Factorization theorems and canonical representations for generating functions of special sums

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To Kush, Cameo, George and Fred. My parents, Sarah and Don, for all their support without which over the years completing this manuscript would never have been possible.

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1 Introduction

"The full beauty of the subject of generating functions emerges only from tuning in on both channels: the discrete and the continuous. See how they make the solution of difference equations into child's play. Then see how the theory of functions of a complex variable gives, virtually by inspection, the approximate size of the solution. The interplay between the two channels is vitally important for the appreciation of the music."

Herbert S. Wilf
Philadelphia, PA
September 1, 1989

When I first embarked on my college adventures as an undergraduate at the University of Illinois in 2004, the first book I checked out from the extensive Altgeld mathematics library stacks was *Generatingfunctionology* by H. S. Wilf. The quote featured at the start of this section is transcribed from the preface to the book. The usage and importance of generating functions in the study of integer sequences is memorably summarized by Wilf by the analogy on the first line to his classic introductory survey of this subject: *A generating function is a clothesline on which we hang up a sequence of numbers for display* [59, §1] (*cf.* [55, 54, 20, 9, 32, 6]).

1.1 Generating functions are an essential tool in discrete mathematics

Generating functions are series expansions summing over a sequence, or arithmetic function, with indeterminate weights in powers of an auxiliary variable. A sequence generating function may be treated formally in the ring of formal power series (e.g., as rigorously motivated in [31]) or may take on conditional meaning as analytic functions of a complex variable depending on the context and use cases in applications. There are many types and flavors of generating functions that we find in practice. For any fixed sequence, $\mathcal{F} := \{f_n\}_{n\geq 0} \subset \mathbb{Q}$, we consider the following types of generating functions[1]:

- The **ordinary generating function (OGF)** of $\mathcal{F}$ (in the variable z) is defined by
$$F(z) := \sum_{n\geq 0} f_n z^n.$$
- The **exponential generating function (EGF)** $\mathcal{F}$ (in the variable z) is defined by
$$\widehat{F}(z) := \sum_{n\geq 0} f_n \frac{z^n}{n!}.$$
- The **Dirichlet generating function (DGF)**, or Dirichlet series, of $\mathcal{F}$ (in the variable s) is defined by
$$D[\mathcal{F}](s) := \sum_{n\geq 1} \frac{f_n}{n^s}.$$

Most of my undergraduate and graduate research in mathematics focuses on studying integer sequences through techniques that facilitate exploration of their properties via transformations of generating functions. One article of mine[2] that was assembled my first semester at Georgia Tech in 2017 provides a short expository introduction to this type of work. The survey article was published in the special issue of the

[1] The notational conventions used to name these generating functions is adapted from the good style in Graham, Knuth and Patashnik [11, *cf.* §7] (*cf.* [18]).
[2] Henceforth, the author of this manuscript, abbreviated MDS.

journal *Axioms* titled *Mathematical Analysis and Applications II* [46]. Other peer-reviewed publications I have authored on this topic since enrolling at Georgia Tech in 2017 include [40, 45, 49, 48, 47].

A significant subtopic I have explored in this area focuses on enumerating certain forms of generalized factorial functions and symbolic product sequences through *Jacobi type continued fractions* (J-fractions) [37, 41, 39, 43, 42]. A J-fraction is a continued fraction that is symbolic in an auxiliary series, or generating function, variable z whose infinite expansion yields the OGF of a sequence, and whose finite convergents approximate the OGF in accuracy as a truncated power series expansion of the OGF of the same sequence. For sequences $\{c_i\}_{i=1}^{\infty}$ and $\{\mathrm{ab}_i\}_{i=2}^{\infty}$, and some typically formal series variable $z \in \mathbb{C}$, we associate the J-fraction expansion with these sequences defined as follows:

$$\begin{aligned} J_\infty(z) &= \cfrac{1}{1 - c_1 z - \cfrac{\mathrm{ab}_2\, z^2}{1 - c_2 z - \cfrac{\mathrm{ab}_3\, z^2}{\cdots}}}, \\ &= 1 + c_1 z + \left(\mathrm{ab}_2 + c_1^2\right) z^2 + \left(2\,\mathrm{ab}_2\, c_1 + c_1^3 + \mathrm{ab}_2\, c_2\right) z^3 \\ &\quad + \left(\mathrm{ab}_2^2 + \mathrm{ab}_2\, \mathrm{ab}_3 + 3\,\mathrm{ab}_2\, c_1^2 + c_1^4 + 2\,\mathrm{ab}_2\, c_1 c_2 + \mathrm{ab}_2\, c_2^2\right) z^4 + \cdots. \end{aligned} \tag{1}$$

Provided that the variable z in the last equation can be restricted to a non-trivial annulus of convergence, the expansions in (1) can be defined as the limit as $h \to \infty$ of the h^{th} convergents to $J_\infty(z)$ defined by

$$\mathrm{Conv}_h(z) := \frac{\mathrm{P}_h(z)}{\mathrm{Q}_h(z)}.$$

The sequences $\{\mathrm{P}_h(z)\}_{h\geq 0}$ and $\{\mathrm{Q}_h(z)\}_{h\geq 0}$ are formed by the finite-degree polynomials in z that satisfy the following recurrence relations:

$$\begin{aligned} \mathrm{P}_h(z) &= (1 - c_h z)\, \mathrm{P}_{h-1}(q, z) - \mathrm{ab}_h\, z^2\, \mathrm{P}_{h-2}(q, z) + [h = 1]_\delta\,, \\ \mathrm{Q}_h(z) &= (1 - c_h z)\, \mathrm{Q}_{h-1}(q, z) - \mathrm{ab}_h\, z^2\, \mathrm{Q}_{h-2}(q, z) + (1 - c_1 z)\, [h = 1]_\delta + [h = 0]_\delta\,. \end{aligned} \tag{2}$$

A Jacobi type J-fraction expansion comes equipped with an extra structure that consitutes the usual rigmarole we can assert by working with continued fractions and their finite convergents [29, *cf.* §3.10] [58]. Manuscripts originally due to the late, great innovator in combinatorial analysis, P. Flajolet, from the 1980's also prove the next property. The following is a result that provides integer congruences for the coefficients in the expansion of the rational functions of z that are formed by the h^{th} convergents to (1) [7, 8]:

$$[z^n] J_\infty(z) \equiv [z^n]\, \mathrm{Conv}_h(z) \qquad \left(\mathrm{mod}\ \ \mathrm{ab}_2\, \mathrm{ab}_3 \times \cdots \times \mathrm{ab}_{h+1}\right), \text{ for any } n \geq 0 \text{ and } h \geq 2.$$

In this manuscript, we will explore another vantage point from which we can use generating functions to find new meaning and interpret generating function based constructions that enumerate integer sequences.

1.2 Lambert series generating functions

In the same way that many combinatorial sequences are described succinctly through the expressions of their OGF or EGF, many arithmetic functions of interest in multiplicative number theory satisfy a structure that is simple to describe by Dirichlet convolution. That is, for any fixed arithmetic functions f and g, we define their *Dirichlet convolution at n* for any integer $n \geq 1$ by

$$(f * g)(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right).$$

We can then appeal to expansions of a special kind of generating function to enumerate the special functions which we wish to study. A Lambert type series, or *Lambert series generating function* (LGF), for an arithmetic function is a special generating function that allows us to capture and enumerate many multiplicative functions f whose descriptions yield a meaningful interpretation of the divisor sums $(f * 1)(n)$ at integers $n \geq 1$.

Definition 1.1. For any arithmetic function $f : \mathbb{Z}^+ \to \mathbb{C}$, we have a corresponding LGF for f defined by the following series expansions:

$$L_f(q) := \sum_{n \geq 1} \frac{f(n)q^n}{1 - q^n} = \sum_{m \geq 1} (f * 1)(m) q^m, |q| < 1. \tag{3}$$

From 2016–2017, my collaborator M. Merca and MDS began collaborative work on overlapping interests from [44, 21] to build on what we have termed as the next so-called *Lambert series factorization theorems* [23, 22, 25, 24]. The divisor sum coefficient expansion on the right-hand-side of (3) reaffirms that there is a natural way to associate an OGF to generate many multiplicatively structured functions using $L_f(q)$ for a fixed arithmetic function f. On the other hand, by taking common denominators of the series terms, these expansions involve the infinite q-Pochhammer symbol. This observation yields a clear connection of the multiplicative functions typically enumerated by these LGF series expansions to the more additive theory of integer partitions.

The basic construction is initially stated in the following form:

$$L_f^{\pm}(q) := \sum_{n \geq 1} \frac{f(n)q^n}{1 \pm q^n} = \frac{1}{(\mp q; q)_\infty} \times \sum_{n \geq 1} \left(\sum_{k=1}^{n} (s_o(n,k) \pm s_e(n,k)) f(k) \right) q^n, |q| < 1.$$

In the last equation, the sequences $s_e(n,k)$ (and $s_o(n,k)$) denote the the number of k's in all partitions of n into an even (and odd, respectively) number of distinct parts for $n \geq k \geq 1$ and where $(a;q)_\infty := \prod_{m \geq 1}(1 - aq^{m-1})$ is the infinite *q-Pochhammer symbol*. We can re-write the last equation as

$$\sum_{n \geq 1} \frac{f(n)q^n}{1 - q^n} = \frac{1}{(q;q)_\infty} \times \sum_{n \geq 1} \left(\sum_{k=1}^{n} s_{n,k} f(k) \right) q^n. \tag{4}$$

The matrices formed by the lower triangular sequence of $s_{n,k}$ are invertible as are the inverse matrices $s_{n,k}^{-1}$. The lower triangular inverse sequence is also related to partition theoretic functions. In particular, we can prove exactly how $s_{n,k}^{-1}$ is related to $p(n)$ as follows:

$$s_{n,k}^{-1} = \sum_{d|n} p(d-k) \mu\left(\frac{n}{d}\right).$$

Remark 1.2. An important takeaway from this factorized representation of any LGF is that the choice of scaling the right-hand-side of the generating function expansion of the series over f is by $(q;q)_\infty^{-1}$. This reciprocal factor providing a multiple of the ordinary generating function for $p(n)$ has a clear-cut, and algebraically natural, relation of both sequences of $s_{n,k}$ and $s_{n,k}^{-1}$ to partition theoretic functions. We explore how to best define a corresponding notion of "canonically best" factorizations corresponding to the generating functions of other special sum types in Section 5. This last section of the dissertation addresses Question 1.4 posed below. The approach to rigorously quantifying our qualitative observations for the LGF case in more generality is introduced by forthcoming discussion in Section 1.6 of the introduction.

1.3 Generalized forms of the factorization theorems

1.3.1 Motivation

Definition 1.3. The *average order* of an arithmetic function f is defined as the arithmetic mean of the summatory function, $F(x) := \sum_{n \leq x} f(n)$, as $f_{\text{ave}}(x) := F(x)x^{-1}$. We typically represent the average order of a function, $f_{\mathrm{ave}}(x)$, in the form of an asymptotic formula where the average order diverges as $x \to \infty$. For example, it is well known that the average order of $\Omega(n)$, which counts the number of prime factors of n (counting multiplicity), is asymptotically dominated by $\log\log x$. The average order of Euler's totient function, $\phi(n)$, grows like $\frac{6x}{\pi^2}$ as $n \to \infty$ [14].

We are motivated by a decomposition of the partial sums of an arithmetic function $f(d)$ whose average order we wish estimate into sums over pairwise disjoint sets of component indices $d \leq x$ that correspond to the indices of summation in each of the three terms on the right-hand-side of the next equation.

$$\sum_{d \leq x} f(d) = \sum_{\substack{1 \leq d \leq x \\ (d,x)=1}} f(d) + \sum_{\substack{d \mid x \\ d>1}} f(d) + \sum_{\substack{1<d \leq x \\ 1<(d,x)<x}} f(d) \tag{5}$$

In evaluating the partial sums of an arithmetic function $f(d)$ over all $d \leq x$, we wish to split the terms in these partial sums into three sets: those d relatively prime to x, the d dividing x (for $d > 1$), and the set of indices d which are neither relatively prime to x nor proper divisors of x. If we let f denote any arithmetic function, we define the remainder terms in our average order expansions from (5) as follows:

$$\widetilde{S}_f(x) = \sum_{d \leq x} f(d) - \sum_{\substack{1 \leq d \leq x \\ (d,x)=1}} f(d) - \sum_{\substack{d \mid x \\ d>1}} f(d). \tag{6}$$

We observe that the divisor sum terms in (6) correspond to the coefficients of powers of q in the Lambert series generating function over f in the form of

$$\sum_{d \mid x} f(d) = [q^x] \left(\sum_{n \geq 1} \frac{f(n) q^n}{1-q^n} \right), x \geq 1.$$

We can also see that the (unscaled) average order sums on the left-hand-side of (5) correspond to a hybrid of nested divisor and relatively prime divisor type sums as

$$\sum_{d \leq x} f(d) = \sum_{m \mid x} \sum_{\substack{k=1 \\ \left(k, \frac{x}{m}\right)=1}}^{\frac{x}{m}} f(km) = \sum_{m \mid x} \sum_{\substack{k=1 \\ (k,m)=1}}^{m} f\left(\frac{kx}{m}\right).$$

Hence, we seek to reconcile the divisibility structure of interesting sequences (and summatory functions) of interest using an enumerative approach by generating functions through a study of generalizations of the original factorization theorems for Lambert series in (4) (see Section 3).

1.3.2 Precise formulations of generating functions for convolution type sequences

We will see many variants of invertible matrix-based factorizations of generating functions for special sums and series in this thesis. We first suggest the next form of the generating function factorization theorems that generalizes (4) to many other applications. We can consider analogous matrix-based factorizations of the generating functions of the sequences of special sums in (7) below provided that these transformations are suitably invertible. That is, we can express generating functions for the sums $s_n(f, \mathcal{A}) := \sum_{k \in \mathcal{A}_n} f(k)$ where we take $\mathcal{A}_n \subseteq [1, n] \cap \mathbb{Z}$ for all $n \geq 1$ as expansions of the following form:

$$\sum_{n \geq 1} \left(\sum_{\substack{k \in \mathcal{A}_n \\ \mathcal{A}_n \subseteq [1,n]}} f(k) \right) q^n = \frac{1}{\mathcal{C}(q)} \times \sum_{n \geq 1} \left(\sum_{k=1}^{n} v_{n,k}(\mathcal{A}, \mathcal{C}) f(k) \right) q^n, \mathcal{C}(0) \neq 0. \tag{7}$$

The matrix entries are generated by

$$v_{n,k}(\mathcal{A},\mathcal{C}) = [q^n]\mathcal{C}(q) \times \sum_{m\geq 1} [k \in \mathcal{A}_m]_\delta \, q^m.$$

These lower triangular coefficients lead to invertible transformations provided that $v_{n,n}(\mathcal{A},\mathcal{C}) \neq 0$ for all $n \geq 1$. We can also naturally consider sums of an arithmetic function weighted by a lower triangular sequence of the form

$$\sum_{n\geq 1} \left(\sum_{k=1}^{n} \mathcal{T}_{n,k} f(k)\right) q^n = \frac{1}{\mathcal{C}(q)} \times \sum_{n\geq 1} \left(\sum_{k=1}^{n} u_{n,k}(\mathcal{T},\mathcal{C}) f(k)\right) q^n. \tag{8}$$

The notable special cases of the so-termed type I and type II sums defined in Section 3 are given by (7) when $\mathcal{A}_{1,n} := \{d : 1 \leq d \leq n, (d,n) = 1\}$ and $\mathcal{A}_{2,n} := \{d : 1 \leq d \leq n, d|(k,n) \text{ for some } 1 \leq k \leq n\}$, respectively.

Question 1.4. What is a "good" (or even canonical) choice of the generating function $\mathcal{C}$ given the definitions of the sets $\{\mathcal{A}_n\}_{n\geq 1}$ that leads to natural formulas for $v_{n,k}(\mathcal{A},\mathcal{C})$ and its inverse matrix?

1.4 Examples: Canonical and illustrative examples of special convolution type sums

The examples cited in the new few subsections provide a survey of interesting applications of the generalized convolution type sequences we consider later in the thesis. We will aperiodically return to these as reference points for comparison with the new results and generalized factorization theorems.

1.4.1 A-convolutions: Restricted classes of Dirichlet convolutions and divisor sums (ACVL type sums)

For each $n \geq 1$, let $A(n) \subseteq \{d : 1 \leq d \leq n, d|n\}$ be a subset of the divisors of n. We say that a natural number $n \geq 1$ is *A-primitive* if $A(n) = \{1, n\}$. Under a list of assumptions so that the resulting A-convolutions are *regular convolutions*, we get a generalized multiplicative Möbius function [36]:

$$\mu_A(p^\alpha) = \begin{cases} 1, & \alpha = 0; \\ -1, & p^\alpha > 1 \text{ is } A\text{-primitive}; \\ 0, & \text{otherwise.} \end{cases}$$

This construction leads to a generalized form of Möbius inversion between the A-convolutions. We can consider A-convolution sums for a fixed set A in the following two flavors for any two arithmetic functions f, g:

$$\begin{aligned} S_{A,1}(f,g;n) &:= \sum_{\substack{d|n \\ d\in A}} f(d) g\left(\frac{n}{d}\right), \\ S_{A,2}(f,g;n) &:= \sum_{\substack{d|n \\ d\in A}} f(d) g\left(\frac{n}{d}\right) \chi_A\left(\frac{n}{d}\right). \end{aligned}$$

We can find an inverse function for f with respect to this A-convolution if and only if $1 \in A$ and $f(1) \neq 0$.

1.4.2 Unitary convolutions: Restricted divisor sums (UCVL type sums)

A so-called *unitary convolution* is a divisor sum in which both the index of summation d and the residual index $\frac{n}{d}$ are relatively prime:

$$(f \odot g)(n) := \sum_{\substack{d|n \\ \left(d, \frac{n}{d}\right)=1}} f(d) g\left(\frac{n}{d}\right).$$

An arithmetic function f is invertible with respect to unitary convolution, i.e., there exists a (unique) function f^{-1} such that $f \odot f^{-1} = \varepsilon \equiv \delta_{n,1}$, if and only if $f(1) \neq 0$. If $f(1) = 1$, then Cohen has given a simple formula to express the inverse of any invertible f with respect to the $\odot$ convolution operator [36]:

$$f^{-1}(n) = \sum_{k=1}^{\omega(n)} (-1)^k \times \sum_{d_1 d_2 \times \cdots \times d_k = n} \left(\prod_{i=1}^{k} f(d_i)\right).$$

1.4.3 K-convolutions: Divisor sums with a kernel weight function (KCVL type sums)

Let the kernel function $K : \mathbb{N} \times \mathbb{N} \to \mathbb{C}$ be defined on all ordered pairs (n, d) such that $n \geq 1$ and $d|n$. We define the K-convolution of two arithmetic functions f, g to be

$$(f *_K g)(n) := \sum_{d|n} f(d) g\left(\frac{n}{d}\right) K(n, d).$$

We can define generating function factorizations of these sums of the form

$$(f *_K g)(n) = [q^n] \frac{1}{\mathcal{C}(q)} \times \sum_{m \geq 1} \left(\sum_{k=1}^{m} e_{n,k}(K, \mathcal{C}; g) f(k)\right) q^m.$$

Similarly, provided reasonable expansions of the function K (e.g., some recurrence relations or other structural properties), we can find inverse functions of an arithmetic function f with respect to K-convolution (see Section 4.1).

Example 1.5 (B-convolutions)**.** Let $\nu_p(n)$ denote the maximum exponent of the prime p in the factorization of n. That is,

$$\nu_p(n) = \begin{cases} \alpha, & \text{if } n \geq 2 \text{ and } p^\alpha || n; \\ 0, & \text{otherwise.} \end{cases}$$

For integers $n \geq d \geq 1$, let the function $B(n, d) := \prod_{p|n} \binom{\nu_p(n)}{\nu_p(d)}$ where the product runs over all prime divisors of n. We have an inversion formula given by

$$f(n) = \sum_{d|n} g(d) B(n, d) \iff g(n) = \sum_{d|n} f(d) \lambda(d) B(n, d),$$

where $\lambda(n) = (-1)^{\Omega(n)}$ is the *Liouville lambda function* [36].

1.4.4 Discrete convolutions with respect to an index set (DCVL type sums)

We can form another variant of the typical discrete convolution of coefficients (or Cauchy product) resulting from the pointwise multiplication of two ordinary generating functions. This generalization involves summations of the form

$$S_{f,g}(\mathcal{A}; n) := \sum_{k \in \mathcal{A}_n} f(k) g(n + 1 - k), \mathcal{A}_n \subseteq \{1, 2, \ldots, n\} \subset \mathbb{Z}^+.$$

Given the relation to a convolution of generating functions (or formal power series), it is no surprise that we can "*encode*" and invertibly "*decode*" auxiliary sequences by multiplying an arbitrary OGF by another OGF and its reciprocal. Hence, discrete convolution sums are in general invertible operations from which we can solve for either f or g.

More generally, we can define sums of the following type for some bivariate kernel weight function, $\mathcal{D} : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{C}$, that is lower triangular and invertible:

$$(f \boxdot_{\mathcal{D}} g)(n) := \sum_{k=1}^{n} f(k)g(n+1-k)\mathcal{D}(n,k), n \geq 1.$$

Sums of this type have familiar expressions by matrix-vector products involving Topelitz matrices, which are themselves common and well studied to express the discrete convolutions of sequences (see Section 5).

1.4.5 Summatory function weighted sums (APT type sums)

For $\mathcal{A}_n \subseteq [1,n] \cap \mathbb{Z}^+$, consider defining sums of the form

$$S_{f,g}(\mathcal{A};n) := \sum_{k \in \mathcal{A}_n} f(k) g\left(\left\lfloor \frac{n}{k} \right\rfloor\right).$$

When the sums defined by $S_{f,g}(\mathcal{A};n)$ are finite, and provided that f is Dirichlet invertible with $f(1) \neq 0$, there are inversion formulas to extract $g(x)$ from these sums in the special case where $\mathcal{A}_n = \{1,2,\ldots,n\}$ [4]. In general, a consequence of *Perron's formula* from complex analysis and analytic number theory provides that if $G(s) := \sum_{n \geq 1} g(n)n^{-s}$ is the DGF of g, and if $F(s) := \int_0^{\infty} f(x)x^{s-1}dx$ is a Mellin transform of f, then we have contour integral representation given on the left-hand-side of

$$\sum_{n \geq 1} f(n) g\left(\frac{x}{n}\right) = \frac{1}{2\pi\imath} \times \int_{c-\imath\infty}^{c+\imath\infty} F(s)G(s)x^s ds, c > \max(\sigma_{a,f}, \sigma_{a,g}).$$

There is another formula for the summatory function of the Dirichlet convolution of any two arithmetic functions f, h of the following form [4]:

$$\sum_{n \leq x} \sum_{d \mid n} f(d) h\left(\frac{n}{d}\right) = \sum_{n \leq x} f(n) H\left(\left\lfloor \frac{x}{n} \right\rfloor\right), x \geq 1.$$

The notation $H(x) := \sum_{n \leq x} h(n)$ used to state the last equation is the summatory function formed by the unweighted partial sums of h.

1.5 Overview of topics in the manuscript

Within this dissertation, we discuss the results in publications based on work completed during 2016–2021 by the author (MDS) and from two-author collaborations with peers about generating function based factorization theorems. These factorization theorem expansions express the coefficients generated by Lambert series and certain restricted GCD type sums that enumerate the sequences of partial sums of any arithmetic function f. We spend several sections recalling the proofs and relevant constructions behind articles published in *Acta Arithmetica*, the *Ramanujan Journal*, the *American Mathematical Monthly* and *INTEGERS*. The articles published in peer-reviewed journals over this time with MDS as the sole author compiled in this manuscript include [44, 47]. The publications that are coauthored work with MDS compiled in this manuscript include [23, 22, 25, 24, 27].

We consider the most general form of the convolution type sums defined by the next $\mathcal{D}$-convolution type sums. For any arithmetic functions f and g, and a lower triangular kernel function $K(n,k)$ that is unambiguously defined for all integers $n \geq k \geq 1$, we define the *$\mathcal{D}$-convolution of f and g at n* by

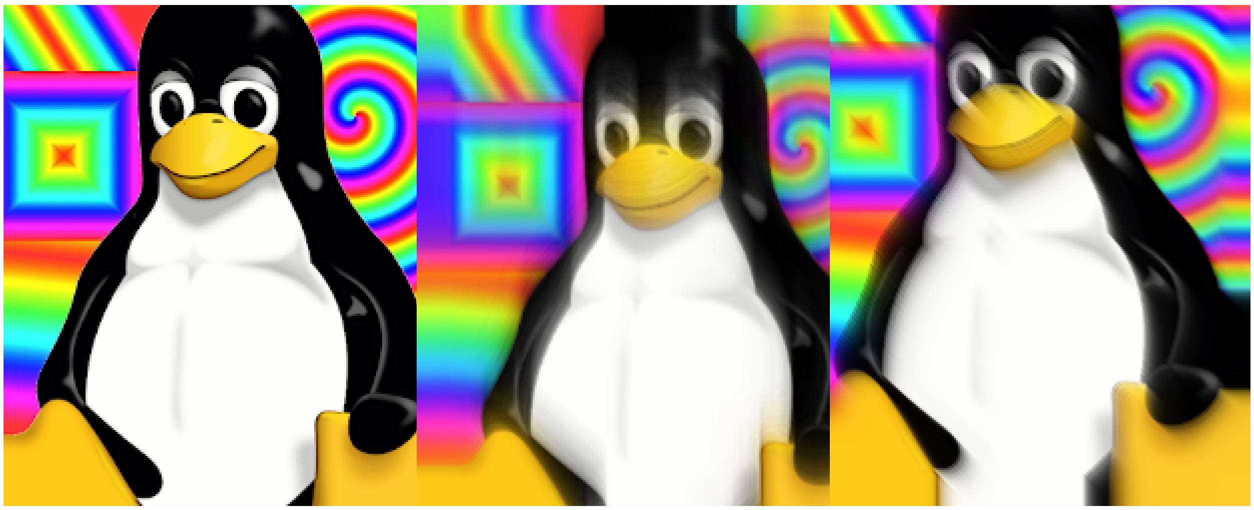
$$(f \boxdot_{\mathcal{D}} g)(n) := \sum_{k=1}^{n} f(k)g(n+1-k)\mathcal{D}(n,k), n \geq 1. \tag{9}$$

In Section 5, we formulate a rigorous answer to Question 1.4 for sums of this most general type and state related conjectures that remain open at the time of this publication. Namely, we will last turn to focus on newer expository results that elaborate on a topical open problem rephrased by Michael Lacey as a remaining loose end from my Ph.D. oral exam presentation at Georgia Tech in August of 2020.

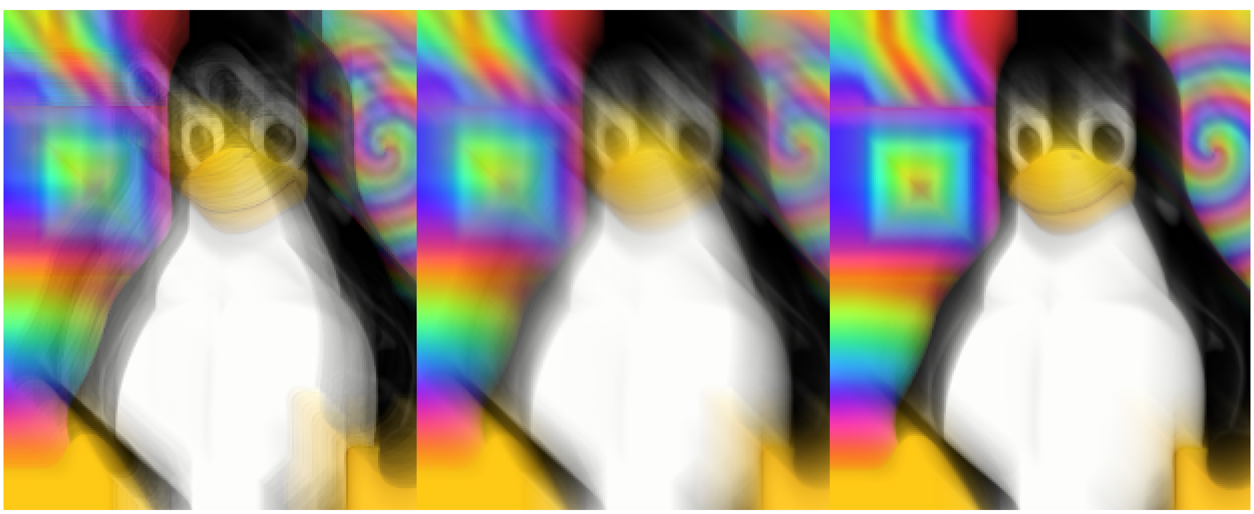
1.6 Rationale for defining "canonically best" generating function factorizations to enumerate generalized classes of convolution type sums

There is a natural question that works its way into the analysis of the prior research and publications by Merca and Schmidt that we have summarized above. One reflection, in hindsight, as to why these seemingly simple expansions related to Lambert series generating functions resulted in so many acceptances in excellent peer-reviewed journals is that they make special, and as at least one reviewer had pointed out "rare", connections between classically multiplicative based constructions and the theory of partitions. Prior to those several publications, only G. E. Andrews and a handful of other authors had found such relations, and none yet it seems so general and clear cut to spot. The choice of factorizing the Lambert series OGF expansions by inserting a multiple of the generating function for the partition function, $p(n) = [q^n](q;q)_\infty^{-1}$, leads to a representation for the matrices $s_{n,k}$ and $s_{n,k}^{-1}$ that both involve special functions that are central to the theory of partitions [2]. The initial relation of the $s_{n,k}$ to partition theoretic constructions was proved independently by Merca in 2017 [21], and seen from a different lense in [44], near the same time we decided to collaborate on generalizing this material.

