FACTORIZATION THEOREMS FOR DIVISOR SUMS OVER RELATIVELY PRIME INTEGERS

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Abstract. We prove new factorizations of the generating functions of a class of divisor sums over $f(d)$ for d relatively prime to n where f is an arbitrary arithmetic function. These series factorizations by matrices and their properties are related to the expansions of Lambert series factorization theorems considered in the references. Applications of the new results proved in the article include new multiple sums for Euler's totient function $\phi(n)$ and the closely-related power totient function $\phi_m(n)$. We also suggest the application of our new results for these divisor sums to Rademacher-type exact series for the partition functions $p(n)$ and $q(n)$ in the concluding remarks to the article.

1. Introduction

1.1. **Factorization theorems.** We construct analogs to the Lambert series factorization theorems proved in [2] for sums over relatively prime divisors of the form

$$\sum_{d:(d,n)=1} f(d) = [q^n]\frac{1}{C_*(q)} \sum_{n\geq 1}\sum_{k=1}^{n} s_{n,k}^{*} f(k)\cdot q^n, \tag{1}$$

where $f(n)$ denotes an arbitrary arithmetic function, typically with $f(1) = 1$, though not by necessity in our initial construction, and where the generating function $C_*(q)$ is combinatorially motivated in our results. In particular, we define the generating function $C_*(q) \equiv (q;q)_\infty$ where the reciprocal of the q-Pochhammer symbol generates Euler's partition function $p(n)$ [5, A000041], and where Euler's pentagonal number theorem is equivalent to the expansion

$$(q;q)_\infty = \sum_{j\geq 0}(-1)^{\lceil j/2\rceil} q^{G_j} = 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + q^{22} + q^{26} - \cdots,$$

for $G_j := \frac{1}{2}\left\lceil \frac{j}{2}\right\rceil \left\lceil \frac{3j+1}{2}\right\rceil$ [5, A001318].

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We use each of these partition function definitions to identify the characteristic expansions of our primary special case of the factorization in (1) defined for an arbitrary arithmetic function f by

$$\sum_{d:(d,n)=1} f(d) = [q^n]\frac{1}{(q;q)_\infty}\sum_{n\geq 1}\sum_{k=1}^{n-1} \widetilde{s}_{n,k} f(k)\cdot q^n. \tag{2}$$

Since the matrix for the sequence $\widetilde{s}_{n,k}$ is not invertible if the index $n=1$ is included, we consider the corresponding inverse sequence, $s_{n,k}^{(-1)}$, to invert the matrices $(s_{i+1,j})_{1\leq i,j<n}$ for integers $n\geq 2$ and then perform an adjustment later in our formulas that employ these sequences. Namely, we re-write (4) in the form of[1]

$$\sum_{d:(d,n)=1} f(d) = [q^n]\left(\frac{1}{(q;q)_\infty}\sum_{n\geq 2}\sum_{k=1}^{n} s_{n,k} f(k)\cdot q^n + f(1)\cdot q\right). \tag{3}$$

The explicit expansion in (3) is the key form of our modified factorization theorem result that we study within this article.

1.2. **Applications and organization of the article.** Our new expansions proved in Section 2 lead to new exact formulas for Euler's totient function $\phi(n)$ [5, A000010] and the power totient function

$$\phi_m(n) = \sum_{d:(d,n)=1} d^m,$$

for any prescribed $m\in\mathbb{C}$. We also suggest applications to new sums for the particular Kloosterman sums involved in the expansion of Rademacher's famous exact formula for $p(n)$. We conclude the article in Section 3 by suggesting additional related applications to other divisor sums involving the greatest common divisor of n of the form

$$\sum_{d|(n,m)} f(d)g(m/d),$$

which are common in number theoretic applications [3, §27.10]. The approach we outline in the article similarly makes the generalized factorization expansions for these sums approachable using the same systematic methodology we have used to develop our new results here. We intend to keep the article brief and focused on proving the key results for our factorizations in (3).

2. Proofs of the main results

2.1. **Inversion relations.** We begin our exploratory analysis here by expanding an inversion formula which is analogous to Möbius inversion for ordinary divisor sums. We prove the following result which is the analog to the sequence inversion relation provided by the Möbius transform in the context of our sums over the integers relatively prime to n [4, *cf.* §2, §3].

