# Math 1552

## *Sections 6.1 and 6.2: Volumes of Revolution*

Math 1552 lecture slides adapted from the course materials
By Klara Grodzinsky (GA Tech, *School of Mathematics*, Summer 2021)

# *Learning Goals*

- Set up and evaluate integrals using the disk method
- Set up and evaluate integrals using the method of cylindrical shells
- Apply the “washer” method to either method above
- Adjust the standard formulas to rotate a region around any horizontal or vertical line

# *Volumes by the Disk Method*

We can find the volume of the solid generated by revolving the region bounded by $y=f(x)$, $x=a$, $x=b$, and the $x$-axis using the basic formulas:

$$V = \pi \int_a^b [f(x)]^2\,dx \quad (\text{revolved about } x\text{-axis})$$

$$V = \pi \int_a^b [g(y)]^2\,dy \quad (\text{revolved about } y\text{-axis})$$

# *Example 1:*

Find the volume of the solid generated by revolving the region bounded by $y=5x$, $x=0$, and $y=5$ about the $y$-axis.

# *Important Notes about Disks:*

- The variable of integration *always* matches the axis of revolution.
- If you revolve about a line other than the $x$- or $y$-axis, you will need to adjust the formula to find the new radius.
- If you revolve a region bounded by two curves, you will need to apply the *washer method*.

# *The Washer Method*

When we revolve a region bounded between two curves, we have an inner and outer radius, and the volume equation is modified to:

$$V = \pi\int_a^b \left[(f(x))^2 - (g(x))^2\right]dx = \pi\int_a^b \left[(top)^2 - (bottom)^2\right]dx$$

$OR$

$$V = \pi\int_a^b \left[(f(y))^2 - (g(y))^2\right]dy = \pi\int_a^b \left[(right)^2 - (left)^2\right]dy$$

# *Example 2:*

Find the volume of the solid generated by revolving the region bounded by

$$y = \sqrt{1 - x^2} \text{ and } x + y = 1$$

about the $x$-axis.

# *Example 3:*

Find the volume of the solid generated by revolving the region bounded by:

$y = \sqrt{x+1}, x = 3, \text{and the } x\text{-axis}$

$\text{about the line } y = -1.$

*<u>Example</u>: Set up the integral to find the volume bounded by*

$y = x + 2$ and $y = x^2, x \geq 0,$

about the $x$-axis.

(A) $V = \pi \int_{-1}^{2} \left[ (x+2)^2 - (x^2)^2 \right] dx$

(B) $V = \pi \int_{0}^{2} \left[ (x+2)^2 - (x^2)^2 \right] dx$

(C) $V = \pi \int_{-1}^{2} \left[ (x^2)^2 - (x+2)^2 \right] dx$

(D) $V = \pi \int_{0}^{2} \left[ (x^2)^2 - (x+2)^2 \right] dx$

# *Volumes by Cylindrical Shells*

We can find the volume of the solid generated by revolving the region bounded by $y=f(x)$, $x=a$, $x=b$, and the $x$-axis using the basic formulas:

$$V = 2\pi \int_a^b x[f(x)]dx \ \ (\text{revolved about } y\text{-axis})$$

$$V = 2\pi \int_a^b y[g(y)]dy \ \ (\text{revolved about } x\text{-axis})$$

# *Notes about the Shell Method:*

- In the shell method, the variable of integration is the *opposite* of the axis of revolution.
- To use the washer method with shells:

$$V = 2\pi\int_a^b x[f(x) - g(x)]dx = 2\pi\int_a^b x[top - bottom]dx$$

$$OR$$

$$V = 2\pi\int_a^b y[f(y) - g(y)]dy = 2\pi\int_a^b y[right - left]dy$$

# *Example 4:*

Find the volume of the solid generated by revolving the region bounded by the curves:

$y = \sin x$, the $x$-axis, and the lines

$x = 0,\ x = \dfrac{\pi}{2}$ about the $y$-axis.

*<u>Example</u>: Set up the integral to find the volume bounded by*

$y = x + 2$ and $y = x^2$

about the line $x = 2$.

