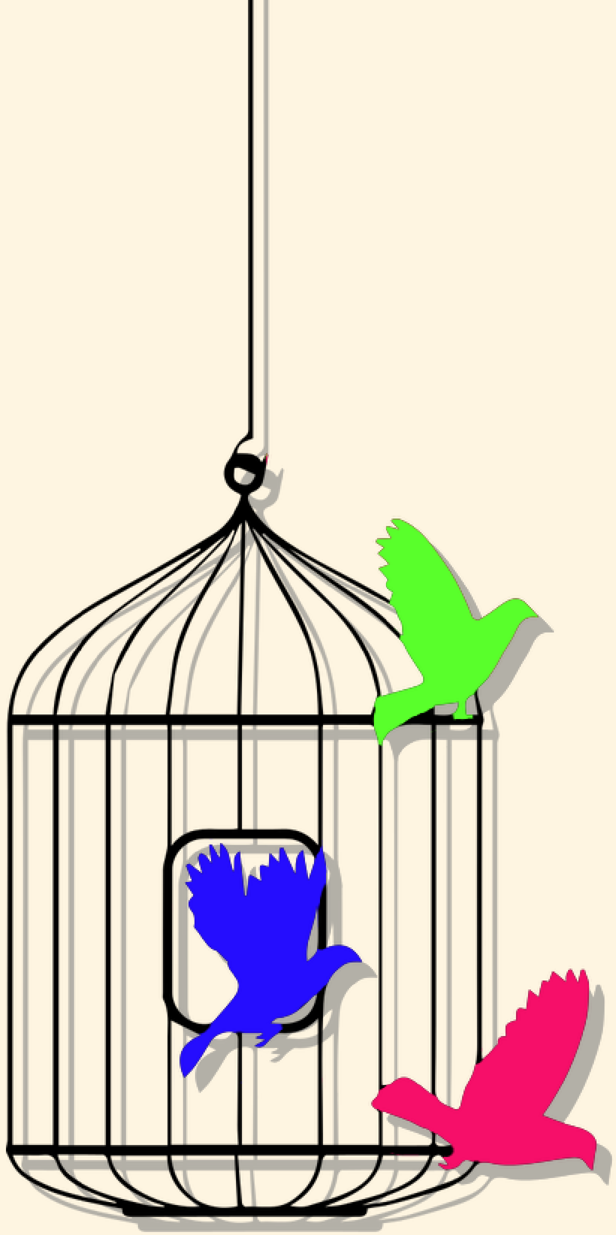

UNITARITY CONSTRAINTS ON 3 HADRON DYNAMICS

Seminar @ UMD – 10/25/18

Maxim Mai
The George Washington University

NSF
Deutsche Forschungsgemeinschaft
DFG

RESULTS

→ unitary 3-body scattering amplitude: $$T_{3\to3} = v\Big(\tau\frac{B}{1+B\tau}\tau - \tau\Big)v$$

EPJA53(2017)

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→ relativistic 3-body quantization condition
EPJA53(2017), PRD97(2018)

$$\det\left(\mathbf{B}^{\Gamma\mathbf{ss}'}_{\mathbf{uu}'} + \frac{\mathbf{2E_s\,L^3}}{\vartheta(\mathbf{s})}\tau_{\mathbf{s}}^{-1}\delta_{\mathbf{ss}'}\delta_{\mathbf{uu}'}\right) = \mathbf{0}$$

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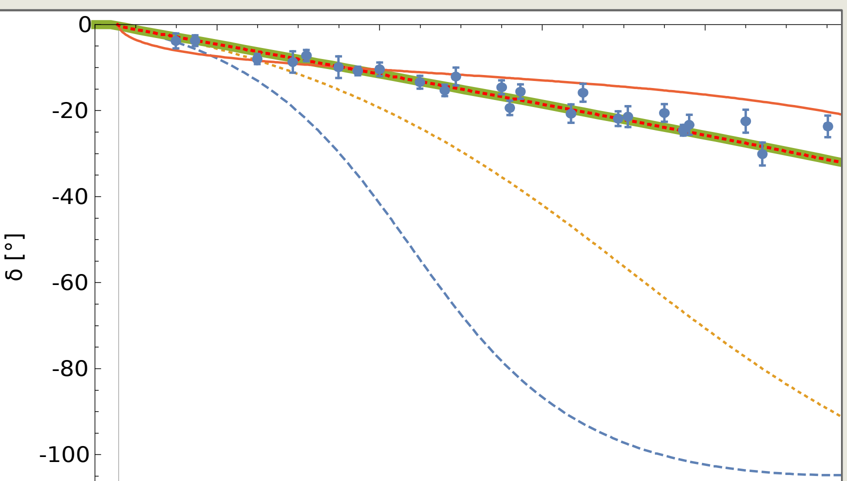
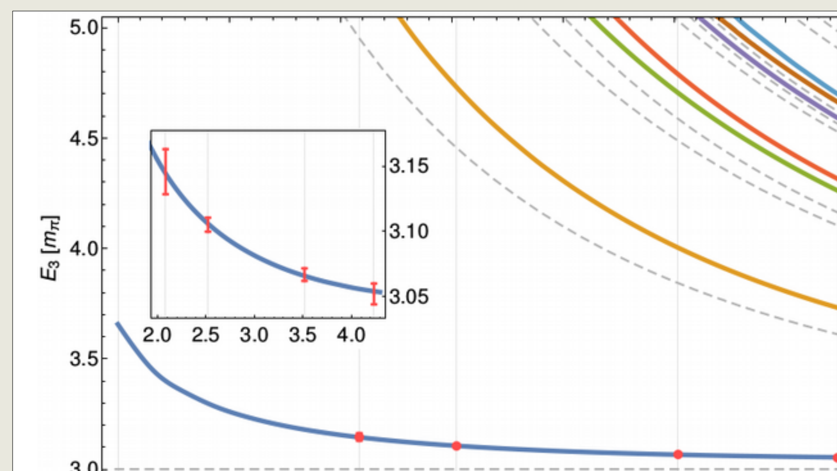
→ application to a physical system

arXiv:1807.04746



experimental $\pi^+\pi^+$ phase-shifts



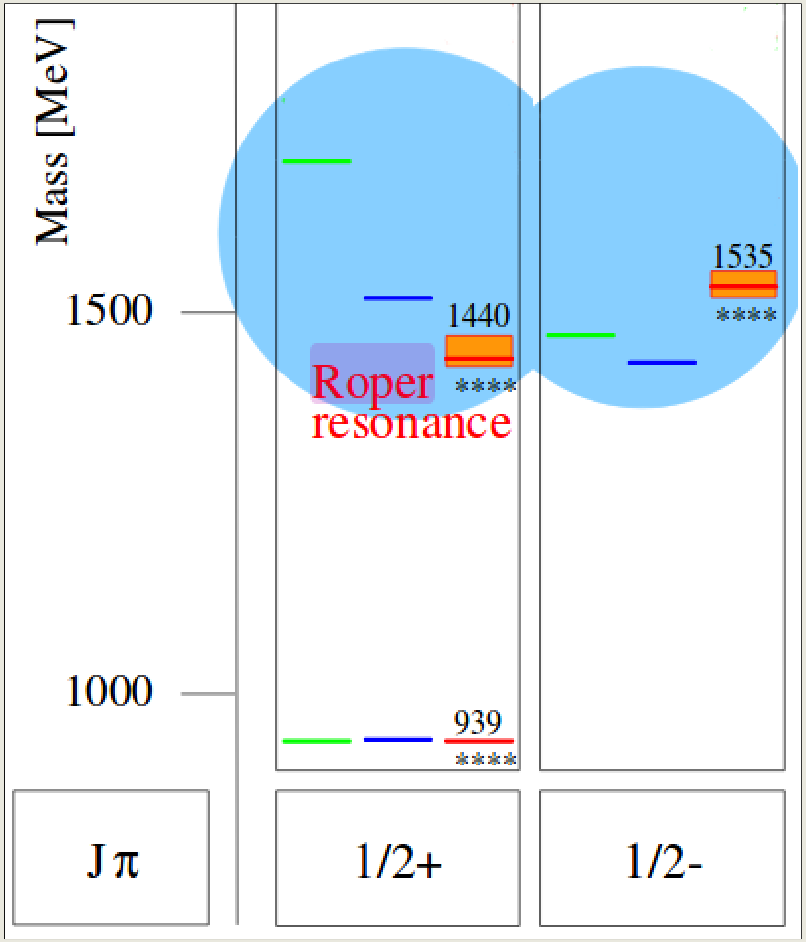
Lattice QCD $\pi^+\pi^+\pi^+$ spectrum

MOTIVATION

Many unsolved questions of QCD involve 3-body channels

EX. 1: Roper-puzzle

Reversed mass pattern vs. Const. Quark Model



Loring et al. EPJA10(2001)

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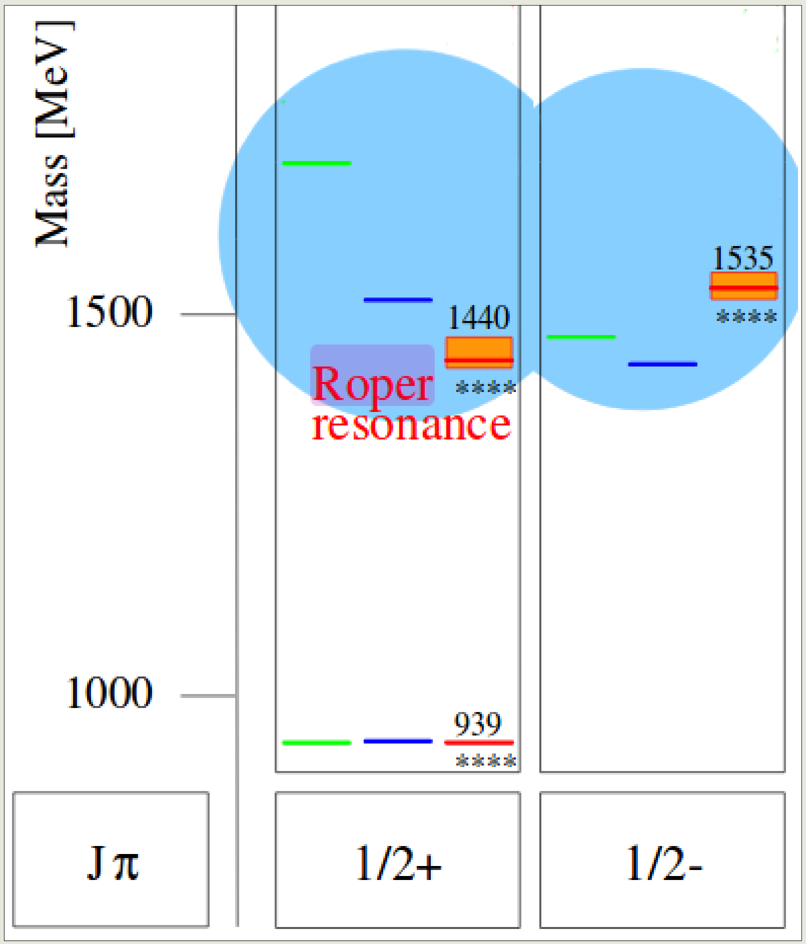

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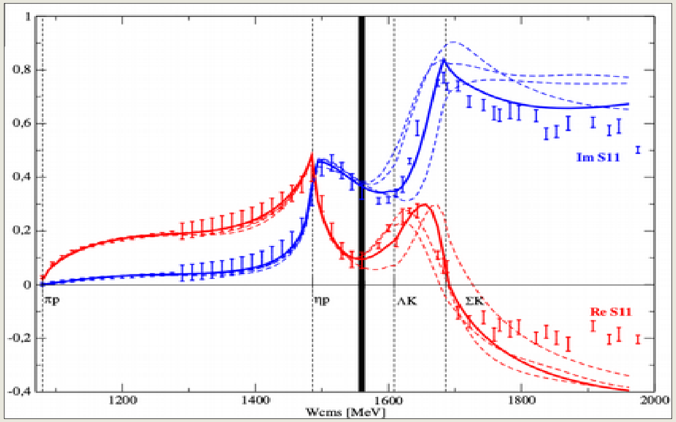
N(1535)1/2⁻ accessed from dyn. approaches, e.g.,



Bruns, MM, Meissner PLB 697(2011)

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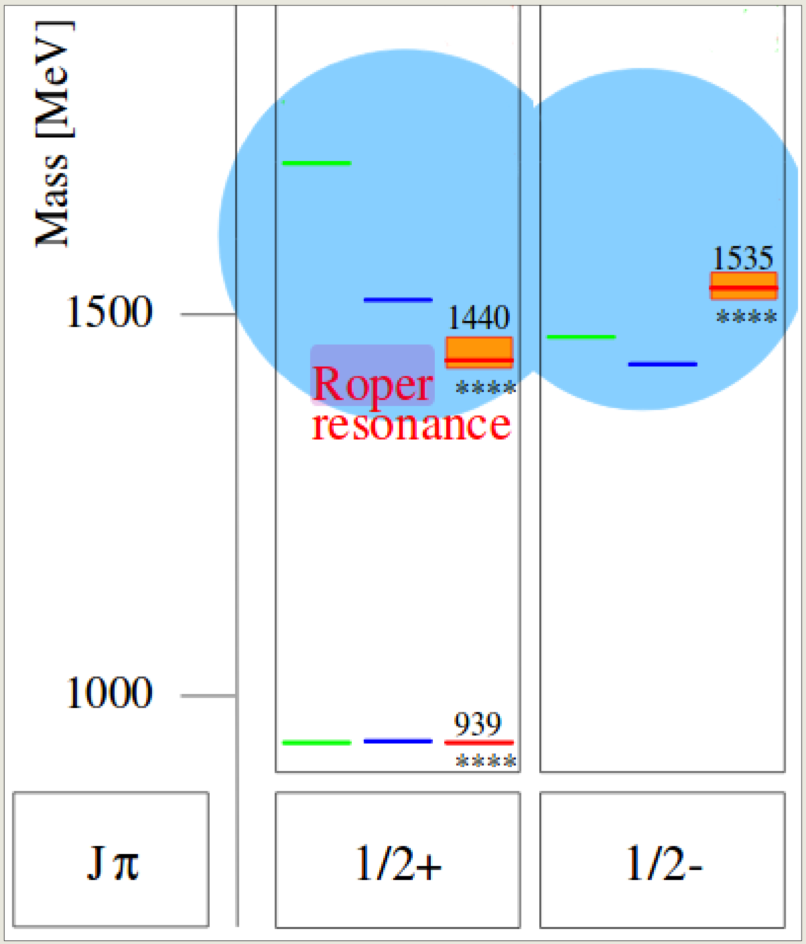

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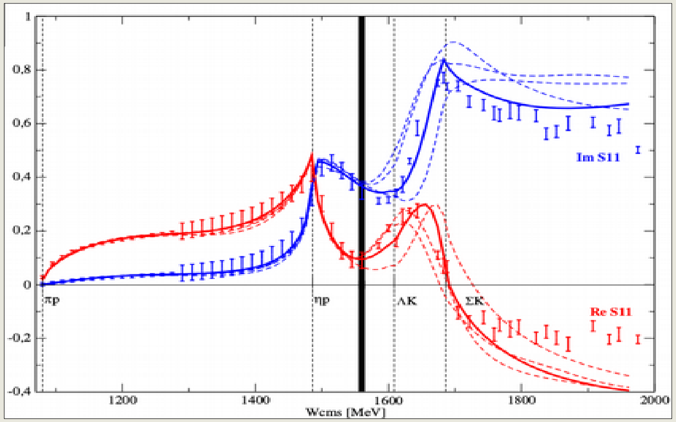

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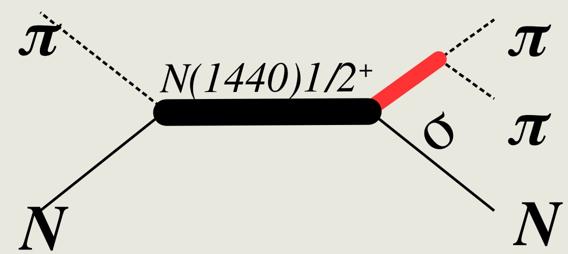
...but ***N(1440)1/2*** $^{+}$ has large branching ratio to $\pi\pi N$



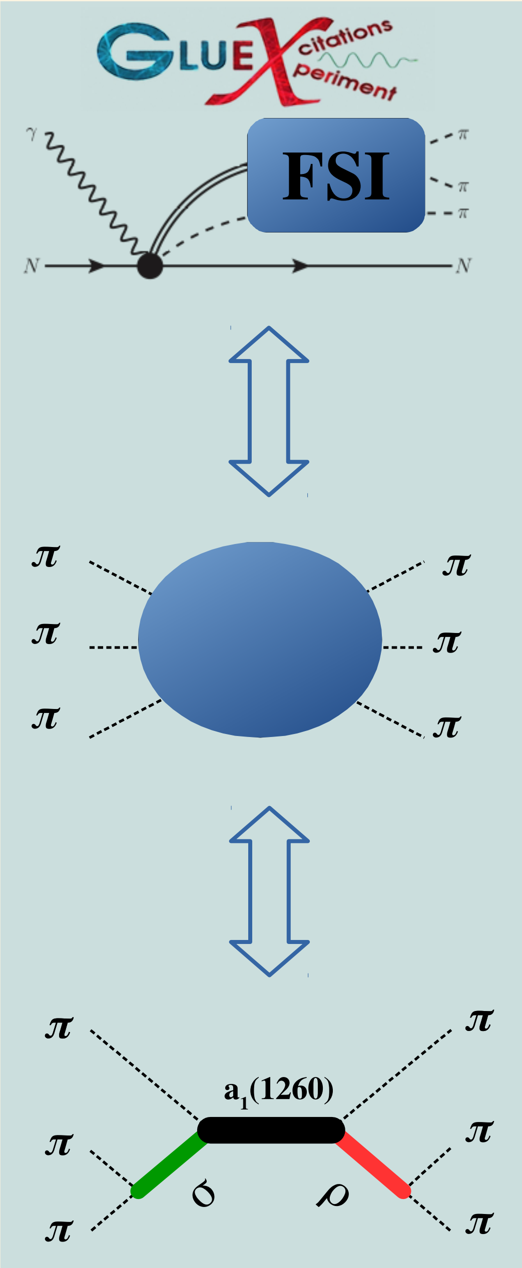
- *1*st *LQCD calculation* w. meson-baryon operators

Lang et al. PRD 95(2017)

Many unsolved questions of QCD involve 3-body channels

EX. 2: $a_1(1260)$

- important for the search for *spin-exotics* (*GlueX, COMPASS, BESIII*)
  - indicator for gluonic degrees of freedom
  - cannot decay in 2 but in 3 pions, such as the $\boldsymbol{a_1(1260)}$



