NEW CONNECTIONS BETWEEN PARTITIONS AND MULTIPLICATIVE FUNCTIONS

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Abstract. Subtle and explicit connections between partitions and special classical functions in multiplicative number theory are decidely hard to come by in the body of literature surrounding the theory of partitions. A more general representative theory which connects special and restricted partition functions to multiplicative functions such as Euler's totient function $\phi(n)$, the Möebius function $\mu(n)$, von Mangoldt's prime characterizing function $\Lambda(n)$, the number of distinct prime factors of n denoted by $\omega(n)$, and the generalized sum-of-divisors functions $\sigma_\alpha(n)$ has been an unapproached topic up until recently. The work of Merca and Schmidt on so-termed "Lambert series factorization theorems" and partition-related matrix products over 2017–2018 has provided a general framework for connecting the theory of partitions with special multiplicative functions from more classical number theory (cf. arXiv papers 1706.00393, 1706.02359, 1712.00611, and 1712.00608). Additional work of Merca and Schmidt published in the Ramanujan Journal and to appear in the American Mathematical Monthly in 2018 respectively provide new identities and a generalization of Stanley's theorem connecting $\mu(n)$ and $\phi(n)$ to the partition functions $p(n)$ and $S_{n,k}^{(r)}$ which denotes the number of k's in all partitions of n into parts of size at least r. We will breifly summarize these recent characteristic factorization theorem approaches, the methodology to their proofs, and a plethora of new identities connecting $p(n)$ and $q(n)$ to special functions as well as new series representations for the Riemann zeta function $\zeta(s)$.

In 2018, Mousavi and Schmidt have worked out related representative matrix-based factorization theorems which are employed to enumerate the special gcd and gcd-divisor-sums of the form $T_f(n) := \sum_{d:(d,n)=1} f(d)$ and $L_{f,g,k}(n) := \sum_{d|(k,n)} f(d)g(n/d)$ for any prescribed arithmetic functions f and g and natural numbers $1 \leq k \leq n$. These sums naturally imply corresponding new identities connecting the partition function $p(n)$ with the classic multiplicative functions $\phi(n)$ and $\mu(n)$, Kloosterman sums, and the cyclotomic polynomials $\Phi_n(q)$. A special case of the series $L_{f,g,k}(n)$ also defines the Ramanujan sums $c_k(n)$ which then leads to even further new identities and partition-related expansions of infinite series for many other special functions and multiplicative divisor sums. In the second part of our talk, we will introduce our new variants of the factorization theorems enumerating these two classes of sums and provide many new identities and infinite series for special functions built on this work.

We feel that these new recent connections between special functions in additive and multiplicative number theory are an important and previously not well studied addition to the theory of partitions which is suitable for a good talk at the conference. We hope that the reviewers of this proposal also find such previously rare connections to be special and worth the extra investigation by Mousavi and Schmidt into generalizations and variants of the work in the references. We also hope that the conference reviewers see the merit in our new work recognizing the role of our factorization theorems in enumerating special gcd and divisor-related sums as both generating functions and matrix products involving partition functions. We hope to see you at the conference later this year!

1. Introduction

Identities and general theory connecting classical multiplicative functions with more additive branches of number theory have been studied in [1, 3, 4, 11, 13, 14]. While there are certainly known relations between partition functions and classical multiplicative functions such as $\phi(n)$ and $\mu(n)$, such identities are decidely more rare in the literature surrounding the theory of partitions. Moreover, a general theory allowing us to connect the partition function $p(n)$ with divisor and gcd sums of the forms $\sum_{d|n} a_d$ and $\sum_{d:(d,n)=1} b_d$ which (at least implicitly) define many special multiplicative functions has up until recently been sparse at best.

