VII

Calculus of Residues

§1. Contour Integration and Residues.

In this chapter we will learn one of the main applications of complex analysis to engineering problems, but we'll concentrate on the math, not the engineering. Cauchy's Theorem says that if f is analytic in a region Ω and if γ is a closed curve in Ω which is homologous to 0, then $\int_\gamma f(z)dz = 0$. What happens if f has an isolated singularity at $a \in \Omega$, but otherwise is analytic? Expanding f in its Laurent series about a, we have

$$f(z) = \sum_{n=-\infty}^{\infty} b_n(z-a)^n$$

$$= \frac{b_{-1}}{z-a} + \frac{d}{dz} \sum_{\substack{n=-\infty \\ n\neq -1}}^{\infty} \frac{b_n}{(n+1)}(z-a)^{n+1}.$$

So if Δ is a disc centered at a and contained in Ω then by the Fundamental Theorem of Calculus,

$$\int_{\partial\Delta} f(z)dz = b_{-1}\int_{\partial\Delta} \frac{dz}{z-a} = 2\pi i b_{-1}, \tag{1.1}$$

provided $\partial\Delta$ is oriented in the positive or counterclockwise direction.

Definition 1.1. *If f is analytic in $\{0 < |z-a| < \delta\}$ for some $\delta > 0$, then the* **residue of** f **at** a, *written* $\mathrm{Res}_a f$, *is the coefficient of $(z-a)^{-1}$ in the Laurent expansion of f about $z = a$.*

Theorem 1.2 (Residue Theorem). *Suppose f is analytic in Ω except for isolated singularities at $a_1, \ldots, a_n$. If γ is a cycle in Ω with $\gamma \sim 0$ and $a_j \notin \gamma$, $j = 1, \ldots, n$, then*

$$\int_\gamma f(z)dz = 2\pi i \sum_k \mathrm{n}(\gamma, a_k)\mathrm{Res}_{a_k} f.$$

Usually the Residue Theorem will be applied to curves γ such that $\mathrm{n}(\gamma, a_k) = 0$ or 1, so that the sum on the right is $2\pi i$ times the sum of the residues of f at points "enclosed" by γ. If f has infinitely many singularities clustering only on $\partial\Omega$ then we can shrink Ω slightly so that it contains only finitely many a_j and still have $\gamma \sim 0$.

Proof. Let Δ_k be a disc centered at a_k, $k = 1, 2, \ldots, n$, such that $\overline{\Delta}_m \cap \overline{\Delta}_k = \emptyset$ if $m \neq k$. Orient $\partial\Delta_k$ in the positive or counterclockwise direction. Then

$$\gamma - \sum_k \mathrm{n}(\gamma, a_k)\partial\Delta_k \sim 0$$

in the region $\Omega \setminus \{a_1, \ldots, a_n\}$. By Cauchy's Theorem

$$\int_\gamma f(z)dz - \sum_{k=1}^{n} \mathrm{n}(\gamma, a_k) \int_{\partial\Delta_k} f(z)dz = 0.$$

Then Theorem 1.2 follows from (1.1). □

Here are some examples illustrating several useful techniques for computing residues.

(1) $f(z) = \dfrac{e^{3z}}{(z-2)(z-4)}$ has a simple pole at $z = 2$ and hence

$$\mathrm{Res}_2 f = \lim_{z\to 2}(z-2)f(z) = \frac{e^6}{-2}.$$

The residue at $z = 4$ can be calculated similarly.

(2) $g(z) = \dfrac{e^{3z}}{(z-2)^2}$. We expand e^{3z} in a series expansion about $z = 2$:

$$g(z) = \frac{e^6 e^{3(z-2)}}{(z-2)^2} = \frac{e^6}{(z-2)^2}\sum_{n=0}^{\infty}\frac{3^n}{n!}(z-2)^n = \frac{e^6}{(z-2)^2} + \frac{3e^6}{z-2} + \ldots,$$

so that

$$\mathrm{Res}_2 g = 3e^6.$$

In this case $\lim(z-2)^2 g(z)$ is not the coefficient of $(z-2)^{-1}$ and $\lim(z-2)g(z)$ is infinite. Of course the full series for e^{3z} was not necessary to compute the residue.