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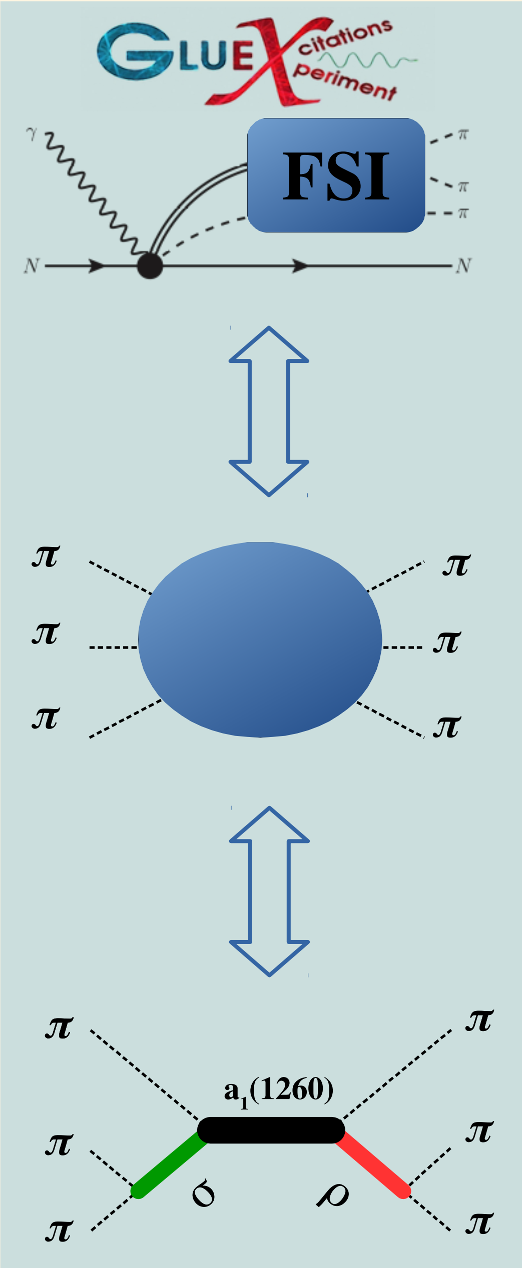
- finite-volume LQCD spectrum — Lang et al. JHEP 1404

  further studies will follow... — Woss et al. JHEP 1807 [HadSpec]

  $\rightarrow$ ***I=2 $\pi\rho$***



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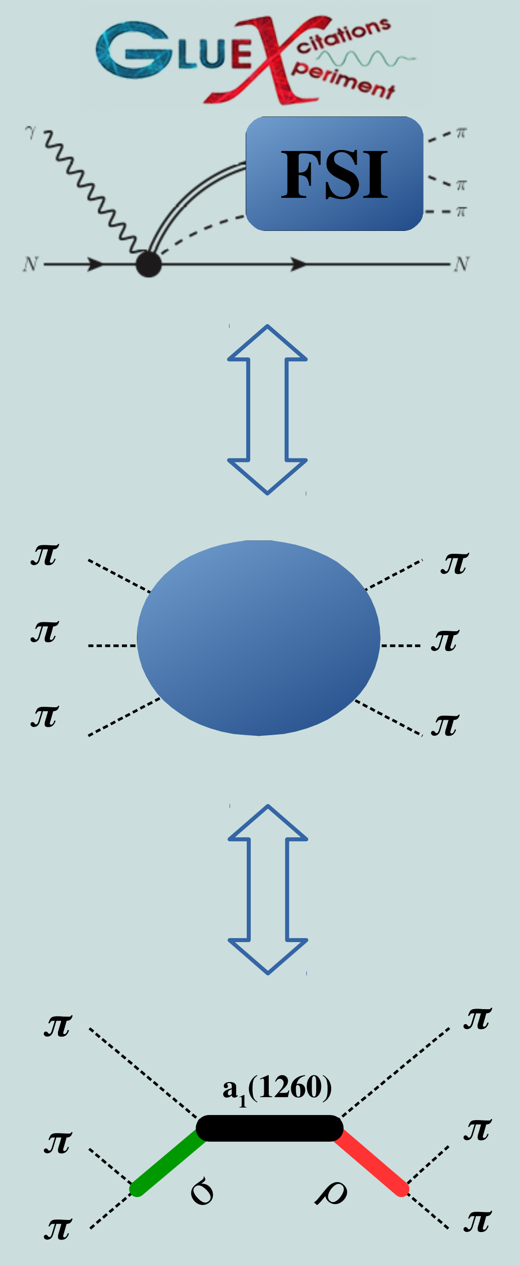
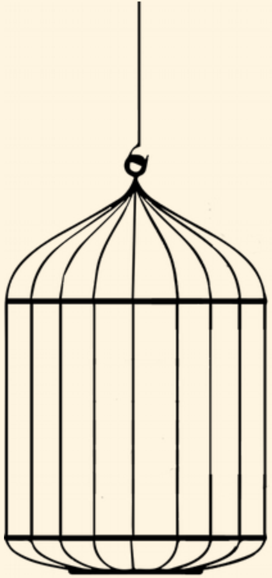
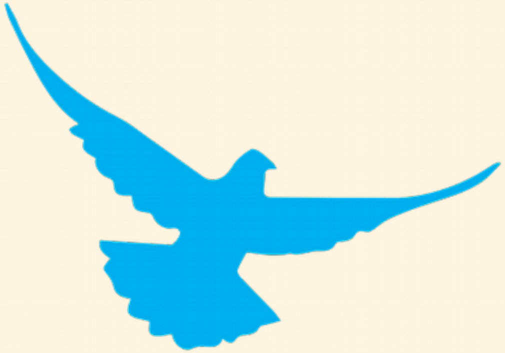
EX. 3: ***X(3872) ↔ $D\bar{D}\pi$*** — Prelovsek,Leskovec PRL111(2013)

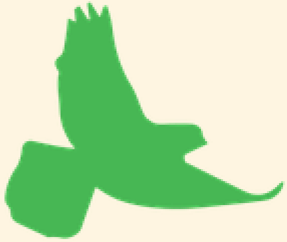
…







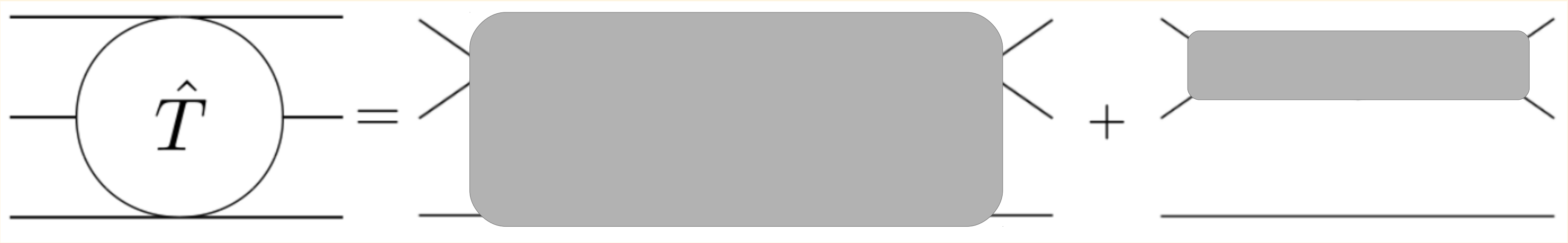
3→3 SCATTERING AMPLITUDE IN INFINITE VOLUME



MM, Hu, Doring, Pilloni, Szczepaniak EPJA53(2017)

T-MATRIX

- 3 asymptotic states (scalar particles of equal mass (m))
- ***<u>Connectedness structure</u>*** of matrix elements



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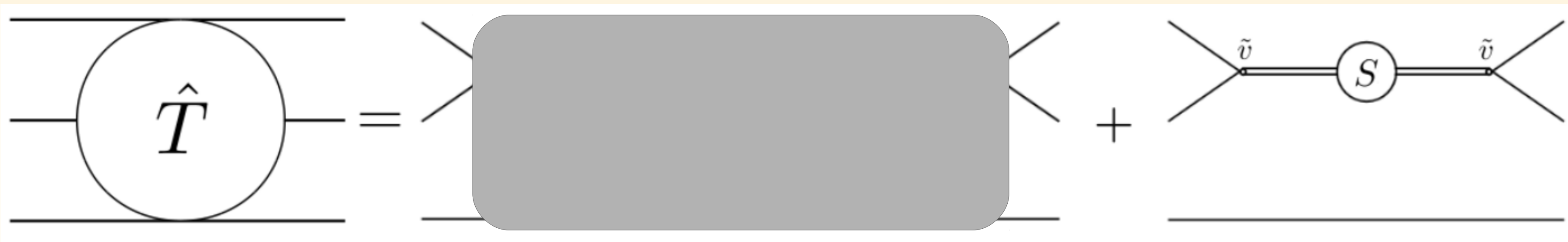

- *isobar*-parametrization of two-body amplitude

Bedaque, Griesshammer (1999)

  - "*isobars*" – ***S(σ)*** for definite QN & correct right-hand-singularities
  - in general a tower of "*isobars*" for ***L=0,1,2,...***
  - coupling to asymptotic states: cut-free-function ***v(q,p)***

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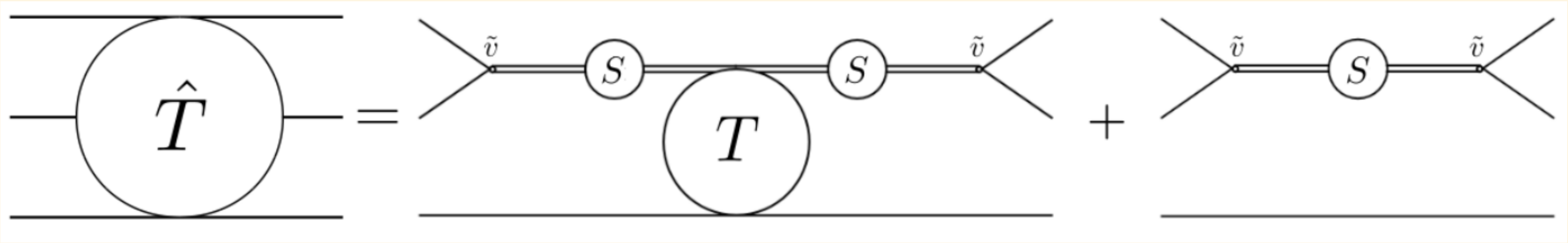

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- Connected part: due to isobar-spectator interaction → ***$T(q_{in},\ q_{out};\ s)$***

  - 3 unknown functions
  - 8 kinematic variables

3-BODY UNITARITY

$$\langle q_1, q_2, q_3|(\hat{T}-\hat{T}^\dagger)|p_1, p_2, p_3\rangle \quad = \quad i \int_P \langle q_1, q_2, q_3|\hat{T}^\dagger|k_1, k_2, k_3\rangle\langle k_1, k_2, k_3|\hat{T}|p_1, p_2, p_3\rangle$$

phase-space integral

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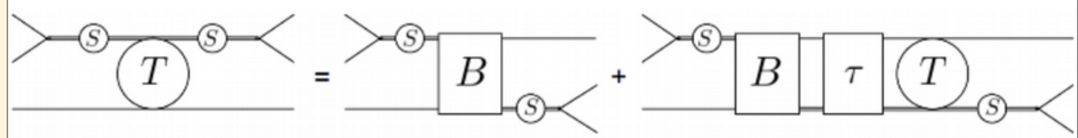

General Ansatz for the isobar-spectator interaction

→ ***B* & *τ*** are **new** unknown functions

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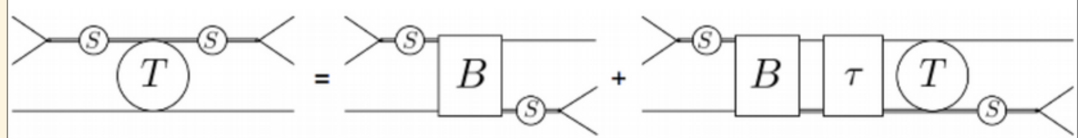

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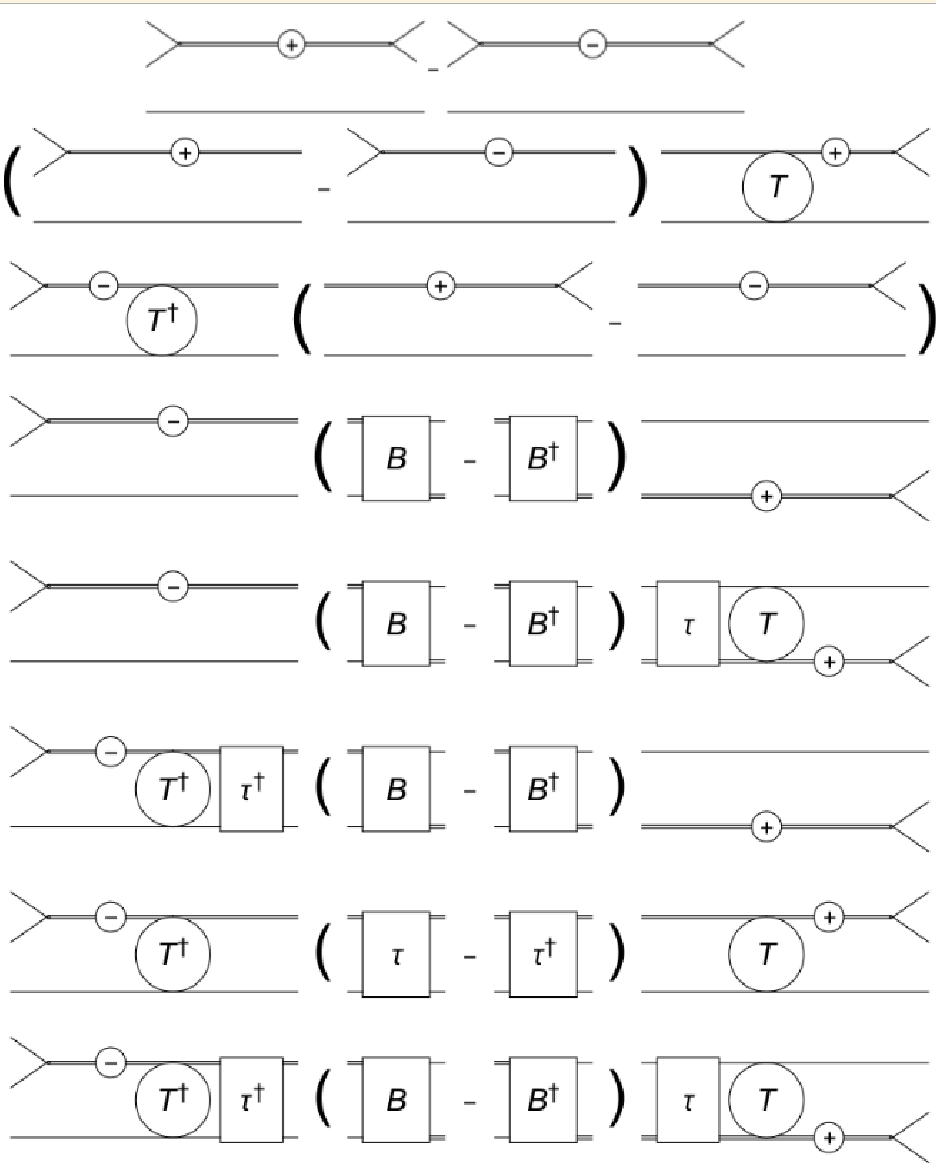

8 topologies

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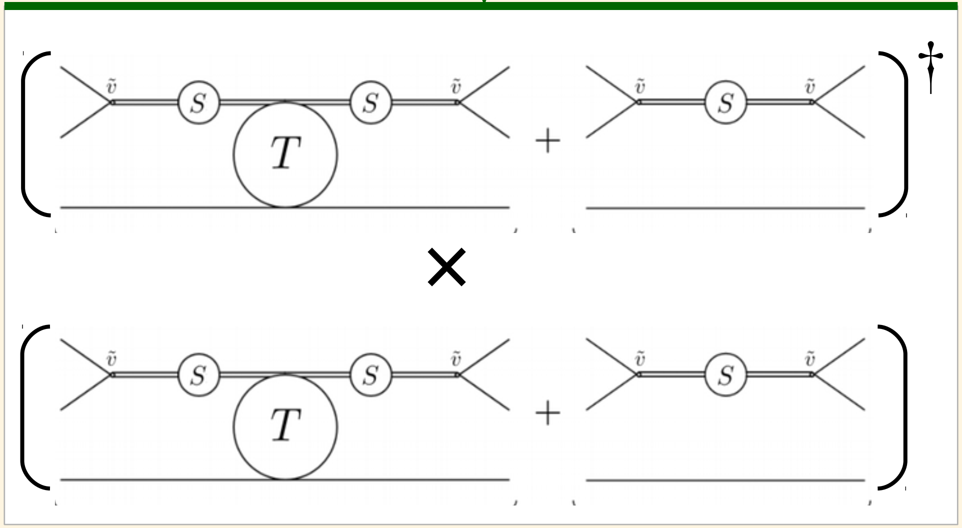
conn. + disc. parts



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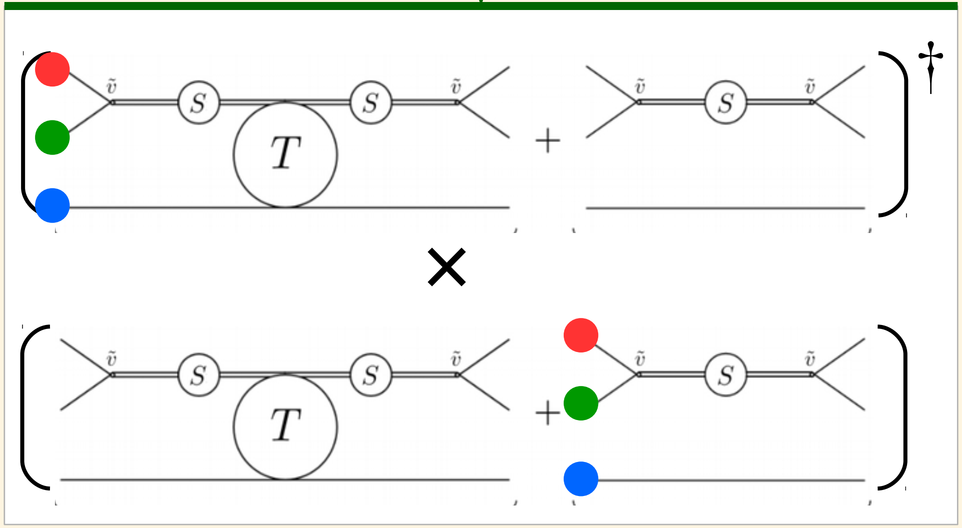

