6241 Final Exam Problem 8

Problem 1. *Show that for a sequence of symmetric iid random variables $X_1, X_2, \dots, X_n, \dots$ with finite moment generating function, and $S_n = \frac{2}{n(n+1)} \sum_{i=1}^n iX_i$, then*

$$\mathbb{P}(|S_n| \geq x) \leq 2\exp\left(-(n+1)I\left(\frac{nx}{n+1}\right)\right) \text{ where } I(x) = \sup_{\lambda\in\mathbb{R}} \left\{\frac{x\lambda}{2} - \int_0^1 \Lambda(\lambda t)dt\right\}$$

where $\Lambda(\lambda) = \log(M(\lambda))$ and $M(\lambda) = \mathbb{E}[e^{\lambda X_1}]$.

Note: here by a symmetric random variable X, we mean that X and $-X$ have the same distributions.

Solution. In the first place, due to the symmetry of the variables, we only need to do this for one side, namely, it suffices to show that for $x \geq 0$,

$$\mathbb{P}(S_n \geq x) \leq \exp\left(-nI(x)\right)$$

Using the standard tool for Large Deviations, we start by writing

$$\begin{aligned}
\mathbb{P}(S_n \geq x) =& \mathbb{P}(e^{n\lambda S_n/2} \geq e^{n\lambda x/2}) \leq \frac{\mathbb{E}[e^{n\lambda S_n}]}{e^{n\lambda x}} \\
=& \exp\left(-n\lambda x/2 + \sum_{k=1}^n \ln(\mathbb{E}[e^{k\lambda X_1/(n+1)}])\right) \\
=& \exp\left(-n\left\{\lambda x/2 - \frac{1}{n}\sum_{k=1}^n \Lambda(k\lambda/(n+1))\right\}\right)
\end{aligned}$$

The key is now that

$$\sum_{k=1}^n \Lambda(k\lambda/(n+1)) \leq (n+1)\int_0^1 \Lambda(t\lambda)dt. \tag{*}$$

To this end notice that because X_1 is symmetric, we will show that Λ is non-decreasing as a function of $\lambda \geq 0$. Indeed

$$\Lambda'(\lambda) = \frac{\mathbb{E}[Xe^{\lambda X}]}{\mathbb{E}[e^{\lambda X}]}.$$

Now, from the symmetry of X.

$$\mathbb{E}[Xe^{\lambda X}] = \frac{1}{2}(\mathbb{E}[Xe^{\lambda X}] + \mathbb{E}[(-X)e^{\lambda(-X)}]) = \mathbb{E}[X\sinh(\lambda X)]$$

with $\sinh(x) = (e^x - e^{-x})/2$. Since $\lambda > 0$, it is easy to see that $X \sinh(\lambda X) \geq 0$, thus Λ is non-decreasing on $[0, \infty)$. In addition, $\Lambda(\lambda) \geq 0$ for $\lambda \geq 0$.

Using this we show (*) by using for each time interval $[k/(n+1), (k+1)/(n+1)]$ for $k = 0, 1, 2 \ldots, n$ that

$$\Lambda(k\lambda/n) \leq (n+1) \int_{k/(n+1)}^{(k+1)/(n+1)} \Lambda(t\lambda) dt$$

Summing these we get (*) and in particular,

$$\mathbb{P}(S_n \geq x) \leq \exp\left(-n\left\{\lambda x/2 - \frac{n+1}{n}\int_0^1 \Lambda(\lambda t) dt\right\}\right) = \exp\left(-(n+1)I\left(\frac{nx}{n+1}\right)\right)$$

$\square$