We can find the appropriate coefficient in the series expansion for e^{3z} by computing the derivative of e^{3z}. More generally, if $G(z)$ is analytic at $z = a$ then

$$\mathrm{Res}_a \frac{G(z)}{(z-a)^n} = \frac{G^{(n-1)}(a)}{(n-1)!}.$$

(3) Another trick that can be used with simple poles, when pole is not already written as a factor of the denominator, is illustrated by the example $h(z) = e^{az}/(z^4+1)$. Then h has simple poles at the fourth roots of -1. If $\omega^4 = -1$, then

$$\mathrm{Res}_\omega h = \lim_{z\to\omega} \frac{(z-\omega)e^{az}}{z^4+1} = \frac{e^{a\omega}}{\lim_{z\to\omega} \frac{z^4+1}{z-\omega}}.$$

Note that the denominator is the limit of difference quotients for the derivative of z^4+1 at $z = \omega$ and hence

$$\mathrm{Res}_\omega \frac{e^{az}}{z^4+1} = \frac{e^{a\omega}}{4\omega^3} = -\frac{\omega e^{a\omega}}{4}.$$

(4) Another method using series is illustrated by the example

$$k(z) = \frac{\pi \cot \pi z}{z^2}$$

To compute the residue of k at $z = 0$, note that $\cot \pi z$ has a simple pole at $z = 0$ and hence k has a pole of order 3, so that

$$\frac{\pi \cot \pi z}{z^2} = \frac{b_{-3}}{z^3} + \frac{b_{-2}}{z^2} + \frac{b_{-1}}{z} + b_0 + \ldots.$$

Then

$$\pi \cos \pi z = \left(\frac{\sin \pi z}{z}\right)(b_{-3} + b_{-2}z + b_{-1}z^2 + b_0 z^3 + \ldots.$$

Inserting the series expansions for cos and sin we obtain

$$\pi(1 - \frac{\pi^2}{2}z^2 + \ldots) = (\pi - \frac{\pi^3}{6}z^2 + \ldots)(b_{-3} + b_{-2}z + b_{-1}z^2 + \ldots,$$

Equating coefficients

$$\pi = \pi b_{-3} \qquad -\frac{\pi^3}{2} = -\frac{\pi^3}{6}b_{-3} + \pi b_{-1},$$

and $\mathrm{Res}_0 k = -\frac{\pi^2}{3}$.

§2. Examples.

In this section we compute some integrals using the Residue Theorem. The emphasis is on the techniques instead of proving general results that can be quoted.

(a) If γ is the circle centered at 0 with radius 3, then

$$\int_\gamma \frac{e^{3z}}{(z-2)(z-4)} dz = -2\pi i \frac{e^6}{2},$$

by the Residue Theorem and Example (1) above.

(b) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^4+1}$.

We construct the contour γ consisting of the interval $[-R, R]$ followed by the semicircle C_R in $\mathbb{H}$ of radius R, with $R > 1$.

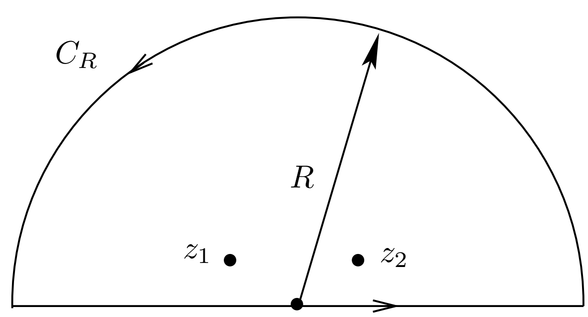

Figure VII.1 Half-disc contour.

