Stability of Local Efficiency in Multiobjective Optimization

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Abstract Analyzing the behavior and stability properties of a local optimum in an optimization problem when small perturbations are added to the objective functions are important considerations in optimization. The tilt stability of a local minimum in a scalar optimization problem is a well studied concept in optimization which is a version of the Lipschitzian stability condition for a local minimum. In this paper, we define a new concept of stability pertinent to the study of multiobjective optimization problems. We prove that our new concept of stability is equivalent to tilt stability when scalar optimizations are available. We then use our new notions of stability to establish new (necessary?) conditions on when strict locally efficient solutions of a multiobjective optimization problem will have small changes when correspondingly small perturbations are added to the objective functions in the representative problems we consider within the article.

1 Introduction

Analyzing the sensitivity and stability of an optimal solution when small perturbations are introduced into an optimization problem is an important topic

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in optimization. In particular, the Lipschitzian stability of locally optimal solutions with respect to small perturbations is a well studied concept in optimization problems. Several variants of Lipschitzian stability have been studied in [1-5]. A well-known notion of Lipschitzian stability is the tilt stability of a local minimizer. Tilt stability is studied by Poliquin and Rockafellar in the context if scalar optimization in [6]. This notion of stability has been extensively studied by several other researchers in [6-10]. These researchers have paved the way for our new methods by identifying necessary and sufficient conditions for the tilt stability status of locally optimal solutions to an optimization problem. The second order subdifferential introduced by Mordukhovich in [11, 12] is similarly employed as a tool for constructing exact criteria to identify the tilt stability status of local minimizers in [6-10].

Robust optimization represents another popular and practical branch of approaches to optimization problems which deal with uncertainty in their parameters. Robustness has been studied from several differing points of view in [13-22]. Considering the multiple notions of characterizing robustness in optimization problems, in this article we benefit from the definition given in [19,20]. This definition states that an efficient point of a multiobjective optimization problem is robust if it remains efficient when small linear terms are added to the objective functions.

Organization of the article. In Section 2, we review preliminary concepts which are assumed in our theories and discussions later in the article. We next introduce a new notion of stability in multiobjective optimization in Section 3 with respect to title stability in scalar optimization. We then obtain relations between our new concept of stability and more traditional concepts of tilt stability. Namely, we show that a tilt-stable local minimizer of the weighted sum problem is a stable locally efficient point of the multiobjective optimization problem. We also present sufficient conditions for stable locally efficient points in our methods based on a few specific properties of the objective functions including linearity, convexity, differentiability, and Lipschitzianity. Furthermore, we determine new necessary and sufficient conditions for robust efficient solutions in our stability analysis. Finally, we demonstrate the significance of the new contributions to stability theory we introduced here and clarify our methodology by elaborating on a short selection of representative examples of our work. We conclude the article in Section 4 with some key remarks and suggestions for future research which builds on the new results we have proved in this article.

2 Preliminaries

In this section, we overview some basics, definitions, and properties of variational analysis, scalar optimization, and multiobjective optimization, which are used in the introduced definitions and properties. For more information, the readers can refer to Borwein and Zhu [23], Rockafellar and Wets [5], and Mordukhovich [4,24].

In mathematical analysis, the δ-neighborhood of a point $\bar{x} \in \mathbb{R}$ is the set of all points in $\mathbb{R}$ which are at a distance less than δ from $\bar{x}$, where $\delta > 0$ and it is presented as follows:

$$B(\bar{x}, \delta) = \{x \in \mathbb{R}^n : \|x - \bar{x}\| < \delta\}.$$

For a set Ω, $cl(\Omega)$ refers to the closure of Ω. The conical hull (positive hull) of set Ω which is a cone is defined as

$$Pos(\Omega) = \{x \in \mathbb{R}^n : \exists m \in \mathbb{N}; \ x = \sum_{i=1}^{m} \lambda_i x_i, \ \lambda_i \geq 0, \ x_i \in \Omega, \ i = 1, ..., m\}.$$

Clearly, for convex sets $\Omega_1, ..., \Omega_k \subseteq \mathbb{R}^n$, we have

$$Pos(\cup_{i=1}^{k} \Omega_i) = \{\sum_{i=1}^{k} \lambda_i x_i : \ \lambda_i \geq 0, \ x_i \in \Omega_i, \ i = 1, 2, ..., k\}.$$

The orders used in this paper to compare $x, y \in \mathbb{R}^n$ are defined as follows:

$$x < y \Leftrightarrow x_i < y_i \ \forall i = 1, 2, ..., n,$$
$$x \leq y \Leftrightarrow x_i \leq y_i \ \forall i = 1, 2, ..., n, \ x \neq y,$$
$$x \leqq y \Leftrightarrow x_i \leq y_i \ \forall i = 1, 2, ..., n.$$

Let $\|.\|$ be a vector norm on $\mathbb{R}^n$ and $\mathbb{R}^p$, then the mapping $\||.\||$ on $\mathbb{R}^{p \times n}$

$$\||A\|| := max_{\|x\|=1}\|Ax\|$$

is called the induced matrix norm corresponding to the norm $\|.\|$, where $\mathbb{R}^{p \times n}$ is the vector space containing all $p \times n$ matrices with real components.
In this paper, we use the induced matrix norm corresponding to $\|.\|_\infty$:

$$\||A\||_\infty := max_{1 \leq i \leq p} \sum_{j=1}^{n} |a_{ij}|.$$

Regarding the definition of the induced matrix norm, there are positive numbers p_1 and p_2 such that

$$p_1 \|A\|_\gamma \leq \||A\||_\infty \leq p_2 \|A\|_\gamma, \quad \forall \gamma \geq 1,$$

where $\|A\|_\gamma = (\sum_{i,j} |a_{ij}|^\gamma)^{\frac{1}{\gamma}}$, for all $\gamma \geq 1$. So, all the results of the paper hold for any arbitrary norm.

2.1 Some Basics of Variational Analysis

Consider the following optimization problem

$$\min_{x \in X} \ g(x), \tag{1}$$

where $g : \mathbb{R}^n \to \mathbb{R}$ is a single real-valued function and $X \subseteq \mathbb{R}^n$ is the set of feasible solutions of (1).

Let

$$\delta_X(x) = \begin{cases} 0, & x \in X, \\ \infty, & x \notin X. \end{cases}$$

In this regard, (1) is transformed to the following problem:

$$\min \ f(x) = g(x) + \delta_X(x), \tag{2}$$

where $f : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$ is an extended real-valued function.

The following proposition expresses a necessary and sufficient condition for minimality of a point, under some assumptions.

Proposition 2.1 *Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\bar{x}$. If there is $\delta > 0$ such that $\bar{x}$ is a minimum solution of problem*

$$\min_{x \in B(\bar{x}, \delta)} \ f(x), \tag{3}$$

then $\nabla f(\bar{x}) = 0$. Conversely, if $f(x)$is convex on $B(\bar{x}, \delta)$ and $\nabla f(\bar{x}) = 0$, then $\bar{x}$ is an optimal solution of Problem (3).

Proof Refer to the proof of the corollary of Theorem 4.1.2. in [25]. $\qquad \square$

In scalar optimization, studying the Lipschitzian stability of locally optimal solutions, when a small linear term is added to the objective, is undoubtedly important. The tilt stability of local minimizers is a version of the Lipschitzian stability, which was introduced by Poliquin and Rockafellar [6] as follows.

Definition 2.1 $\bar{x}$ is a tilt-stable locally minimum point of function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, iff $f(\bar{x})$ is finite and there exist $\delta > 0$ and $\epsilon > 0$ such that the mapping

$$M : v \mapsto \operatorname*{argmin}_{\|x - \bar{x}\| \leq \delta}[f(x) - f(\bar{x}) - \langle v, x - \bar{x} \rangle],$$

is single-valued and Lipschitzian for any $v \in \mathbb{R}^n$ with $\|v\| < \epsilon$ and $M(0) = \{\bar{x}\}$.

Proposition 2.2 *If the mapping M in Definition 2.1 has Lipschitzian behavior for any $v \in \mathbb{R}^n$ with $\|v\| < \epsilon$, then the mapping*

$$m(v) = \{f(x) : x \in M(v)\},$$
$$m : B(0, \epsilon) \subseteq \mathbb{R}^n \mapsto \mathbb{R}$$

has Lipschitzian behavior, as well.

