



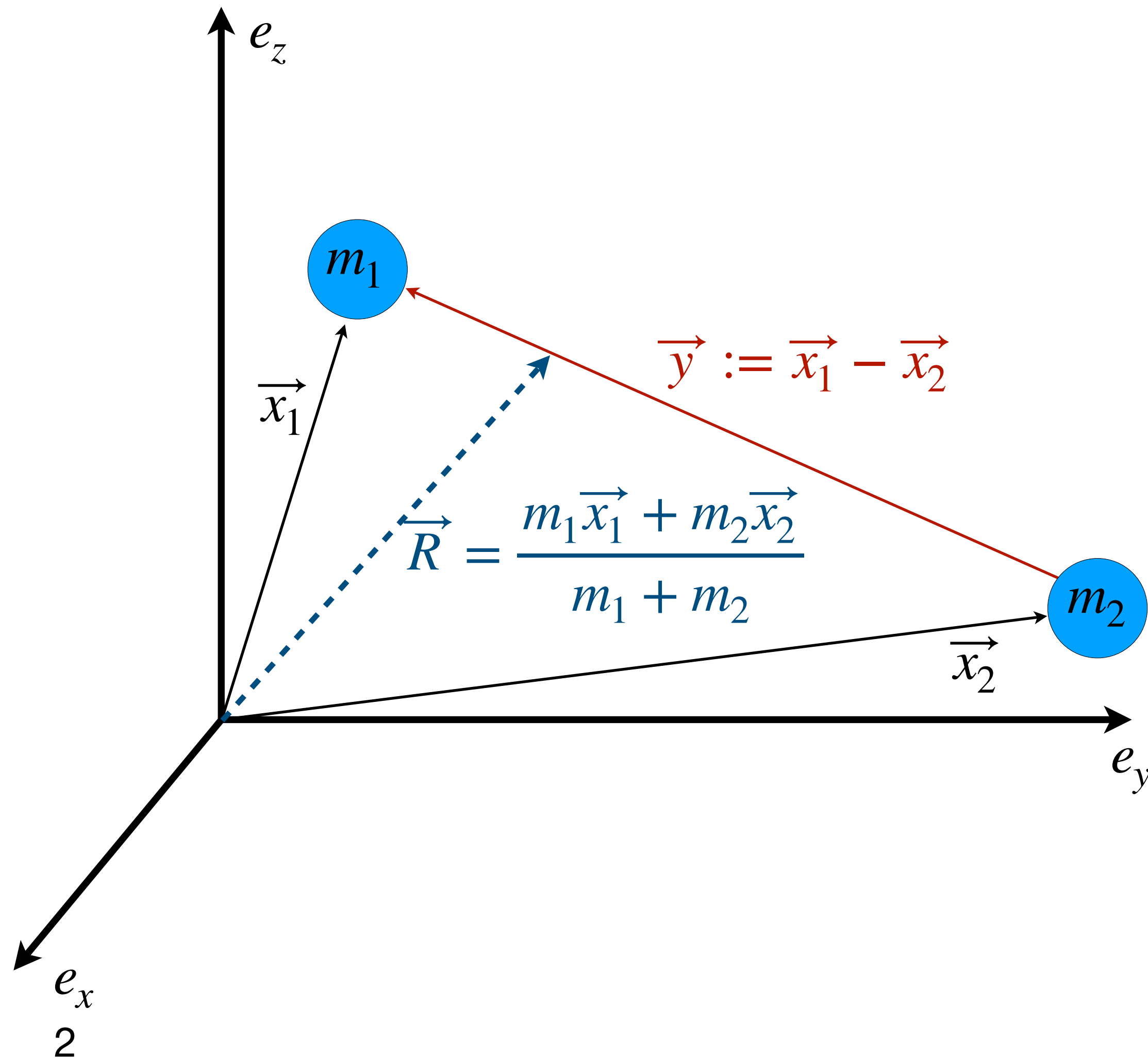
Kepler's laws

Maxim Mai

LECTURE NOTES ONLINE:
maxim-mai.github.io/talks/lecture-01.pdf

Reminder from previous lecture

Setup



Forces & equation of motion

$$\vec{F}_1 = -\vec{F}_2 = \kappa \frac{\vec{y}}{|\vec{y}|^3} \implies \mu \ddot{\vec{y}} = \kappa \frac{\vec{y}}{|\vec{y}|^3}$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2}, \quad \kappa = -Gm_1 m_2$$

Conservation laws

$$\vec{P} := (m_1 + m_2) \dot{\vec{R}} \quad \text{total momentum}$$

$$\dot{\vec{R}} := \dot{\vec{R}} - \dot{\vec{R}}_t \quad \text{center of mass motion}$$