Example 1.6. We can project a sense of distortion (versus similarity) between two tuples of values (as truncated vectors of OGF coefficients) onto an easy to identify image to visualize our intuitions in the LGF series case. This visualization relies on how clearly the projected data (via computation of and convolution with a correlation matrix) allows us to look at *Tux*, the classic good-luck-forebearing Linux penguin, depicted by Figure 1.1 in the form of his traditional Linux kernel emblem. The original Linux penguin image is modified here to assist with clearly depicting distinguishing features in the correlation statistic values computed by the computer program we have used to generate the figure, e.g., to clearly distinguish distortions projected onto the original or a lack thereof. We use a variant of the built-in image processing functions within modern releases of *Mathematica* to generate these images. By inspection of the featured functions, $\mathcal{C}(q)$, in the figure that satisfy $\mathcal{C}(0) \notin \{0, \pm\infty\}$ with integer-valued series coefficients, we conclude that our choice of $\mathcal{C}(q) := (q;q)_\infty$ does indeed appear to be optimal! This is by no means a conclusive proof of the qualitative notion of optimality we have predicted. However, it is a convincing illustration of a heuristic feature that we wish to quantify and make precise for completeness in the last section of the manuscript.



(a) *Original image.* **(b)** *With* $\mathcal{C}(q) := (q;q)_\infty$. **(c)** *With* $\mathcal{C}(q) := (q;q)_\infty^{-1}$.



(d) *With* $\mathcal{C}(q) := (q^2;q^5)_\infty$. **(e)** *With* $\mathcal{C}(q) := (1-q)^{-\frac{3}{2}}$. **(f)** *With* $\mathcal{C}(q) := (1-q)^{-1}$.

Figure 1.1: *Correlation statistics projected onto the image of Tux for various choices of the OGF* $\mathcal{C}(q)$ *(with integer coefficients) to show a visual comparison of how well related the corresponding sequences are. Distortions of the original image indicate a less well correlated set of sequences corresponding to that choice of the* $\mathcal{C}(q)$ *that defines the precise form of the LGF factorization theorem coefficients.*

2 Factorization theorems for Lambert series generating functions

2.1 A detailed introduction to the properties of Lambert series generating functions

Definition 2.1. In our most general setting, we define the *generalized Lambert series* expansion for integers $0 \leq \beta < \alpha$ and any fixed arithmetic f as

$$L_f(\alpha, \beta; q) := \sum_{n \geq 1} \frac{f(n) q^{\alpha n - \beta}}{1 - q^{\alpha n - \beta}}, |q| < 1.$$

The series coefficients of the Lambert series generating function $L_f(\alpha, \beta; q)$ are the divisor sums

$$[q^n] L_f(\alpha, \beta; q) = \sum_{\alpha d - \beta | n} f(d).$$

If we set $(\alpha, \beta) := (1, 0)$, the we recover the classical form of the Lambert series (LGF) construction denoted by the generating functions $L_f(q) \equiv L_f(1, 0; q)$ in (3).

Example 2.2 (Famous LGF expansions)**.** Ramanujan discovered the following remarkable identities [1, §2]:

$$\sum_{n \geq 1} \frac{(-1)^{n-1} q^n}{1 - q^n} = \sum_{n \geq 1} \frac{q^n}{1 + q^n} \tag{10a}$$

$$\sum_{n \geq 1} \frac{n q^n}{1 - q^n} = \sum_{n \geq 1} \frac{q^n}{(1 - q^n)^2} \tag{10b}$$

$$\sum_{n \geq 1} \frac{(-1)^{n-1} n q^n}{1 - q^n} = \sum_{n \geq 1} \frac{q^n}{(1 + q^n)^2} \tag{10c}$$

$$\sum_{n \geq 1} \frac{q^n}{n(1 - q^n)} = \sum_{n \geq 1} \frac{q^n}{1 + q^n} \tag{10d}$$

$$\sum_{n \geq 1} \frac{(-1)^{n-1} q^n}{n(1 - q^n)} = \sum_{n \geq 1} \log\left(\frac{1}{1 - q^n}\right) \tag{10e}$$

$$\sum_{n \geq 1} \frac{\alpha^n q^n}{1 - q^n} = \sum_{n \geq 1} \log(1 + q^n) \tag{10f}$$

$$\sum_{n \geq 1} \frac{n^2 q^n}{1 - q^n} = \sum_{n \geq 1} \frac{q^n}{(1 - q^n)^2} \sum_{k=1}^{n} \frac{1}{1 - q^k}. \tag{10g}$$

We also have the following well-known "classical" examples of Lambert series identities [29, §27.7] [14, §17.10] [4, §11]:

$$\sum_{n \geq 1} \frac{\mu(n) q^n}{1 - q^n} = q, \tag{11a}$$

$$\sum_{n \geq 1} \frac{\phi(n) q^n}{1 - q^n} = \frac{q}{(1 - q)^2}, \tag{11b}$$

$$\sum_{n \geq 1} \frac{n^{\alpha} q^n}{1 - q^n} = \sum_{m \geq 1} \sigma_{\alpha}(n) q^n, \tag{11c}$$

$$\sum_{n \geq 1} \frac{\lambda(n) q^n}{1 - q^n} = \sum_{m \geq 1} q^{m^2}, \tag{11d}$$

$$\sum_{n\geq 1}\frac{\Lambda(n)q^n}{1-q^n}=\sum_{m\geq 1}\log(m)q^m, \tag{11e}$$

$$\sum_{n\geq 1}\frac{|\mu(n)|q^n}{1-q^n}=\sum_{m\geq 1}2^{\omega(m)}q^m, \tag{11f}$$

$$\sum_{n\geq 1}\frac{J_t(n)q^n}{1-q^n}=\sum_{m\geq 1}m^tq^m, \tag{11g}$$

$$\sum_{n\geq 1}\frac{\mu(\alpha n)q^n}{1-q^n}=-\sum_{n\geq 0}q^{\alpha^n}, \alpha\in\mathbb{P}. \tag{11h}$$

There is a natural correspondence between a sequence's OGF, and its Lambert series generating function. Namely, if $\widetilde{F}(q):=\sum_{m\geq 1}f(m)q^m$ is the OGF of f, then

$$L_f(\alpha,\beta;q)=\sum_{n\geq 1}\widetilde{F}(q^{\alpha n-\beta}).$$

The Lambert series over the convolution $(f*g)(n)$ is given by the double sum

$$L_{f*g}(q)=\sum_{n\geq 1}f(n)L_g(q^n), |q|<1.$$

We have by Möbius inversion that the *ordinary generating function* (OGF) of f is given by

$$L_{f*\mu}(q)=\sum_{n\geq 1}f(n)q^n.$$

Higher-order j^{th} derivatives for integer order $j\geq 1$ can be obtained by differentiating the Lambert series expansions termwise in the forms of

$$q^j\times D_q^{(j)}\left[\frac{q^n}{1-q^n}\right]=\sum_{m=0}^{j}\sum_{k=0}^{m}\begin{bmatrix}j\\m\end{bmatrix}\begin{Bmatrix}m\\k\end{Bmatrix}\frac{(-1)^{j-k}k!i^m}{(1-q^i)^{k+1}}, \tag{12a}$$

$$=\sum_{r=0}^{j}\left(\sum_{m=0}^{j}\sum_{k=0}^{m}\begin{bmatrix}j\\m\end{bmatrix}\begin{Bmatrix}m\\k\end{Bmatrix}\binom{j-k}{r}\frac{(-1)^{j-k-r}k!i^m}{(1-q^i)^{k+1}}\right)q^{(r+1)i}. \tag{12b}$$

By the *binomial series* generating functions whose coefficients are given by $[z^n](1-z)^{m+1}=\binom{n+m}{m}$, we find that

$$[q^n]\left(\sum_{n\geq t}\frac{f(n)q^{mn}}{(1-q^n)^{k+1}}\right)=\sum_{\substack{d|n\\ t\leq d\leq \lfloor\frac{n}{m}\rfloor}}\binom{\frac{n}{d}-m+k}{k}f(d),$$

for positive integers $m,t\geq 1$ and $k\geq 0$. This identity leads to explicit closed-form expressions for the coefficients of the generating functions in (12) by so-termed "restricted" divisor sums of the form of the right-hand-side of the last equation [47].

2.2 Relating the multiplicativity of LGF coefficients to the theory of partitions

The results cited next in this subsetion summarize the work in [44, 22]. Let f denote a fixed arithmetic function. For any integers $n\geq 1$, the n^{th} coefficients of the LGF, $L_f(q)$, are generated by the truncated partial sums

$$[q^n]L_f(q)=[q^n]\left(\sum_{m=1}^{N}\frac{f(m)q^m}{1-q^m}\right), 1\leq n\leq N<+\infty.$$

It is then natural to seek representations for the numerator and denominator of the rational generating function in q that generates $(f * 1)(n)$ for any $n \geq 1$. The denominator terms tend to the limiting product given by the infinite *q-Pochhammer symbol*, $(q;q)_\infty = \prod_{i\geq 1}(1-q^i)$. The sequence of numerator polynomials is more complicated to initially express.

1																												
	1																											
1	1	1																										
2	1	1	1																									
4	3	2	1	1																								
5	3	2	2	1	1																							
10	7	5	3	2	1	1																						
12	9	6	4	3	2	1	1																					
20	14	10	7	5	3	2	1	1																				
25	18	13	10	6	5	3	2	1	1																			
41	30	22	15	11	7	5	3	2	1	1																		
47	36	26	19	14	10	7	5	3	2	1	1																	
76	56	42	30	22	15	11	7	5	3	2	1	1																
90	69	51	39	28	21	14	11	7	5	3	2	1	1															
129	97	74	55	41	30	22	15	11	7	5	3	2	1	1														
161	124	94	72	53	40	29	21	15	11	7	5	3	2	1	1													
230	176	135	101	77	56	42	30	22	15	11	7	5	3	2	1	1												
270	212	163	126	95	73	54	41	29	22	15	11	7	5	3	2	1	1											
384	297	231	176	135	101	77	56	42	30	22	15	11	7	5	3	2	1	1										
458	362	281	219	169	130	98	75	55	41	30	22	15	11	7	5	3	2	1	1									
615	482	379	294	229	175	134	101	77	56	42	30	22	15	11	7	5	3	2	1	1								
750	596	468	370	286	224	171	132	99	76	55	42	30	22	15	11	7	5	3	2	1	1							
1001	792	627	490	385	297	231	176	135	101	77	56	42	30	22	15	11	7	5	3	2	1	1						
1187	951	756	601	472	372	289	225	173	133	100	76	56	42	30	22	15	11	7	5	3	2	1	1					
1570	1252	1000	791	626	490	385	297	231	176	135	101	77	56	42	30	22	15	11	7	5	3	2	1	1				
1881	1518	1213	972	770	612	479	378	292	228	174	134	100	77	56	42	30	22	15	11	7	5	3	2	1	1			
2414	1943	1564	1248	997	789	625	489	384	297	231	176	135	101	77	56	42	30	22	15	11	7	5	3	2	1	1		
2907	2358	1901	1532	1225	980	777	616	483	380	294	229	175	134	101	77	56	42	30	22	15	11	7	5	3	2	1	1	
3717	3010	2436	1958	1575	1255	1002	792	627	490	385	297	231	176	135	101	77	56	42	30	22	15	11	7	5	3	2	1	1

Figure 2.1: *The first* 29 *rows of the function* $s_{i,j}^{-1}$ *where the values of Euler's partition function* $p(n)$ *are highlighted in blue and the remaining values of the partition function* $q(n)$ *are highlighted in purple or pink.*

Definition 2.3. For $1 \leq i \leq n$, let

$$a_{n,i} := \sum_{\substack{(k,s):(s+1)i+k(3k\pm 1)/2=n \\ k,s\geq 0}} (-1)^k = \sum_{b=\pm 1} \sum_{s=0}^{\left\lfloor \frac{n}{i} \right\rfloor - 1} (-1)^{\left\lfloor \frac{\sqrt{24(n-(s+1)i)+1}-b}{6} \right\rfloor} \cdot \left[\frac{\sqrt{24(n-(s+1)i)+1}-b}{6} \in \mathbb{Z} \right]_\delta .$$

Let the f-scaled sums of this triangular sequence to be defined as follows:

$$a_f(n) := \sum_{i=1}^{n} a_{n,i} f(i), n \geq 1.$$

The arithmetic function $f(n) := n^\alpha$ for $\alpha > 0$ such that $\sigma_\alpha(n) = (f * 1)(n)$ defines the class of *generalized sum-of-divisors functions* yields the series expansion given by

$$\begin{aligned} \sum_{m\geq 0} a_{n^\alpha}(m) q^m = 1 + q + 2^\alpha q^2 + \left(-1 - 2^\alpha + 3^\alpha\right) q^3 + \left(-1 - 3^\alpha + 4^\alpha\right) q^4 \\ + \left(-1 - 2^\alpha - 3^\alpha - 4^\alpha + 5^\alpha\right) q^5 + \left(3^\alpha - 4^\alpha - 5^\alpha + 6^\alpha\right) q^6 \\ + \left(-3^\alpha - 5^\alpha - 6^\alpha + 7^\alpha\right) q^7 + \cdots . \end{aligned} \tag{13}$$

For any $n \geq 1$, the $n \times n$ square matrix $A_n := (a_{i,j})_{1\leq i,j\leq n}$ is lower triangular with ones on its diagonal, and is hence invertible. It turns out that in the context of rationalizing the partial sums that generate $(f * 1)(n)$ for $n \geq 1$, the formula for the entries of these matrices is key. Observe the next few cases given in Table 2.1, as well as the highlighted suggestions found in Figure 2.1.

n	A_n	A_n^{-1}
1	$[1]$	$[1]$
2	$\begin{bmatrix}1&0\\0&1\end{bmatrix}$	$\begin{bmatrix}1&0\\0&1\end{bmatrix}$
3	$\begin{bmatrix}1&0&0\\0&1&0\\-1&-1&1\end{bmatrix}$	$\begin{bmatrix}1&0&0\\0&1&0\\1&1&1\end{bmatrix}$
4	$\begin{bmatrix}1&0&0&0\\0&1&0&0\\-1&-1&1&0\\-1&0&-1&1\end{bmatrix}$	$\begin{bmatrix}1&0&0&0\\0&1&0&0\\1&1&1&0\\2&1&1&1\end{bmatrix}$
5	$\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\-1&-1&1&0&0\\-1&0&-1&1&0\\-1&-1&-1&-1&1\end{bmatrix}$	$\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\1&1&1&0&0\\2&1&1&1&0\\4&3&2&1&1\end{bmatrix}$

Table 2.1

Theorem 2.4 (Matrix factorizations defining $f(n)$)**.** *Let the intermediate sequence of sums be defined for fixed f by*

$$B_{f*1,m} := (f*1)(m+1) - \sum_{b=\pm 1} \sum_{k=1}^{\lfloor \frac{\sqrt{24m+1}-b}{6} \rfloor} (-1)^{k+1}(f*1)\left(m+1-\frac{k(3k+b)}{2}\right), m \geq 0.$$

For all $n \geq 1$, we have the following matrix factorization equations:

$$\begin{bmatrix} f(1) \\ f(2) \\ \vdots \\ f(n) \end{bmatrix} = A_n^{-1} \begin{bmatrix} B_{f*1,0} \\ B_{f*1,1} \\ \vdots \\ B_{f*1,n-1} \end{bmatrix} \tag{14}$$

Theorem 2.5 (Recurrence relations for $(f*1)(n)$)**.** *For all $n \geq 1$, we have the following recurrence relation:*

$$(f*1)(n+1) = \sum_{b=\pm 1} \left(\sum_{k=1}^{\lfloor \frac{\sqrt{24n+1}-b}{6} \rfloor} (-1)^{k+1}(f*1)\left(n+1-\frac{k(3k+b)}{2}\right)\right) + a_f(n+1).$$

Lemma 2.6 (Partial sums of the Lambert series, $L_f(q)$)**.** *Let $(q;q)_n := (1-q)(1-q^2)\cdots(1-q^n)$ denote the q-*Pochhammer symbol *[29, §17.2], and suppose that the functions* $\mathrm{poly}_{i,m}(f;q)$ *are polynomials in q with coefficients depending on f (for $i=1,2$) each of whose degree is linear in the fixed index m. For a fixed arithmetic function f and for all integers $m \geq 0$ we have that*

$$g_f(m+1) = [q^m]\left(\frac{1}{q} \times \sum_{n=1}^{m+1} \frac{f(n)q^n}{1-q^n}\right) \tag{15a}$$

$$= [q^m] \left(\frac{\frac{1}{q}(q;q)_{m+1} \left(\frac{f(1)q}{1-q} + \frac{f(2)q^2}{1-q^2} + \cdots + \frac{f(n)q^{m+1}}{1-q^{m+1}} \right)}{(1-q)(1-q^2) \times \cdots \times (1-q^{m+1})} \right) \tag{15b}$$

$$= [q^m] \left(\frac{\sum_{1 \leq i \leq m+1} a_f(i) q^i + q^{m+2} \operatorname{poly}_{1,m}(f;q)}{1 + \sum\limits_{b=\pm 1} \sum\limits_{k=1}^{\left\lfloor \frac{\sqrt{24m+25}+1}{6} \right\rfloor} (-1)^k q^{\frac{k(3k+b)}{2}} + q^{m+2} \operatorname{poly}_{2,m}(f;q)} \right). \tag{15c}$$

Proof. To justify (15a), we observe that for all integers $m \geq 1$ and $1 \leq i \leq m$, we have that

$$[q^i] \left(L_f(q) - \sum_{n>m} \frac{f(n)q^n}{1-q^n} \right) = 0,$$

i.e., that the m^{th} partial sums of $L_f(q)$ generate the coefficients $[q^k]L_f(q)$ for each k in the range $1 \leq k \leq m$. This is easy enough to see by considering the numerator multiples, q^n, of the geometric series, $(1-q^n)^{-1}$, in the individual Lambert series terms in the infinite series expansion of $L_f(q)$. The result in (15b) follows immediately from (15a) by combining the terms in the first partial sum, and implies the third result in (15c) in two key ways.

First, the respective form of the denominator terms in (15c) follows from the statement of *Euler's pentagonal number theorem*, which states that [14, §19.9, Thm. 353]

$$\begin{aligned} (q;q)_\infty &= \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n+1)}{2}} = 1 + \sum_{n \geq 1} (-1)^n \left(q^{\frac{k(3k-1)}{2}} + q^{\frac{k(3k+1)}{2}} \right) \\ &= 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + q^{22} + q^{26} - \cdots. \end{aligned}$$

In particular, we see that the pentagonal number theorem shows that

$$[q^i](1-q)(1-q^2) \times \cdots \times (1-q^n) = \begin{cases} (-1)^k, & \text{if } i = \frac{k(3k \pm 1)}{2}; \\ 0, & \text{otherwise,} \end{cases} \quad i \leq n.$$

Since $(1-q^i)$ is a factor of $(q;q)_n$ for all $1 \leq i \leq n$, we see that both of the numerator and denominator of (15b) are polynomials in q, each with degree greater than $m+1$. This implies the correctness of the denominator polynomial form stated in (15c).

Secondly, since the geometric series in q^i is expanded by

$$\frac{1}{1-q^i} = \sum_{s \geq 0} q^{si}, |q| < 1,$$

for integers $i \geq 1$, we have by the definition of $a_f(n)$ given above that the first $m+1$ terms of the numerator expansion in (15c) are correct. Thus since the numerator in (15c) is polynomial in q, we have that this statement holds as well. □

Proof of Theorem 2.5. We use (15c) in Lemma 2.6 to prove our result. If we let $\mathrm{Num}_m(q)$ and $\mathrm{Denom}_m(q)$ denote the numerator and denominator polynomials in (15c), respectively, we see that by definition, $\deg_q \{\mathrm{Num}_m(q)\} < \deg_q \{\mathrm{Denom}_m(q)\}$. Next, for any sequence, $\{f_n\}_{n \geq 0}$, generated by a rational generating function of the form

$$\sum_{n \geq 0} f_n q^n = \frac{a_0 + a_1 q + a_2 q^2 + \cdots + a_{k-1} q^{k-1}}{1 - b_1 q - b_2 q^2 - \cdots - b_k q^k},$$

for some fixed integer $k \geq 1$, we can prove that f_n satisfies at most a k-order finite difference equation with constant coefficients of the form [20, §2.3]

$$f_n = \sum_{i=1}^{\min(k,n)} b_i f_{n-i} + a_n \left[0 \leq n < k\right]_\delta .$$

Then since we define $f(n) = 0$ for all $n < 1$, since the m^{th} partial sums of $L_f(q)$ generate $g_f(i)$ for all $1 \leq i \leq m$ by the lemma, and since $g_f(i) = 0$ for all $i < 1$, we see that (15c) implies our result. □

Proof of Theorem 2.4. The theorem is a consequence of Theorem 2.5. Specifically, by rearranging terms in the result from the previous theorem, we see that

$$A_n \begin{bmatrix} f(1) \\ f(2) \\ \vdots \\ f(n) \end{bmatrix} = \begin{bmatrix} B_{g_f,0} \\ B_{g_f,1} \\ \vdots \\ B_{g_f,n-1} \end{bmatrix}. \tag{i}$$

Then by the definition of $a_{n,i}$, it is easy to see that A_n is lower triangular with ones on its diagonals, and so is invertible for all $n \geq 1$. Thus by applying A_n^{-1} to both sides of (i), we have proved (14) in the statement of the theorem. □

Corollary 2.7 (Recurrence relations for the summatory function of $f * 1$)**.** *Let the average order of the function $g_f := f * 1$ be denoted by $\Sigma_{g_f,x} := \sum_{n \leq x} (f * 1)(n)$. Then for all $n \geq 1$, we have that*

$$\Sigma_{g_f,n+1} = \sum_{b=\pm 1} \left(\sum_{k=1}^{\lfloor \frac{\sqrt{24n+1}-b}{6} \rfloor + 1} (-1)^{k+1} \Sigma_{g_f, n+1-\frac{k(3k+b)}{2}} \right) + \sum_{k=1}^{n} a_f(k+1). \tag{16}$$

Proof of Corollary 2.7. We can show directly by computation that the statement is true for $n = 1$. For some $j \geq 1$, suppose that the hypothesis in (16) is true for $n = j$. Then wee see that

$$\begin{aligned}
\widetilde{\Sigma}_{g_f,j+1} &= \sum_{b=\pm 1} \sum_{k=1}^{\left\lfloor \frac{\sqrt{24j+25}-b}{6} \right\rfloor} (-1)^k \left(\Sigma_{g_f, j+1-\frac{k(3k+b)}{2}} + g_f\left(j + 2 - \frac{k(3k+b)}{2}\right)\right) \\
&\quad + \sum_{k=1}^{j+1} a_f(k+1) \\
&= \Sigma_{g_f,j+1} + \sum_{b=\pm 1} \sum_{k=1}^{\left\lfloor \frac{\sqrt{24j+25}-b}{6} \right\rfloor} g_f\left(j + 2 - \frac{k(3k+b)}{2}\right) + a_f(j+2), \text{by hypothesis} \\
&= \Sigma_{g_f,j+1} + g_f(j+2) \\
&= \Sigma_{g_f,j+2}.
\end{aligned}$$

The second to last of the previous equations follows from Theorem 2.5, the fact that $\left\lfloor \frac{(\sqrt{24n+25}-b)}{6} \right\rfloor \geq \left\lfloor \frac{(\sqrt{24n+1}-b)}{6} \right\rfloor$, and since $g_f(i) = 0$ for all $i < 1$. □

Corollary 2.8. *For integers $x \geq 1$, let the summatory function*

$$\Sigma_{\sigma,x} := \sum_{n \leq x} \sigma(n).$$

The function $\sigma(n) \equiv \sigma_1(n)$ is the ordinary sum-of-divisors function. Then

$$\Sigma_{\sigma,x+1} = \sum_{s=\pm 1} \left(\sum_{0 \leq n \leq x} \sum_{k=1}^{\left\lfloor \frac{\sqrt{24n+25}-s}{6} \right\rfloor} (-1)^{k+1} \frac{k(3k+s)}{2} p(x-n) \right).$$

Corollary 2.9 (Special cases)**.** *Suppose that for any $m \geq 1$, we define the next functions as*

$$\begin{aligned}
B_m(\mu) &:= [m=0]_\delta + \sum_{b=\pm 1} \sum_{k=1}^{\left\lfloor \frac{\sqrt{24m+1}-b}{6} \right\rfloor} (-1)^k \left[m+1-\frac{k(3k+b)}{2} = 1 \right]_\delta \\
B_m(\phi) &:= m+1-\sum_{b=\pm 1} \sum_{k=1}^{\left\lfloor \frac{\sqrt{24m+1}-b}{6} \right\rfloor} (-1)^{k+1} \left(m+1-\frac{k(3k+b)}{2} \right) \\
B_m(\lambda) &:= \left[\sqrt{m+1} \in \mathbb{Z}\right]_\delta - \sum_{b=\pm 1} \sum_{k=1}^{\left\lfloor \frac{\sqrt{24m+1}-b}{6} \right\rfloor} (-1)^{k+1} \left[\sqrt{m+1-\frac{k(3k+b)}{2}} \in \mathbb{Z} \right]_\delta .
\end{aligned}$$

For all $n \geq 1$, we have that

$$\begin{aligned}
\mu(n) &= \sum_{k=1}^{n} \left(\sum_{d|n} p(d-k) \mu\left(\frac{n}{d}\right) \right) B_{k-1}(\mu) \\
\phi(n) &= \sum_{k=1}^{n} \left(\sum_{d|n} p(d-k) \mu\left(\frac{n}{d}\right) \right) B_{k-1}(\phi) \\
\lambda(n) &= \sum_{k=1}^{n} \left(\sum_{d|n} p(d-k) \mu\left(\frac{n}{d}\right) \right) B_{k-1}(\lambda).
\end{aligned}$$

Proof. These identities follow from the well known divisor sum relations providing that $\mu * 1 = \varepsilon$, $\phi * 1 = \mathrm{Id}_1$, and $\lambda * 1 = \chi_{\mathrm{sq}}$, where χ_{sq} is the characteristic function of the squares and $\mathrm{Id}_1(n) = n$. $\square$

Theorem 2.10. *We have that for integers $1 \leq i \leq n$,*

$$a_{n,i} = s_o(n,i) - s_e(n,i) = [q^n](q;q)_\infty \times \frac{q^i}{1-q^i},$$

where $s_o(n,k)$ and $s_e(n,k)$ are respectively the number of k's in all partitions of n into an odd (even) number of distinct parts. Moreover, the entries of inverse matrices, $A_n^{-1} := (a_{n,i}^{-1})_{1 \leq i,j \leq n}$, satisfy

$$a_{n,i}^{-1} = \sum_{d|n} p(d-i) \mu\left(\frac{n}{d}\right).$$

The function $\mu(n)$ is the classical Möbius function and $p(n) = [q^n](q;q)_\infty^{-1}$ is the (ordinary, i.e., Euler) partition function.

Proof of Theorem 2.10. The generating function expression in the first formula follows by rearranging the series. The partition theoretic interpretation of these coefficients is proved by Merca [21, Thm. 1.2]. The original proof due to Merca relies on an expansion of Lambert series by elementary symmetric polynomials.

