Math 1552

Section 8.4:

Trigonometric Substitution (cont.)

Math 1552 lecture slides adapted from the course materials
By Klara Grodzinsky (GA Tech, *School of Mathematics*, Summer 2021)

Review: Trigonometric Substitutions

We use a trig substitution when no other integration method will work, and when the integral contains one of these terms:

$$a^2 - x^2$$

$$x^2 - a^2$$

$$a^2 + x^2$$

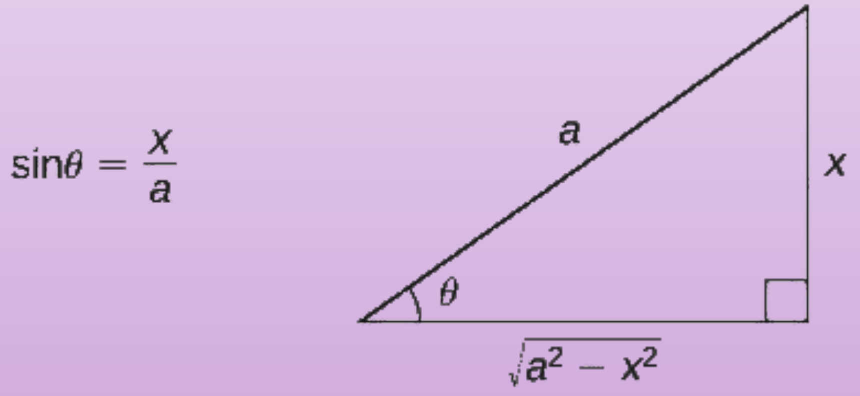
Review of Form 1:

When the integral contains a term of the form

$$a^2 - x^2,$$

use the substitution:

$$\boxed{x = a\sin\theta}$$



Credits for figure: https://math.libretexts.org/Bookshelves/Calculus

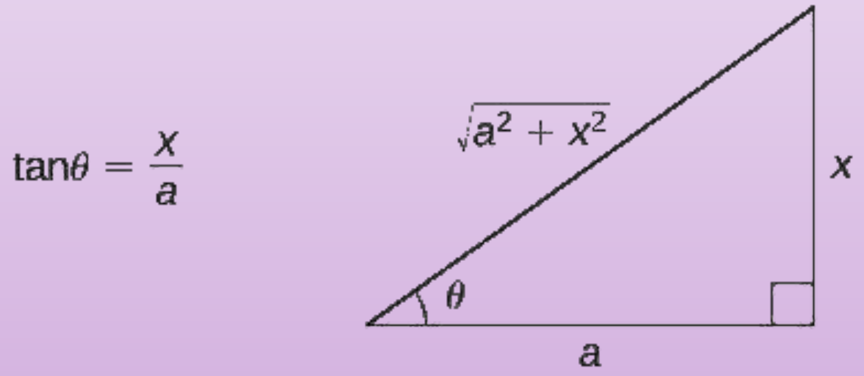
Review of Form 2:

When the integral contains a term of the form

$$a^2 + x^2,$$

use the substitution:

$$\boxed{x = a\tan\theta}$$



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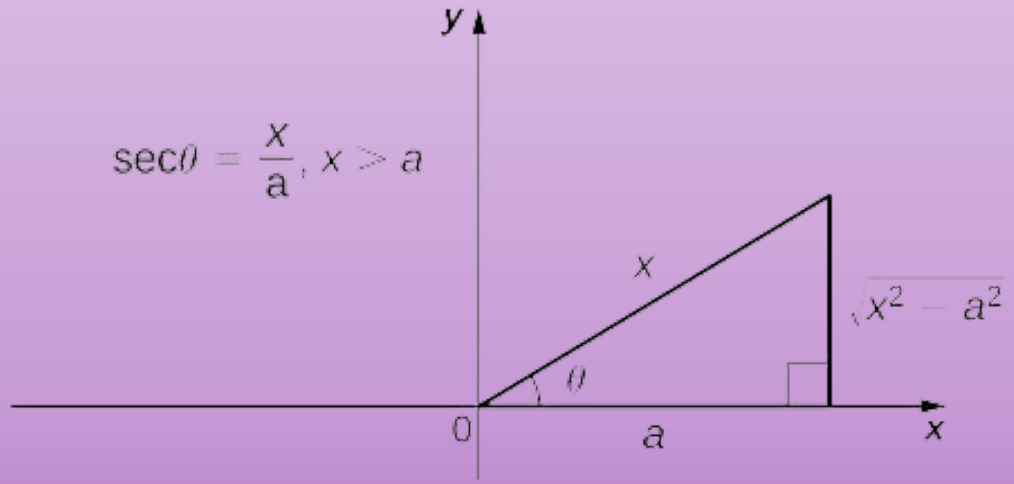
Review of Form 3:

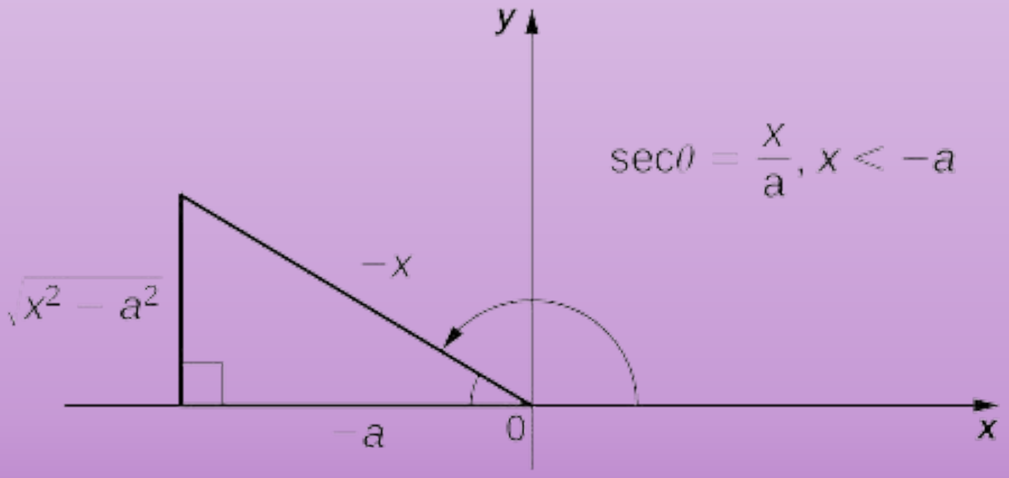
When the integral contains a term of the form

$$x^2 - a^2,$$

use the substitution:

$$\boxed{x = a\sec\theta}$$





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Example 1: Evaluate the integral: $\int \frac{x}{\sqrt{x^2 - 3x + 7}} dx$

Example 2: Evaluate the integral: $\int e^{4x}\sqrt{1+4e^{2x}}dx$

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Section 8.5:

The Method of Partial Fractions

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When to Use Partial Fractions:

Use the method of partial fractions to evaluate the integral of a *rational function* when:

- The degree of the numerator is *less than* that of the denominator.
- The denominator can be *completely factored* into linear and/or irreducible quadratic terms – *NO complex numbers in this class!*

Partial Fractions Procedure:

1. If the leading coefficient of the denominator is not a “1”, factor it out.

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2. If the degree of the numerator is greater than that of the denominator, carry out long division first.

Quick refresher on polynomial long division

Question: What do you when asked to evaluate this integral? $\int \frac{x^3 - 2x^2 - 4}{x - 3} dx$

Short answer: Observe that $x^3 - 2x^2 - 4 = (x-3)(x^2 + x + 3) + \mathbf{5}$ (How?)

(This standard method works for denominator polynomials of degree larger than one.)

Partial Fractions Procedure:

1. If the leading coefficient of the denominator is not a “1”, factor it out.
2. If the degree of the numerator is greater than that of the denominator, carry out long division first.
3. Factor the denominator completely into linear and/or irreducible quadratic terms.

Partial Fractions Procedure:

4. For each linear term of the form $(x-a)^k$, you will have *k* partial fractions of the form:

$$\boxed{\frac{A_1}{x-a}+\frac{A_2}{(x-a)^2}+\frac{A_3}{(x-a)^3}+\ldots+\frac{A_k}{(x-a)^k}}$$

(Note: if *k*=1, there is only one fraction to handle, etc.)

Partial Fractions Procedure:

5. For each irreducible quadratic term of the form $(x^2+bx+c)^m$, you will have *m* partial fractions of the form:

$$\boxed{\frac{A_1x+B_1}{x^2+bx+c}+\frac{A_2x+B_2}{\left(x^2+bx+c\right)^2}+\frac{A_3x+B_3}{\left(x^2+bx+c\right)^3}+\ldots+\frac{A_mx+B_m}{\left(x^2+bx+c\right)^m}}$$

(Note: if *m*=1, there is only one fraction, etc.)

Partial Fractions Procedure:

6. Solve for all the constants A_i and B_i. To solve:
   - Multiply everything by the common denominator.
   - Combine all like terms, then solve equations for all the A_i and B_i terms; OR plug in values to find equations for A_i and B_i terms.
7. Integrate using all the integration methods we have learned.

