AN ASYMPTOTIC EXPANSION FOR A LAMBERT SERIES ASSOCIATED TO THE SYMMETRIC SQUARE L-FUNCTION

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ABSTRACT. Hafner and Stopple proved a conjecture of Zagier, that the inverse Mellin transform of the symmetric square L-function associated to the Ramanujan tau function has an asymptotic expansion in terms of the non-trivial zeros of the Riemann zeta function $\zeta(s)$. Later, Chakraborty, Kanemitsu and the second author extended this phenomenon for any Hecke eigenform over the full modular group. In this paper, we study an asymptotic expansion of the Lambert series

$$y^k \sum_{n=1}^{\infty} \lambda_f(n^2) \exp(-ny)$$
, as $y \to 0^+$,

where $\lambda_f(n)$ is the nth Fourier coefficient of a Hecke eigen form f(z) of weight k over the full modular group.

1. Introduction

Let $\Delta(z) := e^{2\pi i z} \prod_{n=1}^{\infty} (1-e^{2\pi i n z})^{24} = \sum_{n=1}^{\infty} \tau(n) e^{2\pi i n z}$ be the Ramanujan cusp form of weight 12. Around four decades ago, Don Zagier [15, p. 417], [16, p. 271] conjectured that the constant term of the automorphic form $\text{Im}(z)^{12} |\Delta(z)|^2$, that is, the Lambert series

$$a_0(y) := y^{12} \sum_{n=1}^{\infty} \tau(n)^2 \exp(-4\pi ny)$$

has an asymptotic expansion as $y \to 0^+$, and it can be written in terms of the non-trivial zeros of the Riemann zeta function $\zeta(s)$. He also claimed that $a_0(y)$ will have an oscillatory behaviour when $y \to 0^+$. Moreover, his prediction was that, $a_0(y)$ will satisfy

$$a_0(y) \sim A + \sum_{\rho} y^{1-\frac{\rho}{2}} B_{\rho}$$
 as $y \to 0^+$,

where A is some constant and the sum over ρ runs through all non-trivial zeros of $\zeta(s)$, and B_{ρ} are some complex numbers. Under the assumption of the Riemann Hypothesis,

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one can write the above asymptotic expansion as

$$a_0(y) \sim A + y^{3/4} \sum_{n=1}^{\infty} a_n \cos\left(\phi_n - \frac{t_n}{2}\log(y)\right)$$
 as $y \to 0^+$,

where a_n and ϕ_n are some constants. The oscillatory behaviour of $a_0(y)$ is due to the presence of cosine functions in the above asymptotic expansion. This conjecture was finally settled by Hafner and Stople [4] in 2000. In 2017, Chakraborty, Kanemitsu, and the second author [2] extended this phenomenon for Hecke eigenform over the full modular group. Again, Chakraborty et al. [3] also extended it for congruence subgroup. Mainly, they proved that the constant terms of the automorphic form $y^k |f(z)|^2$, where f(z) is a Hecke eigenform of weight k over the full modular group $SL(2, \mathbb{Z})$, that is,

$$y^k \sum_{n=1}^{\infty} |\lambda_f(n)|^2 \exp(-4\pi ny)$$

also has an asymptotic expansion as $y \to 0^+$, and it can be expressed in terms of the non-trivial zeros of the Riemann zeta function. Recently, Banerjee and Chakraborty [1] studied this phenomenon for the Maass cusp forms.

In the present paper, we investigate an asymptotic expansion of the Lambert series

$$y^k \sum_{n=1}^{\infty} \lambda_f(n^2) \exp(-ny)$$

as $y \to 0^+$. Interestingly, we observe that the asymptotic expansion of this Lambert series can also be written in terms of the non-trivial zeros of the Riemann zeta function $\zeta(s)$.

2. Preliminaries

Let k and N be two positive integers and χ be a primitive Dirichlet character modulo N. We define

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\}.$$

We denote $S_k(\Gamma_0(N), \chi)$ be the space of cusp forms of weight k, level N, and Nebentypus character χ . Let $f(z) \in S_k(\Gamma_0(N), \chi)$ be a normalized Hecke eigenform with the Fourier series expansion

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) \exp(2\pi i n z), \quad \forall z \in \mathbb{H}.$$
 (2.1)

The L-function associated to f(z) satisfy following Euler product representation:

$$L(s,f) := \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_{p: \text{ prime}} \left(1 - \lambda_f(p) p^{-s} + \chi(p) p^{k-1-2s} \right)^{-1}$$
$$= \prod_{p: \text{ prime}} (1 - \alpha_p p^{-s})^{-1} (1 - \beta_p p^{-s})^{-1}, \quad \text{Re}(s) > \frac{k+1}{2},$$

where the complex conjugates α_p and β_p satisfy the relations $\alpha_p + \beta_p = \lambda_f(p)$ and $\alpha_p \beta_p = \chi(p) p^{k-1}$. With the help of these complex numbers, Shimura [13, Equation (0.2)] introduced a new *L*-function associated to a Hecke eigenform f(z), namely symmetric square *L*-function, which is defined by

$$L(s, \operatorname{Sym}^{2}(f) \otimes \psi) := \prod_{p: \text{prime}} \left(1 - \psi(p)\alpha_{p}^{2} p^{-s}\right)^{-1} \left(1 - \psi(p)\beta_{p}^{2} p^{-s}\right)^{-1} \left(1 - \psi(p)\alpha_{p}\beta_{p} p^{-s}\right)^{-1},$$

where ψ is a primitive Dirichlet character modulo M. This is one of the important examples of an L-function associated to a GL(3)-automorphic form and its analytic continuation and functional equation has been studied by Shimura. More generally, we can define symmetric power L-function associated to f(z) as follows:

$$L(s, \operatorname{Sym}^{n}(f) \otimes \psi) := \prod_{p: \text{prime } i=0}^{n} \left(1 - \psi(p) \alpha_{p}^{i} \beta_{p}^{n-i}\right)^{-1}.$$