It remains to prove the inverse matrix formula in the second equation. An equivalent statement is to prove that for all $1 \leq k \leq n$,

$$p(n-k) = \sum_{d|n} a_{d,k}^{-1}.$$

Let $r \geq 1$ be a fixed lower index. Consider the expansion of the LGF of the function $f(n) := a_{n,r}^{-1}$. Notice that the coefficients of the LGF factorization satisfy

$$\begin{aligned}
\sum_{d|n} a_{d,r}^{-1} &= \sum_{m=0}^{n} \sum_{j=1}^{n-m} \left(s_o(n-m,j) - s_e(n-m,j)\right) a_{j,r}^{-1} p(m) \\
&= \sum_{m=0}^{n} \delta_{n-r,m} p(m) = p(n-r).
\end{aligned}$$

The second equation in the above lines is a consequence of the matrix inverse orthogonality given by

$$\sum_{j=1}^{m} (s_o(m,j) - s_e(m,j)) a_{j,r}^{-1} = \delta_{m,r}, m \geq r \geq 1.$$

The claimed formula then follows by Möbius inversion on the divisor sum. $\square$

A consequence of the last proof is that the inverse matrix entries have the LGF

$$\sum_{n \geq 1} \frac{a_{n,r}^{-1} q^n}{1-q^n} = \frac{q^r}{(q;q)_\infty}, r \geq 1.$$

Example 2.11 (More special case expansions)**.** For natural numbers $m \geq 0$, let the next component sequences defined in [44, 24] be defined by the formulas

$$\begin{aligned}
B_{\phi,m} &= m+1 - \frac{1}{8}\Biggl(8 - 5 \cdot (-1)^{u_1} - 4\left(-2 + (-1)^{u_1} + (-1)^{u_2}\right) m \\
&\qquad + 2(-1)^{u_1} u_1(3u_1+2) + (-1)^{u_2}(6u_2^2 + 8u_2 - 3)\Biggr) \\
B_{\mu,m} &= [m=0]_\delta + \sum_{b=\pm 1} \sum_{k=1}^{\left\lfloor \frac{\sqrt{24m+25}-b}{6} \right\rfloor} (-1)^k \left[m+1 - \frac{k(3k+b)}{2} = 1\right]_\delta \\
B_{\lambda,m} &= \left[\sqrt{m+1} \in \mathbb{Z}\right]_\delta - \sum_{b=\pm 1} \sum_{k=1}^{\left\lfloor \frac{\sqrt{24m+1}-b}{6} \right\rfloor} (-1)^{k+1} \left[\sqrt{m+1 - \frac{k(3k+b)}{2}} \in \mathbb{Z}\right]_\delta \\
B_{\Lambda,m} &= \log(m+1) - \sum_{b=\pm 1} \sum_{k=1}^{\left\lfloor \frac{\sqrt{24m+1}-b}{6} \right\rfloor} (-1)^{k+1} \log\left(m+1 - \frac{k(3k+b)}{2}\right) \\
B_{|\mu|,m} &= 2^{\omega(m+1)} - \sum_{b=\pm 1} \sum_{k=1}^{\left\lfloor \frac{\sqrt{24m+1}-b}{6} \right\rfloor} (-1)^{k+1} 2^{\omega\left(m+1-\frac{k(3k+b)}{2}\right)} \\
B_{J_t,m} &= (m+1)^t - \sum_{b=\pm 1} \sum_{k=1}^{\left\lfloor \frac{\sqrt{24m+1}-b}{6} \right\rfloor} (-1)^{k+1} \left(m+1 - \frac{k(3k+b)}{2}\right)^t,
\end{aligned}$$

where $u_1 \equiv u_1(m) := \lfloor \frac{(\sqrt{24m+1}+1)}{6} \rfloor$ and $u_2 \equiv u_2(m) := \lfloor \frac{(\sqrt{24m+1}-1)}{6} \rfloor$. Then we have that

$$\phi(n) = \sum_{m=0}^{n-1} \sum_{d|n} p(d-m-1)\mu\left(\frac{n}{d}\right) B_{\phi,m}$$

$$\mu(n) = \sum_{m=0}^{n-1} \sum_{d|n} p(d-m-1)\mu\left(\frac{n}{d}\right) B_{\mu,m}$$

$$\lambda(n) = \sum_{m=0}^{n-1} \sum_{d|n} p(d-m-1)\mu\left(\frac{n}{d}\right) B_{\lambda,m}$$

$$\Lambda(n) = \sum_{m=0}^{n-1} \sum_{d|n} p(d-m-1)\mu\left(\frac{n}{d}\right) B_{\Lambda,m}$$

$$|\mu(n)| = \sum_{m=0}^{n-1} \sum_{d|n} p(d-m-1)\mu\left(\frac{n}{d}\right) B_{|\mu|,m}$$

$$J_t(n) = \sum_{m=0}^{n-1} \sum_{d|n} p(d-m-1)\mu\left(\frac{n}{d}\right) B_{J_t,m}.$$

Remark 2.12 (Related constructions for variants of LGF series)**.** Merca showed another variant of the Lambert series factorization theorem stated in the form of [21, Cor. 6.1]

$$\sum_{n\geq 1} \frac{f(n)q^{2n}}{1-q^n} = \frac{1}{(q;q)_\infty} \times \sum_{n\geq 1} \left(\sum_{k=1}^{\lfloor n/2 \rfloor} \left(s_o(n-k,k) - s_e(n-k,k)\right) f(k) \right) q^n.$$

If we consider the generalized Lambert series formed by taking derivatives of $L_f(q)$ as in [47] in the context of finding new relations between the generalized sum-of-divisors functions, $\sigma_\alpha(n)$, we can similarly formulate new, alternate forms of the factorization theorems unified within this section so far. For example, suppose that $k, m \geq 0$ are integers and consider the factorization theorem resulting from an analysis of the following sums:

$$\sum_{n\geq 1} \frac{f(n)q^{(m+1)n}}{(1-q^n)^{k+1}} = \frac{1}{(q;q)_\infty} \times \sum_{n\geq 1} \left(\sum_{i=1}^{\lfloor \frac{n}{m+1} \rfloor} a_{n-m,i} \times \frac{f(i)}{(1-q^i)^k} \right) q^n.$$

We have by our factorization theorem that the previous series are expanded by

$$\sum_{n\geq 1} \frac{f(n)q^{(m+1)n}}{(1-q^n)^{k+1}} = \frac{1}{(q;q)_\infty} \times \sum_{n\geq 1} \left(\sum_{i=1}^{\lfloor \frac{n}{m+1} \rfloor} \sum_{j=0}^{\lfloor \frac{n-m}{i} \rfloor} \binom{k-1+j}{k-1} a_{n-m-ji,i} f(i) \right) q^n,$$

When $m \geq k$ the series coefficients of these modified Lambert series generating functions are given by

$$\sum_{\substack{d|n \\ d \leq \lfloor \frac{n}{m+1} \rfloor}} \tbinom{\frac{n}{d}-1-m+k}{k} f(d) = \sum_{r=0}^{n} \sum_{i=1}^{\lfloor \frac{n-r}{m+1} \rfloor} \sum_{j=0}^{\lfloor \frac{n-r-m}{i} \rfloor} \binom{k-1+j}{k-1} a_{n-r-m-ji,i} f(i) p(r).$$

2.3 Expansions of generalized LGFs

The results in this section summarize the work from [24]. We extend the factorization theorem results proved above for the classical LGF expansion cases to the generalized Lambert series cases defined in (17). In general, for $\alpha > 1$ when $\bar{f}_n \neq f(n)$ the resulting matrix with entries $s_{n,k}(\alpha, \beta)$ is not invertible. We still arrive at some interesting cases, new identities and relations by considering the LGF series factorizations abstracted to this level of generality.

2.3.1 A few special cases

Proposition 2.13. *For $|q| < 1$, we have that*

$$\sum_{n=1}^{\infty} f(n) \frac{q^{2n-1}}{1-q^{2n-1}} = \frac{1}{(q;q^2)_\infty} \times \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} s_{n,k}^*(2,1) f(k) \right) (-1)^{n-1} q^n,$$

where $s_{n,k}^*(2,1)$ denotes the number of $(2k-1)$'s in all partitions of n into distinct odd parts.

Proof. We consider the identity [21, eq. 2.1], namely

$$\sum_{k=1}^{n} \frac{f(k)x_k}{1-x_k} = \left(\prod_{k=1}^{n} \frac{1}{1-x_k} \right) \times \left(\sum_{k=1}^{n} \sum_{1 \leq i_1 < \ldots < i_k \leq n} (-1)^{k-1} (f(i_1) + \cdots + f(i_k)) x_{i_1} \cdots x_{i_k} \right).$$

By this relation with x_k replaced by the powers q^{2k-1} of the series indeterminate, we get

$$\begin{aligned} &\sum_{k=1}^{n} \frac{f(k)q^{2k-1}}{1-q^{2k-1}} \\ &\quad = \frac{1}{(q;q^2)_n} \times \left(\sum_{k=1}^{n} \sum_{1 \leq i_1 < \ldots < i_k \leq n} (-1)^{k-1} \left(f(i_1) + \cdots + f(i_k) \right) q^{(2i_1-1)+\cdots+(2i_k-1)} \right). \end{aligned}$$

The result follows directly from this identity considering the limiting case as $n \to \infty$. ☐

Example 2.14 (Consequences of the proposition)**.** The result in Proposition 2.13 allows us to derive many specialized identities involving Euler's partition function and various arithmetic functions. For $n \geq 1$ we have that

$$\begin{aligned} \sum_{k=1}^{n} \sum_{2d-1|k} d^x Q(n-k) &= \sum_{k=1}^{n} (-1)^{n-1} k^x s_{n,k}^*(2,1), \\ \sum_{k=1}^{n} \sum_{2d-1|k} \mu(d) Q(n-k) &= \sum_{k=1}^{n} (-1)^{n-1} \mu(k) s_{n,k}^*(2,1), \\ \sum_{k=1}^{n} \sum_{2d-1|k} \varphi(d) Q(n-k) &= \sum_{k=1}^{n} (-1)^{n-1} \varphi(k) s_{n,k}^*(2,1), \\ \sum_{k=1}^{n} \sum_{2d-1|k} \lambda(d) Q(n-k) &= \sum_{k=1}^{n} (-1)^{n-1} \lambda(k) s_{n,k}^*(2,1), \\ \sum_{k=1}^{n} \sum_{2d-1|k} \log(d) Q(n-k) &= \sum_{k=1}^{n} (-1)^{n-1} \log(k) s_{n,k}^*(2,1), \end{aligned}$$

$$\sum_{k=1}^{n} \sum_{2d-1|k} |\mu(d)| Q(n-k) = \sum_{k=1}^{n} (-1)^{n-1} |\mu(k)| s^*_{n,k}(2,1),$$

$$\sum_{k=1}^{n} \sum_{2d-1|k} J_t(d) Q(n-k) = \sum_{k=1}^{n} (-1)^{n-1} J_t(k) s^*_{n,k}(2,1),$$

where $s^*_{n,k}(2,1)$ is defined as in Proposition 2.13, and the partition function $Q(n) := s_e(n) - s_o(n)$ where $s_e(n)$ and $s_o(n)$ denote the numbers of partitions of n into an even (respectively odd) number of parts. We can also similarly express the relations in the previous equations for any special arithmetic function f in the form of

$$\sum_{2d-1|n} f(d) = \sum_{k=0}^{n} \sum_{j=1}^{k} (-1)^{k-1} q(n-k) s^*_{k,j}(2,1) f(j),$$

where the standard partition function $q(n)$ denotes the number partitions of n into (distinct) odd parts. Moreover, since we have a direct factorization of the Lambert series generating function for the *sum of squares function* as in the appendix, we may write

$$\sum_{k=1}^{n} r_2(k) Q(n-k) = \sum_{k=1}^{n} 4(-1)^{k+1} s^*_{n,k}(2,1),$$

using the same notation as above. Similarly, we can invert to expand $r_2(n)$ as the multiple sum

$$r_2(n) = \sum_{k=0}^{n} \sum_{j=1}^{k} 4q(n-k)(-1)^{j+1} s^*_{k,j}(2,1),$$

for all $n \geq 1$.

Definition 2.15. For fixed $\alpha, \beta \in \mathbb{Z}$ such that $\alpha \geq 1$ and $0 \leq \beta < \alpha$, and an arbitrary sequence $\{f(n)\}_{n\geq 1}$, we consider generalized (non-factorized) Lambert series expansions of the following form:

$$L_f(\alpha, \beta; q) := \sum_{n\geq 1} \frac{f(n) q^{\alpha n - \beta}}{1 - q^{\alpha n - \beta}} = \sum_{m \geq 1} \left(\sum_{\alpha d - \beta | m} f(d) \right) q^m, |q^\alpha| < 1. \tag{17}$$

We also consider factorizations of the form

$$L_f(\alpha, \beta; q) = \frac{1}{C(q)} \times \sum_{n\geq 1} \left(\sum_{k=1}^{n} s_{n,k}(\alpha, \beta) \bar{f}_k \right) q^n, \tag{18}$$

where $C(0) \neq 0$, and the intermediate coefficients $\bar{f}_n$ depend only on the $s_{n,k}(\alpha, \beta)$ and on the choice of the input arithmetic function $f(n)$.

Proposition 2.16. *For $|q| < 1$, $0 \leq \beta < \alpha$,*

$$\sum_{n=1}^{\infty} f(n) \frac{q^{\alpha n - \beta}}{1 - q^{\alpha n - \beta}} = \frac{1}{(q^{\alpha - \beta}; q^\alpha)_\infty} \times \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} (s_o(\alpha, \beta; n, k) - s_e(\alpha, \beta; n, k)) f(k) \right) q^n,$$

where $s_o(\alpha, \beta; n, k)$ and $s_e(\alpha, \beta; n, k)$ denote the number of $(\alpha k - \beta)$'s in all partitions of n into an odd (respectively even) number of distinct parts of the form $\alpha k - \beta$.

Proof of Proposition 2.16. The proof follows from [21, eq. 2.1], replacing x_k by $q^{\alpha k - \beta}$. $\square$

Proposition 2.17. *For $|q| < 1$, $0 \leq \beta < \alpha$,*

$$\sum_{n=1}^{\infty} f(n) \frac{q^{\alpha n - \beta}}{1 - q^{\alpha n - \beta}} = (q^{\alpha - \beta}; q^{\alpha})_{\infty} \times \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} s(\alpha, \beta; n, k) f(k) \right) q^n,$$

where $s(\alpha, \beta; n, k)$ denotes the number of $(\alpha k - \beta)$'s in all partitions of n into parts of the form $\alpha k - \beta$.

Proof of Proposition 2.17. We take into account the fact that

$$\frac{q^{\alpha n - \beta}}{1 - q^{\alpha n - \beta}} \times \frac{1}{(q^{\alpha - \beta}; q^{\alpha})_{\infty}},$$

is the generating function for the number of $(\alpha k - \beta)$'s in all partitions of n into parts of the form $\alpha k - \beta$. This generating function interpretation implies our result. □

2.3.2 The generalized factorization matrix entries

Theorem 2.18. *For fixed $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$ such that $\alpha, \gamma \geq 1$, $1 \leq \beta < \alpha$, and $1 \leq \delta < \gamma$, the pair $(\mathcal{C}(q), s_{n,k}[\mathcal{C}](\alpha, \beta, \gamma, \delta))$ in the generalized Lambert series factorization expanded by*

$$L_f(\alpha, \beta, \gamma, \delta; q) := \sum_{n \geq 1} \frac{f(n) q^{\alpha n + \beta}}{1 - q^{\gamma n + \delta}} = \frac{1}{\mathcal{C}(q)} \times \sum_{n \geq 1} \sum_{k=1}^{n} s_{n,k}[\mathcal{C}](\alpha, \beta, \gamma, \delta) f(k) q^n, \tag{i}$$

satisfies

$$s_{n,k}[\mathcal{C}](\alpha, \beta, \gamma, \delta) = [q^n] \left(\mathcal{C}(q) \times \frac{q^{\alpha n + \beta}}{1 - q^{\gamma n + \delta}} \right).$$

Proof of Theorem 2.18. We begin by rewriting (i) in the form of

$$\mathcal{C}(q) \times \sum_{k \geq 1} \frac{f(k) q^{\alpha k + \beta}}{1 - q^{\gamma k + \delta}} = \sum_{k \geq 1} \left(\sum_{n \geq 1} s_{n,k}[\mathcal{C}](\alpha, \beta, \gamma, \delta) q^n \right) f(k).$$

Then if we equate the coefficients of a_k in the previous equation, we see that

$$\mathcal{C}(q) \times \frac{q^{\alpha k + \beta}}{1 - q^{\gamma k + \delta}} = \sum_{n \geq 1} s_{n,k}[\mathcal{C}](\alpha, \beta, \gamma, \delta) q^n,$$

which implies the stated result. □

Corollary 2.19. *Let $\alpha \geq 1$ and $0 \leq \beta < \alpha$ be integers and suppose that $\delta \in \mathbb{Z}$. Suppose that*

$$\sum_{n \geq 1} \frac{f(n) q^n}{1 - q^n} = \frac{1}{\mathcal{C}(q)} \times \sum_{n \geq 0} \sum_{k=1}^{n} s_{n,k}[\mathcal{C}] f(k) q^n,$$

and that

$$\sum_{n \geq 1} \frac{f(n) q^{\alpha n - \beta + \delta}}{1 - q^{\alpha n - \beta}} = \frac{1}{\mathcal{C}(q)} \times \sum_{n \geq 0} \sum_{k=1}^{n} s_{n,k}[\mathcal{C}](\alpha, \beta; \delta) f(k) q^n.$$

Then we have that

$$s_{n,k}[\mathcal{C}](\alpha, \beta; \delta) = s_{n-\delta, \alpha k - \beta}[\mathcal{C}].$$

Proof of Corollary 2.19. The proof follows from the second statement in the last theorem, which provides a generating function for $s_{n,k}[\mathcal{C}]$ for all $n, k \geq 1$. $\square$

Given the difficulty in proving that which find numerically holds for all $n \geq k \geq 1$, we will cite some of the interesting conjectures for properties satisfied by the factorization matrices for a particular degenerate LGF case. Generalizations of this property are not as immediately apparent, though do seem to degrade nicely for $\alpha \geq 3$ based on preliminary computational inspection.

Conjecture 2.20 (Invertible matrix factorizations of degenerate LGF cases). *For $1 \leq k \leq n$, the factorization matrix inverse in the expansion of the degenerate LGF $L_f(1, 0, 2, 1; q)$ satisfies the following relation:*

$$\begin{aligned}
s_{n,k}^{-1}(1,0,2,1) = p(n-k) &- \sum_{i=1}^{n} p\left(\frac{n-i}{2i+1} - k\right)[n \equiv i \bmod 2i+1]_\delta \\
&+ \sum_{m=2}^{n}\sum_{i=1}^{n} p\left(\frac{n-p(m+1)i-p(m-1)}{p(m+1)(2i+1)} - k\right) \times \\
&\quad \times [n \equiv p(m+1)i + p(m-1) \bmod p(m+1)(2i+1)]_\delta .
\end{aligned} \tag{19}$$

Theorem 2.21 (Another generalized factorization theorem). *Define the factorization pair $(\mathcal{C}(q), s_{n,k}[\mathcal{C}](\gamma))$ where $\mathcal{C}(0) \neq 0$ by the requirement that*

$$s_{n,k}^{-1}[\mathcal{C}](\gamma) := \sum_{d|n} [q^{d-k}] \frac{1}{C(q)} \times \gamma\left(\frac{n}{d}\right),$$

for some fixed arithmetic functions $\gamma(n)$ where $\widetilde{\gamma}(n) := \sum_{d|n} \gamma(d)$. We have that the sequence of $\bar{f}_n$ in the notation of (18) is given by the following formula for $n \geq 1$:

$$\bar{f}_n = \sum_{\substack{d|n \\ d \equiv \beta \bmod \alpha}} f\left(\frac{d-\beta}{\alpha}\right) \widetilde{\gamma}\left(\frac{n}{d}\right).$$

Proof of Theorem 2.21. We begin by noticing that

$$\bar{f}_n = \sum_{k=1}^{n} s_{n,k}^{(-1)} \times [q^k]\left(\sum_{d=1}^{k} \frac{f(d) q^{\alpha d+\beta}}{1-q^{\alpha d+\beta}}\right).$$

Then if we let the series coefficients $c_n := [q^n]\mathcal{C}(q)^{-1}$ and set

$$t_{k,d}(\alpha,\beta) := [q^k]\left(\mathcal{C}(q) \times \frac{q^{\alpha d+\beta}}{1-q^{\alpha d+\beta}}\right),$$

where $t_{i,d}(\alpha, \beta) = 0$ whenever $i < \alpha d + \beta$, we have that for each $1 \leq d \leq n$

$$\begin{aligned}
[a_d]\bar{f}_n &= \sum_{k=1}^{n} s_{n,k}^{-1}[\mathcal{C}](\gamma) t_{k,d}(\alpha,\beta) \\
&= \sum_{k=d}^{n}\left(\sum_{r|n} c_{r-k}\gamma\left(\frac{n}{r}\right)\right) t_{k,d}(\alpha,\beta) \\
&= \sum_{r|n}\left(\sum_{i=d}^{r} c_{r-i} t_{i,d}(\alpha,\beta)\right)\gamma\left(\frac{n}{r}\right)
\end{aligned}$$

$$= \sum_{r|n} \left(\sum_{i=0}^{r} c_{r-i} t_{i,d}(\alpha, \beta) \right) \gamma \left(\frac{n}{r} \right),$$

The inner sums in the previous equations are generated by

$$[q^r] \frac{1}{\cancel{\mathcal{C}(q)}} \cdot \frac{q^{\alpha d+\beta}}{1 - q^{\alpha d+\beta}} \cancel{\mathcal{C}(q)} = [\alpha d + \beta | r]_{\delta}.$$

Then we have that for integers $d \geq 1$

$$[f(d)] \bar{f}_n = \sum_{\substack{r|n \\ \alpha d+\beta|r}} \gamma \left(\frac{n}{r} \right),$$

and that

$$\begin{aligned} \bar{f}_n &= \sum_{\substack{d|n \\ d \equiv \beta \bmod \alpha}} f\left(\frac{d-\beta}{\alpha} \right) \sum_{\substack{r|n \\ d|r}} \gamma \left(\frac{n}{r} \right) \\ &= \sum_{\substack{d|n \\ d \equiv \beta \bmod \alpha}} f\left(\frac{d-\beta}{\alpha} \right) \sum_{r|\frac{n}{d}} \gamma \left(\frac{n}{dr} \right). \square \end{aligned}$$

Example 2.22 (A new identity for the sum of squares function)**.** We have that

$$\begin{aligned} &\sum_{\substack{d|n \\ d \text{ odd}}} (-1)^{(d+1)/2} \left(r_2\left(\frac{n}{d} \right) - 4d\left(\frac{n}{2d} \right) \left[\frac{n}{d} \text{ even} \right]_{\delta} \right) \\ &\qquad = \sum_{k=1}^{n} \sum_{d|n} p(d-k)(-1)^{n/d+1} \times [q^k](q;q)_{\infty} \vartheta_3(q)^2, \end{aligned}$$

where $d(n) \equiv \sigma_0(n)$ denotes the divisor function and where the powers of the Jacobi theta function, $\vartheta_3(q) = 1 + 2\sum_{n \geq 1} q^{n^2}$, generate the series over the sums of squares functions, $r_k(n) = [q^n]\vartheta_3(q)^k$.

2.3.3 Factorization theorems for classical LGFs over Dirichlet convolutions

Definition 2.23. Given any two arithmetic functions f and g we define their *Dirichlet convolution*, denoted by $h = f * g$, to be the function [4, §2.6]

$$(f * g)(n) := \sum_{d|n} f(d) g\left(\frac{n}{d} \right), n \geq 1.$$

The function $h^{-1}(n)$ is called the *Dirichlet inverse* of $h(n)$ (or inverse of h with respect to Dirichlet convolution) if $h^{-1} * h = h * h^{-1} = \varepsilon$, where $\varepsilon(n) = \delta_{n,1}$ is the multiplicative identity with respect to Dirichlet convolution. An arithmetic function h has a Dirichlet inverse iff $h(1) \neq 0$. If h^{-1} exists, then it is unique and can be computed recursively by the formula

$$h^{-1}(n) = \begin{cases} \frac{1}{h(1)}, & n = 1; \\ -\frac{1}{h(1)} \times \sum\limits_{\substack{d|n \\ d>1}} h(d) h^{-1}\left(\frac{n}{d} \right), & n \geq 2. \end{cases}$$

n	$f^{-1}(n)$	n	$f^{-1}(n)$	n	$f^{-1}(n)$
1	$\frac{1}{h(1)}$	**4**	$-\frac{h(1)h(4)-h(2)^2}{h(1)^3}$	**7**	$-\frac{h(7)}{h(1)^2}$
2	$-\frac{h(2)}{h(1)^2}$	**5**	$-\frac{h(5)}{h(1)^2}$	**8**	$-\frac{h(2)^3-2h(1)h(4)h(2)+h(1)^2h(8)}{h(1)^4}$
3	$-\frac{h(3)}{h(1)^2}$	**6**	$-\frac{h(1)h(6)-2h(2)h(3)}{h(1)^3}$	**9**	$-\frac{h(1)h(9)-h(3)^2}{h(1)^3}$

Table 2.2: *The first few cases of a Dirichlet invertible h evaluated symbolically. Recall that an arithmetic function h is Dirichlet invertible if and only if $h(1) \neq 0$. When the Dirichlet inverse of the function h exists, it is unique.*

Proposition 2.24 (One possible factorization)**.** *Let f and g denote non-identically-zero arithmetic functions. Suppose that we have an ordinary Lambert series factorization for any prescribed arithmetic function $f(n)$ of the form*

$$\sum_{n\geq 1} \frac{f(n)q^n}{1-q^n} = \frac{1}{\mathcal{C}(q)} \times \sum_{n\geq 1}\left(\sum_{k=1}^{n} s_{n,k}[\mathcal{C}]f(k)\right) q^n, \tag{i}$$

A factorization theorem variant for the Lambert series over the convolution function $h = f * g$ is expanded as follows:

$$\sum_{n\geq 1} \frac{(f*g)(n)q^n}{1-q^n} = \frac{1}{\mathcal{C}(q)} \times \sum_{n\geq 1}\left(\sum_{k=1}^{n} \widetilde{s}_{n,k}[\mathcal{C}](g)f(k)\right) q^n. \tag{ii}$$

The matrix coefficients in the previous equation satisfy

$$\widetilde{s}_{n,k}[\mathcal{C}](g) = \sum_{j=1}^{n} s_{n,kj}[\mathcal{C}]g(j). \tag{iii}$$

Proof of Proposition 2.24. It is apparent by the expansions on the left-hand side of (ii) that there is some sequence of $\widetilde{s}_{n,k}[\mathcal{C}](g)$ depending on the function g that satisfies the factorization of the form in (i) when $a_n \mapsto (f*g)(n)$. For a fixed $k \geq 1$, we begin by evaluating the coefficients of $f(k)$ on the right-hand side of (ii) as follows:

$$\begin{aligned}
[f(k)]\left(\sum_{n\geq 1}\sum_{i=1}^{n} s_{n,i}[\mathcal{C}](f*g)(i)q^n\right) &= \sum_{n\geq 1}\left(\sum_{i=1}^{n}[f(k)]s_{n,i}[\mathcal{C}]\sum_{d\mid i} f(d)g\left(\frac{i}{d}\right)\right) q^n \\
&= \sum_{n\geq 1}\left(\sum_{i=1}^{n} s_{n,i}[\mathcal{C}]g\left(\frac{i}{d}\right)[k|i]_{\delta}\right) q^n \\
&= \sum_{n\geq 1}\left(\sum_{j=1}^{n} s_{n,kj}[\mathcal{C}]g(j)\right) q^n.
\end{aligned}$$

Thus we have that the formula for $\widetilde{s}_{n,k}[\mathcal{C}](g)$ given in equation (iii) holds. □

Discussion 2.25 (Formulating some intuition for the forms of the resulting inverse matrices)**.** The next examples suggest insight that is gathered by computation and experimental mathematics on the inverse matrix entry dataset with *Mathematica*. Consider factorizing the Lambert series generating function of $f * g$:

$$\sum_{n\geq 1} \frac{(f*g)(n)q^n}{1-q^n} = \frac{1}{(q;q)_\infty} \times \sum_{n\geq 1}\left(\sum_{k=1}^{n} \widetilde{s}_{n,k}(g)f(k)\right) q^n.$$

n \ k	1	2	3	4	5	6
1	$\frac{1}{g(1)}$	0	0	0	0	0
2	$-\frac{g(2)}{g(1)^2}$	$\frac{1}{g(1)}$	0	0	0	0
3	$\frac{g(1)^5-g(1)^4g(3)}{g(1)^6}$	$\frac{1}{g(1)}$	$\frac{1}{g(1)}$	0	0	0
4	$\frac{2g(1)^5-g(4)g(1)^4+g(2)^2g(1)^3}{g(1)^6}$	$\frac{g(1)^5-g(1)^4g(2)}{g(1)^6}$	$\frac{1}{g(1)}$	$\frac{1}{g(1)}$	0	0
5	$\frac{4g(1)^5-g(1)^4g(5)}{g(1)^6}$	$\frac{3}{g(1)}$	$\frac{2}{g(1)}$	$\frac{1}{g(1)}$	$\frac{1}{g(1)}$	0
6	$\frac{5g(1)^5-g(2)g(1)^4-g(6)g(1)^4+2g(2)g(3)g(1)^3}{g(1)^6}$	$\frac{3g(1)^5-g(2)g(1)^4-g(3)g(1)^4}{g(1)^6}$	$\frac{2g(1)^5-g(1)^4g(2)}{g(1)^6}$	$\frac{2}{g(1)}$	$\frac{1}{g(1)}$	$\frac{1}{g(1)}$

Table 2.3

n \ k	1	2	3	4	5	6	7	8	9	10
1	1	0	0	0	0	0	0	0	0	0
2	$-g(2)$	1	0	0	0	0	0	0	0	0
3	$1-g(3)$	1	1	0	0	0	0	0	0	0
4	$g(2)^2-g(4)+2$	$1-g(2)$	1	1	0	0	0	0	0	0
5	$4-g(5)$	3	2	1	1	0	0	0	0	0
6	$2g(3)g(2)-g(2)-g(6)+5$	$-g(2)-g(3)+3$	$2-g(2)$	2	1	1	0	0	0	0
7	$10-g(7)$	7	5	3	2	1	1	0	0	0
8	$-g(2)^3+2g(4)g(2)-2g(2)-g(8)+12$	$g(2)^2-g(2)-g(4)+9$	$6-g(2)$	$4-g(2)$	3	2	1	1	0	0
9	$g(3)^2-g(3)-g(9)+20$	$14-g(3)$	$10-g(3)$	7	5	3	2	1	1	0
10	$2g(5)g(2)-4g(2)-g(10)+25$	$-3g(2)-g(5)+18$	$13-2g(2)$	$10-g(2)$	$6-g(2)$	5	3	2	1	1

Table 2.4

Then we can prove that $\widetilde{s}_{n,k}(g)=\sum_{j=1}^{n} s_{n,kj}g(j)$. The pattern that characterizes the inverse matrix entries is less obvious to see immediately. Consider the listing of the first several entries of the inverse matrix sequence, $\widetilde{s}_{n,k}^{-1}(g)$, given in Table 2.3. Looking closely at the entries in the last table suggests that the terms are partition-scaled multiple (m-fold) Dirichlet convolutions related to the indices (n,k). Upon setting $g(1) \mapsto 1$, the special case makes clear the relation of these matrix inverses, the partition function $p(n)$, and the Dirichlet inverse of the function g we observe in Table 2.4.

