GONIII: MORE UNIVERSAL QUATERNARY QUADRATIC FORMS

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ABSTRACT. This is the third paper in a series involving Geometry of Numbers (GoN) methods to provide proofs of representation by positive definite integral quadratic forms. Specifically, we provide here elementary GoN proofs of universality of 105 of the 112 quaternary positive definite integral quadratic forms of square discriminant.

1. INTRODUCTION

This is the third in a sequence of papers exploring applications of **Geometry of Numbers** (GoN) to quadratic forms. The first paper [9] treated primes represented by positive definite binary quadratic forms. The second paper [10] concerned the universality of positive definite quaternary quadratic forms; however, that paper restricted its attention to GoN proofs of universality for the nine such diagonal forms with square discriminant.

In this paper, we again use GoN methods to provide proofs of universality of positive definite quaternary integral quadratic forms. Now, however, we only require that the forms have square discriminant. The work of Conway [6] and Bhargava-Hanke [2] shows there are 112 such forms (a list of all 6436 universal quaternary integral quadratic forms is available at [14]). Of the 112 candidates, 105 have lent themselves to our methods and GoN universality proofs can be given. In light of the nine forms discussed in [10], this paper adds 96 universality statements. Of these 96 forms only 11 are classically integral. Although all the forms treated here come under the aegis of the 290 Theorem (a work of at least ten years in the making, which is at the time of this writing computationally complete but unpublished), it is our understanding that for many of the forms treated here universality was known **only** because of the Bhargava-Hanke 290 Theorem and thus complete universality proofs are appearing here for the first time.

We wish to emphasize that the primary of interest of this work is not the universality theorems themselves but the way in which they are proved. To prove the 290 Theorem, Bhargava-Hanke must analyze the universality of more than 6000 individual forms, and they do so by considering the associated theta series and applying deep and sophisticated techniques from the theory of modular forms. To analyze the Fourier coefficients of the theta-series, Siegel's work on local densities is used to bound the Eisenstein coefficients, and the theory of newforms and Deligne's bounds on Hecke eigenvalues (i.e., the Ramanujan-Petersson Conjecture) are used to bound the cusp coefficients. In constrast, the present method is almost entirely self-contained. The only GoN result which does not receive a full proof here or in [10] is Korkine-Zolotarev's computation of the 4-dimensional Hermite constant γ_4 [12]. In fact, for 95 out of the 96 forms treated here, the upper bound on γ_4 coming from Minkowski's Convex Body Theorem is sufficient. That state-of-the-art universality theorems can be proved by such elementary methods seems truly remarkable...and also somewhat mysterious.

Our techniques prove universality of 78 forms of class number greater than one. To the best of our knowledge all previous applications of GoN methods to representation theorems for integral quadratic forms (including [9] and [10]) treat only class number one forms. One might have guessed that such elementary methods were inherently limited to the class number one case. The present paper shows that the range of applicability of GoN methods is considerably larger. It would be interesting to probe this range more thoroughly, and we hope to do so in the future.

2. Background

We recall the following definitions and results from the theory of quadratic forms. When applicable, references for more detailed explanations are provided.

Let $n \in \mathbb{N}$. An ***n*-ary integral quadratic form** is a homogeneous integral polynomial of degree two of the form

$$q(x) = q(x_1, ..., x_n) \quad = \sum_{1 \le i \le j \le n} a_{ij} x_i x_j \in \mathbb{Z}[x_1, ..., x_n].$$

Equivalently, there is a unique symmetric matrix $A_q \in M_n(\mathbb{Q})$ such that

$$q(x) = x^t A_q x.$$

Under this matrix representation all diagonal entries are integers, while the off-diagonal entries are allowed to be half-integers. We say that q is **non-degenerate** when $\det(A_q) \neq 0$ and henceforth only consider non-degenerate forms. We say that q is **classically integral** if $A_q \in M_n(\mathbb{Z})$. Two n-ary forms q and q' are equivalent over $\mathbb{Z}$ if and only if there exists $M \in GL_n(\mathbb{Z})$ such that $A_{q'} = M A_q M^t$. Then $\det(A_{q'}) = (\det M)^2 \det A_q = \det A_q$. We call this value the **discriminant** of q, denoted $\Delta(q)$. When q is classically integral and non-degenerate, $\Delta(q)$ is a nonzero integer.

Let $q(x)$ be an n-ary integral quadratic form, and let $d \in \mathbb{Z}$. We say that q **represents** d if there exists $\vec{v} \in \mathbb{Z}^n$ such that $q(\vec{v}) = d$. When $q(\vec{v}) \ge 0$ (resp. ≤ 0) for all $\vec{v} \in \mathbb{Z}^n$, we say that q is **positive definite** (resp. negative definite). We say q is **positive universal** (resp. negative universal) if q represents every element of $\mathbb{Z}^{\ge 0}$ (resp. $\mathbb{Z}^{\le 0}$).

Unless otherwise specified, from now on by "form" we mean an "integral, positive definite, quaternary quadratic form" and by "universal" we mean "positive universal".

With respect to positive universality of forms, the following results are fundamental. All counts of forms are made up to integer equivalence.