By the Residue Theorem with $f(z) = 1/(z^4+1)$,

$$\int_{-R}^{R} f(z)dz + \int_{C_R} f(z)dz = 2\pi i(\mathrm{Res}_{z_1} f + \mathrm{Res}_{z_2} f), \tag{1.2}$$

where z_1 and z_2 are the roots of $z^4+1=0$ in the upper half-plane $\mathbb{H}$. Note that

$$\left| \int_{C_R} f(z)dz \right| \leq \int_0^{2\pi} \frac{Rd\theta}{R^4-1} \to 0$$

as $R \to \infty$. Since the integral $\int_{\mathbb{R}}(x^4+1)^{-1}dx$ is absolutely convergent, it equals $\lim_{R\to\infty}\int_{-R}^{R}(x^4+1)^{-1}dx$, so that by (1.2), and Example (3) above

$$\int_{-\infty}^{\infty}\frac{1}{x^4+1}dx = -\frac{2\pi}{4}(z_1+z_2) = \frac{\pi}{\sqrt{2}}.$$

The technique in Example (b) can be used to compute the integral of any rational function with no poles on $\mathbb{R}$ if the degree of the denominator is at least 2 plus the degree of the numerator. This latter condition is needed for the absolute convergence of the integral.

(c) $\displaystyle\int_0^{2\pi}\frac{1}{3+\sin\theta}d\theta.$

Set $z = e^{i\theta}$. Then

$$\int_0^{2\pi}\frac{1}{3+\sin\theta}d\theta = \int_{|z|=1}\frac{1}{(3+\frac{1}{2i}(z-1/z))}\frac{dz}{iz} = \int_{|z|=1}\frac{2dz}{z^2+6iz-1}.$$

The roots of $z^2+6iz-1$ occur at $z_1, z_2 = i(-3\pm\sqrt{8})$. Only $i(-3+\sqrt{8})$ lies inside $|z|=1$. By the Residue Theorem and method for computing residues in Example (1) or Example (3),

$$\int_0^{2\pi}\frac{1}{3+\sin\theta}d\theta = 2\pi i\mathrm{Res}_{i(-3+\sqrt{8})}\frac{2}{z^2+6iz-1} = \frac{2\pi}{\sqrt{8}}.$$

The technique in Example (c) can be used to compute

$$\int_0^{2\pi}R(\cos\theta,\sin\theta)d\theta,$$

where $R(\cos\theta,\sin\theta)$ is a rational function of $\sin\theta$ and $\cos\theta$, with no poles on the unit circle. An integral on the circle as in Example (c), can be converted to an integral on the line using the Cayley transform $z=(i-w)/(i+w)$ of the upper half plane onto the disc. It is interesting to note that you obtain the substitution $x=\tan\frac{\theta}{2}$ which you might have learned in calculus.

(d) $\displaystyle\int_{-\infty}^{\infty}\frac{\cos x}{x^2+1}dx.$

A first guess might be to write $\cos z = (e^{iz} + e^{-iz})/2$, but if $y = \mathrm{Im} z$ then $\cos z \sim e^{|y|}/|z|^2$ for large $|z|$. This won't allow us to find a closed contour where the part off the real line has small contribution to the integral. Instead, we use $e^{iz}/(z^2+1)$ then take real parts of the resulting integral. Using the same contour as in Example (b), we have the estimate

$$\left|\int_{C_R} \frac{e^{iz}}{z^2+1} dz\right| \le \int_{C_R} \frac{e^{-y}}{R^2-1}|dz| \le \frac{\pi R}{R^2-1} \to 0$$

as $R \to \infty$, where $y = \mathrm{Im} z > 0$. By the Residue Theorem and the method in Example (3),

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+1} dx = 2\pi i \sum_{\mathrm{Im} a > 0} \mathrm{Res}_a \frac{e^{iz}}{z^2+1} = 2\pi i \frac{e^{i \cdot i}}{2i} = \frac{\pi}{e}.$$

In this particular case, we did not have to take real parts, because the integral itself is real since $\sin x/(x^2+1)$ is odd.

§3. Fourier, Mellin and Inverse Laplace Transforms.

The technique in Example (d) can be used to compute

$$\int_{-\infty}^{\infty} f(x) e^{i\lambda x} dx \tag{1.3}$$

for $\lambda > 0$, provided f is meromorphic in the closed upper half-plane $\mathbb{H} \cup \mathbb{R}$ with no poles on $\mathbb{R}$ and $|f(z)| \le K/|z|^{1+\varepsilon}$ for some $\varepsilon > 0$ and all large $|z|$ with $\mathrm{Im} z > 0$. If the integral (1.3) is desired for all real λ, then for negative λ use a contour in the lower half-plane, provided f is meromorphic and $|f(z)| \le 1/|z|^{1+\varepsilon}$ in $\mathrm{Im} z < 0$. As the reader can deduce, if f is meromorphic in $\mathbb{C}$ and satisfies this inequality for all $|z|$ large, then f is rational and the degree of the denominator is at least 2 plus the degree of the numerator. The integral in (1.3) is called the **Fourier Transform of** f, as a function of λ.