Proof Suppose that the mapping M is single-valued and Lipschitzian for any $v \in \mathbb{R}^n$ with $\|v\| < \epsilon$. Corresponding to two arbitrary vectors v_1 and v_2 with $\|v_1\|, \|v_2\| < \epsilon$, let $x_1 \in M(v_1)$ and $x_2 \in M(v_2)$. If $x_1 = x_2$ then there is nothing to prove, so consider $x_1 \neq x_2$. Since the mapping M is single-valued, we have

$$f(x_1) + v_1^t x_1 < f(x_2) + v_1^t x_2,$$
$$f(x_2) + v_2^t x_2 < f(x_1) + v_2^t x_1.$$

Without loss of generality, suppose that $f(x_1) \leq f(x_2)$, then because of this and the Lipschitzian property of the mapping M with constant γ we have

$$|m(v_2) - m(v_1)| = |f(x_2) - f(x_1)| = f(x_2) - f(x_1)$$
$$< v_2(x_1 - x_2) < \epsilon\|x_1 - x_2\| \leq \epsilon\gamma\|v_2 - v_1\|.$$

This completes the proof.   □

Most of the properties presented in the paper are constructed based on the generalized differentiation in variational analysis. Here, first-order subdifferentials for extended real-valued functions are presented.

Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $f(\bar{x})$ be finite. The regular subdifferential of the extended real-valued function f at $\bar{x}$ is defined as

$$\hat{\partial}f(\bar{x}) = \{v \in \mathbb{R}^n : \liminf_{x \to \bar{x}} \frac{f(x) - f(\bar{x}) - \langle v, x - \bar{x} \rangle}{\|x - \bar{x}\|} \geq 0\}.$$

Moreover, the Mordukhovich subdifferential of f at $\bar{x}$ is defined as follows:

$$\partial f(\bar{x}) = \limsup_{x \to \bar{x}} \quad \hat{\partial}f(x).$$

If f is a smooth function, then $\hat{\partial}f(\bar{x}) = \partial f(\bar{x}) = \nabla f(\bar{x})$. Furthermore, if f is convex, then

$$\hat{\partial}f(\bar{x}) = \partial f(\bar{x}) = \{v \in \mathbb{R}^n : \ f(x) \geq f(\bar{x}) + \langle v, x - \bar{x} \rangle\}.$$

If f is a nonsmooth and nonconvex function, then we may have $\hat{\partial}f(\bar{x}) = \emptyset$.

For a C^2-smooth function $f(x)$ at point $x = \bar{x}$, a necessary and sufficient condition for $\bar{x}$ being a tilt-stable local optimum has been given in Proposition 1.2 of [6] as follows.

Proposition 2.3 *Let f be a C^2-smooth function. If $\nabla f(\bar{x}) = 0$, then $\bar{x}$ is a tilt-stable local minimum of f if and only if $\nabla^2 f(\bar{x})$ is positive definite.*

By this proposition, the tilt stability of $f(x)$ at $x = \bar{x}$ reduces to the positive definiteness of the Hessian matrix $\nabla^2 f(\bar{x})$ where f is a C^2-smooth function.

2.2 Some Basics of Multiobjective Optimization

Consider the following multiobjective programming (MOP) problem:

$$\min_{x \in X} \ \{f(x) = (f_1(x), \cdots, f_p(x))\}, \tag{4}$$

where $X \subseteq \mathbb{R}^n$ is the feasible set and $f_i(x)$, $i = 1, 2, ..., p$, are the objective functions of the above MOP problem. Usually, there exist conflicts among objective functions in their targets then, usually, there does not exist any feasible solution of an MOP problem that optimizes all objective functions. Therefore, the notions of efficient solutions and weakly efficient solutions are introduced in MOP to replace optimal solutions.

Definition 2.2 A point $\overline{x} \in X$ is a (locally) strictly efficient solution of Problem (4) iff (for some $\delta > 0$) there is no other $(x \in X \cap B(\bar{x}, \delta), x \neq \bar{x})$ $x \in X$, $x \neq \overline{x}$ such that $f(x) \leqq f(\overline{x})$.

Definition 2.3 A point $\overline{x} \in X$ is a (locally) Pareto efficient solution of Problem (4) iff (for some $\delta > 0$) there is no other $(x \in X \cap B(\bar{x}, \delta))$ $x \in X$ such that $f(x) \leq f(\overline{x})$.

Definition 2.4 A point $\overline{x} \in X$ is a (locally) weakly efficient solution of Problem (4) iff (for some $\delta > 0$) there is no other ($x \in X \cap B(\bar{x}, \delta)$) $x \in X$ such that $f(x) < f(\overline{x})$.

We define sets X_{SE}, X_E, and X_{WE} as the sets of strictly, Pareto, and weakly efficient solutions of (4), respectively.

Remark 2.1 Definitions 2.2, 2.3, and 2.4 imply that each strictly efficient solution is a Pareto efficient solution and each Pareto efficient solution is a weakly efficient solution but the reverses are not necessarily true.

The weakly efficient, Pareto efficient, and strictly efficient solutions of Problem (4) can be found by solving the following single-objective optimization problem:

$$\min_{x \in X} \quad \sum_{k=1}^{p} \lambda_k f_k(x), \tag{5}$$

where $\lambda_k \geq 0$, $k = 1, 2, ..., p$, and $\sum_{k=1}^{p} \lambda_k = 1$. The above single-objective optimization problem is called the weighted sum scalarization of the MOP problem (4).

Proposition 2.4 *Suppose that x^0 is an optimal solution of the weighted sum optimization problem (5) with $\lambda \in \mathbb{R}_{\geq}^{p}$. The following statements hold.*
1. If $\lambda \in \mathbb{R}_{\geq}^{p}$ then $x^0 \in X_{wE}$.
2. If $\lambda \in \mathbb{R}_{>}^{\overline{p}}$ then $x^0 \in X_E$.
3. If $\lambda \in \mathbb{R}_{\geq}^{p}$ and x^0 is a unique optimal solution of (5), then $x^0 \in X_{SE}$, where $\mathbb{R}_{\geq}^{p} = \{x = (x_1, x_2, ..., x_p) \neq 0 | x \geq 0\}$ and $\mathbb{R}_{>}^{p} = \{x = (x_1, x_2, ..., x_p) | x > 0\}$.
Conversely, let all f_i, $i = 1, 2, ..., p$, be convex functions and X be a convex set. If $\bar{x} \in X_{WE}$, then there is $\lambda = (\lambda_1, ..., \lambda_p) \in \mathbb{R}_{\geq}^{p}$ such that $\bar{x}$ is an optimal solution of Problem (5).

Proof Refer to [26]. □

3 A New Concept of Stability in Multiobjective Optimization

In this section, we will introduce a new concept of the stability and its properties in multiobjective optimization. Consider the multiobjective optimization problem (4) where $f_k : X \subseteq \mathbb{R}^n \to \mathbb{R}$, $k = 1, 2, ..., p$, are real-valued objective functions. Using the indicator function $\delta_X(x)$, (4) is represented as follows:

$$min \quad F(x) = (f_1(x) + \delta_X(x), f_2(x) + \delta_X(x), ..., f_p(x) + \delta_X(x)).$$

In the above problem, $F : \mathbb{R}^n \to \overline{\mathbb{R}}^p =]-\infty, \infty] \times ... \times]-\infty, \infty]$ is an extended-real-valued multifunction. In this regard, for an index $j \in \{1, 2, ..., p\}$, if $f_j(\bar{x})$ ($\bar{x} \in \mathbb{R}^n$) is not finite, then $F(\bar{x})$ is not finite, and so, point $\bar{x}$ cannot be a candidate for Pareto efficient (weakly efficient) solutions.

In this section, we present a new concept of the stability of locally efficient solutions in multiobjective optimization. This concept is equivalent to the tilt stability when a single-objective function is available ($p = 1$). In this regard, we study the following multiobjective optimization problem:

$$min \ \ F(x) = (f_1(x), f_2(x), ..., f_p(x)) \quad s.t. \ \ x \in cl(B(\bar{x}, \delta)),$$

where $f_i : \mathbb{R}^n \to \overline{\mathbb{R}}, \ i = 1, 2, ..., p$, are extended real-valued functions and $\bar{x}$ is a strictly efficient solution.

To discuss the stability, we perturb $F(x)$ by adding a linear mapping with a sufficiently small norm to the objective functions as follows:

$$min \ \ F(x) + Ax = (f_1(x) + \langle v_1, x \rangle, f_2(x) + \langle v_2, x \rangle, ..., f_p(x) + \langle v_p, x \rangle)$$
$$s.t. \ \ x \in cl(B(\bar{x}, \delta)),$$

where δ is a positive real number and $v_1, v_2, ..., v_p \in \mathbb{R}^n$ are the transpose of the rows of matrix $A_{p \times n}$ with $\|\|A\|\| < \epsilon$ for the sufficiently small number $\epsilon > 0$.