(A) $V = 2\pi\int_{1}^{4} (y+2)\left[(y-2) - \sqrt{y}\right]dy$

(B) $V = 2\pi\int_{-1}^{2} (x-2)\left[(x+2) - x^2\right]dx$

(C) $V = 2\pi\int_{1}^{4} (2-y)\left[(y-2) - \sqrt{y}\right]dy$

(D) $V = 2\pi\int_{-1}^{2} (2-x)\left[(x+2) - x^2\right]dx$

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## *Extra Hints and Problem Solutions on Volumes of Revolution*

# *Example A:*

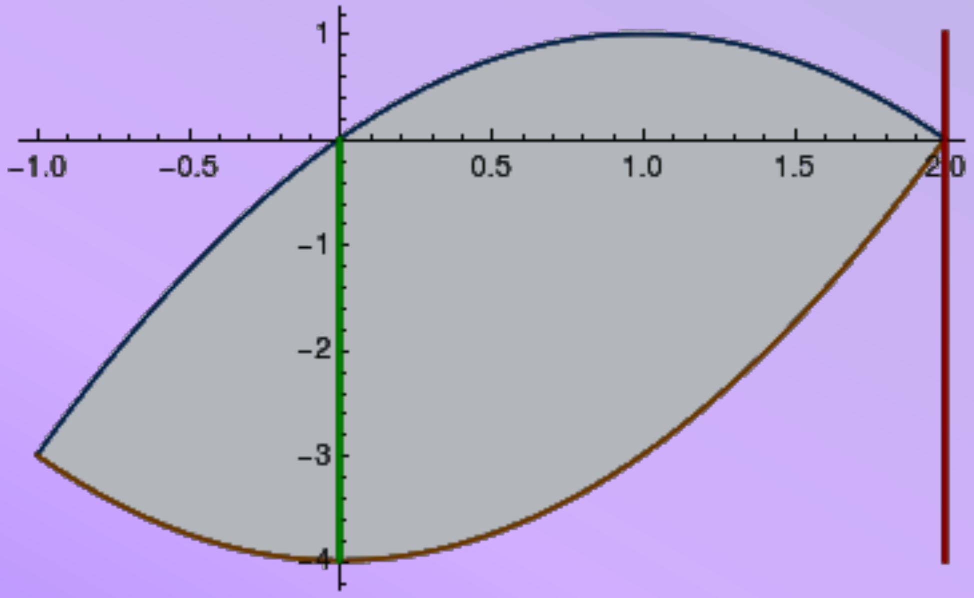
Find the volume of the solid generated by revolving the region bounded by the curves

$$y_1(x) = x^2 - 4$$ (*in orange*)

AND

$$y_2(x) = 2x - x^2$$ (*in blue*)

around the line x=2.




***Use the SHELL METHOD (since we are revolving about a vertical line):***

$$V = 2\pi \times \int_a^b (\textbf{distance to line at x}) \times (\textbf{height of region at x})\, dx$$

$$= 2\pi \times \int_{-1}^{2} (2-x)(4+2x-2x^2)dx$$

$$= 27\pi$$

# *Example B:*

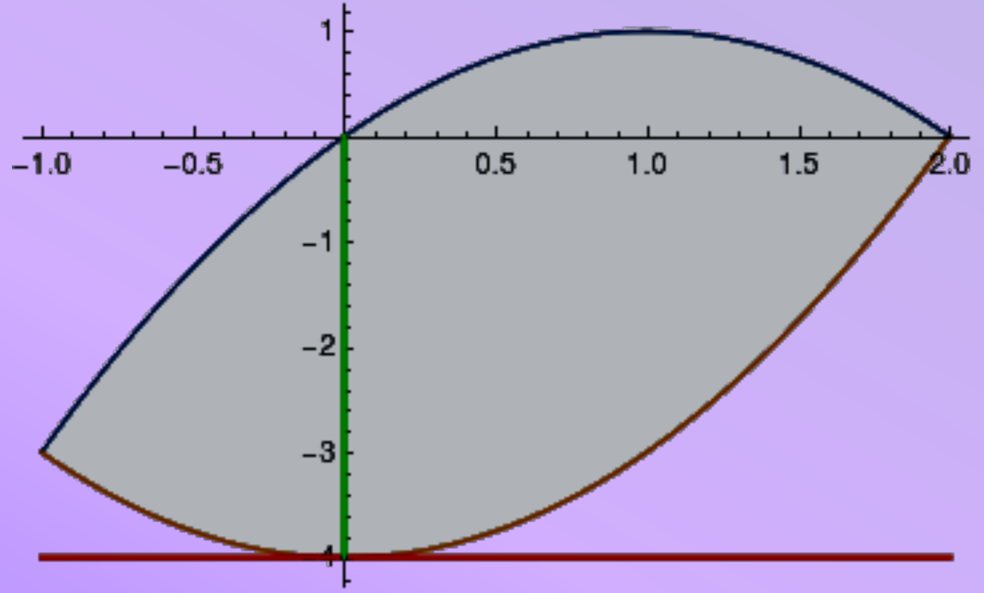
Find the volume of the solid generated by revolving the region bounded by the curves