Definition 2.26. For a fixed arithmetic function g with $g(1) := 1$, let the functions $\mathrm{ds}_{j,g}(n)$ be defined recursively for natural numbers $j \geq 1$ as

$$\mathrm{ds}_{j,g}(n) := \begin{cases} g_{\pm}(n), & \text{if } j=1; \\ \sum\limits_{\substack{d|n \\ d>1}} g(d)\, \mathrm{ds}_{j-1,g}\left(\frac{n}{d}\right), & \text{if } j>1, \end{cases}$$

where $g_{\pm}(n) := g(n)\,[n>1]_{\delta} - \delta_{n,1}$. Let the notation for the *k-shifted partition function* be defined as $p_k(n) := p(n-k)$ for $k \geq 1$. If we set the function $\widetilde{\mathrm{ds}}_{j,g}(n)$ to be the j-fold convolution of g with itself, i.e., that

$$\widetilde{\mathrm{ds}}_{j,g}(n) = \underbrace{(g_{\pm} * g * \cdots * g)}_{j \text{ times}}(n),$$

then we can prove easily by induction that for all $m,n \geq 1$ we have the expansion

$$\mathrm{ds}_{m,g}(n) = \sum_{i=0}^{m-1} \binom{m-1}{i} (-1)^{m-1-i} \widetilde{\mathrm{ds}}_{i+1,g}(n).$$

We then define the following notation for the sums of the variant convolution functions for use in the theorem below for any $n \geq 1$:

$$D_{n,g}(n) := \sum_{j=1}^{n} \mathrm{ds}_{2j,g}(n) = \sum_{m=1}^{\lfloor \frac{n}{2} \rfloor} \sum_{i=0}^{2m-1} \binom{2m-1}{i} (-1)^{i+1} \widetilde{\mathrm{ds}}_{i+1,g}(n).$$

Theorem 2.27 (Inverse matrix sequences)**.** *Suppose that $g(1) \neq 0$, i.e., that g is Dirichlet invertible. When $C(q) = (q;q)_\infty$ we write $s_{n,k}^{-1}[\mathcal{C}](g) \equiv s_{n,k}^{-1}(g)$. For fixed integers $k \geq 1$, the inverse matrix sequences from Proposition 2.24 satisfy*

$$\begin{aligned} \sum_{d|n} s_{n,k}^{-1}(g) &= p_k(n) + \sum_{j=1}^{\Omega(n)} (p_k * \mathrm{ds}_{2j,g})(n) \\ &= p_k(n) + (p_k * D_{n,g})(n), \end{aligned}$$

Equivalently, we have that

$$\begin{aligned} s_{n,k}^{(-1)}(g) &= (p_k * \mu)(n) + \sum_{j=1}^{\Omega(n)} (p_k * \mathrm{ds}_{2j,g} * \mu)(n) \\ &= (p_k * \mu)(n) + (p_k * D_{n,g} * \mu)(n). \end{aligned}$$

Proof of Theorem 2.27. Let the proposed inverse sequence function be defined in the following notation:

$$\widehat{s}_{n,k}^{(-1)}(g) := (p_k * \mu)(n) + (p_k * D_{n,g} * \mu)(n).$$

We begin as in the proof of Theorem 3.2 in [22] to consider the ordinary, non-convolved Lambert series. More precisely, by the expansion in (i) of the proposition we must show that

$$\sum_{d|n} \widehat{s}_{d,k}^{(-1)} := p_k(n) + (p_k * D_{n,g})(n)$$

$$= \sum_{m=0}^{n} \sum_{j=1}^{n-m} \widetilde{s}_{n-m,j}(g) \widehat{s}_{j,k}^{(-1)} p(m).$$

For integers $n, k, i \geq 1$ with $k, i \leq n$, let the coefficient functions, $\rho_{n,k}^{(i)}$ be defined as

$$\rho_{n,k}^{(i)} := \sum_{j=1}^{n} s_{n,ij} \widetilde{s}_{j,k}^{(-1)}.$$

Then for any fixed arithmetic function h, by considering the related expansions of the factorizations in (ii) of the proposition for $\widehat{s}_k * g$, we can prove that

$$t_{n,k}(h) := \sum_{j=1}^{n} s_{n,j} (\widetilde{s}_{n,k}^{(-1)} * h)(j) = \sum_{i=1}^{n} \rho_{n,k}^{(i)} h(i). \tag{i}$$

It remains to show that

$$\sum_{m=0}^{n} t_{n-m,k}(h) p(m) = (p_k * h)(n). \tag{ii}$$

Since we can expand the left-hand side of the previous sum as

$$\sum_{m=0}^{n} \sum_{i=1}^{n-m} \rho_{n-m,k}^{(i)} h(i) p(m) = \sum_{i=1}^{n} h(i) \underbrace{\left(\sum_{m=0}^{n} \rho_{n-m,k}^{(i)} p(m) \right)}_{:= u_{n,k}^{(i)}},$$

to complete the proof of (ii) we need to prove a subclaim that (I) $u_{n,k}^{(i)} = 0$ if $i \not| n$; and (II) if $i|n$ then $u_{n,k}^{(i)} = p\left(\frac{n}{i} - k\right)$.

Proof of Subclaim: For $i := 1$, this is clearly the case since $\rho_{n-m,k}^{(i)} = [n - m = k]_\delta$. For subsequent cases of $i \geq 2$, it is apparent that

$$\rho_{n,k}^{(i)} = \rho_{n-(k-1)i,1}$$

much as in the cases of the tables for the inverse sequences, $s_{n,k}^{(-1)} = (p_k * \mu)(n)$. Finally, we claim that generating functions for the sequences of $u_{n,k}^{(i)}$ for each $i \geq 2$ are expanded in the form of

$$\sum_{n \geq 0} u_{n,k}^{(i)} \cdot q^n = \prod_{j=1}^{i-1} (q^j; q^i)_\infty \times \frac{q^{ik}}{(q;q)_\infty} = q^{ik} \times \sum_{n \geq 0} p(n) q^{in},$$

which we see by comparing coefficients on the right-hand side of the previous equation implies our claim.

Completing the Proof of the Inverse Formula: What we have shown by proving (ii) above is an inverse formula for an ordinary Lambert series factorization over the sequence of $a_j := (\widetilde{s}_{n,k}^{(-1)} * g)(j)$. In particular, by Möbius inversion (ii) shows that we have

$$\begin{aligned} (f * g)(n) = (s_{n,k}^{(-1)} * g)(n) &\iff (f - s_{n,k}^{(-1)}) * g \equiv 0 \\ &\implies f_n = s_{n,k}^{(-1)}, \text{ when } g \not\equiv 0. \end{aligned}$$

More to the point, when we define $f_n := \widetilde{s}_{n,k}^{(-1)}(g)$ where by convenience and experimental suggestion we let

$$\widetilde{s}_{n,k}^{(-1)}(g) = s_{n,k}^{(-1)} * t_{n,k}^{(-1)}(g),$$

for some convolution-wise factorization of this inverse sequence, we can now prove the exact formula for the inverse sequence claimed in the theorem statement. In the forward direction, we suppose that

$$t_{n,k}^{(-1)}(g) = D_{n,g}(n) + \varepsilon(n),$$

where $\varepsilon(n) = \delta_{n,1}$ is the multiplicative identity, and then see from the formulas for $D_{n,g}(n)$ discussed before the claim that $g * (D_{n,g} + \varepsilon) = \varepsilon$, which proves that our inverse formula is correct in this case. Conversely, if we require that

$$s_{n,k}^{(-1)} * t_{n,k}^{(-1)}(g) * g = s_{n,k}^{(-1)}$$

for all n and choices of the function g, we must have that $t_{n,k}^{(-1)} * g = \varepsilon$, and so we see that $t_{n,k}^{(-1)} = D_{n,g} + \varepsilon$ as required. That is to say, we have proved our result using the implicit statement that $t * g = \varepsilon$ if and only if $t = D_{n,g} + \varepsilon$, i.e., that $t = D_{n,g} + \varepsilon$ is the unique function such that $t * g = \varepsilon$ for all $n \geq 1$, a result which we do not prove here and only mention for the sake of brevity. □

Corollary 2.28 (Formulas for the Dirichlet inverse)**.** *For any arithmetic function f defined such that $f(1) = 1$, we have a formula for its Dirichlet inverse function given by*

$$f^{-1}(n) = \sum_{k=1}^{n} ((p_k * \mu)(n) + (p_k * D_{n,f} * \mu)(n)) \times [q^{k-1}] \frac{(q;q)_\infty}{1-q}.$$

Proof of Corollary 2.28. The proof follows from Theorem 2.27 applied in the form of Proposition 2.24. In particular, since $f^{-1} * f = \delta_{n,1}$ by definition, the right-hand side of our Lambert series expansion over the convolved function $a_n := f^{-1} * f$ is given by $q(1-q)^{-1}$. □

Remark 2.29. We also note that given any sequence $b(n)$, we can generate $b(n)$ by the Lambert series over $b * \mu$. This implies that we have recurrence relations for any arithmetic function b defined such that $b(n) = 0$ for all $n < 0$ expanded in the following two forms where $s_{n,k} := [q^n](q;q)_\infty q^k (1-q^k)^{-1}$:

$$b(n) = \sum_{k=1}^{n} (p_k * \mu + p_k * D_{n,\mu} * \mu)(n) \left(b(k) + \sum_{s=\pm 1} \sum_{j=1}^{k} (-1)^j b\left(k - \frac{j(3j+s)}{2}\right)\right),$$

$$b(n) = \sum_{j=1}^{n} \sum_{k=1}^{j} \left(\sum_{i=1}^{\lfloor \frac{j}{k} \rfloor} s_{j,ki} \cdot \mu(i) \right) b(k) p(n-j).$$

Corollary 2.30 (Convolution formulas for arbitrary arithmetic functions)**.** *Suppose that we have any two arithmetic functions f and h and we seek the form of a third function g satisfying $f * g = h * \mu$ for all $n \geq 1$. Then we have a formula for the function g expanded in the form*

$$\begin{aligned} g(n) = \sum_{k=1}^{n} &((p_k * \mu)(n) + (p_k * D_{n,f} * \mu)(n)) \times \\ &\times \left(h(k) + \sum_{s=\pm 1} \sum_{j=1}^{\left\lfloor \frac{\sqrt{24k+1}-s}{6} \right\rfloor} (-1)^j h\left(k - \frac{j(3j+s)}{2}\right)\right). \end{aligned}$$

Proof. This result is an immediate consequence of Proposition 2.24 and the formula for the inverse sequences defined by Theorem 2.27. □

2.4 Hadamard products and derivatives of LGFs

Definition 2.31 (Hadamard products for Lambert series generating functions)**.** For any fixed arithmetic functions f and g, we define the Hadamard product of the two Lambert series over f and g to be the auxiliary Lambert series generating function over the composite function $a_{\mathrm{fg}}(n)$ whose coefficients are given by

$$\sum_{d|n} a_{\mathrm{fg}}(d) = [q^n] \sum_{m\geq 1} \frac{a_{\mathrm{fg}}(m)q^m}{1-q^m} := \underbrace{\left(\sum_{d|n} f_d\right) \times \left(\sum_{d|n} g_d\right)}_{:=\mathrm{fg}(n)}.$$

By Möbius inversion we have that

$$a_{\mathrm{fg}}(n) = \sum_{d|n} \mathrm{fg}(d)\mu\left(\frac{n}{d}\right) = (\mathrm{fg} * \mu)(n).$$

We also define the following function to expand divisor sums over arithmetic functions as ordinary sums for any integers $1 \leq k \leq n$:

$$T_{\mathrm{div}}(n,k) := \begin{cases} 1, & \text{if } k|n; \\ 0, & \text{otherwise.} \end{cases}$$

The results in this subsection are found in the unpublished manuscript [38]. The theorems we present here provide some useful extensions of the factorization theorems and related constructions we have cited in the previous subsections so far. The next theorems in this section define the key matrix sequences, $h_{n,k}(f)$ and $h_{n,k}^{-1}(f)$, in terms of the next factorization of the Lambert series over $a_{\mathrm{fg}}(n)$ in the form of

$$\sum_{n\geq 1} \frac{a_{\mathrm{fg}}(n)q^n}{1-q^n} = \frac{1}{(q;q)_\infty} \times \sum_{n\geq 1}\sum_{k=1}^{n} h_{n,k}(f) g_k q^n, \tag{20}$$

where the matrix entries $h_{n,k}(f)$ are independent of the function g. This expansion is equivalent to defining the factorization expansion by the inverse matrix sequences as

$$g_n = \sum_{k=1}^{n} h_{n,k}^{-1}[q^k]\left((q;q)_\infty \times \sum_{n\geq 1} \frac{a_{\mathrm{fg}}(n)q^n}{1-q^n}\right). \tag{21}$$

Theorem 2.32. *For all integers $1 \leq k \leq n$, we have the following definition of the factorization matrix sequence defining the expansion on the right-hand-side of* (20) *where we adopt the notation $\widetilde{f}(n) := \sum_{d|n} f_d$:*

$$\begin{aligned} h_{n,k}(f) &= T_{\mathrm{div}}(n,k)\widetilde{f}(n) \\ &\quad + \sum_{b=\pm 1} \sum_{j=1}^{\left\lfloor \frac{\sqrt{24(n-k)+1}-b}{6} \right\rfloor} (-1)^j T_{\mathrm{div}}\left(n - \frac{j(3j+b)}{2}, k\right) \widetilde{f}\left(n - \frac{j(3j+b)}{2}\right). \end{aligned}$$

Proof. By the factorization in (20) and the definition of $a_{\mathrm{fg}}(n)$ given above, we have that for $\widetilde{f}(n) = \sum_{d|n} f_d$

$$\begin{aligned} h_{n,k}(f) &= [g_k]\left(\sum_{d|n} f_d\right) \times \sum_{d=1}^{n} g_d T_{\mathrm{div}}(n,d), \\ &= [q^n](q;q)_\infty \times \sum_{n\geq 1} T_{\mathrm{div}}(n,k)\widetilde{f}(n)q^n. \end{aligned}$$

The last equation gives the stated expansion of the sequence by Euler's pentagonal number theorem. The statement of the pentagonal number theorem is that for $|q| < 1$

$$(q;q)_\infty = 1 + \sum_{j \geq 1} (-1)^j \left(q^{\frac{j(3j-1)}{2}} + q^{\frac{j(3j+1)}{2}}\right).$$

The theorem statement follows thusly. □

Theorem 2.33 (Inverse matrix sequences)**.** *For all integers $1 \leq k \leq n$, we have the next definition of the inverse factorization matrix sequence which equivalently defines the expansion on the right-hand-side of* (20)*.*

$$h_{n,k}^{(-1)}(f) = \sum_{d|n} \frac{p(d-k)}{\widetilde{f}(d)} \mu\left(\frac{n}{d}\right).$$

Proof of Theorem 2.33. We expand the right-hand-side of the factorization in (20) for the sequence $g_n := h_{n,r}^{(-1)}(f)$, i.e., the exact inverse sequence, for some fixed $r \geq 1$ as follows:

$$\begin{aligned} \widetilde{f}(n) \times \sum_{d|n} h_{d,r}^{(-1)}(f) &= \sum_{j=0}^{n} \sum_{k=1}^{j} h_{j,k} h_{k,r}^{(-1)} p(n-j) \\ &= \sum_{j=0}^{n} [j = r]_\delta \, p(n-j) = p(n-r). \end{aligned}$$

Then the last equation implies that

$$\sum_{d|n} h_{d,r}^{-1}(f) = \frac{p(n-r)}{\widetilde{f}(n)},$$

which by Möbius inversion implies our stated result. □

Example 2.34 (Applications)**.** If we form the Hadamard product of generating functions of the two Lambert series over Euler's totient function, $\phi(n)$, we obtain the following more exotic-looking sum for our multiplicative function of interest:

$$\phi(n) = \sum_{k=1}^{n} \sum_{d|n} \frac{p(d-k)}{d} \mu\left(\frac{n}{d}\right) \left[k^2 + \sum_{b=\pm 1} \sum_{j=1}^{\left\lfloor \frac{\sqrt{24k-23}-b}{6} \right\rfloor} (-1)^j \left(k - \frac{j(3j+b)}{2}\right)^2\right].$$

We consider the arithmetic function pairs

$$(f, g) := (n^t, n^s), (\phi(n), \Lambda(n)), (n^t, J_t(n)),$$

respectively, and some constants $s, t \in \mathbb{C}$ where $\sigma_\alpha(n)$ denotes the generalized sum-of-divisors function, $\Lambda(n)$ is von Mangoldt's function, $\phi(n)$ is Euler's totient function, and $J_t(n)$ is the Jordan totient function. We then employ the equivalent expansions of the factorization result in (21) to formulate the following "*exotic*" *sums* as consequences of the theorems above:

$$\begin{aligned} n^s = \sum_{k=1}^{n} \sum_{d|n} \frac{p(d-k)}{\sigma_t(d)} \mu\left(\frac{n}{d}\right) \Bigg[& \sigma_t(k)\sigma_s(k) \\ & + \sum_{b=\pm 1} \sum_{j=1}^{\left\lfloor \frac{\sqrt{24k+1}-b}{6} \right\rfloor} (-1)^j \sigma_t\left(k - \frac{j(3j+b)}{2}\right) \sigma_s\left(k - \frac{j(3j+b)}{2}\right)\Bigg], \end{aligned} \tag{22}$$

$$\Lambda(n) = \sum_{k=1}^{n} \sum_{d|n} \frac{p(d-k)}{d} \mu\left(\frac{n}{d}\right) \Bigg[k \log(k)$$
$$+ \sum_{b=\pm 1} \sum_{j=1}^{\left\lfloor \frac{\sqrt{24k-23}-b}{6} \right\rfloor} (-1)^j \left(k - \frac{j(3j+b)}{2}\right) \log\left(k - \frac{j(3j+b)}{2}\right)\Bigg],$$
$$J_t(n) = \sum_{k=1}^{n} \sum_{d|n} \frac{p(d-k)}{d^t} \mu\left(\frac{n}{d}\right) \left[k^{2t} + \sum_{b=\pm 1} \sum_{j=1}^{\left\lfloor \frac{\sqrt{24k-23}-b}{6} \right\rfloor} (-1)^j \left(k - \frac{j(3j+b)}{2}\right)^{2t}\right].$$

By forming a second sum over the divisors of n on both sides of the first equation above, the first more exotic-looking sum for the sum-of-divisors functions leads to an expression for $\sigma_s(n)$ as a sum over the paired pointwise products, $\sigma_t(n)\sigma_s(n)$. We are not aware of another identity like the first equation in (22) relating Hadamard products of the generalized sum-of-divisors functions in a review of surrounding literature.

Corollary 2.35 (The Riemann zeta function)**.** *For fixed $s, t \in \mathbb{C}$ such that $\Re(s) > 1$, we have the following representation of the Riemann zeta function:*

$$\zeta(s) = \sum_{n \geq 1} \sum_{k=1}^{n} \sum_{d|n} \frac{p(d-k)}{\sigma_t(d)} \mu\left(\frac{n}{d}\right) \times \sum_{\substack{j \geq 0 \\ G_j < k}} \frac{(-1)^{\lceil \frac{j}{2} \rceil} \sigma_t(k-G_j)\sigma_s(k-G_j)}{(k-G_j)^s}.$$

The sequence of interleaved pentagonal numbers is denoted by $G_j = \frac{1}{2}\left\lceil \frac{j}{2} \right\rceil \left\lceil \frac{3j+1}{2} \right\rceil$ for $j \geq 0$ [53, A001318].

Proof. These two identities follow as special cases of the theorem in the form of (22) above where we note that the symmetry identity for the generalized sum-of-divisors functions which provides that $\sigma_{-\alpha}(n) = \sigma_\alpha(n) n^{-\alpha}$ for all $\alpha \in \mathbb{R}$. The pentagonal number theorem employed in the inner sums depending on j utilizes the classical expansion

$$(q;q)_\infty = \sum_{j \geq 0} (-1)^{\lceil \frac{j}{2} \rceil} q^{G_j}, |q| < 1.$$

The convergence of these infinite series is guaranteed by our hypothesis that $\Re(s) > 1$. □

In the unpublished manuscript [38], we have other variants of factorization type theorems corresponding to derivatives of Lambert series generating functions and the identities we proved in [47].

3 Factorization theorems for GCD type sums

This section summarizes the work of Mousavi and Schmidt published in the *Ramanujan Journal* in 2020 (completed for review by 2018) [27]. The article provides analogs of the factorization theorems for Lambert series generating functions studied so far to other combinatorially relevant sums. Our new results here provide generating function expansions for the type I and type II sums in the form of matrix-based factorization theorems. These new results characterizing the expansions of the OGFs of these sum types are analogous in many ways to the LGF cases studied in the last section. The matrix products involved in expressing the coefficients of these generating functions for arbitrary arithmetic functions f and g are also closely related to the partition function $p(n)$ (*cf.* the discussion motivated in Section 5).

The known Lambert series factorization theorems proved in the references, and which are summarized in the previous sections on variants above demonstrate the flavor of the matrix-based expansions of these forms for ordinary divisor sums of the form $(f * 1)(n) = \sum_{d|n} f(d)$. Our extensions of these factorization theorem in the context of the forms of the type I and type II sums similarly relate special arithmetic functions from number theory to partition functions and more additive branches of mathematics. The last results proved in Section 3.3 are expanded in the spirit of these matrix factorization constructions using discrete Fourier transforms of functions evaluated at greatest common divisors. We pay special attention to illustrating our new results with many relevant examples and new identities.

Definition 3.1. For any arithmetic functions f and g, the classes of sums termed *type I* and *type II* sums by the authors are defined in respective order as follows:

$$\begin{aligned} T_f(x) &= \sum_{\substack{d=1 \\ (d,x)=1}}^{x} f(d), x \geq 1 \qquad (23) \\ L_{f,g,k}(x) &= \sum_{d|(k,x)} f(d) g\left(\frac{x}{d}\right), x \geq k \geq 1, x \geq 1. \end{aligned}$$

We seek to write the type I, or GCD type sums, denoted by $T_f(x)$ for any arithmetic function f and $x \geq 2$, as the coefficients of the following matrix factorized OGF expansion:

$$T_f(x) = [q^x]\left(\frac{1}{(q;q)_\infty} \times \sum_{n\geq 2}\sum_{k=1}^{n} t_{n,k} f(k) q^n + f(1) q\right). \qquad (24)$$

For an indeterminate parameter w, we seek to factorize the type-II, or Anderson-Apostol like sums, $L_{f,g,k}(n)$, according to the expansions

$$g(x) = [q^x]\left(\frac{1}{(q;q)_\infty} \times \sum_{n\geq 2}\sum_{k=1}^{n} u_{n,k}(f,w)\left(\sum_{m=1}^{k} L_{f,g,m}(k) w^m\right) q^n\right), \ w \in \mathbb{C} \setminus \{0\}. \qquad (25)$$

The sequence $u_{n,k}(f,w)$ is lower triangular and invertible for suitable choices of the indeterminate parameter w. For a fixed $N \geq 1$, we can truncate these sequences after N rows and form the $N \times N$ matrices whose entries are $u_{n,k}(f,w)$ for $1 \leq n,k \leq N$. The corresponding inverse matrices have terms denoted by $u_{n,k}^{(-1)}(f,w)$. That is to say, for $n \geq 2$, these inverse matrices satisfy

$$\sum_{m=1}^{n} L_{f,g,m}(k) w^m = \sum_{k=1}^{n} u_{n,k}^{(-1)}(f,w)[q^k]\left((q;q)_\infty \times \sum_{n\geq 1} g(n) q^n\right), \ w \in \mathbb{C} \setminus \{0\}. \qquad (26)$$

3.1 Factorization theorems for a class of GCD sums (type I sums)

Since the resulting matrices with entries $t_{n,k}$ are lower triangular and invertible, we then obtain that

$$f(n) = \sum_{k=1}^{n} t_{n,k}^{(-1)} [q^k](q;q)_\infty \times \left(\sum_{n \geq 1} T_f(n) q^n\right), n \geq 1. \tag{27}$$

In the next results, the function $\chi_{1,k}(n)$ refers to the principal Dirichlet character modulo k for some $k \geq 1$.

Theorem 3.2 (Exact formulas for the factorization matrix sequences)**.** *Let lower triangular sequence $t_{n,k}$ be defined by the first expansion in* (24) *above. The corresponding inverse matrix coefficients are denoted by $t_{n,k}^{(-1)}$. For integers $n \geq k \geq 1$, the two lower triangular factorization sequences defining the expansion of* (24) *satisfy exact formulas given by*

$$t_{n,k} = \sum_{j=0}^{n} (-1)^{\lceil \frac{j}{2} \rceil} \chi_{1,k}(n+1-G_j) \left[n - G_j \geq 1\right]_\delta, \tag{i}$$

$$t_{n,k}^{(-1)} = \sum_{d=1}^{n} p(d-k) \mu_{n,d}. \tag{ii}$$

We define the sequence of interleaved pentagonal numbers G_j as in the introduction, and the lower tringular sequence $\mu_{n,k}$ as in the Möbius inversion analog proved in Proposition 3.3 (see below).

Before we prove the main theorem, we provide several key examples that apply these results to formulate new expansions of classical number theoretic functions and polynomials in terms of partition functions. First, we obtain the following identities for Euler's totient function based on our new constructions:

$$\phi(n) = \sum_{j=0}^{n} \sum_{k=1}^{j-1} \sum_{i=0}^{j} p(n-j)(-1)^{\lceil \frac{i}{2} \rceil} \chi_{1,k}(j-k-G_i) \left[j-k-G_i \geq 1\right]_\delta + [n=1]_\delta$$

$$\phi(n) = \sum_{\substack{d=1 \\ (d,n)=1}}^{n} \left(\sum_{k=1}^{d+1} \sum_{i=1}^{d} \sum_{j=0}^{k} p(i+1-k)(-1)^{\lceil \frac{j}{2} \rceil} \phi(k-G_j) \mu_{d,i} \left[k - G_j \geq 1\right]_\delta \right).$$

To give another related example that applies to classical multiplicative functions, recall that we have a known representation for the Möbius function given as an exponential sum in terms of powers of the n^{th} primitive roots of unity of the following form [14, §16.6]:

$$\mu(n) = \sum_{\substack{d=1 \\ (d,n)=1}}^{n} \omega_n^d.$$

The *Mertens function*, $M(x)$, is defined as the summatory function over the Möbius function $\mu(n)$ for all $n \leq x$. Using the definition of the Möbius function as one of our type I sums defined above, we have new expansions for the Mertens function given by (*cf.* Corollary 3.26)

$$M(x) = \sum_{1 \leq k < j \leq n \leq x} \left(\sum_{i=0}^{j} p(n-j)(-1)^{\lceil i/2 \rceil} \chi_{1,k}(j-k-G_i) \left[j-k-G_i \geq 1\right]_\delta \omega_n^k \right).$$

Finally, we can form another related polynomial sum of the type indicated above when we consider that the logarithm of the *cyclotomic polynomials* leads to the sums

$$\log \Phi_n(z) = \sum_{\substack{1 \leq k \leq n \\ (k,n)=1}} \log\left(z - \omega_n^k\right)$$

$$= \sum_{1 \le k < j \le n} \left(\sum_{i=0}^{j} p(n-j)(-1)^{\lceil i/2 \rceil} \chi_{1,k}(j-k-G_i) \left[j-k-G_i \ge 1 \right]_\delta \log\left(z - \omega_n^k\right) \right).$$

3.1.1 Inversion relations

We begin our exploration by expanding an inversion formula which is analogous to Möbius inversion for ordinary divisor sums. We prove the following result which is the analog to the sequence inversion relation provided by the Möbius transform in the context of our sums over the integers relatively prime to n [32, *cf.* §2, §3].

Proposition 3.3 (Inversion formula)**.** *For all $n \ge 2$, there is a unique lower triangular sequence, denoted by $\mu_{n,k}$, which satisfies the next lower triangular inversion relation, i.e., so that $\mu_{n,d} = 0$ whenever $n < d$.*

$$g(n) = \sum_{\substack{d=1 \\ (d,n)=1}}^{n} f(d) \qquad \Longleftrightarrow \qquad f(n) = \sum_{d=1}^{n} g(d+1) \mu_{n,d}. \tag{28a}$$

Moreover, if we form the matrix $(\mu_{i,j} \left[j \le i \right]_\delta)_{1 \le i,j \le n}$ for any $n \ge 2$, we have that the inverse sequence satisfies

$$\mu_{n,k}^{(-1)} = \left[(n+1, k) = 1 \right]_\delta \left[k \le n \right]_\delta. \tag{28b}$$

Proof of Proposition 3.3. Consider the $(n-1) \times (n-1)$ matrix

$$(\left[(i, j-1) = 1 \text{ and } j \le i \right]_\delta)_{1 \le i,j < n}, \tag{29}$$

which effectively corresponds to the formula on the left-hand-side of (28a) by applying the matrix to the vector of $[f(1)\ f(2)\ \cdots f(n)]^T$ and extracting the $(n+1)^{th}$ column of the matrix formed by extracting the $\{0,1\}$-valued coefficients of $f(d)$. Since $\gcd(i, j-1) = 1$ for all $i = j$ with $i, j \ge 1$, we see that the matrix (29) is lower triangular with ones on its diagonal. Thus the matrix is non-singular and its unique inverse, which we denote by $(\mu_{i,j})_{1 \le i,j < n}$, leads to the sum on the right-hand-side of the sum in (28a) when we shift $n \mapsto n+1$. The second equation stated in (28b) restates the form of the first matrix of $\mu_{i,j}$ as on the right-hand-side of (28a). ☐

Remark 3.4. We do not know of a comparatively simple closed-form function for the sequence of $\mu_{n,k}$ which is defined recursively by

$$\begin{cases} \mu_{n,k} = 0, & \text{if } n \le 0 \text{ or } k > n; \\ \mu_{n,1} = 1, & \text{if } n = 1; \\ \sum\limits_{\substack{2 \le j \le n+1 \\ (n+1,j)=1}} \mu_{n+1-k,j} = -\chi_{1,k}(n+1), & \text{otherwise.} \end{cases}$$

On the other hand, we can readily see by construction that the sequence and its inverse satisfy

$$\sum_{\substack{d=1 \\ (d,n)=1}}^{n} \mu_{d,k} = 0$$

$$\sum_{\substack{d=1 \\ (d,n)=1}}^{n} \mu_{d,k}^{(-1)} = \phi(n),$$

where $\phi(n)$ is Euler's totient function. The first columns of the $\mu_{n,1}$ appear in the online integer sequences database (OEIS) as the entry [53, A096433].

1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
−1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
−1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	−1	−1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
−1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	−1	−1	1	0	0	0	0	0	0	0	0	0	0	0
1	0	−1	0	−1	0	1	0	0	0	0	0	0	0	0	0	0
−1	0	2	−1	0	0	−1	1	0	0	0	0	0	0	0	0	0
−1	0	0	0	1	0	−1	0	1	0	0	0	0	0	0	0	0
1	0	−1	1	0	−1	1	−1	−1	1	0	0	0	0	0	0	0
−1	0	1	0	0	0	−1	0	0	0	1	0	0	0	0	0	0
1	0	−1	0	0	0	1	0	0	−1	−1	1	0	0	0	0	0
3	0	−2	0	−2	0	2	0	−1	0	−1	0	1	0	0	0	0
−3	0	1	0	3	0	−1	−1	1	0	0	0	−1	1	0	0	0
−1	0	1	0	1	0	−1	0	0	0	0	0	−1	0	1	0	0
1	0	0	0	−2	0	0	1	0	0	1	−1	1	−1	−1	1	0
−3	0	2	0	2	0	−2	0	1	0	0	0	−1	0	0	0	1

$\mu_{n,k}$ for $1 \leq n, k < 18$ (with rows indexed by n and columns by k).

Figure 3.1: *Inversion formula coefficient sequences.*

3.1.2 Exact formulas for the factorization matrices

The next result is key to proving the exact formulas for the matrix sequences, $t_{n,k}$ and $t_{n,k}^{(-1)}$, and their expansions by the partition functions defined in the introduction. We prove the following result first as a lemma which we will use in the proof of Theorem 3.2 given below. The first several rows of the matrix sequence $t_{n,k}$ and its inverse implicit to the factorization theorem in (24) are tabulated in Figure 3.2 for intuition on the formulas we prove in the next proposition and following theorem.