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In our upcoming 2018 article in the *American Mathematical Monthly*, Merca and Schmidt have studied connections between restricted partition functions and Euler's totient function $\phi(n)$. In particular, as another form of Stanley's theorem we have proved that the number of 1's in all partitions of n is equal to [8]

$$\sum_{k=2}^{n+1} \phi(k) S_{n+1,k}^{(2)},$$

where we denote by $S_{n,k}^{(r)}$ the number of partitions of n with smallest part at least r. Similarly, the recent AMM article also proves other relations between $\phi(n)$, $S_{n,k}^{(r)}$, and $p(n)$ including the following result:

$$p(n) = \sum_{k=3}^{n+3} \frac{\phi(k)}{2} S_{n+3} k^{(3)}.$$

In a related article published in the *Ramanujan Journal* in 2018, Merca and Schmidt prove similar relations connecting $p(n)$, $S_{n,k}^{(r)}$, and the Möbius function $\mu(n)$.

2. Recent work on Lambert series factorizations

The recent work of Merca and Schmidt over 2017–2018 builds on a general framework of so-called "*Lambert series factorization theorems*" of the following forms, though other variants of these expansions are possible [5, 6, 7]:

$$\sum_{n\geq 1} \frac{a_n q^n}{1-q^n} = \frac{1}{(q;q)_\infty} \sum_{n\geq 1}\sum_{k=1}^{n} s_{n,k} \cdot a_k \cdot q^n \tag{1}$$

$$\sum_{n\geq 1} \frac{a_n q^{\alpha n+\beta}}{1-q^{\alpha n+\beta}} = \frac{1}{C(q)} \sum_{n\geq 1}\sum_{k=1}^{n} \widetilde{s}_{n,k}(\alpha,\beta) \cdot a_k \cdot q^n,\ \ \alpha \in \mathbb{Z}^+, 1 \leq \beta < \alpha \tag{2}$$

$$\sum_{n\geq 1} \frac{(f*g)(n) q^n}{1-q^n} = \frac{1}{(q;q)_\infty} \sum_{n\geq 1}\sum_{k=1}^{n} \bar{s}_{n,k}(g) \cdot f(k) \cdot q^n. \tag{3}$$

In particular, in [5, 6] we showed that the matrix-product-like expansions in (1) satisfy (i) $s_{n,k} = s_o(n,k) - s_e(n,k)$ where $s_{o/e}(n,k)$ respectively denote the number of k's in all partitions of n into an odd (even) number of distinct parts; and (ii) that the corresponding inverse matrices are expressed by the coefficients

$$s_{n,k}^{(-1)} = \sum_{d|n} p(d-k)\mu(n/d),$$

i.e., so that

$$a_n = \sum_{k=1}^{n}\sum_{d|n} p(d-k)\mu(n/d) \times \sum_{j\geq 0} (-1)^{\lceil \frac{j}{2} \rceil} \left(\sum_{d|k-\frac{1}{2}\lceil \frac{j}{2} \rceil \lceil \frac{3j+1}{2} \rceil} a_d \right). \tag{4}$$

For example, (4) implies new partition identities for the following special Lambert-series-related divisor sum pairs where $\sigma_\alpha(n)$ is the generalized sum-of-divisors function, $\lambda(n)$ is Liouville's lambda function, $\chi_{\mathrm{sq}}(n)$ denotes the characteristic function of the squares, $\Lambda(n)$ is von Mangoldt's function, $\omega(n)$ counts the number of distinct prime divisors of n, and $J_t(n)$ is Jordan's totient function [10, §27]:

$$\left(a_n, \sum_{d|n} a_d\right) = (\mu(n), \delta_{n,1}), (\phi(n), n), (n^\alpha, \sigma_\alpha(n)), (\lambda(n), \chi_{\mathrm{sq}}(n)), (\Lambda(n), \log n),$$
$$(|\mu(n)|, 2^{\omega(n)}), (J_t(n), n^t).$$

