifies boolean-valued conditions and is equivalent to the *Kronecker delta function*, $\delta_{i,j}$, as $[n=k]_\delta \equiv \delta_{n,k}$. Similarly, $[\mathtt{cond} = \mathtt{True}]_\delta \equiv \delta_{\mathtt{cond},\mathtt{True}}$ in the remainder of the article.

Proposition 2.1 (Inversion Formula). *For all $n \geq 2$, there is a unique lower triangular sequence, denoted $\mu_{n,k}$, which satisfies the inversion relation*

$$g(n) = \sum_{d:(d,n)=1} f(d) \qquad \Longleftrightarrow \qquad f(n) = \sum_{d=1}^{n} g(d+1)\mu_{n,d}.$$

Moreover, if we form the matrix $(\mu_{i,j})_{1\leq i,j\leq n}$ for any $n \geq 2$, we have that the inverse sequence satisfies

$$\mu_{n,k}^{(-1)} = [(n+1,k)=1]_{\delta}.$$

Proof. Consider the $(n-1)\times(n-1)$ matrix

$$\left([(i,j)=1]_{\delta}\right)_{1\leq i,j<n},$$

which effectively corresponds to the formula on the left-hand-side of the first equation by applying the matrix to the vector of $[f(1)\ f(2)\ \cdots f(n)]^T$ and extracting the n^{th} column as our stated formula. Since $\gcd(n,n-1)=1$ for all $n>1$, we see that this matrix is lower triangular with ones on its diagonal. Thus the matrix is non-singular and its unique inverse, which we denote by $(\mu_{i,j})_{1\leq i,j<n}$, leads to the sum on the right-hnd-side of the first equation when we shift $n\mapsto n+1$. The second equation restates the form of the first matrix when we perform the shift of $n\mapsto n+1$ as on the right-hand-side of the first equation. $\square$

Figure 1.1 provides a listing of the relevant analogs to the Möbius function in the context of the Möbius transform of the ordinary divisor sum over an arithmetic function from the proposition. We not know of a comparitively simple closed-form function for the sequence of $\mu_{n,k}$ [4, *cf.* A096433]. However, we readily see by construction that the sequence and its inverse satisfy

$$\sum_{d:(d,n)=1} \mu_{d,k} = 0$$
$$\sum_{d:(d,n)=1} \mu_{d,k}^{(-1)} = \phi(n),$$

where $\phi(n)$ is Euler's totient function. The first columns of the corresponding sums in the previous equation performed over the columns index k for fixed n appear in the integer sequences database as the entry [4, A096433].

2.2. **Exact formulas for the factorization matrices.** The next result is key to proving the exact formulas for the matrix sequences, $s_{n,k}$ and $s_{n,k}^{(-1)}$, and their expansions by the partition functions defined in the introduction. We prove it first as a lemma which we will use in the proof of Theorem 2.3 given below.

Lemma 2.2 (A Convolution Identity for Relatively Prime Integers). *For all natural numbers $n\geq 2$ and $k\geq 1$ with $k\leq n$, we have the following expression for the indicator function of whether $(n+1,k)$ forms a pair of relatively prime integers:*

$$\sum_{j=1}^{n} s_{j,k}\,\mathrm{rpp}_2(n-j) = [(n+1,k)=1]_{\delta}.$$

Proof. We begin by noticing that the right-hand-side expression in the statement of the lemma is equal to $\mu_{n,k}^{(-1)}$ by the construction of this sequence in Proposition 2.1. Next, we see that the factorization in (2) is equivalent to the expansion

$$\sum_{d=1}^{n-1} f(d)\mu_{n,d}^{(-1)} = \sum_{j=1}^{n}\sum_{k=1}^{j-1} \operatorname{rpp}_2(n-j)\widetilde{s}_{j,k} \cdot f(k).$$

Since $\mu_{n,k}^{(-1)} = [(n+1,k)=1]_\delta$, we may take the coefficients of $f(k)$ on each side of the previous equation for each $1 \leq k < n$ to establish the claimed result. □

The first several rows of the matrix sequence $s_{n,k}$ and its inverse implicit to the factorization theorem in (3) are tabulated in Figure 2.1 for intuition on the formulas we prove in the next theorem.