Interested readers can see Murty's [9] lecture notes for more information on symmetric power L-function. Upon simplification of the Euler product of the symmetric square L-function, Shimura observed that

$$L(s, \text{Sym}^2(f) \otimes \psi) = L(2s - 2k + 2, \chi^2 \psi^2) \sum_{n=1}^{\infty} \frac{\lambda_f(n^2)\psi(n)}{n^s},$$
 (2.2)

where $L(s, \chi)$ is the usual Dirichlet L-function. In the same paper, Shimura established following important result:

Theorem 2.1. Let us define

$$L^*(s, \operatorname{Sym}^2(f) \otimes \psi) := N^s \pi^{-\frac{3s}{2}} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+1}{2}\right) \Gamma\left(\frac{s-k+2-\lambda_0}{2}\right) L(s, \operatorname{Sym}^2(f) \otimes \psi),$$

where
$$\lambda_0 = \begin{cases} 0, & \text{if } \chi \psi(-1) = 1, \\ 1, & \text{if } \chi \psi(-1) = -1. \end{cases}$$

Then $L^*(s, \operatorname{Sym}^2(f) \otimes \psi)$ can be analytically continued to the complex plane except for simple poles at s = k and at s = k - 1.

Rankin [11] and Selberg [12] independently studied following interesting Dirichlet series associated to the cusp form f(z), namely,

$$RS(s, f \otimes \bar{f}) := \sum_{n=1}^{\infty} \lambda_f^2(n) n^{-s}, \quad \text{Re}(s) > k.$$

This Dirichlet series is known as Rankin-Selberg L-function associated to f(z). For a general construction of the Rankin-Selberg L-function, readers can see the paper of Winnie Li [8]. The Rankin-Selberg L-function and the symmetric square L-function are intimately connected with each other. This connection was established by Shimura, mainly, he observed that the following relation holds:

$$L(s, \operatorname{Sym}^2(f) \otimes \psi) L(s-k+1, \chi \psi) = RS(s, f \otimes \bar{f} \otimes \psi) L(2s-2k+2, \chi^2 \psi^2).$$

For simplicity, now onwards we assume χ and ψ both are trivial characters. Thus, the above relation becomes

$$L(s, \operatorname{Sym}^{2}(f))\zeta(s-k+1) = RS(s, f \otimes \overline{f})\zeta(2s-2k+2).$$

Since χ and ψ both are trivial, one can see that $\lambda_0 = 0$. In this case, Shimura showed that the completed symmetric square *L*-function $L^*(s, \operatorname{Sym}^2(f))$ is entire and satisfy following beautiful functional equation:

$$L^*(s, \operatorname{Sym}^2(f)) = L^*(2k - 1 - s, \operatorname{Sym}^2(f)). \tag{2.3}$$

This functional equation will be one of the crucial ingredients to obtain our main result. The normalized version of the above functional equation can be found in [6].

Now we introduce well-known hypergeometric series. Let a_1, a_2, \dots, a_p and b_1, b_2, \dots, b_q be p+q complex numbers. We define the generalized hypergeometric series by

$$_{p}F_{q}(a_{1},\cdots,a_{p};b_{1},\cdots,b_{q};z) := \sum_{n=0}^{\infty} \frac{(a_{1})_{n}\cdots(a_{p})_{n}}{(b_{1})_{n}\cdots(b_{q})_{n}} \frac{z^{n}}{n!},$$
 (2.4)

where $(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)}$ and b_i 's are not allowed to take non-positive integers. This series converges for all complex values of z if $p \leq q$, and when p = q + 1 it converges for |z| < 1, and in the latter case, Euler showed that it can be analytically continued to the entire complex plane except the branch cut from 1 to $+\infty$.

We define an arithmetic function $B_f(n)$, connected with the symmetric square L-function by the relation:

$$B_f(n) := (a_{\text{Sym}^2(f)} * b)(n), \text{ where } b(n) = \begin{cases} m^{2k-1}, & \text{if } n = m^2, \\ 0, & \text{otherwise.} \end{cases}$$
 (2.5)

One can show that the Dirichlet series associated to $B_f(n)$ is absolutely convergent for Re(s) > k. Now we are ready to state the main theorem.

3. Main Results

Theorem 3.1. Let $f(z) \in S_k(\operatorname{SL}_2(\mathbb{Z}))$ be a Hecke eigenform with the Fourier series expansion (2.1). Assume all the non-trivial zeros of $\zeta(s)$ are simple. Then for any positive real number y, we have

$$\sum_{n=1}^{\infty} \lambda_f(n^2) \exp(-ny) = \frac{\Gamma(k)y^{1-k}}{2\pi^2} \sum_{n=1}^{\infty} \frac{B_f(n)}{n^k} \left[{}_3F_2\left(\frac{k}{2}, \frac{k+1}{2}, 1; \frac{1}{4}, \frac{3}{4}; -\left(\frac{y}{8n\pi}\right)^2\right) - 1 \right] + \mathcal{R}(y),$$