The main message of tilt stability is the Lipschitzian behavior of the solution mapping of perturbed problems, and then the single-valuedness of the local minimum appears as a result of the Lipschitzian property. However, in our definition, we do not use the Lipschitzian behavior of the solution mapping of perturbed problems, because the assumption of Lipschitzian property of the solution mapping implies the uniqueness of the efficient solution of a multiobjective optimization problem, which is not reasonable. Note that, usually, the number of the efficient solutions of a multiobjective optimization problem is uncountable in continuous optimization. In this regard, we use strict efficiency in the concept of the stability of locally efficient solutions in multiobjective optimization instead of unique optimality in the concept of tilt stability in scalar optimization.

Regarding the above introduction, we present a new structure of stability in multiobjective optimization. Throughout the paper, we consider $M_{(\delta, F)}(\bar{x}, A)$ to refer to the following optimization problem:

$$min \ \ F(x) - F(\bar{x}) - A(x - \bar{x})$$
$$s.t. \ \ x \in cl(B(\bar{x}, \delta)), \tag{6}$$

where $\delta > 0$ and A is a $p \times n$ matrix. Here, to extend the notion of tilt stability in scalar optimization to multiobjective optimization, we introduce the following definition.

Definition 3.1 $\bar{x}$ is a stable locally efficient point of the function
$F : \mathbb{R}^n \to \overline{\mathbb{R}}^p$, iff $F(\bar{x})$ is finite and there exist $\delta > 0$, $\epsilon > 0$, $\gamma \geq 0$, and $\lambda \geq 0$ such that
a) $\bar{x}$ is a strictly efficient solution of problem $M_{\delta, F}(\bar{x}, 0_{p \times n})$ and for any $p \times n$ matrix A with $\|\|A\|\| < \epsilon$, there is a strictly efficient solution x of problem $M_{\delta, F}(\bar{x}, A)$ such that
$$\|x - \bar{x}\| \leq \gamma \|\|A\|\|.$$

b) For any two $p \times n$ matrices A_1 and A_2 with $\|\|A_1\|\|, \|\|A_2\|\| < \epsilon$, there are strictly efficient solutions x_1 and x_2 of problems $M_\delta(\bar{x}, A_1)$ and $M_\delta(\bar{x}, A_2)$, respectively, such that

$$\|x_1 - x_2\| \leq \gamma \|\|A_1 - A_2\|\|.$$

c) For any two strictly efficient solutions x_1 and x_2 of problems $M_{\delta,F}(\bar{x}, A_1)$ and $M_{\delta,F}(\bar{x}, A_2)$, respectively, with $\|\|A_1\|\|, \|\|A_2\|\| < \epsilon$ and $\|x_1 - x_2\| \leq \gamma \|\|A_1 - A_2\|\|$, the following relation is true:

$$\|F(x_1) - F(x_2)\| \leq \lambda \|\|A_1 - A_2\|\|.$$

Proposition 3.1 *Stable local efficiency, as defined in Definition* 3.1*, is equivalent to the definition of the tilt stability of a local minimum when* $F : \mathbb{R}^n \to \overline{\mathbb{R}}$.

Proof Strict efficiency in multiobjective optimization when one objective function is available (scalar optimization) is equivalent to the existence of a unique optimal solution. In Definition 3.1, if conditions a) and b) are relaxed for one objective function (scalar optimization), then the Lipschitzian perturbation of the mapping M is seen. For more explanation, suppose $A_1 = v_1$ and $A_2 = v_2$ are $1 \times n$ vectors and also x_1 and x_2 are the optimal solutions of $M_{\delta,F}(\bar{x}, v_1)$ and $M_{\delta,F}(\bar{x}, v_2)$, respectively. Clearly, condition b) in Definition 3.1 is the Lipschitzian property on M. Thus, the stable local efficiency introduced in Definition 3.1 where $p = 1$ implies the tilt stability.
Conversely, by the Lipschitzian property of the mapping M in Definition 2.1 and regarding Proposition 2.2, conditions a), b), and c) of Definition 3.1 hold directly where $p = 1$. $\qquad\square$

Proposition 3.2 *If* $\bar{x}$ *is a stable locally efficient point of* $F(x)$ *with constants* $\delta > 0$, $\epsilon > 0$*, and* $\gamma \geq 0$*, as identified in the above definition, then all relations in the definition are true for any* $0 < \hat{\delta} < \delta$.

Proof Suppose that $0 < \hat{\delta} < \delta$. Without loss of generality, let

$$0 < \epsilon < \begin{cases} \hat{\delta}, & \gamma < 1, \\ \frac{\hat{\delta}}{\gamma}, & \gamma \geq 1. \end{cases}$$

Since there is the strictly efficient solution $\hat{x}$ of problem $M_{\delta,F}(\bar{x}, A)$ such that

$$\|\hat{x} - \bar{x}\| \leq \gamma \|\|A\|\| \leq \gamma\epsilon < \hat{\delta},$$

therefore, $\hat{x}$ is a strictly efficient solution of problem $M_{\hat{\delta},F}(\bar{x}, A)$. $\qquad\square$

Proposition 3.3 *If* $F(x) = (f_1(x), f_2(x), ..., f_p(x))$*, and all functions* f_i *are Lipschitzian with constants* k_i*,* $i = 1, 2, ..., p$*, then for all* x_1 *and* x_2 *with* $\|x_1 - x_2\| \leq \gamma \|\|A_1 - A_2\|\|$*, there is* $\hat{\gamma} \geq 0$ *such that*

$$\|F(x_1) - F(x_2)\| \leq \hat{\gamma} \|\|A_1 - A_2\|\|.$$

Proof For any $\wp \geq 1$ we have

$$\|F(x_1) - F(x_2)\|_\wp \le \sum_{i=1}^{p} |f_i(x_1) - f_i(x_2)| \le \sum_{i=1}^{p} k_i \|x_1 - x_2\| \le$$
$$\gamma \sum_{i=1}^{p} k_i \||A_1 - A_2\||,$$

where $\|.\|_\wp$ is the $L^\wp$-norm. Let $\hat{\gamma} := \gamma \sum_{i=1}^{p} k_i$, and so the proof is completed. $\qquad\square$

Proposition 3.4 *If there is $\delta_1 > 0$ such that each f_i is Lipschitzian on $B(\bar{x}, \delta_1) \cap domF$, then for the stability of the locally efficient solution $\bar{x}$, it is sufficient to check the first and the second conditions of Definition 3.1, where*

$$domF = \{x \in \mathbb{R}^n : F(x) \ is \ finite\}.$$

Proof If conditions a) and b) of Definition 3.1 are true at $\bar{x}$ with constant $\delta > 0$, then according to Proposition 3.2, without loss of generality, we can consider $\delta < \delta_1$. From Proposition 3.3 condition c) holds and the proof is completed. $\qquad\square$

Throughout this paper, we consider $F(x) = (f_1(x), f_2(x), ..., f_p(x))$, where $f_i : \mathbb{R}^n \to \overline{\mathbb{R}}$, $i = 1, 2, ..., p$. Moreover, we suppose that there is $\delta_1 > 0$ such that each f_i is a Lipschitzian function on $B(\bar{x}, \delta_1) \cap domF$. Furthermore, we consider

$$\||A\|| := \||A\||_\infty = max_{1 \le i \le p} \sum_{j=1}^{n} |a_{ij}|,$$

where A is a $p \times n$ matrix. Since there are $\eta_1 > 0$ and $\eta_2 > 0$ such that $\eta_1 \||A\||_\wp \le \||A\||_\infty \le \eta_2 \||A\||_\wp$ for any $\wp \ge 1$, so conditions a) and b) of Definition 3.1 are true for any $\||.\||_\wp$, if we can show that they are true for $\||.\||_\infty$.