Theorem 2.3 (Exact Formulas for the Factorization Matrix Sequences)**.** *For integers $n,k \geq 1$, the two factorization sequences implicit to the expansion of (3) have the next exact formulas given by*

$$s_{n,k} = \sum_{j=0}^{n-k} \operatorname{rpp}(n-k-j)\left[(j+k+1,k)=1\right]_\delta \tag{i}$$

$$s_{n,k}^{(-1)} = \sum_{d=1}^{n} \operatorname{rpp}_2(d-k)\mu_{n,d}. \tag{ii}$$

Proof. It is plain to see by the considerations in our construction of the factorization theorem that both matrix sequences are lower triangular. Thus, we need only consider the cases where $n \leq k$. By a convolution of generating functions, the identity in Lemma 2.2 shows that

$$\sum_{j=k}^{n} \operatorname{rpp}(n-j)\left[(j+1,k)=1\right]_\delta .$$

Then shifting the index of summation in the previous equation implies (i).

To prove (ii), we consider the factorization theorem when $f(n) := s_{n,r}^{(-1)}$ for some fixed $r \geq 1$. Then we expand the result in (3) as

$$\begin{aligned}
\sum_{d:(d,n)=1} s_{d,r}^{(-1)} &= [q^n]\frac{1}{P_\Psi(q)}\sum_{n\geq 1}\sum_{k=1}^{n-1} s_{n,k}\cdot s_{k,r}^{(-1)}\cdot q^n \\
&= \sum_{j=1}^{n}\operatorname{rpp}_2(n-j)\times\sum_{k=1}^{j-1} s_{j,k}s_{k,r}^{(-1)} \\
&= \sum_{j=1}^{n}\operatorname{rpp}_2(n-j)\left[r=j-1\right]_\delta \\
&= \operatorname{rpp}(n-1-r).
\end{aligned}$$

Hence we may perform the inversion by Proposition 2.1 to the left-hand-side sum in the previous equations to obtain our stated result. □

Additional properties of these sequences and their apparent dependence on the shifted partition function $\operatorname{rpp}(n)$ are also suggested by the tables provided in Figure 2.2 for comparision with the next results which we prove rigorously below. In

1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	3	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0
11	4	3	1	1	0	0	0	0	0	0	0	0	0	0	0	0
17	7	6	3	2	1	0	0	0	0	0	0	0	0	0	0	0
24	10	9	4	4	1	1	0	0	0	0	0	0	0	0	0	0
34	14	13	7	7	2	2	1	0	0	0	0	0	0	0	0	0
46	20	19	10	10	3	4	1	1	0	0	0	0	0	0	0	0
61	26	26	14	16	5	7	3	2	1	0	0	0	0	0	0	0
79	35	35	20	22	7	11	4	3	1	1	0	0	0	0	0	0
102	44	46	26	31	10	17	7	6	3	2	1	0	0	0	0	0
129	58	59	35	42	14	23	10	9	4	4	1	1	0	0	0	0
162	71	76	44	54	18	33	14	13	6	7	2	2	1	0	0	0
200	91	96	58	71	24	44	20	19	9	11	3	4	1	1	0	0
243	109	119	71	90	30	58	26	26	12	17	5	7	3	2	1	0
294	134	147	91	114	40	75	35	35	17	24	7	11	4	3	1	1

(i) $s_{n,k}$

1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
−2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
−2	−1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	−1	−2	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	−1	−1	1	0	0	0	0	0	0	0	0	0	0	0	0
0	2	2	−1	−2	1	0	0	0	0	0	0	0	0	0	0	0
2	1	1	1	−2	−1	1	0	0	0	0	0	0	0	0	0	0
−5	0	2	0	1	0	−2	1	0	0	0	0	0	0	0	0	0
−1	−1	−1	−1	3	1	−2	−1	1	0	0	0	0	0	0	0	0
4	−1	−3	2	−1	0	3	−1	−2	1	0	0	0	0	0	0	0
−2	−1	0	−1	2	1	0	0	−1	−1	1	0	0	0	0	0	0
−1	2	1	0	−3	−1	0	2	2	−1	−2	1	0	0	0	0	0
6	0	2	3	−6	−3	3	1	1	1	−2	−1	1	0	0	0	0
−3	2	−6	−3	5	3	−3	−1	0	1	1	0	−2	1	0	0	0
−1	2	0	−4	3	2	−2	−1	0	0	2	1	−2	−1	1	0	0
0	−2	4	6	−4	−2	2	0	−2	0	0	0	3	−1	−2	1	0
−9	−1	1	−2	7	0	−3	−1	0	−1	2	1	0	0	−1	−1	1