Theorem 1.
- (a) *(Ramanujan, Dickson,* [7], [16]*) There are* 54 *diagonal universal forms.*
- (b) *(Conway-Schneeberger, Bhargava* [1], [6]*) Let q be a classically integral form. Then q is universal iff it represents* 1 *through* 15*, and there are* 204 *such forms.*
- (c) *(Bhargava-Hanke,* [2]*) Let q be a form. Then q is universal iff it represents* 1 *through* 290*, and there are* 6436 *such forms.*

Our GoN input is limited to the following result, which was established in [10].

Theorem 2. *(Small Multiple Theorem) Let $q(x)$ be a form of square discriminant. Let $n \in \mathbb{Z}^+$ be squarefree and prime to $2\Delta(q)$.*

- *(Using the Minkowski Convex Body Theorem* [4, § III.2.2, Thm. II]*)* *There exist $x_1, x_2, x_3, x_4, k \in \mathbb{Z}$ such that*
$$q(x_1, x_2, x_3, x_4) = kn$$
and
$$1 \le k \le \left\lfloor \left(\frac{4\sqrt{2}}{\pi}\right)(\Delta q)^{1/4} \right\rfloor.$$
- *(Using Korkine-Zolotarev's Theorem on γ_4* [12] [4, § X.3.2, Cor.]*)* *There exist $x_1, x_2, x_3, x_4, k \in \mathbb{Z}$ such that*
$$q(x_1, x_2, x_3, x_4) = kn$$
and
$$1 \le k \le \lfloor (4\Delta q)^{1/4} \rfloor.$$

Proof. See [10, Thm. 7]. □

Remark 3. *Since $4^{\frac{1}{4}} = 1.4142\ldots < 1.8006\ldots = \frac{4\sqrt{2}}{\pi}$, the second assertion of Theorem 2 is an improvement on the first. On the other hand, the Minkowski Convex Body Theorem is significantly easier to prove than the Korkine-Zolotarev Theorem.*

3. Proving Universality

Theorem 4. *The 105 integral, positive definite, quaternary quadratic forms appearing in Tables I and II of the Appendix are universal.*

Proof. Let $q = \sum_{1 \le i \le j \le n} a_{ij} x_i x_j = x^t A_q x$ be a form in Table I or II. Suppose we can show:
(a) For all squarefree $n \in \mathbb{Z}^+$ with $\gcd(n, 2\Delta_q) = 1$, q represents n.
(b) If q represents $n \in \mathbb{Z}^+$ and $p \mid 2\Delta_q$, then q represents pn.
Then q represents every squarefree positive integer and is thus universal: write $n \in \mathbb{Z}^+$ as ts^2 with t squarefree. There is $\vec{v} \in \mathbb{Z}^4$ with $q(\vec{v}) = t$, so $q(s\vec{v}) = ts^2 = n$.

We now explain how to establish (a) and (b) for q: the method includes computer computation, and an example is provided in the following section.

Establishing (a): By the Small Multiple Theorem (Theorem 2), for all $n \in \mathbb{Z}^+$ there is $\vec{v} \in \mathbb{Z}^4$ such that $q(\vec{v}) = kn$ for some $k \in \{1, 2, ..., \lfloor (4\Delta_q)^{1/4} \rfloor\}$.

If $q(\vec{v}) = kn$, suppose we can find a matrix $A \in M_4(\mathbb{Z})$ such that $q(Ax) = kq(x)$: an identity of quadratic forms. Then $q(A\vec{v}) = kq(\vec{v}) = k^2 n$. If, however, we could show $A\vec{v} \in (k\mathbb{Z})^4$, allowing $\vec{w} = (A\vec{v})/k \in \mathbb{Z}^4$, we would have $q(\vec{w}) = \frac{1}{k^2} q(A\vec{v}) = n$.

The strategy is to use a computer to create a set of such matrices. Since the $A\vec{v} \in (k\mathbb{Z})^4$ condition can be checked modulo k, we have a finite set of vectors to consider. If for each vector we can find a matrix, we will have shown (a). Consider the set of such matrices:
$$O_q(k) = \{A \in M_4(\mathbb{Z}) : q(A\vec{v}) = kq(\vec{v})\}.$$
By [10, Lemma 20], $O_q(k)$ is finite. Here is another algorithm to compute $O_q(k)$:

We create the set of vectors $V_i = \{\vec{v} \in \mathbb{Z}^4 : q(\vec{v}) = ka_{ii}\}$ for $i \in \{1, 2, 3, 4\}$. By positivity of q, the finite set V_i can be enumerated by evaluating $q(\vec{v})$ at all vectors inside a bounded ellipsoid. Let $M = [v_1|v_2|v_3|v_4] \in M_4(\mathbb{Q})$. Then $M \in O_q(k)$ if and only if:

- For all $1 \leq i \leq 4$, $v_i \in V_i$, and
- For all $1 \leq i < j \leq 4$, $v_i^t A_q v_j = k a_{ij}$.

However $O_q(k)$ is not large enough to prove (a) for many of the forms. So we introduce $d \in \mathbb{Z}^+$ to act as a denominator while still allowing the computer to perform integer arithmetic.

$$O_q(k,d) = \{A \in M_4(\mathbb{Z}) : q(Ax) = kd^2 q(x)\}.$$

The map $\mapsto dM$ induces an injection $O_q(k) = O_q(k,1) \hookrightarrow O_q(k,d)$.