Lemma 3.1 *$\hat{x}$ is a strictly efficient solution of problem $M_{(\delta, F)}(A, \bar{x})$ if and only if*

$$\{(x, F(x) - Ax) : \ x \in cl(B(\bar{x}, \delta))\} \cap \{(x, F(\hat{x}) - A\hat{x}) - (0, d) : \ x \in cl(B(\bar{x}, \delta)),$$
$$d \in \mathbb{R}_{\geqq}^p\} = \{(\hat{x}, F(\hat{x}) - A\hat{x})\}. \tag{7}$$

Proof Suppose that $\hat{x}$ is a strictly efficient solution of problem $M_{(\delta, F)}(A, \bar{x})$. So, there is no $x_1 \in cl(B(\bar{x}, \delta))$ with $x_1 \ne \hat{x}$ such that $F(x_1) - Ax_1 \leqq F(\hat{x}) - A\hat{x}$. Therefore, Relation (7) holds, clearly.
Now, suppose that Relation (7) is true and $\hat{x}$ is not a strictly efficient solution of problem $M_{(\delta, F)}(A, \bar{x})$. Then, there is $x_1 \in cl(B(\bar{x}, \delta))$ such that $x_1 \ne \hat{x}$ and $F(x_1) - Ax_1 \leqq F(\hat{x}) - A\hat{x}$. This implies that there is $d \in \mathbb{R}_{\geqq}^m$ such that $F(x_1) - Ax_1 = (F(\hat{x}) - A\hat{x}) - d$. Therefore

$$(x_1, F(x_1) - Ax_1) \in \{(x, F(\hat{x}) - A\hat{x}) - (0, d) : x \in cl(B(\bar{x}, \delta)), \ d \in \mathbb{R}_{\geqq}^p\}.$$

Thus

$$(x_1, F(x_1) - Ax_1) \in \{(x, F(x) - Ax) : x \in cl(B(\bar{x}, \delta))\} \cap$$
$$\{(x, F(\hat{x}) - A\hat{x}) - (0, d) : x \in cl(B(\bar{x}, \delta)), \ d \in \mathbb{R}_{\geqq}^p\}.$$

This shows a contradiction. $\qquad\square$

For function $F : \mathbb{R}^n \to \overline{\mathbb{R}}^p$ and point $\bar{x} \in domF$, let

$$\phi_{(\delta,F)}(A,\bar{x}) := \left\{ \hat{x} \in dom F \cap cl(B(\bar{x},\delta)) : \{(x, F(x) - Ax) | x \in cl(B(\bar{x},\delta))\} \cap \right.$$
$$\left. \{(x, F(\hat{x}) - A\hat{x}) - (0,d) : x \in cl(B(\bar{x},\delta)), d \in \mathbb{R}^p_{\geqq}\} = \{(\hat{x}, F(\hat{x}) - A\hat{x})\} \right\}.$$

$\phi_{(\delta,F)}(A,\bar{x})$ is the set of all strictly efficient solutions of problem $M_{(\delta,F)}(A,\bar{x})$.

Proposition 3.5 $\bar{x} \in dom F$ *is a stable locally efficient point of function* $F : \mathbb{R}^n \to \overline{\mathbb{R}}^p$ *if and only if*
a) there are $\delta > 0$, $\epsilon > 0$, *and* $\gamma \geq 0$ *such that* $\bar{x} \in \phi_{(\delta,F)}(0,\bar{x})$ *and for any matrix* $A_{p \times n}$ *with* $\||A\|| < \epsilon$, *there is* $x \in \phi_{(\delta,F)}(A,\bar{x})$ *such that*

$$\|x - \bar{x}\| \leq \gamma \||A\||.$$

b) For any two $p \times n$ *matrices* A_1 *and* A_2 *with* $\||A_1\||, \||A_2\|| < \epsilon$, *there are* $x_1 \in \phi_{(\delta,F)}(A_1,\bar{x})$ *and* $x_2 \in \phi_{(\delta,F)}(A_2,\bar{x})$ *such that*

$$\|x_1 - x_2\| \leq \gamma \||A_1 - A_2\||.$$

Proof According to the definition of stable local efficiency and Lemma 3.1, the proof is obvious.   □

The following theorem shows a necessary condition for the stability of a locally efficient solution of an MOP problem, where all objective functions are convex.

Proposition 3.6 *Suppose that* $F(x) = (f_1(x), f_2(x), ..., f_p(x))$ *and each* f_i *is convex and finite on some neighborhoods* $B(\bar{x}, \hat{\delta})$ *with* $\hat{\delta} > 0$. *If* $\bar{x}$ *is a stable locally efficient point of* $F(x)$, *then for any* $\delta_1 > 0$ *we have*

$$\cup_{x \in B(\bar{x},\delta_1)} Pos\{\cup_i \partial f_i(x)\} = \mathbb{R}^n.$$

Proof Suppose that $0 \neq d \in \mathbb{R}^n$ and $\delta_1 > 0$ have been given. We show that $d \in \cup_{x \in B(\bar{x},\delta_1)} Pos\{\cup_i \partial f_i(x)\}$. Since $\bar{x}$ is a stable locally efficient point of $F(x)$, there exist $\delta > 0$, $\epsilon > 0$, and $\gamma \geq 0$ such that for any $A_{p \times n}$ with $\||A\|| < \epsilon$ there is the strictly efficient solution x of problem $M_{(\delta,F)}(\bar{x}, A)$ such that

$$\|x - \bar{x}\| \leq \gamma \||A\||.$$

Without loss of generality, consider $\gamma \geq 1$, $\frac{\delta_1}{2} < \epsilon$, and $\delta_1 < \delta < \hat{\delta}$. Let

$$A = \begin{bmatrix} \frac{\delta_1}{2\gamma} \frac{d^t}{\|d\|_1} \\ \frac{\delta_1}{2\gamma} \frac{d^t}{\|d\|_1} \\ \vdots \\ \frac{\delta_1}{2\gamma} \frac{d^t}{\|d\|_1} \end{bmatrix}_{p \times n},$$

in which $\||A\|| = \frac{\delta_1}{2\gamma} < \epsilon$. This means that there is the strictly efficient solution $\hat{x} \in cl(B(\bar{x},\delta))$ of problem $M_{(\delta,F)}(\bar{x}, A)$ such that

$$\|\hat{x} - \bar{x}\| \leq \gamma \||A\|| = \gamma \frac{\delta_1}{2\gamma} = \frac{\delta_1}{2},$$

and so $\hat{x} \in B(\bar{x},\delta)$. Since for any $i \in \{1, 2, ..., p\}$, $f_i(x) - f_i(\bar{x}) - v_i^t(x - \bar{x})$ is a convex function on $cl(B(\bar{x},\delta))$, then, according to Proposition 2.4, there exists vector $\overline{\lambda} \in \mathbb{R}^m_{\geqq}$ such that the optimal solution of problem

$$min \quad \sum_i \overline{\lambda}_i(f_i(x) - f_i(\bar{x}) - v_i^t(x - \bar{x})) = \sum_i \overline{\lambda}_i(f_i(x) - f_i(\bar{x}))$$
$$- \frac{\delta_1}{2\gamma} \frac{1}{\|d\|_1} \sum_i \overline{\lambda}_i \langle d, x - \bar{x} \rangle$$
$$s.t. \quad x \in cl(B(\bar{x}, \delta))$$

is attained at $\hat{x} \in B(\bar{x}, \delta)$. This means that

$$\frac{\delta_1}{2\gamma} \frac{d}{\|d\|_1} \sum_i \overline{\lambda}_i \in \partial(\sum_i \overline{\lambda}_i(f_i(\hat{x}))). \tag{8}$$

Since each f_i is locally Lipschitzian around $\hat{x}$, then, according to Theorem 3.36 of [24], we have

$$\partial(\sum_i \overline{\lambda}_i(f_i(\hat{x}))) \subseteq \sum_i \overline{\lambda}_i \partial f_i(\hat{x}). \tag{9}$$

Regarding (8) and (9) we have $d \in Pos(\cup_i \partial f_i)$. $\qquad \square$

Lemma 3.2 *Suppose that $f : \mathbb{R}^n \mapsto \overline{\mathbb{R}}$ and $\bar{x} \in domf$. If $\bar{x}$ is a tilt-stable local minimum of $f(x)$, then for any $\delta_1 > 0$ we have*

$$\cup_{x \in B(\bar{x}, \delta_1)} Pos\{\partial f(x)\} = \mathbb{R}^n.$$

Proof Suppose that $0 \neq d \in \mathbb{R}^n$ and $\delta_1 > 0$ have been given. We show that $d \in \cup_{x \in B(\bar{x}, \delta_1)} Pos\{\partial f(x)\}$. Since $\bar{x}$ is a tilt-stable local minimum of $f(x)$, there exist $\delta > 0$, $\epsilon > 0$, and $\gamma \geq 0$ such that for any $v \in \mathbb{R}^n$ with $\|v\| < \epsilon$ there is the unique point x such that $M(v) = \{x\}$ and

$$\|x - \bar{x}\| \leq \gamma \|v\|,$$

where $M(.)$ is the mapping defined in Definition 2.1. Without loss of generality, consider $\gamma \geq 1$, $\frac{\delta_1}{2} < \epsilon$ and $\delta_1 < \delta$. Let $v := \frac{\delta_1}{2\gamma} \frac{d}{\|d\|}$, therefore $\|v\| = \frac{\delta_1}{2\gamma} < \epsilon$. This means that there is $\hat{x} \in cl(B(\bar{x}, \delta))$ such that $M(v) = \{\hat{x}\}$ and

$$\|\hat{x} - \bar{x}\| \leq \gamma \|v\| = \gamma \frac{\delta_1}{2\gamma} = \frac{\delta_1}{2}.$$

So $\hat{x} \in B(\bar{x}, \delta_1)$ and according to the definition of the mapping $M(.)$, we have $\frac{\delta_1}{2\gamma} \frac{d}{\|d\|_1} \in \partial f(\hat{x})$, then $d \in \cup_{x \in B(\bar{x}, \delta_1)} Pos(\partial f(x))$. $\qquad \square$

Note that in the above lemma, function f is not necessarily convex.