Lemma 3.5 (A convolution identity)**.** *For all natural numbers $n \geq 2$ and $k \geq 1$ with $k \leq n$, we have the following expression for the principal Dirichlet character modulo k:*

$$\sum_{j=1}^{n} t_{j,k} p(n-j) = \chi_{1,k}(n).$$

Equivalently, we have that

$$\begin{aligned} t_{n,k} &= \sum_{i=0}^{n} (-1)^{\lceil \frac{i}{2} \rceil} \chi_{1,k}(n - G_i) \left[n - G_i \geq k+1 \right]_{\delta} \qquad (30) \\ &= \sum_{b=\pm 1} \left[\sum_{i=0}^{\left\lfloor \frac{\sqrt{24(n-k-1)+1}-b}{6} \right\rfloor} (-1)^{\lceil \frac{i}{2} \rceil} \chi_{1,k}\left(n - \frac{i(3i-b)}{2} \right) \right]. \end{aligned}$$

Proof. We begin by noticing that the right-hand-side expression in the statement of the lemma is equal to $\mu_{n,k}^{(-1)}$ by the construction of the sequence in Proposition 3.3. Next, we see that the factorization in (24) is equivalent to the expansion

$$\sum_{d=1}^{n-1} f(d)\mu_{n,d}^{(-1)} = \sum_{j=1}^{n}\sum_{k=1}^{j} p(n-j)t_{j,k}f(k). \tag{31}$$

Since $\mu_{n,k}^{(-1)} = [(n+1,k)=1]_\delta$, we may take the coefficients of $f(k)$ on each side of (31) for each $1 \leq k < n$ to establish the result we have claimed in this lemma. The equivalent statement of the result follows by a generating function argument applied to the product that generates the left-hand-side Cauchy product in (30). □

Theorem 3.2: Proof of (i). It is plain to see by the considerations in our construction of the factorization theorem that both matrix sequences are lower triangular. Thus, we need only consider the cases where $n \leq k$. By a convolution of generating functions, the identity in Lemma 3.5 shows that

$$t_{n,k} = \sum_{j=k}^{n} [q^{n-j}](q;q)_\infty \, [(j+1,k)=1]_\delta \, .$$

Then shifting the index of summation in the previous equation implies (i). □

Theorem 3.2: Proof of (ii). To prove (ii), we consider the factorization theorem when $f(n) := t_{n,r}^{(-1)}$ for some fixed $r \geq 1$. We then expand (24) as

$$\begin{aligned}
\sum_{\substack{d=1 \\ (d,n)=1}}^{n} t_{d,r}^{(-1)} &= [q^n]\frac{1}{(q;q)_\infty} \times \sum_{n\geq 1}\left(\sum_{k=1}^{n-1} t_{n,k}t_{k,r}^{(-1)}\right) q^n \\
&= \sum_{j=1}^{n} p(n-j)\left(\sum_{k=1}^{j-1} t_{j,k}t_{k,r}^{(-1)}\right) \\
&= \sum_{j=1}^{n} p(n-j)\,[r=j-1]_\delta \\
&= p(n-1-r).
\end{aligned}$$

Hence, we may perform the inversion by Proposition 3.3 to the left-hand-side sum in the previous equations to obtain our stated result. □

Remark 3.6 (Relations to the Lambert series factorization theorems)**.** We notice that by inclusion-exclusion applied to the right-hand-side of (24), we may write our matrices $t_{n,k}$ in terms of the triangular sequence expanded as differences of restricted partitions in the ordinary Lambert series factorizations involving the sequence $s_{n,k} := [q^{n-k}]\frac{(q;q)_\infty}{1-q^k}$. For example, when $k := 12$ we see that

$$\sum_{n\geq 12} [(n,12)=1]_\delta \, q^n = \frac{q^{12}}{1-q} - \frac{q^{12}}{1-q^2} - \frac{q^{12}}{1-q^3} + \frac{q^{12}}{1-q^6}.$$

In general, when $k > 1$ we can expand

$$\sum_{n\geq k} [(n,k)=1]_\delta \, q^n = \sum_{d|k} \frac{q^k\mu(d)}{1-q^d}. \tag{32}$$

Thus, we can relate the triangles $t_{n,k}$ in this article to the $s_{n,k} = [q^n](q;q)_\infty q^k(1-q^k)^{-1}$ for $n \geq k \geq 1$ employed in the expansions from the references as follows:

$$t_{n,k} = \begin{cases} s_{n,k}, & k = 1; \\ \sum_{d|k} \mu(d)s_{n+1-k+d,d}, & k > 1. \end{cases}$$

1													
0	1												
−1	−1	1											
−1	0	0	1										
−1	−1	−2	−1	1									
0	0	0	0	0	1								
0	0	0	−1	−1	−1	1							
1	0	−1	0	−1	−1	0	1						
1	1	1	0	−2	0	−1	−1	1					
1	0	1	0	1	1	−1	0	0	1				
1	1	0	1	1	0	−1	−1	−2	−1	1			
1	0	1	0	1	0	0	0	0	0	0	1		
0	1	1	1	1	0	−1	0	0	−1	−1	−1	1	
0	−1	0	0	−1	−1	2	0	−1	−1	−1	−1	0	1

(i) $t_{n,k}$

1												
0	1											
1	1	1										
1	0	0	1									
4	3	2	1	1								
0	0	0	0	0	1							
5	3	2	2	1	1	1						
4	4	3	1	1	1	0	1					
15	11	8	5	4	2	1	1	1				
−1	−1	−1	1	0	0	1	0	0	1			
32	24	18	12	9	6	4	3	2	1	1		
−6	−4	−3	−1	−1	0	0	0	0	0	0	1	
24	17	13	12	8	7	6	3	2	2	1	1	1

(ii) $t_{n,k}^{(-1)}$

Figure 3.2: *The factorization matrices, $t_{n,k}$ and $t_{n,k}^{(-1)}$, for $1 \leq n, k < 14$ (with rows indexed by n and columns by k).*

3.1.3 Completing the proofs of the main applications

We remark that as in the Lambert series factorization results from the references [21], we have three primary types of expansion identities that we will consider for any fixed choice of the arithmetic function f in the forms of

$$\sum_{\substack{d=1 \\ (d,n)=1}}^{n} f(d) = \sum_{j=1}^{n} \sum_{k=1}^{j-1} p(n-j) t_{j-1,k} f(k) + f(1) \left[n=1\right]_{\delta} \tag{33a}$$

$$\sum_{k=1}^{n-1} t_{n-1,k} f(k) = \sum_{j=1}^{n} \left(\sum_{\substack{d=1 \\ (d,j)=1}}^{j} f(d) \times [q^{n-j}](q;q)_\infty \right) - f(1) \times [q^{n-1}](q;q)_\infty, \tag{33b}$$

and the corresponding inverted formula providing that

$$f(n) = \sum_{k=1}^{n} t_{n,k}^{(-1)} \left(\sum_{\substack{j\geq 0 \\ k+1-G_j>0}} (-1)^{\left\lceil \frac{j}{2} \right\rceil} T_f(k+1-G_j) - [q^k](q;q)_\infty f(1) \right). \tag{33c}$$

Now the applications cited in the introduction follow immediately and require no further proof other than to cite these results for the respective special cases of f.

Example 3.7 (Sum-of-divisors functions)**.** For any $\alpha \in \mathbb{C}$, the expansion identity given in (33c) also implies the following new formula for the generalized sum-of-divisors functions, $\sigma_\alpha(n) = \sum_{d|n} d^\alpha$:

$$\sigma_\alpha(n) = \sum_{d|n} \sum_{k=1}^{d} t_{d,k}^{(-1)} \left(\sum_{\substack{j\geq 0 \\ k+1-G_j>0}} (-1)^{\left\lceil \frac{j}{2} \right\rceil} \phi_{\alpha+1}(k+1-G_j) - [q^k](q;q)_\infty \right).$$

In particular, when $\alpha := 0$ we obtain the next identity for the divisor function $d(n) \equiv \sigma_0(n)$ expanded in terms of Euler's totient function, $\phi(n)$.

$$d(n) = \sum_{d|n} \sum_{k=1}^{d} t_{d,k}^{(-1)} \left(\sum_{\substack{j\geq 0 \\ k+1-G_j>0}} (-1)^{\left\lceil \frac{j}{2} \right\rceil} \phi(k+1-G_j) - [q^k](q;q)_\infty \right).$$

Example 3.8 (Menon's identity and related arithmetical sums)**.** We can use our new results proved in this section to expand new identities for known closed-forms of special arithmetic sums. For example, *Menon's identity* [56] states that

$$\phi(n)d(n) = \sum_{\substack{1\leq k\leq n \\ (k,n)=1}} \gcd(k-1,n),$$

where $\phi(n)$ is Euler's totient function and $d(n) = \sigma_0(n)$ is the divisor function. We can then expand the right-hand-side of Menon's identity as follows:

$$\phi(n)d(n) = \sum_{j=0}^{n} \sum_{k=1}^{j-1} \sum_{i=0}^{j} p(n-j)(-1)^{\left\lceil \frac{i}{2} \right\rceil} \chi_{1,k}(j-k-G_i) \, [j-k-G_i \geq 1]_\delta \gcd(k-1,n).$$

As another application, we show a closely related identity considered by Tóth in [56]. Tóth's identity states that (*cf.* [17]) for an arithmetic function f we have

$$\sum_{\substack{1\leq k\leq n \\ (k,n)=1}} f\left(\gcd(k-1,n)\right) = \phi(n) \times \sum_{d|n} \frac{(\mu * f)(d)}{\phi(d)}.$$

We can use our new formulas to write a gcd-related recurrence relation for f in two steps. First, we observe that the right-hand-side divisor sum in the previous equation is expanded by

$$\sum_{d|n} \frac{(\mu * f)(d)}{\phi(d)} = \frac{1}{\phi(n)} \times \sum_{j=0}^{n} \sum_{k=1}^{j-1} \sum_{i=0}^{j} p(n-j)(-1)^{\left\lceil \frac{i}{2} \right\rceil} \chi_{1,k}(j-k-G_i) \, [j-k-G_i \geq 1]_\delta \, f(\gcd(k-1,n))$$

$$+ f(1) \left[n = 1\right]_{\delta}.$$

Next, by Möbius inversion and noting that the Dirichlet inverse of $\mu(n)$ is $\mu * 1 = \varepsilon$, where $\varepsilon(n) = \delta_{n,1}$ is the multiplicative identity with respect to Dirichlet convolution, we can express $f(n)$ as follows:

$$f(n) = \sum_{d|n} \sum_{r|d} \sum_{j=0}^{r} \sum_{k=1}^{j-1} \sum_{i=0}^{j} \Bigg[p(r-j)(-1)^{\lceil \frac{i}{2} \rceil} \chi_{1,k}(j-k-G_i) \left[j-k-G_i \geq 1\right]_{\delta} \times$$

$$\times f(\gcd(k-1, r)) \frac{\phi(d)}{\phi(r)} \mu\left(\frac{d}{r}\right) \Bigg] + f(1) \cdot \sum_{d|n} \phi(d)\mu(d).$$

We can expand the last right-hand-side term by noting that for f multiplicative [29, §27]

$$\sum_{d|n} f(d)\mu(d) = \prod_{p|n} (1 - f(p)), n \geq 1.$$

Therefore, the last term satisfies

$$\sum_{d|n} \phi(d)\mu(d) = \prod_{p|n} (2 - p), n \geq 1.$$

3.2 Factorization theorems for classes of Anderson-Apostol sums (type II sums)

Recall the notation from Definition 3.1. The sums $L_{f,g,k}(n)$ are sometimes refered to as *Anderson-Apostol sums* named after the authors who first defined these sums (*cf.* [4, §8.3] [3]). Other variants and generalizations of these sums are studied in [35, 16]. There are many number theoretic applications of the periodic sums factorized in this form. For example, the famous expansion of Ramanujan's sum $c_q(n)$ is expressed as the following right-hand-side divisor sum [13, §IX]:

$$c_q(n) = \sum_{\substack{d=1 \\ (d,n)=1}}^{n} \omega_q^{dn} = \sum_{d|(q,n)} d \cdot \mu\left(\frac{q}{d}\right).$$

The applications of our new results to Ramanujan's sum include the expansions

$$\begin{aligned} c_n(x) &= [w^x] \left(\sum_{k=1}^{n} u_{n,k}^{(-1)}(\mu, w) \times \sum_{j \geq 0} (-1)^{\lceil \frac{j}{2} \rceil} \mu(k - G_j) \right) \\ &= \sum_{k=1}^{n} \left(\sum_{d|(n,x)} d \cdot p\left(\frac{n}{d} - k\right) \right) \times \sum_{j \geq 0} (-1)^{\lceil \frac{j}{2} \rceil} \mu(k - G_j), \end{aligned}$$

where the inverse matrices $u_{n,k}^{(-1)}(\mu, w)$ are expanded according to Proposition 3.9. We then immediately have the following new results for the next special expansions of the generalized sum-of-divisors functions when $\Re(s) > 0$:

$$\sigma_s(n) = n^s \zeta(s+1) \times \sum_{i=1}^{\infty} \sum_{k=1}^{i} \left(\sum_{d|(n,i)} d \cdot p\left(\frac{i}{d} - k\right) \right) \sum_{j \geq 0} \frac{(-1)^{\lceil \frac{j}{2} \rceil} \mu(k - G_j)}{i^{s+1}}.$$

3.2.1 Formulas for the inverse matrices

It happens that in the case of the series expansions we defined in (25), the corresponding terms of the inverse matrices $u_{n,k}^{(-1)}(f,w)$ satisfy considerably simpler formulas that the ordinary matrix entries themselves. We first prove a partition-related explicit formula for these inverse matrices as Proposition 3.9 and then discuss several applications of this result.

Proposition 3.9 (Formulas for the inverse matrices of type II sums)**.** *For all $n \geq 1$ and $1 \leq k \leq n$, any fixed arithmetic function f, and $w \in \mathbb{C} \setminus \{0\}$, we have that*

$$u_{n,k}^{(-1)}(f,w) = \sum_{m=1}^{n} \left(\sum_{d|(m,n)} f(d) p\left(\frac{n}{d} - k\right) \right) w^m.$$

Proof of Proposition 3.9. Let $1 \leq r \leq n$ and for some suitably chosen arithmetic function g define

$$u_{n,r}^{(-1)}(f,w) := \sum_{m=1}^{n} L_{f,g,m}(n) w^m. \tag{i}$$

By directly expanding the series on the right-hand-side of (25), we obtain that

$$\begin{aligned} g(n) &= \sum_{j=0}^{n} \left(\sum_{k=1}^{j} u_{j,k}(f,w) u_{k,r}^{(-1)}(f,w) \right) p(n-j) \\ &= \sum_{j=0}^{n} p(n-j) \left[j = r \right]_{\delta} = p(n-r). \end{aligned}$$

Hence the choice of the function g which satisfies (i) above is given by $g(n) := p(n-r)$. The claimed expansion of the inverse matrices then follows. □

Proposition 3.10. *We have the following identity that holds for any arithmetic functions f, g:*

$$L_{f,g,m}(n) = \sum_{k=1}^{n} \sum_{d|(m,n)} f(d) p\left(\frac{n}{d} - k\right) \times \sum_{\substack{j \geq 0 \\ k > G_j}} (-1)^{\lceil \frac{j}{2} \rceil} g\left(k - G_j\right). \tag{34}$$

Proof of Proposition 3.10. Since the coefficients on the left-hand-side of the next equation correspond to a right-hand-side matrix product as

$$[q^n](q;q)_\infty \times \sum_{m \geq 1} g(m) q^m = \sum_{k=1}^{n} u_{n,k}(f,w) \times \sum_{m=1}^{k} L_{f,g,m}(k) w^m,$$

we can invert the matrix product on the right to obtain that

$$\sum_{m=1}^{k} L_{f,g,m}(k) w^m = \sum_{k=1}^{n} \left(\sum_{m=1}^{n} \sum_{d|(n,m)} f(d) p\left(\frac{n}{d} - k\right) w^m \right) \times [q^k](q;q)_\infty \times \sum_{m \geq 1} g(m),$$

so that by comparing coefficients of w^m for $1 \leq m \leq n$, we obtain (34). □

Corollary 3.11 (A new formula to express Ramanujan sums)**.** *For any natural numbers $x, m \geq 1$, we have that*

$$c_x(m) = \sum_{k=1}^{x} \sum_{d|(m,x)} d \cdot p\left(\frac{x}{d} - k\right) \times \sum_{\substack{j \geq 0 \\ k > G_j}} (-1)^{\lceil \frac{j}{2} \rceil} \mu\left(k - G_j\right).$$

Proof of Corollary 3.11. The Ramanujan sums correspond to the special case of Proposition 3.10 where $f(n) := n$ is the identity function and $g(n) := \mu(n)$ is the Möbius function. $\square$

Remark 3.12. We define the following shorthand notation for fixed arithmetic functions f, g, integers $n \geq 1$ and $w \in \mathbb{C} \setminus \{0\}$:

$$\widehat{L}_{f,g}(n; w) := \sum_{m=1}^{n} L_{f,g,m}(n) w^m.$$

In this notation we have that $u_{n,k}^{(-1)}(f, w) = \widehat{L}_{f,g}(n; w)$ when $g(n) := p(n-k)$. Moreover, if we denote by $T_n(x)$ the polynomial $T_n(x) := 1 + x + x^2 + \cdots + x^{n-1} = \frac{1-x^n}{1-x}$, then we have expansions of these sums as convolved ordinary divisor sums by polynomial terms of the form

$$\begin{aligned} \widehat{L}_{f,g}(n; w) &= \sum_{d|n} w^d f(d) T_{\frac{n}{d}}(w^d) g\left(\frac{n}{d}\right) \\ &= (w^n - 1) \times \sum_{d|n} \frac{w^d}{w^d - 1} f(d) g\left(\frac{n}{d}\right). \end{aligned} \tag{35}$$

The Dirichlet inverse of this divisor sums is also not difficult to express, though we will not give its formula here. These sums lead to the expressions for the ordinary matrix entries $u_{n,k}(f, w)$ given by the next corollary.

Corollary 3.13 (A formula for the ordinary matrix entries)**.** *To distinguish notation, let*

$$\widehat{P}_{f,k}(n; w) := \widehat{L}_{f(n),p(n-k)}(n; w),$$

which is an immediate shorthand for the matrix inverse terms $u_{n,k}^{(-1)}(f, w)$ that we will precisely enumerate below. For $n \geq 1$ and $1 \leq k < n$, we have the following formula:

$$\begin{aligned} u_{n,k}(f, w) = -\frac{(1-w)^2}{w^2 \cdot (1-w^n)(1-w^k) \cdot f(1)^2} \Bigg(& \widehat{P}_{f,k}(n; w) \\ + \sum_{m=1}^{n-k-1} \left(\frac{w-1}{w f(1)}\right)^m & \left[\sum_{k \leq i_1 < \cdots < i_m < n} \frac{\widehat{P}_{f,k}(i_1; w) \widehat{P}_{f,i_1}(i_2; w) \widehat{P}_{f,i_2}(i_3; w) \cdots \widehat{P}_{f,i_{m-1}}(i_m; w) \widehat{P}_{f,i_m}(n; w)}{(1-w^{i_1})(1-w^{i_2}) \cdots (1-w^{i_m})} \right] \Bigg) \end{aligned}$$

When $k = n$, we have that

$$u_{n,n}(f, w) = \frac{1-w}{w(1-w^n) f(1)}.$$

Proof. This follows inductively from the inversion relation between the coefficients of a matrix and its inverse. For any invertible lower triangular $n \times n$ matrix $(a_{i,j})_{1 \leq i,j \leq n}$, we can express a non-recursive formula for the inverse matrix entries as follows:

$$a_{n,k}^{(-1)} = \frac{1}{a_{n,n}} \left(-\frac{a_{n,k}}{a_{k,k}} + \sum_{m=1}^{n-k-1} (-1)^{m+1} \left[\sum_{k \leq i_1 < \cdots < i_m < n} \frac{a_{i_1,k} a_{i_2,i_1} a_{i_3,i_2} \cdots a_{i_m,i_{m-1}} a_{n,i_m}}{a_{k,k} a_{i_1,i_1} a_{i_2,i_2} \cdots a_{i_m,i_m}} \right] \right) [k < n]_\delta + \frac{[k = n]_\delta}{a_{n,n}}. \tag{36}$$

The proof of our result is then just an application of the formula in (36) when $a_{n,k} := u_{n,k}^{-1}(f, w)$. While the identity in (36) is not immediately obvious from the known inversion formulas between inverse matrices in the form of

$$a_{n,k}^{(-1)} = \frac{[n = k]_\delta}{a_{n,n}} - \frac{1}{a_{n,n}} \times \sum_{j=1}^{n-k-1} a_{n,j} a_{j,k}^{(-1)},$$

the result is easily obtained by induction on n so we do not prove it here. $\square$

3.2.2 Formulas for simplified variants of the ordinary matrices

In Corollary 3.13 we proved an exact expansion of the ordinary matrix entries $u_{n,k}(f,w)$ by sums of weighted products of the inverse matrices $u_{n,k}^{(-1)}(f,w)$ that is expressed in closed form through Proposition 3.9. We will now develop the machinery needed to more precisely express the ordinary forms of these matrices for general cases of the indeterminate indexing parameter $w \in \mathbb{C} \setminus \{0\}$.

$\frac{1}{\widehat{f}(1)}$	0	0	0	0	0
$-\frac{\widehat{f}(2)}{\widehat{f}(1)^2} - \frac{1}{\widehat{f}(1)}$	$\frac{1}{\widehat{f}(1)}$	0	0	0	0
$\frac{\widehat{f}(2)}{\widehat{f}(1)^2} - \frac{\widehat{f}(3)}{\widehat{f}(1)^2} - \frac{1}{\widehat{f}(1)}$	$-\frac{1}{\widehat{f}(1)}$	$\frac{1}{\widehat{f}(1)}$	0	0	0
$\frac{\widehat{f}(2)^2}{\widehat{f}(1)^3} + \frac{\widehat{f}(2)}{\widehat{f}(1)^2} + \frac{\widehat{f}(3)}{\widehat{f}(1)^2} - \frac{\widehat{f}(4)}{\widehat{f}(1)^2}$	$-\frac{\widehat{f}(2)}{\widehat{f}(1)^2} - \frac{1}{\widehat{f}(1)}$	$-\frac{1}{\widehat{f}(1)}$	$\frac{1}{\widehat{f}(1)}$	0	0
$-\frac{\widehat{f}(2)^2}{\widehat{f}(1)^3} + \frac{\widehat{f}(3)}{\widehat{f}(1)^2} + \frac{\widehat{f}(4)}{\widehat{f}(1)^2} - \frac{\widehat{f}(5)}{\widehat{f}(1)^2}$	$\frac{\widehat{f}(2)}{\widehat{f}(1)^2}$	$-\frac{1}{\widehat{f}(1)}$	$-\frac{1}{\widehat{f}(1)}$	$\frac{1}{\widehat{f}(1)}$	0
$-\frac{\widehat{f}(2)^2}{\widehat{f}(1)^3} + \frac{2\widehat{f}(3)\widehat{f}(2)}{\widehat{f}(1)^3} + \frac{\widehat{f}(4)}{\widehat{f}(1)^2} + \frac{\widehat{f}(5)}{\widehat{f}(1)^2} - \frac{\widehat{f}(6)}{\widehat{f}(1)^2} + \frac{1}{\widehat{f}(1)}$	$\frac{\widehat{f}(2)}{\widehat{f}(1)^2} - \frac{\widehat{f}(3)}{\widehat{f}(1)^2}$	$-\frac{\widehat{f}(2)}{\widehat{f}(1)^2}$	$-\frac{1}{\widehat{f}(1)}$	$-\frac{1}{\widehat{f}(1)}$	$\frac{1}{\widehat{f}(1)}$

Table 3.1: *The simplified matrix entries $\widehat{u}_{n,k}(f,w)$ for $1 \leq n,k \leq 6$ where $\hat{f}(n) = \frac{w^n}{w^n-1} f(n)$ for arithmetic functions f such that $f(1) \neq 0$.*

Remark 3.14 (Simplifications of the matrix terms)**.** Using the formula for the coefficients of $u_{n,k}(f,w)$ in (25) expanded by (35), we can simplify the form of the matrix entries we seek closed-form expressions for in the next calculations. In particular, we make the following definitions for $1 \leq k \leq n$:

$$\begin{aligned} \widehat{f}(n) &:= \frac{w^n}{w^n-1} f(n) \\ \widehat{u}_{n,k}(f,w) &:= (w^k-1) u_{n,k}(f,w). \end{aligned}$$

Then an equivalent formulation of finding the exact formulas for $u_{n,k}(f,w)$ is to find exact expressions expanding the triangular sequence of $\widehat{u}_{n,k}(f,w)$ satisfying

$$\sum_{\substack{j \geq 0 \\ n-G_j > 0}} (-1)^{\left\lceil \frac{j}{2} \right\rceil} g(n-G_j) = \sum_{k=1}^{n} \widehat{u}_{n,k}(f,w) \times \sum_{d|k} \widehat{f}(d) g\left(\frac{n}{d}\right).$$

We will obtain precisely such formulas in the next few results. Table 3.1 provides the first few rows of our simplified matrix entries.

Definition 3.15 (Special forms of multiple convolutions)**.** For $n, j \geq 1$, we define the following nested j-convolutions of the function $\widehat{f}(n)$ [24]:

$$\mathrm{ds}_j(f;n) = \begin{cases} (-1)^{\delta_{n,1}} \widehat{f}(n), & \text{if } j = 1; \\ \sum\limits_{\substack{d|n \\ d>1}} \widehat{f}(d)\, \mathrm{ds}_{j-1}\left(f; \frac{n}{d}\right), & \text{if } j \geq 2. \end{cases}$$

Then we define our primary multiple convolution function of interest as

$$D_f(n) := \sum_{j=1}^{n} \frac{\mathrm{ds}_{2j}(f;n)}{\widehat{f}(1)^{2j+1}}.$$

n	$D_f(n)$	n	$D_f(n)$	n	$D_f(n)$
2	$-\frac{\widehat{f}(2)}{\widehat{f}(1)^2}$	7	$-\frac{\widehat{f}(7)}{\widehat{f}(1)^2}$	12	$\frac{2\hat{f}(3)\hat{f}(4)+2\hat{f}(2)\hat{f}(6)-\hat{f}(1)\hat{f}(12)}{\hat{f}(1)^3} - \frac{3\hat{f}(2)^2\hat{f}(3)}{\hat{f}(1)^4}$
3	$-\frac{\widehat{f}(3)}{\widehat{f}(1)^2}$	8	$\frac{2\widehat{f}(2)\widehat{f}(4)-\widehat{f}(1)\widehat{f}(8)}{\widehat{f}(1)^3} - \frac{\widehat{f}(2)^3}{\widehat{f}(1)^4}$	13	$-\frac{\hat{f}(13)}{\hat{f}(1)^2}$
4	$\frac{\widehat{f}(2)^2-\widehat{f}(1)\widehat{f}(4)}{\widehat{f}(1)^3}$	9	$\frac{\widehat{f}(3)^2-\widehat{f}(1)\widehat{f}(9)}{\widehat{f}(1)^3}$	14	$\frac{2\hat{f}(2)\hat{f}(7)-\hat{f}(1)\hat{f}(14)}{\hat{f}(1)^3}$
5	$-\frac{\widehat{f}(5)}{\widehat{f}(1)^2}$	10	$\frac{2\widehat{f}(2)\widehat{f}(5)-\widehat{f}(1)\widehat{f}(10)}{\widehat{f}(1)^3}$	15	$\frac{2\hat{f}(3)\hat{f}(5)-\hat{f}(1)\hat{f}(15)}{\hat{f}(1)^3}$
6	$\frac{2\widehat{f}(2)\widehat{f}(3)-\widehat{f}(1)\widehat{f}(6)}{\widehat{f}(1)^3}$	11	$-\frac{\hat{f}(11)}{\hat{f}(1)^2}$	16	$\frac{\hat{f}(2)^4}{\hat{f}(1)^5} - \frac{3\hat{f}(4)\hat{f}(2)^2}{\hat{f}(1)^4} + \frac{\hat{f}(4)^2+2\hat{f}(2)\hat{f}(8)}{\hat{f}(1)^3} - \frac{\hat{f}(16)}{\hat{f}(1)^2}$

Table 3.2: *The multiple convolution function $D_f(n)$ for $2 \leq n \leq 16$ where $\widehat{f}(n) := \frac{w^n}{w^n-1} \cdot f(n)$ for an arbitrary arithmetic function f such that $f(1) \neq 0$.*

The first few cases of $D_f(n)$ for $2 \leq n \leq 16$ are computed in Table 3.2. The examples in the table should clarify precisely what multiple convolutions we are defining by the function $D_f(n)$. Namely, a signed sum of all possible ordinary k Dirichlet convolutions of $\widehat{f}$ with itself evaluated at n.

Lemma 3.16. *We claim that for all $n \geq 1$*

$$(D_f * \widehat{f})(n) \equiv \sum_{d|n} f(d) D_f\left(\frac{n}{d}\right) = -\frac{\widehat{f}(n)}{\widehat{f}(1)} + \varepsilon(n).$$

where $\varepsilon(n) \equiv \delta_{n,1}$ is the multiplicative identity function with respect to Dirichlet convolution.

