Section 5.5: Integration by substitution

Math 1552 lecture slides adapted from the course materials
By Klara Grodzinsky (GA Tech, *School of Mathematics*, Summer 2021)

Today’s Learning Goals

- Evaluate integrals using the substitution (usub) method
- Understand how to choose u
- Understand which functions can be evaluated with the substitution method
- The substitution method is a *change of variable* in the integral that simplifies the integrand into **f(u) du** for a function **f** we recognize

Functions we already know how to integrate directly:

Recall the antiderivatives of the following functions we reviewed last week:

$$x^n, \sin(ax), \cos(ax)$$

$$\csc(ax)\cot(ax)$$

$$\sec(ax)\tan(ax)$$

$$\sec^2(ax), \csc^2(ax)$$

$$e^{ax}, b^{ax}$$

$$\frac{1}{1+(ax)^2}, \frac{1}{\sqrt{1-(ax)^2}}$$

Method of u-substitution

This method is the reverse of the chain rule for derivatives:

Let F be an antiderivative of f. Let $u = g(x)$.

$$\int f(g(x)) \cdot g'(x)dx = \int f(u)du = F(u) + C$$

In other words :

$$\int f(stuff) \cdot (stuff)'dx = F(stuff) + C$$

u-substitution with Definite Integrals

To evaluate $\int_a^b f(g(x))g'(x)dx$,

set $u = g(x)$ and *change the limits of integration* to match the new variable:

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Example 1.1: Evaluate. $\int \frac{\cos\left(\sqrt{t}\right)}{\sqrt{t}\sin\left(\sqrt{t}\right)}dt$

Example 1.2: Evaluate. $\int \frac{dx}{x(\ln x)^3}$

Example 1.3: Evaluate. $\int w\sqrt{1+w}dw$

Example 2: Evaluate the integral.

$$\int (\sin 6x) e^{\cos 6x}\, dx$$

(A) $\frac{1}{6} e^{\cos 6x} + C$

(B) $-\frac{1}{6} e^{\cos 6x} + C$

(C) $\frac{1}{6} (\cos 6x) e^{\cos 6x} + C$

(D) $\frac{1}{2} \left(e^{\cos 6x}\right)^2 + C$

Example 3.2:

Evaluate the following indefinite integral: $\int \tan(x)dx$

Example 3.1:

Evaluate the following indefinite integral: $\int \sec(x)dx$

Hint:

Take

$$u = \sec x + \tan x$$

to get that

$$\sec x = \frac{u'}{u}$$

(logarithmic derivative)

Additional Trig Formulas (know how to derive these):

$$\int \tan(u)\,du = \ln\left|\sec u\right| + C$$

$$\int \sec(u)\,du = \ln\left|\sec u + \tan u\right| + C$$

$$\int \cot(u)\,du = \ln\left|\sin u\right| + C$$

$$\int \csc(u)\,du = -\ln\left|\csc u + \cot u\right| + C$$

Extra problems (limits of integration)

Evaluate the following indefinite integral: $\displaystyle\int_0^{\sqrt{\frac{\pi}{4}}} x\cos(x^2)dx$

Challenge problem (foreshadowing trig subs – later)

Hints:

1. See that

$$\cos(u) = \sqrt{1-\sin^2(u)}, u \geq 0$$

2. Write

$$x = \sin(u),$$
$$dx = \cos(u)du$$

1. Use the identity

$$\cos^2(u) = \frac{1}{2}(1+\cos(2u))$$

Evaluate the following indefinite integral: $\int_0^1 \sqrt{1-x^2}dx$

Section 5.6: Area between two curves

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Today's Learning Goals

- Understand what is meant graphically by integrating the difference between two functions (*solve for intersection points between the two curves on the interval*)
- Set up an integral to find the total area bounded between two curves
- Evaluate numerically the area bounded between two curves
- Be able to express the integration in terms of either x or y, depending on the function(s)

