Jacobi Type Continued Fractions for the Ordinary Generating Functions of Generalized Factorial Functions Article Summary

Maxie D. Schmidt
maxieds@gmail.com

Abstract

The article studies a class of generalized factorial functions and symbolic product sequences through Jacobi type continued fractions (J-fractions) that formally enumerate the divergent ordinary generating functions of these sequences. The more general definitions of these J-fractions extend the known expansions of the continued fractions originally proved by Flajolet that generate the rising factorial function, or Pochhammer symbol, $(x)_n$, at any fixed non–zero indeterminate $x \in \mathbb{C}$. The rational convergents of these generalized J-fractions provide formal power series approximations to the ordinary generating functions that enumerate many specific classes of factorial–related integer product sequences.

The article also provides applications to a number of specific identities, new integer congruence relations satisfied by generalized factorial-related product sequences and the r-order harmonic numbers, restatements of classical congruence properties concerning the primality of integer subsequences, among several other notable motivating examples as immediate applications of the new results. In this sense, the article serves as a semi–comprehensive, detailed survey reference that introduces applications to many established and otherwise well–known combinatorial identities, new cases of generating functions for factorial–function–related product sequences, and other examples of the generalized integer–valued multifactorial, or α–factorial, function sequences. The convergent–based generating function techniques illustrated by the particular examples cited within the article are easily extended to enumerate the factorial-like product sequences arising in the context of many other specific applications.

The article is prepared with a more extensive set of computational data and software routines to be tentatively released as open source software to accompany the examples and numerous other applications suggested as topics for future research and investigation within the article (see the notebook file available online at summary-abbrv-working-2015.10.30-v1.nb). It is highly encouraged, and expected, that the interested reader obtain a copy of the summary notebook reference and computational documentation prepared in this format to assist with computations in a multitude of special case examples cited as particular applications of the new results.

1 Introduction to the Article

The article studies continued Jacobi type J–fraction expansions that formally enumerate the ordinary generating functions (OGFs) of many numerical factorial function sequences of combinatorial interest, and in particular, that enumerate the more general cases of the symbolic polynomial sequences for the *rising factorial function*, or *Pochhammer symbol*, $(x)_n$, and the more general cases of the factorial product functions, $p_n(\alpha, R)$, respectively defined by

$$(x)_n := \begin{cases} x(x+1)(x+2)\cdots(x+n-1), & \text{if } n \geq 1; \\ 1 & \text{if } n = 0. \end{cases}$$

$$p_n(\alpha, R) := \begin{cases} x(x+\alpha)(x+2\alpha)\cdots(x+(n-1)\alpha), & \text{if } n \geq 1; \\ 1 & \text{if } n = 0. \end{cases} \tag{1.1}$$

Notable special cases of the numerical product sequences formed as specialized cases of these two symbolic polynomial sequence forms include the classical cases of the *single factorial function*, $n! = (1)_n = p_n(1, n) = p_n(1, 1)$, and the *double factorial function*, $(2n-1)!! = 2^n \cdot (1/2)_n = p_n(-2, n) = p_n(2, 1)$, as well as the more general *multiple factorial*, or *multifactorial*, functions, $n!_{(\alpha)}$, defined recursively by the following equations for each prescribed setting of the parameterized $\alpha \in \mathbb{Z}^+$:

$$n!_{(\alpha)} = \begin{cases} n \cdot (n-\alpha)!_{(\alpha)}, & \text{if } n > 0; \\ 1, & \text{if } -\alpha < n \leq 0; \\ 0, & \text{otherwise.} \end{cases} \tag{1.2}$$

The continued fraction expansions of note in the article generalize Flajolet's known result [9; 10], which provides the infinite J–fraction expansion enumerating the Pochhammer symbol in the form of

$$R_0(x,z) := \sum_{n\geq 0}(x)_n z^n = \cfrac{1}{1-xz-\cfrac{1\cdot xz^2}{1-(x+2)z-\cfrac{2(x+1)z^2}{1-(x+4)z-\cfrac{3(x+2)z^2}{\cdots}}}}. \tag{1.3}$$