(e) $\displaystyle\int_{-\infty}^{\infty} \frac{x \sin \lambda x}{x^2+1} dx = \pi e^{-\lambda}.$

Example (e) cannot be done by the method in Example (d) because the integrand does not decay fast enough to prove $\int_{C_R} |f(z)||dz| \to 0$, where $f(z) = ze^{i\lambda z}/(z^2+1)$. Indeed it is not even clear that the integral in Example (e) exists.

Let $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$ where $\gamma_1 = [-A, B]$, $\gamma_2 = \{B + iy : 0 \le y \le A + B\}$, $\gamma_3 = \{z = x + i(A+B) : B \ge x \ge -A\}$, and $\gamma_4 = \{z = -A + iy : A + B \ge y \ge 0\}$, where γ is oriented as indicated in Figure VII.2

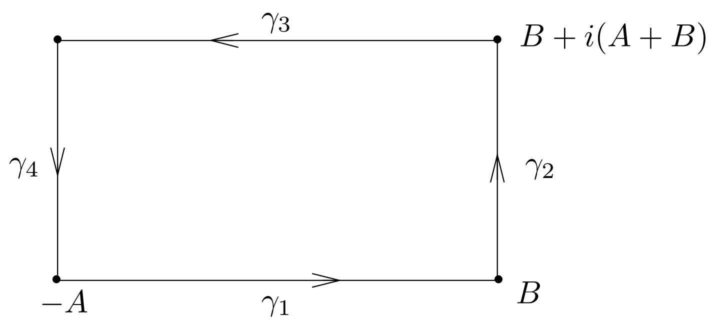

Figure VII.2 Rectangle contour.

To prove convergence of the integral in Example (e), we will let A and B tend to ∞ independently, and use the estimate $|z/(z^2+1)| \le |z|/(|z|^2-1) \le 2/|z|$ when $|z| > 2$. For A and B large,

$$\left|\int_{\gamma_3} \frac{ze^{i\lambda z}}{z^2+1} dz\right| \le \int_{-A}^{B} \frac{2}{A+B} e^{-\lambda(A+B)} dx \le \frac{2e^{-\lambda(A+B)}}{A+B}(A+B) \to 0,$$

as $A + B \to \infty$. Also

$$\left|\int_{\gamma_2} \frac{ze^{i\lambda z}}{z^2+1} dz\right| \le \int_{0}^{A+B} \frac{2}{B} e^{-\lambda y} dy \le \frac{2}{B}\frac{(1 - e^{-\lambda(A+B)})}{\lambda} \to 0,$$

as $B \to \infty$. A similar estimate holds on γ_4 as $A \to \infty$. By the Residue Theorem

$$\lim_{A,B\to\infty} \int_{-A}^{B} \frac{xe^{i\lambda x}}{x^2+1} dx = 2\pi i \operatorname{Res}_i \frac{ze^{i\lambda z}}{z^2+1} = \frac{2\pi i \cdot ie^{-\lambda}}{2i} = i\pi e^{-\lambda}. \tag{1.4}$$

Indeed by our estimates, the integrals over γ_2, γ_3, and γ_4 tend to 0 as A and B tend to ∞ so that the limit on the left side of (1.4) exists and (1.4) holds. Example (e) follows from (1.4) by taking the imaginary parts.

The technique in Example (e) can be used to compute (1.3) with the (weaker) assumption $|f(z)| \le K/|z|$ for large $|z|$.

(f) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x} dx.$

The main difference between Example (e) and Example (f) is that the function $f(z) = e^{iz}/z$ has a simple pole on $\mathbb{R}$. The function $\sin x/x$ is integrable near 0 since $\sin x/x \to 1$ as $x \to 0$, but $f(x)$ is not integrable. However the imaginary part of f is odd so that

$$\lim_{\delta\to 0} \int_{-1}^{-\delta} + \int_{\delta}^{1} \frac{e^{ix}}{x} dx$$

exists.

