Longest pedestrian bridge in the world. Zermatt (Switzerland) 2020

Bridging 3-body systems **in** and **out** of the box

Maxim Mai

maxim-mai.github.io/INT20.pdf

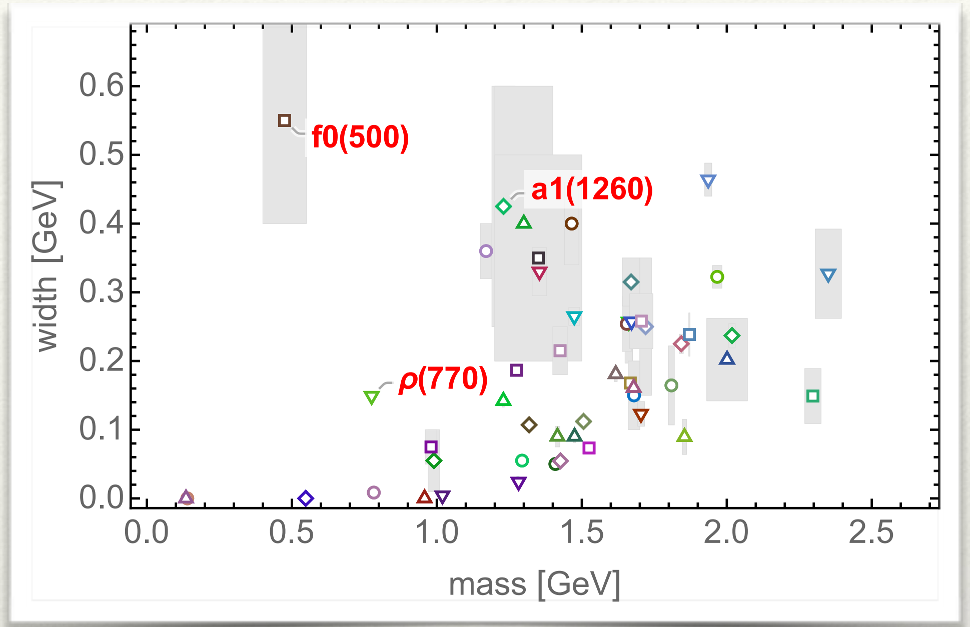
QCD at low energies

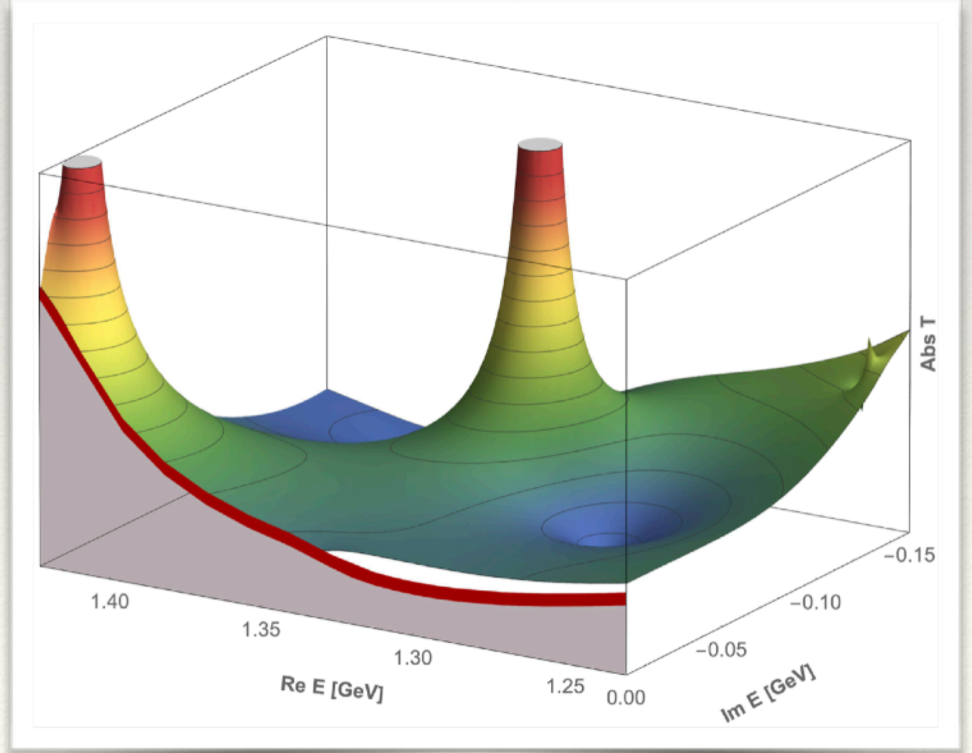
- *Confinement*
- *Intricate spectrum of excited states*
- *Universal parameters of resonances:*

  - *analytic properties of scattering amplitude*

pole position ~ (mass - i · width/2)

residuum ~ (coupling constant)





https://maxim-mai.github.io/INT20.pdf

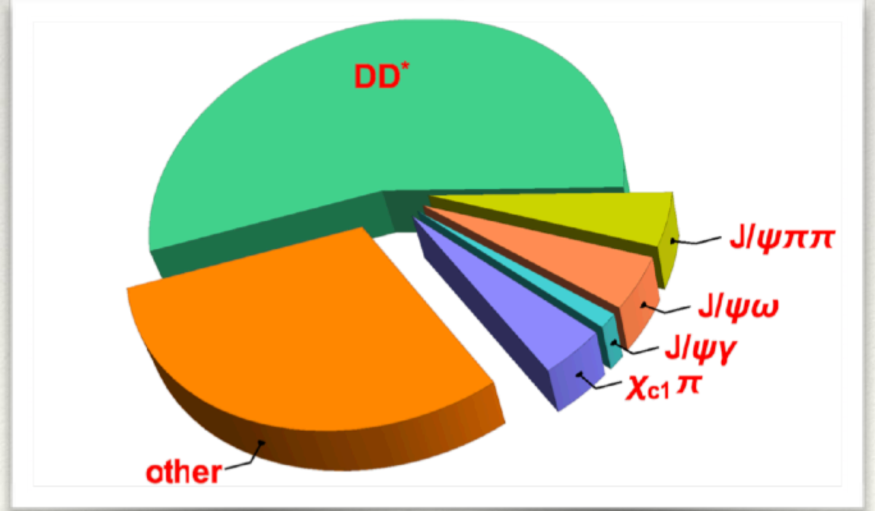
Puzzles

- $a_1(1260)$
  - *does **not** decay into **ππ** but only **πππ** channel*
  - *test channel for the search for **spin-exotics***
  - → ***gluonic d.o.f.** (GlueX, COMPASS)*

TALKS(2.week): M.Mikhasenko/A.Austregesilo

GlueX 201x

- $X(3872)$
  - *puzzling production mechanism*
  - *large BR to DbarDπ*
  - *...similarly for further heavy XYZ exotics*

