Math 1552

Section 8.8:

Improper Integrals (cont.)

Math 1552 lecture slides adapted from the course materials
By Klara Grodzinsky (GA Tech, *School of Mathematics*, Summer 2021)

Example B.1: Evaluate the following integral: $\int_0^1 \frac{\ln(x)}{\sqrt{x}} dx$

Example B.2: Evaluate the following integral: $\displaystyle\int_0^\infty \frac{e^{-\frac{1}{2x}}}{x^2}\,dx$

Example B.3: Evaluate the following integral: $\displaystyle\int_0^\infty \frac{e^x}{e^{2x}+3}\,dx$

Example B.4: Evaluate the following integral: $$\int_1^{\infty} \frac{dx}{x\sqrt{\ln(x)}}$$

Another example:

Evaluate the following integral:

$$\int_1^e \ln(x)dx$$

(What is the geometric interpretation of this integral?)

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Review of integration techniques

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Review of integration methods (so far)

What techniques have we seen so far to evaluate definite and indefinite integrals?

- Direct integration (know/memorize common formulas)
- Substitution, or u-subs
- Integration by parts, or IBP
- Powers and products of trig functions
- Trigonometric substitutions, or trig subs
- Partial fractions

<u>Other topics we covered:</u>

Riemann sums, FTC, areas between curves, L'Hopital's rule, and improper integrals

Hints and suggestions

- Practice picking out the relevant methods of integration on the review sheet (***bring to class next lecture for Q/A☺***)
- When in doubt, try using a u-sub first to simplify the integral
- You may need to combine multiple techniques we have seen, for example, a u-sub followed by IBP and then a term that involves partial fractions (see ***Example R.3*** in the next slides)
- Review key components of each method to study

Method of substitution (u-subs)

This method is the reverse of the chain rule for derivatives:

Let F be an antiderivative of f. Let $u = g(x)$.

$$\int f(g(x)) \cdot g'(x)dx = \int f(u)du = F(u) + C$$

In other words:

$$\int f(stuff) \cdot (stuff)'dx = F(stuff) + C$$

Substitution with Definite Integrals

To evaluate $\int_a^b f(g(x))g'(x)dx$,

set $u = g(x)$ and *change the limits of integration* to match the new variable:

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Example

Evaluate the following integral: $$\int x^2 \sin(x^3 + 5)dx$$

Which integration method to invoke? (Explain)

Example R.1

Evaluate the following integral: $\int x\sqrt{x+10}dx$

Which integration method to invoke? (Explain)

When to Use Partial Fractions:

Use the method of partial fractions to evaluate the integral of a *rational function* when:

- The degree of the numerator is *less than* that of the denominator.
- The denominator can be *completely factored* into linear and/or irreducible quadratics

Partial Fractions Procedure:

1. If the leading coefficient of the denominator is not a “1”, factor it out.
2. If the degree of the numerator is greater than that of the denominator, carry out long division first.
3. Factor the denominator completely into linear and/or irreducible quadratic terms.

Partial Fractions Procedure:

4. For each linear term of the form $(x-a)^k$, you will have *k* partial fractions of the form:

$$\boxed{\frac{A_1}{x-a}+\frac{A_2}{(x-a)^2}+\frac{A_3}{(x-a)^3}+\ldots+\frac{A_k}{(x-a)^k}}$$

(Note: if *k*=1, there is only one fraction to handle, etc.)

Partial Fractions Procedure:

5. For each irreducible quadratic term of the form $(x^2+bx+c)^m$, you will have m partial fractions of the form:

$$\boxed{\frac{A_1x+B_1}{x^2+bx+c}+\frac{A_2x+B_2}{\left(x^2+bx+c\right)^2}+\frac{A_3x+B_3}{\left(x^2+bx+c\right)^3}+\ldots+\frac{A_mx+B_m}{\left(x^2+bx+c\right)^m}}$$

(Note: if *m*=1, there is only one fraction, etc.)

Partial Fractions Procedure:

6. Solve for all the constants A_i and B_i. To solve:
   - Multiply everything by the common denominator.
   - Combine all like terms, then solve equations for all the A_i and B_i terms; OR plug in values to find equations for A_i and B_i terms.
7. Integrate using all the integration methods we have learned.

Example R.2

Evaluate the following integral: $$\int_{-\frac{\pi}{8}}^{\frac{\pi}{4}} \frac{\cos(2t)}{\sin^2(2t) - 3\sin(2t) + 4} dt$$

Which integration method to invoke? (Explain)

Integration by Parts - Summary

Integration by parts comes from the product rule for differentiation.

$$\boxed{\int u \cdot dv = uv - \int v \cdot du}$$

Differentiate *u* to obtain *du*.