where

$$\mathcal{R}(y) = \frac{1}{2y^{k-1}} \sum_{\rho} \frac{\Gamma\left(\frac{\rho}{2} + k - 1\right) L\left(\frac{\rho}{2} + k - 1, \operatorname{Sym}^{2}(f)\right)}{y^{\frac{\rho}{2}} \zeta'(\rho)},$$

and the sum over ρ runs through all the non-trivial zeros of $\zeta(s)$ involves bracketing the terms so that the terms corresponding to ρ_1 and ρ_2 are included in the same bracket if they satisfy

$$|\operatorname{Im}(\rho_1) - \operatorname{Im}(\rho_2)| < \exp\left(-C\frac{\operatorname{Im}(\rho_1)}{\log(\operatorname{Im}(\rho_1))}\right) + \exp\left(-C\frac{\operatorname{Im}(\rho_2)}{\log(\operatorname{Im}(\rho_2))}\right),$$

where C is some positive constant.

The below asymptotic result is an immediate application of the this theorem.

Corollary 3.2. Let N be a positive integer and f(z) be a normalized Hecke eigenform defined as in Theorem 3.1. Assume Riemann Hypothesis and simplicity of the non-trivial zeros of $\zeta(s)$. For $y \to 0^+$, we have

$$y^{k} \sum_{n=1}^{\infty} \lambda_{f}(n^{2}) \exp(-ny) = y^{3/4} \sum_{n=1}^{\infty} b_{n} \cos\left(\delta_{n} - \frac{t_{n}}{2} \log(y)\right) + \sum_{j=1}^{N-1} A_{j} y^{2j+1} + O_{f,k}(y^{2N+1}),$$

where the absolute constants A_j depend only on f and the polar representation of $\Gamma\left(\frac{\rho_n}{2}+k-1\right)L\left(\frac{\rho_n}{2}+k-1,\operatorname{Sym}^2(f)\right)\left(\zeta'(\rho_n)\right)^{-1}$ is considered as $b_n\exp(i\delta_n)$, where $\rho_n=\frac{1}{2}+it_n$ denotes the nth non-trivial zero of $\zeta(s)$.

4. Some Well-known Results

In this section, we mention a few important well-known results that we will use frequently. In his seminal paper, Riemann showed that $\zeta(s)$ can be analytically continued

to the whole complex plane except for a simple pole at s=1 and satisfy following functional equation:

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s). \tag{4.1}$$

The gamma function $\Gamma(s)$ satisfy following duplication formula:

Lemma 4.1. For any complex number s, we have

$$\Gamma(2s) = \frac{\Gamma(s)\Gamma(s+\frac{1}{2}) 2^{2s}}{2\sqrt{\pi}}.$$
(4.2)

The next result, gives an important asymptotic expansion for the gamma function, known as Stirling's formula.

Lemma 4.2. In a vertical strip $c \leq \sigma \leq d$,

$$|\Gamma(\sigma + iT)| = \sqrt{2\pi}|T|^{\sigma - \frac{1}{2}}e^{-\frac{1}{2}\pi|T|}\left(1 + O\left(\frac{1}{|T|}\right)\right), \quad \text{as} \quad |T| \to \infty. \tag{4.3}$$

Lemma 4.3. Let us suppose that there exist a sequence T with arbitrary large absolute value that satisfy $\exp(-C_1 \operatorname{Im}(\rho)/\log(\operatorname{Im}(\rho))) < |T - \operatorname{Im}(\rho)|$ for every non-trivial zeros of $\zeta(s)$, where C_1 is some positive constant. Then

$$\frac{1}{|\zeta(\rho+iT)|} < e^{C_2T},$$

where $0 < C_2 < \pi/4$ is some suitable constant.

Proof. A proof of this lemma can be seen in [14, p. 219].

Lemma 4.4. In a vertical strip $\sigma_0 \leq \sigma \leq d$, we have

$$|L(\sigma + iT, \operatorname{Sym}^2(f))| = O(|T|^{A(\sigma_0)}), \text{ as } |T| \to \infty,$$

where $A(\sigma_0)$ is some constant that depends on σ_0 .

Proof. One can find the proof of this lemma in [5, p. 97].

Now we will introduce an important special function, namely, Meijer G-function. Let m, n, p, q be non-negative integers such that $0 \le m \le q, 0 \le n \le p$. Let a_1, \dots, a_p and b_1, \dots, b_q be p+q complex numbers such that $a_i-b_j \notin \mathbb{N}$ for $1 \le i \le n$ and $1 \le j \le m$. The Meijer G-function [10, p. 415, Definition 16.17] is defined by the line integral:

$$G_{p,q}^{m,n} \begin{pmatrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{pmatrix} z = \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^m \Gamma(b_j - s) \prod_{j=1}^n \Gamma(1 - a_j + s) z^s}{\prod_{j=m+1}^q \Gamma(1 - b_j + s) \prod_{j=n+1}^p \Gamma(a_j - s)} ds, \quad (4.4)$$

where the line of integration L going from $-i\infty$ to $+i\infty$ and it separates the poles of the factors $\Gamma(b_j - s)$ from those of the factors $\Gamma(1 - a_j + s)$. The above integral converges if p + q < 2(m + n) and $|\arg(z)| < (m + n - \frac{p+q}{2})\pi$. Now we shall state Slater's theorem [10, p. 415, Equation 16.17.2], which connects Meijer G-function with the hypergeometric series by the following relation: If $p \le q$ and $b_j - b_k \notin \mathbb{Z}$ for $j \ne k$, $1 \le j, k \le m$, then