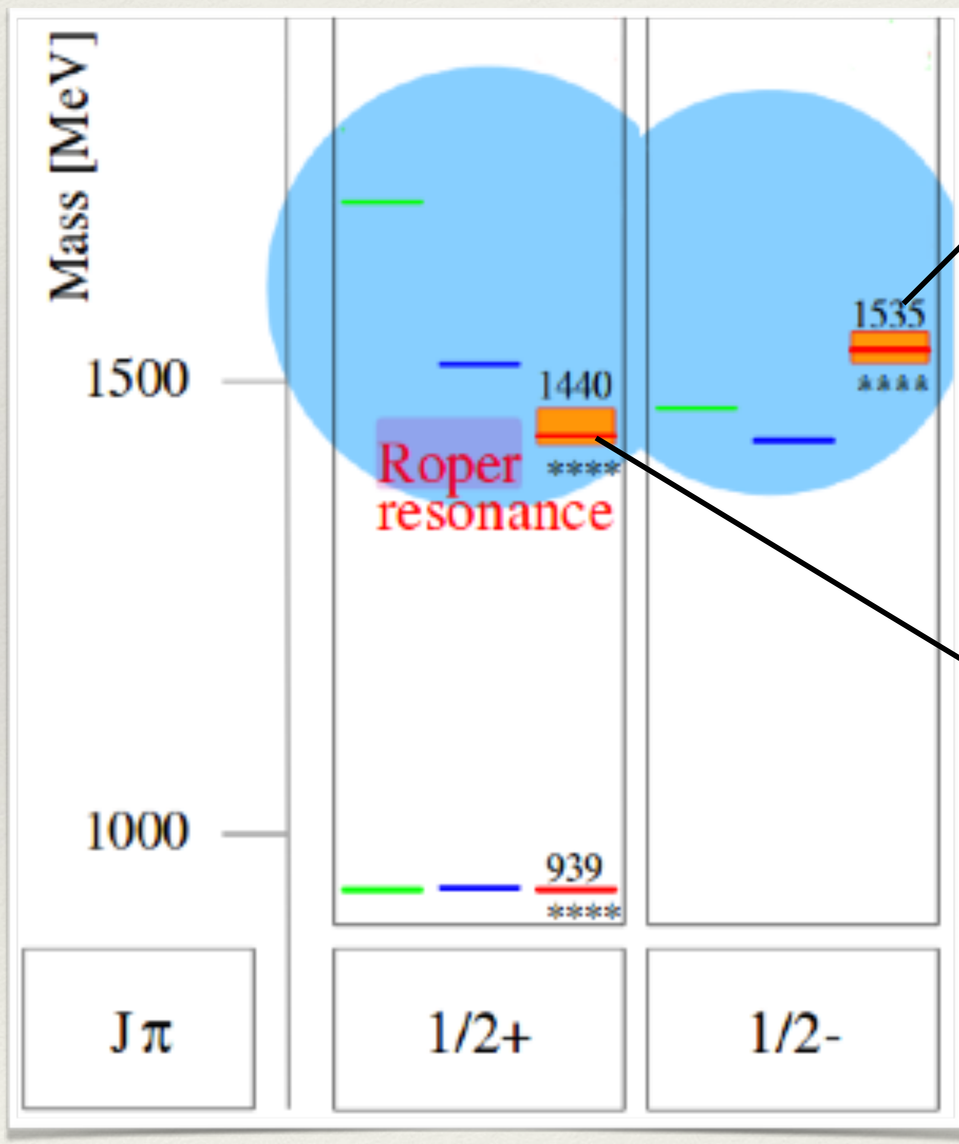


Belle 2003

Puzzles

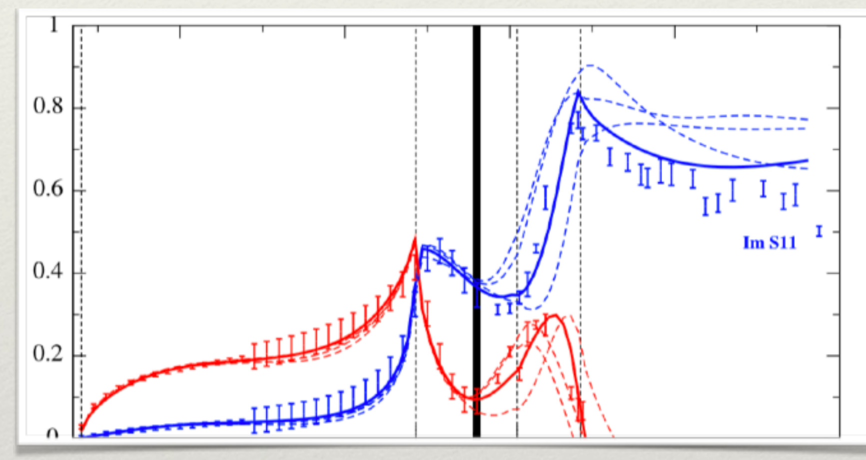
- *Roper-puzzle — N(1440)1/2⁺*

reversed mass pattern cf. constituent Quark Model



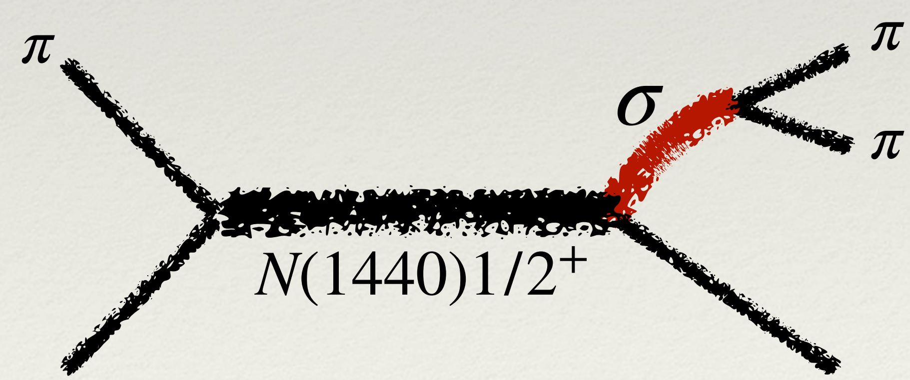
Loring et al. EPJA10 (2001)

***N(1535)1/2⁻** easily accessible from, e.g., chiral unitary approaches*



Bruns, MM, Meißner PLB 697(2011)

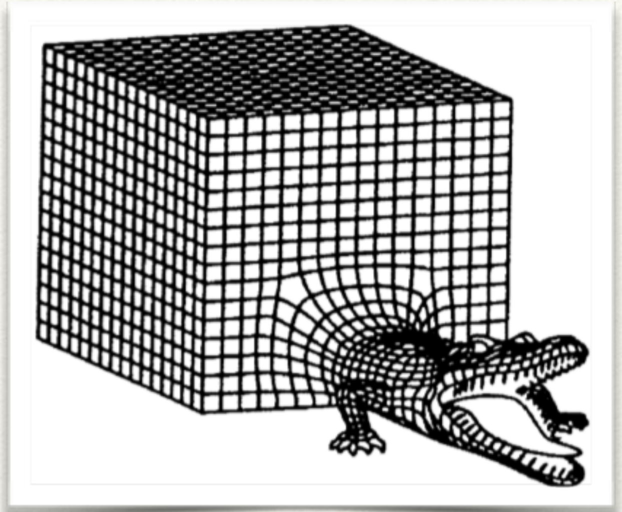
*...but **N(1440)1/2+** has large BR to $\pi\pi N$*



Lattice QCD

- *The only systematic (non-perturbative) approach to QCD*

TALK(1.week): C.Thomas



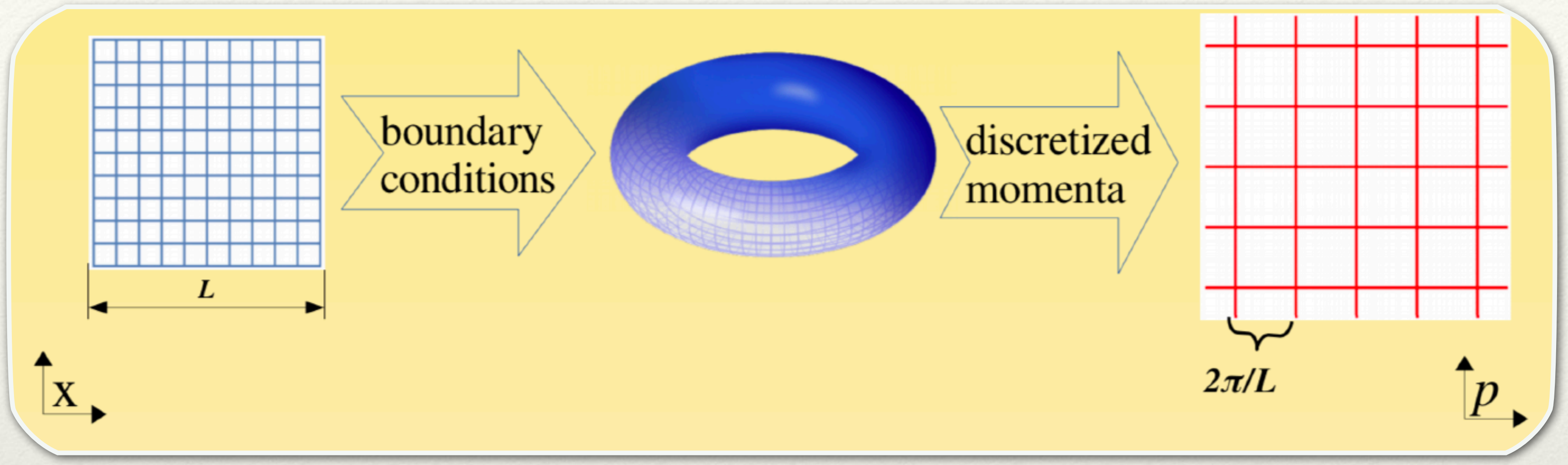
numerical ab-initio calculations

... with some technical hurdles:

1. ***discretized Euclidean space-time***
   - *continuum limit*
   - *often question of LQCD resources*
2. ***unphysical pion mass***
   - *well established techniques from ChPT* — **Gasser/Leutwyler 1984**
   - *extensions to SU(3), baryons, resonances, ...* — **Becher, Meißner, Pelaez…**
3. ***finite volume — "box"***
   - *well understood for 2 hadrons* — **Lüscher 1986**
   - *new developments for 3-hadron systems* — **…this talk**

Lattice QCD

- *Lattice calculations in finite volume*



- *all momenta are discretized*
- *interaction spectrum is **real-valued** and **discrete***

2-body: state of the art

- *2-body case*
  - *well established (**Lüscher's method**)* — **Lüscher (1986)**
  - ***one-to-one mapping** between phase-shifts and energy eigenvalues*
  - *extensions to coupled-channel, spin, …* — **Gottlieb, Rummukainen, Feng, Li, Liu, Döring, Briceño, Rusetsky, Bernard, Meißner…**