Proof of Lemma 3.16. The statement of the lemma is equivalent to showing that

$$\left(D_f + \frac{\varepsilon}{\widehat{f}(1)}\right)(n) = \widehat{f}^{-1}(n). \tag{37}$$

A general recursive formula for the inverse of $\widehat{f}(n)$ is given by [4]

$$\widehat{f}^{-1}(n) = \left(-\frac{1}{\widehat{f}(1)} \times \sum_{\substack{d|n \\ d>1}} \widehat{f}(d)\widehat{f}^{-1}\left(\frac{n}{d}\right)\right)[n>1]_\delta + \frac{1}{\widehat{f}(1)}[n=1]_\delta.$$

This definition is almost how we defined $\mathrm{ds}_j(f;n)$ above. Let's see how to modify this recurrence relation to obtain the formula for $D_f(n)$. We can recursively substitute in the formula for $\widehat{f}^{-1}(n)$ until we hit the point where successive substitutions only leave the base case of $\widehat{f}^{-1}(1) = \widehat{f}(1)^{-1}$. This occurs after $\Omega(n)$ substitutions where $\Omega(n)$ denotes the number of prime factors of n counting multiplicity. We can write the nested formula for $\mathrm{ds}_j(f;n)$ as

$$\mathrm{ds}_j(f;n) = \widehat{f}_{\pm} * \underbrace{\left(\widehat{f} - \widehat{f}(1)\varepsilon\right) * \cdots * \left(\widehat{f} - \widehat{f}(1)\varepsilon\right)}_{j-1 \text{ factors}}(n),$$

where we define $\widehat{f}_{\pm}(n) := \widehat{f}(n)[n>1]_\delta - \widehat{f}(1)[n=1]_\delta$. Next, define the nested k-convolutions $C_k(n)$ recursively by

$$C_k(n) = \begin{cases} \widehat{f}(n) - \widehat{f}(1)\varepsilon(n), & \text{if } k=1; \\ \sum_{d|n}\left(\widehat{f}(d) - \widehat{f}(1)\varepsilon(d)\right) C_{k-1}\left(\frac{n}{d}\right), & \text{if } k \geq 2. \end{cases}$$

Then we can express the inverse of $\widehat{f}(n)$ using this definition as follows:

$$\widehat{f}^{-1}(n) = \sum_{d|n} \widehat{f}(d) \left(\sum_{j=1}^{\Omega(n)} \frac{C_{2k}\left(\frac{n}{d}\right)}{\widehat{f}(1)^{\Omega(n)+1}} - \frac{\varepsilon\left(\frac{n}{d}\right)}{\widehat{f}(1)^2} \right).$$

Then based on the initial conditions for $k = 1$ (or $j = 1$) in the definitions of $C_k(n)$ (and $\mathrm{ds}_j(f; n)$), we see that the function in (37) is in fact the inverse of $\widehat{f}(n)$. □

Proposition 3.17. *For all $n \geq 1$ and $1 \leq k \leq n$, we have that*

$$\sum_{i=0}^{n-1} p(i)\widehat{u}_{n-i,k}(f, w) = D_f \left(\frac{n}{k}\right) [n \equiv 0 \pmod{k}]_\delta + \frac{1}{\widehat{f}(1)} [n = k]_\delta .$$

Proof of Proposition 3.17. We notice that Lemma 3.16 implies that

$$\varepsilon(n) = \left(\left(D_f + \frac{\varepsilon}{\widehat{f}(1)} \right) * \widehat{f} \right)(n),$$

where $\varepsilon(n)$ is the multiplicative identity for Dirichlet convolutions. The last equation implies that

$$g(n) = \left(\left(D_f + \frac{\varepsilon}{\widehat{f}(1)} \right) * \widehat{f} * g \right)(n). \tag{i}$$

Additionally, we know by the expansion of (25) and that $\widehat{u}_{n,n}(f, w) = \widehat{f}(1)^{-1}$ that we have the expansion

$$g(n) = \sum_{k \geq 1} \left(\sum_{j=0}^{n-1} p(j)\widehat{u}_{n-j,k} \right) \sum_{d|k} \widehat{f}(d) g\left(\frac{k}{d}\right). \tag{ii}$$

So we can equate (i) and (ii) to see that

$$\sum_{j=0}^{n-1} p(j)\widehat{u}_{n-j,k} = D_f \left(\frac{n}{k}\right) [k|n]_\delta + \frac{[n = k]_\delta}{\widehat{f}(1)}.$$

This establishes our claim. □

Corollary 3.18 (An exact formula for the ordinary matrices)**.** *For all $n \geq 1$ and $1 \leq k \leq n$, we have that*

$$\widehat{u}_{n,k}(f, w) = \sum_{\substack{j \geq 0 \\ n - G_j > 0}} (-1)^{\left\lceil \frac{j}{2} \right\rceil} \left(D_f \left(\frac{n - G_j}{k} \right) [n - G_j \equiv 0 \pmod{k}]_\delta + \frac{1}{\widehat{f}(1)} [n - G_j = k]_\delta \right).$$

Proof. This is an immediate consequence of Proposition 3.17 by noting that the generating function for $p(n)$ is $(q; q)_\infty^{-1}$ and that

$$(q; q)_\infty = \sum_{j \geq 0} (-1)^{\left\lceil \frac{j}{2} \right\rceil} q^{G_j}.$$ □

3.3 Applications to DTFTs and finite Fourier series expansions

We expand the left-hand-side function $g(x)$ in (25) by considering a new indirect method involving the type II sums $L_{f,g,k}(n)$. The expansions we derive in this section employ results for discrete Fourier transforms of functions of the greatest common divisor studied in [57, 50]. This method allows us to study the factorization forms in (25) where we effectively bypass the complicated forms of the ordinary matrix coefficients $u_{n,k}(f,w)$. Results enumerating the ordinary matrices with coefficients given by $u_{n,k}(f,w)$ are treated in Corollary 3.13 of Section 3.2.1. The discrete Fourier series methods we use to prove our theorems in these sections lead to the next key result proved in Theorem 3.24 which states that for any arithmetic functions f, g we have

$$\sum_{d|k}\sum_{r=0}^{k-1} dL_{f,g,r}(k)e\left(-\frac{rd}{k}\right)\mu\left(\frac{k}{d}\right) = \sum_{d|k}\phi(d)f(d)\left(\frac{k}{d}\right)^2 g\left(\frac{k}{d}\right), k \geq 1,$$

where $e(x) = \exp(2\pi\imath \cdot x)$ is standard notation for the complex exponential function.

The proof of the result given in Theorem 3.24 below builds on several results on discrete Fourier transforms of functions evauated at the greatest common divisor, $(k,n) \equiv \gcd(k,n)$, developed in [57]. For the remainder of this section we take $k \geq 1$ to be fixed and consider the divisor sums of the following form which are periodic with respect to k for any $n \geq 1$:

$$L_{f,g,k}(n) := \sum_{d|(n,k)} f(d)g\left(\frac{n}{d}\right).$$

In [57] these sums are called k-convolutions of f and g. We will first need to discuss some more terminology related to discrete Fourier transforms before moving on.

Definition 3.19. A *discrete Fourier transform* (DFT) maps a (finite) sequence of complex numbers $\{f[n]\}_{n=0}^{N-1}$ onto their associated Fourier coefficients $\{F[n]\}_{n=0}^{N-1}$ defined according to the following reversion formulas relating these sequences:

$$\begin{aligned} F[k] &= \sum_{n=0}^{N-1} f[n]e\left(-\frac{kn}{N}\right), \\ f[k] &= \frac{1}{N} \times \sum_{n=0}^{N-1} F[k]e\left(\frac{kn}{N}\right). \end{aligned}$$

The discrete Fourier transform of functions of the greatest common divisor, which we will employ repeatedly to prove Theorem 3.24 below, is characterized by the formula in the next lemma [57, 50].

Lemma 3.20. *If we take any two arithmetic functions f and g, we can express periodic divisor sums modulo any $k \geq 1$ of the form*

$$s_k(f,g;n) := \sum_{d|(n,k)} f(d)g\left(\frac{k}{d}\right) = \sum_{m=1}^{k} a_k(f,g;m)e^{\frac{2\pi\imath mn}{k}}, n \geq 1. \tag{38a}$$

The discrete Fourier coefficients on the right-hand-side of the previous equation are given by

$$a_k(f,g;m) = \sum_{d|(m,k)} g(d)f\left(\frac{k}{d}\right)\frac{d}{k}. \tag{38b}$$

Proof. For a proof of these relations consult the references [4, §8.3] [29, *cf.* §27.10]. These relations are also related to the gcd-transformations proved in [57, 50]. $\square$

Definition 3.21. The function $c_m(a)$ defined for integers $m, a \geq 1$ by

$$c_m(a) := \sum_{\substack{k=1\\(k,m)=1}}^{m} e\left(\frac{ka}{m}\right),$$

is called *Ramanujan's sum.* Ramanujan's sum is expanded as in the divisor sums in Corollary 3.11 of the last subsection.

Lemma 3.22 (DFT of functions of the greatest common divisor)**.** *Let h be any arithmetic function. For natural numbers $m \geq 1$, the discrete Fourier transform (DFT) of h in the GCD sense is defined by the following function:*

$$\widehat{h}[a](m) := \sum_{k=1}^{m} h\left(\gcd(k,m)\right) e\left(\frac{ka}{m}\right).$$

This form of the DFT of $h(\gcd(n,k))$ (for $k \geq 1$ a free parameter) satisfies $\widehat{h}[a] = h * c_{-}(a)$. The function $\widehat{h}[a]$ is summed explicitly for $n \geq 1$ as the Dirichlet convolution

$$\widehat{h}[a](n) = (h * c_{-}(a))(n) = \sum_{d|n} h\left(\frac{n}{d}\right) c_d(a), n \geq 1, a \in \mathbb{C}.$$

Definition 3.23 (Notation for certain exponential sums)**.** In what follows, for $1 \leq \ell \leq k$ we denote the ℓ^{th} Fourier coefficient with respect to k of the function $L_{f,g,k}(n)$ by $a_{k,\ell}$. This specification is well defined since $L_{f,g,k}(n) = L_{f,g,k}(n+k)$ is periodic with period k over the integers $n \geq 1$. We have an expansion of this function of the form

$$L_{f,g,k}(n) = \sum_{\ell=0}^{k-1} a_{k,\ell} e\left(\frac{\ell n}{k}\right).$$

We can compute the coefficients, $a_{m,\ell}$, directly from $L_{f,g,k}(n)$ according to the formula

$$a_{k,\ell} = \sum_{n=0}^{k-1} L_{f,g,k}(n) e\left(-\frac{\ell n}{k}\right).$$

These Fourier coefficients are given explicitly in terms of f and g by the formulas cited in (38).

Theorem 3.24. *For any arithmetic functions f, g and $k \geq 1$, we have that*

$$\sum_{d|k} \sum_{r=0}^{k-1} d L_{f,g,r}(k) e\left(-\frac{rd}{k}\right) \mu\left(\frac{k}{d}\right) = \sum_{d|k} \phi(d) f(d) \left(\frac{k}{d}\right)^2 g\left(\frac{k}{d}\right), \tag{39}$$

where $\phi(n)$ is Euler's totient function.

Proof of Theorem 3.24. We see that the left-hand-side of (39) corresponds to a divisor sum of the form

$$\sum_{d|k} \sum_{r=0}^{k-1} d L_{f,g,r}(k) e\left(-\frac{rd}{k}\right) \mu\left(\frac{k}{d}\right) = \sum_{d|k} d \cdot \hat{a}_{k,d} \mu\left(\frac{k}{d}\right).$$

The Fourier coefficients $a_{k,d}$ in this expansion are given by (38) [4, §8.3] so that [29, §27.10]

$$\sum_{0 \leq r < k} L_{f,g,r}(k) e\left(-\frac{rd}{k}\right) = \sum_{0 \leq r < k} \sum_{\ell=0}^{r-1} a_{k,\ell} e\left(\frac{\ell r}{k}\right) e\left(-\frac{rd}{k}\right),$$

and where whenever we have that $k|d$, we get that the exponential sum

$$\sum_{0\leq r<k} e\left(-\frac{rd}{k}\right) = k.$$

Then we have that

$$\hat{a}_{k,d} = k \times \sum_{r|(k,d)} g(r) f\left(\frac{k}{r}\right) \frac{r}{k}.$$

The left-hand-side of our expansion then becomes (*cf.* (41) below)

$$\begin{aligned}
\sum_{d|k} d a_{k,d} \mu\left(\frac{k}{d}\right) &= \sum_{d|k}\sum_{r|d} d r g(r) f\left(\frac{k}{r}\right) \mu\left(\frac{k}{d}\right) \\
&= \sum_{d=1}^{k} d\mu\left(\frac{k}{d}\right)\left(\sum_{r|d} r g(r) f\left(\frac{k}{r}\right)\right)[d|k]_{\delta} \\
&= \sum_{r|k} r g(r) f\left(\frac{k}{r}\right)\left(\sum_{d=1}^{k/r} d r \mu\left(\frac{k}{dr}\right)[d|k]_{\delta}\right) \\
&= \sum_{r|k} r^2 g(r) f\left(\frac{k}{r}\right) \phi\left(\frac{k}{r}\right).
\end{aligned}$$

The conclusion follows by interchanging the index of summation by setting $d \leftrightarrow \frac{n}{d}$ and vice versa as inputs to the functions in the last equation. $\square$

Corollary 3.25. *For any $n \geq 1$ and arithmetic functions f, g we have the formula*

$$g(n) = \sum_{d|n}\sum_{j|d}\sum_{r=0}^{d-1} \frac{j L_{f,g,r}(d)}{d^2} e\left(-\frac{rj}{d}\right) \mu\left(\frac{d}{j}\right) y_f\left(\frac{n}{d}\right),$$

where $y_f(n) = (\phi f \operatorname{Id}_{-2})^{-1}(n)$ is the Dirichlet inverse of the function $f(n)\phi(n)n^{-2}$ and $\operatorname{Id}_k(n) := n^k$ for $n \geq 1$ and $k \geq 0$.

Proof of Corollary 3.25. We first divide both sides of the result in Theorem 3.24 by k^2. Then we apply a Dirichlet convolution of the left-hand-side of the formula in Theorem 3.24 with $y_f(n)$ defined as above to obtain the exact expansion for $g(n)$. $\square$

Corollary 3.26 (The Mertens function)**.** *For all $x \geq 1$, the Mertens function, denoted by the partial sums $M(x) := \sum_{n\leq x} \mu(n)$, is expanded by Ramanujan's sum as*

$$M(x) = \sum_{d=1}^{x}\sum_{n=1}^{\left\lfloor \frac{x}{d}\right\rfloor}\sum_{r=0}^{d-1}\left(\sum_{j|d} \frac{j}{d} e\left(-\frac{rj}{d}\right)\mu\left(\frac{d}{j}\right)\right)\frac{c_d(r)}{d} y(n), \tag{40}$$

where $y(n) = (\phi \operatorname{Id}_{-1})^{-1}(n)$ is the Dirichlet inverse of the function $\phi(n)n^{-1}$.

Proof of Corollary 3.26. We begin by citing Theorem 3.24 in the special case corresponding to $L_{f,g,k}(n)$ a Ramanujan sum for $f(n) = n$ and $g(n) = \mu(n)$. Then we sum over the left-hand-side $g(n)$ in Corollary 3.25 to obtain the initial summation identity for $M(x)$ given by

$$M(x) = \sum_{n\leq x}\sum_{d|n}\sum_{j|d}\sum_{r=0}^{d-1} \frac{j}{d^2} c_d(r) e\left(-\frac{rj}{d}\right)\mu\left(\frac{d}{j}\right) y\left(\frac{n}{d}\right). \tag{i}$$

We can then apply the identity that for any arithmetic functions h, u, v we can interchange nested divisor sums as

$$\sum_{k=1}^{n} \sum_{d|k} h(d) u\left(\frac{k}{d}\right) v(k) = \sum_{d=1}^{n} h(d) \left(\sum_{k=1}^{\lfloor \frac{n}{d} \rfloor} u(k) v(dk) \right). \tag{41}$$

Application of this identity to equation (i) leads to the first form for $M(x)$ stated in (40). $\square$

There is a related identity to compare to equation (41) which allows us to interchange the order of summation in the Anderson-Apostol sums of the following form for any natural numbers $x \geq 1$ and arithmetic functions $f, g, h : \mathbb{N} \to \mathbb{C}$:

$$\sum_{d=1}^{x} f(d) \sum_{r|(d,x)} g(r) h\left(\frac{d}{r}\right) = \sum_{r|x} g(r) \sum_{d=1}^{\frac{x}{r}} h(d) f\left(\gcd(x, r) d\right).$$

Corollary 3.27 (Euler's totient function)**.** *For any $n \geq 1$ we have*

$$\frac{\phi(n)}{n} = \sum_{d|n} \sum_{j|d} \sum_{r=0}^{d-1} \frac{j}{d^2} c_d(r) e\left(-\frac{rj}{d}\right) \mu\left(\frac{d}{j}\right).$$

We have the following expansion of the average order sums for $\phi(n)$ given by

$$\sum_{2 \leq n \leq x} \phi(n) = \sum_{d=1}^{x} \sum_{r=0}^{d-1} \sum_{j|d} j e\left(-\frac{rj}{d}\right) \mu\left(\frac{d}{j}\right) \frac{c_d(r)}{2d} \left\lfloor \frac{x}{d} \right\rfloor \left(\left\lfloor \frac{x}{d} \right\rfloor - 1 \right).$$

Proof of Corollary 3.27. We consider the formula in Theorem 3.24 with $f(n) = n$ and $g(n) = \mu(n)$. Since the Dirichlet inverse of the Möbius function is $\mu * 1 = \varepsilon$, we obtain our result by convolution and multiplication by the factor of n. The average order identity follows from the first expansion by applying (41). $\square$

4 Generalized factorization theorems

4.1 K-convolutions: A generalized form of Dirichlet convolutions and divisor sums

Definition 4.1. Consider the next generalization of Dirichlet convolution, $(f * g)(n) := \sum_{d|n} f(d) g\left(\frac{n}{d}\right)$, or divisor sums with respect to some kernel function $K(n,d)$ that is well defined for all divisors $d|n$ of any $n \geq 1$ by the following K-convolution operation:

$$(f *_K g)(n) := \sum_{d|n} f(d) g\left(\frac{n}{d}\right) K(n,d). \tag{42}$$

Typically, we restrict the function K so that $K(n,d) = K\left(n, \frac{n}{d}\right)$ to assure commuativity of the associated convolution operation as $f *_K g = g *_K f$, though this requirement is not a strict necessity for applications. Furthermore, we assume that any admissible kernel function K (subject to the requirements of our definitions above) can be expressed in factored form by $K(n,d) = K_0(n,d) [d|n]_\delta$ for some other suitably defined kernel function K_0 that characterizes the operation. For any fixed finite $1 \leq N < \infty$, we also use the notation

$$\mathcal{K}_0(N) := (K_0(n,d))_{1 \leq n,d \leq N}, \mathcal{K}(N) := (K(n,d))_{1 \leq n,d \leq N}.$$

We note that the sums of this type are expressed by matrix-vector type convolutions defined in Example 4.5 in a later section.

Proposition 4.2 (Generalized Möbius inversion)**.** *For any kernel functions (K, K_0) associated to the definition given in (42) above such that K_0 is invertible, and any arithmetic functions f, g we have that*

$$g(n) = \sum_{d|n} f(d) K_0(n,d) \iff f(n) = \sum_{d|n} \sum_{r|d} g(r) \mu\left(\frac{d}{r}\right) K_0^{-1}(n,d), n \geq 1.$$

Proof of Proposition 4.2. Notice that for fixed $N \geq 1$, with the $N \times N$ matrix $T_{\rm div}(N) := ([j|i]_\delta)_{1 \leq i,j \leq N}$ invertible via the Möbius inversion theorem, we can write a matrix-vector product system of the form

$$\begin{bmatrix} f(1) \\ f(2) \\ \vdots \\ f(N) \end{bmatrix} = \mathcal{K}_0(N) T_{\rm div}(N) \begin{bmatrix} g(1) \\ g(2) \\ \vdots \\ g(N) \end{bmatrix}.$$

By ordinary matrix inversion, the previous equation implies that for any $n \leq N$

$$\begin{aligned} f(n) &= \sum_{d|n} \mu\left(\frac{n}{d}\right) \times \sum_{r=1}^{d} K_0^{-1}(d,r) g(r) \\ &= \sum_{r=1}^{n} K_0^{-1}(n,r) \sum_{d|r} \mu\left(\frac{r}{d}\right) g(r). \end{aligned}$$

Because we can restrict the non-trivial values of $\mathcal{K}_0^{-1}(N)$ to only those $(i,j)^{th}$ entries such that $j|i$, we obtain from the last equation that $f(n) = (K_0^{-1}(n, -) * \mu * g)(n)$ for all integers $n \geq 1$. ☐

In Section 5, we formulate properties that characterize a class of the most general expansions of weighted convolution type sums which we call $\mathcal{D}$-convolutions. Because the inversion relations we find in the divisor sum variants we have used to define the class of K-convlutions are simpler, and represented by products with the classical Möbius function, we consider properties of the latter sums in this section before formalizing the general case constructions.

Theorem 4.3 (Factorization theorems for K-convolutions)**.** *For integers $n \geq 1$, let the triangular generating functions be defined as*

$$\mathcal{H}_{K,n}(q) := \sum_{k \geq 1} K_0(n,k) q^{nk}.$$

Suppose that given a fixed K-convolution kernel pair (K, K_0), and any arithmetic function f, we define the LGF analog for K-convolutions as

$$\mathcal{L}_{K,f}(q) := \sum_{n \geq 1} \mathcal{H}_{K,n}(q) f(n).$$

For all $n \geq 1$, $[q^n]\mathcal{L}_{K,f}(q) = (f *_K 1)(n)$. We define a factorized form of these generating functions in the form of

$$\mathcal{L}_{K,f}(q) := \prod_{j \geq 1} \mathcal{H}_{K,j}(q) \times \sum_{n \geq 1} \left(\sum_{k=1}^{n} \hat{S}_K(n,k) f(k) \right) q^n,$$

so that we have the following results that for all integers $n \geq m \geq 1$:

(i) $\hat{S}_K(n,m) = [q^m]\mathcal{H}_{K,m}(q) \times \prod_{j \geq 1} \mathcal{H}_{K,j}^{-1}(q)$;

(ii) $\hat{S}_K^{-1}(n,m) = \sum_{d|n} \sum_{s|d} \mu\left(\frac{d}{s}\right) K_0(n,d) \times [q^{s-r}] \prod_{j \geq 1} \mathcal{H}_{K,j}$.

Remarks on Theorem 4.3. Some partition theoretic analogs that we should expect from the LGF factorization theorem cases appear in [4, §14.10]. In general, the structure and exact combinatorial interpretation of the matrix coefficients, $\hat{s}_K(n,k)$, are more complicated at determined by the series coefficients of the products of the $\prod_{j \geq 1} \mathcal{H}_{K,j}(q)$. While we do not state a precise formula here, the analogs to the LGF factorization theorems for the Dirichlet convolutions $f * g$ with $g(1) \neq 0$ from Section 2.3.3, must assume that the arithmetic function g is invertible with respect to the operation of K-convolution. These generalized factorization theorem results, like the work proved in the preceeding section by Merca and Schmidt, involve expansions that are stated in terms of the inverse function of g with respect to K-convolution. The prior LGF series cases we defined in Section 2 correspond to choosing $K_0(n,k) \equiv 1$ for all $1 \leq k \leq n$.

Proof of Theorem 4.3. To prove that the coefficients of $\mathcal{L}_{K,f}(n)$ are given by the K-concolution formula stated above, we consider the following expansions for any fixed $n \geq 1$:

$$\begin{aligned}
[q^n]\mathcal{L}_{K,f}(n) &= \sum_{j \geq 1} f(j) \times [q^n]\mathcal{H}_{K,n}(q), \\
&= \sum_{j \geq 1} f(j) \left(\sum_{\substack{m \geq 1 \\ mj=n}} K_0(n,m) \right), \\
&= \sum_{j|n} f(j) K_0\left(n, \frac{n}{j}\right), \\
&= \sum_{j|n} f(j) K_0(n,j).
\end{aligned}$$

The expansion in (i) follows from

$$\begin{aligned}
\hat{S}(n,m) &= [q^n][f(m)]\mathcal{L}_{K,f}(q) \times \prod_{j \geq 1} \mathcal{H}_{K,j}(q)^{-1}, \\
&= [q^n]\mathcal{H}_{k,m} \times \prod_{j \geq 1} \mathcal{H}_{K,j}(q)^{-1}.
\end{aligned}$$

The formula in (ii) is somewhat more complicated to show. For any fixed $r \geq 1$ and $n \geq r$, let the function $\bar{f}(n) := \hat{S}_K^{-1}(n, r)$. By orthogonality relations on the right multiplication of a matrix by its inverse, we see that

$$\sum_{k=1}^{n} \hat{S}_K(n, k)\bar{f}(k) = [n = r]_\delta \, , n \geq 1.$$

The last equation implies that

$$\begin{aligned}(\bar{f} *_K 1)(n) &= \sum_{d|n} \hat{S}_K^{-1}(d, r) K_0(n, d), \\ &= [q^n]\mathcal{L}_{K,\bar{f}}(q) = [q^n]q^r \times \prod_{j\geq 1} \mathcal{H}_{K,j}(q).\end{aligned}$$

The inversion theorem in Proposition 4.2 applied to the local functions f and g given by $f(n) \equiv \bar{f}(n) = \hat{S}^{-1}(n, r)$ and $g(n) \equiv [q^{n-r}] \prod_{j\geq 1} \mathcal{H}_{K,j}(q)$ implies the formula in (ii) correct. □

4.2 Topelitz matrix constructions to express discrete convolution sum types

Definition 4.4 (Topelitz matrices)**.** For any sequence $\{t_n\}_{n\geq 1} \subseteq \mathbb{C}$ such that $t_1 \neq 0$, we define its associated $N \times N$ *Topelitz matrix* to be the lower triangular operator

$$T_N \equiv T_N(t) := \begin{bmatrix} t_1 & 0 & \cdots & 0 \\ t_2 & t_1 & \cdots & 0 \\ \vdots & & \ddots & \\ t_N & \cdots & & t_1 \end{bmatrix}$$

Topelitz matrices define a structure by which we can express the discrete convolution of any two sequences (arithmetic functions) via a matrix-vector product. Namely, we have that for any $N \times 1$ vector $\vec{g} := (g_1, \ldots, g_N)^T$

$$(T_N(f) \cdot \vec{g})_n = \sum_{k=1}^{n} f_k g_{n+1-k}.$$

Example 4.5. The K-convolution sums from Definition 4.1 are naturally and concisely expressed by Topelitz matrices in the form of

$$(\mathcal{K} \cdot T_N(f) \cdot \vec{g})_n = \sum_{d|n} f(d) g\left(\frac{n}{d}\right) K_0(n, d),$$

where for K_0 an invertible tranformation, we have that $\mathcal{K}^{-1} = \left(\mu\left(\frac{n}{d}\right)[d|n]_\delta\right)_{1\leq n,d\leq N} \cdot \mathcal{K}_0^{-1}$. Note that we have the special case of the Topelitz matrix for the constant unitary function given by $T_N(1) = ([i \leq j]_\delta)_{1\leq i,j\leq N}$ and such that its inverse is expressed by shift matrices:

$$T_N(1)^{-1} \equiv \left(I - U^T\right)^{-1} = (\delta_{i,j} - \delta_{i+1,j})_{1\leq i,j\leq N}, N \geq 1.$$

Remark 4.6 (Möbius inversion formulas on posets)**.** It happens that we can draw comparisons with these summation types, their corresponding generating factorizations by invertible matrices, and other inversion identities to constructions that arise in *incidence alegbras*, of which the classically defined $\mu(n)$ is the special case where $\mu(n, d) = \mu\left(\frac{n}{d}\right)$. We recall that Rota studied generalized forms of the Möbius function for posets $\mathcal{P}$ endowed with a partial ordering $\leq$ [33]. Within this context, we define

$$\mu(s, s) = 1, \forall s \in \mathcal{P}, \mu(s, u) = -\sum_{s\leq t<u} \mu(s, t), \forall s < u \in \mathcal{P},$$

where we have an inversion relation for all $t \in \mathcal{P}$ of the following form:

$$g(t) = \sum_{s \leq t} f(s) \iff f(t) = \sum_{s \leq t} g(s)\mu(s,t).$$

4.3 Functional equations for generating functions of triangular sequences

Example 4.7 (The Binomial transform)**.** Suppose that we have a sequence, $\{f_n\}_{n\geq 0}$, and its (formal) *ordinary generating function* (OGF) is given by $F(z) := \sum_{n\geq 0} f_n z^n$. Then a standard method for generating the summatory functions of the f_n is to scale by a factor of the geometric series as

$$[z^n]\frac{F(z)}{1-z} = \sum_{n=0}^{x} f_n.$$

In fact, we can actually go farther with a so-called *binomial transform* of generating functions to express

$$[z^n]\frac{F\left(-\frac{z}{1-z}\right)}{1-z} = \sum_{n=0}^{x} \binom{x}{n}(-1)^n f_n.$$

Other generating function transformations can be used to generate finite sums of a sequence scaled by another lower triangular sequence, such as the Stirling numbers using the *Stirling transform* to note, in a similar manner.