$$G_{p,q}^{m,n} \begin{pmatrix} a_1, \cdots, a_p \\ b_1, \cdots, b_q \end{pmatrix} z$$

$$= \sum_{k=1}^m A_{p,q,k}^{m,n}(z) {}_p F_{q-1} \begin{pmatrix} 1 + b_k - a_1, \cdots, 1 + b_k - a_p \\ 1 + b_k - b_1, \cdots, *, \cdots, 1 + b_k - b_q \end{pmatrix} (-1)^{p-m-n} z$$

$$, \qquad (4.5)$$

where * indicates that the entry $1 + b_k - b_k$ is omitted and

$$A_{p,q,k}^{m,n}(z) := \frac{z^{b_k} \prod_{j=1, j \neq k}^m \Gamma(b_j - b_k) \prod_{j=1}^n \Gamma(1 + b_k - a_j)}{\prod_{j=m+1}^q \Gamma(1 + b_k - b_j) \prod_{j=n+1}^p \Gamma(a_j - b_k)}.$$

5. Proof of theorem 3.1

First, we show that the Mellin transform of the Lambert series $\sum_{n=1}^{\infty} \psi(n) \lambda_f(n^2) \exp(-ny)$ is equals to

$$\frac{\Gamma(s)L(s, \operatorname{Sym}^2(f) \otimes \psi)}{L(2s - 2k + 2, \chi^2 \psi^2)} \quad \text{for } \operatorname{Re}(s) > k.$$

That is, for Re(s) > k, we write

$$\int_0^\infty \sum_{n=1}^\infty \psi(n)\lambda_f(n^2) \exp(-ny)y^{s-1}dy = \sum_{n=1}^\infty \psi(n)\lambda_f(n^2) \int_0^\infty \exp(-ny)y^{s-1}dy$$
$$= \Gamma(s) \sum_{n=1}^\infty \psi(n)\lambda_f(n^2)n^{-s}$$
$$= \frac{\Gamma(s)L(s, \operatorname{Sym}^2(f) \otimes \psi)}{L(2s - 2k + 2, \chi^2\psi^2)},$$

in the last step we have used the identity (2.2). Therefore, by inverse Mellin transform, we can see that, for y > 0,

$$\sum_{n=1}^{\infty} \psi(n)\lambda_f(n^2) \exp(-ny) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)L(s, \operatorname{Sym}^2(f) \otimes \psi)}{L(2s - 2k + 2, \chi^2\psi^2)} y^{-s} ds, \tag{5.1}$$

where Re(s) = c > k. We have already mentioned, for simplicity of the calculation, we assume χ and ψ are trivial characters. Thus, the above equation (5.1) becomes

$$\sum_{n=1}^{\infty} \lambda_f(n^2) \exp(-ny) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)L(s, \text{Sym}^2(f))}{\zeta(2s - 2k + 2)} y^{-s} ds.$$
 (5.2)

Now we shall analyse the poles of the integrand function. Note that $\Gamma(s)L(s, \operatorname{Sym}^2(f))$ is an entire function since $L^*(s, \operatorname{Sym}^2(f))$ is entire as we are dealing with trivial character χ . In general, $L^*(s, \operatorname{Sym}^2(f))$ may not be an entire function. Assuming Riemann Hypothesis, one can see that the integrand function has infinitely many poles on $\operatorname{Re}(s) = k - \frac{3}{4}$. Furthermore, the integrand function has simple poles at k - n for $n \geq 2$ due to the trivial zeros of $\zeta(2s - 2k + 2)$. Consider the following rectangular contour \mathcal{C} : [c - iT, c + iT], [c + iT, d + iT], [d + iT, d - iT], and [d - iT, c - iT], where k - 2 < d < k - 1 and T is a large positive real number. We can observe that the integrand function has finitely many poles inside this contour \mathcal{C} due to the non-trivial zeros ρ of $\zeta(2s - 2k + 2)$ with $|\operatorname{Im}(\rho)| < T$ and the poles at k - n, for $n \geq 2$, are lying outside the contour. Therefore, employing Cauchy residue theorem, we have

$$\frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\Gamma(s)L(s, \operatorname{Sym}^{2}(f))}{\zeta(2s - 2k + 2)} y^{-s} ds = \mathcal{R}_{T}(y), \tag{5.3}$$

where $\mathcal{R}_T(y)$ denotes the residual term that includes finitely many terms that are supplied by the non-trivial zeros ρ of $\zeta(2s-2k+2)$ with $|\operatorname{Im}(\rho)| < T$. We denote two vertical integrals as

$$V_1(T,y) := \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{\Gamma(s)L(s, \text{Sym}^2(f))}{\zeta(2s - 2k + 2)} y^{-s} ds,$$

$$V_2(T,y) := \frac{1}{2\pi i} \int_{d-iT}^{d+iT} \frac{\Gamma(s)L(s, \text{Sym}^2(f))}{\zeta(2s - 2k + 2)} y^{-s} ds,$$

and the horizontal integrals are denoted as

$$H_1(T,y) := \frac{1}{2\pi i} \int_{c+iT}^{d+iT} \frac{\Gamma(s)L(s, \operatorname{Sym}^2(f))}{\zeta(2s - 2k + 2)} y^{-s} ds,$$

$$H_2(T,y) := \frac{1}{2\pi i} \int_{d-iT}^{c-iT} \frac{\Gamma(s)L(s, \operatorname{Sym}^2(f))}{\zeta(2s - 2k + 2)} y^{-s} ds.$$