(ii) $s_{n,k}^{(-1)}$

Figure 2.1. *The factorization matrices, $s_{n,k}$ and $s_{n,k}^{(-1)}$, for $1 \leq n,k < 18$*

particular, we have the following identities for the first few special case columns of the sequence $s_{n,k}$ which follow from the identity in (i) of Theorem 2.3:

$$\begin{aligned}
s_{n,1} &= \sum_{i=0}^{n-1} \mathrm{rpp}(n-k-i) \\
s_{n,2} &= \sum_{i=0}^{\lfloor \frac{n-2}{2} \rfloor} \mathrm{rpp}(n-k-2i) \\
s_{n,3} &= \sum_{i=0}^{n-3} \left(\mathrm{rpp}(n-k-3i) + \mathrm{rpp}(n-k-3i-1)\right) \\
s_{n,4} &= \sum_{s\in\{0,2\}} \sum_{i=0}^{n-4} \mathrm{rpp}(n-k-4i-s) \\
s_{n,5} &= \sum_{s\in\{0,1,2,3\}} \sum_{i=0}^{n-5} \mathrm{rpp}(n-k-5i-s) \\
s_{n,6} &= \sum_{s\in\{0,4\}} \sum_{i=0}^{n-6} \mathrm{rpp}(n-k-6i-s) \\
s_{n,7} &= \sum_{s\in\{0,1,2,3,4,5\}} \sum_{i=0}^{n-7} \mathrm{rpp}(n-k-7i-s) \\
s_{n,8} &= \sum_{s\in\{0,2,4,6\}} \sum_{i=0}^{n-8} \mathrm{rpp}(n-k-8i-s).
\end{aligned}$$

More generally, we have that

$$s_{n,k} = \sum_{s\in(\mathbb{Z}/k\mathbb{Z})^*} \sum_{i=0}^{n-8} \mathrm{rpp}(n+1-k-ki-s).$$

2.3. **Completing the proofs of the main applications.** We note that as in the Lambert series factorization results from the references [**?**], we have three primary expansion types of identities based on the form of (2) and (3) that we may consider for a prescribed choice of the arithmetic function f in the forms of

$$\begin{aligned}
\sum_{d:(d,n)=1} f(d) &= \sum_{j=1}^{n} \sum_{k=1}^{j-1} \mathrm{rpp}_2(n-j) s_{j-1,k} f(k) \\
\sum_{k=1}^{n-1} s_{n-1,k} f(k) &= \sum_{j=1}^{n} \sum_{d:(d,j)=1} \mathrm{rpp}(n-j) f(d),
\end{aligned}$$

and the corresponding inverted formula providing that

$$f(n) = \sum_{k=1}^{n+1} s_{n,k-1}^{(-1)} \times \sum_{j=2}^{k} \mathrm{rpp}_2(k-j) \sum_{d:(d,j)=1} f(d).$$

In particular, we formalize the known expansions in the previous equations to the modified cases of the factorization theorem in (3) according to the next corollary.

Now the applications cited in the introduction follow immediately and require no further proof than to cite these results in the special case where $f(n) \equiv 1$ for all n. We provide other similar corollaries of the factorization theorem results for the sake of completeness as the next examples.

Corollary 2.4 (Euler's Totient Function)**.** *For all $n \geq 1$, we have the following two identities for Euler's totient function:*

$$\begin{aligned}\phi(n) &= \sum_{j=1}^{n}\sum_{k=1}^{j-1}\sum_{i=0}^{j-1-k} \operatorname{rpp}_2(n-j)\operatorname{rpp}(j-1-k-i)\left[(i+k+1,k)=1\right]_\delta \\ \phi(n) &= \sum_{d:(d,n)=1}\left(\sum_{k=1}^{d+1}\sum_{i=1}^{d}\sum_{j=2}^{k}\operatorname{rpp}(k-j)\operatorname{rpp}_2(i+1-k)\mu_{d,i}\phi(j)\right).\end{aligned}$$

Corollary 2.5 (Identities for the Function $\phi_m(n)$)**.** *For all $n \geq 1$ and any fixed $m \in \mathbb{Z}$, we have that*

$$\begin{aligned}\phi_m(n) &= \sum_{j=1}^{n}\sum_{k=1}^{j-1}\sum_{i=0}^{j-1-k} \operatorname{rpp}_2(n-j)\operatorname{rpp}(j-1-k-i)\left[(i+k+1,k)=1\right]_\delta \cdot k^m \\ \phi_m(n) &= \sum_{d:(d,n)=1}\left(\sum_{k=1}^{d+1}\sum_{i=1}^{d}\sum_{j=2}^{k}\operatorname{rpp}(k-j)\operatorname{rpp}_2(i+1-k)\mu_{d,i}\phi_m(j)\right).\end{aligned}$$

Remark 2.6 (Other Forms of the Modified Divisor Sums)**.** The previous corollary leads to the special case sums involving Euler's totient function in the form of[2]

$$\phi_1(n) = \frac{n}{2}\phi(n).$$

We also notice that by a generating function argument, we may expand the second partition function sequence, $\operatorname{rpp}_2(n)$, directly in terms of the more well-known partitions $\operatorname{rpp}(n)$ as

$$\operatorname{rpp}_2(n) = \sum_{m=1}^{n}\sum_{i_1+\cdots+i_m=n-m}\binom{n-m}{i_1,\ldots,i_m}(-1)^m\operatorname{rpp}(i_1+1)\cdots\operatorname{rpp}(i_m+1).$$

TODO: $A_k(n)$ / Kloosterman sum; and see examples in [NISTHB]
TODO: Relate to divisor sums?