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[1] Notation: *Iverson's convention* compactly specifies boolean-valued conditions and is equivalent to the *Kronecker delta function*, $\delta_{i,j}$, as $[n=k]_\delta \equiv \delta_{n,k}$. Similarly, $[\texttt{cond}=\texttt{True}]_\delta \equiv \delta_{\texttt{cond},\texttt{True}}$ in the remainder of the article.

1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
−1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
−1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	−1	−1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
−1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	−1	−1	1	0	0	0	0	0	0	0	0	0	0	0
1	0	−1	0	−1	0	1	0	0	0	0	0	0	0	0	0	0
−1	0	2	−1	0	0	−1	1	0	0	0	0	0	0	0	0	0
−1	0	0	0	1	0	−1	0	1	0	0	0	0	0	0	0	0
1	0	−1	1	0	−1	1	−1	−1	1	0	0	0	0	0	0	0
−1	0	1	0	0	0	−1	0	0	0	1	0	0	0	0	0	0
1	0	−1	0	0	0	1	0	0	−1	−1	1	0	0	0	0	0
3	0	−2	0	−2	0	2	0	−1	0	−1	0	1	0	0	0	0
−3	0	1	0	3	0	−1	−1	1	0	0	0	−1	1	0	0	0
−1	0	1	0	1	0	−1	0	0	0	0	0	−1	0	1	0	0
1	0	0	0	−2	0	0	1	0	0	1	−1	1	−1	−1	1	0
−3	0	2	0	2	0	−2	0	1	0	0	0	−1	0	0	0	1

$\mu_{n,k}$ *for* $1 \leq n, k < 18$

Figure 1.1. *Inversion formula coefficient sequences*

Proposition 2.1 (Inversion Formula)**.** *For all $n \geq 2$, there is a unique lower triangular sequence, denoted $\mu_{n,k}$, which satisfies the inversion relation*

$$g(n) = \sum_{d:(d,n)=1} f(d) \qquad \Longleftrightarrow \qquad f(n) = \sum_{d=1}^{n} g(d+1)\mu_{n,d}.$$

Moreover, if we form the matrix $(\mu_{i,j})_{1 \leq i,j \leq n}$ for any $n \geq 2$, we have that the inverse sequence satisfies

$$\mu_{n,k}^{(-1)} = [(n+1,k)=1]_\delta .$$

Proof. Consider the $(n-1) \times (n-1)$ matrix

$$([(i,j)=1]_\delta)_{1 \leq i,j < n},$$

which effectively corresponds to the formula on the left-hand-side of the first equation by applying the matrix to the vector of $[f(1)\ f(2)\ \cdots f(n)]^T$ and extracting the n^{th} column as our stated formula. Since $\gcd(n, n-1) = 1$ for all $n > 1$, we see that this matrix is lower triangular with ones on its diagonal. Thus the matrix is non-singular and its unique inverse, which we denote by $(\mu_{i,j})_{1 \leq i,j<n}$, leads to the sum on the right-hand-side of the first equation when we shift $n \mapsto n+1$. The second equation restates the form of the first matrix when we perform the shift of $n \mapsto n+1$ as on the right-hand-side of the first equation. □

Figure 1.1 provides a listing of the relevant analogs to the Möbius function in the context of the Möbius transform of the ordinary divisor sum over an arithmetic function from the proposition. We not know of a comparatively simple closed-form function for the sequence of $\mu_{n,k}$ [5, *cf.* A096433]. However, we readily see by construction that the sequence and its inverse satisfy

$$\sum_{d:(d,n)=1} \mu_{d,k} = 0$$
$$\sum_{d:(d,n)=1} \mu_{d,k}^{(-1)} = \phi(n),$$

where $\phi(n)$ is Euler's totient function. The first columns of the corresponding sums in the previous equation performed over the columns index k for fixed n appear in the integer sequences database as the entry [5, A096433].

2.2. **Exact formulas for the factorization matrices.** The next result is key to proving the exact formulas for the matrix sequences, $s_{n,k}$ and $s_{n,k}^{(-1)}$, and their expansions by the partition functions defined in the introduction. We prove it first as a lemma which we will use in the proof of Theorem 2.4 given below.