Definition 3.1. *If f is continuous on $\gamma \setminus \{a\}$, where γ is a smooth curve (γ' is continuous), then the Cauchy Principal Value of the integral of f along γ is defined by*

$$PV \int_{\gamma} f(z)dz = \lim_{\delta\to 0} \int_{\gamma\cap\{|z-a|\geq\delta\}} f(z)dz,$$

provided the limit exists.

Note that we have deleted points in a ball *centered* at a. If the limit exists for all balls containing a, then the usual integral of f exists. For example

$$PV \int_{-1}^{1} \frac{\cos x}{x} dx = 0,$$

because the integrand is odd, but the integral itself does not exist.

Proposition 3.2. *Suppose f is meromorphic in $\{\text{Im} z \geq 0\}$, such that*

$$|f(z)| \leq \frac{K}{|z|}$$

when $\text{Im} z \geq 0$ and $|z| > R$. Suppose also that all poles of f on $\mathbb{R}$ are simple. If $\lambda > 0$ then

$$PV \int_{-\infty}^{\infty} f(x)e^{i\lambda x}dx = 2\pi i \sum_{\text{Im} a>0} \text{Res}_a e^{i\lambda z} f(z) + 2\pi i \sum_{\text{Im} a=0} \frac{1}{2}\text{Res}_a e^{i\lambda z} f(z). \tag{1.5}$$

Part of the conclusion of Proposition 3.2 is that the integral exists.

Proof. Note that f has at most finitely many poles in $\{\text{Im} z \geq 0\}$ because $|f(z)| \leq K/|z|$, so that both sums in the statement of Proposition 3.2 are finite. Construct a contour similar to the rectangle in Figure VII.2, but avoiding the poles on $\mathbb{R}$ using small semicircles C_j of radius $\delta > 0$ centered at each pole $a_j \in \mathbb{R}$. See Figure VII.3.

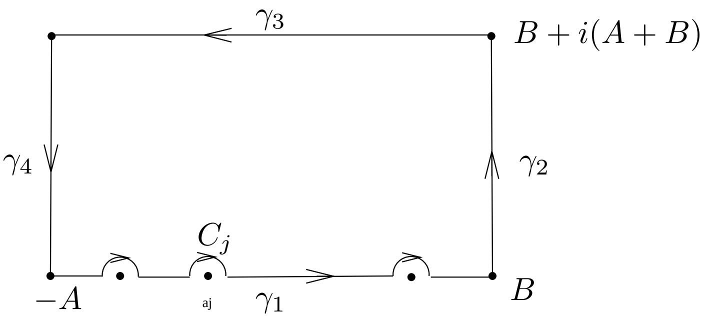

Figure VII.3 Contour avoiding poles on $\mathbb{R}$.

The integral of $f(z)e^{i\lambda z}$ along the top and sides of the contour tend to 0 as $A, B \to \infty$ as in the previous example.

The semi-circle C_j centered at a_j can be parameterized by $z = a_j + \delta e^{i\theta}$, $\pi > \theta > 0$ so that if

$$f(z)e^{i\lambda z} = \frac{b_j}{z - a_j} + g_j(z)$$

where g_j is analytic in an neighborhood of a_j then

$$\int_{C_j} f(z)e^{i\lambda z} dz = \int_\pi^0 \frac{b_j}{\delta e^{i\theta}} \delta i e^{i\theta} d\theta + \int_{C_j} g(z) dz.$$

Because g_j is continuous at a_j and the length of C_j is $\pi\delta$, we have

$$\lim_{\delta \to 0} \int_{C_j} f(z)e^{i\lambda z} dz = -i\pi b_j.$$

By the Residue Theorem

$$PV \int_{\mathbb{R}} f(z)e^{i\lambda z} dz - i\pi \sum_j b_j = 2\pi i \sum_{\mathrm{Im} a > 0} \mathrm{Res}_a f(z)e^{i\lambda z}$$

and (1.5) holds. □

One way to remember the conclusion of Proposition 3.2 is to think of the real line as cutting the pole at each a_j in half. The integral contributes half of the residue of f at a_j.