Now one of the main aim is to show that the contribution of the horizontal integrals vanish as $T \to \infty$. One can write

$$H_1(T,y) = \frac{1}{2\pi i} \int_c^d \frac{\Gamma(\sigma + iT)L(\sigma + iT, \operatorname{Sym}^2(f))}{\zeta(2\sigma - 2k + 2 + 2iT)} y^{-\sigma - iT} d\sigma.$$

Thus,

$$|H_1(T,y)| \ll \int_c^d \frac{|\Gamma(\sigma+iT)||L(\sigma+iT,\operatorname{Sym}^2(f))|}{|\zeta(2\sigma-2k+2+2iT)|} y^{-\sigma} d\sigma.$$

Use Lemmas 4.2, 4.3, and 4.4, to derive that

$$|H_1(T,y)| \ll |T|^C \exp\left((C_2T - \frac{\pi}{4}|T|\right),$$

where C and C_2 are some constants with $0 < C_2 < \pi/4$. This immediately implies that $H_1(T, y)$ goes to zero as $T \to \infty$. Similarly we can show that $H_2(T, y)$ also vanishes as $T \to \infty$. Now allowing $T \to \infty$ in (5.3), using (5.2), we have

$$\sum_{n=1}^{\infty} \lambda_f(n^2) \exp(-ny) = \mathcal{R}(y) + \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \frac{\Gamma(s)L(s, \operatorname{Sym}^2(f))}{\zeta(2s - 2k + 2)} y^{-s} ds, \tag{5.4}$$

where $\mathcal{R}(y) = \lim_{T\to\infty} \mathcal{R}_T(y)$ is the residual function consisting infinitely many terms. Assuming simplicity hypothesis, that is, all the non-trivial zeros of $\zeta(s)$ are simple, one can show that

$$\mathcal{R}(y) = \sum_{\rho} \lim_{s \to \frac{\rho}{2} + k - 1} \left(s - \frac{\rho}{2} - k + 1 \right) \frac{\Gamma(s)L(s, \text{Sym}^{2}(f))}{\zeta(2s - 2k + 2)} y^{-s}$$

$$= \frac{1}{2y^{k-1}} \sum_{\rho} \frac{\Gamma\left(\frac{\rho}{2} + k - 1\right)L\left(\frac{\rho}{2} + k - 1, \text{Sym}^{2}(f)\right)}{\zeta'(\rho)y^{\frac{\rho}{2}}}, \tag{5.5}$$

where the sum over ρ runs through all non-trivial zeros of $\zeta(s)$. Now we shall try to simplify the left vertical integral:

$$V_2(y) = \lim_{T \to \infty} V_2(T, y) = \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \frac{\Gamma(s)L(s, \text{Sym}^2(f))}{\zeta(2s - 2k + 2)} y^{-s} ds.$$
 (5.6)

First, we shall make use of the functional equation of the symmetric square L-function. Mainly, employing (2.3) and with the help of the duplication formula for the gamma function (4.2), one can obtain

$$V_{2}(y) = \frac{1}{2\pi^{3k-1}} \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \frac{\Gamma\left(\frac{2k-1}{2} - \frac{s}{2}\right) \Gamma\left(k - \frac{s}{2}\right) \Gamma\left(\frac{k+1}{2} - \frac{s}{2}\right)}{\Gamma\left(\frac{2-k}{2} + \frac{s}{2}\right) \zeta(2s - 2k + 2)} \times L(2k - 1 - s, \operatorname{Sym}^{2}(f)) \left(\frac{yN^{2}}{2\pi^{3}}\right)^{-s} ds.$$
 (5.7)

Replace s by 2s - 2k + 2 in (4.1) to see

$$\zeta(2s - 2k + 2) = \frac{\pi^{2s - 2k + 2}}{\sqrt{\pi}} \frac{\Gamma\left(\frac{2k - 2s - 1}{2}\right)}{\Gamma(1 - k + s)} \zeta(2k - 2s - 1). \tag{5.8}$$

Substituting (5.8) in (5.7) and simplifying, we have

$$V_{2}(y) = \frac{1}{2\pi^{k+\frac{1}{2}}} \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \frac{\Gamma\left(\frac{2k-1}{2} - \frac{s}{2}\right) \Gamma\left(k - \frac{s}{2}\right) \Gamma\left(\frac{k+1}{2} - \frac{s}{2}\right) \Gamma(1 - k + s)}{\Gamma\left(\frac{2-k}{2} + \frac{s}{2}\right) \Gamma\left(\frac{2k-1}{2} - s\right) \zeta(2k - 2s - 1)} \times L(2k - 1 - s, \operatorname{Sym}^{2}(f)) \left(\frac{y}{2\pi}\right)^{-s} ds.$$

At this juncture, we would like to shift the line of integral and to do that we change the variable, namely, 2k - 1 - s = w, then we obtain

$$V_{2}(y) = \frac{1}{2\pi^{k+\frac{1}{2}}} \frac{1}{2\pi i} \int_{d'-i\infty}^{d'+i\infty} \frac{\Gamma\left(\frac{w}{2}\right) \Gamma\left(\frac{w+1}{2}\right) \Gamma\left(\frac{w}{2} + \frac{2-k}{2}\right) \Gamma(k-w)}{\Gamma\left(\frac{1+k}{2} - \frac{w}{2}\right) \Gamma\left(w + \frac{1-2k}{2}\right)} \times \frac{L(w, \operatorname{Sym}^{2}(f))}{\zeta(2w - 2k + 1)} \left(\frac{y}{2\pi}\right)^{w-2k+1} dw, \tag{5.9}$$