Lemma 2.2 (A Convolution Identity for Relatively Prime Integers)**.** *For all natural numbers $n \geq 2$ and $k \geq 1$ with $k \leq n$, we have the following expression for the indicator function of whether (n,k) forms a pair of relatively prime integers:*

$$\sum_{j=1}^{n} s_{j,k} p(n-j) = [(n,k)=1]_{\delta}\,.$$

Proof. We begin by noticing that the right-hand-side expression in the statement of the lemma is equal to $\mu_{n,k}^{(-1)}$ by the construction of this sequence in Proposition 2.1. Next, we see that the factorization in (4) is equivalent to the expansion

$$\sum_{d=1}^{n-1} f(d)\mu_{n,d}^{(-1)} = \sum_{j=1}^{n}\sum_{k=1}^{j} p(n-j)s_{j,k}\cdot f(k).$$

Since $\mu_{n,k}^{(-1)} = [(n+1,k)=1]_{\delta}$, we may take the coefficients of $f(k)$ on each side of the previous equation for each $1 \leq k < n$ to establish the claimed result. □

The first several rows of the matrix sequence $s_{n,k}$ and its inverse implicit to the factorization theorem in (3) are tabulated in Figure 2.1 for intuition on the formulas we prove in the next theorem.

Proposition 2.3 (A Generating Function for $s_{n,k}$)**.** *For all $n,k \geq 1$ we have that*

$$s_{n,k} = [q^n](q;q)_{\infty}\left(\frac{q^{k+1}}{1-q} + \sum_{m=1}^{k} \sum_{\substack{p_1,\ldots,p_m | k \\ p_i \text{prime}}} \frac{(-1)^m q^{p_1\cdots p_m + k}}{1-q^{p_1\cdots p_m}}\right).$$

Proof. We consider the coefficients of $f(k)$ for a fixed $k \geq 1$ in the series in (3) multiplied through by a factor of $(q;q)_{\infty}$. In particular, we see that

$$[f(k)]\sum_{n\geq 1}\sum_{k=1}^{n-1} s_{n,k} f(k)\cdot q^n = \sum_{n\geq 1} s_{n,k} q^n,$$

1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0
−1	1	0	0	0	0	0	0	0	0	0	0	0	0
−1	−1	1	0	0	0	0	0	0	0	0	0	0	0
−1	0	0	1	0	0	0	0	0	0	0	0	0	0
0	−1	−2	−1	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0	0	0	0
1	0	0	−1	−1	−1	1	0	0	0	0	0	0	0
1	0	−1	0	−1	−1	0	1	0	0	0	0	0	0
1	1	1	0	−2	0	−1	−1	1	0	0	0	0	0
1	0	1	0	1	1	−1	0	0	1	0	0	0	0
1	1	0	1	1	0	−1	−1	−2	−1	1	0	0	0
0	0	1	0	1	0	0	0	0	0	0	1	0	0
0	1	1	1	1	0	−1	0	0	−1	−1	−1	1	0

(i) $s_{n,k}$

1	0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0	0	0	0
1	0	0	1	0	0	0	0	0	0	0	0	0
4	3	2	1	1	0	0	0	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0	0	0
5	3	2	2	1	1	1	0	0	0	0	0	0
4	4	3	1	1	1	0	1	0	0	0	0	0
15	11	8	5	4	2	1	1	1	0	0	0	0
−1	−1	−1	1	0	0	1	0	0	1	0	0	0
32	24	18	12	9	6	4	3	2	1	1	0	0
−6	−4	−3	−1	−1	0	0	0	0	0	0	1	0
24	17	13	12	8	7	6	3	2	2	1	1	1

(ii) $s_{n,k}^{(-1)}$

Figure 2.1. *The factorization matrices,* $s_{n,k}$ *and* $s_{n,k}^{(-1)}$*, for* $1 \leq n, k < 14$

and that on the left-hand-side we have that

$$
\begin{aligned}
[f(k)](q;q)_\infty \sum_{n\geq 1} \sum_{d:(d,n)=1} f(d) \cdot q^n &= (q;q)_\infty \sum_{n \geq 1} [(n,k)=1]_\delta \, q^n \\
&= (q;q)_\infty \left(\frac{q^{k+1}}{1-q} - \frac{q^k}{1-q^k} + \operatorname{duplicates}(q) \right),
\end{aligned}
$$

where we may expand the right-hand-side terms in the last equation using inclusion and exclusion over the paired products of the prime divisors of k. These equations imply that we have a generating function for $s_{n,k}$ at each fixed k of the form specified in the statement of the proposition. Thus we have proved our result. $\square$

