1 Summary of Results

Let $x \in \mathbb{N}$, and $\alpha \in \mathbb{C}$. The author has developed new formulas for the sum of divisors function

$$\sigma_\alpha(x) = \sum_{d|x} d^\alpha.$$

They note that $\sigma_\alpha(x)$ is the coefficient of q^x for

$$\sum_{n=1}^{x} \frac{n^\alpha q^n}{1-q^n},$$

and so devise a method to express each term $\frac{q^n}{1-q^n}$ in terms of sums of special derivatives of cyclotomic polynomials. The author then extracts the coefficient of q^x from each sum, which implies a formula for $\sigma_\alpha(x)$.

Once the formula for $\sigma_\alpha(x)$ is proved, applications are discussed for asymptotics involving the average order of the divisor function, and for the study of perfect numbers. Comparisons are also made to other exact formulas, including the Hardy–Ramanujan–Rademacher integer partition formula, and similar more restrictive partition functions.

2 Importance

Given the importance of $\sigma_\alpha(x)$ for elementary number theory, it is interesting to note that few exact formulas for $\sigma_\alpha(x)$ have been developed. The existence of such a formula is therefore potentially valuable.

However, there is a difficulty with implementation and questions of efficiency. The referee has been unable to properly implement and test the formula. Moreover, the formula contains the sum

$$\sum_{d|x} \tau_x^{\alpha+1}(d),$$

in which each divisor of x must be known to begin with. Given the comparable complexity of computing τ_x^α, this raises the question of whether it would be easier to simply compute $\sigma_\alpha(x)$ directly, without resorting to the computed formula.

This important objection notwithstanding, the formula may still be of considerable interest. As in the study of the primes, or even of the integer partition function $p(n)$ discussed by the author toward the end of the paper, whether or not a given formula is of optimal efficiency is not necessarily the sole criterion for importance.

3 Key Difficulties For the Paper

By far, the single most important objection that can be levelled against the paper is the abundancy of errors. Very likely, most of these errors are simply typos, but they are serious enough that they present difficulties for the referee to properly test the main result.

As an example, in Section 1.2: *Factoring partial sums into irreducibles*, Definition 1.1, the author provides the following identity for $n \geq 1$:

$$\Pi_n(q) = \sum_{j=0}^{n-2} \frac{(n-1-j)q^j(1-q)}{(1-q^n)} = \frac{(n-1)+nq-q^n}{1-q}.$$

Of course, it must be an error to suppose that $n \geq 1$; more likely, the author meant that $n \geq 2$. More important, however, as may easily be checked, is the fact that this formula is incorrect. Moreover, at various points, the author implies that $\Pi_n(q) = \sum_{j=0}^{n-2} \frac{(n-1-j)q^j(1-q)}{(1-q^n)}$ is equal to the following expressions:

- $\frac{nq^n}{1-q^n} + n - \frac{1}{1-q}$ (First equation given in Section 1.3): correct,
- $\frac{nq^{n-1}}{q^n-1} - \frac{1}{q-1}$ (Footnote to Section 1.3): incorrect,
- $\frac{(n-1)+nq-q^n}{(1-q)^2} \frac{1-q}{(1-q^n)}$ (Section 2.1, in the case that n is a prime): incorrect.

It is difficult to know exactly which identity the author is attempting to refer to.

There is an incorrect property given in Section 1.3 for cyclotomic polynomials, i.e. that $\Phi_{2p}(q) = \Phi_p(q)$. The correct property is that $\Phi_{2p}(q) = \Phi_p(-q)$.

Moreover, Table 1.1 examines formula expressions of $\frac{nq^n}{1-q^n} + n - \frac{1}{1-q}$ in terms of special derivatives of cyclotomic polynomials. Column 3 gives such expressions, and Column 4 gives a reduced expression; but many of these cyclotomic polynomials in Column 4 are not fully reduced. For example, $\Phi_6(q)$ is given, even though such a term can be reduced still further to $\Phi_3(-q)$.

More directly related to the central results of the paper, the function $\tau_x^\alpha(x)$ is defined in Section 1.3.1, Definition 1.2 as the following:

$$\tau_x^\alpha(i) = \sum_{s \in \mathbb{S}_{x,i}} \mu(s/i) i s^{\alpha-1},$$

with $\mathbb{S}_{x,i}$ a subset of the integers from 12 to n satisfying certain properties. There is a slight typographical issue with using i as an integer, given that α is a complex number; perhaps a better index here would be j. But of far greater importance is the fact that this formula does not match with the computed values of τ_x^α given in Table 2.1.

Of greatest importance, however, is the main formula itself. Each of these errors might be overlooked if the formula could be successfully implemented and tested. The referee has attempted to implement the formula with Mathematica, but the result does not closely compare with the correct value of $\sigma_\alpha(x)$. The referee strongly suggests that the author attempt to implement and test the formula before attempting publication again.

4 Recommendation

The referee finds that the author's result is interesting and of potential significance. However, the errors in the paper are significant enough that the referee cannot yet recommend publication. By no means should the author abandon this interesting paper. Rather, the author needs to return to the paper, to more carefully express the equations that are intended for the main result. The referee strongly recommends that the author use a computer algebra package, such as Sage, Mathematica, or Maple, to actually implement and test the formula. Such an implementation would not only give additional evidence to the veracity of the formula, but it would also allow the author to carefully check for any errors that may have been made.