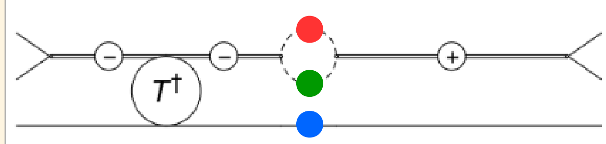

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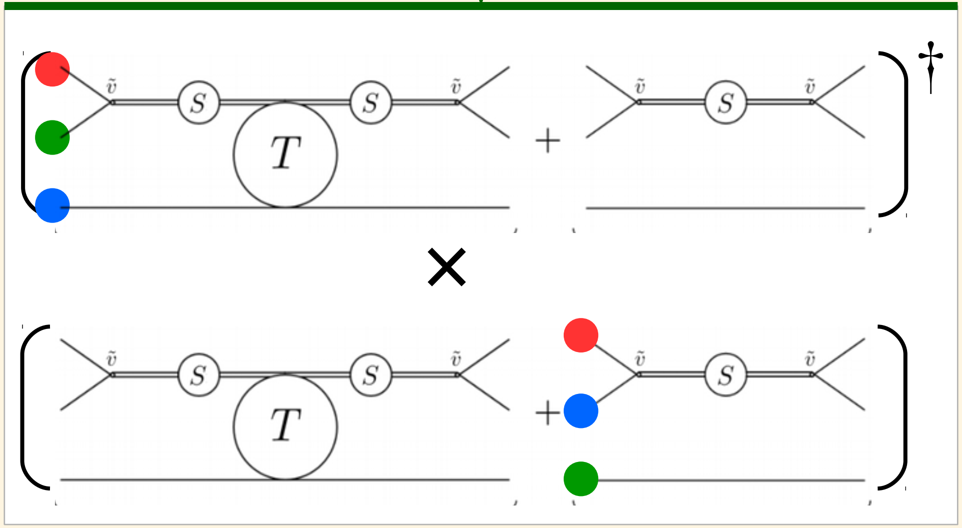

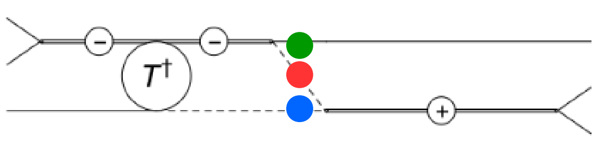

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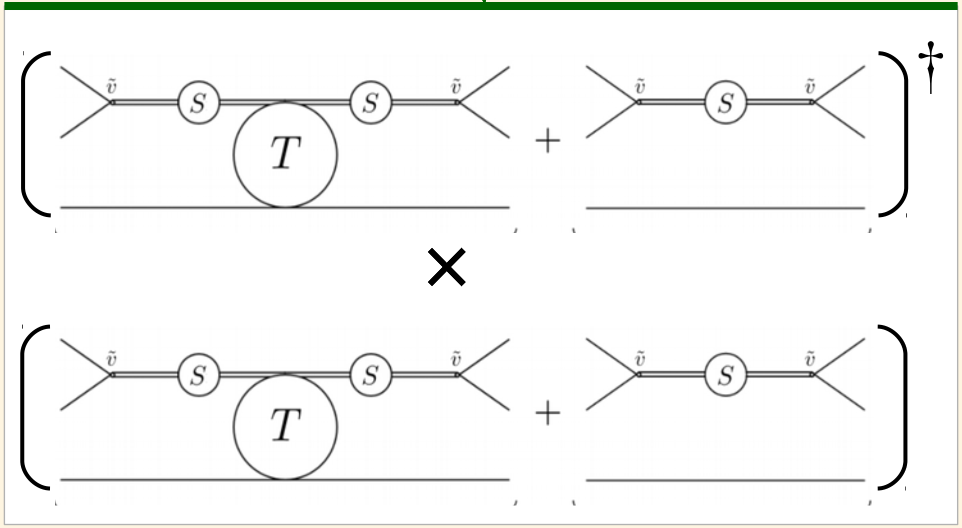

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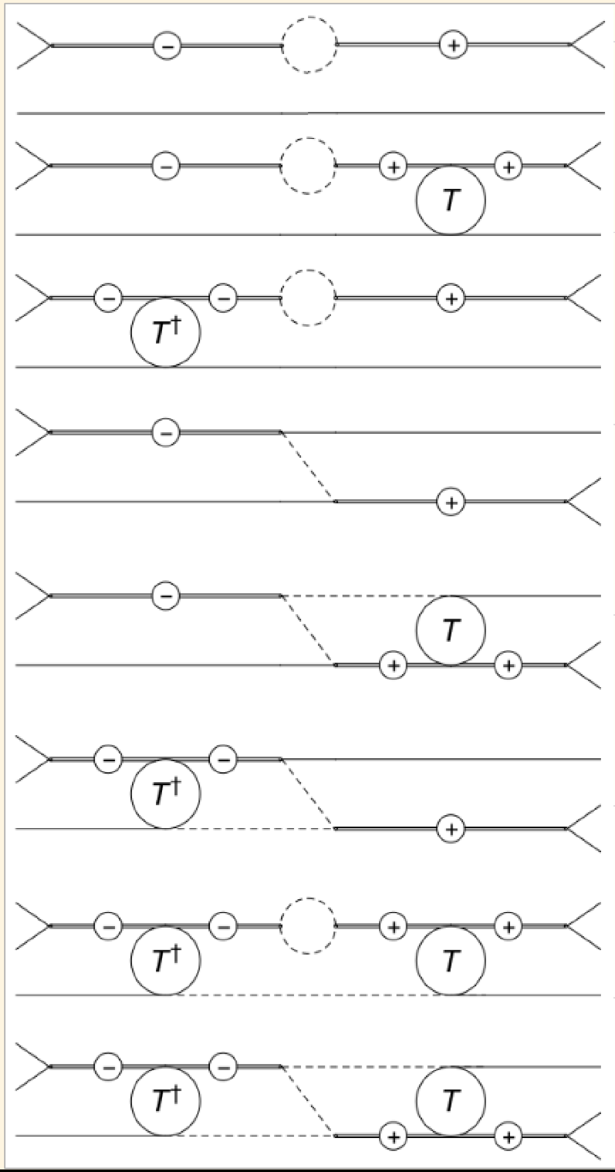

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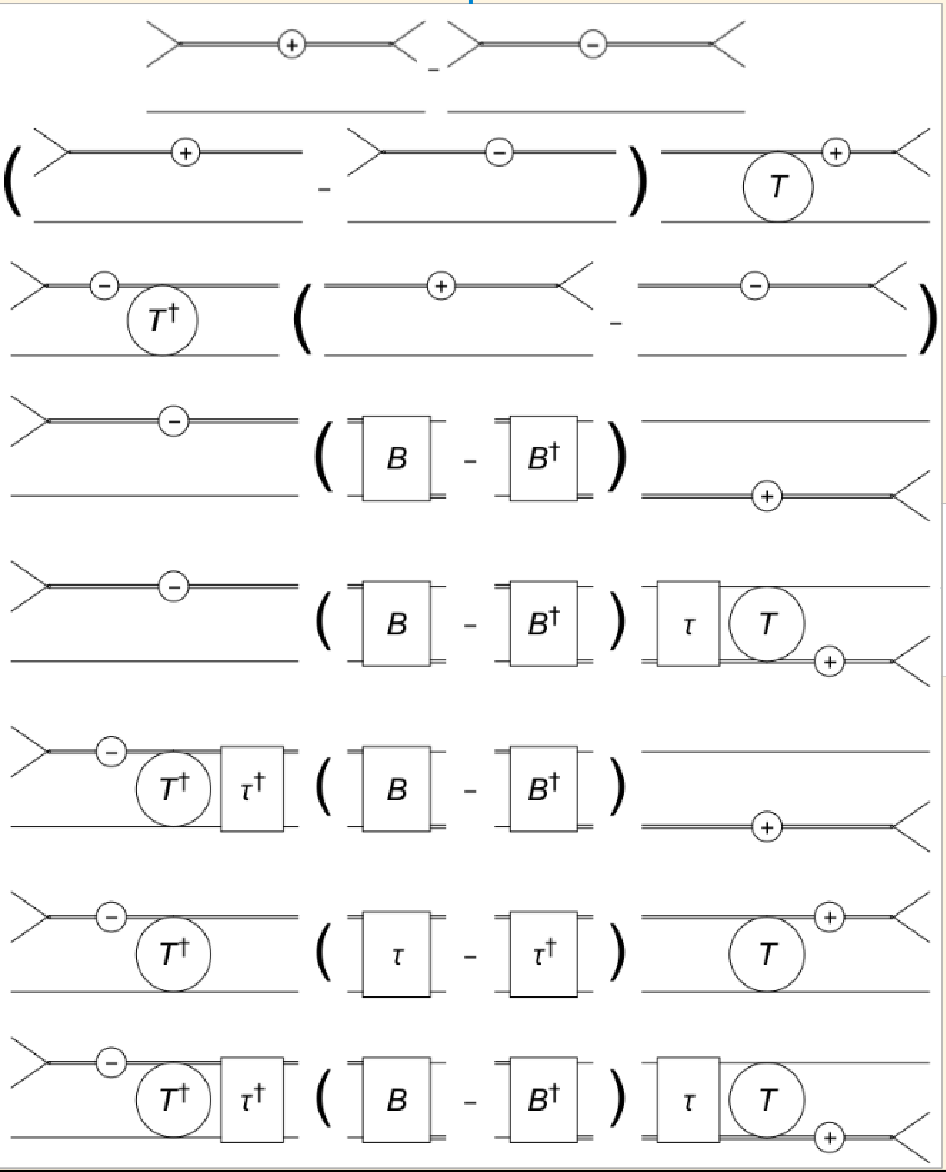

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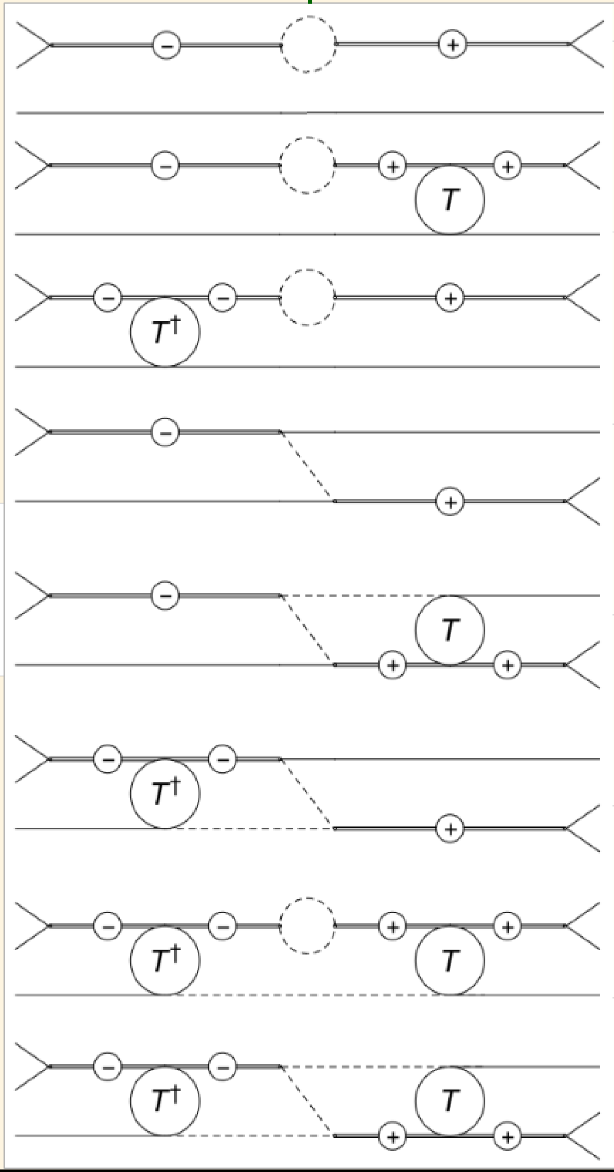

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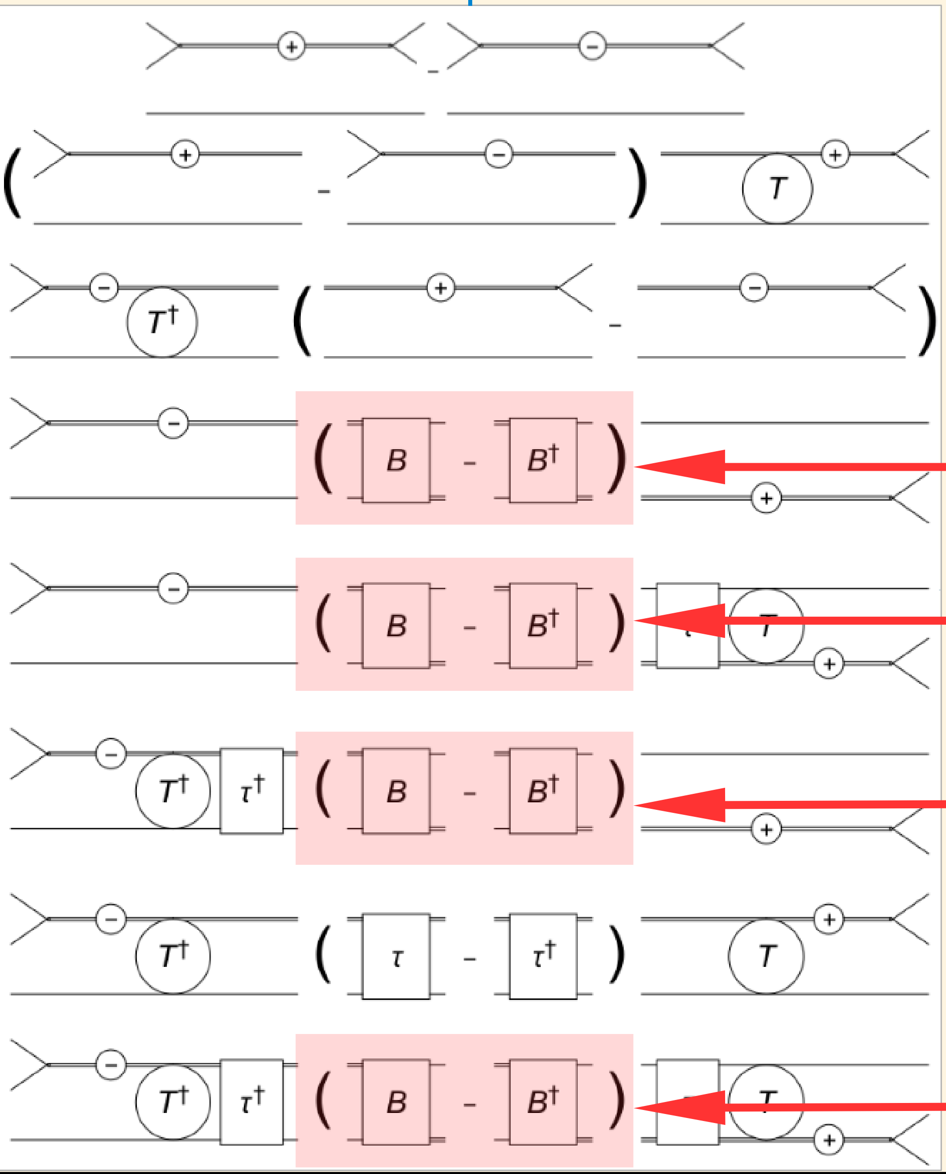

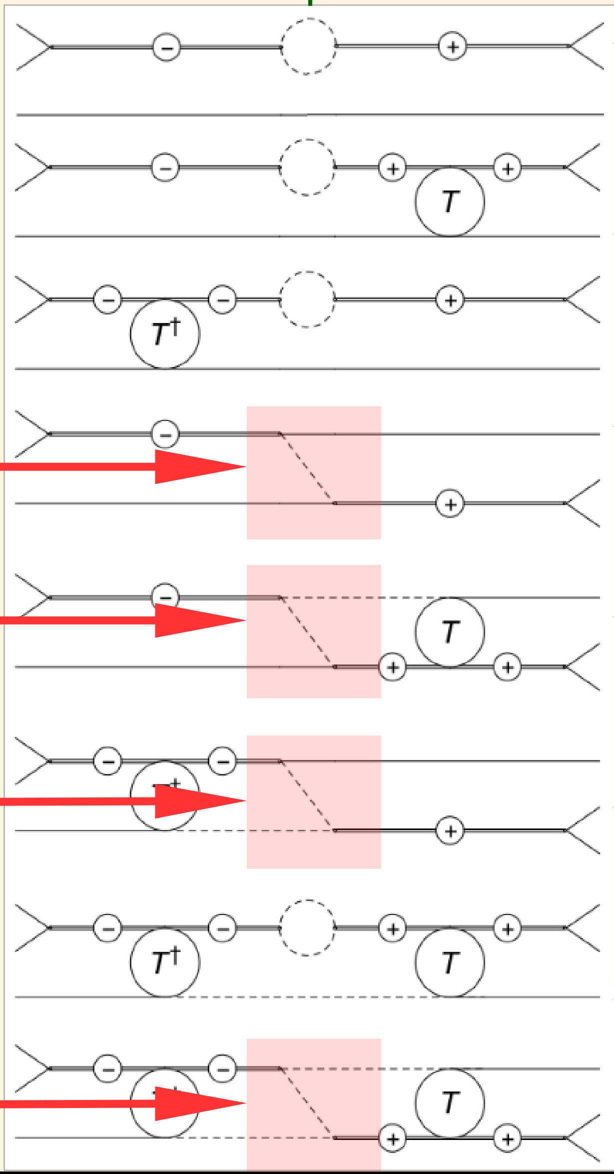

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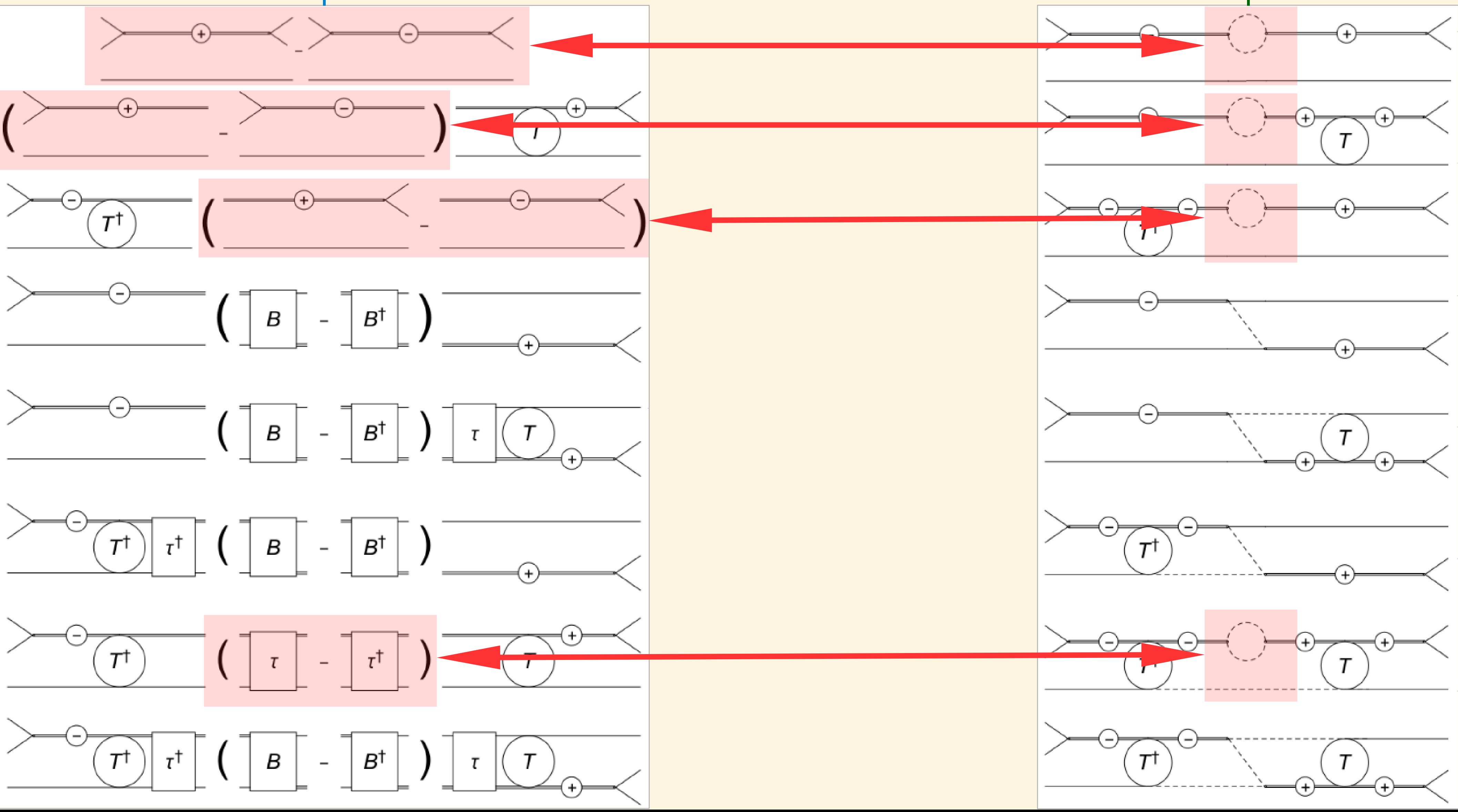