We have $q(A\vec{v}) = kd^2 q(\vec{v}) = k^2 d^2 n$. We need $A\vec{v} \in (kd\mathbb{Z})^4$ to set $\vec{w} = \frac{\vec{v}}{kd} \in \mathbb{Z}^4$ and $q(\vec{w}) = n$. To check these conditions requires only considering the reduction of the coordinates of vectors modulo kd, so it suffices to look at the finite collection of vectors in $(\mathbb{Z}/kd\mathbb{Z})^4$.

But we do not need to consider every one of these vectors in $(\mathbb{Z}/kd\mathbb{Z})^4$: since $q(\vec{v}) = kn$ we only need consider vectors such that $q(\vec{v}) \equiv 0 \pmod{k}$. We call such vectors **admissible**, and the set of all admissible vectors for a fixed k and d is given by

$$A_q(k,d) = \{\vec{v} \in (\mathbb{Z}/kd\mathbb{Z})^4 : q(\vec{v}) \equiv 0 \pmod{k}\}.$$

For an admissible vector $\vec{v} \in A_q(k,d)$, we say that a matrix, $A \in O_q(k,d)$, **reduces** $\vec{v}$ if $A\vec{v} \in (kd\mathbb{Z})^4$. We say that $O_q(k,d)$ **covers** $A_q(k,d)$ if for every $\vec{v} \in A_q(k,d)$ there exists a $A_{\vec{v}} \in O_q(k,d)$ such that $A_{\vec{v}}$ reduces $\vec{v}$.

The problem has been reduced to a computer search to find a d such that $O_q(k,d)$ covers $A_q(k,d)$. For all the forms in Tables I and II, the computer search was successful.

Establishing (b): fix a prime p such that $p \mid 2\Delta_q$ and a vector $\vec{v} \in \mathbb{Z}^4$. This time we wish to find a matrix $A \in M_4(\mathbb{Z})$ such that $q(A\vec{v}) = pq(\vec{v})$. We again consider matrices in $O_q(p,d)$. Now all vectors in $(\mathbb{Z}/pd\mathbb{Z})^4$ are admissible. We say a matrix $A \in O_q(p,d)$ **multiplies** a vector $\vec{v} \in (\mathbb{Z}/pd\mathbb{Z})^4$ if $A\vec{v} \in (d\mathbb{Z})^4$, which then gives $\vec{x} = A\frac{\vec{v}}{d} \in \mathbb{Z}^4$ and $q(\vec{x}) = q\left(A\frac{\vec{v}}{d}\right) = \frac{1}{d^2} q(A\vec{v}) = pq(\vec{v})$.

We are again reduced to a computer search, and upon finding a d for all (q,p) pairs required, we have shown that if q represents n then q represents pn. This search was successful for all the forms in Tables I and II.

This completes the proof. □

Remark 5. *For* 104 *of the* 105 *forms of Theorem 4, using the first assertion of the Small Multiple Theorem – coming from the Minkowski bound – either does not change the computations at all or does not significantly lengthen them. However for*

$$q = x_1^2 + 2x_2^2 + x_2x_3 + 4x_3^2 + 31x_4^2,$$

the last form in Table II, using the first assertion of the Small Multiple Theorem requires consideration of $k = 7$*, and our computation has not terminated for this value. If we use* *the second assertion of the Small Multiple Theorem – coming from the Korkine-Zolotarev bound – then $k = 7$ does not need to be considered.*

4. An Example

We will now illustrate all steps of the algorithm with a particular form:

$$q(x) = x_1^2 + x_1x_2 + 2x_2^2 + 3x_3^2 + 3x_3x_4 + 6x_4^2.$$

This form is not classically integral, has class number 3 and discriminant $\frac{441}{16}$. Applying the Small Multiple Theorem for n satisfying $(n, 42) = 1$, we obtain $k \leq 3$. For the cases $q(\vec{v}) = 3n$ or $q(\vec{v}) = 2n$, we must prove the existence of a reduction to a representation of n by q.

For $k = 3$ (i.e., assuming $q(\vec{v}) = 3n$), using a computer search we find that a denominator of 1 that all that is required. That is, we only need to consider vectors $\vec{v} \in (\mathbb{Z}/3\mathbb{Z})^4$. Moreover, setting

$$M = \begin{pmatrix} 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

one quickly observes that $qM(x) = 3q(x)$. Noting that M reduces all admissible vectors $v \in A_q(1,3)$:

$$\begin{aligned}
M\,(0,\ 0,\ 0,\ 0)^t &= (0,\ 0,\ 0,\ 0)^t \\
M\,(0,\ 0,\ 0,\ 1)^t &= (0,\ 3,\ 0,\ 0)^t \\
M\,(0,\ 0,\ 0,\ 2)^t &= (0,\ 6,\ 0,\ 0)^t \\
M\,(0,\ 0,\ 1,\ 0)^t &= (3,\ 0,\ 0,\ 0)^t \\
M\,(0,\ 0,\ 1,\ 1)^t &= (3,\ 3,\ 0,\ 0)^t \\
M\,(0,\ 0,\ 1,\ 2)^t &= (3,\ 6,\ 0,\ 0)^t \\
M\,(0,\ 0,\ 2,\ 0)^t &= (6,\ 0,\ 0,\ 0)^t \\
M\,(0,\ 0,\ 2,\ 1)^t &= (6,\ 3,\ 0,\ 0)^t \\
M\,(0,\ 0,\ 2,\ 2)^t &= (6,\ 6,\ 0,\ 0)^t
\end{aligned}$$

we see that a representation of $3n$ by q can be reduced.