The corresponding generalized forms of these continued fraction expansions that formally enumerate the product sequences, $p_n(\alpha, R)$, from (1.1) have h^{th} convergent functions, denoted by $\mathrm{Conv}_h(\alpha, R; z)$, defined exactly and component wise as

$$\mathrm{Conv}_h(\alpha,R;z) = \cfrac{1}{1-R\cdot z-\cfrac{\alpha R\cdot z^2}{1-(R+2\alpha)\cdot z-\cfrac{2\alpha(R+\alpha)\cdot z^2}{1-(R+4\alpha)\cdot z-\cfrac{3\alpha(R+2\alpha)\cdot z^2}{\cfrac{\cdots}{1-(R+2(h-1)\alpha)\cdot z}}}}}$$

$$\mathrm{Conv}_h(\alpha,R;z) := \frac{\mathrm{FP}_h(\alpha,R;z)}{\mathrm{FQ}_h(\alpha,R;z)} = \sum_{n=0}^{2h-1} p_n(\alpha,R)z^n + \sum_{n=2h}^{\infty} \widetilde{e}_{h,n}(\alpha,R)z^n, \tag{1.4}$$

where the remaining series coefficients, $\widetilde{e}_{h,n}(\alpha, R)$, in the truncated power series approximations to the typically divergent OGFs for these functions denote "error terms," which can then be arbitrarily improved by increasing the parameter input of $h \geq 2$. Moreover, the zeros of the well–known special functions and the classical orthogonal polynomial sequences studied in the references [3; 11] related to these rational convergent functions at each fixed, finite h suggest additional more algebraic approaches to exactly representing the factorial and multiple factorial functions providing the underlying motivation for the more combinatorial and number theoretic applications considered within the article.

2 Examples of the New Results Proved in the Article

2.1 Generating Function Identities for Special Combinatorial Sequences

The primary approach within the article to the special functions and special sequences involves convergent–based generating function identities applied to a wide range of specific notable examples. For example, when $n, \alpha \in \mathbb{Z}^+$ and for $0 \leq d < \alpha$, the multiple factorial functions, $(\alpha n - d)!_{(\alpha)}$ and $n!_{(\alpha)}$, are enumerated as follows:

$$\begin{aligned}
(\alpha n-d)!_{(\alpha)} &= \underbrace{(-\alpha)^n\cdot\left(\frac{d}{\alpha}-n\right)_n}_{p_n(-\alpha,\alpha n-d)} = [z^n]\,\mathrm{Conv}_n(-\alpha,\alpha n-d;z) \\
&= \underbrace{\alpha^n\cdot\left(1-\frac{d}{\alpha}\right)_n}_{p_n(\alpha,\alpha-d)} = [z^n]\,\mathrm{Conv}_n(\alpha,\alpha-d;z) \\
n!_{(\alpha)} &= \left[z^{\lfloor (n+\alpha-1)/\alpha\rfloor}\right]\mathrm{Conv}_n(-\alpha,n;z) \\
&= [z^n]\left(\sum_{0\leq d<\alpha} z^{-d}\cdot\mathrm{Conv}_n(\alpha,\alpha-d;z^\alpha)\right) \\
&= [z^{n+\alpha-1}]\left(\frac{1-z^\alpha}{1-z}\times\mathrm{Conv}_n(-\alpha,n;z^\alpha)\right).
\end{aligned}$$

<u>*Multifactorial Generating Function Identities*</u>

Similarly, specific examples of convergent–based generating function identities that enumerate the *central binomial coefficients* and several other notable special case examples related to the binomial coefficients when $n \geq 1$ include

$$\binom{2n}{n} = \frac{2^{2n}}{n!}\times(1/2)_n = [z^n][x^0]\left(e^{2x}\,\mathrm{Conv}_n\left(2,1;\frac{z}{x}\right)\right)$$

<u>*Central Binomial Coefficient Identities*</u>

$$= \frac{2^n}{n!} \times (2n-1)!! = [z^n][x^1]\left(e^{2x}\operatorname{Conv}_n\left(-2, 2n-1; \frac{z}{x}\right)\right)$$

$$\binom{2n}{n} = \frac{2^{2n}\,(1)_n\left(\frac{1}{2}\right)_n}{(n!)^2} = [x_1^0 x_2^0 z^n]\left(\operatorname{Conv}_n\left(2, 2; \frac{z}{x_2}\right)\operatorname{Conv}_n\left(2, 1; \frac{x_2}{x_1}\right) I_0\left(2\sqrt{x_1}\right)\right)$$

$$= \frac{(2n)!!(2n-1)!!}{(n!)^2} = [x_1^0 x_2^0 z^n]\left(\operatorname{Conv}_n\left(-2, 2n; \frac{z}{x_2}\right)\operatorname{Conv}_n\left(-2, 2n-1; \frac{x_2}{x_1}\right) I_0\left(2\sqrt{x_1}\right)\right)$$