$$y_1(x) = x^2 - 4$$ (*in orange*)

AND

$$y_2(x) = 2x - x^2$$ (*in blue*)

around the line y=-4.




***Use the WASHER METHOD***

***(be careful to add +4 to each of the functions before squaring):***

$$V = 2\pi \times \int_a^b (\textbf{radius of washer at x})^2 \, dx$$

$$= 45\pi$$

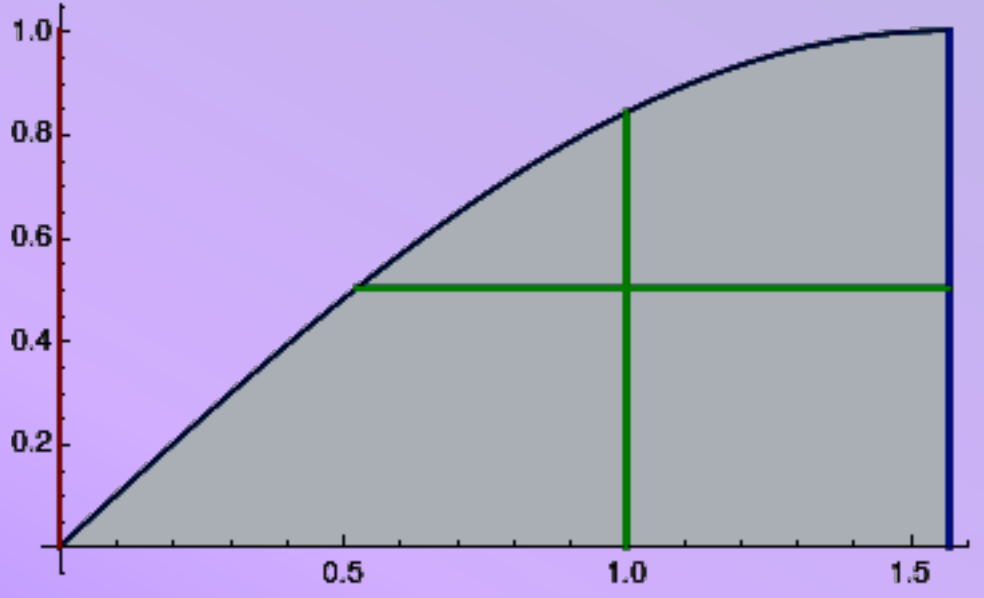
# *Example C:*

Find the volume of the solid generated by revolving the region bounded by the curve

$$y = \sin(x)$$

and the x-axis and the lines $x = 0, \dfrac{\pi}{2}$

about the y-axis.




***SHELL METHOD SETUP (Vertical Slices):*** $V = 2\pi \times \displaystyle\int_0^{\frac{\pi}{2}} x\sin(x)dx$

***WASHER METHOD SETUP (Horizontal Slices):*** $V = \pi \times \displaystyle\int_0^1 \left[\frac{\pi^2}{4} - \left(\sin^{-1}(y)\right)^2\right] dx$