Next we address the case where $q(\vec{x}) = 2n$. This time a denominator of 2 suffices. There are 160 admissible vectors and we need to consider vectors in $(\mathbb{Z}/4\mathbb{Z})^4$.

$$\begin{aligned}
O_q(2,2) \quad = \quad & \left\{ M_1 = \begin{pmatrix} 0 & -2 & 0 & -6 \\ 1 & 1 & -3 & 0 \\ 0 & -2 & 0 & 2 \\ 1 & 1 & 1 & 0 \end{pmatrix}, M_2 = \begin{pmatrix} 0 & -4 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 2 & 0 \end{pmatrix}, M_3 = \begin{pmatrix} 0 & -4 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & -1 & -3 \end{pmatrix}, \right. \\
& \left. M_4 = \begin{pmatrix} 0 & -4 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & -2 & -2 \end{pmatrix}, M_5 = \begin{pmatrix} 0 & 4 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 2 & 0 \end{pmatrix}, M_6 = \begin{pmatrix} 0 & -2 & 0 & 6 \\ 1 & 1 & 3 & 0 \\ 0 & -2 & 0 & -2 \\ 1 & 1 & -1 & 0 \end{pmatrix} \right\}.
\end{aligned}$$

For all $M_i \in O_q(2,2)$, we have $q(M_i x) = 8q(x)$. For each of the six matrices we provide an example of an admissible vector that it covers:

$$\begin{aligned}
M_0(0,0,0,2)^t &= (-12,0,4,0)^t\\
M_1(0,0,0,1)^t &= (0,0,4,0)^t\\
M_2(0,0,1,1)^t &= (0,0,4,-4)^t\\
M_3(0,0,1,3)^t &= (0,0,-4,-8)^t\\
M_4(0,1,0,0)^t &= (4,0,0,0)^t\\
M_5(0,1,1,1)^t &= (4,4,-4,0)^t.
\end{aligned}$$

Similarly all other 154 admissible tuples are covered by one of the six matrices above.

Now it remains to show that if q represents a positive integer n then it represents $2n$, $3n$, and $7n$. In each case there is an integer matrix that allows multiplication. Specifically:

$$P_2 = \begin{pmatrix} 0 & 2 & 0 & 0\\ -1 & -1 & 0 & 0\\ 0 & 0 & 0 & 2\\ 0 & 0 & -1 & -1 \end{pmatrix}, P_3 = \begin{pmatrix} 0 & 0 & 3 & 0\\ 0 & 0 & 0 & 3\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 \end{pmatrix}, P_7 = \begin{pmatrix} 1 & -2 & -3 & -3\\ 0 & 2 & 0 & -3\\ 0 & -1 & 2 & -1\\ 1 & 1 & 0 & 1 \end{pmatrix}.$$

Note that for each i, $P_i \in M_4(\mathbb{Z})$, and hence for all $\vec{v} \in \mathbb{Z}^4$, $P_i\vec{v} \in \mathbb{Z}^4$. Moreover, for each i, $q(P_i x) = i \cdot q(x)$. This completes the proof of the universality of q.

5. Local Success of the Method

In this section we will discuss the success of the method. At every level $d \in \mathbb{Z}^+$, we say that the method succeeds if the finite set $A_q(k,d)$ of admissible vectors is covered by the finite set $O_q(k,d)$ of matrices: i.e., if every $\vec{v} \in A_q(k,d)$ is reduced by at least one $A \in O_q(k,d)$. If not, then we move on to $A_q(k,d')$ for a larger value of d' (up to the limits of our computational power). We show here that the method necessarily **succeeds locally** in the following sense: given any $\vec{v} \in A_q(k,d')$, there is a lift $\tilde{\vec{v}}$ of $\vec{v}$ to $A_q(k,dd')$ such that $\tilde{\vec{v}}$ is reduced by some $\tilde{A} \in A_q(k,dd')$.

Although the above statement in terms of congruence classes is a natural one when analyzing the method of proof of Theorem 4, we will actually prove a stronger result concerning integer vectors. In turn, by clearing denominators, this integral result follows quickly from a result about rational quadratic forms. The result for rational forms uses one of the key facts in the basic theory of algebraic quadratic forms: the isometry group of a nondegenerate quadratic form acts transitively on the set of vectors on which the quadratic form takes any fixed nonzero value. To make a short, clean proof of a slightly more general result, we have decided to make use of a basic property of Pfister forms (see [13] for details).