$$\frac{(6n)!}{(3n)!} = \frac{6^{6n}\cancel{(1)_n}\cancel{\left(\frac{2}{6}\right)_n}\cancel{\left(\frac{3}{6}\right)_n} \times \left(\frac{1}{6}\right)_n\left(\frac{3}{6}\right)_n\left(\frac{5}{6}\right)_n}{3^{3n}\cancel{(1)_n}\cancel{\left(\frac{1}{3}\right)_n}\cancel{\left(\frac{2}{3}\right)_n}}$$ *Binomial Coefficient Related Identities*

$$= 24^n \times 6^n\,(1/6)_n \times 2^n\,(1/2)_n \times 6^n\,(5/6)_n$$

$$= [x_2^0 x_1^0 z^n]\left(\operatorname{Conv}_n\left(6, 5; \frac{24z}{x_2}\right)\operatorname{Conv}_n\left(2, 1; \frac{x_2}{x_1}\right)\operatorname{Conv}_n\left(6, 1; x_1\right)\right)$$

$$= 8^n \times (6n-5)!_{(6)}\,(6n-3)!_{(6)}\,(6n-1)!_{(6)}$$

$$= [x_2^0 x_1^0 z^n]\left(\operatorname{Conv}_n\left(-6, 6n-5; \frac{8z}{x_2}\right)\operatorname{Conv}_n\left(-6, 6n-3; \frac{x_2}{x_1}\right)\operatorname{Conv}_n\left(-6, 6n-1; x_1\right)\right).$$

Similar identities expanded in the article are combined to form generating function identities for other special factorial function sequences, including sums of the squares and cubes of the double and triple factorial functions, among many other particular examples of combinatorial interest.

2.2 Techniques for Performing Approximate Formal Laplace–Borel Transforms

An example of another new convergent–based generating function technique is employed in many places within the article to perform the function of the approximate formal Laplace–Borel transform, which is typically defined as an integral transformation on the complete, non–approximate form of a sequence generating function. The transformation effectively transforms a known exponential generating function (EGF) of an arbitrary sequence into its corresponding OGF representation. The next examples provide several representative concrete applications of this approximate generating function transformation technique using the particularly "nice" properties of the rational convergents, $\operatorname{Conv}_h(\alpha, R; z)$, that enumerate the single factorial function, $n!$, at any $n \geq 1$.

For example, we can generate the double factorial function in the following form by applying this new convergent–function–based technique:

$$(2n-1)!! = \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!} \cdot k \cdot (2k-3)!!$$ *Combinatorial Sums*

$$= (n-1)! \times [x_2^n][x_1^0]\left(\frac{x_2}{(1-x_2)} \times \operatorname{Conv}_n\left(2, 1; \frac{x_2}{x_1}\right) \times (x_1+1)e^{x_1}\right)$$

$$= [x_1^0 x_2^0 x_3^{n-1}]\left(\operatorname{Conv}_n\left(1, 1; \frac{x_3}{x_2}\right)\operatorname{Conv}_n\left(2, 1; \frac{x_2}{x_1}\right) \times \frac{(x_1+1)}{(1-x_2)} \cdot e^{x_1}\right).$$

Related challenges are posed in the statements of several other finite sum identities involving the double factorial function cited in the references [4; 12]. Convergent–based generating functions for the *subfactorial function*, denotes by $!n$, or the alternately the *number of derangements of* n, denoted $n¡$, include the next identities expanded as

$$!n = n! \times \sum_{i=0}^{n} \frac{(-1)^i}{i!} = [x^0 z^n]\left(\frac{e^{-x}}{(1-x)} \times \operatorname{Conv}_n\left(1, 1; \frac{z}{x}\right)\right)$$ *Subfactorial OGF Identities*

$$= \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}k! = [z^n x^n]\left(\frac{(x+z)^n}{(1+z)} \times \operatorname{Conv}_n\left(1, 1; x\right)\right).$$

Similarly, the next sums provide another summation–based recursive formula for the subfactorial function derived from the known exponential generating function, $\widehat{D}_{n¡}(z) = e^{-z} \cdot (1-z)^{-1}$, for this sequence [13, §5.4] [6, §4.2].

$$!n = n! - \sum_{i=1}^{n}\binom{n}{i}!(n-i)$$ *Basic Subfactorial Recurrence*

$$= n! \times \left(1 - \sum_{i=1}^{n} \frac{1}{i!} \cdot \frac{!(n-i)}{(n-i)!}\right)$$

$$= n! \times \left(1 - [x_1^0 x_2^0 x_3^n]\left((e^{x_3}-1)\operatorname{Conv}_n\left(1,1;\frac{x_3}{x_2x_1}\right)\frac{e^{x_2-x_1}}{(1-x_1)}\right)\right)$$
$$= [x_x^0 x_2^0 x_3^0 z^n]\left(\operatorname{Conv}_n\left(1,1;\frac{z}{x_3}\right)\left(\frac{1}{(1-x_3)} - \operatorname{Conv}_n\left(1,1;\frac{x_3}{x_2x_1}\right)\frac{e^{x_2-x_1}\cdot(e^{x_3}-1)}{(1-x_1)}\right)\right).$$