TALK(1.week): C.Thomas

  - *Well studied example:* **ππ**-scattering

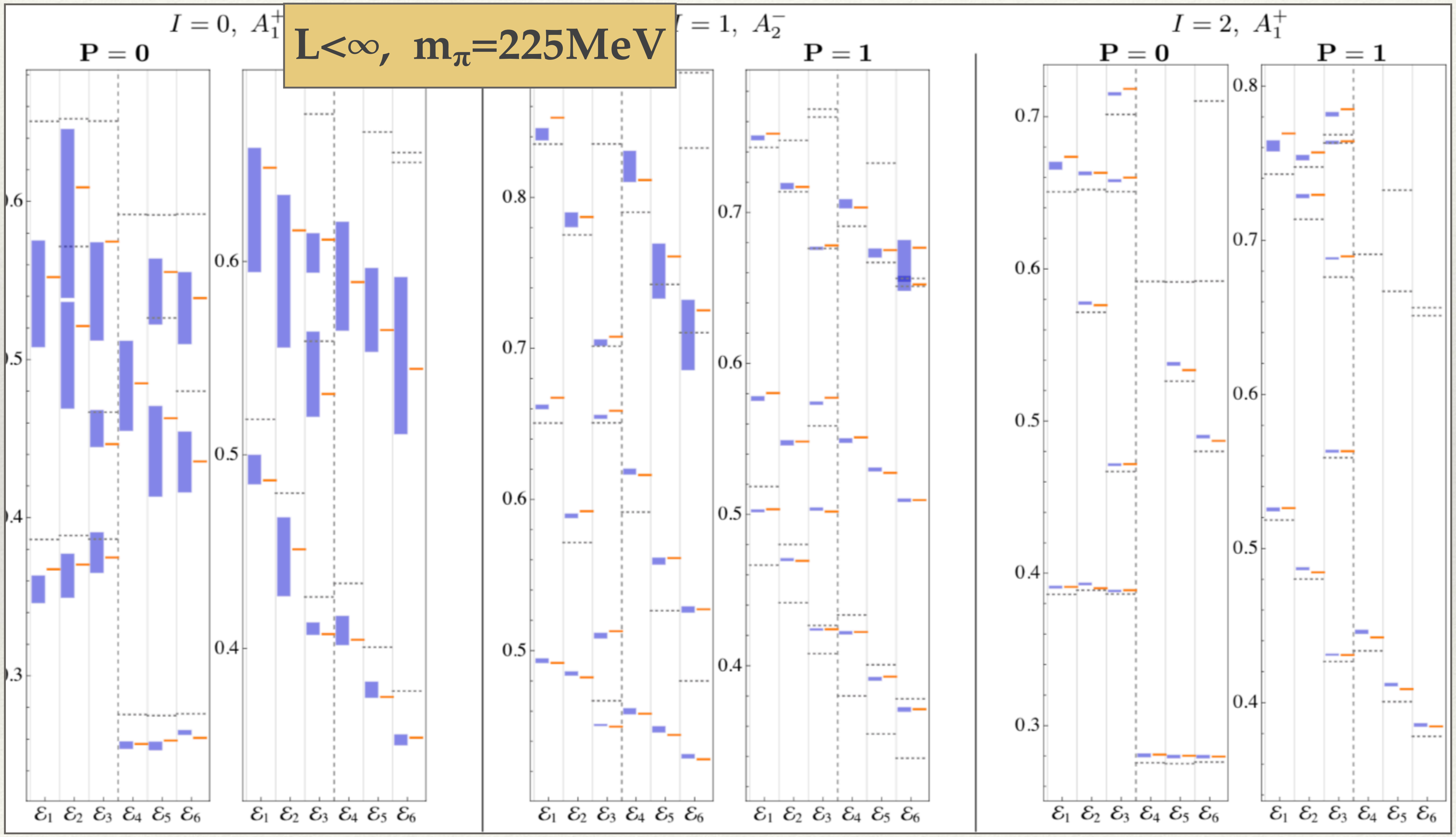
NPLQCD;
HadSpec;
ETMC;
GW-LAT;
CP-PACS;
….

2-body: state of the art

- *cross-channel description of all* **ππ** *interaction channels* ***(N_f= 2)***

GW-LAT: Guo et al. (2016) Guo et al. (2018) Culver et al. (2019) MM et al.(2019)

Phys.Rev.D 100 (2019)



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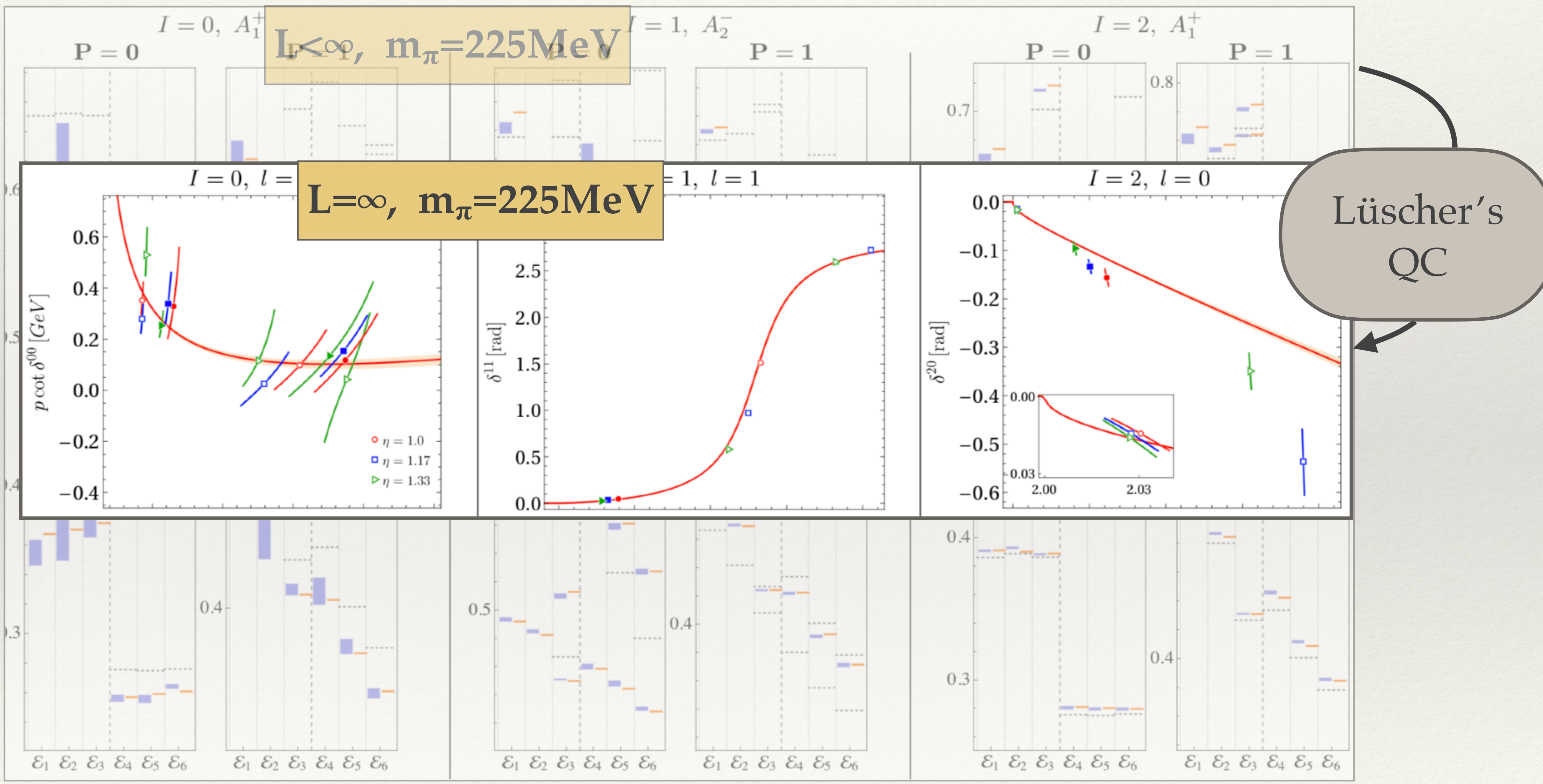

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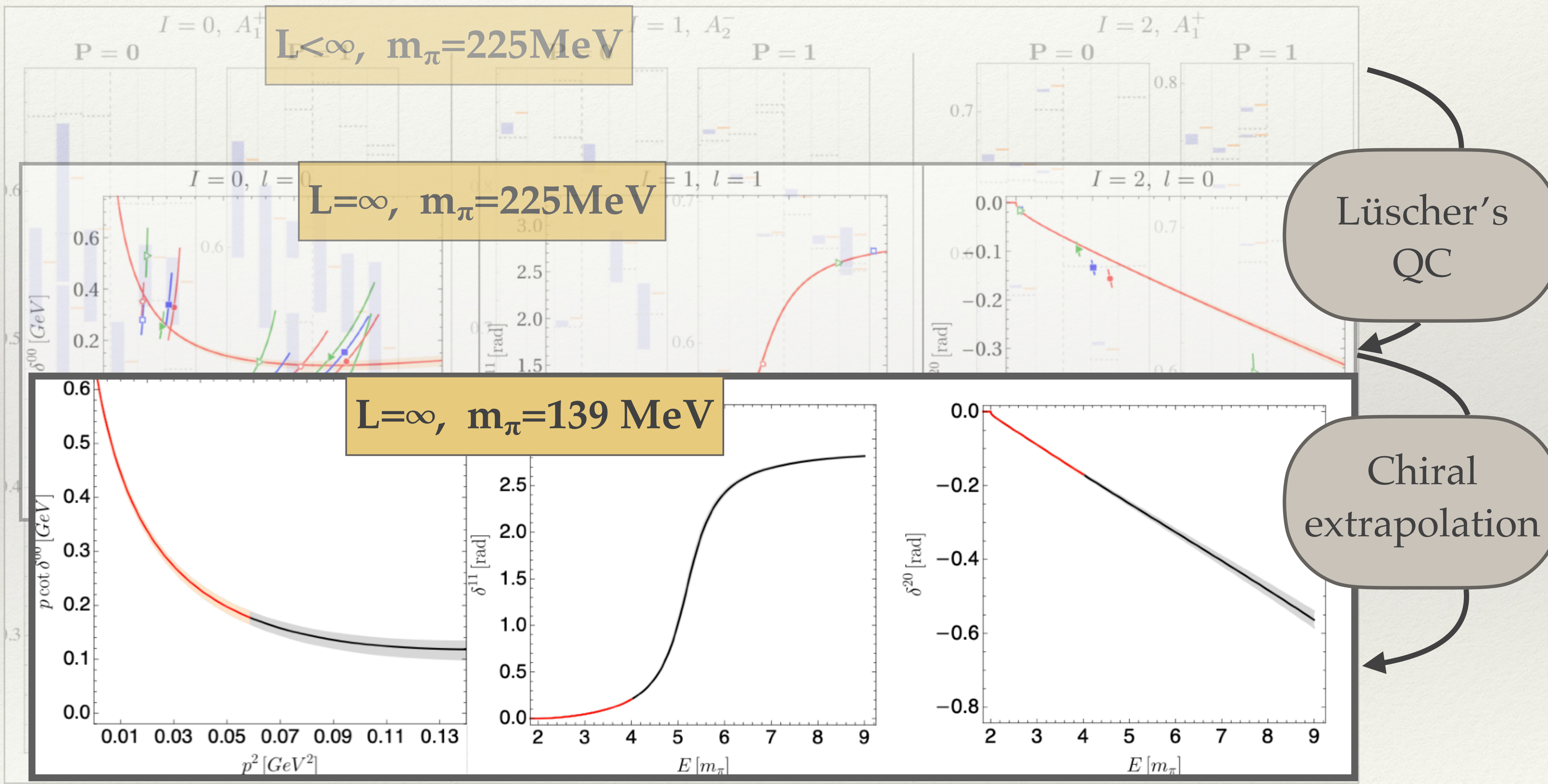