Theorem 6. *Let K be a field of characteristic different from 2, and let $q_{/K}$ be a nondegenerate n-ary Pfister form of square discriminant. For $k, p \in K^\times$, suppose q represents p and kp. Then for all $\vec{v}, \vec{w} \in K^n$ with $q(\vec{v}) = p$ and $q(\vec{w}) = kp$, there is $M \in \mathrm{GL}_n(K)$ with $M\vec{w} = k\vec{v}$ and $q(Mx) = kq(x)$.*

Proof. Putting

$$O_q(k) = \{M \in GL_n(K) \mid q(Mx) = kq(x)\},$$

we must show that there is $M \in O_q(k)$ such that $M\vec{w} = k\vec{v}$. Since q is a Pfister form, by [13, Thm. X.1.8] q is a **round form**: if we define

$$D(q)^\bullet := \{q(x) \mid x \in K^n\} \setminus \{0\}$$

and

$$G(q) := \{c \in K^\times \mid cq \cong q\},$$

then $D(q)^\bullet = G(q)$. Thus $D(q)^\bullet$ is a subgroup of $K^\times$, so $p, kp \in D(q)^\bullet \implies k \in D(q)^\bullet = G(q)$: there is $M_1 \in GL_n(K)$ such that $q(M_1 x) = kq(x)$ for all x. Taking $x = \vec{v}$, we get

$$q(M_1\vec{v}) = kq(\vec{v}) = kp = q(\vec{w}).$$

By [13, Prop. I.4.7], there is $M_2 \in O(q) = O_q(1)$ with $M_2M_1\vec{v} = \vec{w}$. Put $M = M_2M_1$. Then

$$M\vec{v} = \vec{w}$$

and

$$q(Mx) = q(M_2M_1x) = q(M_1x) = kq(x),$$

so $M \in O_k(q)$. □

Corollary 7. *Let q_Z be a positive quaternary quadratic form with square discriminant.*
a) For $k, p \in \mathbb{Z} \setminus \{0\}$, suppose q integrally represents $1, p, kp$. Then for all $\vec{v}, \vec{w} \in \mathbb{Z}^4$ with $q(\vec{v}) = p$ and $q(\vec{w}) = kp$, there is $M \in M_4(\mathbb{Q})$ such that $M\vec{w} = k\vec{v}$ and $q(M\vec{x}) = kq(\vec{x})$.
b) If q is positive universal, reduction always succeeds locally: for all $k, p \in \mathbb{Z} \setminus \{0\}$ and $\vec{w} \in \mathbb{Z}^4$ with $q(\vec{w}) = kp$, there is $d \in \mathbb{Z}^+$ and $A \in O_q(k, d)$ with $A\vec{w} \in (kd\mathbb{Z})^4$.
(Thus if $\vec{v} = \frac{\vec{w}}{kd}$, then $\vec{v} \in \mathbb{Z}^4$ and $q(\vec{v}) = p$.)

Proof. a) A nondegenerate quaternary quadratic form over a field of characteristic different from 2 is a Pfister form if and only if it has a diagonal representation $\langle 1, a, b, ab\rangle$ if and only if it represents 1 and has square discriminant. Thus Theorem 6 applies to $q_{/\mathbb{Q}}$: there is $M \in M_4(\mathbb{Q})$ such that $M\vec{w} = k\vec{v}$.

b) Since q is positive universal, there is $\vec{v} \in \mathbb{Z}^4$ with $q(\vec{v}) = p$. Applying part a), we get $M \in M_4(\mathbb{Q})$ with $q(Mx) = kq(x)$ and $M\vec{w} = k\vec{v}$. Let d be the greatest common denominator of the entries of M, and put

$$A = dM, \ \vec{u} = \frac{A\vec{w}}{kd}.$$

Then $A \in M_4(\mathbb{Z})$ and

$$q(Ax) = q(dMx) = d^2q(Mx) = kd^2q(x),$$

so $A \in O_q(k, d)$, and finally

$$A\vec{w} = dM\vec{w} = kd\vec{v} \in (kd\mathbb{Z})^4.$$ □

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7. Appendix

The 112 universal positive definite integral quaternary quadratic forms referenced in the introduction are separated into three tables below: Table I consists of the classically integral forms for which GoN proofs of universality now exist (note that this table includes all forms from [10]); Table II consists of not-classically integral forms for which GoN proofs of universality now exist; Table III consists of the seven forms for which our proof of universality algorithm has not terminated.

The coefficients for a particular quadratic form appear in the first column of each table; for a particular form $q(x) = \sum_{1 \le i \le j \le 4} a_{ij} x_i x_j$ this entry will read

$$\langle a_{11}, a_{12}, a_{13}, a_{14}, a_{22}, a_{23}, a_{24}, a_{33}, a_{34}, a_{44} \rangle.$$

Reading across, the tables then provide the class number $h(q)$ (see [15]), and $\mathcal{D}(q)$ (the discriminant of the unique quaternion algebra over $\mathbb{Q}$ with norm q). The next three columns relate to the Small Multiples Theorem, and have the heading k-value. As all forms considered had k-values bounded by 5, the entries mark the denominators needed for reduction. The remaining five columns have the heading prime values; these five rows correspond to the primes for which multiplication matrices must be produced (i.e., those dividing $2\Delta(q)$), and the smallest denominator for each multiplication matrix. As $p = 2$ is a required check on all forms, the denominator is placed in the 2-column; for any other necessary primes (of which there are at most two), d_i is the denominator associated to multiplication by p_i.

