Factorization theorems and canonical representations for generating functions of special sums

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Overview of topics

- Gentle introduction to sequence generating functions (OGFs)
- Motivate certain "factorized" forms of OGFs for special sums
- Examples and main results from publications
- Topics on the frontier of these research topics
- Questions from the committee and audience

Generating functions are essential tools in discrete mathematics

- For a sequence, $\mathcal{F} := \{f_n\}_{n \geq 0} \subset \mathbb{C}$, we define its **ordinary generating function** (OGF) to be

$$F(z) := \sum_{n \geq 0} f_n z^n.$$

- **Notation:** For $n \geq 0$, $[z^n]F(z) := f_n$ (**coefficient extraction**)
- **Good concise explanation:** *A generating function is a clothesline on which we hang up a sequence of numbers for display* (Wilf, [23])
- We can treat $F(z)$ using complex analysis or may work with it formally (e.g., disregard convergence; see [10])
- Usually only consider integer sequences (or rational ones over $\frac{f_n}{n!}$)

Focus of the thesis is on peer-reviewed publications from 2017–2021 (since enrolling at GT)

- Primary publications summarized in the thesis: [17, 20, 5, 4, 7, 6, 8]
- Publications focused on **Jabobi-type continued fractions (J-fractions)**: [11, 14, 12, 16, 15]
- Other related peer-reviewed publications: [19, 13, 18, 22, 21, 20]

Motivating series expansions for the OGFs of special sums (LGFs)

- For arithmetic functions f and g, we define their **Dirichlet convolution at n** by

$$(f * g)(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right), \text{ integers } n \geq 1.$$

- A **Lambert series generating function** (LGF) is an OGF that allows us to generate multiplicative functions expressed via divisor sums of the form $(f * \mathbb{1})(n)$:

$$L_f(q) := \sum_{n \geq 1} \frac{f(n) q^n}{1 - q^n} = \sum_{m \geq 1} (f * \mathbb{1})(m) q^m.$$

- OGF relation: $F(q) = L_{f*\mu}(q)$ (for $\mu(n)$ the **Möbius function**)

Examples: Some number theoretic function LGFs

$$\sum_{n\geq 1} \frac{\mu(n)q^n}{1-q^n} = q \tag{1a}$$

$$\sum_{n\geq 1} \frac{\phi(n)q^n}{1-q^n} = \frac{q}{(1-q)^2}, |q| < 1 \tag{1b}$$

$$\sum_{n\geq 1} \frac{n^\alpha q^n}{1-q^n} = \sum_{m\geq 1} \sigma_\alpha(n)q^n, \alpha \in \mathbb{R} \tag{1c}$$

$$\sum_{n\geq 1} \frac{\lambda(n)q^n}{1-q^n} = \sum_{m\geq 1} q^{m^2} \tag{1d}$$

$$\sum_{n\geq 1} \frac{\Lambda(n)q^n}{1-q^n} = \sum_{m\geq 1} \log(m)q^m. \tag{1e}$$

Definitions – Some standard notation

- **Iverson's convention**: The symbol $[\mathrm{cond}]_\delta \in \{0, 1\}$ is one if and only if cond is true (cf. [2])
- The **greatest common divisor (GCD)**: $(n, m) \equiv \gcd(n, m)$
- The **(infinite) q-Pochhammer symbol**: $(a; q)_\infty := \prod_{m \geq 1}(1 - aq^{m-1})$
- The **(Euler) partition function**: The number of (unordered) partitions of n is $p(n) := [q^n](q; q)_\infty^{-1}$, with $p(0) := 1$, for integers $n \geq 0$
- The sequences $s_e(n, k)$ (and $s_o(n, k)$) denote the the number of k's in all partitions of n into an even (and odd, respectively) number of distinct parts for integers $1 \leq k \leq n$

Factorization theorems for LGF series

- Overlapping ideas in publications by MDS and M. Merca: [17, 3]
- Coauthored work over the next few years: [5, 4, 7, 6]
- Key idea is to re-write the LGF series as in the following LHS expansion:

$$\sum_{n\geq 1} \frac{f(n)q^n}{1-q^n} = \frac{1}{(q;q)_\infty} \times \sum_{n\geq 1} \left(\sum_{k=1}^{n} s_{n,k} f(k)\right) q^n \qquad \textbf{(LGF-FT)}$$