Example 1: Evaluate the integral: $\int \frac{x^3+4x^2}{2x^2+8x-10}\,dx$

Example 2: Evaluate the integral: $\int \frac{x^2 - 1}{x(x^2 + 1)^2} \, dx$

Example 3: Evaluate the integral: $\int \frac{2x-1}{x^2(x-2)^2}\,dx$

Example 4: Evaluate the definite integral: $\displaystyle\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\sin\vartheta}{\cos^2\vartheta+\cos\vartheta-2}\,d\vartheta$

Challenge Problem I:

Evaluate the following integral (sketch key steps):

$$\int \frac{dx}{\sqrt{x} + \sqrt[3]{x}}$$

Hint: Use the substitution $u^6 = x, 6u^5du = dx$

Challenge Problem II:

Evaluate the following integral (sketch key steps): $\displaystyle\int \frac{dx}{1+x^4}$

Hint: Write $x^4 + 1 = (x^2+1)^2 - 2x^2$, then factorize the quadratic and apply partial fractions.

Review Question: Which of the following integrals would you evaluate using partial fractions? Why?

$$(A)\int \frac{x}{4-x^2}\,dx$$

$$(B)\int \frac{x^2-2}{x^2(x-3)^2}\,dx$$

$$(C)\int \frac{x}{1+x^4}\,dx$$

$$(D)\int \frac{x+1}{x^3+6x^2+9x}\,dx$$

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Section 4.5:
L'Hopital's Rule

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Today’s Learning Goals

- Understand which forms are indeterminate
- Apply L’Hopital’s Rule to evaluate limits
- Rewrite limits in forms appropriate to applying L’Hopital’s Rule

Indeterminate Forms

$$\frac{0}{0}, \frac{\infty}{\infty}$$

$$1^{\infty}, 0^{0}, \infty^{0}$$

$$0 \cdot \infty, \infty - \infty$$

Which of the following limits
does NOT contain an indeterminate form? Why?

A. $\lim_{x\to\infty}(x+1)^{3x}$

B. $\lim_{x\to 0^+} x^{6x}$

C. $\lim_{x\to\infty} x^2 e^{-x}$

D. $\lim_{x\to 0^+}(\cos x)^{\frac{1}{x}}$

L'Hopital's Rule

Let *f* and *g* be two functions. Then IF:

a) *f* and *g* are differentiable,

b) $\frac{f(x)}{g(x)}$ has the indeterminate form of $\frac{0}{0}$ OR $\frac{\infty}{\infty}$

c) $\lim_{x \to c} \frac{f'(x)}{g'(x)} = L$

THEN: $$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)} = L$$

Example 1.1: Use L'Hopital's rule to evaluate the following limit.

$$\lim_{x \to \infty} \frac{e^x + x^2}{e^x + x}$$

Example 1.2:

Use L'Hopital's rule to evaluate the following limit.

$$\lim_{x \to 0^+} [\sin(x) \cdot \ln(x)]$$

Evaluate the limit: $$\lim_{x \to 0} \frac{3^x - 1}{4^x - 1}$$

A. 0
B. 1
C. ln(3/4)
D. (ln3)/(ln4)

Example 2.1: Use L'Hopital's rule and logarithms to evaluate the following limit.

$$\lim_{x \to 0^+} x^{\frac{1}{\ln(5x)}}$$

Logarithm rule: $\lim_{x \to a} f(x) = \lim_{x \to a} e^{\ln(f(x))} = \exp\left(\lim_{x \to a} \ln(f(x))\right)$

Example 2.2: Use L'Hopital's rule and logarithms to evaluate the following limit.

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x$$

Logarithm rule: $$\lim_{x\to a} f(x) = \lim_{x\to a} e^{\ln(f(x))} = \exp\left(\lim_{x\to a}\ln(f(x))\right)$$

Evaluate the limit: $$\lim_{x \to 0^+} (1+2x)^{\frac{1}{x}}$$

A. e^2

B. $e^{1/2}$

C. 1

D. Infinity

Compendia of Common Limits (memorize)

1) If $x > 0$, then $\lim_{n\to\infty} x^{1/n} = 1.$

2) If $|x| < 1$, then $\lim_{n\to\infty} x^{n} = 0.$

3) If $\alpha > 0$, then $\lim_{n\to\infty} \frac{1}{n^{\alpha}} = 0.$

4) $\lim_{n\to\infty} \frac{x^n}{n!} = 0$

5) $\lim_{n\to\infty} \frac{\ln(n)}{n} = 0$

6) $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$

7) $\lim_{n\to\infty} n^{1/n} = 1$

Extra Problem I: Evaluate the following limit:

$$\lim_{w \to -6} \frac{\sin(2\pi w)}{w^2 - 36}$$

Extra Problem II: Evaluate the following limit:

$$\lim_{x \to 0^+} \frac{\sin(2x)}{2x}$$

Extra Problem III: Evaluate the following limit:

$$\lim_{x \to \frac{1}{2}^+} \left(x - \frac{1}{2}\right) \tan(\pi x)$$

Bonus Practice Problems: Evaluate each of the following limits:

(In class: practice verifying that we get an indeterminate form in each case)

- $$\lim_{x\to\infty} \frac{x^2-2}{2x^2-3x+1}$$

  Hint: Multiply through by $1 = \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$, and then take limits

- $$\lim_{t\to+\infty} \left[t \cdot \ln\left(1+\frac{8}{t}\right)\right]$$
- $$\lim_{x\to 0^+} \frac{3^x-4^x}{x^2-2x}$$
- $$\lim_{x\to+\infty} \left[\sqrt{x^2+2}-\sqrt{x+2}\right]$$

  Hint: Multiply through by the *conjugate* $1 = \frac{\sqrt{x^2+2}+\sqrt{x+2}}{\sqrt{x^2+2}+\sqrt{x+2}}$, to simplify the numerator first

- $$\lim_{x\to 0} x^{3x}$$

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Section 8.8:

Improper Integrals

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Today’s Learning Goals

- Be able to identify when an integral is improper
- Rewrite an improper integral as a limit
- Understand the meaning of convergence and divergence as relating to integration
- Evaluate improper integrals

Improper integrals

A definite integral is improper if:

- The function has a vertical asymptote at $x=a$, $x=b$, or at some point c in the interval (a,b).
- One or both of the limits of integration are infinite (positive or negative infinity).

Which of the following integral(s) is (are) improper? Why / which case?

$$1)\int_{0}^{\frac{\pi}{4}} \tan(2x)dx$$

$$2)\int_{-1}^{1} \frac{x-3}{x^2-2x-3}dx$$

$$3)\int_{0}^{\frac{\pi}{2}} \cos(x)dx$$

$$4)\int_{0}^{3} \frac{x-2}{x^2-6x+8}dx$$

Convergence of an Integral

- If an improper integral evaluates to a **finite number**, we say it ***converges***.
- If the integral evaluates to $\pm\infty$ or to, $\infty - \infty$, we say the integral ***diverges***.

Case 1: At Least One Infinite Limit

Redefine the integral into one of the following.

$$(i)\int_a^{\infty} f(x)dx = \lim_{b\to\infty}\int_a^{b} f(x)dx$$

$$(ii)\int_{-\infty}^{b} f(x)dx = \lim_{a\to-\infty}\int_a^{b} f(x)dx$$

$$(iii)\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_0^{\infty} f(x)dx$$

and now use parts (i) and (ii).

Example 1.1: Evaluate the integral: $$\int_{-\infty}^{0} \frac{dx}{1+x^2}$$

Example 1.2:

Evaluate the integral: $$\int_0^\infty x^3 e^{-x^2}\,dx$$

Case 2: f(c)$\rightarrow\infty$ Between *a* and *b*

- Case 2 occurs when f has a vertical asymptote on the interval [*a*,*b*].
- Redefine the integral into one of the following.

$$(i)\text{ If } f(a)\text{ DNE, then}: \int_a^b f(x)dx = \lim_{c\to a^+}\int_c^b f(x)dx$$

$$(ii)\text{ If } f(b)\text{ DNE, then}: \int_a^b f(x)dx = \lim_{c\to b^-}\int_a^c f(x)dx$$

$$(iii)\text{ If } f(c)\text{ DNE, where } a<c<b, \text{ then}:$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

and now use parts (i) and (ii).

Example 2.1: Evaluate the integral: $\int_{\frac{\pi}{2}}^{\pi} \tan(x)dx$

Example 2.2: Evaluate the integral: $\displaystyle\int_{-1}^{32} \frac{dx}{x^5}$

Example 3: Find the area of the region bounded by

$$y = e^{-x}, \text{ the } x\text{-axis, and } x \geq 0$$

Bonus Problems on Improper Integrals

Evaluate each of the next integrals (*if time permits*).

- $$\int_0^1 \frac{\ln(x)}{\sqrt{x}}dx$$
- $$\int_0^\infty \frac{e^{-\frac{1}{2x}}}{x^2}dx$$
- $$\int_0^\infty \frac{e^x}{e^{2x}+3}dx$$
- $$\int_1^e \frac{dx}{x\sqrt{\ln(x)}} \text{ (converges)}$$
- $$\int_e^\infty \frac{dx}{x\sqrt{\ln(x)}} \text{ (diverges)}$$