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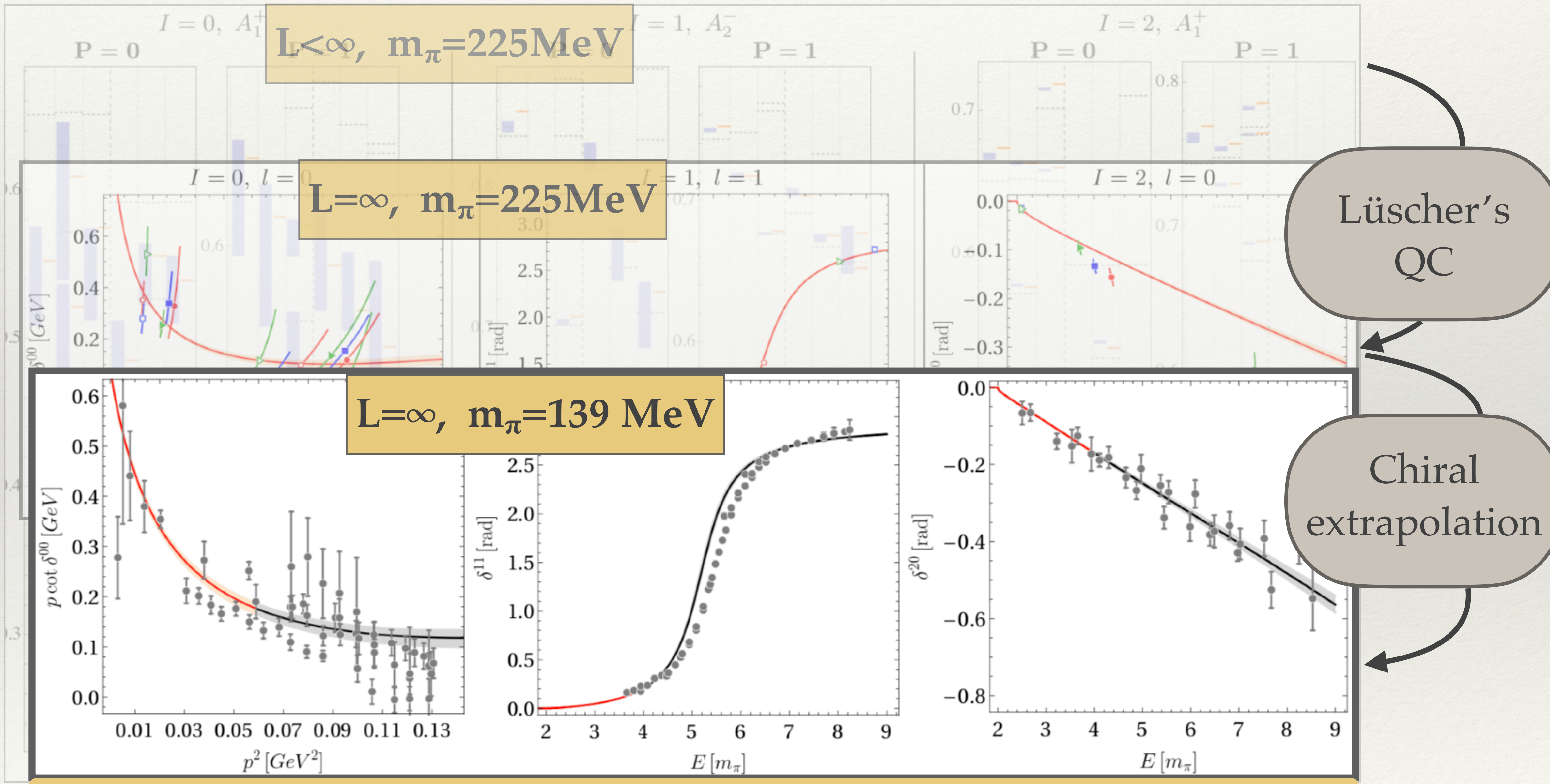

A THEORIST DREAM: *NO FIT!*

FULL FLEDGED THEOR. PREDICTION VS EXP.

3-body: state of the art

- *Challenges*

  - complex kinematics (8 variables)

  - complex interaction structure (sub-channels)

TALK (2. week): A. Jackura

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- *Progress towards lattice applications*

  - *theoretical developments & numerical investigation*

Rusetsky, Polejaeva, Sharpe, Meißner, Davoudi, Hansen, Guo, Briceño, MM, Döring, Hammer, Blanton, Griesshammer, Wu, Bedaque, Romero-López, Pang

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Review: Hansen,Sharpe (2019)

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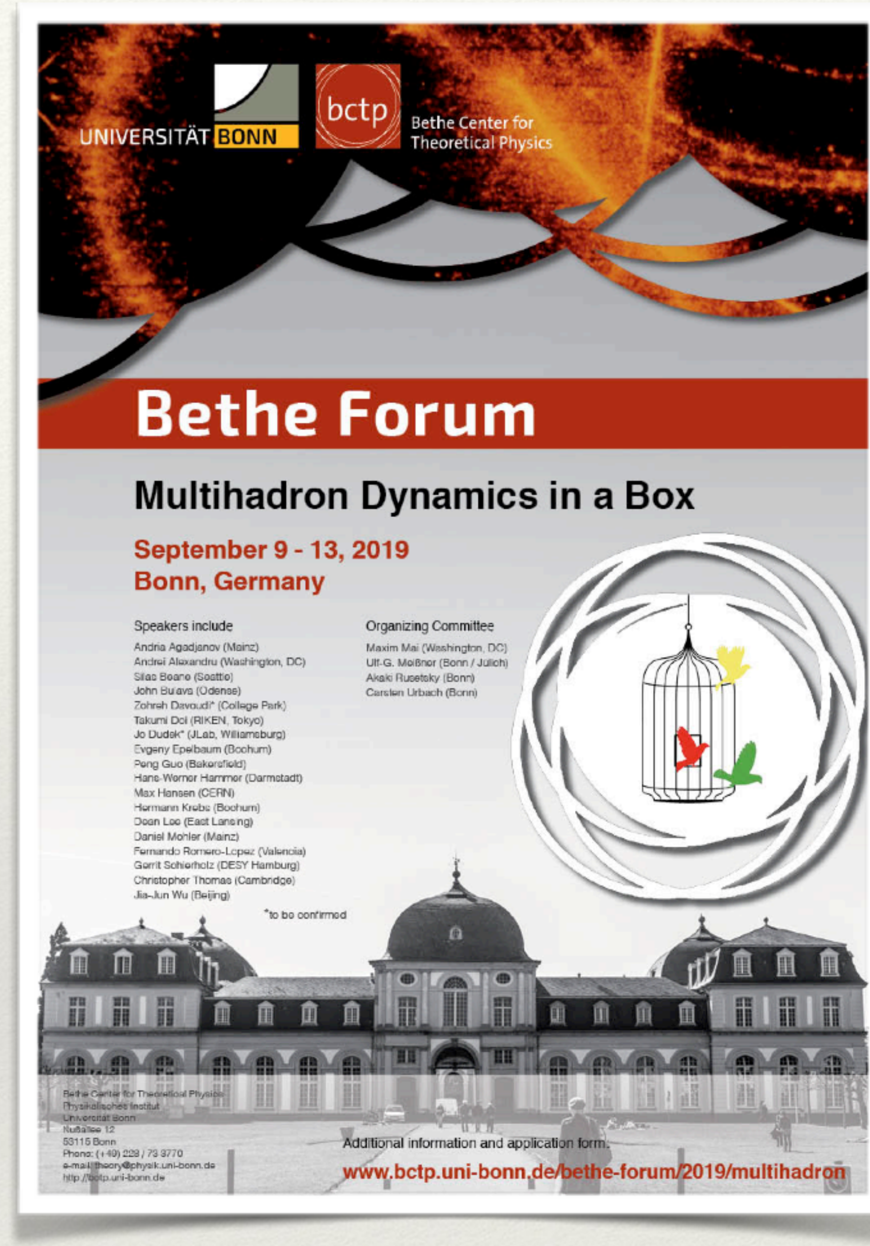
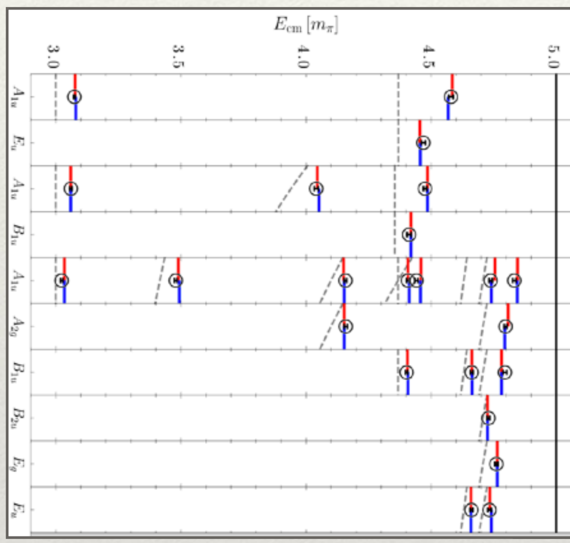
  - first application to Lattice QCD results