DISPERSION RELATION

- The only imaginary parts required by 3-body unitarity:

$$\mathrm{Disc}\, B(u) = 2\pi i \, \frac{\delta\left(E_Q - \sqrt{m^2+\mathbf{Q}^2}\right)}{2\sqrt{m^2+\mathbf{Q}^2}} v^2$$

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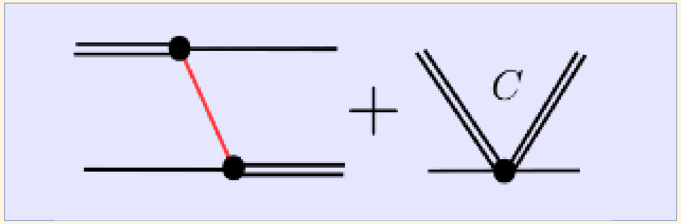

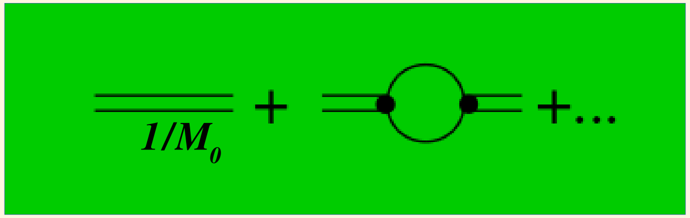

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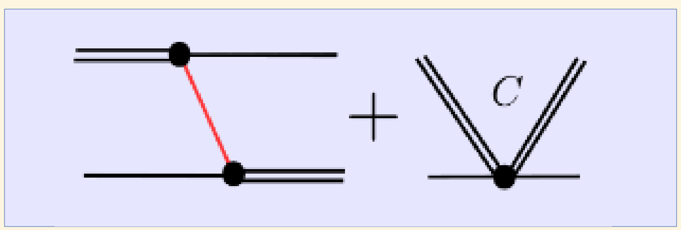
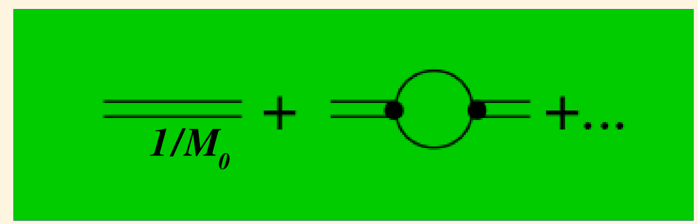
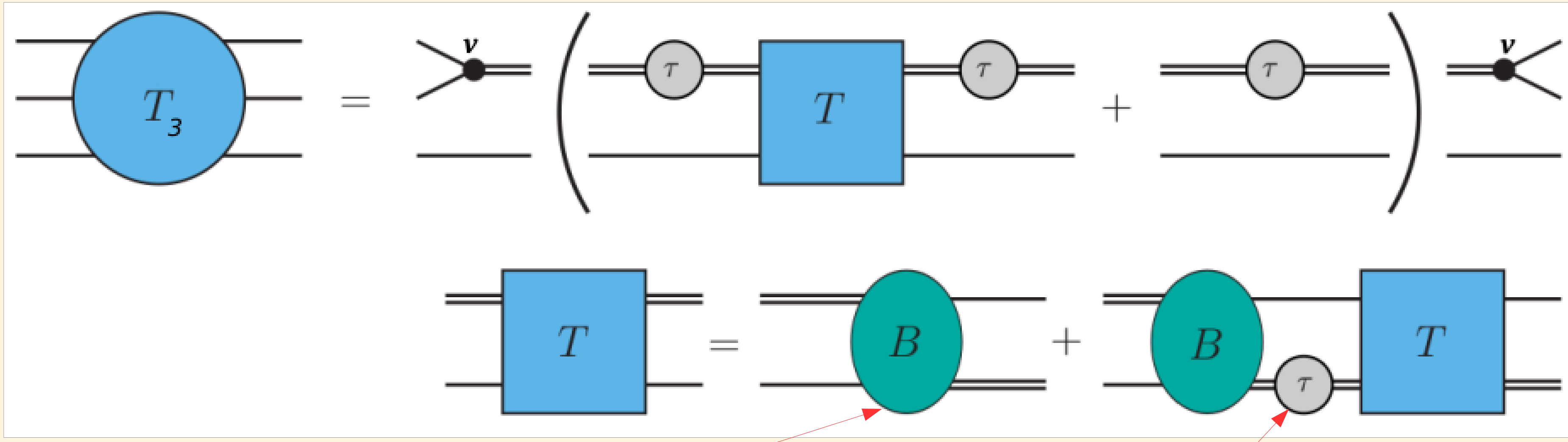




- One-meson-exchange diagram emerges automatically ↔ **UNITARITY**
- Real constants M_0 & C are free → input from data required

SCATTERING EQUATION

- 3-dimensional relativistic integral equation



e.g. $v=\lambda$

- un-subtracted dispersion relation

$$B(s) = \frac{-\lambda^2}{2\sqrt{m^2+\mathbf{Q}^2}\left(E_Q - \sqrt{m^2+\mathbf{Q}^2}+i\epsilon\right)} + C$$

- one-π exchange in TOPT → **R E S U L T** !

e.g. $v=\lambda$

- twice subtracted dispersion relation in invariant mass - $\sigma(k)$

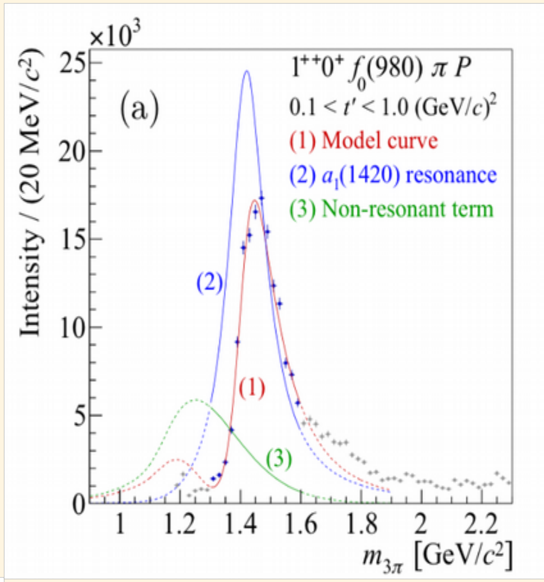
$$\frac{1}{\tau} = \sigma(k) - M_0^2 - \int \frac{d^3\ell}{(2\pi)^3}\frac{\lambda^2}{2E_\ell(\sigma(k)-4E_\ell^2+i\epsilon)}$$

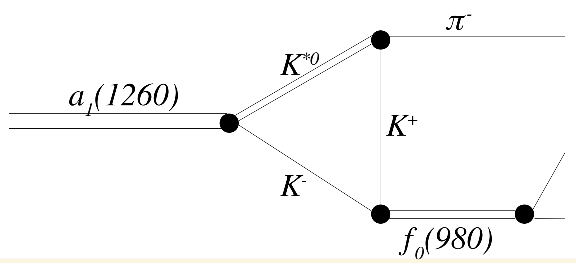
- in the rest-frame of isobar (*Lorentz invariance!*)

RECENT DEVELOPMENTS

- Interesting application: $\boldsymbol{a_1(1420)}$
  - observed in *COMPASS@CERN* in $\boldsymbol{f_0(980)\pi}$ final state
  - one explanation:
    log-like behavior of the "*triangle-diagram*"

Mikhasenko,Ketzer,Sarantsev(2015)
Aceti,Dai,Oset(2016)





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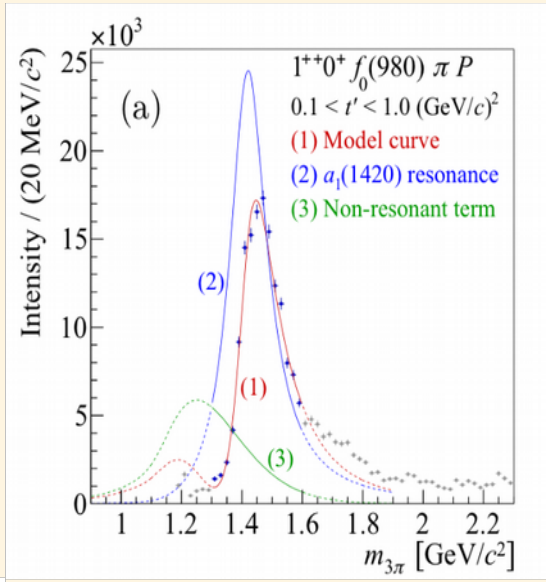
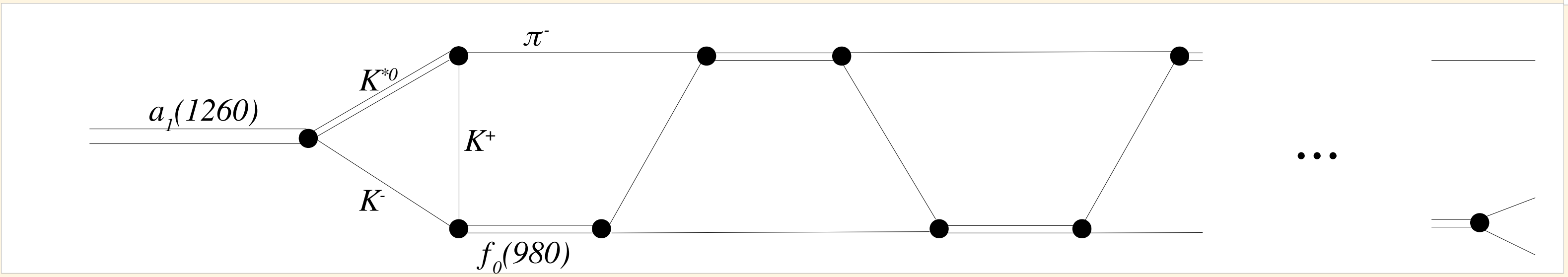




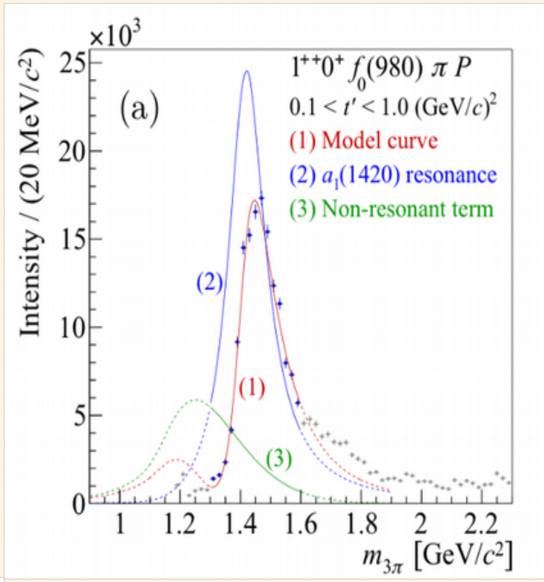
- **Q:** Does such a feature exist in full 3b-unitary FSI?

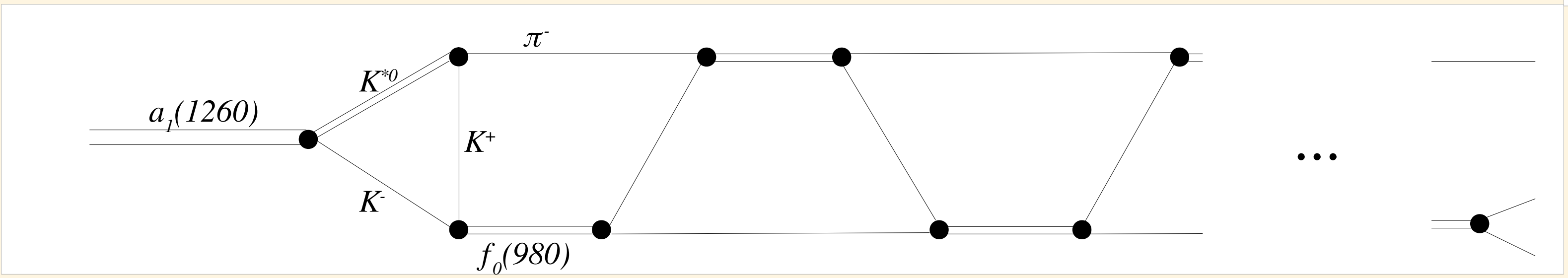
Sadisavan et al. (in progress)

RECENT DEVELOPMENTS

- Interesting application: ***a_1(1420)***
  - observed in *COMPASS@CERN* in ***f_0(980)π*** final state
  - one explanation:
  log-like behavior of the "*triangle-diagram*"

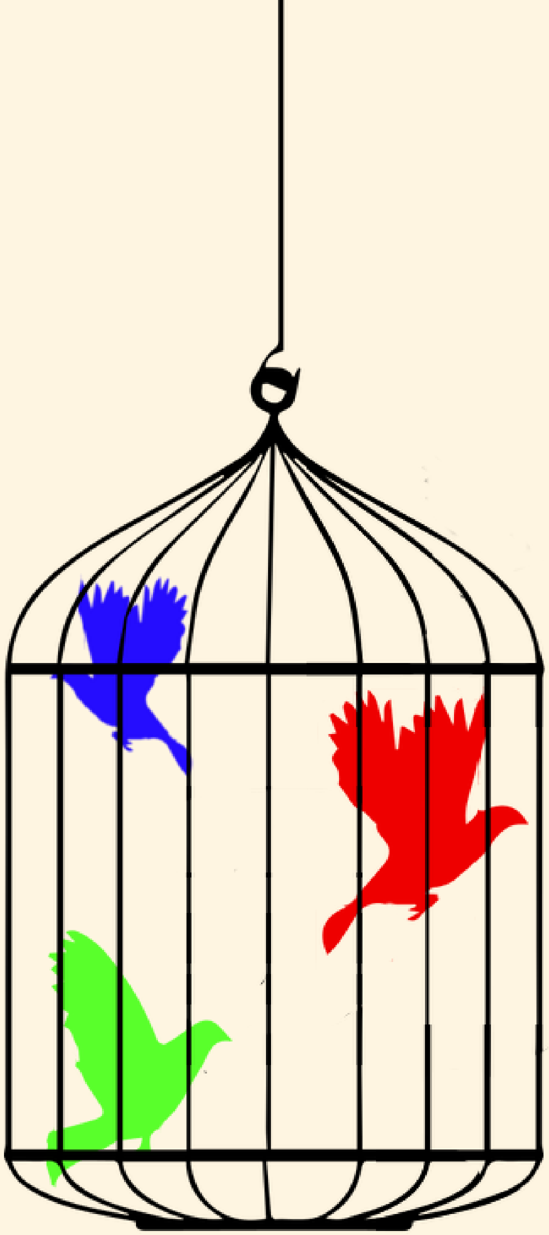
Mikhasenko,Ketzer,Sarantsev(2015)
Aceti,Dai,Oset(2016)





  - **Q:** Does such a feature exist in full 3b-unitary FSI? Sadasivan et al.(in progress)

- Recent theoretical progress Jackura et al.(2018)
  - Re(△) depends on the treatment of OPE – singularities in unphys. regions...



3→3 SCATTERING IN A BOX

M.M., Doring **EPJA53(2017)**

LATTICE QCD

- the only systematic approach from first principles
- new interesting developments for 3-hadron systems

Lang et al. JHEP 1404

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continuum limit

Chiral extrapolations

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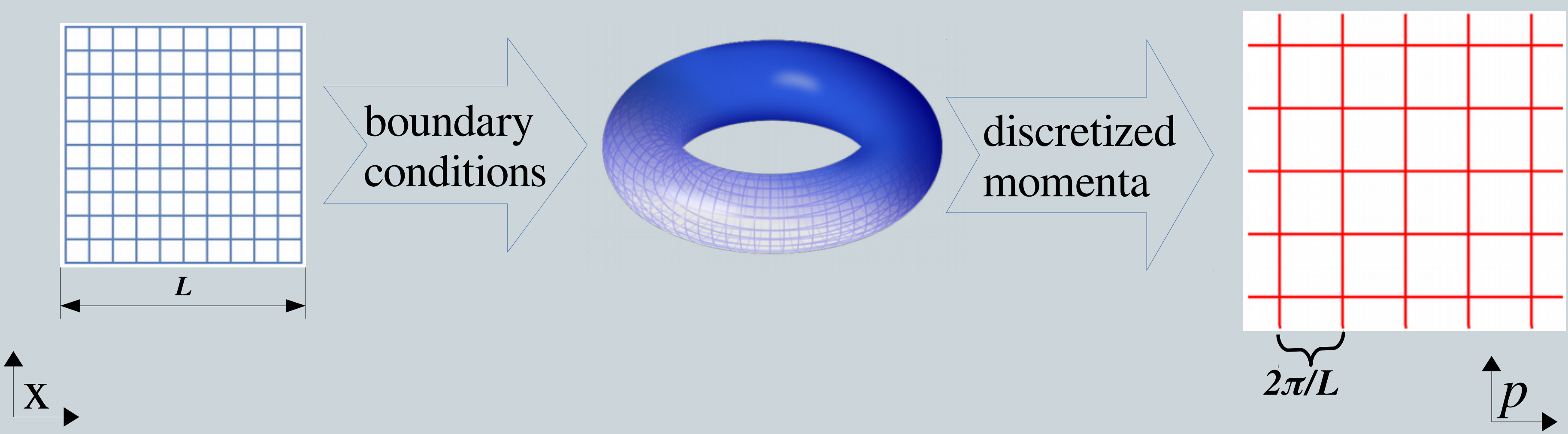

☞ momenta & spectra are discretized! → **<u>Quantization Condition</u>**

OVERVIEW

- **2-body case**
  - well understood
  - multi-channels, spin, ...