Example 3.1 Consider linear function $f(x) := c^t x : \mathbb{R}^n \mapsto \mathbb{R}$. For any $\bar{x} \in \mathbb{R}^n$ and $\delta_1 > 0$ we have

$$\partial f(x) = c \quad \forall x \in B(\bar{x}, \delta_1),$$

and $\cup_{x \in B(\bar{x}, \delta_1)} Pos\{\partial f(x)\} \neq \mathbb{R}^n$. So, there exists no tilt-stable local minimum of $f(x)$.

Proposition 3.7 *Let $F(x) := (f_1(x), f_2(x), ..., f_p(x))$, where $f_i : \mathbb{R}^n \longmapsto \overline{\mathbb{R}}$, $i = 1, 2, ..., p$. $\bar{x} \in domF$ is a stable locally efficient point of $F(x)$ if and only if there is an index set $I := \{i_1, i_2, ..., i_k\} \subseteq \{1, 2, ..., p\}$ such that $\bar{x}$ is a stable locally efficient point of function $G(x) := (f_{i_1}(x), f_{i_2}(x), ..., f_{i_k}(x))$.*

Proof ($\Rightarrow$) By letting $I := \{1, 2, ..., p\}$, the proof is clear.

($\Leftarrow$) Conversely, let $\bar{x}$ be a stable locally efficient point of $G(x)$, so, there are $\delta > 0$, $\epsilon > 0$, and $\gamma \geq 0$ such that the conditions of Definition 3.1 are satisfied. Since $\bar{x}$ is a strictly efficient solution of problem $M_{\delta,G}(\bar{x}, 0_{|I|\times n})$, then according to Corollary 2.32 of [26], $\bar{x}$ is a strictly efficient solution of problem $M_{\delta,F}(\bar{x}, 0_{p\times n})$. Now consider an arbitrary matrix $A_{p\times n}$ with $\||A|\| < \epsilon$. Without loss of generality suppose that $i_1 \leq i_2 \leq ... \leq i_k$. Let matrix $B_{|I|\times n}$ be as follows:

$$b_l := a_{i_l}, \quad l = 1, 2, ...k,$$

where b_l and a_{i_l}, $l = 1, 2, ..., k$, are the rows of matrices B and A, respectively. Because of the stable local efficiency of $\bar{x}$ for $G(x)$ and inequality $\||B|\| \leq \||A|\| < \epsilon$, there is a strictly efficient solution x of problem $M_{\delta,G}(\bar{x}, B)$ such that

$$\|x - \bar{x}\| \leq \gamma\||B|\| \leq \gamma\||A|\|.$$

Furthermore, according to Corollary 2.32 of [26], x is a strictly efficient solution of problem $M_{\delta,F}(\bar{x}, A)$ and so, condition a) is satisfied for $F(x)$ at $\bar{x}$.

Now, consider $p \times n$ matrices A_1 and A_2 with $\||A_1|\|, \||A_2|\| < \epsilon$. Similar to the above definition for constructing B from A, $|I| \times n$ matrices B_1 and B_2 are defined corresponding to A_1 and A_2, respectively. There are strictly efficient solutions x_1 and x_2 of problems $M_{\delta,G}(\bar{x}, B_1)$ and $M_{\delta,G}(\bar{x}, B_2)$, respectively, and so according to Corollary 2.32 of [26], x_1 and x_2 are strictly efficient solutions of problems $M_{\delta,F}(\bar{x}, A_1)$ and $M_{\delta,F}(\bar{x}, A_2)$, respectively, such that

$$\|x_1 - x_2\| \leq \gamma\||B_1 - B_2|\| \leq \gamma\||A_1 - A_2|\|,$$

which completes the proof.   □

Corollary 3.1 *If $\bar{x}$ is a tilt-stable local minimum of one of the functions f_i, $i = 1, 2, ..., p$, then $\bar{x}$ is a stable locally efficient point of $F(x)$.*

Proof The proof is obtained directly from Proposition 3.7 and Proposition 3.3.
                                                                              □

Now, we present a necessary and sufficient condition of the stability of locally efficient solutions for multiobjective linear optimization using robustness which has been defined by Georgiev et al. [19] for linear multiobjective optimization and has been extended by Zamani et al. [20] for multiobjective optimization as follows.

Definition 3.2 A point $\bar{x} \in X$ is a robust efficient solution of problem

$$min \quad F(x) \quad s.t. \quad x \in X, \tag{10}$$

iff $\bar{x}$ is an efficient solution of Problem (10) and there is $\epsilon > 0$ such that for any $p \times n$ matrix A with $\||A|\| < \epsilon$, the point $\bar{x}$ is an efficient solution of problem

$$min \quad F(x) + Ax \quad s.t. \quad x \in X.$$

Now, we show that any robust efficient point is a stable point.

Proposition 3.8 *If $\bar{x}$ is a robust efficient solution of Problem (10) with constant radius ϵ, then $\bar{x}$ is a strictly efficient point of Problem (10).*

Proof By contradiction, suppose that $\bar{x}$ is a robust efficient solution of Problem (10) with constant ϵ, while it is not a strictly efficient point of Problem (10). So, there is $\hat{x} \in X$ such that $\hat{x} \neq \bar{x}$ and $F(\hat{x}) = F(\bar{x})$.
Let

$$
A := \begin{bmatrix} -\frac{\epsilon}{2}\frac{(\hat{x}-\bar{x})^t}{\|\hat{x}-\bar{x}\|_1} \\ -\frac{\epsilon}{2}\frac{(\hat{x}-\bar{x})^t}{\|\hat{x}-\bar{x}\|_1} \\ \vdots \\ -\frac{\epsilon}{2}\frac{(\hat{x}-\bar{x})^t}{\|\hat{x}-\bar{x}\|_1} \end{bmatrix}_{p \times n}.
$$

Now, we have

$$
F(\hat{x}) - F(\bar{x}) + A(\hat{x}-\bar{x}) = A(\hat{x}-\bar{x}) = \begin{bmatrix} -\frac{\epsilon}{2}\frac{\|\hat{x}-\bar{x}\|^2}{\|\hat{x}-\bar{x}\|_1} \\ -\frac{\epsilon}{2}\frac{\|\hat{x}-\bar{x}\|^2}{\|\hat{x}-\bar{x}\|_1} \\ \vdots \\ -\frac{\epsilon}{2}\frac{\|\hat{x}-\bar{x}\|^2}{\|\hat{x}-\bar{x}\|_1} \end{bmatrix}_{p \times n} < 0. \qquad (11)
$$

Furthermore,

$$
\|A\| = \frac{\epsilon}{2} < \epsilon. \qquad (12)
$$

Regarding (11) and (12) we have a contradiction with the assumption that $\bar{x}$ is a robust efficient solution with radius ϵ. $\qquad \square$

Lemma 3.3 *If $\bar{x}$ is a robust efficient solution of Problem (10) with constant radius ϵ, then for any matrix $A_{p \times n}$ with $\|\|A\|\| = \epsilon_1 < \epsilon$, $\bar{x}$ is a robust efficient solution of problem*

$$
min \quad F(x) - Ax \quad s.t. \quad x \in X.
$$

Proof Let $\epsilon_2 := \epsilon - \epsilon_1 > 0$. For any matrix $B_{p \times n}$ with $\|\|B\|\| < \epsilon_2$, $\bar{x}$ is an efficient solution of problem

$$
\begin{aligned} min \quad & F(x) - Ax + Bx = F(x) + (B-A)x \\ s.t. \quad & x \in X, \end{aligned}
$$

because $\bar{x}$ is a robust efficient solution of Problem (10) with constant ϵ and also

$$
\|\| - A + B\|\| \leq \|\|A\|\| + \|\|B\|\| < \epsilon_1 + \epsilon_2 = \epsilon.
$$

This completes the proof. $\qquad \square$

Corollary 3.2 *If $\bar{x}$ is a robust efficient point of Problem (10) with constant radius ϵ, then for any matrix A with $\|A\| < \epsilon$, $\bar{x}$ is a strictly efficient point of problem*

$$min \quad F(x) + Ax$$
$$s.t. \quad x \in cl(B(\bar{x}, \delta)).$$

Proof With regard to Proposition 3.8 and Lemma 3.3 the proof is clear. $\quad\square$

Corollary 3.3 *If there is $\delta > 0$ such that $\bar{x}$ is a robust efficient point of problem*

$$min \quad F(x)$$
$$s.t. \quad x \in cl(B(\bar{x}, \delta)),$$

then $\bar{x}$ is a stable locally efficient point.