- For any $N \geq 1$, the matrices $(s_{n,k})_{1\leq n,k\leq N}$ are lower triangular with ones on the diagonal (and hence invertible)

Factorization theorems for LGF series (cont'd)

- We prove: $s_{n,k} = s_o(n,k) - s_e(n,k)$
- We prove: $s_{n,k}^{-1} = \sum_{d|n} p(d-k)\mu\left(\frac{n}{d}\right)$
- **Interpretations:** Interesting new ties between OGFs for multiplicative functions and the more additive theory of partitions
- **Key questions to keep in mind for later:**
  - Why was the factor of $(q;q)_\infty^{-1}$ in the OGF factorization in equation **(LGF-FT)** so natural?
  - Collecting common denominators of the partial sums of the RHS yields this OGF factor in the limiting case (algebraic rationale for the choice)
  - Is there a deeper underlying principle to explain why this factorized form should be the most natural?

LGF factorization theorems – Other results

- Let the **(normalized) average order** of the function f be defined by
$$\Sigma_f(x) := \sum_{1\leq n\leq x} f(n), \text{ for } x \geq 1.$$
- Let $a_f(n) := \sum_{1\leq k\leq n} s_{n,k} f(k)$ where the lower triangular $s_{n,k}$ are the same as in **(LGF-FT)**
- **Theorem:** For all $n \geq 1$
$$\Sigma_{f*\mathbb{1}}(n+1) = \sum_{b=\pm 1} \sum_{k=1}^{\left\lfloor \frac{\sqrt{24n+1}-b}{6} \right\rfloor + 1} (-1)^{k+1} \Sigma_{f*\mathbb{1}}\left(n+1-\frac{k(3k+b)}{2}\right) + \sum_{1\leq k\leq n} a_f(k+1).$$

LGFs – Other results (cont'd)

- **Notation:** $\sigma(n) \equiv \sigma_1(n) := \sum_{d|n} d$ is the **(ordinary) sum-of-divisors function**
- **Corollary:** For all $x \geq 1$

$$\Sigma_\sigma(x+1) = \sum_{s=\pm 1} \left(\sum_{0 \leq n \leq x} \sum_{k=1}^{\left\lfloor \frac{\sqrt{24n+25}-s}{6} \right\rfloor} (-1)^{k+1} \frac{k(3k+s)}{2} p(x-n) \right)$$

- **Compare to classical bounds:** $\Sigma_\sigma(x) = \frac{\pi^2 x^2}{12} \times \left(1 + O\left(\frac{\log x}{x}\right)\right)$
- **Improvement (Walfisz, 1964):**

$$\Sigma_\sigma(x) = \frac{\pi^2 x^2}{12} \times \left(1 + O\left(\frac{(\log x)^{\frac{2}{3}}}{x}\right)\right)$$

Branching out from LGFs I

- **Motivation:** We find that

$$\Sigma_f(x) = \sum_{\substack{d|x \\ d>1}} f(d) + \sum_{\substack{1\leq d\leq x \\ (d,x)=1}} f(d) + \sum_{\substack{1<d\leq x \\ 1<(d,x)<x}} f(d), \text{ for } x \geq 1,$$

- The first summations are generated by LGFs as

$$\sum_{\substack{d|x \\ d>1}} f(d) = [q^n]L_f(q) - f(1), \text{ for any } n \geq 1.$$

- What about the next two sum terms?

Branching out from LGFs II

- Generalized factorization theorems for GCD-type sums in [8]
- For integers $1 \leq k \leq x$, we define

$$T_f(x) = \sum_{\substack{d=1 \\ (d,x)=1}}^{x} f(d), \qquad \text{(Type I Sums)}$$

$$L_{f,g,k}(x) = \sum_{d|(k,x)} f(d) g\left(\frac{x}{d}\right). \qquad \text{(Type II Sums)}$$

- The factorization theorems considered are now of the form

$$T_f(x) = [q^x]\left(\frac{1}{(q;q)_\infty} \times \sum_{n\geq 2}\sum_{k=1}^{n} t_{n,k} f(k) q^n + f(1)q\right),$$

$$g(x) = [q^x]\left(\frac{1}{(q;q)_\infty} \times \sum_{n\geq 2}\sum_{k=1}^{n} u_{n,k}(f,w)\left(\sum_{m=1}^{k} L_{f,g,m}(k) w^m\right) q^n\right),$$

- Here, $w \in \mathbb{C} \setminus \{0\}$ is a non-zero indeterminate parameter inserted to force invertibility of the sequences $u_{n,k}(f,w)$ above

Examples I: What other types of sums might we want to generate?