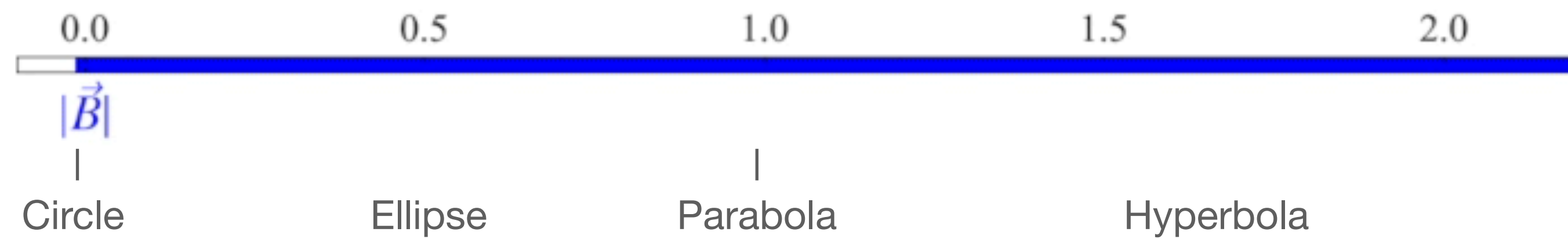
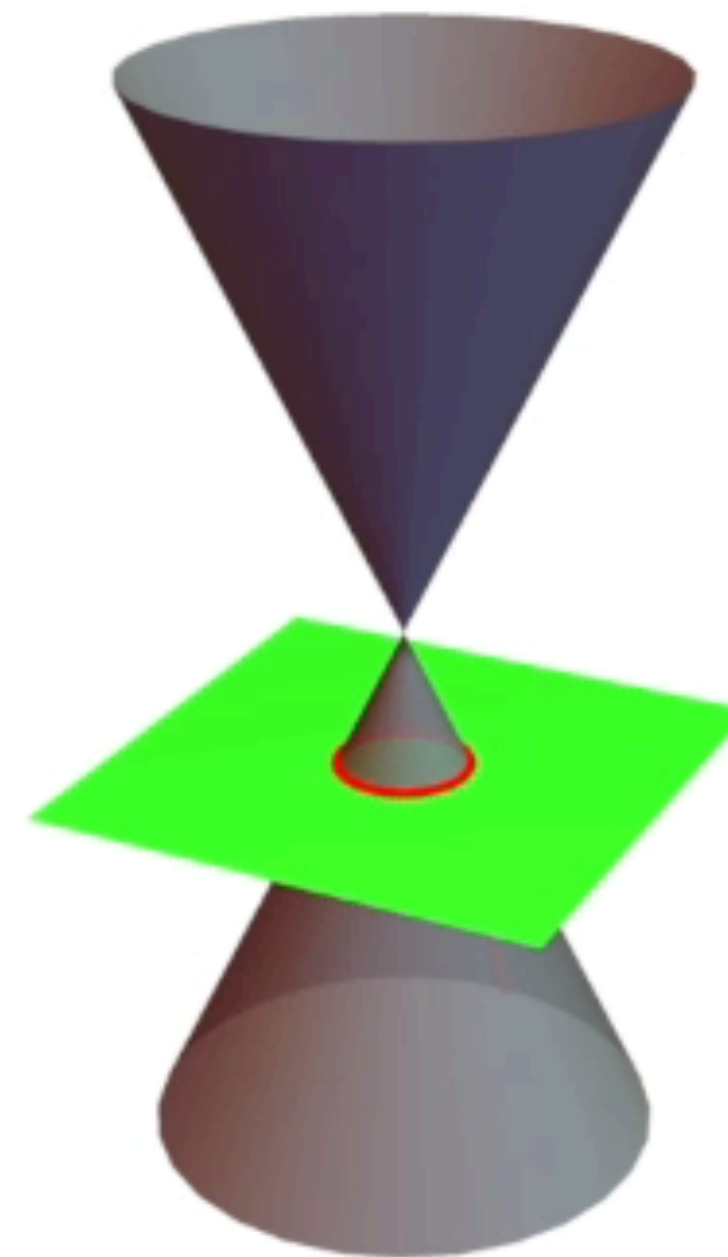
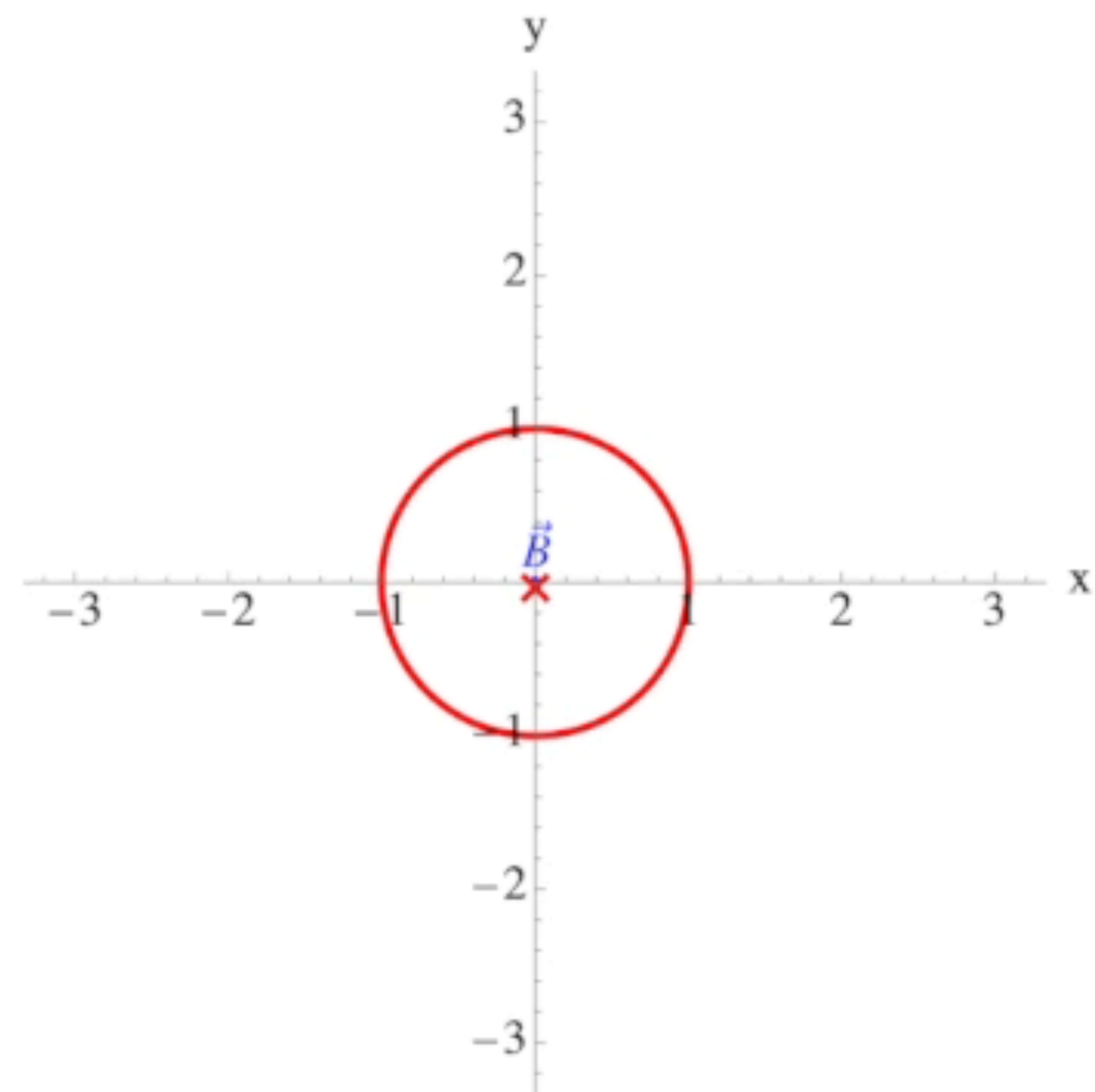
$$\vec{L}' := \mu \vec{y} \times \dot{\vec{y}} \quad \text{relative angular momentum}$$