As a concrete example, consider the form $q(\vec{x}) = x_1^2 + x_1x_2 + 2x_2^2 + 3x_3^2 + 3x_3x_4 + 6x_4^2$

highlighted earlier in this document. Because this form is not classically integral, it appears in Table II. As noted above, $h(q) = 3$. The Small Multiple Theorem gave $k \leq 3$. For $k = 2$, the smallest denominator needed to reduce was 2; for $k = 3$, the smallest denominator needed to reduce was 1; $k = 5$ was unnecessary to check. Moreover, as the discriminant was $3^2 \cdot 7^2/2^4$ the primes for which multiplication matrices must be produced are $2, p_1 = 3, p_2 = 7$; in each case an integral (i.e., denominator 1) multiplication matrix existed. Therefore, the corresponding row in Table II will read:

			k value			prime values				
q	$h(q)$	$\mathcal{D}(q)$	2	3	5	2	p_1	d_1	p_2	d_2
$\langle 1,1,0,0,2,0,0,3,3,6\rangle$	3	3	2	1	$*$	1	3	1	7	1

A final note: an $*$ indicates a computation that was unnecessary to compute (as in the case above where reducing by $k = 5$ was unnecessary). In Table III, an X indicates a necessary step that has not yet terminated.

Table I: Classically Integral Forms

			k value			prime values				
q	$h(q)$	$\mathcal{D}(q)$	2	3	5	2	p_1	d_1	p_2	d_2
$\langle 1,0,0,0,1,0,0,1,0,1\rangle$	1	2	*	*	*	1	*	*	*	*
$\langle 1,0,0,0,1,0,0,1,0,4\rangle$	1	2	2	*	*	2	*	*	*	*
$\langle 1,0,0,0,1,0,0,2,0,2\rangle$	1	2	2	*	*	1	*	*	*	*
$\langle 1,0,0,0,2,0,2,2,2,2\rangle$	1	2	2	*	*	2	*	*	*	*
$\langle 1,0,0,0,1,0,0,2,2,5\rangle$	1	2	3	3	*	3	3	3	*	*
$\langle 1,0,0,0,1,0,0,3,0,3\rangle$	1	3	2	1	*	1	3	1	*	*
$\langle 1,0,0,0,2,2,0,2,0,3\rangle$	1	2	1	1	*	1	3	1	*	*
$\langle 1,0,0,0,1,0,0,2,0,8\rangle$	2	2	4	4	*	4	*	*	*	*
$\langle 1,0,0,0,2,0,0,2,0,4\rangle$	1	2	2	1	*	1	*	*	*	*
$\langle 1,0,0,0,2,0,0,3,2,3\rangle$	1	2	4	4	*	4	*	*	*	*
$\langle 1,0,0,0,1,0,0,2,2,13\rangle$	3	2	5	5	*	5	5	5	*	*
$\langle 1,0,0,0,2,0,2,3,2,5\rangle$	3	2	5	5	*	5	5	5	*	*
$\langle 1,0,0,0,2,2,0,3,0,5\rangle$	1	5	2	1	*	1	5	1	*	*
$\langle 1,0,0,0,2,0,0,3,0,6\rangle$	2	2	2	2	*	1	3	1	*	*
$\langle 1,0,0,0,2,0,2,4,0,5\rangle$	2	2	6	6	*	6	3	6	*	*
$\langle 1,0,0,0,2,0,2,4,4,6\rangle$	2	2	6	6	*	6	3	6	*	*
$\langle 1,0,0,0,2,0,2,3,2,9\rangle$	5	2	7	7	*	7	7	7	*	*
$\langle 1,0,0,0,2,2,0,4,0,7\rangle$	3	7	2	3	*	1	7	1	*	*
$\langle 1,0,0,0,2,0,0,4,0,8\rangle$	2	2	4	4	4	1	*	*	*	*
$\langle 1,0,0,0,2,0,0,5,0,10\rangle$	2	5	8	8	1	1	5	1	*	*

Table II: Not-Classically Integral Forms

			k value			prime values				
q	$h(q)$	$\mathcal{D}(q)$	2	3	5	2	p_1	d_1	p_2	d_2
$\langle 1,0,0,1,1,0,1,1,1,1\rangle$	1	2	*	*	*	1	*	*	*	*