In our example

$$PV \int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx = \pi i, \tag{1.6}$$

so that by taking imaginary parts

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi.$$

Note that $\sin x/x \to 1$ as $x \to 0$ so that $\int \sin x/x dx$ exists as an ordinary Riemann integral, if we extend the function $\sin x/x$ to equal 1 at $x = 0$. For this reason we can drop the PV in front of the integral.

Mellin Transforms are a useful tool in Physics.

(g) **Mellin Transforms** $\displaystyle\int_0^{\infty} \frac{x^\alpha}{x^2+1} dx,$

where $0 < \alpha < 1$. Here we define $z^\alpha = e^{\alpha \log z}$ in $\mathbb{C} \setminus [0, +\infty)$ where $0 < \arg z < 2\pi$ and set $f(z) = 1/(z^2+1)$. Part of the difficulty here is constructing a closed contour that will give the desired integral. Consider the "keyhole" contour γ consisting of a portion of a large circle C_R of radius R and a portion of a small circle C_δ of radius δ, both circles centered at 0, along with two line segments between C_δ and C_R at heights $\pm\varepsilon$, oriented as indicated in Figure VII.4.

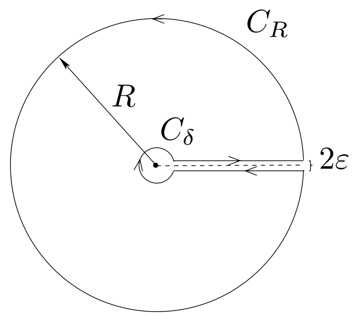

Figure VII.4 Keyhole contour.

By the Residue Theorem, for R large and δ small,

$$\int_\gamma z^\alpha f(z) dz = 2\pi i(\mathrm{Res}_i z^\alpha f(z) + \mathrm{Res}_{-i} z^\alpha f(z))$$

$$= 2\pi i\left(\frac{e^{\alpha \log i}}{2i} + \frac{e^{\alpha \log -i}}{-2i}\right) = \pi(e^{\frac{i\pi\alpha}{2}} - e^{i\frac{3\pi\alpha}{2}}). \tag{1.7}$$

We will first let $\varepsilon \to 0$, then $R \to \infty$ and $\delta \to 0$. Even though the integrals along the horizontal lines are in opposite directions, they do not cancel as $\varepsilon \to 0$. For $\varepsilon > 0$

$$\lim_{\varepsilon \to 0} (x + i\varepsilon)^\alpha f(x + i\varepsilon) = e^{\alpha \log |x|} f(x),$$

and

$$\lim_{\varepsilon\to 0}(x-i\varepsilon)^\alpha f(x-i\varepsilon)=e^{\alpha(\log|x|+2\pi i)}f(x),$$

because of our definition of $\log z$. Thus the integral over the horizontal line segments tends to

$$\int_\delta^R (1-e^{2\pi i\alpha})x^\alpha f(x)dx. \tag{1.8}$$

For R large

$$\left|\int_{C_R} z^\alpha f(z)dz\right|\leq \int_0^{2\pi}\frac{R^\alpha}{R^2-1}Rd\theta\to 0, \tag{1.9}$$

as $R\to\infty$. Similarly

$$\left|\int_{C_\delta} z^\alpha f(z)dz\right|\leq \int_0^{2\pi}\frac{\delta^\alpha}{1-\delta^2}\delta d\theta\to 0, \tag{1.10}$$

as $\delta\to 0$. By (1.7), (1.8), (1.9), and (1.10)

$$\int_0^\infty \frac{x^\alpha}{x^2+1}dx=\pi\frac{e^{\frac{i\pi\alpha}{2}}-e^{i\frac{3\pi\alpha}{2}}}{1-e^{2\pi i\alpha}}=\frac{\pi}{2\cos\alpha\pi/2}.$$

This line of reasoning works for meromorphic f satisfying $|f(z)|\leq C|z|^{-2}$ for large $|z|$ and with at worst a simple pole at 0. The function z^α can be replaced by other functions which are not continuous across $\mathbb{R}$, such as $\log z$. In this case real parts of the integrals along $[0,\infty)$ will cancel, but the imaginary parts will not. Mellin transforms are used in applications to signal processing, image filtering, stress analysis and other areas.