$$H' := \frac{\mu}{2} |\dot{\vec{y}}|^2 \quad \text{inner energy}$$

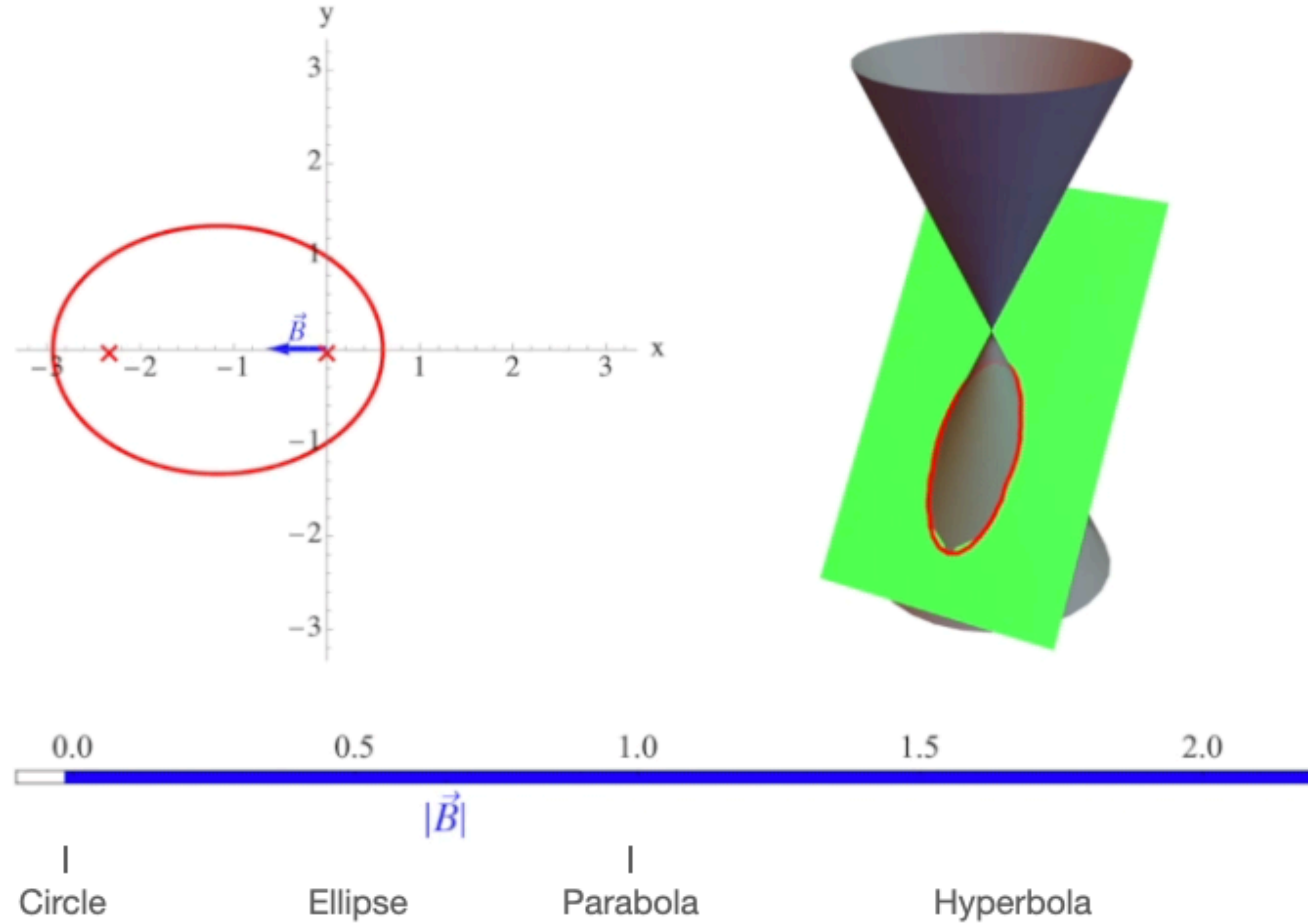
$$\vec{B} := \frac{1}{\kappa} \dot{\vec{y}} \times \vec{L}' + \frac{\vec{y}}{|\vec{y}|} \quad \text{Runge - Lenz vector}$$

$\Rightarrow 11$ independent conserved quantities