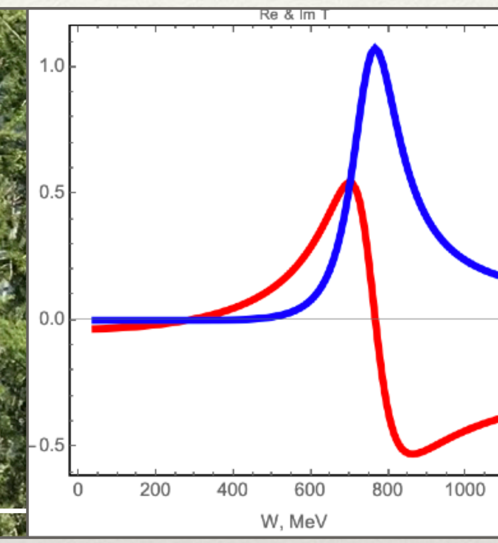


MM, Döring (2018)
Blanton, Romero-López, Sharpe (2019)
MM, Alexandru, Culver, Döring (2019)
Culver, MM, Brett, Döring, Alexandru (2019)
Fischer et al. [ETMC](2020)



$E_{cm}\,[m_\pi]$
3.0
3.5
4.0
4.5
5.0

QUANTIZATION CONDITION



Re & Im T
1.0
0.5
0.0
-0.5
0
200
400
600
800
1000
W, MeV

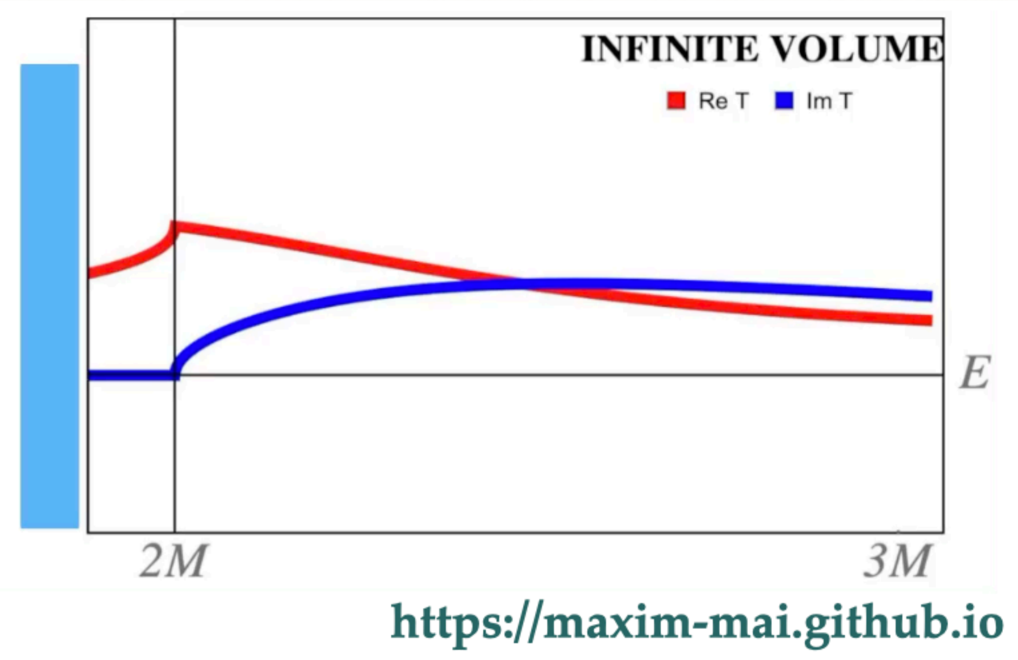
Quantization Condition

- ***Lessons from the 2-body case***

  1. ***Unitarity*** *fixes the imaginary part of the scattering amplitude:*

$$\frac{1}{T_2(s)} = \underbrace{\frac{1}{K_2(s)}}_{\in \Re} - \underbrace{\int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{2E_\mathbf{k}(s - 4E_\mathbf{k}^2 + i\epsilon)}}_{\in \mathbb{C} \;\; if \;\; s>4M^2}$$

  2. *Generated by the continuum of* ***the on-shell configurations***



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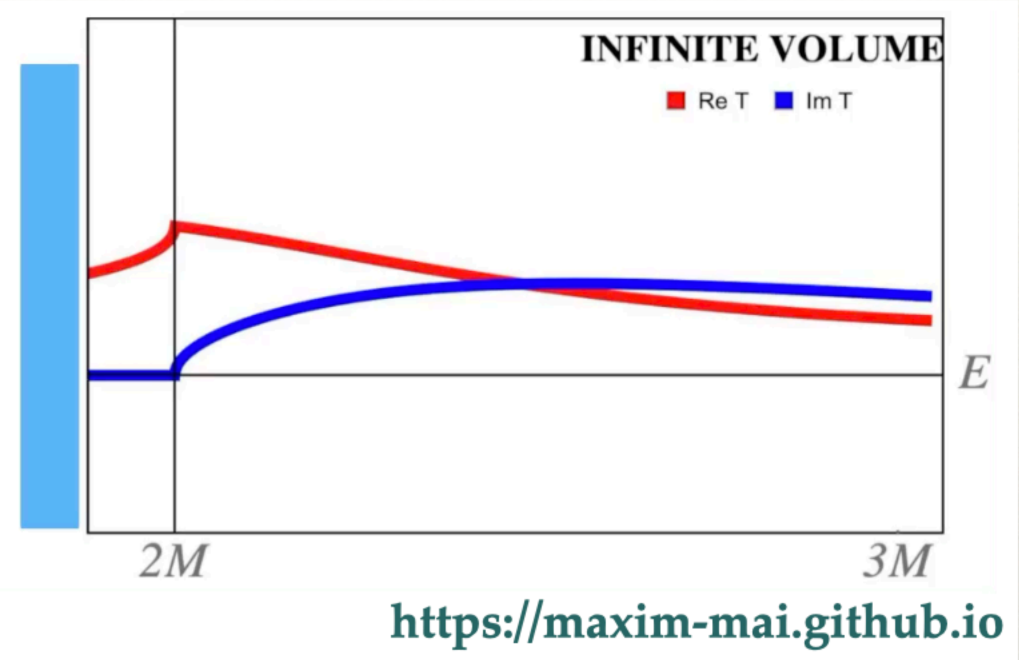

- ***What changes in a box?***

  1. *Discrete set of allowed momenta* ***k***

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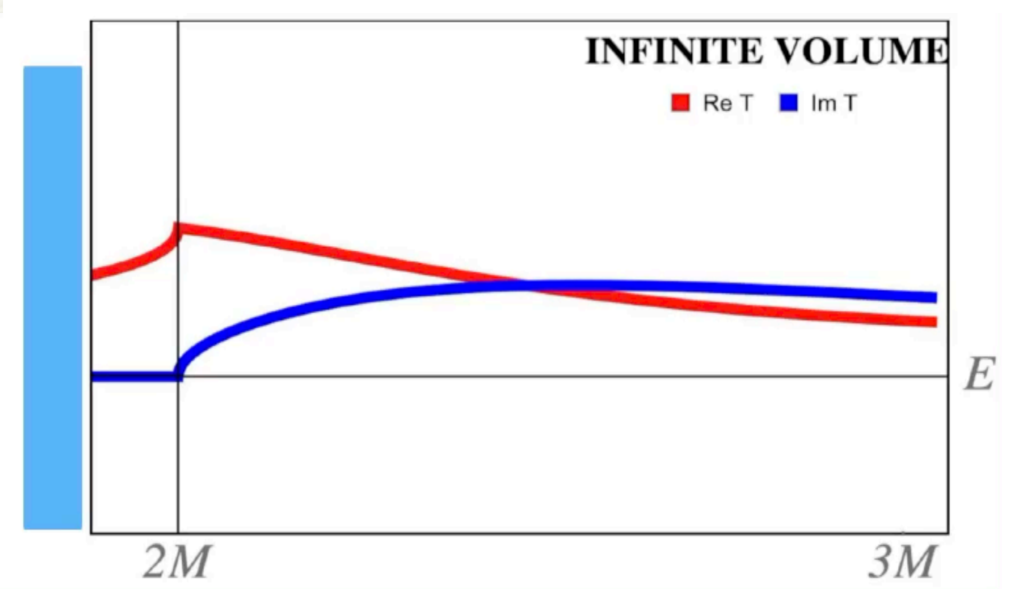