Example ($\mathcal{A}$-Set convolutions, ACVL)

For each $n \geq 1$, let $A(n) \subseteq \{1 \leq d \leq n : d|n\}$ be a subset of the divisors of n. We say that n is A-primitive if $A(n) \equiv \{1, n\}$. Let the set of A-primitive positive integers be denoted by

$$\mathcal{A} := \{n \geq 1 : n \text{ is } A\text{-primitive}\}.$$

Then we may consider the following invertible convolutions:

$$S_{1,\mathcal{A}}(f, g; n) := \sum_{\substack{d|n \\ d \in \mathcal{A}}} f(d) g\left(\frac{n}{d}\right),$$

$$S_{2,\mathcal{A}}(f, g; n) := \sum_{\substack{d|n \\ d, \frac{n}{d} \in \mathcal{A}}} f(d) g\left(\frac{n}{d}\right).$$

Examples II: What other types of sums might we want to generate?

Example (Unitary convolutions, UCVL)

The *unitary convolution* of f and g at integers $n \geq 1$ is defined by

$$(f \odot g)(n) := \sum_{\substack{d \mid n \\ \left(d, \frac{n}{d}\right)=1}} f(d) g\left(\frac{n}{d}\right).$$

Examples III: What other types of sums might we want to generate?

Example ($\mathcal{D}$-Kernel convolutions, DCVL)

Suppose that $\mathcal{D} : \mathbb{Z}^+ \times \mathbb{Z}^+ \rightarrow \mathbb{C}$ is an **invertible** and **lower triangular kernel function**: I.e., $\mathcal{D}(n, k) = 0$ whenever $k > n$ and $\mathcal{D}(n, n) \neq 0$ for all $n \geq 1$. We want to study a generalized class of $\mathcal{D}$-convolution type sums of the form

$$(f \boxdot_{\mathcal{D}} g)(n) := \sum_{1 \leq k \leq n} f(k) g(n+1-k) \mathcal{D}(n, k), \text{ for integers } n \geq 1.$$

Definitions of generalized factorization theorems

- **Option 1:** For $n \geq 1$ and multiplier OGFs such that $\mathcal{C}(0) \neq 0$

$$\sum_{\substack{k \in A_n \\ A_n \subseteq [1,n) \bigcup \{n\}}} f(k) := [q^n] \left(\frac{1}{\mathcal{C}(q)} \times \sum_{\substack{n \geq 1 \\ 1 \leq k \leq n}} v_{n,k}(\mathcal{A}, \mathcal{C}) f(k) q^n \right)$$

- **Option 2:** For $n \geq 1$, weights $\mathcal{T}_{j,j} \neq 0$ for all $j \geq 1$, and multiplier OGFs such that $\mathcal{C}(0) \neq 0$

$$\sum_{1 \leq k \leq n} \mathcal{T}_{n,k} f(k) := [q^n] \left(\frac{1}{\mathcal{C}(q)} \times \sum_{\substack{n \geq 1 \\ 1 \leq k \leq n}} u_{n,k}(\mathcal{T}, \mathcal{C}) f(k) q^n \right)$$

- These generalized sum types allow us to consider (weighted) forms of the special case LGF and GCD-type OGF factorizations we have seen so far **(Explain, clarify quantifiers)**

A good open question to ask (LGF case)

- **Recall:** The original factorization theorem expansions for the LGF case are of the form

$$L_f(q) := \frac{1}{(q;q)_\infty} \times \sum_{n\geq 1} \left(\sum_{k=1}^{n} s_{n,k} f(k)\right) q^n.$$

- We proved: $s_{n,k} = s_{n,k} = s_o(n,k) - s_e(n,k)$
- We proved: $s_{n,k}^{-1} = \sum_{d|n} p(d-k)\mu\left(\frac{n}{d}\right)$
- The matrix entries and inverses are expressed in terms of partition theoretic functions (cf. [1])
- This gives new connections between functions in multiplicative number theory and the theory of partitions

Generalizing "canonically best" OGF factorizations (cont'd)

- The view of the OGF, $\mathcal{C}(q) := (q;q)_\infty$, in the LGF case being "*optimal*" (*or somehow encoding the most meaningful hidden information about this sum type*) is inherently **qualitative**
- How can we precisely define a corresponding **quantitative** metric with which we can express the intuition from the special case?