$\langle 1,1,0,0,1,0,0,1,1,1\rangle$	1	3	*	*	*	1	3	1	*	*
$\langle 1,1,1,0,1,1,0,1,0,2\rangle$	1	2	*	*	*	2	*	*	*	*
$\langle 1,1,1,0,1,1,1,2,2,2\rangle$	1	5	1	*	*	1	5	1	*	*
$\langle 1,0,0,1,1,0,1,1,1,3\rangle$	1	2	3	*	*	3	3	3	*	*
$\langle 1,1,0,0,1,0,0,1,0,3\rangle$	1	3	2	*	*	2	3	1	*	*
$\langle 1,1,1,0,1,1,1,1,1,5\rangle$	1	2	3	*	*	3	3	3	*	*
$\langle 1,0,1,1,1,1,1,2,1,2\rangle$	1	3	1	*	*	1	3	1	*	*
$\langle 1,1,0,0,1,0,0,2,2,2\rangle$	1	2	1	*	*	1	3	1	*	*
$\langle 1,0,1,0,1,0,1,2,0,2\rangle$	1	7	1	*	*	1	7	1	*	*
$\langle 1,1,1,0,1,1,0,1,0,8\rangle$	2	2	4	*	*	4	*	*	*	*
$\langle 1,0,0,1,1,0,1,2,2,3\rangle$	2	2	4	*	*	4	*	*	*	*
$\langle 1,1,0,0,1,0,1,2,0,3\rangle$	1	2	4	*	*	4	*	*	*	*
$\langle 1,0,0,1,1,0,1,1,1,7\rangle$	2	2	5	*	*	5	5	5	*	*
$\langle 1,1,1,0,1,1,1,1,1,13\rangle$	2	2	5	*	*	5	5	5	*	*
$\langle 1.1,1,0,2,2,-1,2,1,3\rangle$	1	5	1	*	*	1	5	1	*	*
$\langle 1,0,0,1,2,2,0,2,2,3\rangle$	2	2	5	*	*	5	5	5	*	*
$\langle 1,1,0,0,1,0,1,2,2,5\rangle$	2	2	5	*	*	5	5	5	*	*
$\langle 1,0,0,0,2,1,-1,2,1,2\rangle$	2	5	2	*	*	2	5	2	*	*
$\langle 1,0,1,1,1,1,1,3,1,3\rangle$	1	2	1	*	*	1	5	1	*	*
$\langle 1,1,1,0,1,1,0,2,0,5\rangle$	2	5	2	*	*	2	5	2	*	*
$\langle 1,0,1,0,1,0,1,3,0,3\rangle$	3	11	2	*	*	1	11	1	*	*
$\langle 1,1,0,0,1,0,0,2,0,6\rangle$	2	2	2	2	*	2	3	1	*	*
$\langle 1,0,0,0,1,0,1,3,3,4\rangle$	2	3	4	4	*	4	3	4	*	*
$\langle 1,0,1,1,2,2,2,3,0,3\rangle$	2	2	2	6	*	2	3	1	*	*
$\langle 1,0,1,0,1,1,0,3,2,4\rangle$	2	2	6	3	*	6	3	3	*	*
$\langle 1,0,0,1,2,0,0,2,2,3\rangle$	2	2	6	4	*	6	3	3	*	*
$\langle 1,1,1,0,2,1,0,2,0,3\rangle$	2	3	4	1	*	4	3	4	*	*
$\langle 1,1,1,0,2,1,2,2,2,4\rangle$	1	3	2	1	*	1	3	1	*	*
$\langle 1,1,0,1,2,1,1,2,2,4\rangle$	1	13	1	1	*	1	13	1	*	*
$\langle 1,1,0,0,1,0,1,2,2,9\rangle$	3	2	7	7	*	7	7	7	*	*
$\langle 1,0,1,1,2,0,2,3,0,3\rangle$	3	2	1	7	*	1	7	7	*	*
$\langle 1,0,1,1,2,1,-1,3,2,3\rangle$	2	7	2	2	*	2	7	1	*	*
$\langle 1,0,1,0,1,0,0,2,0,7\rangle$	2	7	4	4	*	2	7	1	*	*
$\langle 1,0,1,0,1,1,0,3,1,5\rangle$	3	2	1	7	*	1	7	7	*	*
$\langle 1,1,0,0,2,0,0,2,2,4\rangle$	3	7	2	2	*	1	7	1	*	*
$\langle 1,0,0,1,2,2,0,2,2,5\rangle$	3	2	7	7	*	7	7	7	*	*
$\langle 1,0,0,1,2,1,0,2,0,4\rangle$	3	5	2	2	*	1	3	1	5	1
$\langle 1,1,1,1,2,1,0,2,0,5\rangle$	3	3	5	5	*	5	3	5	5	5
$\langle 1,1,0,0,2,1,1,3,1,3\rangle$	2	5	3	3	*	3	3	3	5	3
$\langle 1,1,1,1,2,2,0,2,1,6\rangle$	2	5	3	3	*	3	3	3	5	3
$\langle 1,1,0,0,2,0,2,3,3,4\rangle$	3	3	5	5	*	5	3	5	5	5
$\langle 1,1,1,0,2,2,-1,3,2,5\rangle$	3	17	2	2	*	1	17	1	*	*
$\langle 1,0,1,0,2,0,2,3,3,5\rangle$	4	2	9	9	*	9	3	9	*	*
$\langle 1,0,1,1,2,2,0,3,2,5\rangle$	4	2	9	9	*	9	3	9	*	*
$\langle 1,0,1,0,2,2,2,3,1,5\rangle$	1	2	1	3	*	1	3	1	*	*
$\langle 1,1,1,0,2,1,2,3,3,6\rangle$	3	19	2	2	*	1	19	1	*	*