where k < d' = Re(w) < k+1 as k-2 < d = Re(s) < k-1. One can easily check that the symmetric square L-function $L(w, \text{Sym}^2(f))$ and $\zeta(2w-2k+1)$ are both absolutely convergent on the line Re(w) = d'. Therefore, we write

$$\frac{L(w, \text{Sym}^{2}(f))}{\zeta(2w - 2k + 1)} = \sum_{n=1}^{\infty} \frac{a_{Sym^{2}(f)}(n)}{n^{w}} \sum_{n=1}^{\infty} \frac{n^{2k-1}}{n^{2w}}$$

$$= \sum_{n=1}^{\infty} \frac{B_{f}(n)}{n^{w}}, \qquad (5.10)$$

where $B_f(n)$ is defined as in (2.5). Implement (5.10) in (5.9), and interchange the order of integration and summation to derive

$$V_2(y) = \frac{1}{2\pi^{k+\frac{1}{2}}} \left(\frac{y}{2\pi}\right)^{1-2k} \sum_{n=1}^{\infty} B_f(n) I_{k,y}(n), \tag{5.11}$$

where

$$I_{k,y}(n) := \frac{1}{2\pi i} \int_{d'-i\infty}^{d'+i\infty} \frac{\Gamma\left(\frac{w}{2}\right) \Gamma\left(\frac{w+1}{2}\right) \Gamma\left(\frac{w}{2} + \frac{2-k}{2}\right) \Gamma(k-w)}{\Gamma\left(\frac{1+k}{2} - \frac{w}{2}\right) \Gamma\left(w + \frac{1-2k}{2}\right)} \left(\frac{y}{2n\pi}\right)^{w} dw.$$

Now one of our main goals shall be to evaluate this line integral explicitly, if possible. First, replace $w \to 2w$,

$$I_{k,y}(n) := \frac{1}{2\pi i} \int_{\frac{d'}{2} - i\infty}^{\frac{d'}{2} + i\infty} \frac{\Gamma(w) \Gamma\left(w + \frac{1}{2}\right) \Gamma\left(w + \frac{2-k}{2}\right) \Gamma(k - 2w)}{\Gamma\left(\frac{1+k}{2} - w\right) \Gamma\left(2w + \frac{1-2k}{2}\right)} \left(\frac{y}{2n\pi}\right)^{2w} 2 \, \mathrm{d}w. \quad (5.12)$$

To simplify more we use duplication formula for the gamma function. Mainly, we have to use following two identities:

$$\Gamma(k-2w) = \frac{2^{k-2w}}{2\sqrt{\pi}} \Gamma\left(\frac{k}{2} - w\right) \Gamma\left(\frac{1+k}{2} - w\right),\tag{5.13}$$

$$\Gamma\left(2w + \frac{1 - 2k}{2}\right) = \frac{2^{2w + \frac{1 - 2k}{2}}}{2\sqrt{\pi}}\Gamma\left(w + \frac{1 - 2k}{4}\right)\Gamma\left(w + \frac{3 - 2k}{4}\right). \tag{5.14}$$

Invoking (5.13) and (5.14) in (5.12) we arrive at

$$I_{k,y}(n) := \frac{1}{2\pi i} \int_{\frac{d'}{2} - i\infty}^{\frac{d'}{2} + i\infty} \frac{\Gamma(w) \Gamma(w + \frac{1}{2}) \Gamma(w + \frac{2-k}{2}) \Gamma(\frac{k}{2} - w) 2^{2k - 4w - \frac{1}{2}}}{\Gamma(w + \frac{1-2k}{4}) \Gamma(w + \frac{3-2k}{4})} \left(\frac{y}{2n\pi}\right)^{2w} 2 dw,$$

$$= \frac{2^{2k + \frac{1}{2}}}{2\pi i} \int_{\frac{d'}{2} - i\infty}^{\frac{d'}{2} + i\infty} \frac{\Gamma(w) \Gamma(w + \frac{1}{2}) \Gamma(w + \frac{2-k}{2}) \Gamma(\frac{k}{2} - w)}{\Gamma(w + \frac{1-2k}{4}) \Gamma(w + \frac{3-2k}{4})} \left(\frac{y}{8n\pi}\right)^{2w} dw. \quad (5.15)$$

To write this integral in terms of the Meijer G-function, we shall analyse the poles of the integrand function. We know the poles of $\Gamma(w)$ are at $0, -1, -2, \cdots$; poles of $\Gamma(w+1/2)$ are at $-1/2, -3/2, -5/2, \cdots$; and the poles of $\Gamma(w+\frac{2-k}{2})$ are at $k/2-1, k/2-2, k/2-3, \cdots$; whereas the poles of $\Gamma(\frac{k}{2}-w)$ are at $k/2, k/2+1, k/2+2, \cdots$. Therefore, we can not write the integral (5.15) in terms of the Meijer G-function since the line of integration $(d'/2) \in (k/2, (k+1)/2)$ does not separate the poles of the gamma factors $\Gamma(w)\Gamma(w+\frac{1}{2})\Gamma(w+\frac{2-k}{2})$ from the poles of the gamma factor $\Gamma(\frac{k}{2}-w)$. Hence, we construct a new line of integration (d'') with $d'' \in (k/2-1, k/2)$ so that it separates the poles of the gamma factors $\Gamma(w)\Gamma(w+\frac{1}{2})\Gamma(w+\frac{2-k}{2})$ from the poles of the gamma factor $\Gamma(\frac{k}{2}-w)$. Now consider the contour C' consisting of the line segments [d'-iT,d'+iT],[d'+iT,d''+iT],[d''+iT,d''-iT,], and [d''-iT,d'-iT], where T is some large positive real number, and employ Cauchy residue theorem to obtain

