6241 Final Exam

Due: Monday, Dec. 10th, 2018

Select and solve 3 problems from this list

Problem 1. *Show that a random variable X such that*

$$\mathbb{E}[e^{\lambda X}] \le e^{2|\lambda|^3} \text{ for any } \lambda \in [-1,1]$$

satisfies $X = 0$ almost surely.

Problem 2. *Give an example of random variables X_n which converges to 0 almost surely, by such that $\sum_{n\ge1} \mathbb{P}(|X_n| \ge \epsilon) = \infty$ for any $\epsilon > 0$.*

Problem 3.

1. *If the moment generating function of a random variable X, $M_X(\lambda) = \mathbb{E}[e^{\lambda X}]$ is defined for $|\lambda| < \delta$ with some $\delta > 0$, then*

$$\mathbb{P}(|X| \ge x) \le (M_X(c) + M_X(-c))e^{-cx} \text{ for any } x > 0 \text{ and } \delta > c > 0.$$

2. *Show that, if for a random variable X there exist a $c, C > 0$, such that*

$$\mathbb{P}(|X| \ge x) \le Ce^{-cx} \text{ for any } x > 0$$

then $\mathbb{E}[e^{\lambda X}] < \infty$ for $|\lambda| < c$.

3. *If the moment generating function $M_X(\lambda) = \mathbb{E}[e^{\lambda X}]$ exists for $|\lambda| < \delta$, with $\delta > 0$, then for all $\alpha > 0$, $\mathbb{E}[|X|^\alpha] < \infty$. Justify that*

$$M_X(\lambda) = \sum_{k\ge0} \frac{\mathbb{E}[X^k]\lambda^k}{k!}.$$

We also have

$$\frac{d^k}{d\lambda^k} M_X(\lambda)\bigg|_{\lambda=0} = \mathbb{E}[X^k].$$

Problem 4.

1. *Assume that X, Y are two bounded random variables. If for any integer numbers $m, n \ge 0$, we have*

$$\mathbb{E}[X^m Y^n] = \mathbb{E}[X^m]\mathbb{E}[Y^n] \qquad (*)$$

show that X and Y must be independent. Notice that the reverse is simply trivial (namely that for X, Y independent, (*) is satisfied).

2. Show the same for random variables X, Y such that for some constant $c > 0$,

$$\mathbb{P}(|X| \geq x) + \mathbb{P}(|Y| \geq x) \leq e^{-cx} \text{ for all } x > 0. \tag{1}$$

Precisely, show that for any $m, n \geq 0$ integer numbers, $X^m Y^n$ is integrable. In addition, if for any such $m, n \geq 0$

$$\mathbb{E}[X^m Y^n] = \mathbb{E}[X^m]\mathbb{E}[Y^n]$$

then X and Y must be independent.

3. Assume that X and Y are random variables such that their moment generating functions $M_X(\alpha) = \mathbb{E}[e^{\alpha X}]$ and $M_Y(\beta) = \mathbb{E}[e^{\beta Y}]$ are finite for all $|\alpha|, |\beta| < \delta$ for some $\delta > 0$. Prove that if

$$\mathbb{E}[e^{\alpha X + \beta Y}] = \mathbb{E}[e^{\alpha X}]\mathbb{E}[e^{\beta Y}] \text{ for all } |\alpha|, |\beta| < \delta$$

then X and Y must be independent.

Problem 5. *For a sequence $X_1, X_2, \ldots, X_n$ of exponential iid random variables, find a sequence of constants a_n, b_n, c_n, d_n such that*

$$Y_n = a_n \max\{X_1, X_2, \ldots, X_n\} + b_n \text{ and } Z_n = c_n \min\{X_1, X_2, \ldots, X_n\} + d_n$$

converge in distribution and find the limiting distributions. Anything interesting here?

Problem 6. *Assume $X_1, X_2, \ldots, X_n, \ldots$ are iid $N(0,1)$. Show that for any $\lambda > 1/2$,*

$$\frac{1}{n^\lambda} \sum_{i=1}^n X_i \xrightarrow[n\to\infty]{a.s.} 0.$$

Problem 7. *Show that for any sequence $X_1, X_2, \ldots, X_n, \ldots$ of iid random variables,*

$$\frac{1}{n^2} \sum_{i=1}^n iX_i \xrightarrow[n\to\infty]{a.s.} \frac{m}{2} \iff X_1 \text{ is integrable and } \mathbb{E}[X_1] = m.$$

Problem 8. *Show that for a sequence of symmetric iid random variables $X_1, X_2, \ldots, X_n, \ldots$ with finite moment generating function, and $S_n = \frac{2}{n(n+1)} \sum_{i=1}^n iX_i$, then*

$$P(|S_n| \geq x) \leq 2\exp\left(-nI(x)\right) \text{ where } I(x) = \sup\left\{\frac{x\lambda}{2} - \int_0^1 \Lambda(\lambda t)dt : \lambda \geq 0\right\}$$

where $\Lambda(\lambda) = \log(M(\lambda))$ and $M(\lambda) = \mathbb{E}[e^{\lambda X_1}]$.

Note: here by a symmetric random variable X, we mean that X and $-X$ have the same distributions.