Proposition 4.8. *Fix any sequence $\{f_n\}_{n\geq 1}$ and some lower triangular sequence $g_{n,k}$. Let the associated sequence of sums be defined by*

$$S_f[g](n) := \sum_{k=1}^{n} g_{n,k} f_k, n \geq 1.$$

Let the column generating functions of g be defined as absolutely convergent series when $|q| < \sigma_{g,a}$ for some $\sigma_{g,a} > 1$ for integers $k \geq 1$ as follows:

$$G_k(q) := \sum_{n\geq k} g_{n,k} q^n.$$

Suppose that the OGF of $S_f[g](n)$ is given by

$$\widetilde{S}_f[g](q) := \sum_{n\geq 1} S_f[g](n) q^n,$$

and that $F(q), \widehat{F}(q)$ denote the OGF and EGF of $\{f_n\}_{n\geq 0}$, respectively. We have the following formulas for special case series types that provide explicit relations between these OGFs:

(A) If $G_k(q) = H_1(q)H_2(q)^k$ for some fixed functions $H_i(q)$, then

$$\widetilde{S}_f[g](q) = H_1(q)H_2(2)\left(F\left(H_2(q)\right) - f_0\right).$$

(B) If $G_k(q) = H_3(q)\frac{H_4(q)^k}{k!}$ for some component functions $H_i(q)$, then

$$\widetilde{S}_f[g](q) = H_3(q)\left(\widehat{F}\left(H_4(q)\right) - f_0\right).$$

Proof. The proofs are nearly trivial provided that the OGFs $G_k(q)$ are absolutely convergent so that we can interchange the order of summation. With this assumption, each respective claim follows by summing a geometric or exponential series. □

Definition 4.9. The *Hadamard product* of two ordinary generating functions $F(q)$ and $G(q)$, respectively enumerating the sequences of $\{f_n\}_{n\geq 0}$ and $\{g_n\}_{n\geq 0}$ is defined by

$$(F \circ G)(q) := \sum_{n\geq 0} f_n g_n q^n, \quad \text{for } |q| < \sigma_{c,f}\sigma_{c,g},$$

where $\sigma_{c,f}$ and $\sigma_G c, g$ denote the radii (abscissa) of convergence of each respective generating function. Analytically, we have an integral formula and corresponding coefficient extraction formula for the Hadamard product of two generating functions when $F(q)$ is expandable in a fractional (Pusieux) series respectively given by [6, §1.12(V); Ex. 1.30, p. 85] [54, §6.3]

$$\begin{aligned}(F \circ G)(q^2) &= \frac{1}{2\pi}\int_0^{\pi} F(qe^{\imath t})G(qe^{-\imath t})dt \\ (F \circ G)(q) &= [x^0]F\left(\frac{q}{x}\right)G(x).\end{aligned}$$

Theorem 4.10 (An integral formula for generalized sums)**.** *We adopt the notation for the summation and column type OGFs from Proposition 4.8. Let the bivariate generating function*

$$G(w,q) := \sum_{k\geq 1} G_k(q)w^k.$$

We have that

$$S_f[g](n) = [q^n]\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} F\left(\sqrt{q}e^{\imath t}\right)G(1,\sqrt{q}e^{-\imath t})dt\right).$$

Proof. The proof of this theorem is trivial. □

Example 4.11. Suppose that we set $g_{n,k} := [(n,k)=1]_\delta\,[n\geq k]_\delta$. We have seen that

$$G_k(q) = q^k \times \sum_{d|k} \frac{\mu(d)}{1-q^d}.$$

Theorem 4.10 implies that the sums

$$S_f[g](n) := \sum_{\substack{d\leq n\\(d,n)=1}} f(d),$$

are generated as the coefficients of q^n in the expansion [19, §1]

$$\begin{aligned}\frac{1}{2\pi}\times\int_{-\pi}^{\pi}\left(\sum_{n,k\geq 1}\sum_{d|k}\frac{\mu(d)f_n\sqrt{q}^{n+k}}{1-\sqrt{q}^d e^{-\imath dt}}e^{\imath(n-k)t}\right)dt &= \sum_{\substack{n,k\geq 1\\m\geq 0}}\sum_{d|k}\mu(d))f_n\,[n=md+k]_\delta \\ &= \sum_{k\geq 1}\sum_{0\leq m<k}\sum_{d|(k,k-m)}\mu(d)\times\sum_{n\geq 1} f_{nk+m}q^{nk+m} \\ &= \sum_{k\geq 1}\sum_{\substack{1\leq m\leq k\\(m,k)=1}}\sum_{0\leq r<k} F\left(qe^{\frac{2\pi\imath r}{k}}\right)\frac{e^{\frac{2\pi\imath m}{k}}}{k}.\end{aligned}$$

This argument also shows that

$$\phi_m(n) = \sum_{1\leq k\leq n} k^m\left(\sum_{d|(k,n-k)}\mu(d)\right), n\geq 1.$$

More generally, for any arithmetic function f we have that

$$\sum_{\substack{d\leq n\\(d,n)=1}} f(d) = \sum_{1\leq k\leq n} f(k)\left(\sum_{d|(k,n-k)} \mu(d)\right), n \geq 1.$$

4.4 Definitions of generalized kernel-based discrete convolution type sums

For a bivariate kernel function $\mathcal{D} : \mathbb{N}^2 \to \mathbb{C}$, we define the next class of convolution type sums, or $\mathcal{D}$-convolution type sums, according to the formula

$$(f \boxdot_{\mathcal{D}} g)(n) := \sum_{k=1}^{n} f(k)g(n+1-k)\mathcal{D}(n,k), n \geq 1. \tag{43}$$

In the section ahead, we are able to connect these special convolution type sums that form a widely reaching class of applications through particular specializations of the $(f, g; \mathcal{D})$.

Definition 4.12. We say that a kernel, or weight function $\mathcal{D} : (\mathbb{Z}^+)^2 \to \mathbb{C}$ is *lower triangular* if $\mathcal{D}(n,k) = 0$ for all $k > n \geq 1$. We say that this kernel function is invertible provided that

$$\det\left[(\mathcal{D}(n,k))_{1\leq n,k\leq N}\right] \neq 0, \forall N \geq 1.$$

Suppose that $\mathcal{D}$ is an invertible, lower triangular kernel function, and that the arithmetic function g is invertible with respect to $\mathcal{D}$-convolution, i.e., defined such that $g(1) \neq 0$. Then we can express the OGF of these sums according to the following parameterized invertible matrix based factorizations for $n \geq 1$:

$$(f \boxdot_{\mathcal{D}} g)(n) = [q^n]\frac{1}{\mathcal{C}(q)} \times \sum_{n\geq 1}\left(\sum_{k=1}^{n} s_{n,k}(g; \mathcal{C}, \mathcal{D})f(k)\right)q^n, \text{ for } \mathcal{C}(0) \neq 0. \tag{44}$$

Proposition 4.13 (Inversion)**.** *Suppose that $\mathcal{D}$ is a kernel function for a convolution sequence defined in* (43) *that is both invertible and satisfies $\mathcal{D}(n,n) \neq 0$ for all $n \geq 1$. Let the corresponding lower triangular sequence of entries for its inverse matrix be denoted by $\mathcal{D}^{-1}(n,k)$. Then for any $n \geq 1$ we have that*

$$g(n) = (f \boxdot_{\mathcal{D}} 1)(n) \iff f(n) = \sum_{k=1}^{n} g(k)\mathcal{D}^{-1}(n,k).$$

Proof. By construction, we suppose that $1 \leq N < +\infty$ and that we have two N-dimensional vectors, $\vec{f} := [f(1), \ldots, f(N)]^T$ and $\vec{g} := [g(1), \ldots, g(N)]^T$. It follows that

$$\vec{g} = \mathcal{D}(N) \cdot \vec{f} \implies \vec{f} = \mathcal{D}(N)^{-1} \cdot \vec{g},$$

where $\mathcal{D}(N) := (\mathcal{D}(n,k))_{1\leq n,k\leq N}$. □

Suppose that $\mathcal{D}(n,k)$ is an invertible, lower triangular kernel function. We say that an arithmetic function g is invertible with respect to $\mathcal{D}$-convolution if there exists a (left) inverse function $g^{-1}[\mathcal{D}](n)$ such that for all integers $n \geq 1$ we have that $(g^{-1}[\mathcal{D}] \boxdot_{\mathcal{D}} g)(n) = \delta_{n,1}$. We can restrict ourselves to the cases where we take $\mathcal{D}$ to be *symmetric* with respect to $\mathcal{D}$-convolution: That is, where we have that $\mathcal{D}(n,k) = \mathcal{D}(n, n+1-k)$ for all $1 \leq k \leq n$. In these cases, we have that the left and corresponding right inverse functions of any g with respect to $\mathcal{D}$-convolution are identical when they exist.

Proposition 4.14 (Inverses with respect to $\mathcal{D}$-convolution)**.** *An arithmetic function g is invertible with respect to $\mathcal{D}$-convolution for a fixed invertibly lower triangular and symmetric kernel $\mathcal{D}$ if and only if $g(1) \neq 0$. When the function $g^{-1}[\mathcal{D}](n)$ exists, it is unique, and can be computed exactly by recursion via the following formula:*

$$g^{-1}[\mathcal{D}](n) = \begin{cases} \frac{1}{\mathcal{D}(1,1)g(1)}, & n = 1; \\ -\frac{1}{\mathcal{D}(n,n)g(1)} \times \sum\limits_{1 \leq k < n} g^{-1}[\mathcal{D}](k) g(n+1-k) \mathcal{D}(n,k), & n \geq 2. \end{cases}$$

Moreover, provided that $g^{-1}[\mathcal{D}]$ exists, we have that

$$g^{-1}[\mathcal{D}](n) = \mathcal{D}^{-1}(n,1) \times [q^{n-1}] \left(\sum_{n \geq 0} g(n+1) q^n \right)^{-1}, n \geq 1.$$

Proof. Fix any arithmetic function g with $g(1) \neq 0$. The recursive formula follows by a rearrangement of the terms in the equation $(g^{-1}[\mathcal{D}] \boxdot_{\mathcal{D}} g)(n) = \delta_{n,1}$. To prove the exact formula, we see that we can set up an invertible matrix vector system of the form $A_{\mathcal{D},g}(N) \cdot \vec{f} = \vec{b}$ and solve for $\vec{f}$ where $A_{\mathcal{D},g}(N) := (\mathcal{D}(n,k) g(n+1-k) [k \leq n]_{\delta})_{1 \leq n,k \leq N}$, $\vec{f} = [g^{-1}[\mathcal{D}](1), \ldots, g^{-1}[\mathcal{D}](N)]^T$, and $\vec{b} = [1, 0, \ldots, 0]^T$ for any $N \geq 1$. Then we have that

$$g^{-1}[\mathcal{D}](n) = (A_{\mathcal{D},g}^{-1}(N))_{n,1}, \forall N \geq n \geq 1.$$

Notice that for $\mathcal{D}(N) := (\mathcal{D}(n,k))_{1 \leq n,k \leq N}$, we can write

$$A_{\mathcal{D},g}(N) = \mathcal{D}(N) \cdot \begin{bmatrix} g(1) & 0 & 0 & \cdots & 0 \\ g(2) & g(1) & 0 & \cdots & 0 \\ g(3) & g(2) & g(1) & \cdots & 0 \\ & & & \vdots & 0 \\ g(N) & g(N-1) & g(N-2) & \cdots & g(1) \end{bmatrix},$$

where the right-hand-side matrix involving g is an invertible Topelitz matrix. Thus by inversion, we see that our claimed formula is correct. ☐

n	$g^{-1}[\mathcal{D}](n)$
1	$\frac{1}{\mathcal{D}(1,1)g(1)}$
2	$-\frac{\mathcal{D}(2,2)g(2)}{\mathcal{D}(1,1)\mathcal{D}(2,1)g(1)^2}$
3	$\frac{\mathcal{D}(2,2)\mathcal{D}(3,2)g(2)^2}{\mathcal{D}(1,1)\mathcal{D}(2,1)\mathcal{D}(3,1)g(1)^3} - \frac{\mathcal{D}(3,3)g(3)}{\mathcal{D}(1,1)\mathcal{D}(3,1)g(1)^2}$
4	$-\frac{\mathcal{D}(2,2)\mathcal{D}(3,2)\mathcal{D}(4,2)g(2)^3}{\mathcal{D}(1,1)\mathcal{D}(2,1)\mathcal{D}(3,1)\mathcal{D}(4,1)g(1)^4} + \frac{(\mathcal{D}(2,1)\mathcal{D}(3,3)\mathcal{D}(4,2)+\mathcal{D}(2,2)\mathcal{D}(3,1)\mathcal{D}(4,3))g(3)g(2)}{\mathcal{D}(1,1)\mathcal{D}(2,1)\mathcal{D}(3,1)\mathcal{D}(4,1)g(1)^3} - \frac{\mathcal{D}(4,4)g(4)}{\mathcal{D}(1,1)\mathcal{D}(4,1)g(1)^2}$

Table 4.1: *A table of inverse functions $g^{-1}[\mathcal{D}]$ for any fixed arithmetic function g such that $g(1) \neq 0$, and any fixed invertible, lower triangular kernel function satisfying $\mathcal{D}(n,n) \neq 0$ for all $n \geq 1$, is computed symbolically in the listings above.*

In what follows, we adopt the notation that $c_n(\mathcal{C}) := [q^n]\mathcal{C}(q)$ and $p_n(\mathcal{C}) := [q^n]\mathcal{C}(q)^{-1}$ for any $n \geq 0$ and any OGF $\mathcal{C}(q)$ such that $\mathcal{C}(0) \neq 0$.

Theorem 4.15 (Generalized factorization theorems for $\mathcal{D}$-convolution)**.** *Suppose that g is any arithmetic function that is invertible with respect to $\mathcal{D}$-convolution for some fixed invertible, lower triangular kernel function $\mathcal{D}(n, k)$. The matrices with entries given by $s_{n,k}(g; \mathcal{C}, \mathcal{D})$ in (44) are invertible and satisfy the following formulas for $1 \leq k \leq n$:*

$$s_{n,k}(g; \mathcal{C}, \mathcal{D}) = \sum_{j=1}^{n} c_{n-j}(\mathcal{C}) g(j+1-k) \mathcal{D}(j, k)$$
$$s_{n,k}^{-1}(g; \mathcal{C}, \mathcal{D}) = \sum_{j=1}^{n} g^{-1}[\mathcal{D}](n+1-j) p_{j-k}(\mathcal{C}) \mathcal{D}(n, j).$$

Proof. The formula for the ordinary matrix entries is obvious upon multiplying both sides of (44) by the OGF, $\mathcal{C}(q)$, and then extracting the coefficients of q^n and $f(k)$ in the resulting expansion. The proof of the inverse matrix formulas is routine, but less obvious. Since $s_{n,k}(g; \mathcal{C}, \mathcal{D})$ is lower triangular with non-zero entries when $n = k$ for all $n \geq 1$, it forms a sequence of invertible square matrices taking determinants over $1 \leq n, k \leq N$ for each fixed $N \geq 1$. Consider the special case of the $\mathcal{D}$-convolution sums where $f(n) := s_{n,k}^{-1}(g; \mathcal{C}, \mathcal{D})$, an inverse sequence that we know is unique for each fixed pair (f, g), for integers $k \geq 1$. By the orthogonality relations between the lower triangular ordinary and inverse matrices, we can see that

$$p_{n-k}(\mathcal{C}) = \sum_{j=1}^{n} s_{n,k}^{-1}(g; \mathcal{C}, \mathcal{D}) g(n+1-j) \mathcal{D}(n, j).$$

Since g is invertible with respect to $\mathcal{D}$-convolution, we recover our claimed formula for the inverse matrix entries. $\square$

5 Canonical representations of factorization theorems for special sums

The material we present in this concluding section of the thesis is not exhaustive, nor conclusive. Rather it serves to motivate a discussion of rigorously formulating "best possible" factorization theorems and relationships between application-dependent convolution type sequences. These so-termed "canonical" expressions that arise in other important applications and future useful constructions based on our work from Section 2. A few conjectures are presented in the last subsection below that suggest a loose application-tied partition theoretic interpretation behind the ideal correlation for factorization theorems we have for the class of more general convolution type sums defined in equation (43).

5.1 Correlation statistics to quantify the notion of a "canonically best" property

There is a vast body of modern literature in number theory that motivates semi-standardized ways to quantify relationships between functions and sequences we study via correlation based statistics. There is historically relevant literature about using statistical analysis to motivate studying number theoretic objects. For example, the non-trivial zeros of the Riemann zeta function have been related and bounded via pair correlation formulas. Moreover, this topic continues to be a active and fruitful way of understanding this complicated subject matter. We recall from [28] that results in analytic number theory that make sense of the distribution of the non-trivial zeros of $\zeta(s)$ originated in the work of Montgomery. Subsequent follow-up work that collectively builds on Montgomery's contributions in the context of L-functions, Gaussian Unitary Ensemble (or GUE), applications in random matrix theory and their associated correlation statistics is famously due to Hejal, Rudnick, Sarnak and Odlyzko.

We posit by extension that using correlation metrics, or so-called sequence correlation *statistics*, to precisely define and rigorously formulate what we consider to be best possible attainable relationships for the factorization theorems given in (44). In general, we can study the so-called *sequence vector correlation* (including the information theoretic cross-correlation statistics seen below) that relate more general sequences and vectors of real and rational numbers. In our case, we need to identify and prove optimal representations for our notion of the "best possible", or optimal. e.g., canonical view point, for how we should express the OGF factorization theorems as they are identified in (44). We then set out to precisely construct formulas that can be maximized (minimized) with respect to all possible one-dimensional sequences in a way that captures the qualitatively meaningful relationships between the sequences from the LGF case. The goal is to do this in a very general setting that reveals underlying hidden relationships charaterizing any particular class of $\mathcal{D}$-convolution type sums in analog to the observations of natural relationships between multiplicative number theory and the partition functions from the LGF case witnessed in Section 2.

Example 5.1 (A model starting point)**.** The exact bounded ranges we can expect for cross-correlation coefficients to express a numerical index between vectors in our problem context are, in general, variable and subject to the qualitative interpretation which we have to reason about separately to ensure a good model fit. If we wish to normalize the range to be within $[-1, 1]$, there is the standardized definition of the (non-central, or non-centralized) *Pearson correlation coefficient.* It is defined as the numerical statistic relating any two N-tuples, $\vec{a} := (a_1, \ldots, a_N), \vec{b} := (b_1, \ldots, b_N) \in \mathbb{Q}^N$ for any fixed $N \geq 1$, given by

$$\operatorname{PearsonCorr}(N; \vec{a}, \vec{b}) := \frac{1}{N} \times \frac{\sum\limits_{j=1}^{N} a_j b_j}{\sqrt{\sum\limits_{1 \leq i,j \leq N} a_i^2 b_j^2}} \in [-1, 1].$$

Notation 5.2. We again define the shorthand sequence notation of $c_n(\mathcal{C}) := [q^n]\mathcal{C}(q)$ and $p_n(\mathcal{C}) := [q^n]\mathcal{C}(q)^{-1}$ for any $\mathcal{C}(q)$ such that $\mathcal{C}(0) \neq 0$. We are going to adapt the non-centralized Pearson cross-correlation formula by choosing our correlation statistic to be computed according to the following sums:

$$\begin{aligned}
\operatorname{Corr}(n;\mathcal{C},\mathcal{D}) &:= \frac{1}{n} \times \frac{\sum\limits_{k=1}^{n} |c_k(\mathcal{C})\mathcal{D}^{-1}(n,k)|}{\sqrt{\left(\sum\limits_{k=1}^{n} c_k(\mathcal{C})^2\right)\left(\sum\limits_{k=1}^{n} \mathcal{D}^{-1}(n,k)^2\right)}} \\
\operatorname{Corr}(\mathcal{C},\mathcal{D}) &:= \sum_{n\geq 1} \operatorname{Corr}(n;\mathcal{C},\mathcal{D}).
\end{aligned} \tag{45}$$

Question 5.3 (The crux of our correlation statistic optimization problem)**.** For a fixed lower triangular, invertible kernel function $\mathcal{D}$, we need to identify a concrete candidate OGF, $\mathcal{C}(q)$, so that

$$0 \leq \operatorname{Corr}(\mathcal{C},\mathcal{D}) < +\infty,$$

is maximized or minimized (and finite) over all possible input functions $\mathcal{C}(q) \in \mathbb{Q}[[q]]$ such that $\mathcal{C}(0) \neq 0$, or alternately $\mathcal{C}(q) \in \mathbb{Z}[[q]]$ with $\mathcal{C}(0) = 1$. Note that this criteria and the corresponding maximization procedure is always independent of the arithmetic functions f, g input to the weighted $\mathcal{D}$-convolution sums, $f \boxdot_{\mathcal{D}} g$.

Example 5.4 (Finding optimal statistics for the LGF case)**.** We will make a somewhat arbitrary decision that works well in practice to define

$$f(n,k) := \begin{cases} \frac{1}{n} \times \frac{c_k(\mathcal{C})\mathcal{D}^{-1}(n,k)}{\left(\sum\limits_{m\leq n} c_m(\mathcal{C})^2\right)^{\frac{1}{2}}\left(\sum\limits_{m\leq n} \mathcal{D}^{-1}(n,m)^2\right)^{\frac{1}{2}}}, & \text{if } 1 \leq k \leq n \leq N; \\ 0, & \text{otherwise,} \end{cases}$$

Notice that in the cases we next look at for the LGF example, we have that

$$\sum_{m\leq n} \mathcal{D}^{-1}(n,m)^2 = \sum_{d|n} \mu^2(d) = 2^{\omega(n)}, n \geq 1.$$

We then want to optimize the minimal bounded formulas

$$\lim_{N\to\infty} \sum_{n\leq N} \sum_{k=1}^{n} f(n,k) = \lim_{N\to\infty} \sum_{n\leq N} \frac{(|c_{-}(\mathcal{C})| * |\mu|)(n)}{n\rho_{\mathcal{C}}(n)\left(\sqrt{2}\right)^{\omega(n)}} \in [0,1],$$

over all $\mathcal{C}(q)$ such that $\mathcal{C}(0) \neq 0$ and with

$$\rho_{\mathcal{C}}(n) := \left(\sum_{0\leq m\leq n} c_m(\mathcal{C})^2\right)^{\frac{1}{2}}.$$

We see that minimizing the reciprocal of the limiting series in the previous equation leads to a maximal possible bound on the cross-correlation statistics we defined in (45). The preliminary numerical results we cite for this case in Section 5.2 below is able to numerically predict how closely this statistic for the LGF case comes to attaining the theoretically maximal correlation statistic in limiting cases. These series approximations can be made very accurate for certain classes of OGFs that commonly arise in applications.

5.2 Maximal correlation bounds for the LGF case

Since the task of identifying the target limiting cross-correlation statistic in (45) is substantially complicated in the general case, we first look at the problem of optimality for the LGF case. For each such OGF $\mathcal{C}(q)$, we define

$$\mathrm{Corr}_{\mathrm{LGF}}(\mathcal{C}) = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{\mu^2(k)}{k} \times \sum_{j\leq \left\lfloor \frac{n}{k}\right\rfloor} \frac{|c_j(\mathcal{C})|}{j\rho_{\mathcal{C}}(jk)(\sqrt{2})^{\omega(jk)}} = \sum_{j,k\geq 1} \frac{\mu^2(j)|c_k(\mathcal{C})|}{(jk)\rho_{\mathcal{C}}(jk)(\sqrt{2})^{\omega(jk)}}, \tag{46}$$

where we define the partial variance of $\mathcal{C}$ to be

$$\rho_{\mathcal{C}}(N) := \sqrt{\sum_{1\leq i\leq N} c_i(\mathcal{C})^2}, N \geq 1.$$

The previous doubly infinite series for the LGF correlation statistic is non-trivial to tightly bound from above and below because we have, in general, for $j, k \geq 1$ that

$$\omega(jk) = \omega(j) + \omega\left(\frac{k}{(k,j)}\right).$$

That is, the function $\omega(n)$ is only *strongly* (as opposed to *completely*) *additive.*

Definition 5.5. Fix any $0 < \delta < +\infty$. Provided an input test OGF $\mathcal{C}(q)$, we set

$$A_0(\mathcal{C}, \delta) := \lim_{N\to\infty} \frac{1}{N^{\delta}} \times \sqrt{\sum_{1\leq n\leq N} c_n(\mathcal{C})^2}.$$

We are naturally interested in finding the optimal, or so-termed "canonically best" correlation coefficient that corresponds to an explicit OGF $\mathcal{C}(q)$. We have already noticed that taking $\mathcal{C}(q) := (q;q)_{\infty}$ leads to very interesting relationships between the matrices in the Lambert series factorization theorems. Conjecture 5.12 given at the end of this section is suggestive of why the expected optimal OGF witness that attains the maximal correlation statistic, the function $\mathcal{C}(q)$, has series coefficients that satisfy (by the *pentagonal number theorem*)

$$\delta = \sup\left\{\rho > 0 : A_0(\mathcal{C}, \rho) > 0\right\} \equiv \frac{1}{4}.$$

Hence, we are interested in considering OGFs $\mathcal{C}(q)$ with integer coefficients such that the maximal $\delta > 0$ in the definition for which $A_0(\mathcal{C}, \delta) > 0$ is given by $\delta := \frac{1}{4}$.

Using the same construction as the special case where $\mathcal{C}(q) := (q;q)_{\infty}$, OGF forms whose coefficients are in $\{0, \pm 1\}$ such that the corresponding $\delta = \frac{1}{4}$ can be seen as often having non-zero coefficients, $|c_n(\mathcal{C})|$, for $n \in \{p(n)\}_{n=-\infty}^{\infty} \setminus \{0\}$ where $\hat{p}(n) = \frac{n(an+b)}{2}$ for integers $a \geq 3$ and $1 \leq b < a$. As we can see through the next OGF examples, generating functions of this form are natural to consider in partition theoretic applications [14, §19.9]:

$$\begin{aligned}
(q;q)_{\infty} &= \prod_{n\geq 0}\left\{(1-q^{3n+1})(1-q^{3n+2})(1-q^{3n+3})\right\} &&= \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2} \qquad \text{(A)} \\
(-q;q^2)_{\infty}^2(q^2;q^2)_{\infty} &= \prod_{n\geq 0}\left\{(1+q^{2n+1})^2(1-q^{2n+2})\right\} &&= \sum_{n=-\infty}^{\infty} q^{n^2} \\
(q;q^2)_{\infty}^2(q^2;q^2)_{\infty} &= \prod_{n\geq 0}\left\{(1-q^{2n+1})^2(1-q^{2n+2})\right\} &&= \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}
\end{aligned}$$

$$(q;q^5)_\infty(q^4;q^5)_\infty(q^5;q^5)_\infty = \prod_{n\geq 0}\left\{(1+q^{5n+1})(1-q^{5n+4})(1-q^{5n+5})\right\} \quad = \sum_{n=-\infty}^{\infty}(-1)^n q^{n(5n+3)/2} \quad \text{(B)}$$

$$(q^2;q^5)_\infty(q^3;q^5)_\infty(q^5;q^5)_\infty = \prod_{n\geq 0}\left\{(1+q^{5n+2})(1-q^{5n+3})(1-q^{5n+5})\right\} \quad = \sum_{n=-\infty}^{\infty}(-1)^n q^{n(5n+1)/2} \quad \text{(C)}$$

In these cases, we have that $A \equiv A_0(\mathcal{C},\delta) = 2\left(\frac{2}{a}\right)^{\frac{1}{4}}$ with $a = 3$ (A), 1, 1, 5 (B) and 5 (C), respectively. A comparison of these generating functions extending the visual projection of a correlation matrix onto the penguin image is provided below for reference with the OGFs labeled (A), (B) and (C) corresponding to the images in Figure 5.1 ordered from left to right.

Figure 5.1

Whenever (a,b) are integers such that $a \geq 1$ and $1 \leq b < a$, let

$$\mathcal{C}_{a,b}(q) := \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n(an+b)}{2}}.$$

We compute the special case values of $\mathrm{Corr}_{\mathrm{LGF}}(\mathcal{C}_{a,b})$ in Figure 5.2.

(a,b)	$(1,0)$	$(3,1)$	$(5,1)$	$(5,3)$	$(7,1)$	$(7,3)$	$(7,5)$
$\mathrm{Corr}_{\mathrm{LGF}}(\mathcal{C}_{a,b})$	0.76336	0.920081	0.0624965	0.672979	0.0108645	0.0374214	0.612813
(a,b)	$(11,1)$	$(11,3)$	$(11,5)$	$(11,7)$	$(11,9)$	$(13,1)$	$(13,3)$
$\mathrm{Corr}_{\mathrm{LGF}}(\mathcal{C}_{a,b})$	0.00158547	0.00264543	0.00470973	0.0202822	0.587536	0.000748935	0.00107876
(a,b)	$(13,5)$	$(13,7)$	$(13,9)$	$(13,11)$	$(17,15)$	$(23,21)$	$(29,27)$
$\mathrm{Corr}_{\mathrm{LGF}}(\mathcal{C}_{a,b})$	0.00182387	0.0046692	0.0188231	0.583074	0.58239	0.569502	0.56664

Figure 5.2

Lemma 5.6. *We have that the average order*

$$\mathbb{E}\left[\omega(\hat{p}(x)) + \omega(\hat{p}(-x))\right] \sim \left(\frac{a+4}{a^2}\right)(\log\log x + B),$$

where $B \approx 0.261497$ is the Mertens constant *from Mertens' second theorem. That is, the average order of $\omega(n)$ over the two distinct integer-valued polynomials we get by expanding $p(\pm n)$ over all non-zero integers n is approximately a constant times $\mathbb{E}[\omega(x)]$ up to error terms that vanish as $x \to \infty$.*

Remark 5.7. The function $\omega(n)$ stays near its average order with a limiting centrally normal tendency as proved by the Erdős-Kac theorem that states [15, §1.7] [26, *cf.* §7.4]

$$\frac{1}{x} \times \#\left\{n \leq x : \frac{\omega(n) - \log\log n}{\sqrt{\log\log n}} \leq z\right\} = \Phi(z) + o(1), z \in \mathbb{R}, \text{ as } x \to \infty.$$

We also have that uniformly for $0 < r \leq 1$

$$\#\{n \leq x : \omega(n) \leq r \log\log x\} \ll x(\log x)^{r-1-r\log r}, \text{ as } x \to \infty,$$

and that uniformly for $1 \leq r \leq R < 2$

$$\#\{n \leq x : \omega(n) \geq r \log\log x\} \ll_R x(\log x)^{r-1-r\log r}, \text{ as } x \to \infty.$$

Our intuition is hence to notice that $\omega(k)$ is so universally centered at its average order for almost every positive integer $k \geq 3$. Suppose that for $j, k \geq 16$ (so that the first n such that $\log\log(n) > 1$ is $n := \lceil e^e \rceil = 16$) we replace the terms in (46) involving $\omega(jk)$ with the corresponding average order formula from Lemma 5.6 evaluated at $x := jk$. We will denote this modified series by $\widehat{\operatorname{Corr}}_{\mathrm{LGF}}(\mathcal{C}_{a,b})$. The following table (in comparison to the one given above) is suggestive of the regularity of these series and hence may be an approach towards obtaining tight bounds on the actual LGF correlation statistics as we observe in Figure 5.3.