Lüscher (1986)

Gottlieb,Rummukainen,Feng,Li,Liu,

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    Polejaeva/Rusetsky (2012)
    Briceño/Hansen/Sharpe (2014,2015,2016,2017,2018)

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Requirements:

- 3-body systems involve *(resonant)* two-body sub-amplitudes
- extrapolations between different energies *(problem of underdetermination)*
- all possible intermediate on-shell configurations must be identified and included to ensure all power-law finite-volume effects are taken account of

$\Longrightarrow$ This work: ***3body Quantization Condition from 3-body unitarity in isobar formulation***
MM/Doring(2017)

A WAY TO DERIVE 2-BODY QC

2-body Unitarity

$$T(E) = \frac{1}{K^{-1}(E) + i\Phi(E)}$$

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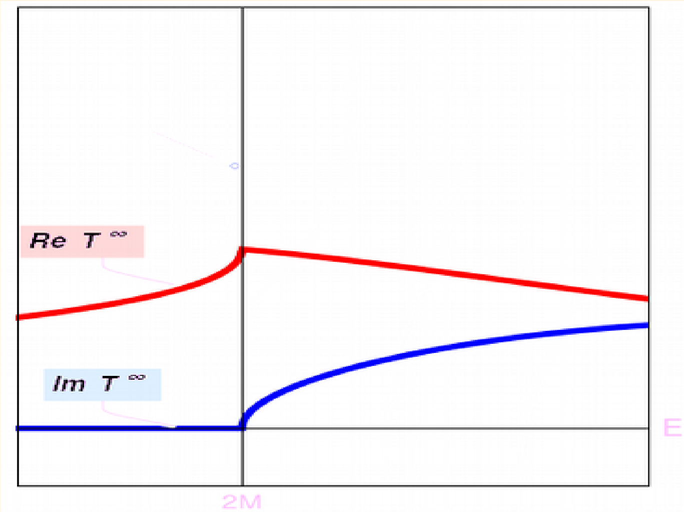

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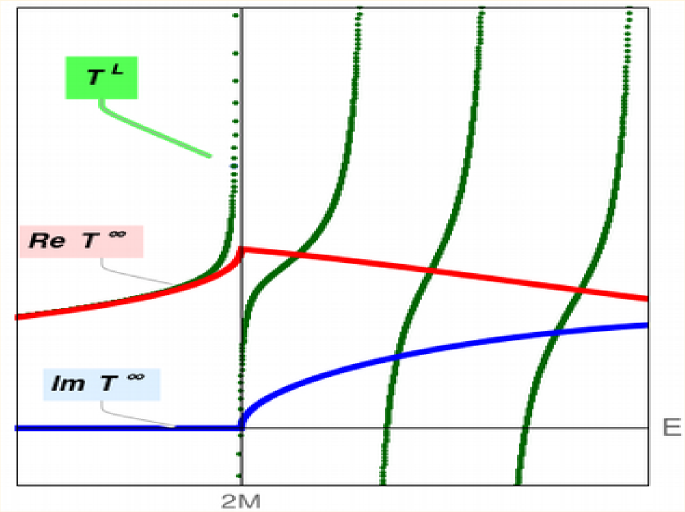

Discretization

Power-law fin.-vol. corrections

$$T^{FV}(E) = \frac{1}{K^{-1}(E) + \frac{2}{\sqrt{\pi}L} Z_{00}(E, L)}$$

- Regular summation theorem applies for ***E<2M***
- For ***E>2M***: ***T(E)*** can be singular
- **<u>LSZ formalism:</u> energy eigenvalues ↔ pole-positions**

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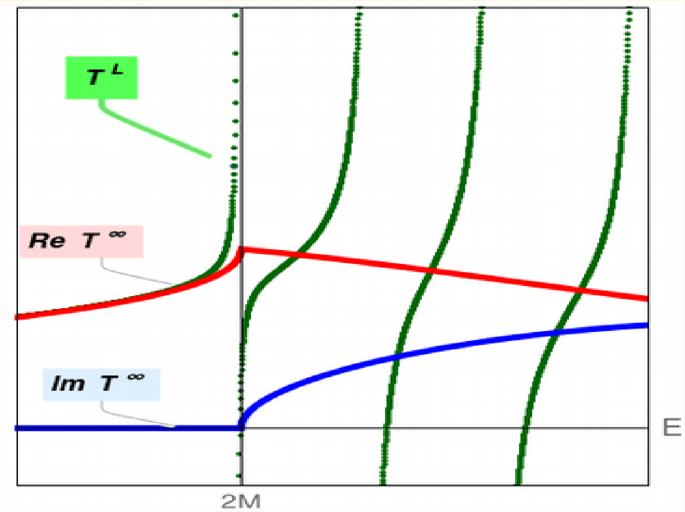

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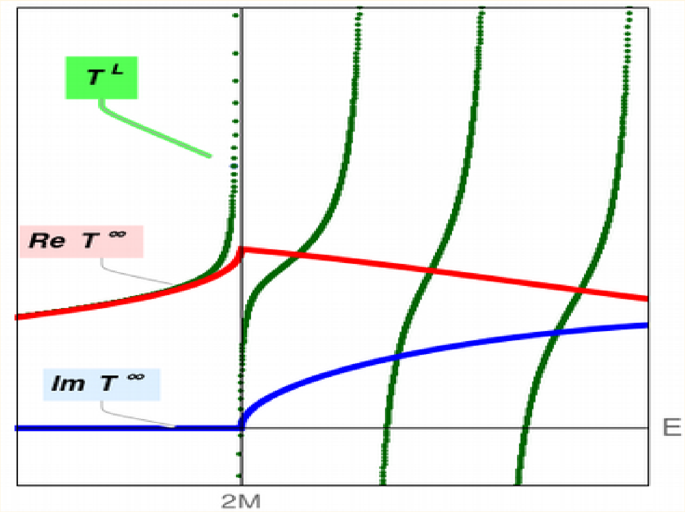

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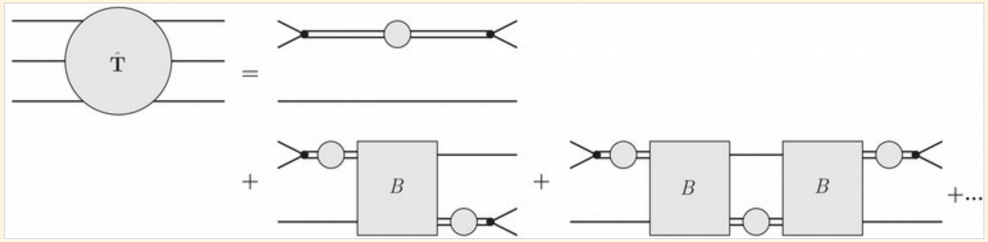

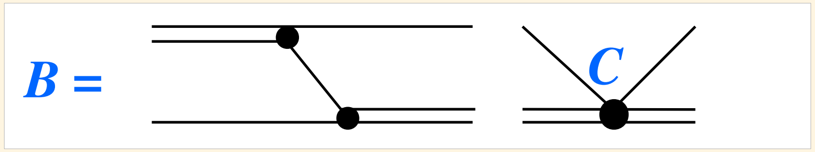

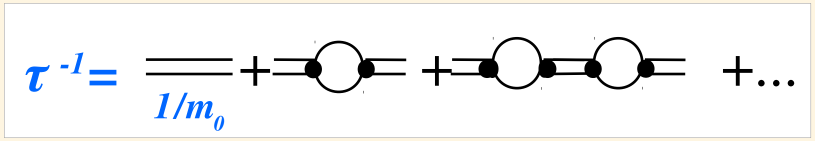

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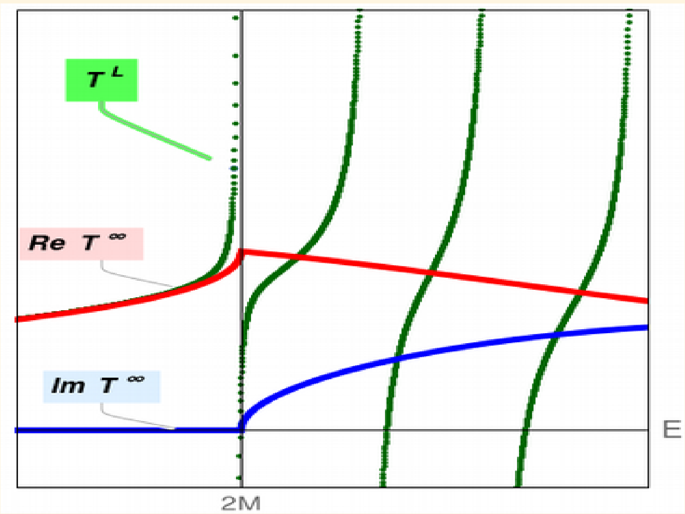

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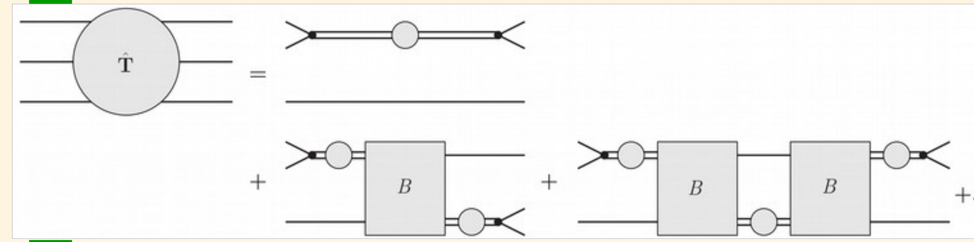

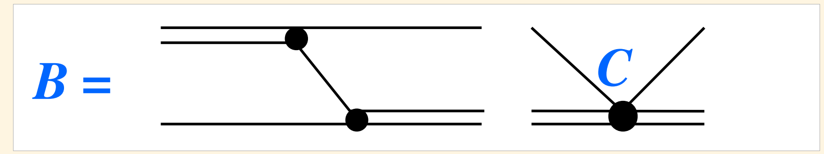

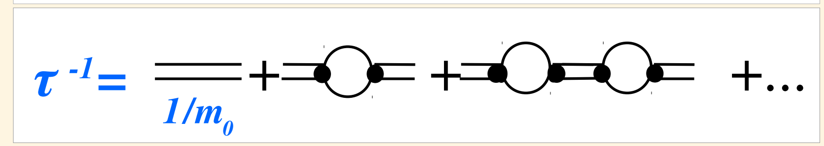

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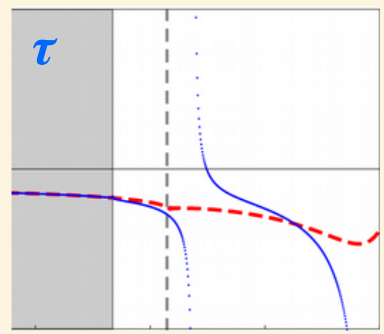

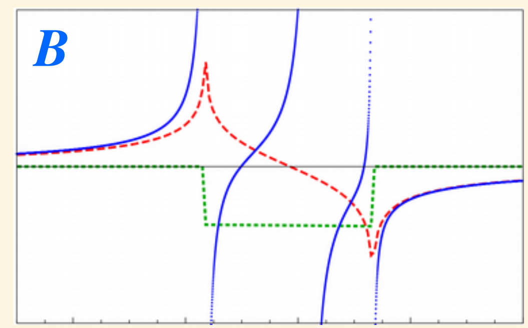

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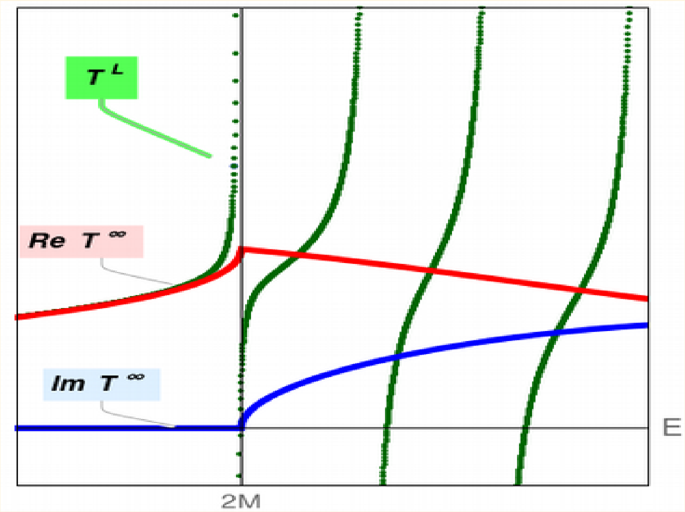

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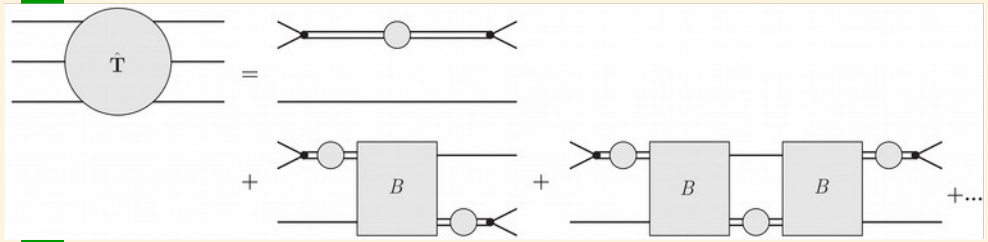

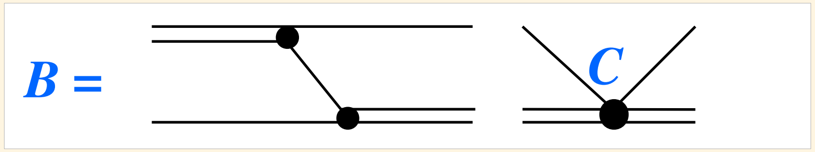

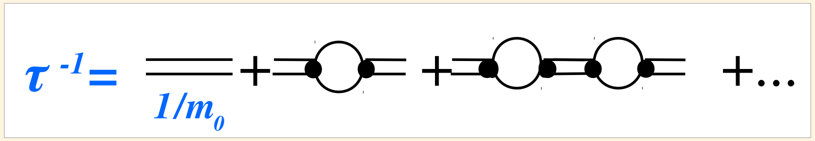

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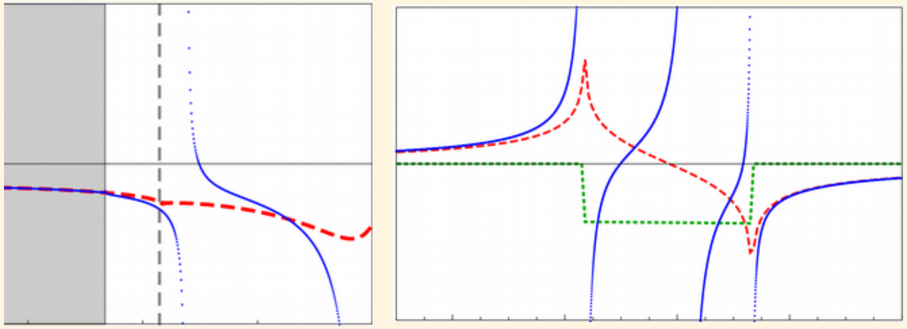


Discretization

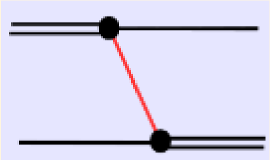
$$T_3^{FV} = v\Big(\tau \frac{B}{1 + B\tau}\tau - \tau\Big)v$$

Quantization condition

$$\boxed{\mathrm{Det}\left(\mathbf{B(E)} + \tau(\mathbf{E})^{-1}\right) = \mathbf{0}}$$

PROJECTION TO IRREPS

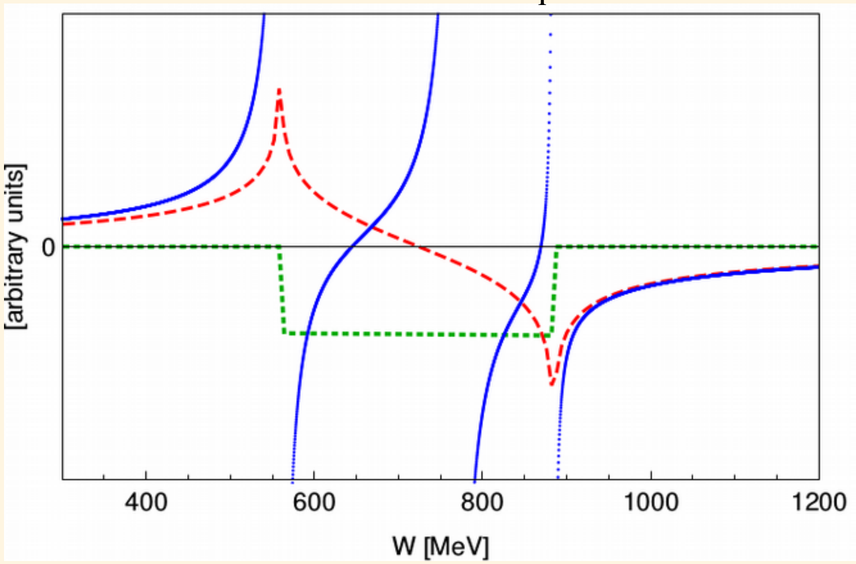
- High-dimensional problem
- *B* (OPE potential) is singular!