Proof Regarding Corollary 3.2, for any matrix A with $\|A\| < \epsilon$, $\bar{x}$ is a strictly efficient solution of problem

$$min \quad F(x) + Ax$$
$$s.t. \quad x \in cl(B(\bar{x}, \delta)),$$

so the conditions of Definition 3.1 are satisfied. $\quad\square$

Proposition 3.9 *Let X be a closed and convex set, f_i $(i = 1, ..., p)$ be convex and $\bar{x}$ be an efficient solution of Problem (10). $\bar{x}$ is a robust efficient solution of (10) if and only if $T_X(\bar{x}) \cap G(\bar{x}) = \{0\}$, where $G(\bar{x})$ is the set of all non-ascent directions defined as*

$$G(\bar{x}) := \{d \in \mathbb{R}^n : \quad d^t \xi_i \leq 0, \quad \forall \xi_i \in \partial f_i(\bar{x}), \ i = 1, ..., p\},$$

and $T_X(\bar{x})$ is the tangent cone of X at $\bar{x}$ defined as

$$T_X(\bar{x}) := \{d \in \mathbb{R}^n : \exists(\{x_i\} \subseteq X, \{t_i\} \subseteq \mathbb{R}); t_i \downarrow 0, \tfrac{x_i - \bar{x}}{t_i}\}.$$

Proof Refer to the proof of Theorem 3.3 of [20]. $\quad\square$

Proposition 3.10 *Suppose that $F(x)$ is a multiobjective linear function in a neighborhood of $\bar{x}$. $\bar{x}$ is a stable locally efficient point of $F(x)$ if and only if there is $\delta > 0$ such that $\bar{x}$ is a strictly efficient solution of problem $M_{\delta,F}(\bar{x}, 0)$.*

Proof ($\Rightarrow$) From Definition 3.1 of stable local efficiency, we conclude that if $\bar{x}$ is a stable locally efficient point of $F(x)$, then there exists $\delta > 0$ such that $\bar{x}$ is a strictly efficient solution of $M_{\delta,F}(\bar{x}, 0)$.
($\Leftarrow$) Conversely, let $F(x) = Cx$ on some neighborhoods of $\bar{x}$, where C is a $p \times n$ matrix. Since $\partial f_i(\bar{x}) = \{\nabla f_i(\bar{x})\} = \{c_i\}$, $i = 1, ..., p$, where c_i is the ith row of C, and $\bar{x}$ is a strictly efficient solution of problem $M_{\delta,F}(\bar{x}, 0)$,

$$G(\bar{x}) = \{d \in \mathbb{R}^n : \quad c_i^t d \leq 0, \ i = 1, ..., p\} = \{0\}.$$

According to Proposition 3.9, point $\bar{x}$ is a robust efficient solution of problem

$$min \quad F(x)$$
$$s.t. \quad x \in cl(B(\bar{x}, \delta)). \tag{13}$$

Thus, by Corollary 3.3 we conclude that $\bar{x}$ is a stable locally efficient point.
$\quad\square$

Remark 3.1 Consider the following multiobjective linear programming problem:

$$\begin{aligned} min \ Cx \\ s.t. \quad Hx \geq 0. \end{aligned} \tag{14}$$

All strict locally efficient solutions of (14) are strict (globally) efficient solutions of (14), where C and H are two $p \times n$ and $k \times n$ matrices, respectively. Regarding Propositions 3.9 and 3.10, $\bar{x}$ is a stable locally efficient solution of (14) if and only if $\bar{x}$ is a robust efficient solution of (14). To see more examples for robust efficiency in multiobjective linear programming refer to [19]. Those examples can also be used for stable local efficiency in multiobjective linear programming.

As mentioned in the preliminaries of multiobjective optimization, MOP problems can be converted to scalar optimization problems using some scalarization techniques. One of these scalarization techniques is the weighted sum approach introduced in the previous section. Here, using the weighted sum approach, we introduce some characterizations of the stability of the locally efficient solutions of an MOP problem and the tilt stability of the locally optimal solutions of the weighted sum problem.

Proposition 3.11 *Suppose that $F : \mathbb{R}^n \mapsto \overline{\mathbb{R}}^p$ and $\bar{x} \in domF$. If there is $\lambda \in \mathbb{R}^p_\geq$ such that $\bar{x}$ is a tilt-stable local minimum of the function $\sum_{i=1}^p \lambda_i f_i(x)$, then $\bar{x}$ is a stable locally efficient point of $F(x)$.*

Proof Suppose that there is $0 \neq \lambda \in \mathbb{R}^p_\geq$ such that $\bar{x}$ is a tilt-stable local minimum of the function $\sum_{i=1}^p \lambda_i f_i(x)$. Without loss of generality, suppose that $\sum_{i=1}^p \lambda_i = 1$(by letting $\lambda_i := \frac{\lambda_i}{\sum_{i=1}^p \lambda_i}$).
Therefore, there are $\epsilon > 0$ and $\delta > 0$ such that for any $v \in \mathbb{R}^n$ with $\|v\| < \epsilon$, the mapping

$$M : v \mapsto \underset{\|x - \bar{x}\| \leq \delta}{argmin} \ \{\textstyle\sum_{i=1}^p \lambda_i f_i(x) - \sum_{i=1}^p \lambda_i f_i(\bar{x}) - \langle v, x - \bar{x}\rangle\}$$

is single-valued and Lipschitzian. For any matrix $A_{p \times n}$ with $\||A\|| < \epsilon$, let $v = \sum_{i=1}^p \lambda_i v_i$, where v_i is the transpose of the ith row of A. Thus

$$\|v\| = \|\lambda_1 v_1 + \lambda_2 v_2 + ... + \lambda_p v_p\| \leq |\lambda_1|\|v_1\| + |\lambda_2|\|v_2\| + ... + |\lambda_p|\|v_p\|$$
$$\leq \||A\|| \textstyle\sum_{i=1}^p \lambda_i < \epsilon.$$

Since the mapping

$$M : v \mapsto \underset{\|x - \bar{x}\| \leq \delta}{argmin} \ \{\textstyle\sum_{i=1}^p \lambda_i f_i(x) - \sum_{i=1}^p \lambda_i f_i(\bar{x}) - \langle v, x - \bar{x}\rangle\}$$

is single-valued and Lipschitzian with constant $\gamma \geq 0$, there is $x \in cl(B(\bar{x}, \delta))$ such that $M(v) = \{x\}$. Regarding Part 3 of Proposition 2.4, x is a strictly efficient solution of problem $M_{\delta, F}(\bar{x}, A)$, and also we have

$$\|x - \bar{x}\| \leq \gamma\|v\| \leq \gamma(|\lambda_1|\|v_1\| + |\lambda_2|\|v_2\| + ... + |\lambda_p|\|v_p\|) \leq \gamma\||A\|| \textstyle\sum_{i=1}^p \lambda_i = \gamma\||A\||.$$

Similar to the above discussions, for any two $p \times n$ matrices A_1 and A_2 with $|\|A_1\||, |\|A_2\|| < \epsilon$, there are strictly efficient solutions x_1 and x_2 of problems $M_{\delta,F}(\bar{x}, A_1)$ and $M_{\delta,F}(\bar{x}, A_2)$, respectively, such that

$$
\begin{aligned}
\|x_1 - x_2\| &\leq \gamma \|v_1 - v_2\| = \gamma \| \sum_{i=1}^{p} \lambda_i (v_{i1} - v_{i2})\| \\
&\leq \gamma (|\lambda_1| \|(v_{11} - v_{12})\| + |\lambda_2| \|(v_{21} - v_{22})\| + ... + |\lambda_p| \|(v_{p1} - v_{p2})\|) \\
&\leq \gamma \|\|A_1 - A_2\|\| \sum_{i=1}^{p} \lambda_i = \gamma \|\|A_1 - A_2\|\|.
\end{aligned}
$$

This completes the proof.   □

Note that Corollary 3.1 can also be obtained as a corollary of Proposition 3.11. The converse of the above theorem is not valid in the general case. The following example shows this inconsistency.