The generalizations of these factorization theorems expanded in (2) are similarly studied in [7]. These extensions of the original form of the Lambert series factorizations given in (1) allow us, for example, to obtain new identities of the form

$$\sum_{k=1}^{n} \left(\sum_{2d-1|k} a_d \right) (s_e(n-k) - s_o(n-k)) = \sum_{k=1}^{n} (-1)^{n-1} a_k \cdot s_{2,-1}(n,k)$$

$$r_2(n) = \sum_{k=0}^{n}\sum_{j=1}^{k} 4(-1)^{j+1}\left(s_e(n-k)-s_o(n-k)\right)s_{2,-1}(n,k),$$

and

$$\begin{aligned}\sum_{\substack{d\mid n\\ d \text{ odd}}} &(-1)^{(d+1)/2}\left(r_2\left(\frac{n}{d}\right) - 4\cdot d\left(\frac{n}{2d}\right)\left[\frac{n}{d}\text{ even}\right]_\delta\right)\\ &= \sum_{k=1}^{n}\sum_{d\mid n} p(d-k)(-1)^{n/d+1}\cdot [q^k](q;q)_\infty \vartheta_3(q)^2,\end{aligned}$$

where $s_{2,-1}(n,k)$ denotes the number of all $(2k-1)$'s in all partitions of n into distinct odd parts, $s_{e/o}(n)$ is the number of of partitions of n into even (odd) parts, $r_2(n)$ is the sum-of-squares function, and $[R(n)]_\delta$ denotes Iverson's convention which is equal to 1 if and only if $R(n)$ is true and is zero otherwise.

Additionally, [7] studies the variant of (1) and (2) defined in (3) above corresponding to the Lambert series enumerting the Dirichlet convolutions $(f*g)(n)$ for any arithmetic functions, $f,g:\mathbb{N}\to\mathbb{C}$. Specifically, we obtain that

$$\begin{aligned}\bar{s}_{n,k}(g) &= \sum_{j=1}^{n} s_{n,kj}\cdot g(j)\\ \bar{s}_{n,k}^{(-1)}(g) &= \left(s_{n,k}^{(-1)} * (\varepsilon * D_{n,g})\right)(n),\end{aligned}$$

where $\varepsilon(n) = \delta_{n,1}$ is the multiplicative identity for Dirichlet convolutions and the function $D_{n,g}(n) = \sum_{j=1}^{n} \mathrm{ds}_{2j,g}(n)$ denotes the sum over the nested convolutions defined by

$$\mathrm{ds}_{j,g}(n) := \begin{cases} g_\pm(n), & \text{if } j=1;\\ \sum\limits_{\substack{d\mid n\\ d>1}} g(d)\,\mathrm{ds}_{j-1,g}(n/d), & \text{if } j>1.\end{cases}$$

In 2017–2018, Schmidt has obtained new results for variations of the Lambert series factorization theorems in the context of Hadmard products and derivatives of the Lambert series generating functions defined on the left-hand-side of (1). New identities which are then obtained in [12] include the next new identities involving special multiplicative functions.

$$\begin{aligned}\phi(n) &= \sum_{k=1}^{n}\sum_{d\mid n}\frac{p(d-k)}{d}\mu(n/d)\left[k^2 + \sum_{b=\pm 1}\sum_{j=1}^{\left\lfloor\frac{\sqrt{24k-23}-b}{6}\right\rfloor}(-1)^j\left(k-\frac{j(3j+b)}{2}\right)^2\right]\\ n^s &= \sum_{k=1}^{n}\sum_{d\mid n}\frac{p(d-k)}{\sigma_t(d)}\mu(n/d)\Bigg[\sigma_t(k)\sigma_s(k)\\ &\quad + \sum_{b=\pm 1}\sum_{j=1}^{\left\lfloor\frac{\sqrt{24k+1}-b}{6}\right\rfloor}(-1)^j\sigma_t\left(k-\frac{j(3j+b)}{2}\right)\sigma_s\left(k-\frac{j(3j+b)}{2}\right)\Bigg]\\ \Lambda(n) &= \sum_{k=1}^{n}\sum_{d\mid n}\frac{p(d-k)}{d}\mu(n/d)\Bigg[k\log(k)\\ &\quad + \sum_{b=\pm 1}\sum_{j=1}^{\left\lfloor\frac{\sqrt{24k-23}-b}{6}\right\rfloor}(-1)^j\left(k-\frac{j(3j+b)}{2}\right)\log\left(k-\frac{j(3j+b)}{2}\right)\Bigg]\\ \omega(n) &= \log_2\left[\sum_{k=1}^{n}\sum_{j=1}^{k}\left(\sum_{d\mid k}\sum_{i=1}^{d}p(d-ji)\right)s_{n,k}\cdot|\mu(j)|\right]\end{aligned}$$