The interpretation of the previous generating function formula for $s_{n,k}$ is related to the sequence of $\bar{s}_{n,k}$ from the original Lambert series factorization defined in [2, §1]. In particular, the original factorization sequence corresponds to

$$\bar{s}_{n,k} = [q^n]\frac{q^k}{1-q^k}(q;q)_\infty = s_o(n,k) - s_e(n,k),$$

where $\bar{s}_{n,k} = s_o(n,k) - s_e(n,k)$ denotes the difference of the number of k's in all partitions of n into an odd (even) number of distinct parts. Then we see that our new sequence of $s_{n,k}$ is given in terms of the sequence in from the previous equation, and in turn its interpretation as the difference of two important partition functions, as

$$s_{n,k} = [q^{n-k}](q;q)_\infty \frac{q}{1-q} + \sum_{m=1}^{k} \sum_{\substack{p_1,\ldots,p_m \mid k \\ p_i \,\mathrm{prime}}} (-1)^m \bar{s}_{n-k,p_1\cdots p_m}.$$

The next theorem provides alternate exact formulas for the sequence $s_{n,k}$ and its inverse.

Theorem 2.4 (Exact Formulas for the Factorization Matrix Sequences)**.** *For integers $n,k \geq 1$, the two factorization sequences implicit to the expansion of* (3) *have the next exact formulas given by*

$$s_{n,k} = \sum_{j=0}^{n} (-1)^{\lceil j/2 \rceil} \left[(n-k-G_j,k)=1\right]_\delta \left[n-k-G_j \geq 1\right]_\delta \tag{i}$$

$$s_{n,k}^{(-1)} = \sum_{d=1}^{n} p(d-k)\mu_{n,d}, \tag{ii}$$

where we define the sequence of interleaved pentagonal numbers to be $G_j := \frac{1}{2}\left\lceil \frac{j}{2} \right\rceil \left\lceil \frac{3j+1}{2} \right\rceil$ [5, A001318].

Proof. It is plain to see by the considerations in our construction of the factorization theorem that both matrix sequences are lower triangular. Thus, we need only consider the cases where $n \leq k$. By a convolution of generating functions, the identity in Lemma 2.2 shows that

$$\sum_{j=k}^{n} [q^{n-j}](q;q)_\infty \cdot \left[(j+1,k)=1\right]_\delta.$$

Then shifting the index of summation in the previous equation implies (i).

To prove (ii), we consider the factorization theorem when $f(n) := s_{n,r}^{(-1)}$ for some fixed $r \geq 1$. Then we expand the result in (3) as

$$\begin{aligned}
\sum_{d:(d,n)=1} s_{d,r}^{(-1)} &= [q^n]\frac{1}{(q;q)_\infty} \sum_{n\geq 1}\sum_{k=1}^{n-1} s_{n,k} \cdot s_{k,r}^{(-1)} \cdot q^n \\
&= \sum_{j=1}^{n} p(n-j) \times \sum_{k=1}^{j-1} s_{j,k} s_{k,r}^{(-1)}
\end{aligned}$$

$$\begin{aligned} &= \sum_{j=1}^{n} p(n-j) \left[r = j-1\right]_{\delta} \\ &= p(n-1-r). \end{aligned}$$

Hence we may perform the inversion by Proposition 2.1 to the left-hand-side sum in the previous equations to obtain our stated result. ☐

2.3. **Completing the proofs of the main applications.** We note that as in the Lambert series factorization results from the references [2], we have three primary expansion types of identities based on the form of (4) and (3) that we may consider for a prescribed choice of the arithmetic function f in the forms of

$$\begin{aligned} \sum_{d:(d,n)=1} f(d) &= \sum_{j=1}^{n} \sum_{k=1}^{j-1} p(n-j) s_{j-1,k} f(k) \\ \sum_{k=1}^{n-1} s_{n-1,k} f(k) &= \sum_{j=1}^{n} \sum_{d:(d,j)=1} [q^{n-j}](q;q)_{\infty} \cdot f(d), \end{aligned}$$

and the corresponding inverted formula providing that

$$f(n) = \sum_{k=1}^{n+1} s_{n,k-1}^{(-1)} \times \sum_{j=2}^{k} p(k-j) \sum_{d:(d,j)=1} f(d).$$

In particular, we formalize the known expansions in the previous equations to the modified cases of the factorization theorem in (3) according to the next corollary. Now the applications cited in the introduction follow immediately and require no further proof than to cite these results in the special case where $f(n) \equiv 1$ for all n. We provide other similar corollaries of the factorization theorem results for the sake of completeness as the next examples.