☞ **Project to irreps of cubic group:**

$$\Gamma\ \{A_1, A_2, E, T_1, T_2\}$$

S-wave infinite volume vs. A_1^+ finite volume



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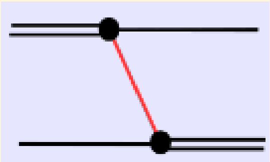

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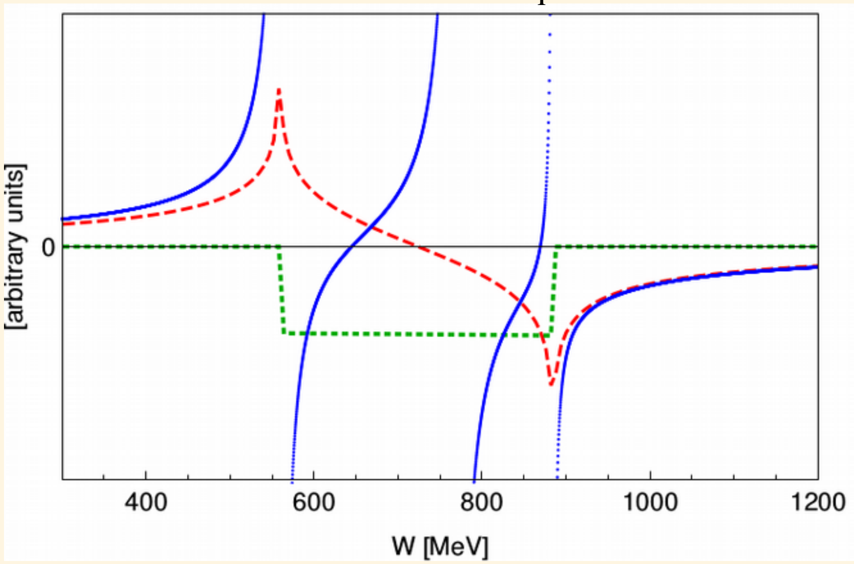


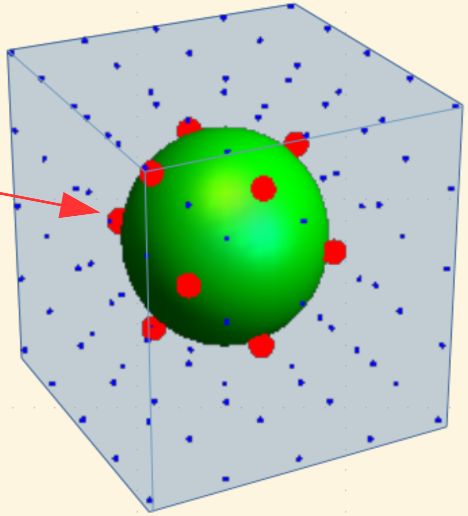
1) Separation of variables

– shells = sets of points related by O_h

2) Find the ONB of functions on each shell

– $$f^s(\hat{\mathbf{p}}_j) = \sqrt{4\pi}\sum_{\Gamma\alpha}\sum_{u} f_u^{\Gamma\alpha s}\chi_u^{\Gamma\alpha s}(\hat{\mathbf{p}}_j)$$

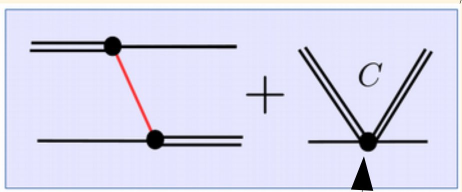
– *infinite volume analog:* PWA



Döring, Hammer, MM, Pang, Rusetsky, Wu (2018)

PROJECTED 3-BODY QC

$$\mathrm{Det}\left(\mathbf{B}_{\mathbf{uu}'}^{\Gamma\mathbf{ss}'}(\mathbf{W^2})+\frac{\mathbf{2E_s\,L^3}}{\vartheta(\mathbf{s})}\tau_{\mathbf{s}}(\mathbf{W^2})^{-\mathbf{1}}\delta_{\mathbf{ss}'}\delta_{\mathbf{uu}'}\right)=\mathbf{0}$$



Fix to 3→3 data

W – total energy

s/s’ - shell index

u/u’ - basis index

ϑ – multiplicity

L – lattice volume

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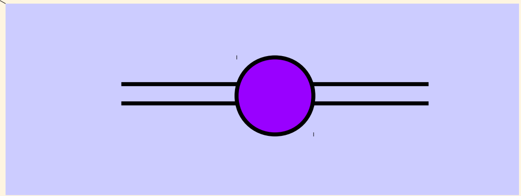

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$T_{22} = v\ \tau\ v$

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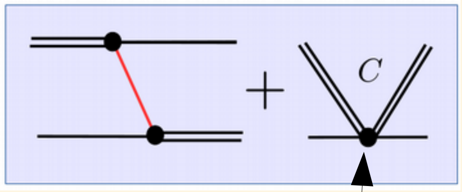

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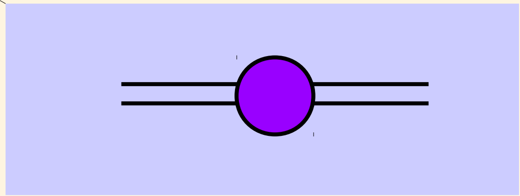

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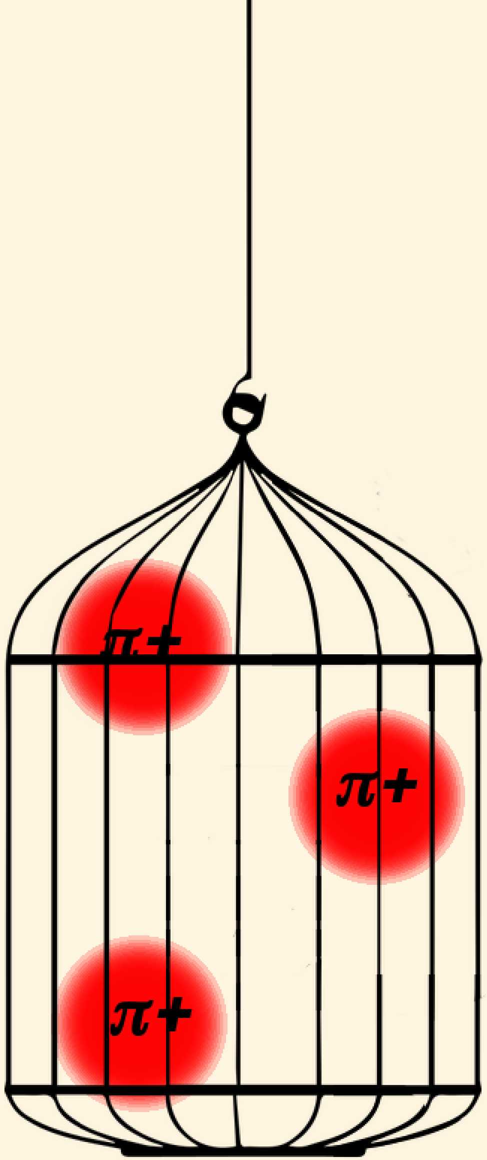
- **relativistic 3-body quantization condition**
- determinant in ***u, s***; diagonal in ***Γ***
- Not a Lüscher-like equation (“left”: infinite volume, “right”: finite volume)

  Instead: Fix parameters to lattice eigenvalues → evaluate infinite-volume amplitude

  Same work-flow as in many 2-body coupled-channel fits (e.g., Doring et al., EPJA (2012))



FINITE-VOLUME SPECTRUM OF $\pi^+\pi^+\pi^+$

MM, Doring [arXiv:1807.04746]

LQCD results from NPLQCD Detmold et al.(2008)

PHYSICAL SYSTEM

$\pi^+\pi^+$ & $\pi^+\pi^+\pi^+$ ground level @ L=2.5 fm &m_π=291/352/491/591 MeV Detmold et al.(2008)

- ☞ ***Q: does the "isobar" picture hold for repulsive channel?***
- ☞ ***Bonus Q: chiral extrapolation in 3body system?***

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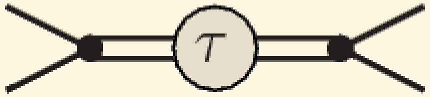
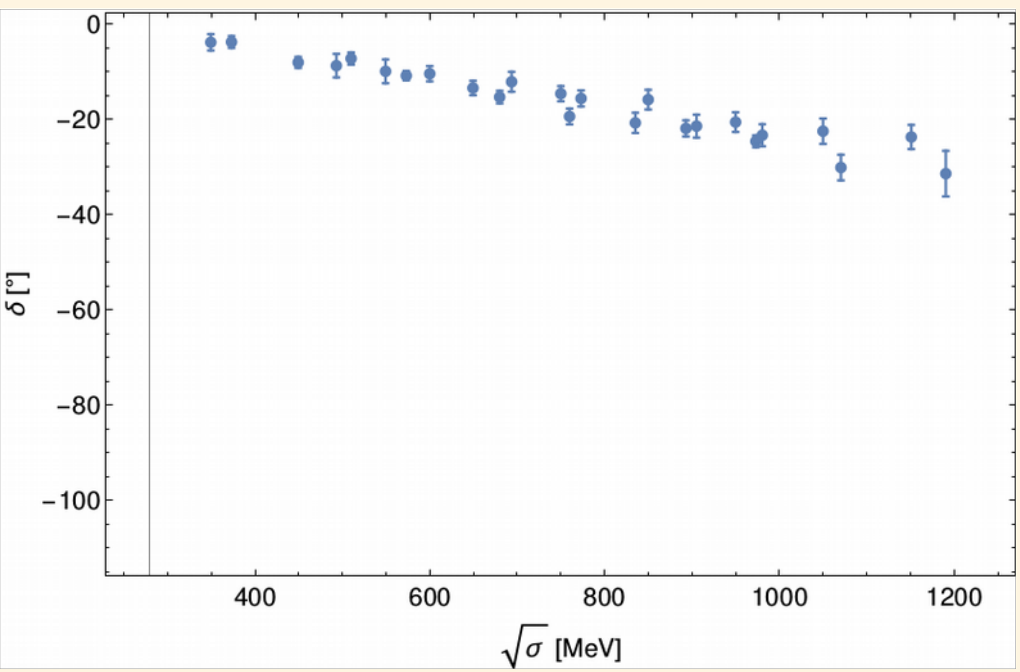
- one-channel problem: $\boldsymbol{\pi\pi}$-system in S-wave, I=2
- 2-body amplitude consistent with 3-body one



$$T_2 = v\frac{1}{\sigma - M_0^2 - \Sigma}v$$



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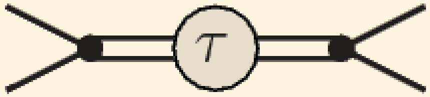
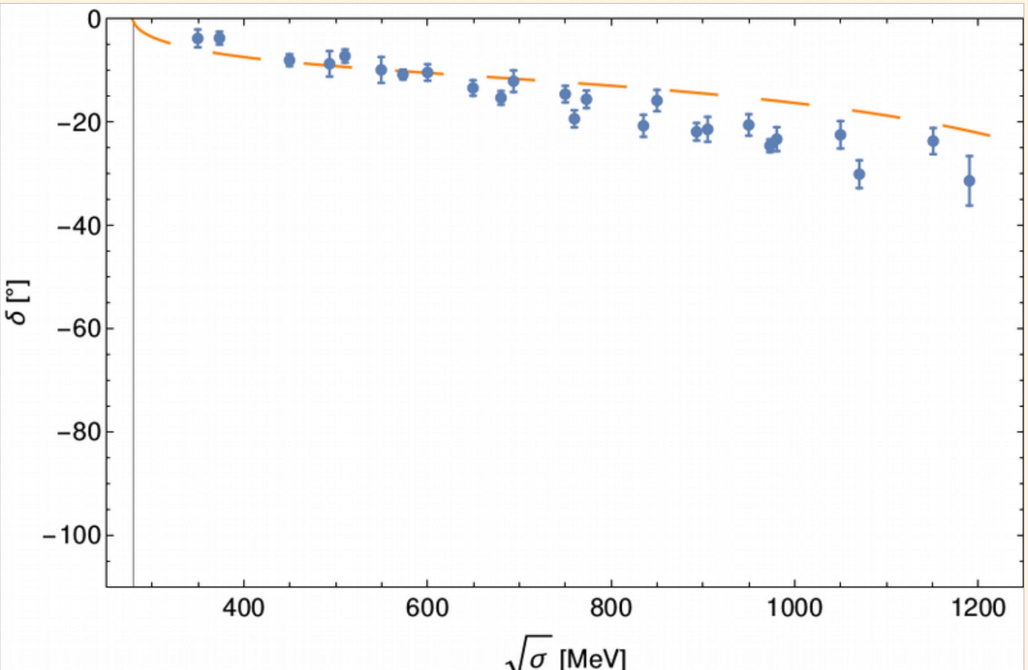


$$T_2 = \frac{-\lambda^2/(32\pi)}{\sigma - M_0^2 - \sum_\pm \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \frac{\lambda^2}{4E_{\boldsymbol{k}}\sqrt{\sigma}(\sqrt{\sigma}\pm 2E_{\boldsymbol{k}})}}$$



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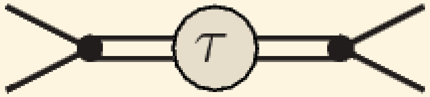

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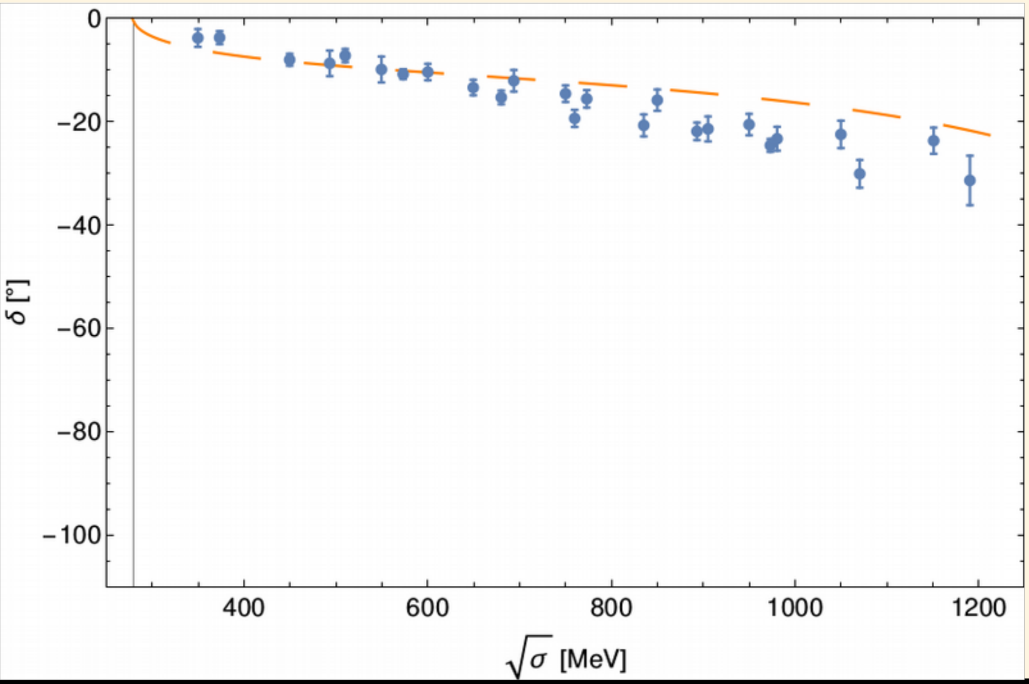

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☹ incorrect m_π behavior!

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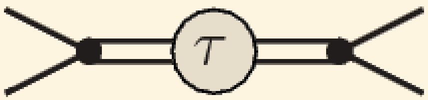

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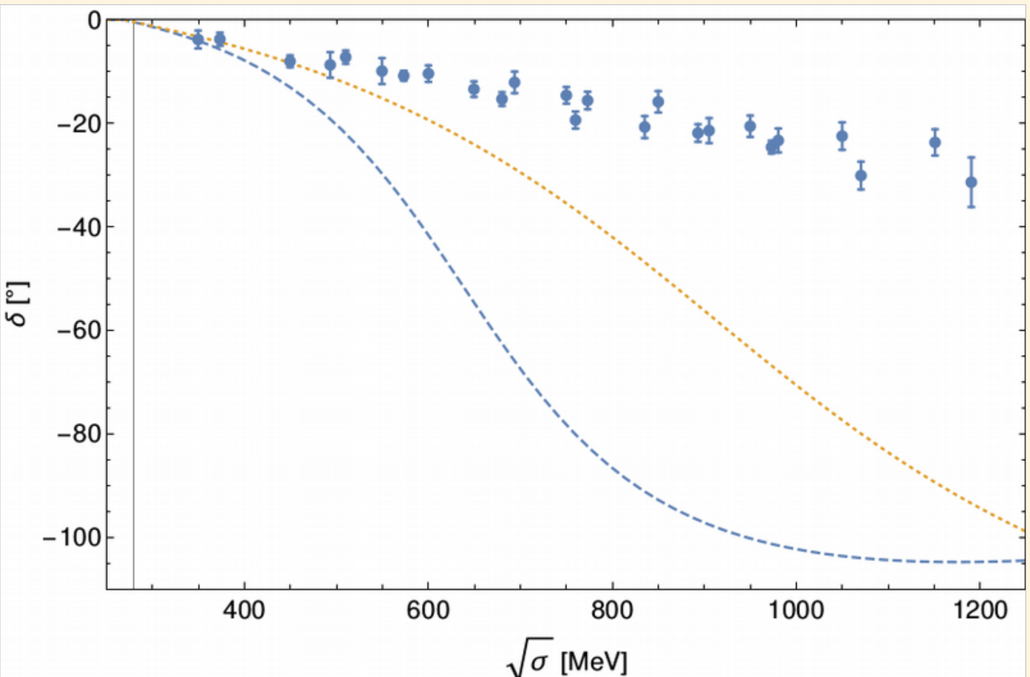

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☹ incorrect $\boldsymbol{m_\pi}$ behavior!