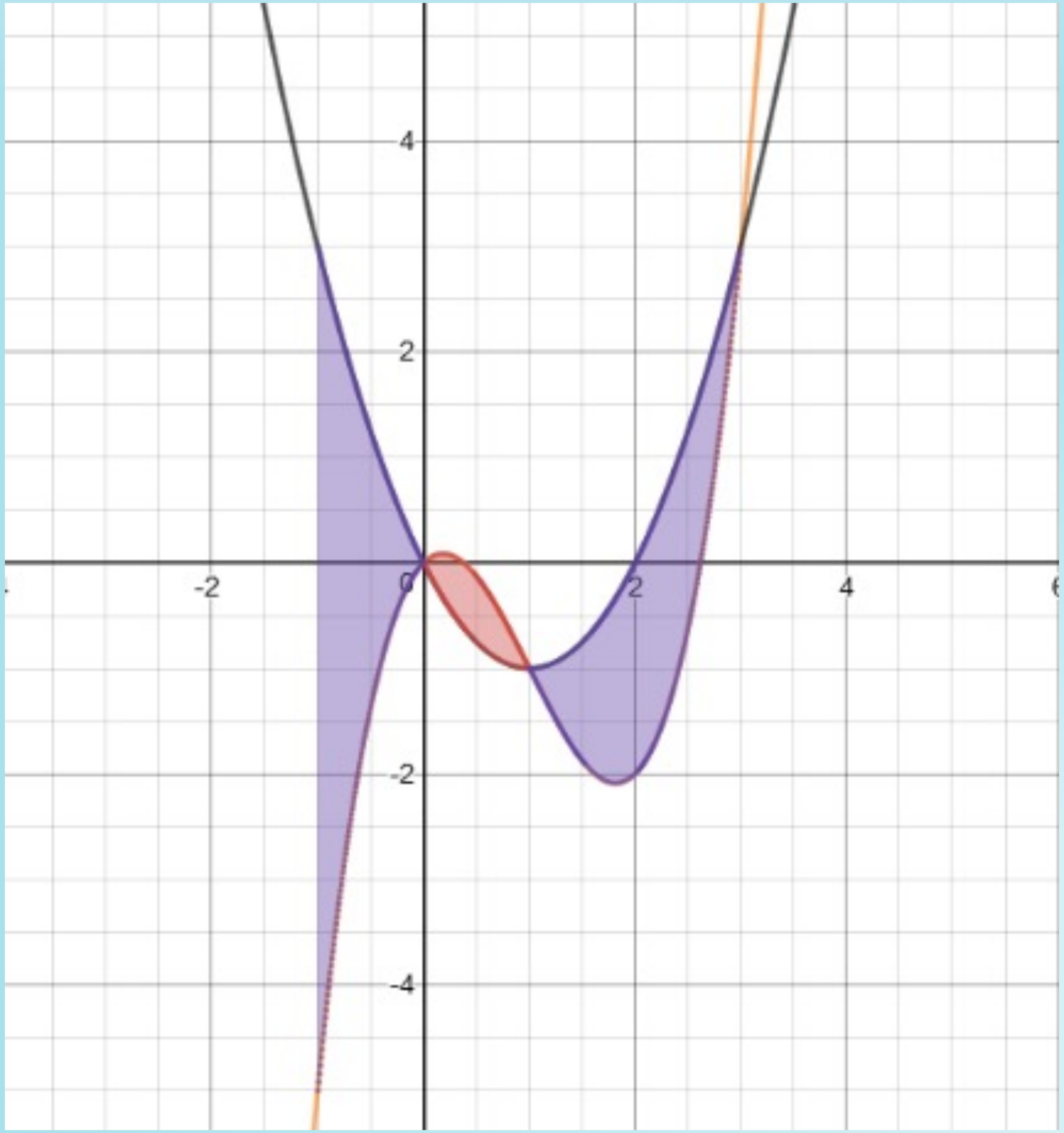
Area Between Two Curves

- To find the area between two curves, written as functions of *x*:

$$A=\int_a^b |f(x)-g(x)|\,dx=\int_a^b (top-bottom)\,dx$$

- To find the area between two curves, written as functions of *y*:

$$A=\int_a^b |f(y)-g(y)|\,dy=\int_a^b (right-left)\,dy$$



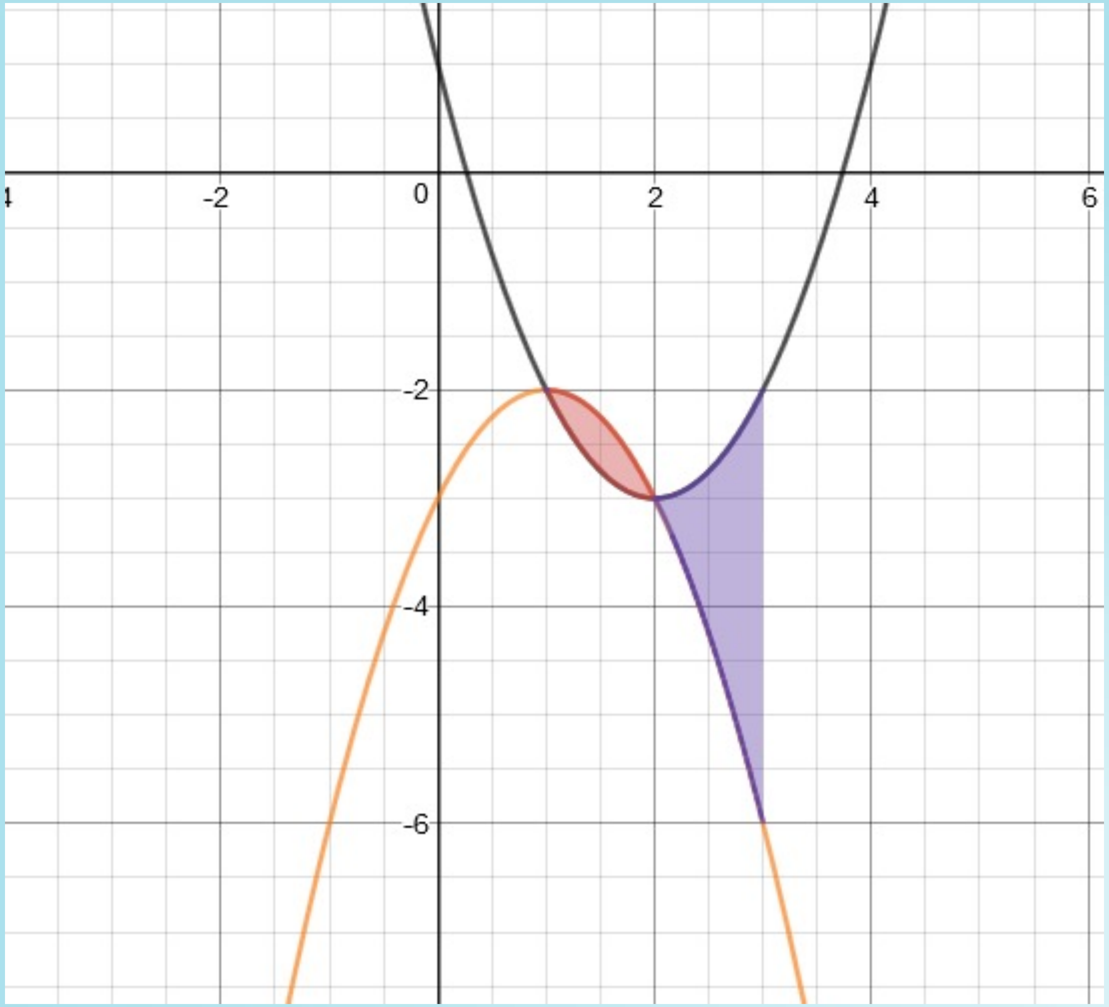
4
2
-2
-4
-2
0
2
4

Steps to Evaluating Area

1. Where do the curves intersect? Break up the interval $[a,b]$ into sub-intervals based on points of intersection.
2. For each subinterval, which function is bigger?
3. Integrate *top-bottom* or *right-left* on each subinterval.

Example 1:

Find the area bounded by the curves $y=-x^2+2x-3$ and $y=x^2-4x+1$ and the lines $x=1$ and $x=3$.

