Probability Exam Review Sheet

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1 Sets and sigma algebras

- $$\limsup_{n\to\infty} A_n = \\ =$$

- Three defining properties of a σ-algebra:

  (i)

  (ii)

  (iii)

- For $A \subset B$, we have that $\mathbb{P}(B \setminus A) =$,
  since $\mathbb{P}(B) =$.

- Fatou's lemma for probabilities:

2 Independence and characterizations

- Independence characterization: $\mathbb{P}(A \cap B) =$.

- Independence proof: $\mathbb{P}\left(A^C \cap B\right) = \mathbb{P}(B \setminus A) = \mathbb{P}(B) - \mathbb{P}(A \cap B) = ...$

- Kolmogorov's $0-1$ Law:

  (i)

  (ii)

- The Borel-Cantelli lemma:

  (i)

  (ii)

- If $X_1, \ldots, X_n$ are pairwise orthogonal, then
  $\mathrm{Var}(a_1 X_1 + \cdots + a_n X_n) =$.

3 Common distributions and their properties

- If $X \sim \mathrm{Bernoulli}(p)$, then $E[X] =$
  and $\mathrm{Var}[X] =$.

- If $X_i \sim \mathrm{Bernoulli}(p)$, then $E[X_1 + \cdots + X_n] =$
  and $\mathrm{Var}[X_1 + \cdots + X_n] =$.

- If $X \sim \mathrm{Binomial}(n, p)$, then $E[X] =$
  and $\mathrm{Var}[X] =$.

- If $X \sim \text{Poisson}(\lambda)$, then $E[X] =$ and $\text{Var}[X] = E[X^2] - E[X]^2 =$.
- If $X_i \sim N(\mu_i, \sigma_i^2)$, then $Y := \sum_{i=1}^n X_i \sim$.
- If $X_1 \sim N(0,1)$, then $Z_n := X_1 + \cdots + X_n$ has the same distribution as

4 Expectation and uniform integrability

- If $X := \chi_A$, then $E[X] =$.
- Fatou's lemma for expectation:
- DCT for expectation:
- Markov's inequality (and its proof):
- Chebyshev's inequality:
- Jensen's inequality for expectation:
- The generalized Chebyshev inequality:
- $(X_i)_{i \in \mathcal{I}}$ is a *uniformly integrable* sequence if

5 Characteristic functions of a random variable

- The characteristic function of a random variable X is defined as $\phi_X(t) =$ which is also given by the Fourier transformation integral $\phi_X(t) =$
- The characteristic function $\phi_X(t)$ is a ________________________ continuous function of t.
- If $Y := aX + b$ for $a, b \in \mathbb{R}$, then $\phi_Y(t) =$.
- If two random variables have the same characteristic function, then they have the ____.
- If $X \sim N(0,1)$ and $Y \sim N(\mu, \sigma^2)$, then Y has the same distribution as and $\phi_Y(t) =$.
- If the characteristic functions of X_n converge pointwise to that of X, then
- If X, Y are independent, then $\phi_{X+Y}(t) =$.
- If $E[|X|^k] < \infty$, then $\phi_X(t)$ has k continuous derivatives, and for all $0 \le j \le k$, $\phi^{(j)}(0) =$.

6 Modes of convergence

- Definition of convergence everywhere (pointwise):

- Definition of almost sure convergence:

- $X_n \xrightarrow{a.s} X \iff$
  An important corollary of this is that:

- Convergence in probability definition:

- Definition of convergence in the mean (L^p convergence):

- Definition of weak convergence (convergence in distribution):

- Flowchart of implications (convergence hierarchy):

- Proof that almost everywhere convergence implies convergence of a subsequence:

- Hölder's inequality:

- Cauchy-Schwarz inequality:

- Dual (conjugate) exponents:

7 Laws of large numbers and the central limit theorem

- State the WLLN:

- If (X_i) are independent, $E[X_i] = 0$, and $\sup_{i\geq 1} E[X_i^{2k}] < \infty$, then $S_n \xrightarrow{\mathbb{P}, L^{2k}, a.s} 0$. Moreover, we can show that
  $\mathbb{P}\left(|\bar{S}_n| \geq \varepsilon\right) \leq$.

- State the SLLN:

- Real analysis fact: If $\sum_{n\geq 1} \frac{x_n}{n} < \infty$, then

- State Kolmogorov's convergence by variance criterions:

- State the CLT:

8 Large deviations and concentration inequalities

- *Large deviations*: If $X_1, \ldots, X_n$ are iid with mean $\mu = 0$ and $E[e^{\alpha|X_1|}] < \infty$ for some $\alpha > 0$, then
  $\mathbb{P}(|S_n| > \varepsilon) \leq$,
  where the *rate function*
  $I(\varepsilon) :=$.

- *Concentration inequality*: If $X_1, \cdots, X_n$ are iid such that $a \leq X_i \leq b$ for all i, then
  $\mathbb{P}\left(|\bar{S}_n - \mu| > \varepsilon\right) \leq$.

- Important derivation using Cauchy-Schwarz:
  $\left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} a_i\right)^2 \leq$.

9 Generalized normal distributions

- $E[Z^n] =$.

- We write that $X := (X_1, \ldots, X_d) \sim N(\mu, C)$ with mean $\mu \in \mathbb{R}^d$ and (symmetric) covariance matrix C if
  $f_X(x_1, \ldots, x_d) =$.

- $X \sim N(\mu, C) \iff X =$
  for $Z = (Z_1, \ldots, Z_d)$, Z_i iid, and $Z_1 \sim N(0, 1)$.

- If $X \sim N(\mu, C)$, and A is a linear map, then
  $Y := AX \sim$.

- If $X_i \sim N(\mu_i, C_i)$, then
  $\sum_{i=1}^{n} a_i X_i \sim$.

10 Misc

- $E[X] = \lim_{\lambda \to 0}$.