Generalizing "canonically best" OGF factorizations (cont'd)

- **Idea (first approximation):** For $1 \times N$ vectors $\vec{a} := (a_1, \ldots, a_n)$ and $\vec{b} := (b_1, \ldots, b_N)$, one standard way to evaluate how well matched these vectors are is given by the **(normalized) correlation statistic**

$$\operatorname{PCorr}\left(\vec{a}, \vec{b}\right) := \frac{\frac{1}{N} \times \sum\limits_{1 \leq j \leq N} a_j b_j}{\sqrt{\left(\sum\limits_{1 \leq i \leq N} a_i^2\right)\left(\sum\limits_{1 \leq j \leq N} b_j^2\right)}} \in [-1, 1] \quad (\textbf{PC-STAT})$$

- **Idea (refinement):** Use the correlation statistic in **(PC-STAT)** with infinite sequences in place of the N-vectors; These sequences should depend on (reflect key features of) the series coefficients of $\mathcal{C}(q)^{\pm 1}$ and $\mathcal{D}(n, k)$ (or $\mathcal{D}^{-1}(n, k)$) – *Precise definitions in the thesis manuscript*

Visualizing the LGF case – High-level procedure I

- Can we visualize the notion of an optimally correlated OGF,
$$\mathcal{C}(q) := 1 + \sum_{n \geq 1} c_n(\mathcal{C}) q^n \in \mathbb{Z}[[q]],$$
to see if taking $\mathcal{C}(q) := (q;q)_\infty$ is really the best? **(YES!)**
- **Notation:** Let the set of **(unsigned) pentagonal numbers** be defined as follows: $\mathcal{N}_{\mathrm{Pent}} := \{G_j : j \geq 0\}$ where
$$G_j := \frac{1}{2} \left\lceil \frac{j}{2} \right\rceil \left\lceil \frac{3j+1}{2} \right\rceil \mapsto \{0, 1, 2, 5, 7, 12, 15, 22, \ldots\}$$
- For $1 \leq k \leq n$, let the correlation component
$$f_{\mathrm{LGF}}[\mathcal{C}](n,k) := \frac{\frac{1}{n} \times [k \in \mathcal{N}_{\mathrm{Pent}}]_\delta \times \mu^2\left(\frac{n}{k}\right) [k|n]_\delta}{\sqrt{2^{\omega(n)} \times \sum_{0 \leq k \leq n} [k \in \mathcal{N}_{\mathrm{Pent}}]_\delta}}.$$

Visualizing the LGF case – High-level procedure II

- Then for $N \gg 1$ (as large as possible, computationally), form the **correlation matrix**

$$\overleftrightarrow{\mathrm{CorrM}}(N) := \left(f_{\mathrm{LGF}}[\mathcal{C}](n,k)\,[k \leq n]_{\delta}\right)_{1 \leq n,k \leq N}.$$

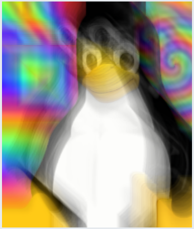
- Pick a clear target image (**Tux** penguin, below), and partition its pixels into a $N \times N$ grid
- Convolve the N-sized pixels with the prospective correlation matrix, $\overleftrightarrow{\mathrm{CorrM}}(N)$. We should observe the following qualitative trends:
  - Less distortion (e.g., clearer results) indicates good (high) correlation
  - More distortion (e.g., blurrier results) indicates poor (low) correlation

Visualizing the LGF case (cont'd)

(a) *Original image.*

(b) $\mathcal{C}(q) := (q;q)_\infty$.

(c) $\mathcal{C}(q) := (q;q)_\infty^{-1}$.



(d) $\mathcal{C}(q) := (q^2;q^5)_\infty$.

(e) $\mathcal{C}(q) := (1-q)^{-\frac{3}{2}}$.

(f) $\mathcal{C}(q) := (1-q)^{-1}$.