Example 3.2 Consider the following multiobjective optimization problem:

$$
\begin{aligned}
&min \ F(x) = (f_1(x), f_2(x)) \\
&s.t. \ \ x \in \mathbb{R},
\end{aligned}
$$

where

$$
(f_1(x), f_2(x)) := \begin{cases} (x, -x), & -1 \leq x \leq 1, \\ (x, -1), & x \geq 1, \\ (-1, -x), & x \leq -1. \end{cases}
$$

$\bar{x} = 0$ is a strictly efficient solution of problem $M_{\frac{1}{2}, F}(0, 0_{2 \times 1})$. Furthermore, $F(x)$ is a multi-linear function on $B(0, 1)$. With regard to Proposition 3.10, $\bar{x}$ is a stable locally efficient point of $F(x)$. Moreover, for any $\lambda = (\lambda_1, \lambda_2) \geq 0$ and $-1 \leq x \leq 1$, let

$$
h(x) := \lambda_1 f_1(x) + \lambda_2 f_2(x) = \lambda_1 x - \lambda_2 x = (\lambda_1 - \lambda_2)x.
$$

Consider $\delta := 0.5$, therefore,

$$
\partial h(x) = \{\nabla h(x)\} = \{\lambda_1 - \lambda_2\}, \quad \forall x \in cl(B(0, 0.5)).
$$

Thus

$$
\cup_{x \in cl(B(0,\delta))} Pos\{\partial h(x)\} \neq R.
$$

Regarding Lemma 3.2, we conclude that $\bar{x} = 0$ is not a tilt-stable local minimum for all weights $\lambda_i \geq 0$, $i = 1, 2$.

Proposition 3.12 *Let $F : \mathbb{R}^n \to \overline{\mathbb{R}}^p$ and all f_i be finite on some neighborhoods of $\bar{x}$, where $i \in I \subseteq \{1, 2, ..., p\}$. Suppose that there are $\lambda \in \mathbb{R}_{\geq 0}^{|I|}$ and $\epsilon > 0$ such that $\phi^{-1}(v) = \{x \mid v \in \sum_{i \in I} \lambda_i \partial f_i(x)\}$ is a Lipschitzian single-valued mapping for any $v \in \mathbb{R}^n$, with $\|v\| < \epsilon$. If $\phi^{-1}(0) = \{\bar{x}\}$, then $\bar{x}$ is a stable locally efficient point of $F(x)$.*

Proof Since for all $i \in I$, f_i are locally Lipschitzian on some neighborhoods of $\bar{x}$, according to Theorem 3.36 of [24], for any x in these neighborhoods of $\bar{x}$ we have

$$
\partial (\sum_{i \in I} \lambda_i f_i(x)) \subseteq \sum_{i \in I} \lambda_i \partial f_i(x).
$$

Moreover, regarding Theorem 3.4 of [27], $\bar{x}$ is a tilt-stable local minimum of $\sum_{i \in I} \lambda_i f_i(x)$. Using Propositions 3.7 and 3.11 of the present paper, we conclude that $\bar{x}$ is a stable locally efficient point of $F(x)$.   □

Proposition 3.13 *Let $F : \mathbb{R}^n \to \overline{\mathbb{R}}^p$ and $\bar{x} \in dom F$. Suppose that there are index set $I \subseteq \{1, 2, ..., p\}$ and $\delta > 0$ such that f_i are C^2-smooth on $cl(B(\bar{x}, \delta))$ and $\nabla^2 f_i(\bar{x})$ are semi-positive definite for any $i \in I$ and also there is $j \in I$ such that $\nabla^2 f_j(\bar{x})$ is positive definite. If there exist $\lambda \in \mathbb{R}_{>}^{|I|}$ and $\hat{\delta} > 0$ such that $\bar{x}$ is an optimal solution of the following problem*

$$\begin{aligned} \min \quad & \sum_{i \in I} \lambda_i f_i(x) \\ s.t. \quad & \|x - \bar{x}\| \leq \hat{\delta}, \end{aligned}$$

then $\bar{x}$ is a stable locally efficient point of $F(x)$.

Proof Without loss of generality consider $\hat{\delta} \leq \delta$, then according to Propositions 2.1 and 2.3, $\bar{x}$ is a tilt-stable local minimum of function $\sum_{i \in I} \lambda_i f_i(x)$. Therefore, Propositions 3.7 and 3.11 show that $\bar{x}$ is a stable locally efficient point of $F(x)$.   □

The converse of Proposition 3.13 does not hold in general. Consider the following example.

Example 3.3 Consider $F : \mathbb{R}^2 \to \mathbb{R}^3$ as follows:

$$F(x_1, x_2) = (f_1(x), f_2(x), f_3(x)) := (x_1^2, x_2^2, (x_1 + 1)^2 + (x_2^2 + 1)^2).$$

Let $\lambda := (0.5, 0.5, 0)$, then $x = (0, 0)$ is a tilt-stable locally optimal point of $g(x) = \sum_{i=1}^3 \lambda_i f_i(x) = 0.5(x_1^2 + x_2^2)$. Using Proposition 3.11, $x = (0, 0)$ is a stable locally efficient point of $F(x)$.

Moreover, $f_1(0, 0)$ and $f_2(0, 0)$ are positive semi-definite and $f_3(0, 0)$ is positive definite. Furthermore, there are no $\hat{\delta} > 0$ and $\lambda \in \mathbb{R}_{\geq}^3$ with $\lambda_3 > 0$ such that the optimal solution of the following problem is determined at $x = (0, 0)$,

$$\begin{aligned} \min \quad & \sum_{i=1}^3 \lambda_i f_i(x) \\ s.t. \quad & \|x\| \leq \hat{\delta}, \end{aligned} \tag{15}$$

because $\sum_{i=1}^3 \lambda_i \nabla f_i(0, 0) = \lambda_3 [2, 2]^t \neq 0$. Therefore, we conclude that the converse of Proposition 3.13 is not valid in general.

Proposition 3.14 *For vector function F, let all f_i be C^2-smooth on some neighborhoods $B(\bar{x}, \delta)$. Suppose that there is an index set $I \subseteq \{1, ..., p\}$ such that $\nabla^2 f_i(\bar{x})$ are positive definite for all $i \in I$ and $f_j(x)$ are linear functions on $B(\bar{x}, \delta)$ for all $j \in I^c$. Then, $\bar{x}$ is a stable locally efficient point of $F(x)$ if and only if there are $\lambda \in \mathbb{R}_{\geq}^p$ and $\hat{\delta} > 0$ such that the optimal solution of problem*

$$\begin{aligned} min \quad & \sum_i \lambda_i f_i(x) \\ s.t. \quad & x \in cl(B(\bar{x}, \hat{\delta})) \end{aligned}$$

is attained uniquely at $\bar{x}$.

Proof ($\Leftarrow$) Suppose that $\hat{\delta} < \delta$ and there is $\lambda \geq 0$ such that the optimal solution of the following problem is attained uniquely at $\bar{x}$:

$$
\begin{aligned}
&min \sum_i \lambda_i f_i(x) \\
&s.t. \quad x \in cl(B(\bar{x}, \hat{\delta})).
\end{aligned}
\tag{16}
$$

We claim that there is $j \in I$ such that $\lambda_j > 0$. By contradiction, suppose that $\lambda_j = 0$ for all $j \in I$. Let $I_1 := \{i \in I^c : \lambda_i > 0\} := \{i_1, i_2, ..., i_k\}$ and $G(x) := (f_{i_1}, f_{i_2}, ..., f_{i_k})$. $G(x)$ is a linear mapping on $cl(B(\bar{x}, \hat{\delta}))$, and so, the objective function of Problem (16) is a parametric scalar linear function with parameters λ_i, $i \in I_1$. Furthermore, $\bar{x} \in B(\bar{x}, \delta)$, which implies that (16) has another optimal solution in addition to $\bar{x}$. Thus, the optimal solution of (16) is not attained uniquely at $\bar{x}$ which is a contradiction. Therefore, there exists index $j \in I$ such that $\lambda_j \neq 0$, and so $\sum_i \lambda_i \nabla^2 f_i(\bar{x})$ is positive definite. By Propositions 2.1 and 2.3, $\bar{x}$ is a tilt-stable local minimum of $\sum_i \lambda_i f_i(\bar{x})$, and by Proposition 3.11, $\bar{x}$ is a stable locally efficient point of $F(x)$.