Corollary 2.5 (Euler's Totient Function)**.** *For all $n \geq 1$, we have the following two identities for Euler's totient function:*

$$\begin{aligned} \phi(n) &= \sum_{j=0}^{n} \sum_{k=1}^{j-1} \sum_{i=0}^{j} p(n-j)(-1)^{\lceil i/2 \rceil} \left[(j-k-G_i, k) = 1\right]_{\delta} \left[j-k-G_i \geq 1\right]_{\delta} + \left[n=1\right]_{\delta} \\ \phi(n) &= \sum_{d:(d,n)=1} \left(\sum_{k=1}^{d+1} \sum_{i=1}^{d} \sum_{j=0}^{k} p(i+1-k)(-1)^{\lceil j/2 \rceil} \phi(k-G_j) \mu_{d,i} \left[k-G_j \geq 1\right]_{\delta} \right). \end{aligned}$$

Corollary 2.6 (Identities for the Function $\phi_m(n)$)**.** *For all $n \geq 2$ and any fixed $m \in \mathbb{Z}$, we have that*

$$\begin{aligned} \phi_m(n) &= \sum_{j=0}^{n} \sum_{k=1}^{j-1} \sum_{i=0}^{j} p(n-j)(-1)^{\lceil i/2 \rceil} \left[(j-k-G_i, k) = 1\right]_{\delta} \left[j-k-G_i \geq 1\right]_{\delta} \cdot k^m \\ \phi_m(n) &= \sum_{d:(d,n)=1} \left(\sum_{k=1}^{d+1} \sum_{i=1}^{d} \sum_{j=0}^{k} p(i+1-k)(-1)^{\lceil j/2 \rceil} \phi_m(k-G_j) \mu_{d,i} \left[k-G_j \geq 1\right]_{\delta} \right). \end{aligned}$$

Remark 2.7 (Applications to Dedekind and Kloosterman-Type Sums)**.** For $K := \pi\sqrt{2/3}$, Rademacher proved that Euler's partition function has an exact expansion given by [3, §27.14(iii)]

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k \geq 1} \sqrt{k} \cdot A_k(n) \times \frac{d}{dt}\left[\frac{\sinh(K\sqrt{t}/k)}{\sqrt{t}}\right]_{t=n-\frac{1}{24}},$$

where $A_k(n)$ is of the form of the divisor sums we are focused on in this article as

$$A_k(n) = \sum_{h:(h,k)=1} \exp\left(\pi\imath \cdot s(h,k) - 2\pi\imath nh/k\right),$$

for the Dedekind sum variant $s(h,k)$ expanded by

$$s(h,k) = \sum_{r=1}^{k-1} \frac{r}{k}\left(\frac{hr}{k} - \left\lfloor \frac{hr}{k} \right\rfloor - \frac{1}{2}\right).$$

Another famous formula providing an application of our factorizations for the divisor sums we consider in this article provides exact and a corresponding asymptotic approximation for the special function $q(n) = (-q;q)_\infty$ [5, A000009] which counts the number of partitions of n into distinct parts. More precisely, we have the exact sum expansion for this function given by [3, §26.10(vi)]

$$q(n) = \pi \sum_{k \geq 1} \frac{B_{2k-1}(n)}{(2k-1)\sqrt{24n+1}} I_1\left(\frac{\pi}{2k-1}\sqrt{\frac{24n+1}{72}}\right),$$

where $I_1(z)$ is a modified Bessel function and where we define

$$\begin{aligned} B_k(n) &= \sum_{h:(h,k)=1} \exp\left(\pi\imath \cdot f(h,k) - 2\pi\imath nh/k\right) \\ f(h,k) &= \sum_{j=1}^{k} \left[\!\left[\frac{2j-1}{2k}\right]\!\right] \times \left[\!\left[\frac{h(2j-1)}{k}\right]\!\right], \end{aligned}$$

for

$$[\![x]\!] = \begin{cases} x - \lfloor x \rfloor - \frac{1}{2}, & x \in \mathbb{Q} \setminus \mathbb{Z} \\ 0, & x \in \mathbb{Z}. \end{cases}$$

Other applications of our new factorization results are suggested in the concluding remarks on future research topics given in the next section.