Generalizing "canonically best" OGF factorizations (cont'd)

- Based on inspection, the visual **Tux** examples for the LGF case, seem to conform to the "*ideal*" case expactation:
  That is, we cannot do better than to choose $\mathcal{C}(q) := (q;q)_\infty$ (as we had defined to be qualitatively optimal)
- For more general convolution sum types, we seek to **maximize** (**minimize**) the series

$$\operatorname{Corr}(\mathcal{C},\mathcal{D}) := \sum_{n\geq 1} \frac{\frac{1}{n} \times \sum\limits_{k=1}^{n} |c_k(\mathcal{C})\mathcal{D}^{-1}(n,k)|}{\sqrt{\left(\sum\limits_{k=1}^{n} c_k(\mathcal{C})^2\right)\left(\sum\limits_{k=1}^{n} \mathcal{D}^{-1}(n,k)^2\right)}}$$

Generalizing "canonically best" OGF factorizations (cont'd)

- How best to approach finding the optimal OGF? What if we restrict to integer coefficients with $\mathcal{C}(0) := 1$?
- This is an open topic; Some preliminary conjectures and discussion are given in the last section of the thesis.
- **Historical notes on correlation statistic tactics:**
  - There is literature documenting and motivating the use of statistical analysis to study number theoretic objects
  - Montgomery: Pair correlation to study the non-trivial zeros of $\zeta(s)$
  - Hejal, Rudnick, Sarnak and Odlyzko, respectively, built on HM's work to apply statistical analysis (correlation statisitics) to L-functions, the Gaussian Unitary Ensemble (GUE) and in applications to random matrix theory
  - See the survey in [9]

Concluding remarks

The End

Questions?

Comments?

Feedback?

Thank you for your time!

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Extra slides

Extra slides and references

LGFs – Other results (cont'd)

- The function h^{-1} is called the **Dirichlet inverse of h** if $h * h^{-1} = h^{-1} * h = \varepsilon$ where $\varepsilon(n) = \delta_{n,1}$ is the multiplicative identity with respect to Dirichlet convolution
- The function h^{-1} exists and is unique iff $h(1) \neq 1$.
- When h^{-1} exists, it is computed recursively via the formula

$$h^{-1}(n) = \begin{cases} \frac{1}{h(1)}, & n = 1; \\ -\frac{1}{h(1)} \times \sum\limits_{\substack{d|n \\ d>1}} h(d) h^{-1}\left(\frac{n}{d}\right), & n \geq 2. \end{cases}$$

LGFs – Other results (cont'd)

n	$h^{-1}(n)$	**n**	$h^{-1}(n)$	**n**	$h^{-1}(n)$
1	$\frac{1}{h(1)}$	**4**	$-\frac{h(1)h(4)-h(2)^2}{h(1)^3}$	**7**	$-\frac{h(7)}{h(1)^2}$
2	$-\frac{h(2)}{h(1)^2}$	**5**	$-\frac{h(5)}{h(1)^2}$	**8**	$-\frac{h(2)^3-2h(1)h(4)h(2)+h(1)^2h(8)}{h(1)^4}$
3	$-\frac{h(3)}{h(1)^2}$	**6**	$-\frac{h(1)h(6)-2h(2)h(3)}{h(1)^3}$	**9**	$-\frac{h(1)h(9)-h(3)^2}{h(1)^3}$

LGFs – Other results (cont'd)

- For fixed f, g and any OGF $\mathcal{C}(q)$ with $\mathcal{C}(0) \neq 0$, we define

$$\sum_{n \geq 1} \frac{f(n)q^n}{1-q^n} = \frac{1}{\mathcal{C}(q)} \times \sum_{n \geq 1} \left(\sum_{k=1}^{n} s_{n,k}[\mathcal{C}] f(k) \right) q^n, \qquad \text{(i)}$$

and let

$$\sum_{n \geq 1} \frac{(f * g)(n)q^n}{1-q^n} = \frac{1}{\mathcal{C}(q)} \times \sum_{n \geq 1} \left(\sum_{k=1}^{n} \widetilde{s}_{n,k}[\mathcal{C}](g) f(k) \right) q^n. \qquad \text{(ii)}$$

- We can prove: $\widetilde{s}_{n,k}[\mathcal{C}](g) = \sum_{j=1}^{n} s_{n,kj}[\mathcal{C}] g(j)$