Conic section



Conic section



Reality check #1

Orbit of a star (S2) around the center of Milky Way



- 20y of data by VLT and UCLA-Keck

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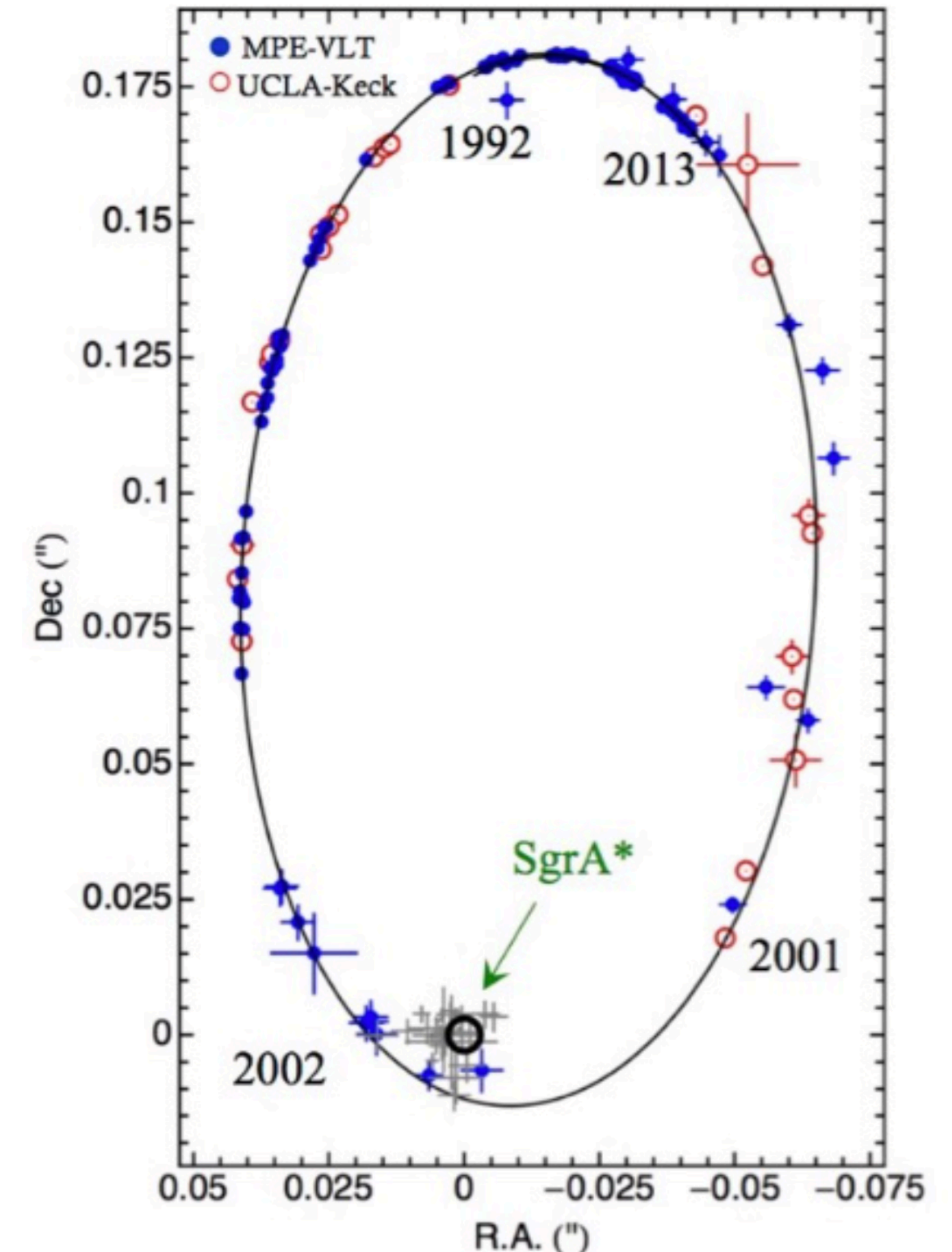