(h) Inverse Laplace Transforms

The **Laplace Transform** of a function F defined on $(0,+\infty)$ is given by

$$\mathcal{L}(f)(z)=\int_0^\infty f(t)e^{-zt}dt.$$

The Laplace Transform can be used to convert a differential equation for a function f into an algebra problem whose solution is the Laplace transform of f. It roughly converts differentiation into multiplication. Indeed, an integration by parts for reasonably behaved functions f shows that $\mathcal{L}(f')=z\mathcal{L}(f)-f(0)$, and a similar result holds for higher order derivatives. The algebra problem is generally easy to solve, but to find the solution to the original differential equation, we need find a function whose Laplace transform is the solution to the algebra problem. In other words, given a function F, we'd like to find f so that $\mathcal{L}(f) = F$. In this example, we will find an integral equation for the inverse Laplace Transform.

Theorem 3.3. *Suppose F is analytic in $\{\text{Re}z > a\}$ and satisfies*

$$|F(z)| \le \frac{C}{(1+|z|)^p}$$

when $\text{Re}z > a$, for some constant C and some $p > 1$. For $b > a$ define

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(b+iy)e^{(b+iy)t}dy.$$

Then the definition of f does not depend on b, $|f(t)| \le C_1 e^{bt}$ where C_1 is a constant depending only on C and p, and

$$\mathcal{L}(f)(\zeta) = F(\zeta),$$

for $\text{Re}\zeta > a$.

Proof. For $t > 0$, $z = b + iy$, and $b > a$,

$$|F(z)e^{zt}| \le \frac{Ce^{bt}}{(1+|y|)^p}$$

and so the integral defining f converges and $|f(z)| \le C_1 e^{bt}$ for $\text{Re}z = b > a$, where C_1 depends only on p and C. Thus the integral defining $\mathcal{L}(f)$ converges absolutely in $\text{Re}z > a$ and by Cauchy's Theorem, it is independent of the choice of $b > a$. By Fubini's Theorem, we can write

$$\mathcal{L}(f)(\zeta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(b+iy)\int_0^{\infty} e^{(b+iy)t}e^{-\zeta t}dtdy = -\frac{1}{2\pi i}\int_{\text{Re}z=b}\frac{F(z)}{z-\zeta}dz,$$

where the integral along $\{\text{Re}z = b\}$ is in the upward direction. By the assumptions on the growth of $|F(z)|$, if C_R is the semi-circle in the right half-plane with radius R, centered at b, then

$$\int_{C_R}\left|\frac{F(z)}{\zeta - z}\right||dz| \le \frac{C}{R^p}\pi R \to 0,$$

as $R\to\infty$. Applying the Residue Theorem to the integral over the closed contour consisting of C_R together with the portion of $\{\mathrm{Re}z=b\}$ between the endpoints of C_R, and letting $R\to\infty$, we obtain

$$\mathcal{L}(f)(\zeta)=F(\zeta),$$

for $\mathrm{Re}\zeta>b$, and since $b>a$ was arbitrary, this holds for $\mathrm{Re}z>a$. □

§4. Series via Residues.

(i) $\displaystyle\sum_{n=0}^{\infty}\frac{1}{n^2+1}$.

We can turn this game around. Instead of using sums to compute integrals, we can use integrals to compute sums. Set $f(z)=\frac{1}{z^2+1}$ and consider the meromorphic function $f(z)\pi\cot\pi z$. Because $\pi\cot\pi z$ has a simple pole at $z=0$ with residue 1, and because $\cot\pi(z-n)=\cot\pi z$, for any integer n, $\pi\cot\pi z$ has a simple pole with residue 1 at each integer. Because the poles are simple, $f(z)\pi\cot\pi z$ has a simple pole with residue $f(n)$ at $z=n$. We consider the contour integral of $f(z)\pi\cot\pi z$ around the square S_N with vertices $(N+\frac{1}{2})(\pm1\pm i)$. The function $\pi\cot\pi z$ is uniformly bounded on S_N, independent of N. To see this write