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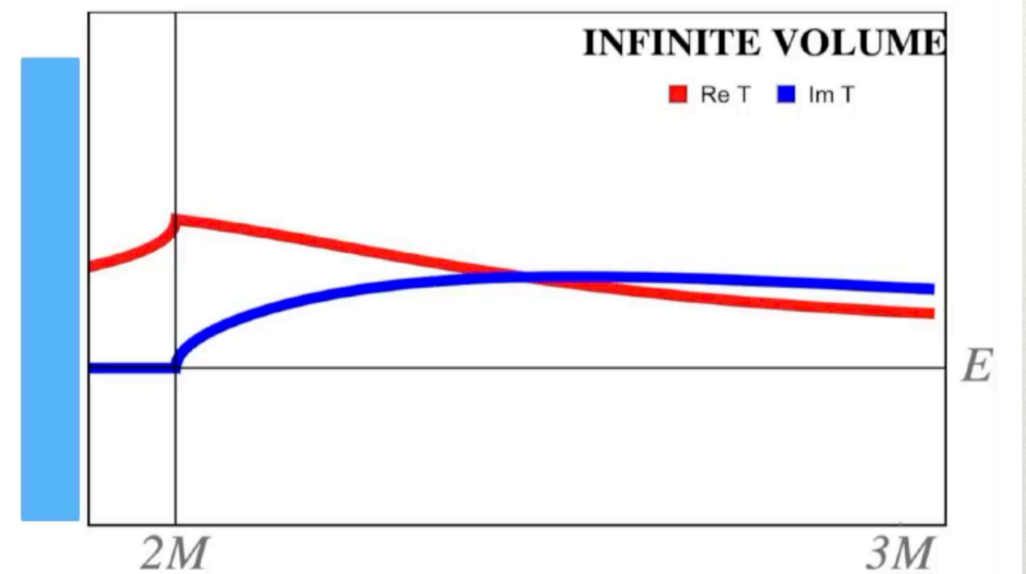

https://maxim-mai.github.io

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- ***Inverse problem (from poles to complex-valued functions)***

Double lim: $W^{-1}(E) = \lim_{\epsilon\to 0}\lim_{L\to\infty} W_L^{-1}(E+i\epsilon)$

Agadjanov,Döring,MM,Meißner,Rusetsky (2016)
Hansen,Meyer,Robaina(2017)
Guo(2020)Briceño,Guererro,Hansen,Sturzu(2020)
…

https://maxim-mai.github.io/INT20.pdf

3-body Unitarity

- *Follow the recipe for 3-body:*
  - *Unitary scattering amplitude*

MM, Hu, Döring, Pilloni, Szczepaniak EPJA53 (2017)

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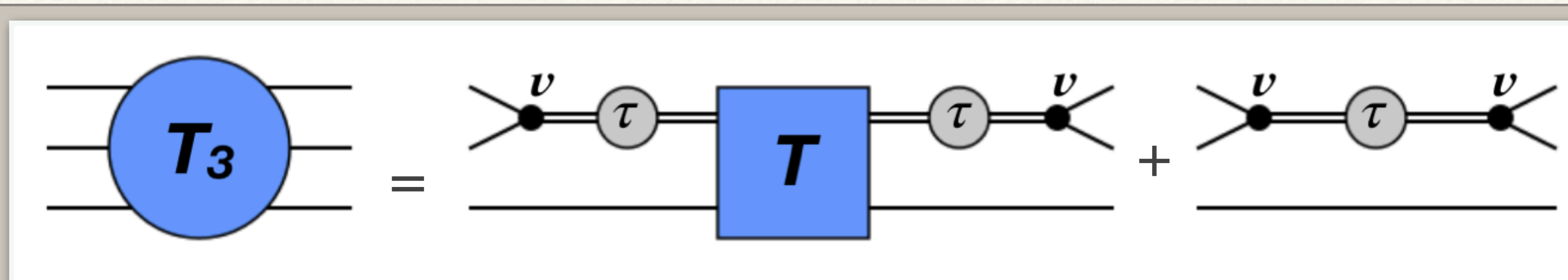
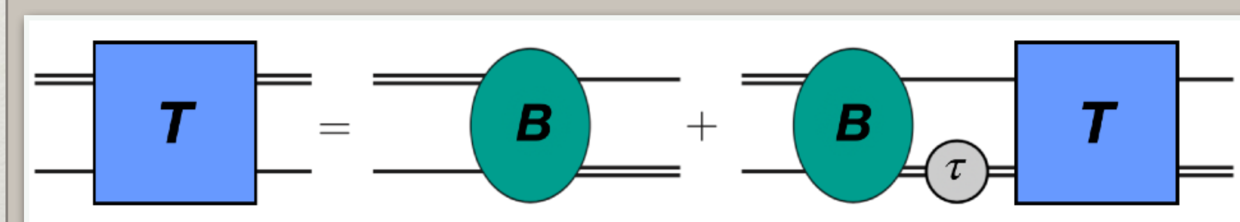


connected part

- general relativistic Int.-Eq. with kernel ***B***



disconnected part

- spectator/tower of functions **τ(Minv)** with correct right-hand-singularities
- coupling **v(q,p)** is cut-free

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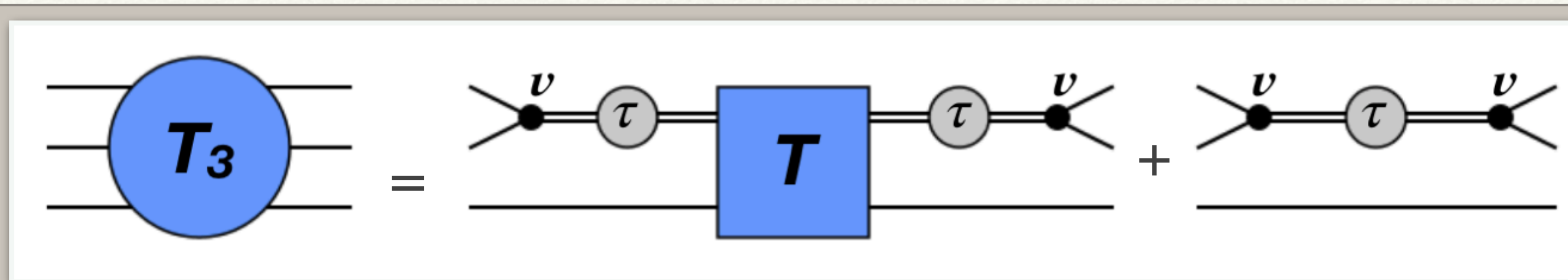

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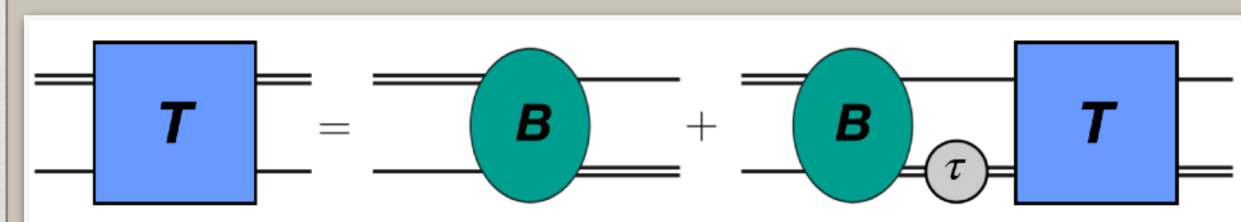

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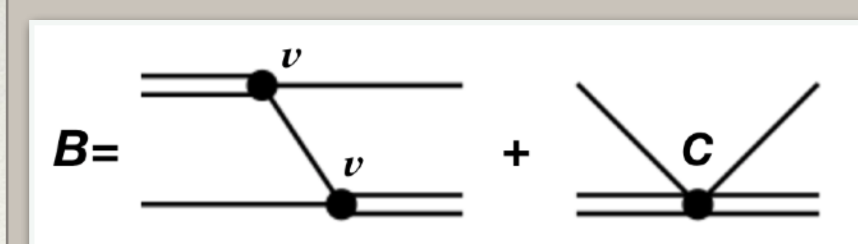
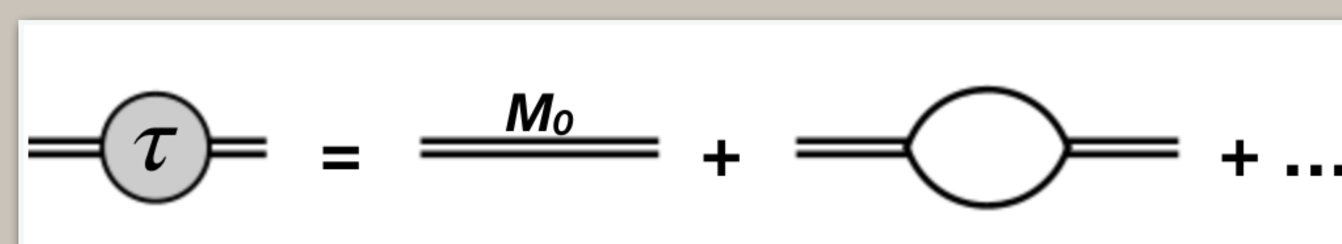
B & τ from 3-body unitarity





- *analytic properties and generalization*

Jackura et al. [JPAC] EPJ C79 (2019)