If we further let the modified generating function, $\widetilde{B}_{a,b,u}(w,x)$, be defined as

$$\widetilde{B}_{a,b,u}(w,x) := \sum_{n\geq 0}\left(\sum_{m=0}^{n}((am+b)e^{(am+b)x} - e^x)u^m\right)w^n$$
$$= \frac{be^{bx} + (a-b)e^{(a+b)x}uw}{(1-w)(uwe^{ax}-1)^2} - \frac{e^x - 2e^{(a+1)x}uw + e^{(2a+1)x}u^2w^2}{(1-w)(1-uw)(uwe^{ax}-1)^2},$$

we also readily check that the following generating functions enumerate the Bernoulli–polynomial–related generalized sums of powers sequences given by

$$S_p(a,b;u,n) := \sum_{0\leq m<n}(am+b)^p u^m \qquad \underline{\textit{Generalized Sums of Powers Sequences}}$$
$$= \frac{u^n-1}{u-1} + [w^{n-1}][z^{p-1}x^0]\left(\operatorname{Conv}_p\left(-1,p-1;\frac{z}{x}\right)\widetilde{B}_{a,b,u}(w,x)\right)$$
$$= \frac{u^n-1}{u-1} + [w^{n-1}][x^0z^{p-1}]\left(\operatorname{Conv}_p\left(1,1;\frac{z}{x}\right)\widetilde{B}_{a,b,u}(w,x)\right).$$

The subsections of the article derive several other related identities that enumerate the sequences of binomials of the form, 2^p-1, and more generally m^p-1 for any fixed $m>0$.

2.3 Convergent–Based Generating Function Identities for Superfactorial Sequences and the Barnes G–Function

The *Barnes G–function*, $G(z)$, which satisfies the characteristic "*double–gamma–like*" functional equation given by

$$G(z+1) = \Gamma(z)G(z) + \delta_{z,1}, \qquad \underline{\textit{Barnes G–Function}}$$

for $z\geq 1$, is related to the so–termed "ordinary" superfactorial function product $S_1(n) := 1!\times 2!\times\cdots n! = G(n+2)$. Generalized definitions for the superfactorial functions, $S_{\alpha,d}(n)$, when $n\geq 1$, $\alpha\in\mathbb{Z}^+$, and for any fixed $0\leq d<\alpha$ are generated as the diagonal coefficients of the convergent–based OGF approximations in the following forms:

$$S_{\alpha,d}(n) := \prod_{j=1}^{n}(\alpha j - d)!_{(\alpha)},\ n\geq 1, \alpha\in\mathbb{Z}^+, 0\leq d<\alpha \tag{2.1}$$
$$S_{\alpha,d}(n) = [x_1^{-1}x_2^{-1}\cdots x_{n-1}^{-1}x_n^n]\left(\prod_{i=0}^{n-2}\operatorname{Conv}_n\left(-\alpha,\alpha(n-i)-d;\frac{x_{n-i}}{x_{n-i-1}}\right)\times\operatorname{Conv}_n(-\alpha,\alpha-d;x_1)\right)$$
$$= [x_1^{-1}x_2^{-1}\cdots x_{n-1}^{-1}x_n^n]\left(\prod_{i=0}^{n-2}\operatorname{Conv}_n\left(\alpha,\alpha-d;\frac{x_{n-i}}{x_{n-i-1}}\right)\times\operatorname{Conv}_n(\alpha,\alpha-d;x_1)\right).$$

Additionally, the Barnes G–function and its generalizations have deep connections in other areas of mathematics discussed in the references [1; 7]. For rational $z>2$, the ordinary cases of these functions are related to the generalized superfactorial function products defined and enumerated as in (2.1) above. Particular special case examples simplified by *Mathematica* that are cited within the article include the products