- 20y of data by VLT and UCLA-Keck

⇒ **perfectly elliptic orbit:**

$$T=15.2y, l=46\text{Deg}, e=0.87, a=0.119''$$

⇒ **conformation of a Black Hole (SgrA*) scenario**



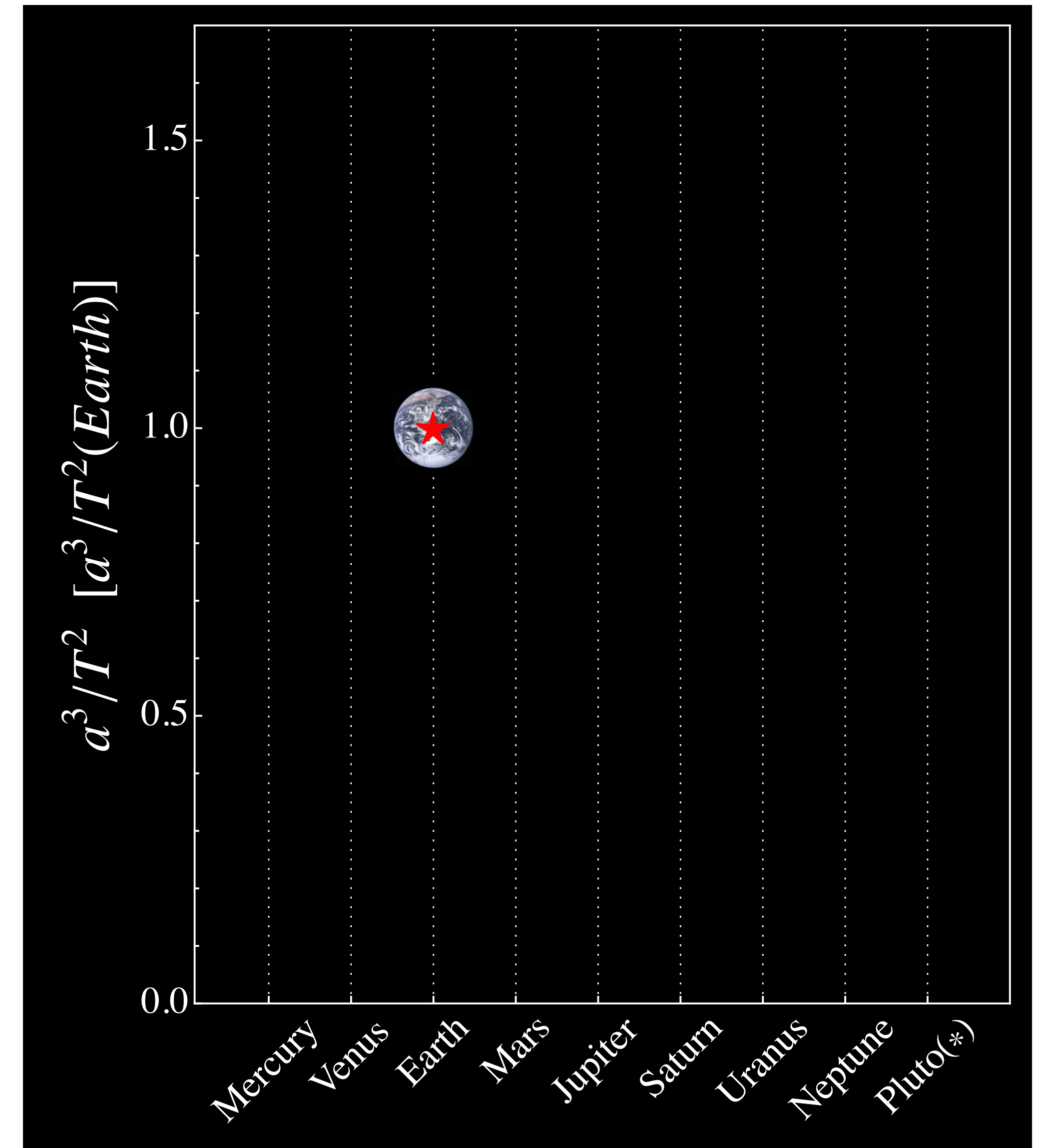
Reality check #2

3rd Kepler's law in solar system:

- 8 planets orbiting Sun ($M_{\text{planet}} \ll M_{\text{sun}}$)

$$\frac{a^3}{T^2} = \frac{|\kappa|}{4\pi^2\mu} \approx \frac{GM_{\text{sun}}}{4\pi^2} = \text{const}$$

- normalize a^3/T^2 to Earth's values



Data: <https://nssdc.gsfc.nasa.gov/planetary/factsheet/>

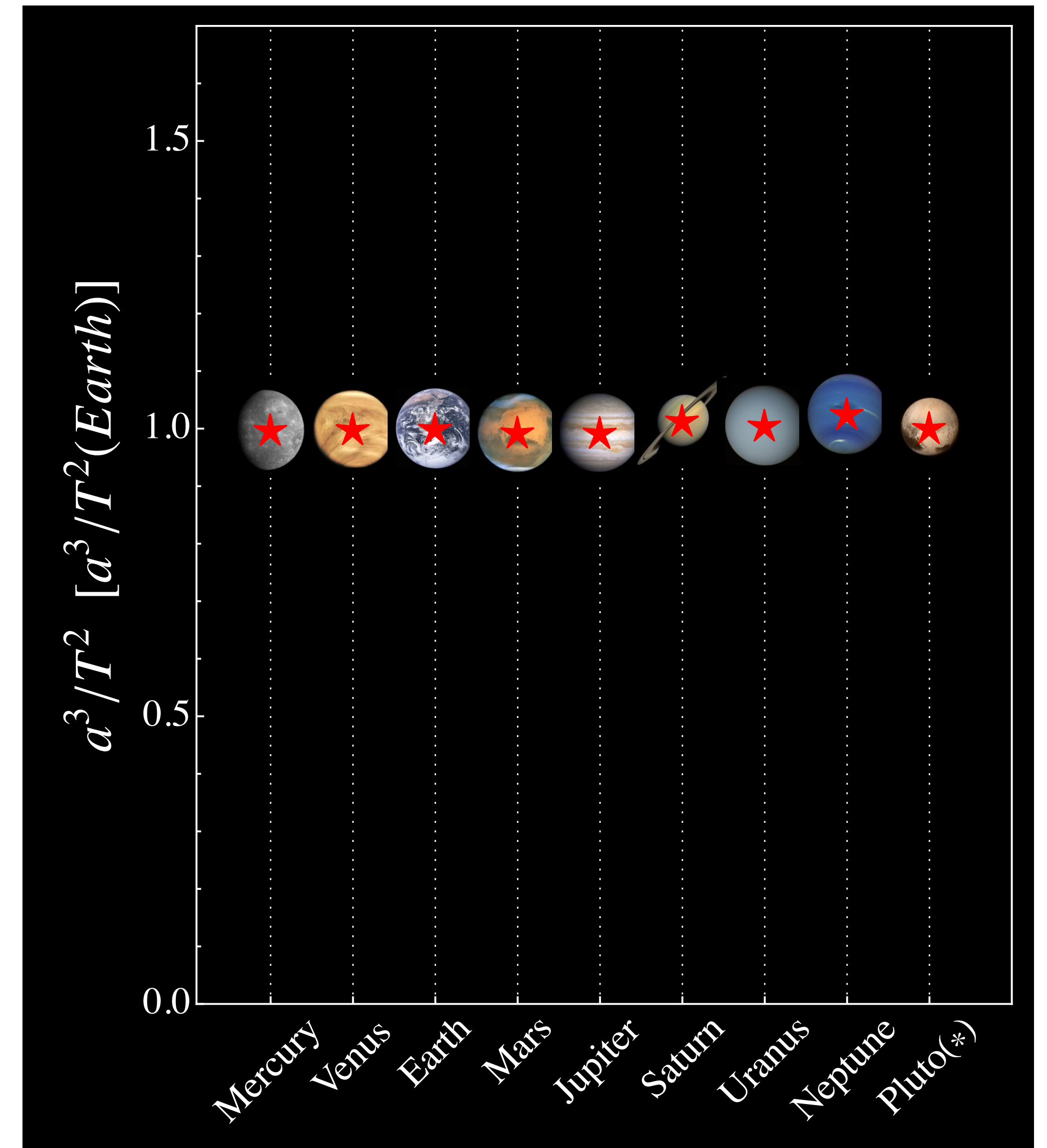
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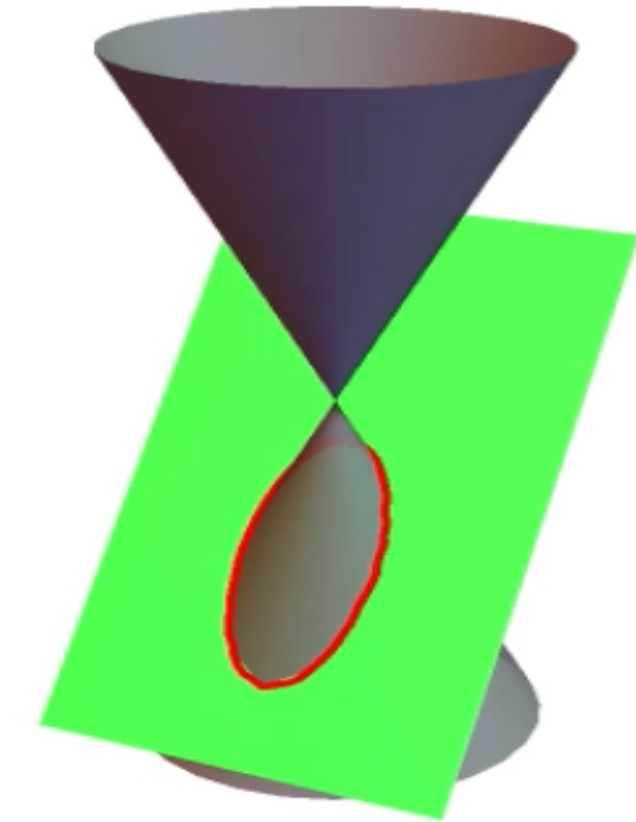
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Summary