3. Conclusions

3.1. **Summary.** We have proved several expansions of factorization theorems and properties for divisor sums over relatively prime integers of the form

$$\sum_{d:(d,n)=1} f(d),$$

where the arithmetic function f is arbitrary in our expansions. The special cases of $f(n) := 1$ and $f(n) := n^m$ for some $m \in \mathbb{C}$ define the famous variants of the totient functions $\phi(n)$ and $\phi_m(n)$ [3, §]. These results are analogous in many respects to the expansions of the Lambert series factorization theorems proved in [2] among other possible references. In fact, in Proposition 2.3 and the remarks shortly thereafter, we proved relations of the implicit factorization sequence $s_{n,k}$ in (3) to the Lambert series factorization sequence $\bar{s}_{n,k} = [q^n]q^k/(1-q^k)(q;q)_\infty$.

The results in this article and in [2] allow us to give an exact formula for the more complicated divisor sum

$$T_f(n) = \sum_{d=1}^{n} f(d) - \left(\sum_{d|n} f(d) + \sum_{d:(d,n)=1} f(d)\right),$$

for all $n \geq 1$. The previous sum consists of all $f(d)$ such that d contains a factor of n, but where $d \not| n$. In some sense these factorization theorems provide a form of an indicator function for the sequences of such integers. The factorization of the sum consisting of $f(d)$ at all divisors d of n corresponds to the series coefficients of a Lambert series generating function summed over the function f in the form of

$$\sum_{d|n} f(d) = [q^n]\left(\sum_{m\geq 1} \frac{f(m)q^m}{1-q^m}\right).$$

The properties of the sums of this form and their many curious properties are explored in [2] and in the joint work by M. Merca and M. D. Schmidt in 2017.

3.2. **Future research topics and applications.** We consider the related factorizations for divisor sums of the form

$$\sum_{m=1}^{n}\left(\sum_{d|(n,m)} f(d)g(m/d)\right) w^m = [q^n]\frac{1}{(q;q)_\infty}\sum_{n\geq 1}\sum_{k=1}^{n} t_{n,k}(g)f(k)\cdot q^n, \tag{4}$$

i.e., so that by keeping track of the coefficients of powers of w, we can keep track of the inner divisor sums on the left-hand-side of the previous equation. It happens (and this is not too difficult to prove) that we can express the sequence of $t_{n,k}$ in terms of sums over the sequence $s_{n,k}$ from our factorizations in (3). In particular, we have that (TODO)

$$t_{n,k} = \sum_{m=1}^{\lfloor n/k\rfloor} s_{n+1-m,m}g(m)w^{mk}.$$

There are many number theoretic applications of sums factorized in this form. For example, Ramanujan's sum $c_k(n)$ is expressed as the divisor sum [1, §IX]

$$c_k(n) = \sum_{h:(h,n)=1} e^{2\pi\imath hm/n} = \sum_{d|(n,k)} d\cdot\mu(k/d).$$

We also have the modified form of an expansion of Ramanujan's tau function given by the following equation for integers $m, n \geq 1$ [1, §IX, §X] [3, §27.10]:

$$\tau(m)\tau(n) = \sum_{d|(m,n)} d^{11}\tau\left(\frac{mn}{d^2}\right).$$

Even more generally, if we take the two arithmetic functions f and g to be arbitrary, we can express the periodic divisor sums of the form [3, §27.10]

$$s_k(f,g;n) = \sum_{d|(n,k)} f(d)g(k/d),$$

as the Fourier series

$$s_k(f,g;n) = \sum_{m=1}^{k} a_k(f,g;m)\cdot e^{2\pi\imath mn/k},$$

where the inner sum terms in the last equation are given by the corresponding divisor sum

$$a_k(f,g;m) = \sum_{d|(m,k)} g(d)f(k/d) \cdot \frac{d}{k}.$$

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