$$S_{2,1}(n) := \prod_{j=1}^{n}(2j-1)!! = \frac{A^{3/2}}{2^{1/24}e^{1/8}\pi^{1/4}}\cdot\frac{2^{n(n+1)/2}}{\pi^{n/2}}\times G\left(n+\frac{3}{2}\right)$$
$$S_{3,1}(n) := \prod_{j=1}^{n}(3j-1)!!! = 3^{n(n-1)/2}\left(\frac{2\cdot G\left(\frac{5}{3}\right)}{G\left(\frac{8}{3}\right)}\right)^n\times\frac{G\left(n+\frac{5}{3}\right)}{G\left(\frac{5}{3}\right)} \qquad \underline{\textit{Special Case Products}}$$
$$S_{4,1}(n) := \prod_{j=1}^{n}(4j-1)!!!! = 4^{n(n-1)/2}\left(\frac{3\cdot G\left(\frac{7}{4}\right)}{G\left(\frac{11}{4}\right)}\right)^n\times\frac{G\left(n+\frac{7}{4}\right)}{G\left(\frac{7}{4}\right)},$$

where $A \approx 1.2824271$ denotes *Glaisher's constant* [19, §5.17], and where the particular constant multiples in the previous equation correspond to the special case values, $\Gamma(1/2) = \sqrt{\pi}$ and $G(3/2) = A^{-3/2} 2^{1/24} e^{1/8} \pi^{1/4}$ [1].

2.4 New Congruences for the Stirling Numbers, r–Order Harmonic Number Sequences, and Multiple Factorial Functions

A few representative examples of the new congruences for the double and triple factorial functions obtained from the statements of the propositions in the article include the following particular expansions for integers $p_1, p_2 \geq 2$, and where $0 \leq s \leq p_1$ and $0 \leq t \leq p_2$ assume some prescribed values over the non–negative integers (see the computations contained in the reference [23]):

$$\begin{aligned}
(2n-1)!! &\equiv \sum_{i=0}^{n} \binom{p_1}{i} 2^n (-2)^{(s+1)i} \left(1/2 - p_1\right)_i \left(1/2\right)_{n-i} && \pmod{2^s p_1} \\
&\equiv \sum_{i=0}^{n} \binom{p_1}{i} (-2)^n 2^{(s+1)i} \left(1/2 + n - p_1\right)_i \left(1/2 - n + i\right)_{n-i} && \pmod{2^s p_1} \\
(3n-1)!!! &\equiv \sum_{i=0}^{n} \binom{p_2}{i} 3^n (-3)^{(t+1)i} \left(1/3 - p_2\right)_i \left(2/3\right)_{n-i} && \pmod{3^t p_2} \\
&\equiv \sum_{i=0}^{n} \binom{p_2}{i} (-3)^n 3^{(t+1)i} \left(1/3 + n - p_2\right)_i \left(1/3 - n + i\right)_{n-i} && \pmod{3^t p_2}.
\end{aligned}$$

For comparison with known results for the Stirling numbers of the first kind modulo 2 expanded as in the result from the reference provided below, several particular cases of new congruences for the Stirling numbers of the first kind, $\left[{n \atop m}\right] := [z^n][R^m] \operatorname{Conv}_h(1, R; z)$ for $1 \leq m \leq n \leq 2h$, derived from the approximate convergent functions given as examples modulo 2 in the article include

$$\begin{aligned}
\begin{bmatrix} n \\ 1 \end{bmatrix} &\equiv \frac{2^n}{4} [n \geq 2]_\delta + [n = 1]_\delta && \pmod{2} \\
\begin{bmatrix} n \\ 2 \end{bmatrix} &\equiv \frac{3 \cdot 2^n}{16} (n-1) [n \geq 3]_\delta + [n = 2]_\delta && \pmod{2} \\
\begin{bmatrix} n \\ 3 \end{bmatrix} &\equiv 2^{n-7} (9n - 20)(n-1) [n \geq 4]_\delta + [n = 3]_\delta && \pmod{2} \\
\begin{bmatrix} n \\ 4 \end{bmatrix} &\equiv 2^{n-9} (3n - 10)(3n-7)(n-1) [n \geq 5]_\delta + [n = 4]_\delta && \pmod{2} \\
\begin{bmatrix} n \\ 5 \end{bmatrix} &\equiv 2^{n-13} (27n^3 - 279n^2 + 934n - 1008)(n-1) [n \geq 6]_\delta + [n = 5]_\delta && \pmod{2} \\
\begin{bmatrix} n \\ 6 \end{bmatrix} &\equiv \frac{2^{n-15}}{5} (9n^2 - 71n + 120)(3n-14)(3n-11)(n-1) [n \geq 7]_\delta + [n = 6]_\delta && \pmod{2},
\end{aligned}$$

where

$$\begin{bmatrix} n \\ m \end{bmatrix} \equiv \binom{\lfloor n/2 \rfloor}{m - \lceil n/2 \rceil} = [x^m] \left(x^{\lceil n/2 \rceil} (x+1)^{\lfloor n/2 \rfloor} \right) \qquad \pmod{2},$$

for all $n \geq m \geq 1$ [28, §4.6]. For comparison, the termwise expansions of the row generating functions, $(x)_n$, for the Stirling number triangle considered modulo 3 with respect to the non–zero indeterminate x similarly imply the next properties of these coefficients for any $n \geq m > 0$.