INTERMEZZO: $a_1(1260)$

- *Recent efforts to study 3-body production beyond the "isobar approximation"*

P. Magalhães, A. C. dos Reis et al., PRD84 (2011);
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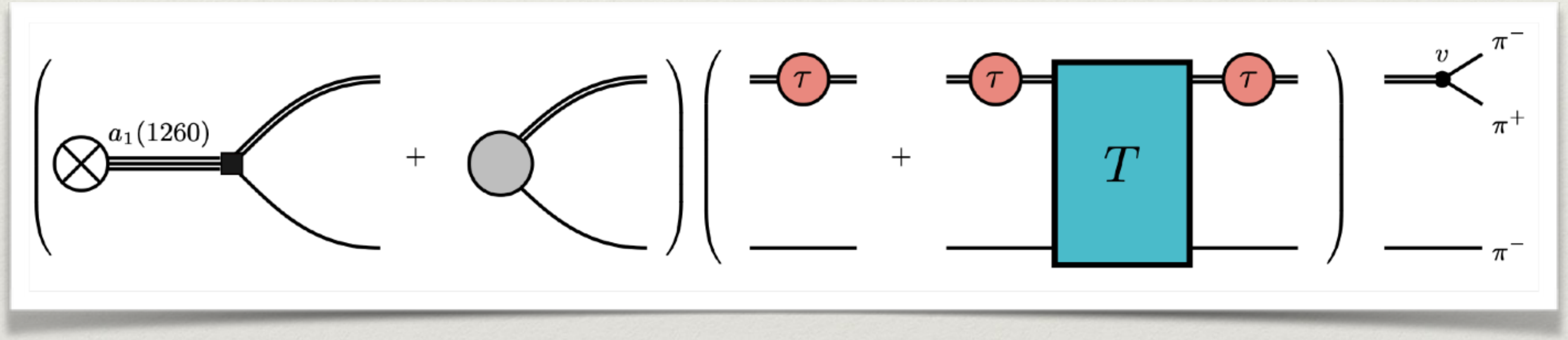
- ***Here****: Full solution of three-body equation with* ***exact*** *three-body unitarity*

Sadasivan,MM,Akdag,Döring *Phys.Rev.D* 101 (2020)



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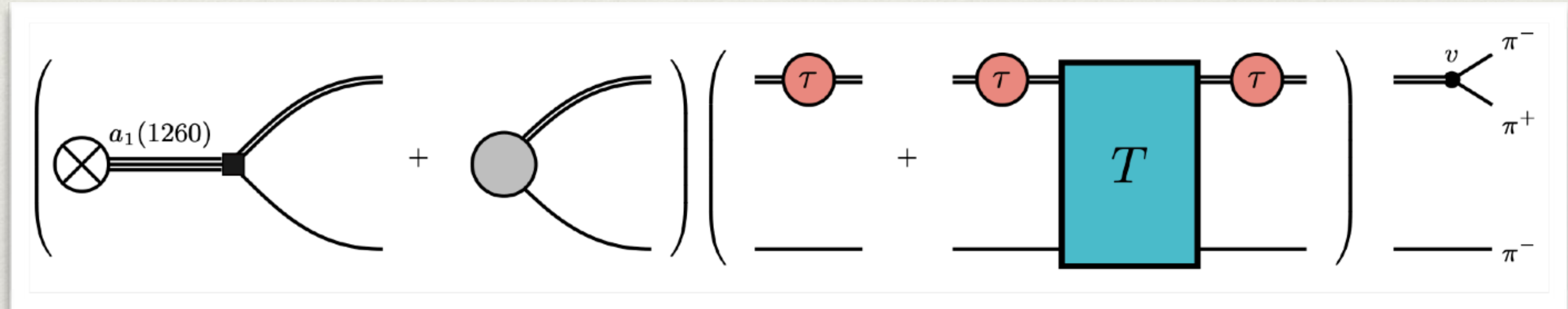

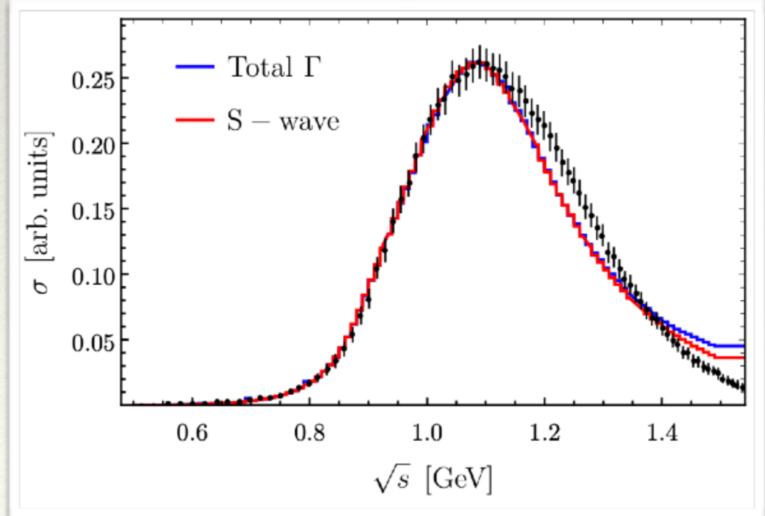

fix free parameters

Data: ALEPH coll. [hep-ex/0506072]

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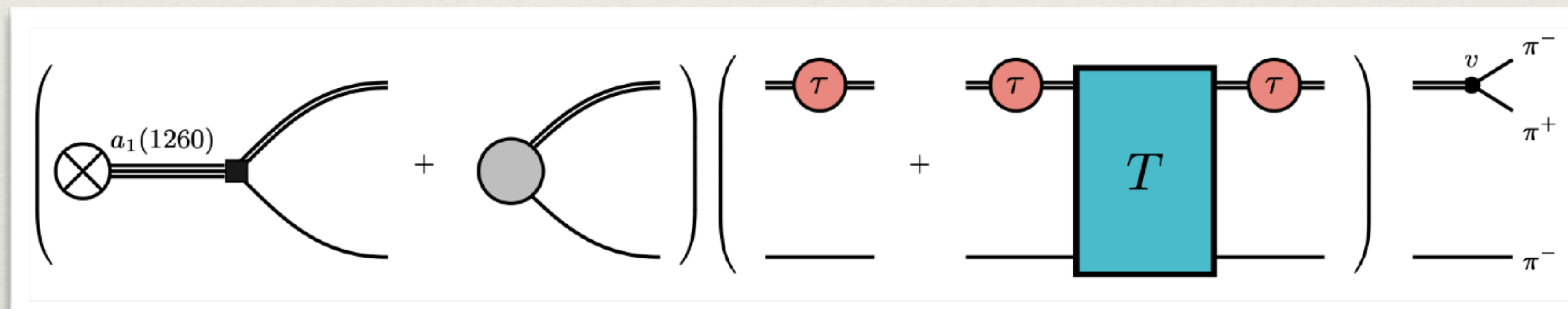
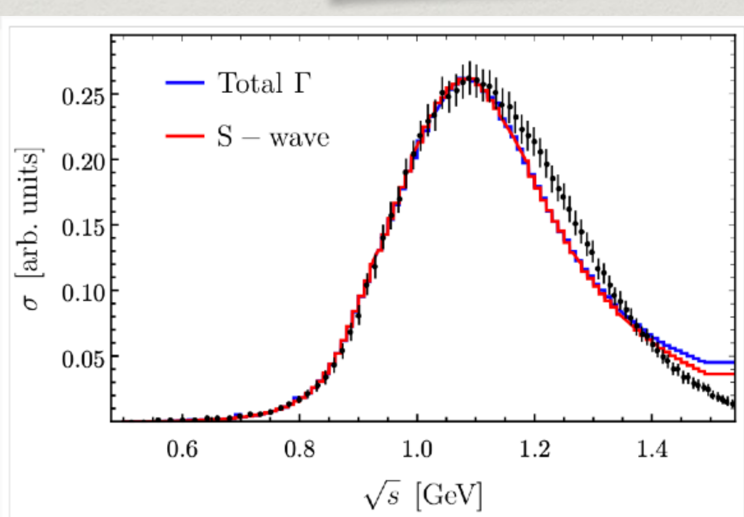
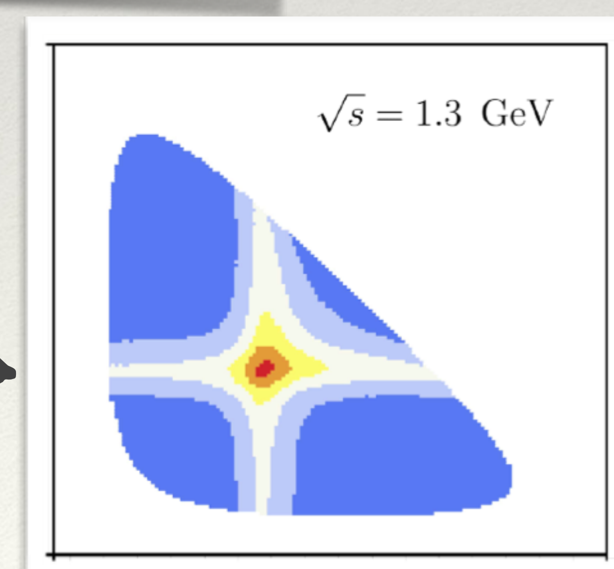




fix free parameters

predict Dalitz Plots



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Back to LQCD: Relativistic 3b-QC

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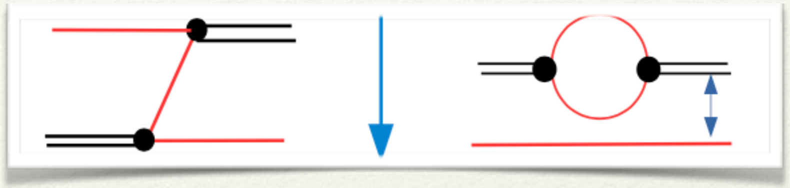