$$\frac{1}{2\pi i} \int_{\mathcal{C}'} F_k(w) = \operatorname{Res}_{s = \frac{k}{2}} F_k(w), \tag{5.16}$$

where

$$F_k(w) = \frac{\Gamma(w)\Gamma\left(w + \frac{1}{2}\right)\Gamma\left(w + \frac{2-k}{2}\right)\Gamma\left(\frac{k}{2} - w\right)}{\Gamma\left(w + \frac{1-2k}{4}\right)\Gamma\left(w + \frac{3-2k}{4}\right)} \left(\frac{y}{8n\pi}\right)^{2w} dw.$$

Again, with the help of Stirling's formula for the gamma function (4.3), one can show that the horizontal integrals tend to zero as T tends to infinity. Therefore, letting $T \to \infty$ in (5.16) and calculating the residual term and substituting it in (5.15), we will have

$$I_{k,y}(n) = \frac{2^{2k+\frac{1}{2}}}{2\pi i} \int_{d''-i\infty}^{d''+i\infty} F_k(w) dw - \frac{2^{2k+\frac{1}{2}}\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)} \left(\frac{y}{8n\pi}\right)^k.$$
 (5.17)

Now we shall try to write the line integral along (d'') in-terms of the Meijer G-function and to do that we reminisce the definition of the Meijer G-function (4.4). We consider m=1, n=3, p=3, q=3 with $a_1=1, a_2=1/2, a_3=k/2$; and $b_1=k/2, b_2=(1+2k)/4, b_3=(3+2k)/4$. One can easily check that $a_i-b_j \notin \mathbb{N}$ for $1 \le i \le n, 1 \le j \le m$

and the inequality p + q < 2(m + n) also satisfied. Hence, one can write

$$\frac{1}{2\pi i} \int_{d''-i\infty}^{d''+i\infty} F_k(w) dw = G_{3,3}^{1,3} \begin{pmatrix} 1, \frac{1}{2}, \frac{k}{2} \\ \frac{k}{2}, \frac{1+2k}{4}, \frac{3+2k}{4} \mid \left(\frac{y}{8n\pi}\right)^2 \end{pmatrix}.$$
 (5.18)

Utilize Slater's theorem (4.5) to write the above Meijer G-function in-terms of the hypergeometric function:

$$G_{3,3}^{1,3} \left(\begin{array}{c} 1, \frac{1}{2}, \frac{k}{2} \\ \frac{k}{2}, \frac{1+2k}{4}, \frac{3+2k}{4} \end{array} \middle| z \right) = \frac{z^{\frac{k}{2}} \Gamma\left(\frac{k}{2}\right) \Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{3}{4}\right)} {}_{3}F_{2}\left(\frac{k}{2}, \frac{k+1}{2}, 1; \frac{1}{4}, \frac{3}{4}; -z\right). \tag{5.19}$$

Substituting $z = \left(\frac{y}{8n\pi}\right)^2$ in (5.19) and together with (5.18) and (5.17), we achieve

$$I_{k,y}(n) = \frac{2^{2k + \frac{1}{2}} \Gamma\left(\frac{k}{2}\right) \Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{3}{4}\right)} \left(\frac{y}{8n\pi}\right)^k \left[{}_{3}F_{2}\left(\frac{k}{2}, \frac{k+1}{2}, 1; \frac{1}{4}, \frac{3}{4}; -\left(\frac{y}{8n\pi}\right)^2\right) - 1\right]. \tag{5.20}$$

Finally, substituting (5.20) in (5.11) and together with (5.4), (5.5), and (5.6), we complete the proof of Theorem 3.1.

Proof of Corollary 3.2. With the help of the definition (2.4) of the hypergeometric series, for any positive integer N, we have

$$_{3}F_{2}\left(\frac{k}{2}, \frac{k+1}{2}, 1; \frac{1}{4}, \frac{3}{4}; -\left(\frac{y}{8n\pi}\right)^{2}\right) - 1 = \sum_{j=1}^{N-1} C_{j}\left(\frac{y}{n}\right)^{2j} + O_{k}\left(\left(\frac{y}{n}\right)^{2N}\right), \text{ as } y \to 0^{+},$$

$$(5.21)$$

where $C_j = (-1)^j \frac{(\frac{k}{2})_j (\frac{k+1}{2})_j}{(\frac{1}{4})_j (\frac{3}{4})_j (8\pi)^{2j}}$. Now invoke (5.21) in Theorem 3.1 to derive that

$$y^{k} \sum_{n=1}^{\infty} \lambda_{f}(n^{2}) \exp(-ny) = \frac{\Gamma(k)}{2\pi^{2}} \sum_{j=1}^{N-1} C_{j} y^{2j+1} \sum_{n=1}^{\infty} \frac{B_{f}(n)}{n^{k+2j}} + O_{k} \left(y^{2N+1} \sum_{n=1}^{\infty} \frac{B_{f}(n)}{n^{k+2N}} \right) + y^{k} \mathcal{R}(y)$$

$$= \sum_{j=1}^{N-1} A_{j} y^{2j+1} + O_{f,k} \left(y^{2N+1} \right) + y^{k} \mathcal{R}(y), \qquad (5.22)$$