Kepler's laws (two-body with central forces)

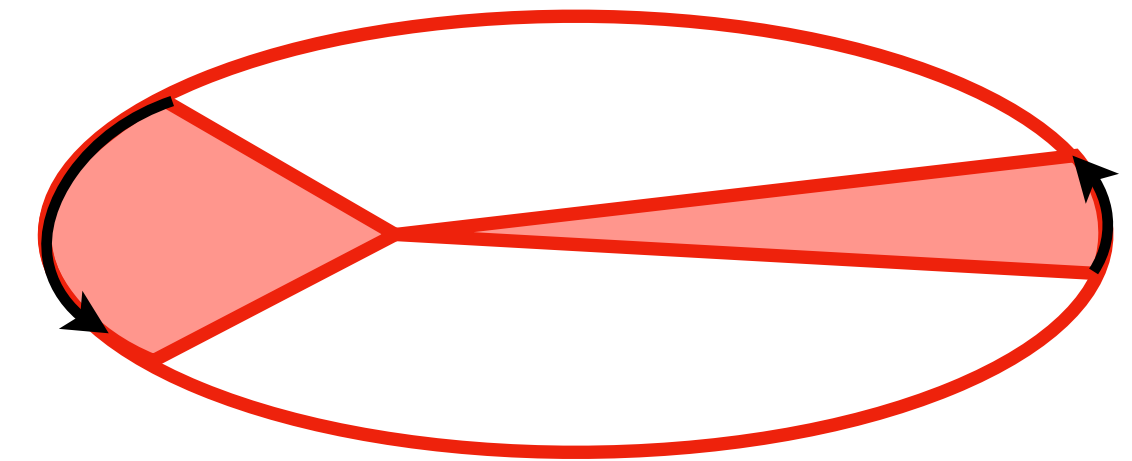
1. Motion is one-dimensional >> conic sections

$$|\vec{y}| = \frac{|\vec{L}'|^2}{\kappa\mu} \frac{1}{|\vec{B}| \cos \varphi - 1}$$



2. Area per time is constant

$$\frac{dA}{dt} = \frac{|\vec{L}'|}{2\mu}$$



3. Ratio of cubes of major axis to squares of rotation period is constant

$$\frac{a^3}{T^2} = \frac{|\kappa|}{4\pi^2\mu}$$

KEPLER'S LAWS

Introduction: (say)

- Mockup lecture by Maxim Mai as part of Application for a tenure track position.
- all materials are online

• What will we learn:

- universal laws of two-body motion with central force. (planetary motion)
- historically an important test of Newtonian dynamics.
- Kepler was also active in Graz.