The results proved in [12] also include the following which connect the partition function with new series expansions for the Riemann zeta function $\zeta(s)$ for any $\Re(s) > 1$ where $G_j := \frac{1}{2}\left\lceil \frac{j}{2} \right\rceil \left\lceil \frac{3j+1}{2} \right\rceil \mapsto \{0,1,2,5,7,12,22,26,35,\ldots\}$ denotes the sequence of interleaved pentagonal numbers:

$$\zeta(s) = \sum_{n\geq 1}\sum_{k=1}^{n}\sum_{d|n} \frac{p(d-k)}{\sigma_t(d)}\mu(n/d) \times \sum_{j:G_j<k} (-1)^{\lceil j/2\rceil} \frac{\sigma_t(k-G_j)\sigma_s(k-G_j)}{(k-G_j)^s}$$

$$\zeta(s) = \sum_{n\geq 1}\sum_{k=1}^{n}\sum_{d|n} \frac{d^t\cdot p(d-k)}{\sigma_t(d)}\mu(n/d) \times \sum_{j:G_j<k} (-1)^{\lceil j/2\rceil} \frac{\sigma_t(k-G_j)\sigma_s(k-G_j)}{(k-G_j)^{s+t}}.$$

3. Conference talk proposal

In this talk we will summarize the recent results of Merca and Schmidt described in the references and in the previous section and discuss new, even more recent work of Mousavi and Schmidt considering connections between $p(n)$ and gcd-sums and divisor sums of the form

$$\begin{aligned} T_f(n) &:= \sum_{d:(n,d)=1} f(d) \\ L_{f,g,k}(n) &:= \sum_{d|(n,k)} f(d)g(n/d), \end{aligned} \tag{5}$$

for any prescribed arithmetic functions f and g and natural numbers $1 \leq k \leq n$. Notable special cases of the gcd-sums and and divisor sums defined in the previous two equations include identities expanding $\phi(n)$, $\mu(n)$, and the Ramanujan sums $c_k(n)$ which are involved in many notable infinite series for special functions [2]. The primary objects of study in the expansions of special multiplictive functions by the gcd-related sums in (5) are the corresponding factorization theorem analogs to those first defined in (1)–(3) as

$$\begin{aligned} T_f(n) &= [q^n]\left(\frac{1}{(q;q)_\infty}\sum_{n\geq 2}\sum_{k=1}^{n} t_{n,k} f(k)\cdot q^n + f(1)\cdot q\right) \\ \sum_{m=1}^{k} L_{f,g,m}(n)w^m &= [q^n]\left(\sum_{n\geq 2}\sum_{k=1}^{n} u_{n,k}(g,w) f(k)\cdot q^n\right) \\ g(x) &= [q^x]\left(\frac{1}{(q;q)_\infty}\sum_{n\geq 2}\sum_{k=1}^{n} v_{n,k}(f,w)\left[\sum_{m=1}^{k} L_{f,g,m}(k)w^m\right]\cdot q^n\right),\ w\in\mathbb{C}. \end{aligned}$$

Thus our new work this year provides an effective generalization of the recent work of Merca and Schmidt summarized in the introduction to the divisor sum variants in (5). Examples of the gcd-sum identities we are able to obtain includes the expansions