where $A_j = \frac{\Gamma(k)}{2\pi^2} C_j \sum_{n=1}^{\infty} \frac{B_f(n)}{n^{k+2j}}$ are computable finite constants since the Dirichlet series associated to $B_f(n)$ is absolutely convergent for Re(s) > k. Assuming Riemann hypothesis and using the fact that the non-trivial zeros appears with conjugate pairs

to write the residual term as

$$y^{k}\mathcal{R}(y) = \frac{y}{2} \sum_{\substack{\rho_{n} = \frac{1}{2} + it_{n}, \\ t_{n} > 0}} 2 \operatorname{Re} \left(\frac{\Gamma\left(\frac{\rho_{n}}{2} + k - 1\right) L\left(\frac{\rho_{n}}{2} + k - 1, \operatorname{Sym}^{2}(f)\right)}{y^{\frac{\rho_{n}}{2}} \zeta'(\rho_{n})} \right)$$

$$= y^{3/4} \sum_{\substack{\rho_{n} = \frac{1}{2} + it_{n}, \\ t_{n} > 0}} b_{n} \cos\left(\delta_{n} - \frac{t_{n}}{2} \log(y)\right), \tag{5.23}$$

here we have considered $b_n \exp(i\delta_n)$ as the polar representation of $\Gamma\left(\frac{\rho_n}{2} + k - 1\right) L\left(\frac{\rho_n}{2} + k - 1\right)$, Sym²(f)) $(\zeta'(\rho_n))^{-1}$. Employ (5.23) in (5.22) to the complete the proof.

6. Concluding Remarks

We have seen that the constant terms of the automorphic form $y^k | f(z)|^2$, that is, the Lambert series $y^k \sum_{n=1}^{\infty} |\lambda_f(n)|^2 \exp(-4\pi ny)$, where $\lambda_f(n)$ is the nth Fourier coefficient of a Hecke eigenform f(z) of weight k over $\mathrm{SL}_2(\mathbb{Z})$, has an asymptotic expansion in terms of the non-trivial zeros of $\zeta(s)$. Recently, authors [7] studied an asymptotic expansion of a Lambert series associated to a Hecke eigenform and the Möbius function. Inspired from these works, in the present paper, we established an exact formula for the Lambert series $y^k \sum_{n=1}^{\infty} \lambda_f(n^2) \exp(-ny)$, and we found that the main term can be expressed in terms of the non-trivial zeros of $\zeta(s)$, and the error term is expressed in terms of the hypergeometric function ${}_3F_2(a,b,c;d;z)$. It would be an interesting problem to study a more general Lambert series $y^k \sum_{n=1}^{\infty} |\lambda_f(n)|^N \exp(-ny)$ for $N \geq 3$. It would also be a challenging problem to classify automorphic forms for which constant terms will have an asymptotic expansion in terms of the non-trivial zeros of $\zeta(s)$.

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References

- [1] S. Banerjee and K. Chakraborty, Asymptotic behaviour of a Lambert series á la Zagier: Maass case, *Ramanujan J.* **48** (2019), 567–575.
- [2] K. Chakraborty, S. Kanemitsu, and B. Maji, Modular-type relations associated to the Rankin-Selberg L-function, Ramanujan J. 42 (2017), 285–299.
- [3] K. Chakraborty, A. Juyal, S. D. Kumar, and B. Maji, An asymptotic expansion of a Lambert series associated to cusp forms. *Int. J. Number Theory* **14** (2018), 289–299.

- [4] J. Hafner and J. Stopple, A heat kernel associated to Ramanujan's tau function, *Ramanujan J.* 4 (2000), 123–128.
- [5] H. Iwaniec, E. Kowalski, Analytic number theory, American Mathematical Society Colloquium Publications, vol. 53, 2004.
- [6] H. Iwaniec and P. Michel, The second moment of the symmetric square L-functions. *Ann. Acad. Sci. Fenn. Math.* **26** (2001), 465–482.
- [7] A. Juyal, B. Maji, and S. Sathyanarayana, An exact formula for a Lambert series associated to a cusp form and the Möbius function, *Ramanujan J.* (2021). https://doi.org/10.1007/s11139-020-00375-7
- [8] W. Li, L-series of Rankin type and their functional equations, Mathematische Annalen 244, 135–166 (1979).
- [9] R. Murty, Applications of Symmetric Power L-functions, Lectures on automorphic L-functions, 203–283, Fields Inst. Monogr., 20, 2004.
- [10] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, C. W. Clark, eds., NIST Handbook of Mathematical Functions, Cambridge University Press, Cambridge, 2010.
- [11] R. A. Rankin, Contributions to the theory of Ramanujan's function $\tau(n)$ and similar arithmetical functions, I, II, *Proc. Cambridge Philos. Soc.* **35** (1939), 351–372.
- [12] A. Selberg, Bemerkungen über eine Dirichletsche Reihe, die mit der Theorie der Modulformen nahe verbunden ist, Arch. Math. Naturvid. 43 (1940), 47–50.
- [13] G. Shimura, On the holomorphy of certain Dirichlet series, *Proc. London Math. Soc.*, Ser. **31** (1975), 79–98.
- [14] E. C. Titchmarsh, The Theory of the Riemann Zeta-function, Clarendon Press, Oxford, 1986.
- [15] D. Zagier, The Rankin-Selberg method for automorphic functions which are not of rapid decay, J. Fac. Sci. Univ. Tokyo IA Math. 28 (1981), 415–437.
- [16] D. Zagier, "Introduction to modular forms," From number theory to physics (Les Houches, 1989), Springer, 1992, 238–291.

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