• We begin with by repeating results of last lecture. \Rightarrow

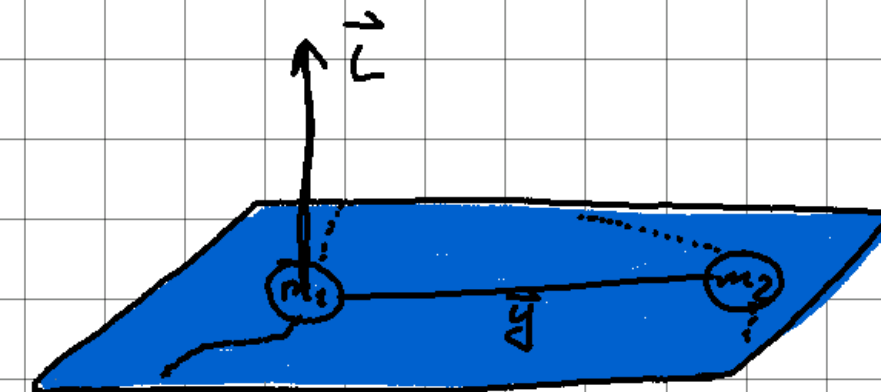
SLIDE 2

- setup
- relative coordinates are useful
- constraints on motion \Rightarrow 1d

• Can we learn something about the motion? (Blackboard)

$$1) (\vec{y}, \vec{L}) = 0 \quad \& \quad (\dot{\vec{y}}, \vec{L}) = 0$$

\Rightarrow relation motion takes place in a plane



O's Kepler Law

$$2) (\vec{B}, \vec{L}) = 0$$

R-L vector also lies in the plane

Both \vec{B} & $\dot{\vec{y}}$ lie in the plane...

$$3) (\vec{B}, \dot{\vec{y}}) = \frac{1}{\kappa} (\dot{\vec{y}}, \dot{\vec{y}} \times \vec{L}) + \frac{(\dot{\vec{y}}, \dot{\vec{y}})}{|\dot{\vec{y}}|}$$

$$= \frac{1}{\kappa} (\vec{L}, \dot{\vec{y}} \times \dot{\vec{y}}) + |\dot{\vec{y}}|$$

$$(\vec{a}, \vec{b} \times \vec{c}) = (\vec{c}, \vec{a} \times \vec{b}) = \dots$$

$$= \frac{1}{\kappa} |\vec{L}|^2 + |\dot{\vec{y}}| \rightarrow \text{examine degrees of freedom}$$

$$= \cos \varphi \cdot |\vec{B}| |\dot{\vec{y}}| \quad \text{for } \varphi: \angle(\vec{B}, \dot{\vec{y}})$$

$$\Rightarrow |\dot{\vec{y}}| = \frac{|\vec{L}|^2 / (\kappa m)}{|\vec{B}| \cos \varphi - 1}$$

1st Kepler Law

• motion is 1-dimensional (as expected)

• $|\dot{\vec{y}}|(\varphi) \rightarrow$ circle for $B=0$ ($\kappa < 0$)
 \rightarrow in general conic section

SLIDE 3

- We know now the geometrical form of motion, but what is with the time evolution?

Consider dA infinitesimal area drawn by \vec{y}

$$dA = \frac{1}{2} \left| \vec{y} \times d\vec{y} \right| = \frac{1}{2} \left| \vec{y} \times \frac{d\vec{y}}{dt} dt \right|$$

$$= \frac{1}{2} \left| \vec{y} \times \frac{d\vec{y}}{dt} \right| \cdot dt$$

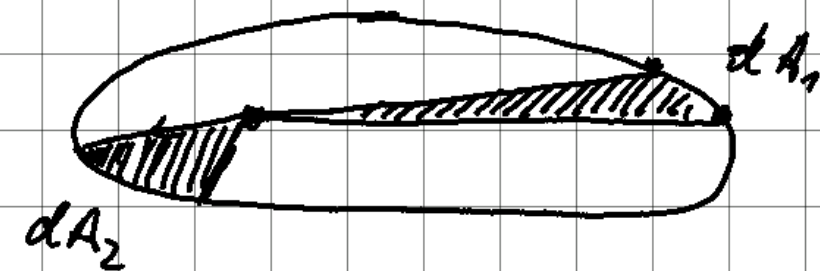
\Rightarrow per time increment dt :

$$\frac{dA}{dt} = \frac{dt}{dt} \cdot \frac{1}{2} \left| \vec{y} \times \frac{d\vec{y}}{dt} \right| \quad \text{looks like } |\vec{L}'|$$

$$\vec{L}' = \mu \vec{y} \times \dot{\vec{y}} = \mu \frac{dt}{dt} \vec{y} \times \frac{d\vec{y}}{dt}$$

$$\Rightarrow \boxed{\frac{dA}{dt} = \frac{1}{2\mu} |\vec{L}'| = \text{const}} \quad \text{2nd Kepler's law}$$

e.g. ellipse



- Full area (ellipse)

$$A = \int_0^T \frac{dA}{dt} \cdot dt = \frac{1}{2\mu} |\vec{L}'| \cdot T \quad (2KL)$$

$$A = \pi a \cdot b = \pi \cdot \left(a \frac{|\vec{L}'|^2}{\mu k} \right)^{3/2} \quad (\text{geometry})$$

$$\Rightarrow \boxed{\frac{a^3}{T^2} = \frac{k}{4\pi^2 \mu}} \quad \text{3rd Kepler's law}$$

HOW UNIVERSAL ARE THESE LAWS \rightarrow

SLIDES 4+