($\Rightarrow$) Conversely, suppose that $\bar{x}$ is a stable locally efficient point of $F(x)$, so there is $\bar{\delta} < \delta$ such that $\bar{x}$ is a strictly efficient solution of problem

$$
\begin{aligned}
&min \; F(x) \\
&x \in cl(B(\bar{x}, \bar{\delta})).
\end{aligned}
\tag{17}
$$

Without loss of generality suppose that $I := \{1, 2, ..., k\}$ and $I^c := \{k+1, k+2, ..., p\}$.
Let

$$
\begin{aligned}
A^t &:= [\nabla f_1(\bar{x}) \; \nabla f_2(\bar{x}) \; ... \; \nabla f_k(\bar{x})], \\
B^t &:= [\nabla f_{k+1}(\bar{x}) \; \nabla f_{k+2}(\bar{x}) \; ... \; \nabla f_p(\bar{x})].
\end{aligned}
\tag{18}
$$

Since $\bar{x}$ is a strictly efficient point of Problem (17) and f_j is a linear function on $cl(B(\bar{x}, \bar{\delta}))$ for any $j \in I^c$, we conclude that the system

$$
Ad < 0, \quad Bd \leqq 0, \quad d \in \mathbb{R}^n
\tag{19}
$$

has no solution. So, regarding Theorem 3.22 of [26], there are $y^1 \in \mathbb{R}^k_{\geq}$ and $y^2 \in \mathbb{R}^{p-k}_{\geqq}$ such that

$$
A^t y^1 + B^t y^2 = 0.
\tag{20}
$$

Define $\lambda \in \mathbb{R}^p_{\geq}$ as follows:

$$
\lambda_i := \begin{cases} y^1_i, 1 \leq i \leq k, \\ y^2_{i-k}, \quad\quad k+1 \leq i \leq p. \end{cases}
\tag{21}
$$

So, considering (20) and (21) we have

$$
\sum_i \lambda_i \nabla f_i(\bar{x}) = 0.
\tag{22}
$$

In addition, since there is an index $1 \leq j \leq k$ such that $\lambda_j > 0$ and $\nabla^2 f_j(\bar{x})$ is positive definite, then $\nabla^2(\sum_i \lambda_i f_i(\bar{x}))$ is positive definite and by Proposition

2.3 we conclude that $\bar{x}$ is a tilt-stable local minimum of problem $\sum_i \lambda_i f_i(x)$. The proof is completed by the definition of tilt stability. $\square$

Note that in Proposition 3.14 the existence of index set I such that $\nabla^2 f_i(\bar{x})$ are positive definite for all $i \in I$ is essential. This is shown by Example 3.2. Regarding Example 3.2, $\bar{x} = 0$ is a stable locally efficient point of $F(x)$ while there are no $\lambda \in \mathbb{R}^2_{\geq}$ and $\delta > 0$ such that $\bar{x} = 0$ is the unique optimal solution of Problem (16).

Note that in Proposition 3.14 the condition that each f_j, $j \in I^c$ be linear on $B(\bar{x}, \delta)$ cannot change to the semi-positive definiteness of $\nabla^2 f_j(\bar{x})$, $j \in I^c$.

Example 3.4 To show the shortcoming of substituting semi-positive definiteness for linearity in Proposition 3.14, we consider $F(x) = (-x, x^4, (x-1)^2)$. $\bar{x} = 0$ is a unique optimal point of the weighted sum problem (16) with weights $\lambda_1 = \lambda_3 = 0$, $\lambda_2 = 1$ and arbitrary $\hat{\delta} > 0$. Each f_i is convex, $\nabla^2 f_1(0)$ and $\nabla^2 f_2(0)$ are semi-positive definite, and also $\nabla^2 f_3(0)$ is positive definite. We will show that $\bar{x} = 0$ is not a stable locally efficient point of F(x). By contradiction, suppose that $\bar{x} = 0$ is a stable locally efficient point of F(x); so, there are $1 > \delta$ and $\epsilon > 0$ such that conditions a) and b) of Definition 3.1 hold for any matrix A with $\|A\| < \epsilon$. Moreover, there is $k \in N$ such that $\frac{1}{k^{\frac{1}{3}}} < \delta$ and $\frac{1}{k} < \epsilon$. For any $m \in \mathbb{N}$ with $m \geq k$, let

$$A_m := \begin{bmatrix} 0 \\ -\frac{1}{m} \\ 0 \end{bmatrix},$$

then $\|A_m\| < \epsilon$. The set of strictly efficient points of $M_{(\delta,F)}(\bar{x}, A_m)$ is $[(\frac{1}{4m})^{\frac{1}{3}}, \delta]$. Now we have

$$\|x - 0\| \geq \frac{1}{4m^{\frac{1}{3}}},$$

for any strictly efficient point of $M_{(\delta,F)}(\bar{x}, A_m)$. In addition, there is no $\gamma \geq 0$ such that $\frac{1}{4m^{\frac{1}{3}}} \leq \gamma \|A_m\| = \frac{\gamma}{m}$ for any $m \in \mathbb{N}$ with $m \geq k$, and this contradicts condition a) of Definition 3.1.

Proposition 3.15 *If the conditions of Proposition 3.14 hold, then $\bar{x}$ is a stable locally efficient point of $F(x)$ if and only if there is $\delta > 0$ such that $\bar{x}$ is an efficient point of problem $M_{\delta,F}(\bar{x}, 0)$.*

Proof ($\Leftarrow$) Suppose that $\bar{x}$ is an efficient point of problem $M_{\delta,F}(\bar{x}, 0)$ and f_j are linear functions on $cl(B(\bar{x}, \bar{\delta}))$ for any $j \in I^c$, then we conclude the system

$$Ad < 0, \quad Bd \leqq 0, \quad d \in \mathbb{R}^n \tag{23}$$

has no solution, where A and B are as defined in (18). Similar to the proof of Proposition 3.14, the result is obtained.

($\Rightarrow$) The converse statement is obtained directly by the definition of the stability of F(x). $\square$

Proposition 3.16 *For vector function F, let each f_i be convex and C^2-smooth on some neighborhoods of $\bar{x} \in domF$ and also $\nabla^2 f_i(\bar{x})$ be positive definite for*

all i. $\bar{x}$ is a stable locally efficient point of $F(x)$ if and only if there is $\delta > 0$ such that $\bar{x}$ is a weakly efficient solution of problem $M_{\delta, F}(\bar{x}, 0)$.

Proof($\Leftarrow$) Without loss of generality, suppose that all f_i are convex and C^2-smooth on $cl(B(\bar{x}, \delta))$. Since $\bar{x}$ is a weakly efficient solution of problem

$$
\begin{aligned}
&min\ F(x)\\
&s.t.\quad x \in cl(B(\bar{x}, \delta)),
\end{aligned}
$$

there is $\lambda \geq 0$ such that the optimal value of the following problem is attained at $\bar{x}$,

$$
\begin{aligned}
&min\ \sum_{i \in I} \lambda_i f_i(x)\\
&s.t.\quad x \in cl(B(\bar{x}, \delta)),
\end{aligned}
$$

and so $\sum_i \lambda_i \nabla f_i(\bar{x}) = 0$. Furthermore, there is $j \in \{1, 2, ..., p\}$ such that $\lambda_j > 0$. Since $\nabla^2 f_i(\bar{x})$ are positive definite for all i, $\sum_i \lambda_i \nabla^2 f_i(\bar{x})$ is positive definite. Regarding Proposition 2.3, $\bar{x}$ is a tilt-stable local minimum of $\sum_i \lambda_i f_i(x)$, and so according to Proposition 3.11, $\bar{x}$ is a stable locally efficient point of $F(x)$.

($\Rightarrow$) The converse statement is obtained directly by the definition of the stability of F(x). $\qquad\square$

4 Conclusions

Tilt stability of a local minimum is a well-known concept to investigate the sensitivity and stability of an optimal solution. This concept is a version of the Lipschitzian stability of a local minimum in scalar optimization. In this paper, a new concept of stability in multiobjective optimization is introduced. In this regard, some perturbations are added to the objective functions as small linear terms. Furthermore, some propositions are established to identify the stability status of a strictly efficient solution. Also, the equivalency of the introduced stability concept and tilt stability is proved, when the set of objective functions is a singleton. In other words, the proposed concept can be considered as an extension of tilt stability in scalar optimization to multiobjective optimization.

Using some nonlinear scalarization methods for the proposed concept of stability and robustness can be considered as an interesting research for future.

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