3. Factorizations for a generalized divisor sum

We consider the related factoizations for divisor sums of the form [2, §TODO]

$$\sum_{m=1}^{n}\left(\sum_{d|(n,m)} f(d)g(k/d)\right)w^m = [q^n]\frac{1}{P_\Psi(q)}\sum_{n\geq 1}\sum_{k=1}^{n} t^*_{n,k} f(k)\cdot q^n, \tag{4}$$

i.e., so that by keeping track of the coefficients of powers of w, we can keep track of the inner divisor sums on the left-hand-side of the previous equation.

Proposition 3.1.

Proof. □

[2] Dear Mircea: Can you figure out a creative partition-wise way to generalize this result?

4. Conclusions

TODO: Note the alternate form of the divisor sums ...

Acknowledgments. The authors thank the referees for their helpful insights and comments on preparing the manuscript. The author also thanks Mircea Merca for suggestions on the article and for collaboration on the prior articles on factorization theorems.

References

[1] G. H. Hardy and E. M. Wright. *An Introduction to the Theory of Numbers*. Oxford University Press, 2008.
[2] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark. *NIST Handbook of Mathematical Functions*. Cambridge University Press, 2010.
[3] J. Riordan, *Combinatorial Identities*, Wiley, 1968.
[4] N. J. A. Sloane, The Online Encyclopedia of Integer Sequences, 2017, https://oeis.org/.

E-mail address: maxieds@gmail.com, mschmidt34@gatech.edu

Georgia Institute of Technology, School of Mathematics, 117 Skiles Building, 686 Cherry Street NW, Atlanta, GA 30332, USA

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	3	3	1	1	0	0	0	0	0	0	0	0	0	0	0	0
17	4	5	1	2	0	0	0	0	0	0	0	0	0	0	0	0
24	7	7	3	4	0	1	0	0	0	0	0	0	0	0	0	0
34	10	12	4	6	0	2	0	0	0	0	0	0	0	0	0	0
46	14	16	7	10	1	4	1	1	0	0	0	0	0	0	0	0
61	20	23	10	15	1	7	1	1	0	0	0	0	0	0	0	0
79	26	31	14	21	3	11	3	3	1	1	0	0	0	0	0	0
102	35	41	20	30	4	16	4	5	1	2	0	0	0	0	0	0
129	44	53	26	39	6	23	7	7	2	4	0	1	0	0	0	0
162	58	69	35	53	9	32	10	12	3	7	0	2	0	0	0	0
200	71	86	44	67	12	43	14	16	5	11	1	4	1	1	0	0
243	91	109	58	87	17	57	20	23	7	17	1	7	1	1	0	0

(i) $s_{n,k} - \mathrm{rpp}(n-1-k)$

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	2	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0
4	3	3	1	1	0	0	0	0	0	0	0	0	0	0	0	0
6	4	5	3	2	1	0	0	0	0	0	0	0	0	0	0	0
7	6	7	4	4	1	1	0	0	0	0	0	0	0	0	0	0
10	7	10	6	7	2	2	1	0	0	0	0	0	0	0	0	0
12	10	13	9	10	3	4	1	1	0	0	0	0	0	0	0	0
15	12	17	11	15	5	7	3	2	1	0	0	0	0	0	0	0
18	15	22	16	20	7	11	4	3	1	1	0	0	0	0	0	0
23	18	27	19	27	9	17	7	6	3	2	1	0	0	0	0	0
27	23	33	25	35	13	23	10	9	4	4	1	1	0	0	0	0
33	27	41	30	44	16	32	14	13	6	7	2	2	1	0	0	0
38	33	50	38	55	21	42	20	19	9	11	3	4	1	1	0	0
43	38	60	45	68	25	54	25	26	12	17	5	7	3	2	1	0
51	43	71	56	83	33	68	34	35	17	24	7	11	4	3	1	1

(ii) $s_{n,k} - s_{n-k,k}$

Figure 2.2. *Auxiliary properties of the factorization matrix sequence, $s_{n,k}$, and its relation to the partitions of n into pairwise relatively prime parts,* $\mathrm{rpp}(n)$