$$\begin{aligned}
\begin{bmatrix} n \\ m \end{bmatrix} &\equiv [x^m] \left(x^{\lceil n/3 \rceil} (x+1)^{\lceil (n-1)/3 \rceil} (x+2)^{\lfloor n/3 \rfloor} \right) && \pmod{3} \\
&\equiv \sum_{k=0}^{m} \binom{\lceil (n-1)/3 \rceil}{k} \binom{\lfloor n/3 \rfloor}{m - k - \lceil n/3 \rceil} \times 2^{\lceil n/3 \rceil + \lfloor n/3 \rfloor - (m-k)} && \pmod{3}
\end{aligned}$$

The next few particular examples of the special case congruences satisfied by the Stirling numbers of the first kind modulo 3 are also similarly obtained from the generalized convergent functions studied in the article expanded in the following forms:

$$\begin{bmatrix} n \\ 1 \end{bmatrix} \equiv \sum_{j = \pm 1} \tfrac{1}{36} \left(9 - 5j\sqrt{3}\right) \times \left(3 + j\sqrt{3}\right)^n [n \geq 2]_\delta + [n = 1]_\delta \qquad \pmod{3}$$

$$\begin{bmatrix} n \\ 2 \end{bmatrix} \equiv \sum_{j=\pm 1} \tfrac{1}{216}\left((44n-41)-(25n-24)\cdot j\sqrt{3}\right)\times\left(3+j\sqrt{3}\right)^n [n\geq 3]_\delta + [n=2]_\delta \pmod{3}$$

$$\begin{aligned}\begin{bmatrix} n \\ 3 \end{bmatrix} &\equiv \sum_{j=\pm 1} \tfrac{1}{15552}\left((1299n^2-3837n+2412)-(745n^2-2217n+1418)\cdot j\sqrt{3}\right)\times \\ &\quad \times\left(3+j\sqrt{3}\right)^n [n\geq 4]_\delta + [n=3]_\delta \pmod{3}\end{aligned}$$

$$\begin{aligned}\begin{bmatrix} n \\ 4 \end{bmatrix} &\equiv \sum_{j=\pm 1} \tfrac{1}{179936}\left((6409n^3-383778n^2+70901n-37092)\right. \\ &\quad \left. -(3690n^3-22374n^2+41088n-21708)\cdot j\sqrt{3}\right)\times \\ &\quad \times\left(3+j\sqrt{3}\right)^n [n\geq 5]_\delta + [n=4]_\delta\,. \pmod{3}\end{aligned}$$

Additional congruences for the Stirling numbers of the first kind modulo 4 and modulo 5 are straightforward to expand by related formulas with exact algebraic expressions for the roots of the third–degree and fourth–degree equations.

Similarly, since we can solve for any sequence of r–order harmonic numbers of the form $(n!)^r \times H_n^{(r)}$ for integers $r \geq 1$, the same rational generating function techniques employed in obtaining the previous results lead to new congruences for these sequences modulo any fixed integer $p > 1$. The congruences for the classical Stirling number cases and corresponding r–order harmonic number sequences expanded within the article lead to generalizations of these results briefly suggested in the working summary notebook file prepared to accompany the article [23]. Additionally, given the more classical and modern number theoretic interest in congruences involving these harmonic number sequences, for example in studying known identities related to the *Wolstenholme prime* sequence, a more purely algebraic approach to the predictable forms of the rational convergent functions involved in generating these sequences is suggested as a topic for future study.

3 Other Applications and Future Research Topics

One of the primary open topics touched on at the end of the paper is the applications to the classically–phrased congruence relations derived from Wilson's theorem which provide necessary and sufficient conditions on the primality of pairs, tuples, and several other special subsequences cited as noteworthy examples in the last section of the article. For example, the sequence of *Wilson primes*, or the subsequence of odd integers $p \geq 5$ satisfying $n^2 \mid (n-1)!+1$, is characterized through each of the following additional divisibility requirements placed on the expansions of the single factorial function implicit to Wilson's theorem:

$$\underbrace{\sum_{i=0}^{n-1}\binom{n^2}{i}^2(-1)^i i!(n-1-i)!}_{\equiv\ (n-1)!\ \pmod{n^2}} \equiv -1 \pmod{n^2}$$