Find *v* by taking an antiderivative of *dv*.

$$(fg)' = f'g + fg' \implies$$
$$f(x)g(x) = \int f'(x)g(x)dx + \int f(x)g'(x)dx$$

Example R.3

Evaluate the following integral: $$\int_0^1 \ln\left(1 + x^{1/4} + x^{1/2}\right) dx$$

Which integration method to invoke? (Explain)

Antiderivatives of powers and products of trig functions

$$(*)\sin^2 x + \cos^2 x = 1$$

$$(*)1 + \tan^2 x = \sec^2 x$$

$$(*)\sin^2 x = \frac{1}{2}\left[1 - \cos(2x)\right]$$

$$(*)\cos^2 x = \frac{1}{2}\left[1 + \cos(2x)\right]$$

$$(*)\sin(2x) = 2\sin x \cos x$$

$$\sin x \cos y = \frac{1}{2}\left[\sin(x-y) + \sin(x+y)\right]$$

$$\sin x \sin y = \frac{1}{2}\left[\cos(x-y) - \cos(x+y)\right]$$

$$\cos x \cos y = \frac{1}{2}\left[\cos(x-y) + \cos(x+y)\right]$$

$$\tan^2 x + 1 = \sec^2 x$$

$$1 + \cot^2 x = \csc^2 x$$

What to expect with powers / products of trig functions:

For integrals of the form

$$\int \cos^n(x)\sin^m(x)dx$$

OR

$$\int \tan^n(x)\sec^m(x)dx$$

we need to apply appropriate trig identities from the last slide to handle respective separate cases of n and m even or odd.

Apply other identities for integrals of the form

$$\int \cos(ax)\cos(bx)dx$$

OR

$$\int \sin(ax)\sin(bx)dx$$

OR

$$\int \cos(ax)\sin(bx)dx$$

Example R.4

Evaluate the following integral: $\int \sin^2(2x)dx$

Which integration method to invoke? (Explain)

Example R.5

Evaluate the following integral: $$\int_{-\frac{1}{3}}^{\frac{1}{6}} \sin^2(\pi x)\cos^5(\pi x)dx$$ *(Sketch solution)*

Which integration method to invoke? (Explain)

Trigonometric Substitutions (trig subs)

We use a trig substitution when no other integration method will work, and when the integral contains one of these types of terms:

$$a^2 - x^2$$

$$x^2 - a^2$$

$$a^2 + x^2$$

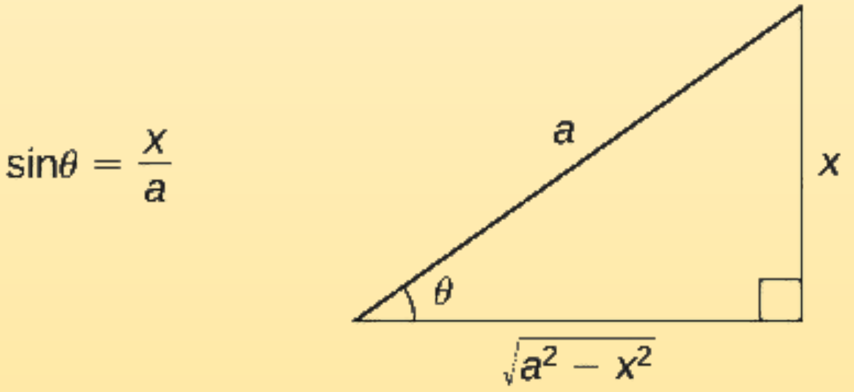
Trig subs - Form 1:

When the integral contains a term of the form

$$a^2 - x^2,$$

use the substitution:

$$\boxed{x = a\sin\theta}$$



Credits for figure: https://math.libretexts.org/Bookshelves/Calculus

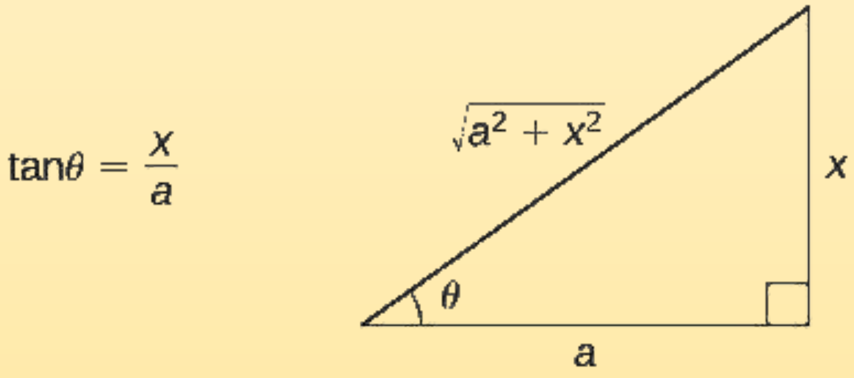
Trig subs - Form 2:

When the integral contains a term of the form

$$a^2 + x^2,$$

use the substitution:

$$\boxed{x = a\tan\theta}$$



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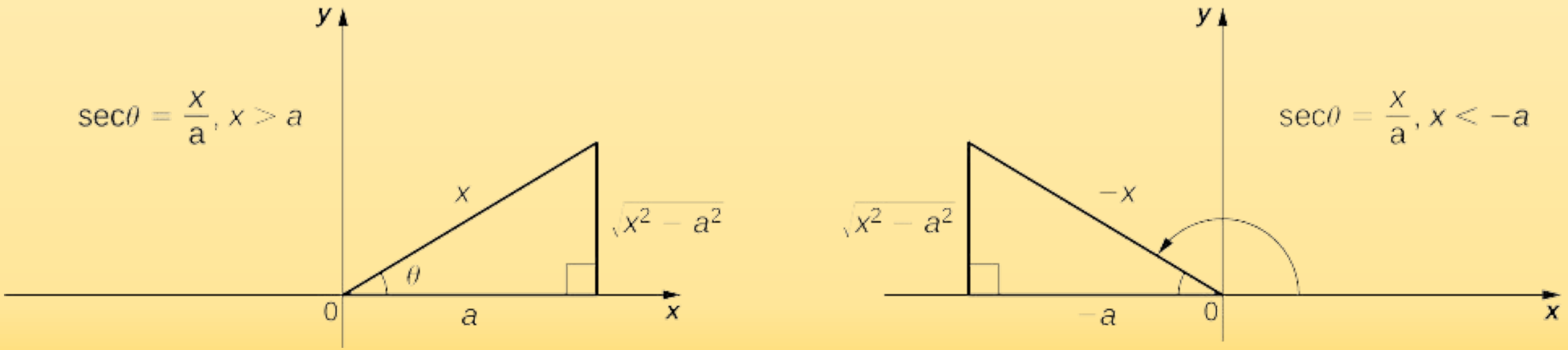
Trig subs - Form 3:

When the integral contains a term of the form

$$x^2 - a^2,$$

use the substitution:

$$\boxed{x = a\sec\theta}$$



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Example R.6

Evaluate the following integral: $\int \sqrt{25+x^2}dx$

Which integration method to invoke? (Explain)

Improper integrals

A definite integral is improper if:

- The function has a vertical asymptote at $x=a$, $x=b$, or at some point c in the interval (a,b).
- One or both of the limits of integration are infinite (positive or negative infinity).

Convergence of an Integral

- If an improper integral evaluates to a **finite number**, we say it ***converges***.

- If the integral evaluates to $\pm\infty$ or to, $\infty - \infty$, we say the integral ***diverges***.

Case 1: At Least One Infinite Limit

Redefine the integral into one of the following.

$$(i)\int_a^{\infty} f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx$$

$$(ii)\int_{-\infty}^{b} f(x)dx = \lim_{a\to-\infty}\int_a^b f(x)dx$$

$$(iii)\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_0^{\infty} f(x)dx$$

and now use parts (i) and (ii).

Case 2: f(c)→∞ Between a and b

- Case 2 occurs when f has a vertical asymptote on the interval [*a*,*b*].
- Redefine the integral into one of the following.

$$(i)\ \text{If } f(a) \text{ DNE, then}: \int_a^b f(x)dx = \lim_{c\to a^+} \int_c^b f(x)dx$$

$$(ii)\ \text{If } f(b) \text{ DNE, then}: \int_a^b f(x)dx = \lim_{c\to b^-} \int_a^c f(x)dx$$

$$(iii)\ \text{If } f(c) \text{ DNE, where } a<c<b, \text{ then}:$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

and now use parts (i) and (ii).

Example R.7

Evaluate the following integral: $$\int_0^{\frac{1}{2}} \left[\pi\left(x - \frac{1}{2}\right)\sec^2(\pi x) + \tan(\pi x)\right] dx$$

Which integration method to invoke? (Explain)

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Review for the Midterm Exam

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Let’s open things up for general questions and specific questions on the review sheet?

(List and enumerate problems)