2) Chiral NLO & K-matrix

PHYSICAL SYSTEM

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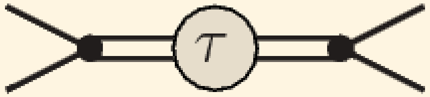

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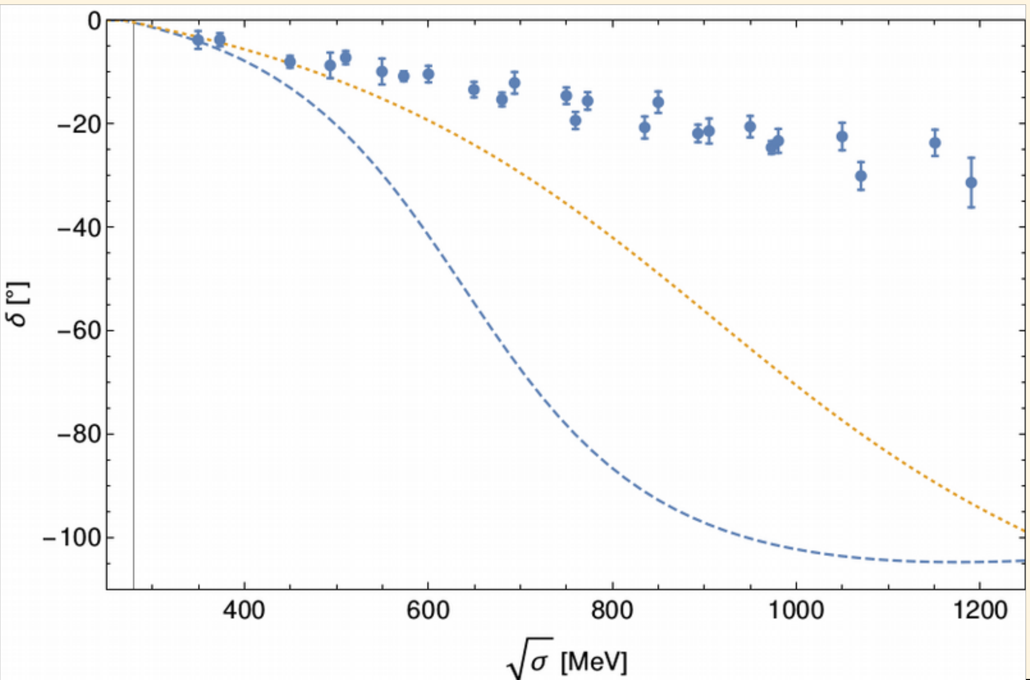

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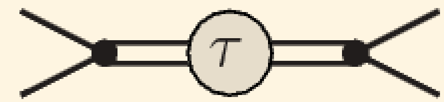

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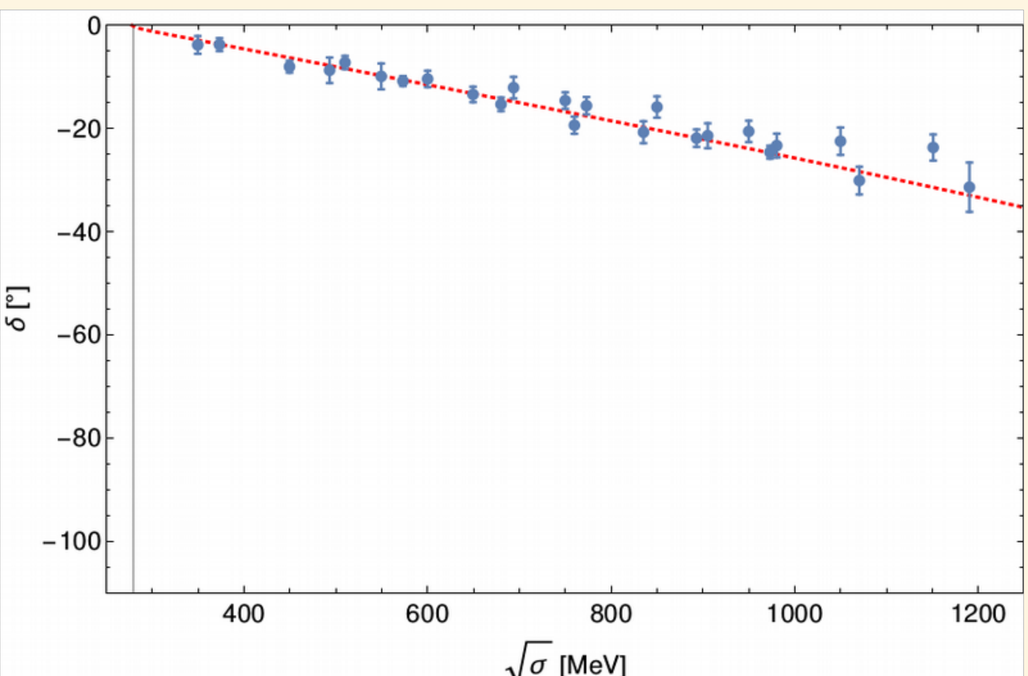

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Truong(1988)

😃 correct $\boldsymbol{\sigma}$ & $\boldsymbol{m_\pi}$ behavior

😃 parameters known

Gasser/Leutwyler(1984)

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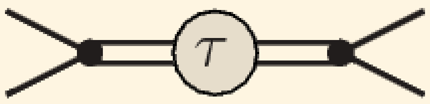

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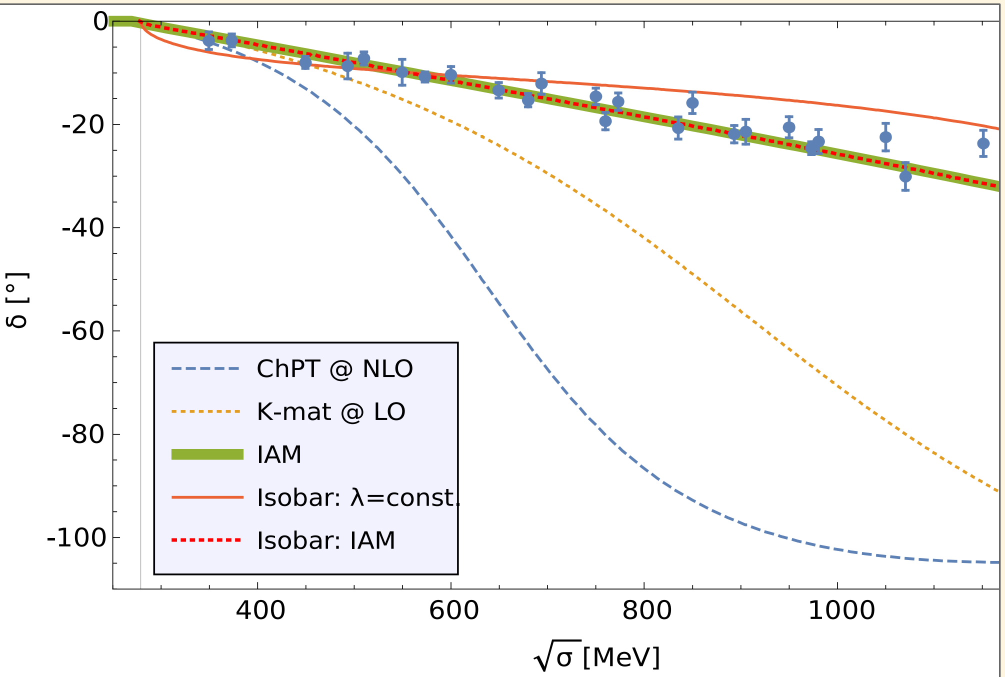

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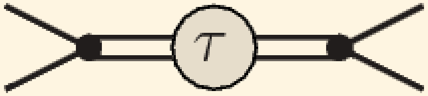

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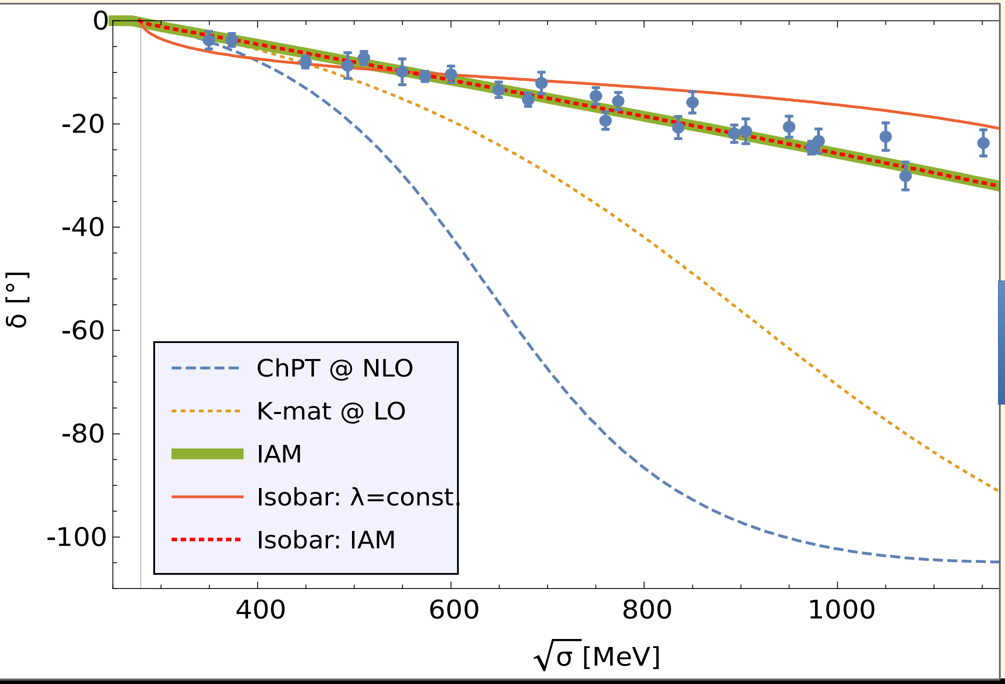
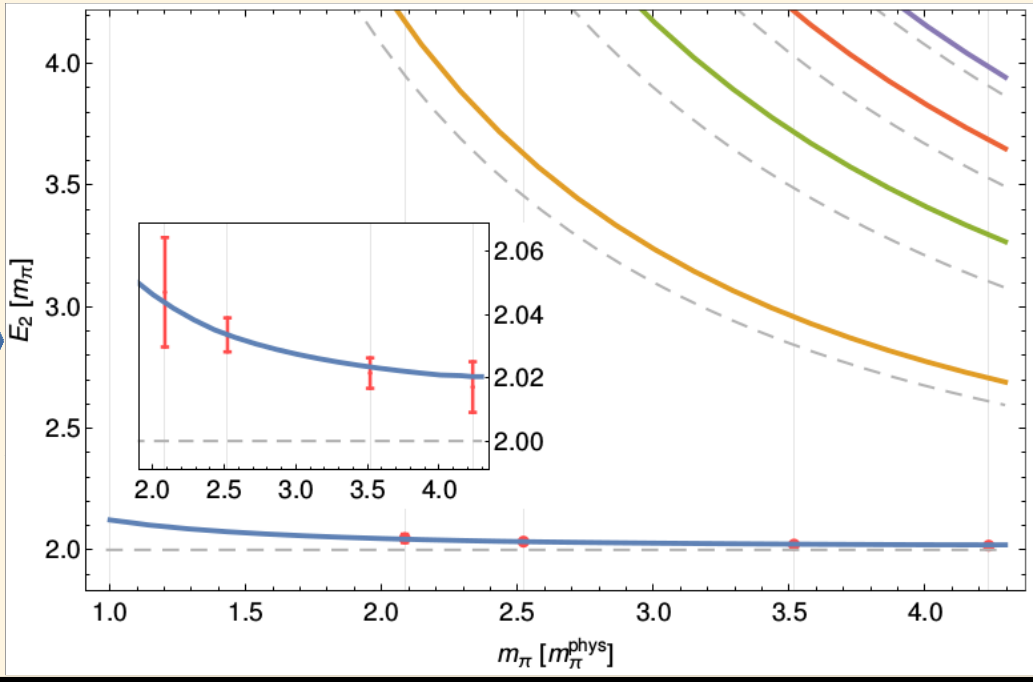
inf-vol input for T_2 → discretize (Lüscher) → predicted fin-vol. spectrum





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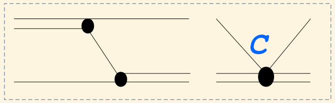
Remaining unknown: ***C***

- *genuine (momenta-dependent) 3-body "force"*
- *simplest case:* $\boxed{C_{\boldsymbol{qp}} = c\,\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q})}$

3-BODY QUANTIZATION CONDITION

$$\mathrm{Det}\left(\mathbf{B}^{\Gamma\mathbf{ss}'}_{\mathbf{uu}'}(\mathbf{W^2}) + \frac{\mathbf{2E_s\,L^3}}{\vartheta(\mathbf{s})}\tau_{\mathbf{s}}(\mathbf{W^2})^{-1}\delta_{\mathbf{ss}'}\delta_{\mathbf{uu}'}\right) = \mathbf{0}$$



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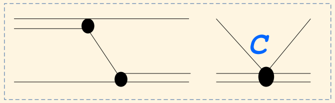

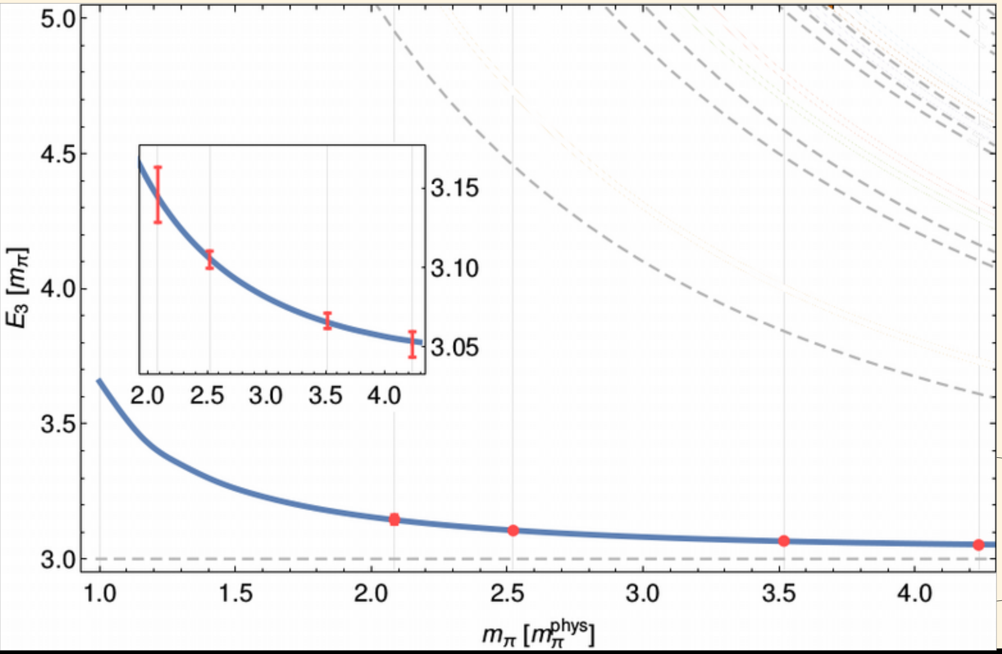

Fit *c* to NPLQCD ground state level
→ *$c=0.2\pm1.5\cdot10^{-10}$*

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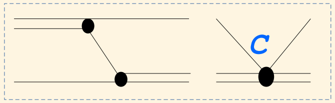

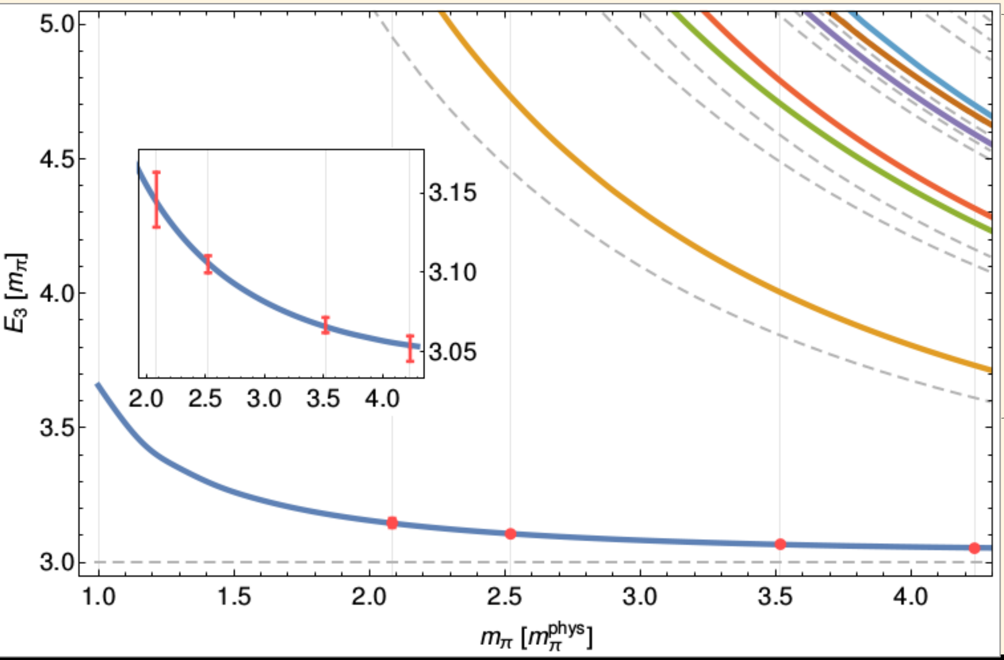

Predict exited spectrum:

→ novel pattern

1/1 of interacting/non-interacting lvls

shifted block wise

→ all genuine poles are simple

→ chiral extrapolation to phys. point

SUMMARY

3-body amplitude in infinite volume

- Parametrization via 2-body sub-channel amplitudes ↔ "isobars"
- Relativistic integral equation
- Equivalent to Khuri-Treiman equations*
- Phenomenological applications in progress…

Analysis of finite-volume spectra

- discretization & projection to irreps of $\mathbf{O_h}$:

  a relativistic 3-body quantization condition
- numerical (ρ-like) toy-examples explored

First num. appl. to physical system: $\pi^+\pi^+\pi^+$

- (excited spectrum) of $\pi^+\pi^+$ & $\pi^+\pi^+\pi^+$ predicted
- ground level compared with NPLQCD results
- new 3-body fin-vol. spectrum features & chiral extrapolation...

OUTLOOK

- include: spin isobars, multiple isobars, unequal masses
- further practical studies: ***a_1(1260)***, ***Roper*** ...