(a,b)	$(1,0)$	$(3,1)$	$(5,1)$	$(5,3)$	$(7,1)$	$(7,3)$	$(7,5)$
$\widehat{\operatorname{Corr}}_{\mathrm{LGF}}(\mathcal{C}_{a,b})$	0.76336	0.920081	0.0624965	0.672979	0.0108645	0.0374214	0.612813
(a,b)	$(11,1)$	$(11,3)$	$(11,5)$	$(11,7)$	$(11,9)$	$(13,1)$	$(13,3)$
$\widehat{\operatorname{Corr}}_{\mathrm{LGF}}(\mathcal{C}_{a,b})$	0.00158547	0.00264543	0.00470973	0.0202822	0.587536	0.000748935	0.00107876
(a,b)	$(13,5)$	$(13,7)$	$(13,9)$	$(13,11)$	$(17,15)$	$(23,21)$	$(29,27)$
$\widehat{\operatorname{Corr}}_{\mathrm{LGF}}(\mathcal{C}_{a,b})$	0.00182387	0.0046692	0.0188231	0.583074	0.58239	0.569502	0.56664

Figure 5.3

Proof of Lemma 5.6. Suppose that we are evaluating the following sum:

$$E_{a,b}(x) = \frac{1}{\sqrt{x}} \times \sum_{\substack{n \geq 1 \\ \frac{n(an\pm b)}{2} \leq x}} \left[\omega\left(\frac{n(an+b)}{2}\right) + \omega\left(\frac{n(an-b)}{2}\right)\right].$$

We can perform a change of variable in the form of $v = \frac{n(an\pm b)}{2}$ so that $n = \frac{\sqrt{b^2+8av} \mp b}{2a}$ and $dn = \frac{2dv}{\sqrt{8av+b^2}}$. We know that the average order of the original function $\omega(n)$ is given by [14, §22.10]

$$\mathbb{E}[\omega(n)] = \frac{1}{x} \times \sum_{n \leq x} \omega(n) = \log\log n + B + o(1).$$

Then we have by the Abel summation formula, taking the summatory function, $A(t) := t(\log\log t + B + o(1))$, that

$$\begin{aligned}
E_{a,b}(x) &= \frac{4}{\sqrt{x}} \times \sum_{v \leq x} \frac{\omega(v)}{\sqrt{8av+b^2}} \\
&\sim \frac{4}{\sqrt{x}} \left(\frac{(\log\log x + B)x}{\sqrt{8ax+b^2}} + \int_3^x \frac{4av(\log\log v + B)}{(8av+b^2)^{\frac{3}{2}}} dv \right) \\
&\sim \frac{1}{2}\sqrt{\frac{8}{a}}(\log\log x + B) + \frac{16a}{(8a)^{\frac{3}{2}}\sqrt{x}} \times \int_3^x \frac{\log\log v + B}{\sqrt{v}} dv \\
&= \frac{1}{2}\sqrt{\frac{8}{a}}(\log\log x + B) + \frac{32a}{(8a)^{\frac{3}{2}}\sqrt{x}} \left((\log\log x + B)\sqrt{x} - 2\,\mathrm{Ei}\left(\frac{\log x}{2}\right)\right) \\
&\sim \frac{1}{2}\sqrt{\frac{8}{a}}\left(1 + \frac{4}{a}\right)(\log\log x + B). \qquad (*)
\end{aligned}$$

In determining the main term in transition from the second to last equation above, we have used that

$$\log\log x - \log\log 2 + 1 - \frac{3\log x}{8} \leq \mathrm{Ei}\left(\frac{\log x}{2}\right) \leq \log\log x - \log\log 2 + 1 - \frac{3\log x}{8} + \frac{11(\log x)^2}{144}.$$

The main term for $\frac{2E_{a,b}(x)}{\sqrt{8a}}$ in $(*)$ above corresponds to the average order formula we seek to evaluate since $n \leq \frac{\sqrt{8ax+b^2} \mp b}{2a}$ so that the correct scalar multiple in front of the average sum is similar to $\frac{2}{\sqrt{8ax}}$. $\square$

5.3 Conjectures on canonically best factorization theorems

5.3.1 Partition theoretic conjectures

Given the way in which we have chosen to expand the factorizations of $L_f(q)$ as

$$(f * 1)(n) = [q^n]L_f(q) = \frac{1}{\mathcal{C}(q)} \times \sum_{n \geq 1} \sum_{k=1}^{n} s_{n,k}[\mathcal{C}] f(k) q^n,$$

the form of the invertible, lower triangular matrices with entries given by the $s_{n,k}[\mathcal{C}]$ are independent of any arithmetic f that defines these expansions. Moreover, these matrices are completely determined by the choice of the reciprocal generating function factor of $\mathcal{C}(q)$ subject only to the requirement that $\mathcal{C}(0) \neq 0$. It follows that seeking an interpretation of the canonically "best possible" choice of this function as corresponding to the choice of taking $\mathcal{C}(q) := (q;q)_\infty$, a criteria which we define qualitatively as inducing an unexpected, or particularly revealing substructure to the left-hand-side divisor sums, $f * 1$, is also independent of any fixed arithmetic f that defines $L_f(q)$.

Definition 5.8 (The Euler transform)**.** We observe a property called the *Euler transform* which nicely suggests motivation for why the partition function $p(n)$ arises so naturally in this class of LGF examples. Namely, we borrow the canonical integer sequence transformation identified by Bernstein and Sloane (circa 2002) called `EULER` [5]. It states that if two arithmetic functions a_n, b_n with a corresponding former OGF defined by $A(x) := \sum_{n \geq 1} a_n x^n$ are related by the identity

$$1 + \sum_{n \geq 1} b_n x^n = \prod_{i \geq 1} \frac{1}{(1-x^i)^{a_i}} \equiv \exp\left(\sum_{k \geq 1} \frac{A(x^k)}{k}\right),$$

then we can explicitly relate these sequence by introducing an intermediate divisor sum, denoted by c_n. In particular, if $c_n := \sum_{d|n} d \cdot a_d$, then we have that

$$a_n = \frac{1}{n} \times \sum_{d|n} c_d \mu\left(\frac{n}{d}\right), n \geq 1.$$

Results from elementary number theory due to Euler show that the partition function $p(n)$ is related to the (ordinary) sum-of-divisors function, $\sigma(n) \equiv \sigma_1(n) := \sum_{d|n} d$, through the following recurrence relation:

$$n \cdot p(n) = \sum_{0 \leq k < n} \sigma_1(n-k)p(k), n \geq 1.$$

Since $p(0) = 1$, the resulting ODE for the generating functions that relate these two sequences shows that

$$p(n) = [q^n] \exp\left(\sum_{k \geq 1} \frac{\sigma_1(k) q^k}{k}\right).$$

On the other hand, when we take the constant sequence $a_n \equiv 1, \forall n \geq 1$, the product expanded through the `EULER` transformation we defined above corresponds to the infinite q-Pochhammer function product, $(q;q)_\infty^{-1}$, which again generates $p(n)$ for all $n \geq 0$. Since $\sigma_{-\alpha}(n) = \sigma_\alpha(n) n^{-\alpha}$ for all $n \geq 1$ and any real parameter $\alpha > 0$, taking the exponential of the sum over the OGF $A(q)$ yields that

$$\prod_{i \geq 1} (1-q^i)^{-1} = \exp\left(\sum_{k \geq 1} \frac{q^k}{k(1-q^k)}\right) = \exp\left(\sum_{k \geq 1} \frac{\sigma_1(k) q^k}{k}\right).$$

Thus, we reason that the fundamental relation for $p(n)$ to the multiplicative divisor sums $\sigma_1(n)$ explains why the partition function arises here in the LGF case. We can look to the LGF special case for clues to see a good first order heuristic that we can use to measure how closely related the matrix and inverse matrix coefficients are for a fixed $\mathcal{D}$-convolution summation type. We clearly must define our metric to qunatify this heuristic so that is depends only on the kernel function $\mathcal{D}$, and the OGF $\mathcal{C}(q)$, and is always (of course) independent of f, g.

Conjecture 5.9. *An optimal OGF, $\mathcal{C}(q)$, that maximizes the correlation coefficients in (45), is given by*

$$\mathcal{C}(q) := \prod_{k \geq 1} \left(\sum_{n \geq 0} \mathcal{D}(n+k-1, k) q^n\right)^{-1}.$$

Conjecture 5.10. *The LGF OGF matchings we saw in Section 5.3.1 by applying the Euler transform of sequences suggests an optimal selection of $\mathcal{C}(q)$ satisfies the following expansions:*

$$\mathcal{C}(q) = \exp\left(-\sum_{n \geq 1} \sum_{k=1}^{n} k\mathcal{D}(n,k) \frac{q^n}{n}\right) = \prod_{n \geq 1} \left(1 + q \times \sum_{k=1}^{n} k\mathcal{D}(n,k)\right)^{-1}.$$

Conjecture 5.11 (Equivalence of problems). *The cross-correlation statistic*

$$\operatorname{Corr}_1(\mathcal{C}, \mathcal{D}) := \sum_{n \geq 1} \frac{\frac{1}{n} \times \sum_{1 \leq k \leq n} c_k(\mathcal{C}) \mathcal{D}^{-1}(n,k)}{\sqrt{\sum_{1 \leq k \leq n} c_k(\mathcal{C})^2 \times \sum_{1 \leq k \leq n} \mathcal{D}^{-1}(n,k)^2}},$$

is maximized (minimized) over all possible OGFs $\mathcal{C}(q)$ if and only if

$$\operatorname{Corr}_2(\mathcal{C},\mathcal{D}) := \sum_{n\geq 1} \frac{\frac{1}{n} \times \sum\limits_{1\leq k\leq n} p_k(\mathcal{C})\mathcal{D}(n,k)}{\sqrt{\sum\limits_{1\leq k\leq n} p_k(\mathcal{C})^2 \times \sum\limits_{1\leq k\leq n} \mathcal{D}(n,k)^2}},$$

is maximized (minimized) over all such OGFs.

5.3.2 Other conjectures

The values of certain signed sums are often modeled as a $\{\pm 1\}$-valued random walk on the integers whose height after the x^{th} step is taken to be approximately $M(x)$ where the probabilities of moving by ± 1 at any given step are randomized [52, 34]. We know that the expectation of the absolute height at x of a prototypical random walk of this type is asymptotically $C\sqrt{x}$ for $C > 0$ an absolute constant. Namely, suppose that $\{X_i\}_{i\geq 1}$ is a sequence of independent random variables defined such that $\mathbb{P}[X_i = -1] = \mathbb{P}[X_i = +1] = \frac{1}{2}$ for all $i \geq 1$. We can form the sums $Y_n := \sum_{i\leq n} X_i$ for $n \geq 1$. By computation we have that $\mathbb{E}[Y_n] = 0$. At the same time, we can show that

$$\sigma_{Y_n} := \sqrt{\mathbb{E}[Y_n^2] - \mathbb{E}[Y_n]^2} = \sqrt{n}.$$

This follows since

$$Y_{n+1}^2 = (Y_n + X_n)^2 \implies \mathbb{E}[Y_{n+1}^2] = \mathbb{E}[Y_n^2] + 1 \implies \mathbb{E}[Y_n^2] = n, \text{ for all } n \geq 1.$$

An interpretation of the combined first and second moment analysis is that we should expect the random walk modeled by Y_n to be approximately zero-valued most of the time but with an expected spread in actual values of as much as $\sqrt{n}$ [51]. The *law of the iterated logarithm* more precisely implies that $Y_n = O\left(\sqrt{n \log\log n}\right)$ for all sufficiently large n.

Since the coefficients enumerated by a LGF for any function f are $f * 1$, and we know by elementary number theory that $(\mu * 1)(n) = (1 * \mu)(n) = \delta_{n,1}$ (the multiplicative identity function with respect to Dirichlet convolution), we have that inversion of the convolution sums of this type is performed by Dirichlet convolution with $\mu(n)$ (*cf.* Möbius inversion). The summatory function, or partial sums of $\mu(n)$ are defined by $M(x) := \sum_{n\leq x} \mu(n)$ for any $x \geq 1$. The values of $M(x)$ are often modeled by a similar random walk whose values are $\{0, \pm 1\}$-valued according to the distribution of $\mu(n) \mapsto \pm 1$. This is often viewed as a case of the ± 1-valued random walk above with a different leading constant factor on the variance.

The Riemann Hypothesis is equivalent to proving that

$$M(x) = O\left(x^{\frac{1}{2}+\epsilon}\right), \text{ for all } 0 < \epsilon < \frac{1}{2}.$$

The dependence of the divisors $d|n$ over which we sum f to compute $(f * 1)(n)$ at each $n \geq 1$ is deeply connected to the distribution of the primes. We assert that the conventional interpretation of the primes as randomly determined in the sense of the ± 1-valued random walk model from above plays a pivotal role in the maximal correlation statistic that relates the kernel $\mathcal{D}^{-1}(n,k) = \mu\left(\frac{n}{k}\right)[k|n]_\delta$ to an optimal OGF, $\mathcal{C}(q) = (q;q)_\infty$ (as we have predicted it should be). The next conjecture cuts precisely to the crux of the matter with respect to why we seem to witness an optimal correlation statistic in the LGF case when $\mathcal{C}(q)$ satisfies $\rho_{\mathcal{C}}(n)^2 \asymp \sqrt{n}$.

Conjecture 5.12. *For a fixed lower triangular, invertible kernel function $\mathcal{D}(n,k)$, let*

$$M_{\mathcal{D}}(x) := \sum_{n\leq x} \mathcal{D}^{-1}(n,1), x \geq 1.$$

Suppose that

$$\delta = \left(\inf\left\{\rho > 0 : M_{\mathcal{D}}(x) = O\left(x^{\rho+\varepsilon}\right), \forall \varepsilon > 0\right\}\right)^2.$$

An optimal OGF, $\mathcal{C}(q)$, that witnesses the maximum possible value of $\operatorname{Corr}(\mathcal{C}, \mathcal{D})$ *satisfies $\rho_{\mathcal{C}}(n) \asymp n^{\delta}$ as $n \to \infty$. That is, $\rho_{\mathcal{C}}(n)$ is bounded above and below by absolute constant multiples of n^{δ} for all sufficiently large n.*

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Glossary of notation and conventions

Symbol	Definition
$a_k(f,g;n)$	Discrete Fourier coefficients of the periodic divisor sums $s_k(f,g;n)$ defined as symbol $s_k(f,g;n)$ in this glossary. The precise definition of these sums is given by $a_k(f,g;n) = \sum_{d\mid(k,n)} g(d) f\left(\frac{n}{d}\right) \frac{d}{k}$.
$a_{k,\ell}$	Sequence of coefficients that are defined explicitly in the discrete Fourier series expansion of the type II sums $L_{f,g,k}(x)$. These coefficients are implicitly defined by Definition 3.23 by the sums $L_{f,g,k}(n) = \sum_{\ell=0}^{k-1} a_{k,\ell} \cdot e\left(\frac{\ell n}{k}\right)$, where $e(x)$ is the shorthand for the complex exponential terms in the exponential sums we define in the article.
$\Delta^k[f](n)$	We denote by the operator $\Delta^k[f]$ at $n \geq 1$ the following: $\Delta^k[f](n) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k f(n-k)$.
$B_n(x)$, B_n	The Bernoulli polynomials and Bernoulli numbers $B_n = B_n(0)$. These polynomials can be used via Faulhaber's formula, among others, to generate the integral k^{th} power sums $\sum_{d\mid n} \phi_k(d)(n/d)^k = 1^k + 2^k + \cdots + n^k$.
$\lceil x \rceil$	The ceiling function $\lceil x \rceil := x + 1 - \{x\}$ where $0 \leq \{x\} < 1$ denotes the fractional part of $x \in \mathbb{R}$.
$\chi_{1,k}(n)$	The principal Dirichlet character modulo k, i.e., the indicator function of the natural numbers which are relatively prime for $n, k \geq 1$, $\chi_{1,k}(n) = [(n,k) = 1]_\delta$.
$C_k(n)$	Sequence of nested k-convolutions of an arithmetic function f with itself. The precise definition of this sequence is given by $$C_k(n) = \begin{cases} \widehat{f}(n) - \widehat{f}(1)\varepsilon(n), & \text{if } k = 1; \\ \sum_{d\mid n} \left(\widehat{f}(d) - \widehat{f}(1)\varepsilon(d)\right) C_{k-1}(n/d), & \text{if } k \geq 2, \end{cases}$$ where the symbol $\hat{f}(n)$ is defined in glossary entry $\hat{f}(n)$.
$[q^n]F(q)$	The coefficient of q^n in the power series expansion of $F(q)$ about zero.
$\int_{C-\imath T}^{C+\imath T} f(s)ds$, $\int_{C-\imath\infty}^{C+\imath\infty} f(s)ds$	Denotes a complex contour integral typcially defined for some $C > \sigma_{f,c}$, the abcissa of convergence of f in the complex plane. We have that $\int_{C-\imath\infty}^{C+\imath\infty} f(s)ds = \lim_{T\to\infty} \int_{C-\imath T}^{C+\imath T} f(s)ds$.
$c_q(n)$	Ramanujan's sum, $c_q(n) := \sum_{d\mid(q,n)} d\mu\left(\frac{q}{d}\right)$.
$d_k(n)$	The generalized k-fold divisor function $d_k(n) = 1_{*_k}(n)$ whose DGF is $\zeta(s)^k$. Note that the divisor function $d(n) \equiv d_1(n)$.
$D_f(n)$	Function related to the Dirichlet inverse of a function f. More precisely, this function is defined by the sum $D_f(n) := \sum_{j=1}^{n} \frac{\mathrm{ds}_{2j}(f;n)}{\hat{f}(1)^{2j+1}}$, where this definition involves the glossary symbols $\mathrm{ds}_j(f;n)$ and $\hat{f}(n)$. Lemma 3.16 relates this function to the Dirichlet inverse of the function $\hat{f}(n)$.
$d(n)$	The ordinary divisor function, $d(n) := \sum_{d\mid n} 1$.

Symbol	Definition
$\mathrm{ds}_j(f;n)$	Summands in the formula for the Dirichlet inverse of an arithmetic function. The precise definition of this function is given by $$\mathrm{ds}_j(f;n)=\begin{cases}(-1)^{\delta_{n,1}}\widehat{f}(n), & \text{if } j=1;\\ \sum\limits_{\substack{d\mid n\\ d>1}}\widehat{f}(d)\,\mathrm{ds}_{j-1}\left(f;\frac{n}{d}\right), & \text{if } j\geq 2,\end{cases}$$ where the fixed function $\hat{f}$ is defined by glossary symbol $\hat{f}(n)$.
$\mathrm{DFT}[f](k)$	The discrete Fourier transform (DFT) of f at k. We use this transformation in Section 3.3 of the article.
$\mathrm{DTFT}[f](k)$	The discrete time Fourier transform (DTFT) of f at k, also denoted by $F[k]$.
$\varepsilon(n)$	The multiplicative identity with respect to Dirichlet convolution, $\varepsilon(n)=\delta_{n,1}$.
$e(x)$	The complex exponential function, $e(x):=\exp(2\pi\imath x)$.
$f*C_-(m)$	This notation indicates that the index over which we perform the Dirchlet convolution is given by the dash parameter, $(f*C_-(m))(n):=\sum_{d\mid n}f(d)C_{\frac{n}{d}}(m)$.
$f*C_k(-)$	This notation indicates that the index over which we perform the Dirchlet convolution is given by the dash parameter, $(f*C_k(-))(n):=\sum_{d\mid n}f(d)C_k\left(\frac{n}{d}\right)$.
$*; f*g$	The Dirichlet convolution of f and g, $f*g(n):=\sum\limits_{d\mid n}f(d)g\left(\frac{n}{d}\right)$, for $n\geq 1$. This symbol for the discrete convolution of two arithmetic functions is the only notion of convolution of functions we employ within the article.
$\hat{f}(n)$	A shorthand notation for scaled arithmetic function terms $\hat{f}(n):=w^n(w^n-1)^{-1}f(n)$ for some non-zero indeterminate w.
$f^{-1}(n)$	The Dirichlet inverse of f with respect to convolution defined recursively by $f^{-1}(n)=-\frac{1}{f(1)}\sum\limits_{\substack{d\mid n\\ d>1}}f(d)f^{-1}\left(\frac{n}{d}\right)$ provided that $f(1)\neq 0$.
$F[k]$	Discrete Fourier transform coefficients.
$\lfloor x\rfloor$	The floor function $\lfloor x\rfloor:=x-\{x\}$ where $0\leq\{x\}<1$ denotes the fractional part of $x\in\mathbb{R}$.
f_{*_j}	Sequence of nested j-convolutions of an arithmetic function f with itself for integers $j\geq 1$. We define $f_{*_0}(n)=\delta_{n,1}$, the multiplicative identity with respect to Dirichlet convolution.
$f_\pm(n)$	For any arithmetic function f, we define $f_\pm(n)=f(n)\left[n>1\right]_\delta-f(1)\left[n=1\right]_\delta$, i.e., the function that has identical values as f for all $n\geq 2$, and whose initial value is $f_\pm(1):=-f(1)$ when $n=1$.
$\gamma(n)$	The squarefree kernel of n, $\gamma(n):=\prod_{p\mid n}p$.
G_j	Denotes the interleaved (or generalized) sequence of pentagonal numbers defined explictly by the formula $G_j:=\frac{1}{2}\left\lceil\frac{j}{2}\right\rceil\left\lceil\frac{3j+1}{2}\right\rceil$. The sequence begins as $\{G_j\}_{j\geq 0}=\{0,1,2,5,7,12,15,22,26,35,40,51,\ldots\}$.
$\mathrm{Id}_k(n)$	The power-scaled identity function, $\mathrm{Id}_k(n):=n^k$ for $n\geq 1$.

Symbol	Definition
$\mathbb{1}_{\mathbb{S}}$, $\chi_{\texttt{cond}(x)}$	We use the notation $\mathbb{1}, \chi : \mathbb{N} \to \{0,1\}$ to denote indicator, or characteristic functions. In paticular, $\mathbb{1}_{\mathbb{S}}(n) = 1$ if and only if $n \in \mathbb{S}$, and $\chi_{\texttt{cond}}(n) = 1$ if and only if n satisfies the condition `cond`.
$[n = k]_\delta$	Synonym for $\delta_{n,k}$ which is one if and only if $n = k$, and zero otherwise.
$[\texttt{cond}]_\delta$	For a boolean-valued `cond`, $[\texttt{cond}]_\delta$ evaluates to one precisely when `cond` is true, and zero otherwise.
$\vartheta_i(z,q)$, $\vartheta_i(q)$	For $i = 1,2,3,4$, these are the classical Jacobi theta functions where $\vartheta_i(q) \equiv \vartheta_i(0,q)$.
$J_t(n)$	The Jordan totient function $J_t(n) = n^k \times \prod_{p\mid n}(1 - p^{-t})$ satisfies $\sum_{d\mid n} J_t(d) = n^t$.
$\Lambda(n)$	The von Mangoldt lambda function $\Lambda(n) = \sum_{d\mid n} \log(d)\mu\left(\frac{n}{d}\right)$.
$\lambda(n)$	The Liouville lambda function $\lambda(n) = (-1)^{\Omega(n)}$.
$\lambda_k(n)$	The arithmetic function defined by $\lambda_k(n) = \sum_{d\mid n} d^k \lambda(d)$.
$\operatorname{lcm}(m,n)$, $[m,n]$	The least common multiple of m and n.
$L_{f,g,k}(x)$	The type II Anderson-Apostol sum over the arithmetic functions f,g, $L_{f,g,k}(x) := \sum\limits_{d\mid(k,x)} f(d)g\left(\frac{x}{d}\right)$.
$\operatorname{lsb}(n)$	The least significant bit of n in the base-2 expansion of n.
$\gcd(m,n); (m,n)$	The greatest common divisor of m and n. Both notations for the GCD are used interchangably within the article.
$\mu(n)$	The Möbius function.
$\mu_{n,k}$	The corresponding invertible sequence is an analog to the role of the Möbius function in Möbius inversion. In this case these inversion coefficients are defined such that $$g(n) = \sum_{\substack{d=1 \\ (d,n)=1}}^n f(d) \quad \iff \quad f(n) = \sum_{d=1}^n g(d+1)\mu_{n,d}.$$ See Proposition 3.3 for the relation of this sequence (and its inverse) to the factorizations of type I sums.
$\mu_{n,k}^{(-1)}$	Inverse matrix sequence of $\mu_{n,k}$.
$M(x)$	The Mertens function which is the summatory function over $\mu(n)$ denoted by the partial sums $M(x) := \sum\limits_{n \leq x} \mu(n)$.
OGF	*Ordinary generating function.* Given a sequence $\{f_n\}_{n\geq 0}$, its OGF (or sometimes called ordinary power series, OPS) enumerates the sequence by powers of a typically formal variable z: $F(z) := \sum_{n\geq 0} f_n z^n$. For $z \in \mathbb{C}$ within some radius or abcissa of convergence for the series, asymptotic properties can be extracted from the closed-form representation of F, and/or the original sequence terms can be recovered by performing an inverse Z-transform on the OGF.
ω_a	A primitive a^{th} root of unity $\omega_a = \exp\left(\frac{2\pi\imath}{a}\right)$ for integers $a \geq 1$.
$\omega(n)$, $\Omega(n)$	If $n = p_1^{\alpha_1} \times \cdots \times p_r^{\alpha_r}$ is the prime factorization of n into distinct prime powers, then $\omega(n) = r$ and $\Omega(n) = \alpha_1 + \cdots + \alpha_r$.

Symbol	Definition
$\phi_k(n)$	Generalized totient function, $\phi_k(n) := \sum_{\substack{1 \leq d \leq n \\ (d,n)=1}} d^k$.
$\phi(n)$	Euler's classical totient function, $\phi(n) := \sum_{\substack{1 \leq d \leq n \\ (d,n)=1}} 1$.
$\Phi_n(z)$	The n^{th} cyclotomic polynomial in z defined by $\Phi_n(z) := \prod_{\substack{1 \leq k \leq n \\ (k,n)=1}} (z - e^{2\pi\imath k/n})$.
$p(n)$	The partition function generated by $p(n) = [q^n] \prod_{n \geq 1} (1-q^n)^{-1}$.
$\pi(x)$	The prime counting function denotes the number of primes $p \leq x$.
$\sum_{p \leq x}$	Unless otherwise specified by context, we use the index variable p to denote that the summation is to be taken only over prime values within the summation bounds.
$P(s)$	For complex s with $\Re(s) > 1$, we define $P(s) = \sum_{p \text{ prime}} p^{-s}$.
$\Psi_k(n)$	The k^{th} Dedekind totient function, $\Psi_k(n) := n^k \times \prod_{p \mid n}(1 + p^{-k})$.
$\psi_k(n)$	The arithmetic function defined by $\psi_k(n) := \sum_{d \mid n} d^k \mu^2\left(\frac{n}{d}\right)$.
$(a;q)_n$, $(q)_n$	The q-Pochhammer symbol defined as the product $(a;q)_\infty := \prod_{n \geq 1} (1 - aq^{n-1})$. We adopt the notation that $(q)_n \equiv (q;q)_n$ and that $(a;q)_\infty$ denotes the limiting case for $\lvert q \rvert < 1$ as $n \to \infty$.
$(q;q)_\infty$	The infinite q-Pochhammer symbol defined as the product $(q;q)_\infty := \prod_{n \geq 1} (1-q^n)$ for $\lvert q \rvert < 1$.
$(a_1, \ldots, a_r; q)_n$	We use the common shorthand that $(a_1, \ldots, a_r; q)_n = \prod_{i=1}^{r} (a_i; q)_n$.
$r_k(n)$	The sum of k squares function denotes the number of integer solutions to $n = x_1^2 + \cdots + x_k^2$. A generating function is given by $r_k(n) = [q^n]\vartheta_3(q)^k$.
$\sigma_\alpha(n)$	The generalized sum-of-divisors function, $\sigma_\alpha(n) := \sum_{d \mid n} d^\alpha$, for any $n \geq 1$ and $\alpha \in \mathbb{C}$.
$s_k(f,g;n)$	Shorthand for the periodic (modulo k) divisor sums expanded by the functions listed in $a_k(f,g;n)$ of this glossary. The precise expansion and corresponding finite Fourier series expansion of this function is given by $s_k(f,g;n) = \sum_{d \mid (n,k)} f(d)g(k/d) = \sum_{m=1}^{k} a_k(f,g;m) e^{\frac{2\pi\imath mn}{k}}$.
$s_{n,k}$	Matrix coefficients in Lambert series type factorizations. These coefficients are defined precisely as the coefficients of the generating function $[q^n](q;q)_\infty q^k(1-q^k)^{-1}$ for $k \geq 1$ where $(q;q)_\infty$ is the infinite q-Pochhammer symbol.
$\left[{n \atop k}\right]$, $\left\{{n \atop k}\right\}$	The Stirling numbers of the first and second kinds, respectively. Alternate notation for these triangles is given by $s(n,k) = (-1)^{n-k}\left[{n \atop k}\right]$ and $S(n,k) = \left\{{n \atop k}\right\}$.
$\sum_{n \leq x}'$	We denote by $\sum_{n \leq x}' f(n)$ the summatory function of f at x minus $\frac{f(x)}{2}$ if $x \in \mathbb{Z}$.
$\tau(n)$	The function defined by $\tau(n) := [x^{n-1}] \prod_{m \geq 1} (1-x^m)^{24}$.
$T_f(x)$	The type I sum over an arithmetic function f, $T_f(n) := \sum_{\substack{d \leq x \\ (d,x)=1}} f(d)$.

Symbol	Definition
$t_{n,k}$	The matrix sequence involved in the generating function expansions of the type I sums defined as $$T_f(x) = [q^x]\left(\frac{1}{(q;q)_\infty} \times \sum_{n\geq 2}\sum_{k=1}^{n} t_{n,k} f(k) q^n + f(1)q\right)$$
$t_{n,k}^{(-1)}$	Inverse matrix of the sequence $t_{n,k}$.
$\hat{u}_{n,k}(f,w)$	Matrix coefficients defined in terms of an indeterminate parameter w as $\hat{u}_{n,k}(f,w) := (w^k-1)u_{n,k}(f,w)$.
$u_{n,k}(f,w)$	The matrix sequence defined in the expansion of the generating functions for the type II sums as $$g(x) = [q^x]\left(\frac{1}{(q;q)_\infty} \times \sum_{n\geq 2}\sum_{k=1}^{n} u_{n,k}(f,w)\left(\sum_{m=1}^{k} L_{f,g,m}(k) w^m\right) q^n\right), w \in \mathbb{C}\backslash\{0\}.$$
$u_{n,k}^{(-1)}(f,w)$	Inverse matrix terms of the sequence $u_{n,k}(f,w)$.
$y_f(n)$	The function $y_f(n)$ denotes the Dirichlet inverse of the function $h(n) := f(n)\phi(n)n^{-2}$ where $\phi(n)$ is Euler's totient function and f is any invertible arithmetic function such that $f(1) \neq 0$. This function is used to express the result in Corollary 3.25. A special case, denoted by $y(n)$, corresponding to the case where $f(n) \equiv n$ is employed in stating Corollary 3.26 in Section 3.3.
$\zeta(s)$	The Riemann zeta function, defined by $\zeta(s) := \sum_{n\geq 1} n^{-s}$ when $\Re(s) > 1$, and by analytic continuation to the entire complex plane with the exception of a simple pole at $s = 1$.