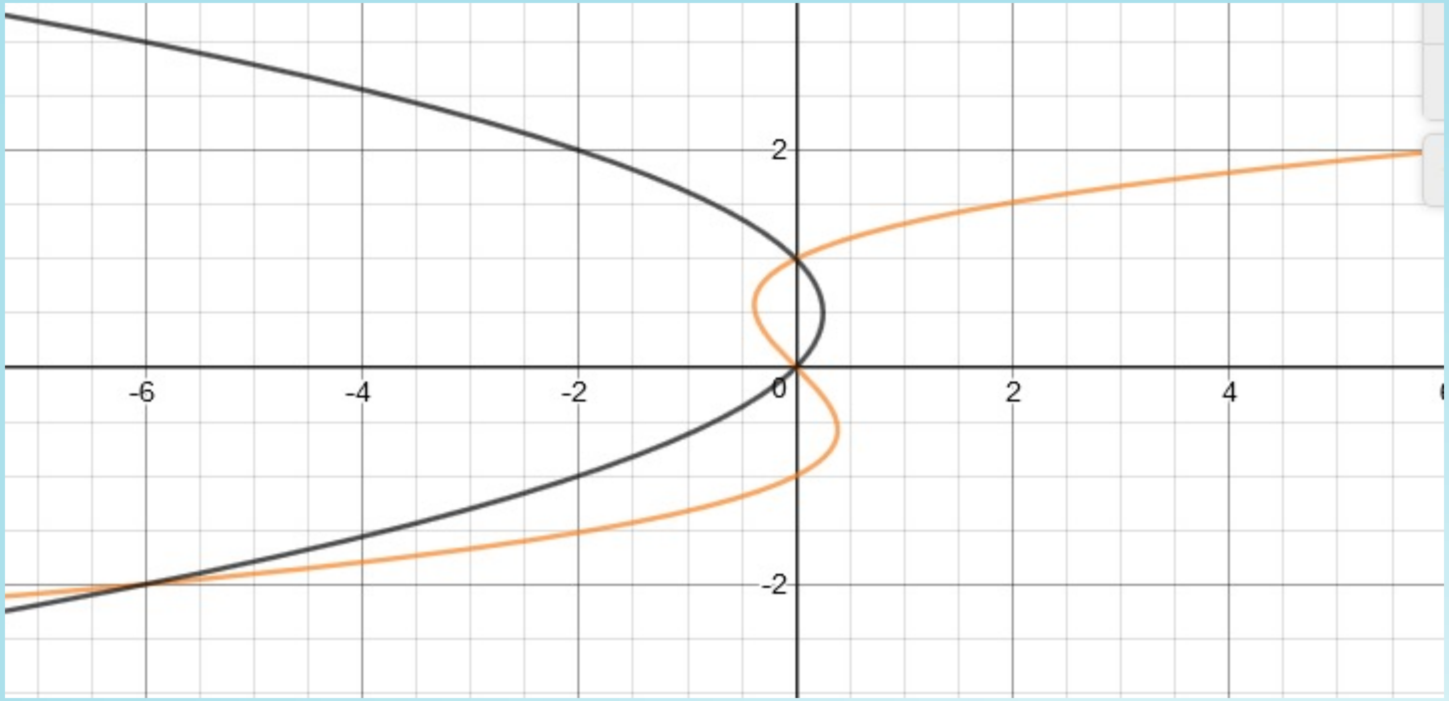


-4
-2
0
2
4
6
-2
-4
-6

Example 2:

Find the area of the region bounded by

$$x + y - y^3 = 0 \text{ and } x - y + y^2 = 0.$$



Example 3:

Find the area bounded by the curves

$$x = y^2 \text{ and } x = \sqrt{y}.$$

A. 0

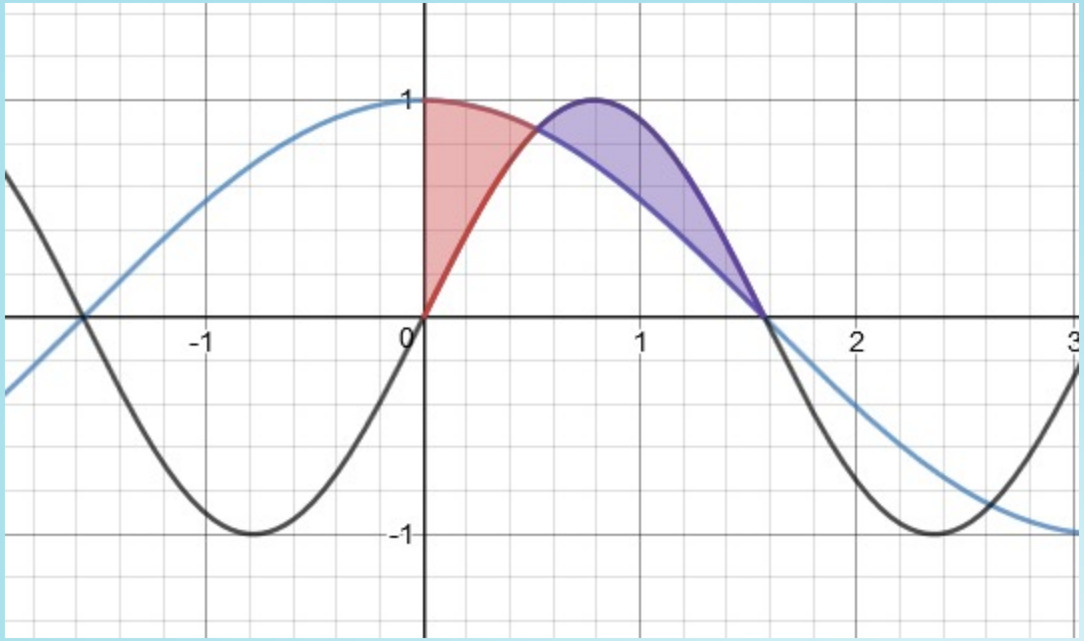
B. 1/3

C. 2/3

D. 1

Example 4:

Find the area of the region bounded by the curves $y = \cos x$ and $y = \sin(2x)$ on $\left[0, \frac{\pi}{2}\right]$.