$$\begin{aligned} \phi_m(n) &= \sum_{j=0}^{n}\sum_{k=1}^{j-1}\sum_{i=0}^{j} p(n-j)(-1)^{\lceil i/2\rceil}\left[(j-k-G_i,k)=1\right]_\delta \left[j-k-G_i\geq 1\right]_\delta \cdot k^m \\ \phi_m(n) &= \sum_{d:(d,n)=1}\left(\sum_{k=1}^{d+1}\sum_{i=1}^{d}\sum_{j=0}^{k} p(i+1-k)(-1)^{\lceil j/2\rceil}\phi_m(k-G_j)\mu_{d,i}\left[k-G_j\geq 1\right]_\delta\right) \\ \mu(n) &= \sum_{1\leq k<j\leq n}\left(\sum_{i=0}^{j} p(n-j)(-1)^{\lceil i/2\rceil}\left[(j-k-G_i,k)=1\right]_\delta \left[j-k-G_i\geq 1\right]_\delta e^{2\pi\imath k/n}\right) \\ \log\Phi_n(q) &= \sum_{1\leq k<j\leq n}\left(\sum_{i=0}^{j} p(n-j)(-1)^{\lceil i/2\rceil}\left[(j-k-G_i,k)=1\right]_\delta \left[j-k-G_i\geq 1\right]_\delta \log\left(q-e^{2\pi\imath k/n}\right)\right), \end{aligned}$$

where $\mu_{n,k}$ denotes the inverse matrix of terms defined by $\mu_{n,k}^{(-1)} := [(n+1,k)=1]_\delta$, $\phi_m(n) = \sum_{d:(d,n)=1} d^m$ is a generalized form of Euler's totient function $\phi(n) = \phi_0(n)$, and $\Phi_n(q)$ is a cyclotomic polynomial.

We feel that these new recent connections between special functions in additive and multiplicative number theory are an important and previously not well studied addition to the theory of partitions which is suitable for a good talk at the conference. We hope that the reviewers of this proposal also find such previously rare connections to be special and worth the extra investigation by Mousavi and Schmidt into generalizations and variants of the work in the references. We also hope that the conference reviewers see the merit in our new work recognizing the role of our factorization theorems in enumerating special gcd and divisor-related sums as both generating functions and matrix products involving partition functions. We hope to see you at the conference later this year. Thank you for reading this proposal!

References

[1] Alladi, K., Erdős, P.: On an additive arithmetic function, *Pacific J. Math.* **71**(2), 275–294 (1977).

[2] G. H. Hardy, *Ramanujan: Twelve lectures on subjects suggested by his life and work*, AMS Chelsea Publishing, 1999.

[3] Jameson, M., Schneider, R.: Combinatorial applications of Möbius inversion, *Proc. Amer. Math. Soc.* **142**(9), 2965–2971 (2014).

[4] M. Merca, The Lambert series factorization theorem, *The Ramanujan Journal*, pp. 1–19 (2017).

[5] M. Merca and M. D. Schmidt, Generating special arithmetic functions by Lambert series factorizations, 2017, https://arxiv.org/abs/1706.00393, to appear in *Contributions to Discrete Mathematics*.

[6] M. Merca and M. D. Schmidt, New Factor Pairs for Factorizations of Lambert Series Generating Functions, 2017, https://arxiv.org/abs/1706.02359, submitted.

[7] M. Merca and M. D. Schmidt, Factorization Theorems for Generalized Lambert Series and Applications, 2017, https://arxiv.org/abs/1712.00611, submitted.

[8] Merca, M., Schmidt, M.D., A Partition Identity Related to Stanley's Theorem, *Amer. Math. Monthly* (accepted, to appear).

[9] Merca, M., Schmidt, M.D., The partition function $p(n)$ in terms of the classical Möbius function, *Ramanujan J.* (2018).

[10] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark. *NIST Handbook of Mathematical Functions*. Cambridge University Press, 2010.

[11] M. D. Schmidt, New recurrence relations and matrix equations for arithmetic functions generated by Lambert series, 2017, https://arxiv.org/abs/1701.06257, *Acta Arith.* (2018).

[12] M. D. Schmidt, Factorization Theorems for Hadamard Products and Higher-Order Derivatives of Lambert Series Generating Functions, 2017, https://arxiv.org/abs/1712.00608, submitted.

[13] Schneider, R.: Arithmetic of partitions and the q-bracket operator, *Proc. Amer. Math. Soc.* **145**(5), 1953–1968 (2017)

[14] Wakhare, T.: Special classes of q-bracket operators, *Ramanujan J.* (2017). https://doi.org/10.1007/s11139-017-9956-8.