$$\underbrace{\sum_{i=0}^{n-1}\binom{n^2}{i}(n^2-n)^{\underline{i}}\times(-1)^{n-1-i}(n-1)^{\underline{n-1-i}}}_{\equiv (n-1)!\ \pmod{n^2}} \equiv -1 \pmod{n^2}$$

$$\sum_{s=0}^{n-1}\sum_{i=0}^{s}\sum_{v=0}^{i}\left(\sum_{k=0}^{n-1}\sum_{m=0}^{k}\tbinom{n^2}{k}\tbinom{m}{s}\tbinom{i}{v}\tbinom{n^2+v}{v}\left[{k \atop m}\right]\left\{{s \atop i}\right\}(-1)^{i-v}(n-1)^{\underline{n-1-k}}(-n)^{m-s}i!\right) \equiv -1. \pmod{n^2}$$

If we further seek to determine new properties of the odd primes of the form $p := n^2+1 \geq 5$, obtained from adaptations of the new forms given by these sums, the second consequence of Wilson's theorem provided in (**??**) above leads to an analogs requirement expanded in the forms of the next equations [20, §3.4(D)].

$$\begin{aligned} n^2+1 \text{ prime} &\iff \left(\textstyle\sum_{i=0}^{n^2/2}\binom{n^2+1}{i}\left(-(n^2+1)\right)_i\left(\frac{1}{2}(n^2-2i)\right)!\right)^2 \equiv (-1)^{n^2/2+1} \pmod{n^2+1} \\ &\iff \left(\textstyle\sum_{i=0}^{n^2/2}\binom{n^2+1}{i}\binom{i-n^2-2}{i}i!\left(\frac{1}{2}(n^2-2i)\right)!\right)^2 \equiv (-1)^{n^2/2+1} \pmod{n^2+1}\end{aligned}$$

Lastly, several results characterizing the *twin primes*, *cousin primes*, and *sexy prime* pairs are stated through analogous summation–based identities related to generalized harmonic number sequences. The next congruences characterizing the twin primes are representative of the results cited as particular applications in the article.

$$
\begin{aligned}
& 2n+1, 2n+3 \text{ odd primes} && && (3.1) \\
& \iff 2\left(\sum_{i=0}^{n} \binom{(2n+1)(2n+3)}{i}^2 (-1)^i i!(n-i)!\right)^2 + (-1)^n(10n+7) && \equiv 0 \pmod{(2n+1)(2n+3)} \\
& \iff 4\left(\sum_{i=0}^{2n} \binom{(2n+1)(2n+3)}{i}^2 (-1)^i i!(2n-i)!\right) + 2n+5 && \equiv 0 \pmod{(2n+1)(2n+3)}.
\end{aligned}
$$

The multiple sum expansions of the single factorial functions in the congruences given in the previous two examples also yield similar restatements of the pair of congruences in (3.1) providing that for some $n \geq 1$, the odd integers, $(p_1, p_2) := (2n+1, 2n+3)$, are both prime whenever either of the following divisibility conditions hold:

$$
\begin{aligned}
& 2 \times \underbrace{\left(\sum_{\substack{0 \leq i \leq s \leq n \\ 0 \leq m \leq k \leq n}} \binom{(2n+1)(2n+3)}{i}\binom{(2n+1)(2n+3)}{k}\binom{m}{s}\begin{bmatrix} k \\ m \end{bmatrix}\begin{Bmatrix} s \\ i \end{Bmatrix}(-1)^{s+k} i! \times n^{\underline{n-k}}(n+1)^{m-s}\right)^2}_{C_{(2n+1)(2n+3),n}(-1,n)} \qquad \underline{\textit{Twin Prime Pairs}} \\
& \quad + (-1)^n(10n+7) \equiv 0 \pmod{(2n+1)(2n+3)} \\
& 4 \times \underbrace{\left(\sum_{\substack{0 \leq v \leq i \leq s \leq 2n \\ 0 \leq m \leq k \leq 2n}} \binom{(2n+1)(2n+3)}{k}\binom{(2n+1)(2n+3)+v}{v}\binom{i}{v}\binom{m}{s}\begin{bmatrix} k \\ m \end{bmatrix}\begin{Bmatrix} s \\ i \end{Bmatrix}(-1)^{s-i+v+k} i! \times (2n)^{\underline{2n-k}}(2n+1)^{m-s}\right)}_{C_{(2n+1)(2n+3),2n}(-1,2n)} \\
& \quad + (2n+5) \equiv 0 \pmod{(2n+1)(2n+3)}.
\end{aligned}
$$

The treatment of the modular congruence identities involved in these few notable example cases in the last several examples and in the remarks above, is by no means exhaustive, but serves to demonstrate the utility of this approach in formulating several new forms of non–trivial prime number results with many notable applications. The results provided by symbolic summation software routines such as those found in *Mathematica*'s `Sigma` package suggest new forms of known Wolstenholme–prime–like congruences and the sums analogous in form to Apéry–like congruences suggested by the previous few particular special case examples. The expansions of these congruence forms in terms of generalized harmonic number sequences remains an open topic worthy of further study from multiple angles.