Section 8.2: Integration by parts

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Review Question: Evaluate the integral.

$$\int x\left(\frac{1}{3}\right)^{x^2} dx$$

$$(A)\ -\frac{1}{2\ln 3}\left(\frac{1}{3}\right)^{x^2} + C$$

$$(B)\ \frac{1}{2(x^2+1)}\left(\frac{1}{3}\right)^{x^2+1} + C$$

$$(C)\ -\frac{1}{\ln 3}\left(\frac{1}{3}\right)^{x^2} + C$$

$$(D)\ \frac{\ln 3}{2}\cdot\left(\frac{1}{3}\right)^{x^2} + C$$

Learning Goals

- Identify which functions can be solved using the method of integration by parts
- Understand how to choose the values of "u" and "dv"
- Evaluate integrals using integration by parts

Formula for Integration by Parts

Integration by parts comes from the product rule for differentiation.

$$\boxed{\int u \cdot dv = uv - \int v \cdot du}$$

Differentiate *u* to obtain *du*.

Find *v* by taking an antiderivative of *dv*.

$$(fg)' = f'g + fg' \implies$$

$$f(x)g(x) = \int f'(x)g(x)dx + \int f(x)g'(x)dx$$

Rules to Apply Integration by Parts

- The original integral CANNOT be evaluated by a normal *u*-substitution alone.
- Begin by rewriting the original function as the product of two pieces, *u* and *dv*.
- We must be able to integrate *dv*!
- The new integral should be easier than the original problem. If not, try a different choice for *u* and *dv*.

When to use Integration by Parts

Use integration by parts to evaluate the integrals of:

- Inverse functions
- Logarithmic functions
- Functions that are combinations of more than one type of function (i.e., polynomials, trigonometric, exponential, logarithmic functions)
- **Note:** We can combine IBP with the methods we have learned so far (e.g., start with a u-sub and then apply IBP after simplifying)
- After practice, you should be able to spot IBP type integrals quickly

Hints about IBP techniques

- **<u>DO NOT</u>** use tables, or tabular integration methods, you have seen before in this class!
- Start with a blank slate of parameters you need to find organized like the following:

$$\left\{ \begin{array}{rr} u = & dv = \\ du = & v = \end{array} \right\}$$

- Be prepared to apply IBP more than once, e.g., to evaluate $\int x^2 e^x \, dx$
- If nothing else works, you can always take $dv = 1 \cdot dx$
- We will see many examples in the next slides

Order in which to choose *u*

Choose *u* according to the *ILATE* rule:

- **I** – Inverse Functions $\sin^{-1}(x), \cos^{-1}(x), \tan^{-1}(x)$
- **L** – Logarithmic Functions $\ln(x), \log(x), \log_b(x) \text{ for } b > 0$
- **A** – Algebraic Expressions (polynomials, rational functions, etc.) $1, x, x^2$
- **T** – Trigonometric Functions $\sin(x), \cos(x), \tan(x)$
- **E** – Exponential Functions $e^x, e^{-2x}, 3^x$

Tip: In the event of a “tie” in the *ILATE* rule, pick u to be the simplest of the two functions.

Example 1 (inverse functions):

Evaluate the integral $\int \sin^{-1}(x)dx.$

What should we choose for the value of *u* in the integral

Hint:

$$\sin(2x) = 2\sin(x)\cos(x)$$

$$\int x\sin(x)\cos(x)dx\,?$$

A. x

B. sin(x)

C. cos(x)

D. sin(x)cos(x)

Example 2:

Evaluate the integral: $\int x \sin(x) \cos(x) dx.$

Example 3: Evaluate the integral: $\int (\ln x)^2 dx$

What should we choose for the value of *u* in the integral

$$\int \sin[\ln(x)]dx\,?$$

A. sin(x)
B. ln(x)
C. sin[ln(x)]
D. dx

Example 4:

Key Idea:
We will do IBP twice and then solve a system for the original integral
(after a substitution)

Evaluate the integral: $\int \sin[\ln(x)]dx.$

Example 6:

Evaluate the integral: $\int x^4 \ln(x)dx$

Other examples of the IBP method to try:

Practice which functions to take as u and dv (u-sub first?):

$$\int u^5 e^{u^3} du$$

$$\int x\sqrt{x+1}dx$$

Other examples of the IBP method to try:

Practice which functions to take as u and dv (u-sub first?):

$$\int x^7 \sqrt{3x^4 + 5}\,dx$$

$$\int x^3 \cos(x^2)\,dx$$

Other examples of the IBP method to try:

Practice which functions to take as u and dv (u-sub first?):

$$\int x \sec^2(x)dx$$