Problem 9. *Find the rate function of the large deviation of a sequence of iid random variables with Poisson distribution with parameter $a > 0$.*

Problem 10. *If $X_1, X_2, \ldots$ is a sequence of independent random variables so that $X_n \sim N(\mu_n, \sigma_n^2)$, then the following are equivalent:*

1. $\sum_{n=1}^{\infty} X_n$ *converges almost surely*

2. $\sum_{n=1}^{\infty} \mu_n$ *and* $\sum_{n=1}^{\infty} \sigma_n^2$ *are convergent series.*

Problem 11. *If* X_n *and* Y_n *are independent random variables such that* X_n *converges in distribution to* X *and* Y_n *converges in distribution to* Y*, then for any continuous function* $f : \mathbb{R}^2 \to \mathbb{R}$*,* $f(X_n, Y_n)$ *converges in distribution to* $f(X, Y)$*, where* X *and* Y *are assumed independent.*

Problem 12. *Assume that* $X_n \sim Poisson(n)$ *and* $Y_m \sim Poisson(m)$ *are independent random variables. Show that*

$$\frac{(X_n - n) - (Y_m - m)}{\sqrt{n+m}}$$

converges in distributions to $N(0, 1)$ *when both* $n, m \to \infty$*.*

Problem 13. *Take* $Y_1, Y_2, \dots, Y_n, \dots$ *to be iid random variables with* $P(Y_i = -1) = P(Y_i = 1) = 1/2$ *and construct* $X_n = Y_n Y_{n+1}$*. Find the variance of* $S_n = \sum_{i=1}^{n} X_i$ *and show that*

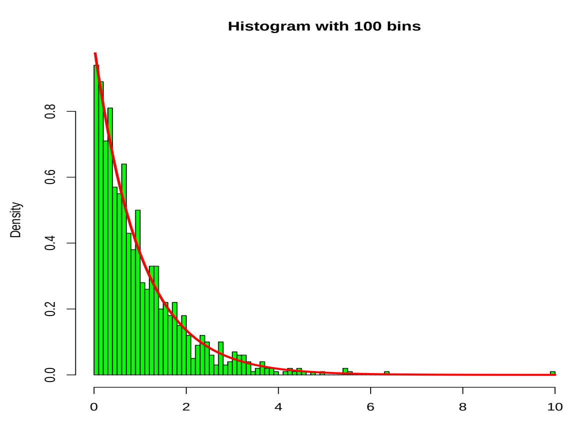
$$\lim_{n\to\infty} \mathbb{E}[\hat{S}_n^k] = \mathbb{E}[Y^k] \ \ \textit{where}\, Y \sim N(0,1)$$

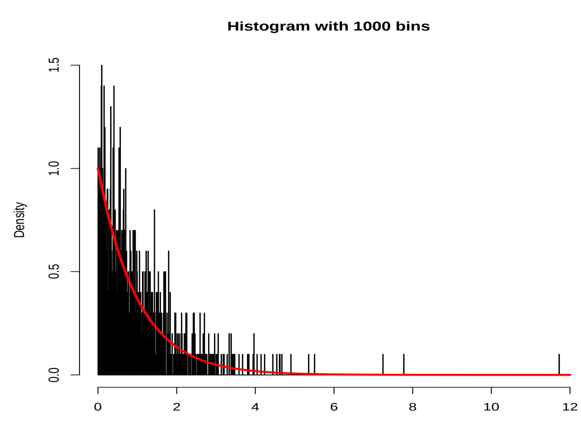
and $\hat{S}_n = \frac{S_n}{\sqrt{Var(S_n)}}$.

Problem 14. *Let* $n \geq 1$ *be a fixed number and* $X_1, X_2, \dots X_n$ *be iid geometric random variables with parameter* p*, i.e.* $P(X_i = k) = pq^{k-1}$*, for* $k \geq 1$*, and* $q = 1 - p$*. Let* $Y_p = X_1 + X_2 + \cdots + X_n$*. Show that when* $p \to 0$*,* pY_p *converges in distribution to a* Γ *distribution and determine which.*

Problem 15. *We generate a matrix of* n *columns and* k *rows such that the entries are all iid uniform on* $[0, 1]$*. Let* $X_j = n \max_{i=1,\dots,n} U_{m,j}$ *for* j *from* 1 *to* k*. Now we fire up R, simulate this and get the picture we saw in class and which is given below. The histogram of the points* X_m *while the red curve is the the density curve of the exponential random variable.*

Explain what mathematical results justify the proximity of the histogram to the density of the exponential curve. Comment on the two pictures generated with $l = 100$ *bins and also* $l = 1000$ *bins.*





The R script is given below

```
# the number of samples
n=1000
# number of simulations
k=1000
# number of bins
l=100

y=c() # this collects the minimum times n
m=c() # m collects the minima
for (j in 1:k){
# generate a row of uniform random variables
  x=runif(n)
# now take the minimum
  w=min(x)
# collects all the minima
  m=c(m,w)
# this multiplies the x by n
  y=c(y,n*w)
}

# this is the histogram in probability form (the total area is 1)
hist(y,breaks=,prob=T,col='green',main=paste("Histogram with", l, "bins"))
# this is simply the exponential density plot
curve(dexp(x), from=0, to=ceiling(max(y)),add=T, col="red")
```