* Ian Aitchison, private communication

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THANK YOU

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BACKUP

Cancellations:

→ fin. vol. normalization of δ-distribution!

$$\bar{T}^{A_1^+}_{nm}(s) = \tau_n(s) T^{A_1^+}_{nm}(s) \tau_m(s) - 2E_n \tau_n(s) \frac{L^3}{\vartheta(n)} \delta_{nm}$$

$$T^{A_1^+}_{nm}(s) = B^{A_1^+}_{nm}(s) - \frac{1}{L^3} \sum_{x \in set_8} \vartheta(x) B^{A_1^+}_{nx}(s) \frac{\tau_x(s)}{2E_x} T^{A_1^+}_{xm}(s)$$

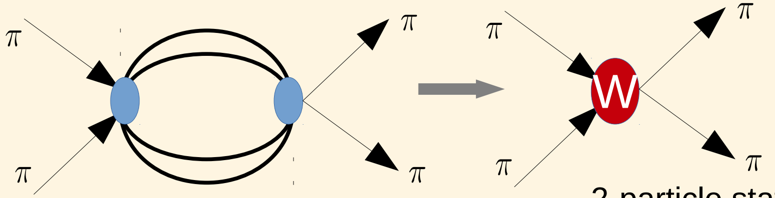
$B^{A_1^+}$ singular at $W^+ = E_m + E_n + E(\boldsymbol{q}_{nj} + \boldsymbol{p}_{mi})$

τ_m^{-1} singular at $W^{\pm\pm} = E_m \pm E((2\pi/L)\boldsymbol{y}) \pm E((2\pi/L)\boldsymbol{y} + \boldsymbol{p}_{mi})$ for $\boldsymbol{y} \in \mathbb{Z}^3$

– when isobar-momenta are discretized in the 3-body cms momenta

$$\tau = \sigma(k) - M_0^2 - \frac{1}{(2\pi)^3} \int d^3\boldsymbol{\ell} \frac{\lambda^2}{2E_\ell(\sigma(k) - 4E_\ell^2 + i\epsilon)}$$

Optical potential: The formal rewriting of a complicated scattering problem



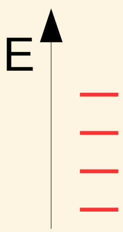
n-particle states
m coupled channel

2-particle state
1 channel
BUT
complex potential
with full dynamics

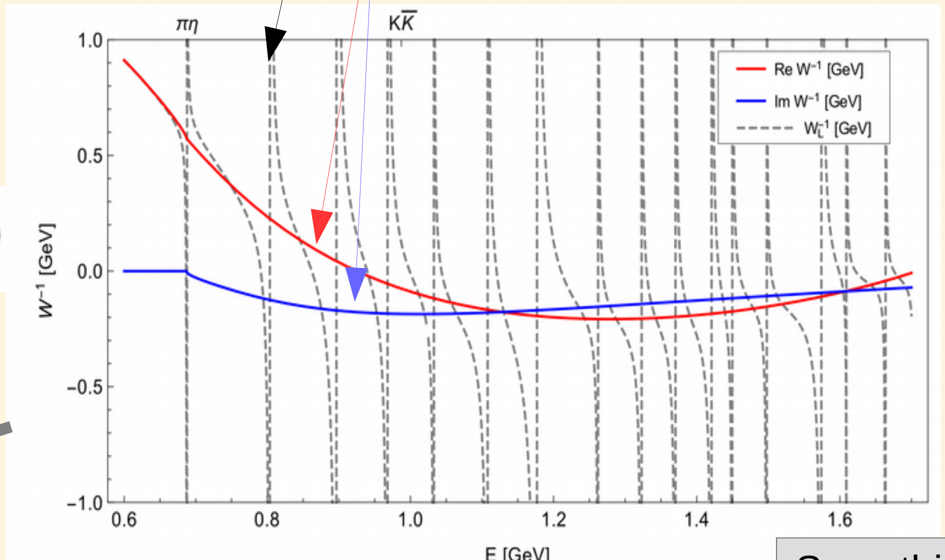
- Measured finite-volume optical potential
- Poles/functional form contain full multi-channel/ multi-particle dynamics
- How to efficiently measure this function → later

How to reconstruct true OP (complex) from finite volume OP (real)?

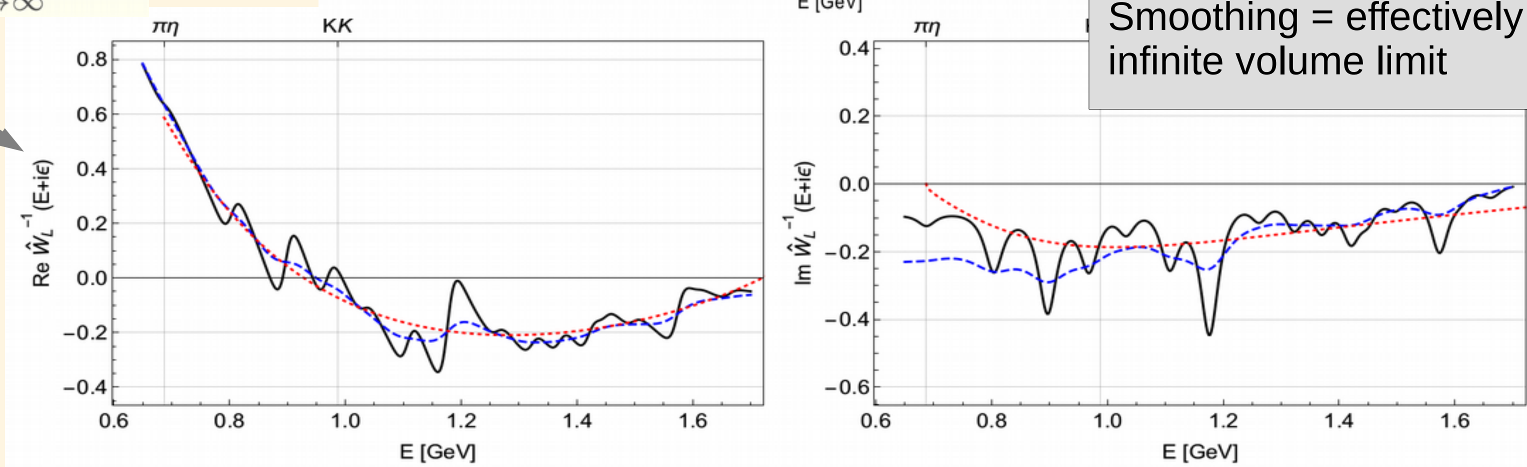
Lattice: measure eigenvalues, map to the optical potential



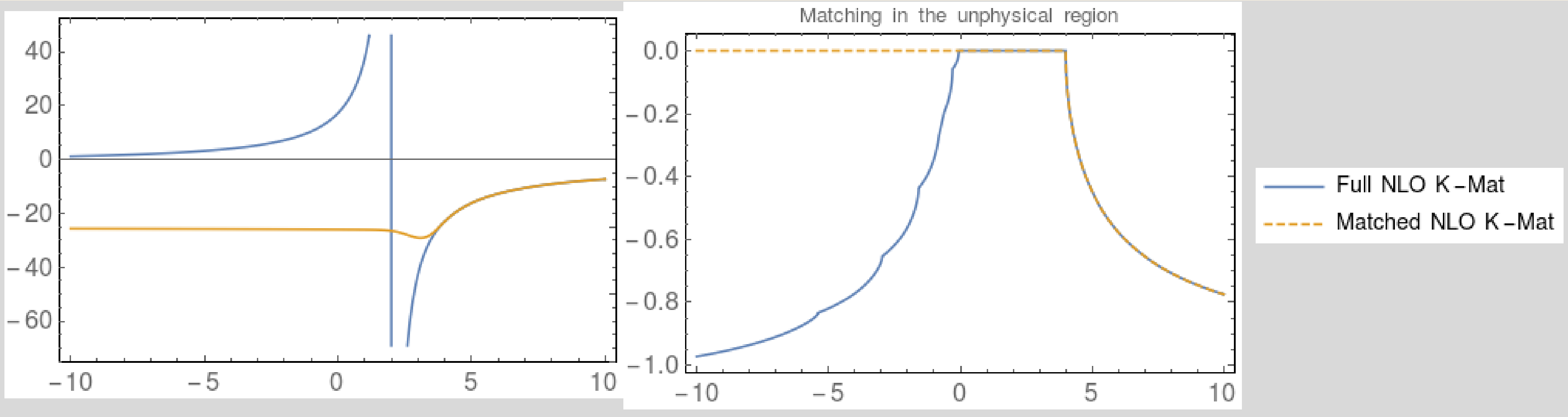
$$W_L^{-1}(E) \doteq \frac{2}{\sqrt{\pi}L} Z_{00}(1; q^2_{K\bar{K}})$$

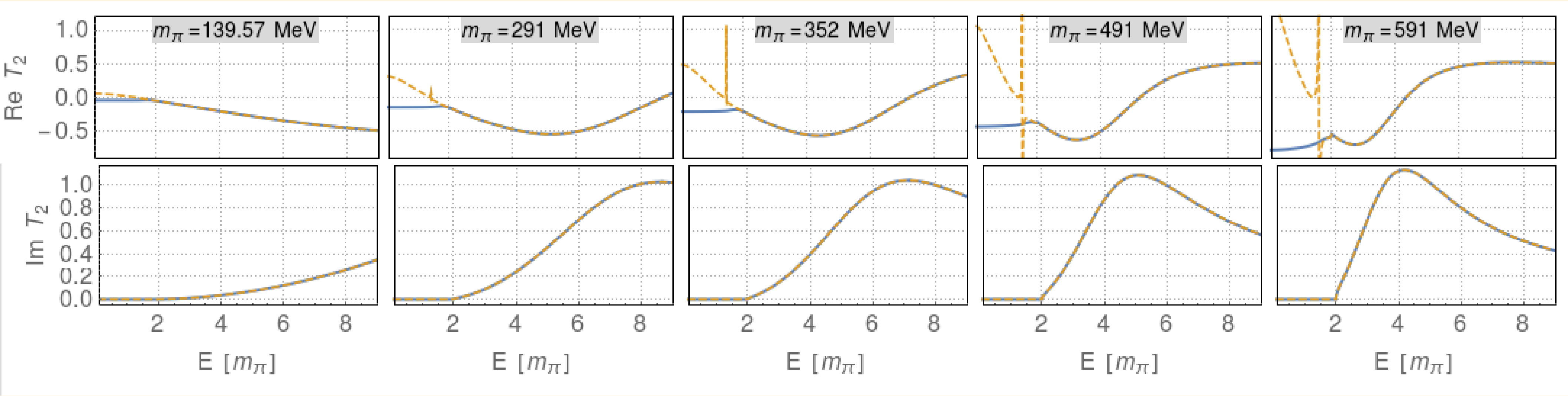


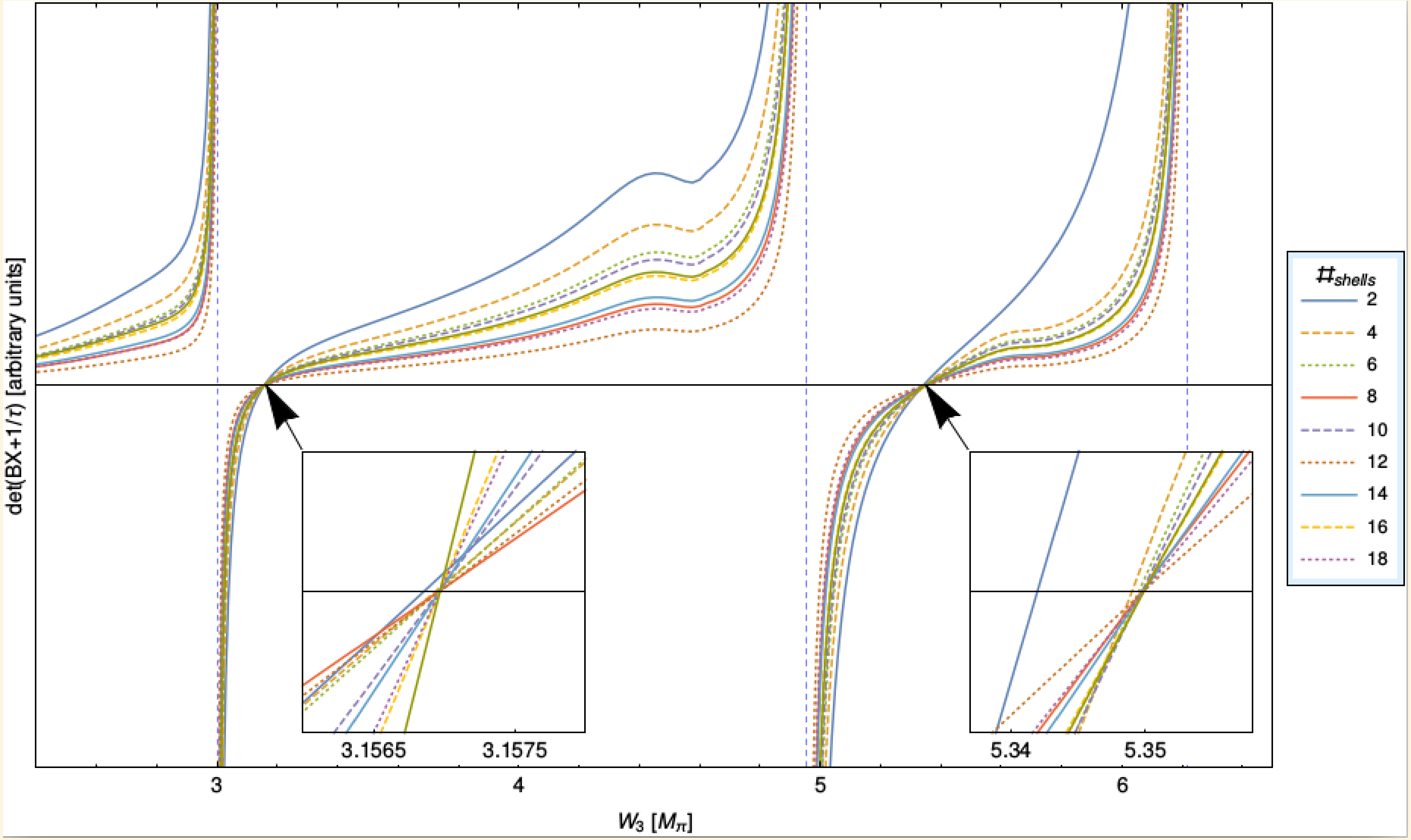
$$W^{-1} = \lim_{\epsilon\to 0} \lim_{L\to\infty} W_L^{-1}$$

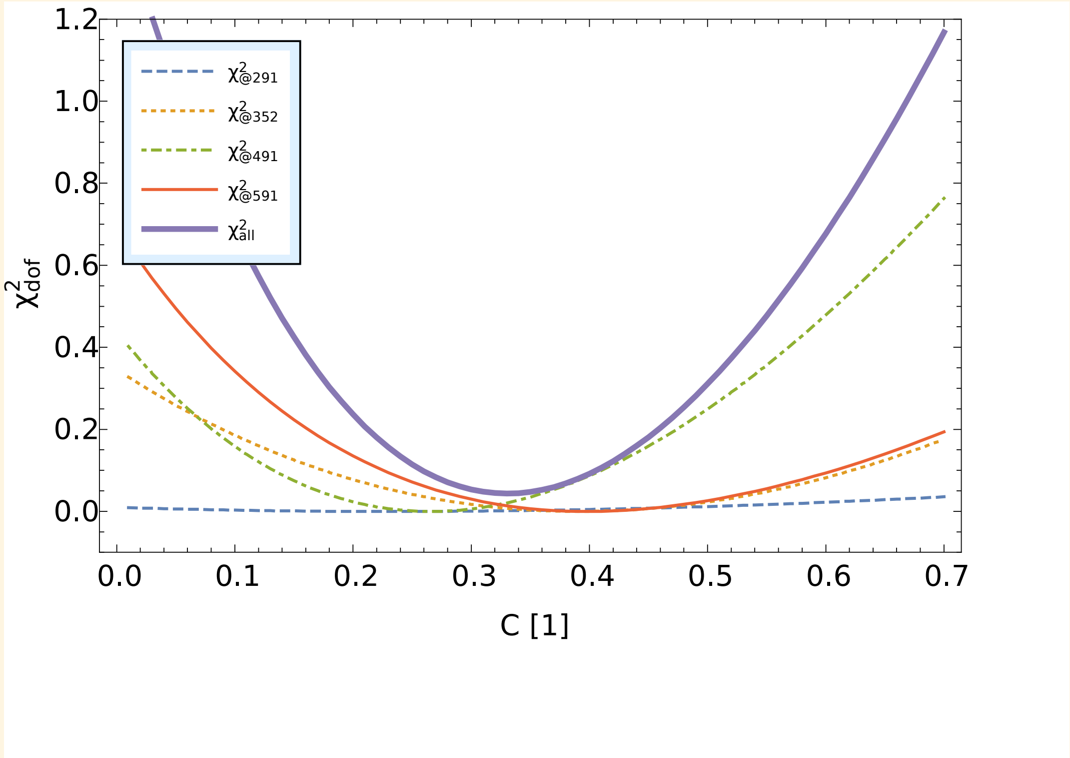


Smoothing = effectively taking infinite volume limit


Matching in the unphysical region
Full NLO K-Mat
Matched NLO K-Mat


m_π = 139.57 MeV
m_π = 291 MeV
m_π = 352 MeV
m_π = 491 MeV
m_π = 591 MeV
Re T_2
Im T_2
E [m_π]


det(BX+1/τ) [arbitrary units]
W_3 [M_π]
3
4
5
6
3.1565
3.1575
5.34
5.35
#shells
2
4
6
8
10
12
14
16
18



1.2
1.0
0.8
0.6
0.4
0.2
0.0
χ^2_{dof}
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
C [1]
$\chi^2_{@291}$
$\chi^2_{@352}$
$\chi^2_{@491}$
$\chi^2_{@591}$
χ^2_{all}

m_π [MeV]	139.57	291	352	491	591
E_2^1 $[m_\pi]$	$2.1228^{+0.0068}_{-0.0069}$	$2.0437^{+0.0071}_{-0.0086}$	$2.0334^{+0.0076}_{-0.0086}$	$2.0233^{+0.0105}_{-0.0098}$	$2.0204^{+0.0200}_{-0.0106}$
Refs. [24, 25]	–	**2.0471(27)(65)**	**2.0336(22)(22)**	**2.0215(16)(13)**	**2.0171(16)(19)**
E_2^2 $[m_\pi]$	–	–	$3.6245^{+0.0746}_{-0.0299}$	$2.9556^{+0.0728}_{-0.0263}$	$2.7045^{+0.0827}_{-0.0271}$
E_2^3 $[m_\pi]$	–	–	–	$3.7114^{+0.1482}_{-0.0737}$	$3.2911^{+0.1241}_{-0.0688}$
E_2^4 $[m_\pi]$	–	–	–	–	$3.6802^{+0.0707}_{-0.0902}$
E_2^5 $[m_\pi]$	–	–	–	–	$3.9829^{+0.0500}_{-0.0299}$
E_3^1 $[m_\pi]$	$3.6564^{+0.1014}_{-0.0847}$	$^*3.1444^{+0.0171}_{-0.0192}$	$^*3.1058^{+0.0091}_{-0.0147}$	$^*3.0655^{+0.0029}_{-0.0095}$	$^*3.0537^{+0.0048}_{-0.0119}$
Refs. [24, 25]	–	**3.1458(49)(125)**	**3.1050(27)(27)**	**3.0665(26)(22)**	**3.0516(27)(53)**
E_3^2 $[m_\pi]$	–	–	$4.7301^{+0.1577}_{-0.1027}$	$4.0031^{+0.0196}_{-0.1836}$	$3.7315^{+0.0309}_{-0.0742}$
E_3^3 $[m_\pi]$	–	–	–	$4.7043^{+0.0126}_{-0.5923}$	$4.2621^{+0.0001}_{-0.1739}$
E_3^4 $[m_\pi]$	–	–	–	$4.7890^{+0.0506}_{-0.1722}$	$4.3155^{+0.0837}_{-0.1341}$
E_3^5 $[m_\pi]$	–	–	–	–	$4.5913^{+0.0001}_{-0.1995}$
E_3^6 $[m_\pi]$	–	–	–	–	$4.6634^{+0.0001}_{-0.1070}$
E_3^7 $[m_\pi]$	–	–	–	–	$4.6995^{+0.0001}_{-0.0661}$