$\langle 1,0,0,1,2,0,0,2,2,7\rangle$	6	2	10	10	*	10	5	5	*	*
$\langle 1,0,1,1,2,0,0,4,3,4\rangle$	3	5	8	8	*	2	5	4	*	*
$\langle 1,0,0,0,2,1,-1,2,1,7\rangle$	4	5	4	4	*	4	5	4	*	*
$\langle 1,1,1,0,2,2,0,2,0,10\rangle$	3	5	8	8	*	2	5	4	*	*
$\langle 1,0,1,1,2,0,0,3,2,5\rangle$	6	2	10	10	*	10	5	10	*	*
$\langle 1,1,0,0,2,1,0,3,0,5\rangle$	4	5	4	4	*	4	5	4	*	*
$\langle 1,0,1,0,2,2,0,3,2,6\rangle$	6	2	10	10	*	10	5	5	*	*
$\langle 1,1,0,0,2,0,0,3,3,6\rangle$	3	3	2	1	*	1	3	1	7	1
$\langle 1,0,1,0,1,0,0,3,0,11\rangle$	5	11	8	8	*	4	11	1	*	*
$\langle 1,0,1,0,2,0,2,3,0,6\rangle$	1	2	1	1	*	1	11	1	*	*
$\langle 1,0,1,1,2,2,0,3,0,7\rangle$	6	2	11	11	*	11	11	11	*	*
$\langle 1,0,0,1,2,2,0,5,5,5\rangle$	6	2	11	11	*	11	11	11	*	*
$\langle 1,0,0,1,2,1,0,3,0,6\rangle$	6	23	4	4	*	1	23	1	*	*
$\langle 1,0,1,1,2,0,0,3,2,7\rangle$	3	2	12	12	*	12	3	12	*	*
$\langle 1,0,1,0,2,2,0,3,0,8\rangle$	6	2	12	12	*	12	3	12	*	*
$\langle 1,0,1,1,2,2,2,5,1,5\rangle$	6	2	36	36	*	12	3	12	*	*
$\langle 1,1,0,0,2,1,0,2,0,13\rangle$	4	13	12	12	*	12	13	12	*	*
$\langle 1,0,1,0,2,0,2,3,3,9\rangle$	8	2	39	13	*	39	13	13	*	*
$\langle 1,0,1,0,2,2,2,5,4,6\rangle$	8	2	39	13	*	39	13	13	*	*
$\langle 1,0,1,0,2,2,2,3,0,10\rangle$	8	2	13	13	*	13	13	13	*	*
$\langle 1,0,0,1,2,2,0,5,1,5\rangle$	8	2	13	13	*	13	13	13	*	*
$\langle 1,0,0,0,2,1,-1,5,3,5\rangle$	4	13	12	12	*	12	13	12	*	*
$\langle 1,0,1,0,2,0,0,5,4,6\rangle$	12	2	14	14	*	14	7	7	*	*
$\langle 1,0,1,0,2,2,0,5,2,6\rangle$	12	2	42	14	*	14	7	7	*	*
$\langle 1,0,1,1,2,1,2,4,1,8\rangle$	6	29	6	6	*	1	29	1	*	*
$\langle 1,0,1,0,2,2,2,3,1,13\rangle$	6	2	15	15	*	15	3	15	5	15
$\langle 1,0,1,1,2,2,0,3,2,13\rangle$	5	2	5	15	*	5	3	5	5	5
$\langle 1,0,1,1,2,0,2,5,3,7\rangle$	6	2	15	15	*	15	3	15	5	15
$\langle 1,0,0,1,2,2,2,5,1,7\rangle$	6	2	15	15	*	15	3	15	5	15
$\langle 1,0,0,1,2,2,0,5,3,7\rangle$	5	2	5	15	*	5	3	5	5	5
$\langle 1,0,1,1,2,2,0,5,2,7\rangle$	6	2	15	15	*	15	3	15	5	15
$\langle 1,0,0,1,2,1,0,4,0,8\rangle$	6	31	12	6	10	1	31	1	*	*
$\langle 1,0,1,1,2,0,2,5,4,9\rangle$	14	2	1	17	17	1	17	17	*	*
$\langle 1,0,1,0,2,2,2,5,1,9\rangle$	3	2	1	3	3	1	17	1	*	*
$\langle 1,0,1,1,2,1,2,5,1,10\rangle$	4	37	18	20	20	1	37	1	*	*
$\langle 1,0,1,0,2,0,2,5,0,10\rangle$	4	2	1	3	3	1	19	1	*	*
$\langle 1,0,0,0,2,1,0,3,0,23\rangle$	14	23	24	24	24	8	23	1	*	*
$\langle 1,0,0,0,2,1,0,4,0,31\rangle$	19	31	72	60	60	48	31	1	*	*

On the forms that have not finished, a total of six months of CPU time was spent on 19 different modern processors. These were stopped when substantial improvements were made on the implementation of the algorithms. Together with some heuristic speed-ups, the new code was parallelized to be able to run on all available processors. All of the work done by the original run of the old algorithm was verified by the new code over the period of a week running on 48 processors. The new code continued to run for 5 months checking every d value sequentially up to $d = 230$. After this more selective runs were done for denominators up to 350. At this point running even a single denominator for a single k-value of a single form took several days on 24 processors.

Table III: Remaining Universal Forms

			k value			prime values				
q	$h(q)$	$\mathcal{D}(q)$	2	3	5	2	p_1	d_1	p_2	d_2
$\langle 1,0,1,1,2,2,2,3,0,9\rangle$	4	2	X	*	*	X	*	*	*	*
$\langle 1,0,0,0,2,2,2,5,4,5\rangle$	2	2	X	X	*	2	3	2	*	*
$\langle 1,0,1,0,2,0,0,3,1,9\rangle$	11	2	X	X	*	X	7	14	*	*
$\langle 1,0,1,0,2,2,0,3,1,11\rangle$	11	2	X	X	*	X	7	7	*	*
$\langle 1,0,0,0,2,2,2,5,0,6\rangle$	5	2	X	X	*	1	7	7	*	*
$\langle 1,0,0,0,2,0,2,4,0,13\rangle$	6	2	X	X	*	10	5	5	*	*
$\langle 1,0,0,0,2,0,2,4,4,14\rangle$	6	2	X	X	*	10	5	5	*	*