LGFs – Other results (cont'd)

Table of the inverse matrices, $\widetilde{s}_{n,k}^{-1}[\mathcal{C}](g)$:

	1	**2**	**3**	**4**
1	1	0	0	0
2	$-g(2)$	1	0	0
3	$1-g(3)$	1	1	0
4	$g(2)^2-g(4)+2$	$1-g(2)$	1	1
5	$4-g(5)$	3	2	1
6	$2g(3)g(2)-g(2)-g(6)+5$	$-g(2)-g(3)+3$	$2-g(2)$	2
7	$10-g(7)$	7	5	3
8	$-g(2)^3+2g(4)g(2)-2g(2)-g(8)+12$	$g(2)^2-g(2)-g(4)+9$	$6-g(2)$	$4-g(2)$
9	$g(3)^2-g(3)-g(9)+20$	$14-g(3)$	$10-g(3)$	7
10	$2g(5)g(2)-4g(2)-g(10)+25$	$-3g(2)-g(5)+18$	$13-2g(2)$	$10-g(2)$

(Special case where $g(1) := 1$ for simplicity.)

Factorization theorems for LGFs – Variants I

- **Notation:** When $C(q) = (q;q)_\infty$ we write $s_{n,k}^{-1}[\mathcal{C}](g) \equiv s_{n,k}^{-1}(g)$
- **Notation:** Let the function $p_k(n) := p(n-k)$
- For $n \geq 1$, let

$$f^{-1}(n) := \left(D_{n,f} + \frac{\varepsilon}{f(1)}\right)(n).$$

  (The function $D_{n,f}(n)$ can be defined recursively by partial sums of multiple convolutions of f with itself.)
- **Theorem:** We can prove that

$$\sum_{d|n} s_{n,k}^{-1}(g) = p_k(n) + (p_k * D_{n,g})(n),$$

Factorization theorems for LGFs – Variants II

- We also considered factorization theorems for Hadamard products:

$$\sum_{d|n} a_{\mathrm{fg}}(d) := \underbrace{\left(\sum_{d|n} f_d\right) \times \left(\sum_{d|n} g_d\right)}_{:=\mathrm{fg}(n)}.$$

where

$$\sum_{n\geq 1} \frac{a_{\mathrm{fg}}(n) q^n}{1-q^n} = \frac{1}{(q;q)_\infty} \times \sum_{n\geq 1} \sum_{k=1}^{n} h_{n,k}(f) g_k q^n,$$

Factorization theorems for LGFs – Variants III

- **Notation:** Let $\widetilde{f}(n) := \sum_{d|n} f_d$
- We prove:

$$h_{n,k}(f) = \widetilde{f}(n)\,[k|n]_\delta + \sum_{j=1}^{\left\lfloor \frac{\sqrt{24(n-k)+1}-b}{6} \right\rfloor} (-1)^j \widetilde{f}\left(n - \frac{j(3j+b)}{2}\right) \left[k|n - \frac{j(3j+b)}{2}\right]_\delta$$

- We prove: $h_{n,k}^{-1}(f) = \sum_{d|n} \frac{p(d-k)}{\widetilde{f}(d)} \mu\left(\frac{n}{d}\right)$

Factorization theorems for LGFs – Variants IV

- **Corollaries:** We have so-termed "*exotic*" sums of the form

$$\phi(n) = \sum_{k=1}^{n} \sum_{d|n} \frac{p(d-k)}{d} \mu\left(\frac{n}{d}\right) \left[k^2 + \sum_{b=\pm 1} + \sum_{j=1}^{\left\lfloor \frac{\sqrt{24k-23}-b}{6} \right\rfloor} (-1)^j \left(k - \frac{j(3j+b)}{2}\right)^2\right]$$

$$n^s = \sum_{k=1}^{n} \sum_{d|n} \frac{p(d-k)}{\sigma_t(d)} \mu\left(\frac{n}{d}\right) \left[\sigma_t(k)\sigma_s(k) + \sum_{b=\pm 1} \sum_{j=1}^{\left\lfloor \frac{\sqrt{24k+1}-b}{6} \right\rfloor} (-1)^j \sigma_t\left(k - \frac{j(3j+b)}{2}\right) \sigma_s\left(k - \frac{j(3j+b)}{2}\right)\right].$$