References

[1] V. S. Adamchik, Contributions to the theory of the Barnes function, 2003, http://arxiv.org/abs/math/0308086v1.

[2] C. Aebi and G. Cairns, Generalizations of Wilson's theorem for double–, hyper–, sub–, and superfactorials, *The American Mathematical Monthly* **122**(5) (2015), 433–443.

[3] S. Ahmed, Properties of the zeros of confluent hypergeometric functions, *Journal of Approximation Theory* (34) (1980), 335–347.

[4] D. Callan, A combinatorial survey of combinatorial identities for the double factorial, 2009, http://arxiv.org/abs/0906.1317.

[5] P. A. Clement, Congruences for sets of primes, *Amer. Math. Monthly* **56**(1) (1949), 23–25.

[6] L. Comtet, *Advanced Combinatorics: The Art of Finite and Infinite Expressions*, D. Reidel Publishing Company, 1974.

[7] R. Diaz and E. Pariguan, On hypergeometric functions and k–Pochhammer symbol, 2005, http://arxiv.org/abs/math/0405596v2.

[8] P. Flajolet and R. Sedgewick, *Analytic Combinatorics*, Cambridge University Press, 2009 (Third printing 2010).

[9] P. Flajolet, Combinatorial aspects of continued fractions, *Discrete Mathematics* **32** (1980), 125–161.

[10] P. Flajolet, On congruences and continued fractions for some classical combinatorial quantities, *Discrete Mathematics* **41**(2) (1982), 145–153.

[11] W. Gautschi and C. Giordano, Luigi Gatteschi's work on asymptotics of special functions and their zeros, *Numer Algor* (2008).

[12] H. Gould and J. Quaintance, Double fun with double factorials, *Mathematics Magazine* **85**(3) (2012), 177–192.

[13] R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics: A Foundation for Computer Science*, Addison-Wesley, 1994.

[14] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Oxford University Press, 2008 (Sixth Edition).

[15] D. E. Knuth, Convolution polynomials, *The Mathematica J.* **2**(4) (1992), 67–78.

[16] D. E. Knuth, *The Art of Computer Programming: Fundamental Algorithms*, Vol. 1, Addison-Wesley, 1997.

[17] S. K. Lando, *Lectures on Generating Functions*, American Mathematical Society, 2002.

[18] C. Lin and L. Zhipeng, On Wilson's theorem and Polignac conjecture, *Math. Medley* **6**(1) (2005).

[19] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, editors, *NIST Handbook of Mathematical Functions*, Cambridge University Press, 2010.

[20] P. Ribenboim, *The New Book of Prime Number Records*, Springer, 1996.

[21] S. Roman, *The Umbral Calculus*, Dover, 1984.

[22] M. D. Schmidt, Generalized j–factorial functions, polynomials, and applications, *J. Integer Seq.* **13**(10.6.7) (2010).

[23] M. D. Schmidt, *Mathematica summary notebook, supplementary reference, and computational documentation*, 2015, Source code and detailed computational documentation to appear online. Temporary access to the summary notebook file for review of the article is available online at the *Google Drive* link: `summary-abbrv-working-2015.10.30-v1.nb`.

[24] C. Schneider, Symbolic summation assists combinatorics, *Sem. Lothar. Combin.* **56** (2007), 1–36.

[25] N. J. A. Sloane, The Online Encyclopedia of Integer Sequences, 2010, `http://oeis.org`.

[26] J. Spanier and K. B. Oldham, *An Atlas of Functions*, Taylor & Francis / Hemisphere, Bristol, PA, USA, 1987.

[27] R. P. Stanley, *Enumerative Combinatorics*, Vol. 2, Cambridge, 1999.

[28] H. S. Wilf, *Generatingfunctionology*, Academic Press, 1994.

[29] Wolfram Functions Site, *Introduction to the factorials and binomials*, 2015, `http://funtions.wolfram.com/GammaBetaErf/Pochhammer/introductions/FactorialBinomials/ShowAll.html`.