# *Example D:*

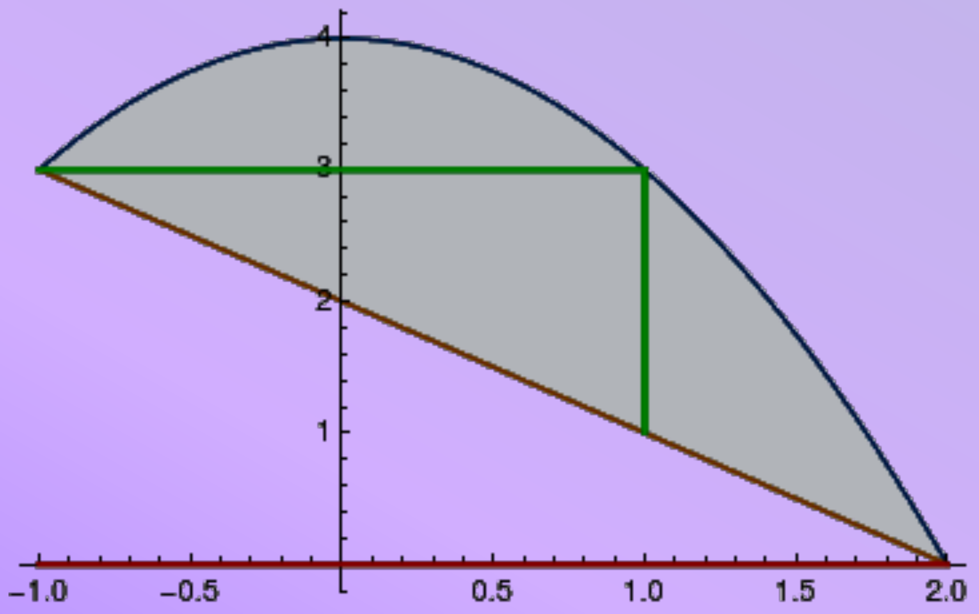
Find the volume of the solid generated by revolving the region bounded by the curves

$$y_1 = x^2 - 4$$

AND

$$y_2 = 2x - x^2$$

is revolved about the x-axis.




***SHELL METHOD SETUP (Vertical Slices):*** $V = 2\pi \times \int_2^4 y\left[\sqrt{4-y} - (2-y)\right] dy$

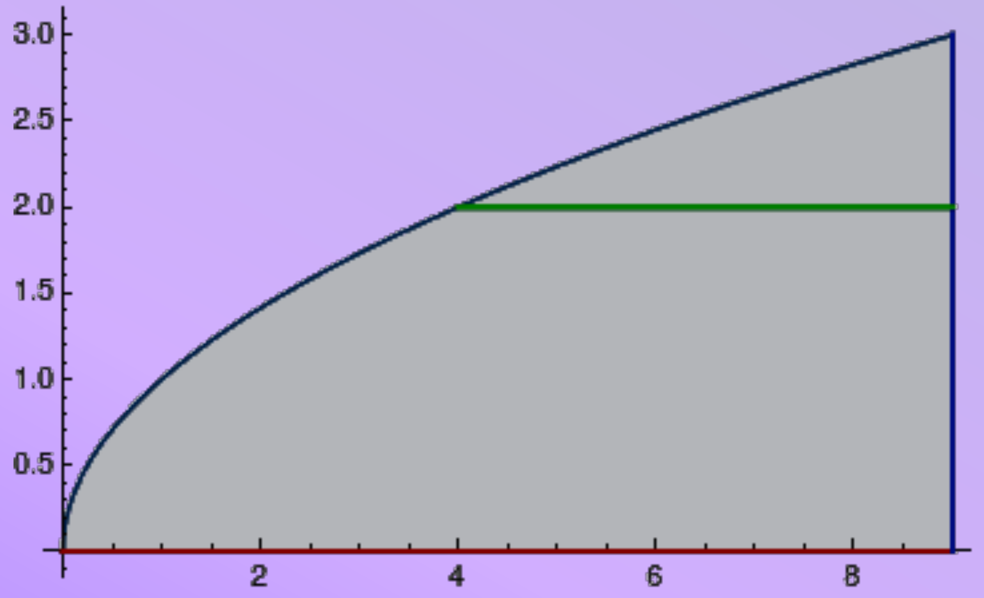
***WASHER METHOD SETUP (Horizontal Slices):*** $V = \pi \times \int_{-1}^{2} \left[(4-x^2)^2 - (2-x)^2\right] dx$

# *Example E:*

Find the volume of the solid generated by revolving the region bounded by the curve

$$y = \sqrt{x}$$

and the x-axis and the line x=9 is revolved about the x-axis.




***SHELL METHOD SETUP (Vertical Slices):***

$$V = 2\pi \times \int_0^3 y(9 - y^2)dy$$

***WASHER METHOD SETUP (Horizontal Slices):***

$$V = \pi \times \int_0^9 (\sqrt{x})^2\, dx$$