5 Applications

Proposition 5.1. *We have the following variants of the DGFs $\mathcal{F}_*(s)$ for the respective functions $f(n) \in \{\phi(n), J_k(n), \phi_m(n), \sigma_\alpha(n)\}$:*

$$\sum_{n\geq 1} \phi(n)^{-s} = \zeta(s) \times \prod_p \left(1 - p^{-s} + (p-1)^{-s}\right), \mathrm{Re}(s) > 1,$$

$$\sum_{n\geq 1} J_k(n)^{-s} = \zeta(ks) \times \prod_p \left(1 - p^{-ks} + (p^{ks}-1)^{-s}\right), \mathrm{Re}(ks) > 1,$$

$$\sum_{n\geq 1} \phi_m(n)^{-s} = \left(\frac{\zeta((m+1)s)}{m+1} + \frac{\zeta(ms)}{2} + \sum_{2\leq j\leq m} \frac{B_j m^{j-1}}{j!} \zeta((m+1-j)s)\right) \times \prod_p \left(2 - p^k\right)^s, \mathrm{Re}(s) > 1,$$

$$\sum_{n\geq 1} \sigma_\alpha(n)^{-s} = \prod_p \left(1 + (p^\alpha - 1)^s \times \sum_{r\geq 1} (p^{\alpha(r+1)} - 1)^{-s}\right), \mathrm{Re}(\alpha s) > 1.$$

Proof. □

5.1 Problem type setup for Ernie to look at:

We are given an integer-valued multiplicative function $f \geq 1$ with $f(1) = 1$ and where we assume $f(n) \geq 2$ for all $n \geq 2$. We form the modified zeta functions over the sets of $f(n)$ as follows:

$$\mathcal{F}_{1,*}(s) := 1 + \sum_{n\geq 2} f(n)^{-s}, \mathrm{Re}(s) > \sigma_f^*.$$

Want to look at the coefficients of the following DGFs and then compute solid asymptotics for their summatory functions:

$$\mathcal{F}_*(s,z) := \left[\frac{1}{2 - \mathcal{F}_{1,*}(s)}\right]^z := \sum_{n\geq 1} \frac{\widehat{f}_z(n)}{n^s}.$$

Note that

$$\mathcal{F}_*(s,z) = \prod_{n\geq 2} \left(1 - f(n)^{-s}\right)^{-z}.$$

The big question I have is how to use the multiplicative structure, and say a "nice enough" (Euler product definitely) DGF for $\mathcal{F}(s)$, then use bounds on this DGF to derive a sane main and error term bound for the sums

$$\widehat{F}_z(x) := \sum_{n\leq x} \widehat{f}_z(n).$$

Note that the analogous summatory functions of the coefficients of the DGF $\zeta(s)^z$ are known, and derived using a somewhat intricate contour argument in Montgomery and Vaughan. The basic bound relies on an inductive hyperbola method, and then more importantly on the known special property of the zeta function that $\zeta(s) = \frac{s}{s-1}\left(1 + O(|s-1|)\right)$.

N.b.: This construction is not a trinket problem. It effectively provides a generalization of the method I used to characterize the Mertens function with an underlying probability distribution (weighted by summands of $\lambda(n)$, which makes it properly still a hard problem) derived from strongly additive functions. I can do this to sum $F^{-1}(x) := \sum_{n\leq x} f^{-1}(n)$ for generally positive, Dirichlet invertible f, and then characterize these summatory functions with a limiting probability measure underneath. The proof so far is inexact in so much as the method still needs the good bounds on the sums $\widehat{F}_z(x)$ to proceed from formalism to an exact formula for the limiting CLT-like (Erdős-Kac analog) distributions. It is nonetheless a very interesting, and definitely challenging problem that I need input on. If you feel like talking even more, I can show you the work I already have done and the specific strongly additive functions associated with any such multiplicative f.