$$\cot\pi z=i\frac{e^{2\pi iz}+1}{e^{2\pi iz}-1}.$$

The linear fractional transformation $(\zeta+1)/(\zeta-1)$ is bounded in the region $|\zeta-1|>\delta$ and $|e^{2\pi iz}-1|>1-e^{-2\pi N}$ on S_N, as can be seen by considering the horizontal and vertical segments separately. Since $|f(z)|\le C|z|^{-2}$, we have

$$\int_{S_n}f(z)\pi\cot\pi z dz\to0.$$

By the Residue Theorem

$$0=\mathrm{Res}_i f(z)\pi\cot\pi z+\mathrm{Res}_{-i}f(z)\pi\cot\pi z+\sum_{-\infty}^{\infty}f(n),$$

and hence

$$\sum_{n=0}^{\infty}\frac{1}{n^2+1}=\frac{\pi}{2}\left[\frac{e^{\pi}+e^{-\pi}}{e^{\pi}-e^{-\pi}}\right]-1.$$

This technique can be used to compute

$$\sum_{n=-\infty}^{\infty} f(n),$$

provided f is meromorphic with $|f(z)| \leq C|z|^{-2}$ for $|z|$ large. If some of the poles of f occur at integers, then the residue calculation at those poles is slightly more complicated because the pole of $f(z)\pi \cot \pi z$ will not have order 1. If only the weaker estimate $|f(z)| \leq C|z|^{-1}$ holds, then f has a removable singularity at ∞ and so $g(z) = f(z) + f(-z)$ satisfies $|g(z)| \leq C|z|^{-2}$ for large $|z|$. Applying the techique to g, we can find the symmetric limit

$$\lim_{N\to\infty} \sum_{n=-N}^{N} f(n).$$

§5. Exercises.

A

1. Find all non-zero residues of the following functions:
   (a) $\dfrac{z-1}{(z+1)^2(z-2)}$
   (b) $\dfrac{z^2-2z}{(z+1)^2(z^2+4)}$
   (c) $e^z \csc^2 z$
   (d) ze^{-1/z^2}
   (e) $\dfrac{\cot \pi z}{z^6}$

2. Let C be the circle of radius 3 centered at 0, oriented in the positive sense. Find

$$\int_C \frac{e^{\lambda z}}{(z+4)(z-1)^2(z^2+4z+5)} dz.$$

3. Suppose that f and g are analytic in a neighborhood of the closed unit disc. Let $\{a_n\}$ denote the zeros of f, including multiplicity, and suppose that $|a_n| < 1$ for all n. Prove:

$$\int_0^{2\pi} \frac{e^{i\theta} g(e^{i\theta}) f'(e^{i\theta})}{f(e^{i\theta})} \frac{d\theta}{2\pi} = \sum_n g(a_n).$$

B

In all of the problems below, be sure you have defined your functions carefully. Prove all claims about integrals. Draw a picture of any contour you use.

4. Find the Fourier transform of $x^3/(x^2+1)^2$. (Note that the integral is not absolutely convergent, so part of the problem is to prove that the integral converges, i.e. let the limits of integration tend to $\pm\infty$ independently.)

5. For $0 < \alpha < 1$, find

$$\int_0^\infty \frac{x^\alpha}{x(x+1)}dx.$$

6. Find the inverse Laplace transform of

$$F(z) = \frac{3z^2+12z+8}{(z+2)^2(z+4)(z-1)}$$

using the residue theorem.

7. Verify:

$$\sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32}.$$

8. Compute

$$\zeta(6) = \sum_{n=1}^\infty \frac{1}{n^6}$$

by calculus of residues. You can use your answer to problem 1(e). (Remark: This process can be used to compute $\zeta(2n)$. The value of ζ at an odd integer is another story.)

9. By means of the calculus of residues, evaluate

$$\int_0^\infty \frac{\sqrt{x}\log x}{(1+x^2)}dx.$$

10. Find:

$$\int_0^\infty \frac{x^3+8}{x^5+1}dx,$$

using a contour integral of $(\log z)(z^3+8)/(z^5+1)$.

11. Find

$$\int_0^1 \frac{1}{\sqrt{x(1-x)}}dx.$$

Put a “dog bone” around the interval $[0,1]$ and add a large circle. Carefully define the integrand so that it is analytic at ∞, then the integral over the large circle can be found from the series expansion at ∞. Redo the problem by first making the substitution $x = 1/w$.