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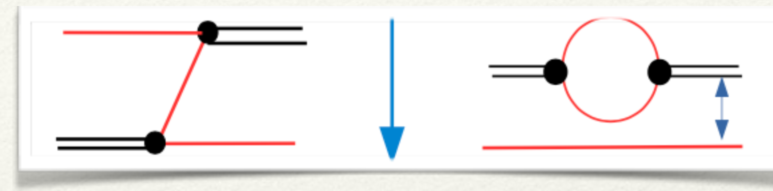


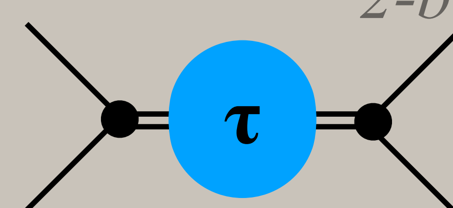
Three-body relativistic unitary formalism — "TRUF"

aka. Finite Volume Unitarity — "FVU" MM, Döring(2017)(2018)

$$\left\langle v(p_1, p_2) \left[\frac{1}{B(s) + C(s) + (2L^3 E_{\mathbf{p}}) 1/\tau_{LP}(s)} \right]_{\mathbf{p}_3 \mathbf{q}_3} v(q_1, q_2) \right\rangle = \infty$$

OPE and 3-body force

2-body sub-channel



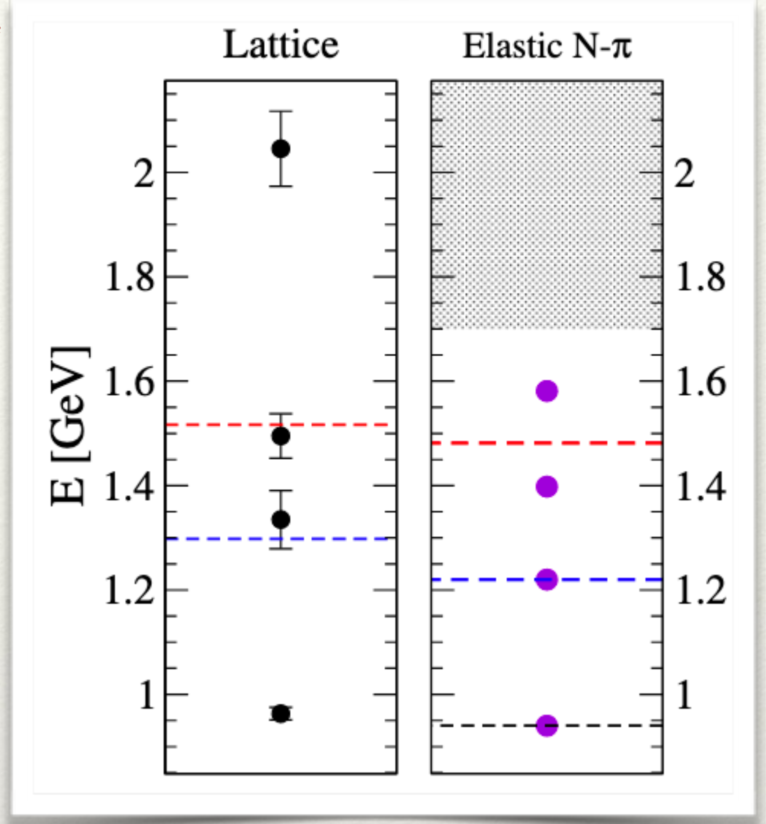
$$= \frac{1}{K_2(\sigma)} - \frac{1}{L^3} \sum_{\mathbf{k} \in \mathbb{Z}^3} \frac{1}{2E_{\mathbf{k}}(\sigma - 4E_{\mathbf{k}}^2)}$$

- *matrix equation in spectator momenta <=> inherently different to the 2-body case*

Lattice studies of 3b systems

- *I=3* **πππ** — Beane et al. [NPLQCD] PRL100 (2008)
- *X(3872)* — Prelovsek, Leskovec PRL111 (2013)
- *a_1(1260)* — Lang et al. JHEP 1404
- *N(1440)1/2$^+$* — Lang et al. PRD 95(2017)
- *I=2* **πρ** — Woss et al. JHEP 1807 [Hadspec]
- **πω/πφ** — Woss et al (2019)
- *I=3* **πππ** — Hörz/Hanlon PRL123 (2019); Culver/MM/Brett/Döring/Alexandru (2019); Fisher et al. (2020)

... more to expect



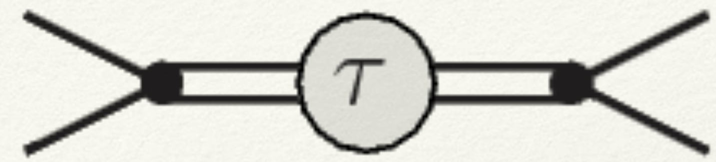
Lang et al.(2017)

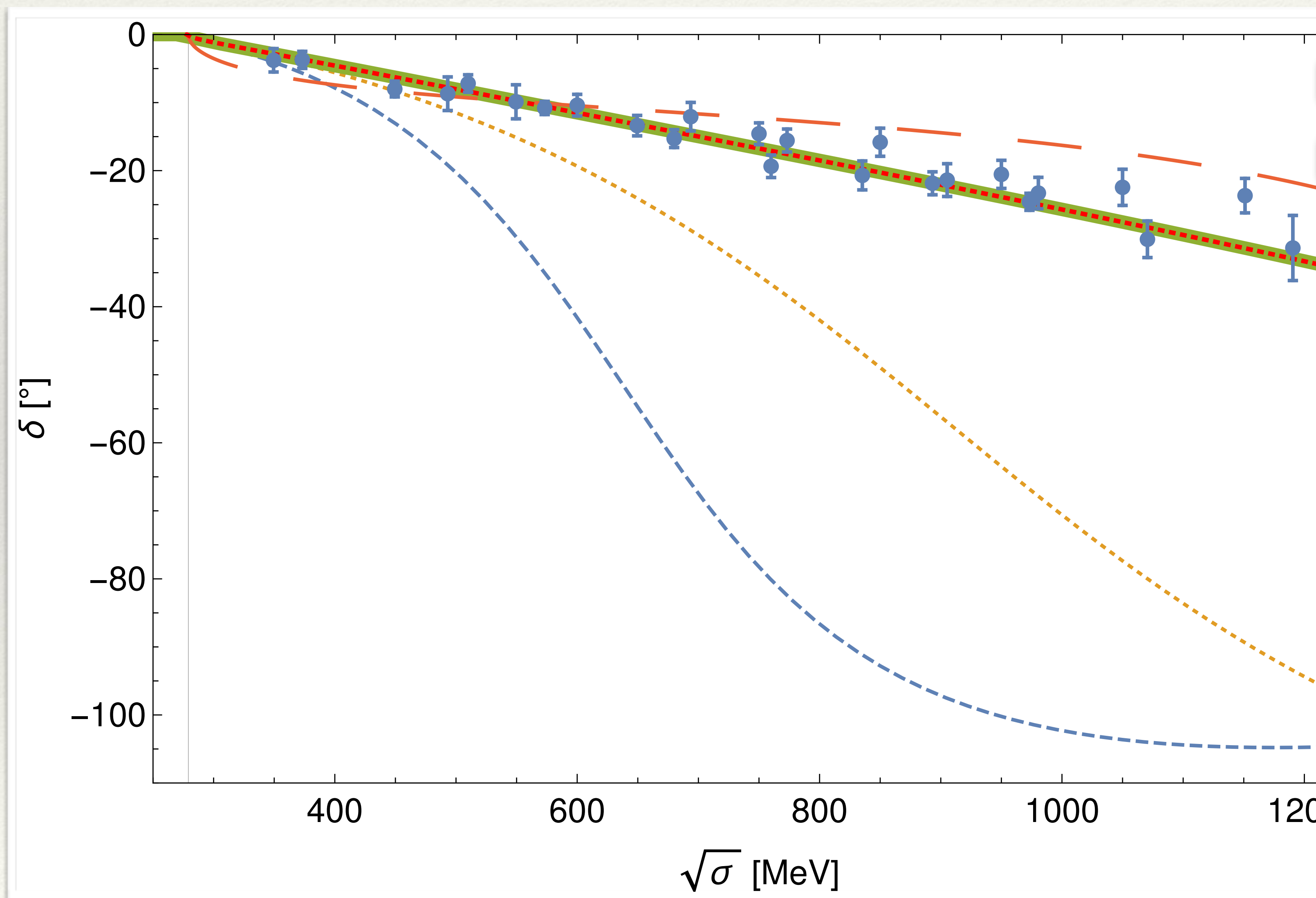
= fin.-vol. analysis accomplished

Three positive pions (1)

MM, Döring PRL 122 (2018)

- *2-body sub-channel*
  - *One-channel problem: $\pi^+\pi^+$ system in S-wave*
  - *How to parametrize the scattering amplitude?*





$$T_2 = \frac{-\lambda^2/(32\pi)}{\sigma - M_0^2 - \sum_\pm \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{\lambda^2}{4E_{\boldsymbol{k}}\sqrt{\sigma}(\sqrt{\sigma}\pm 2E_{\boldsymbol{k}})}}$$

$$T_2 = \frac{T_2^2}{T_2 - T_4} = \frac{1}{K_{IAM}^{-1} - ip_{cms}}$$ 👍

Truong 86, Pelaez,…

$$T_2 = \frac{1}{K_{ChPT@LO}^{-1} - ip_{cms}}$$

$$T_2 = T_{ChPT\ @NLO}$$

Gasser/Leutwyler 82/83

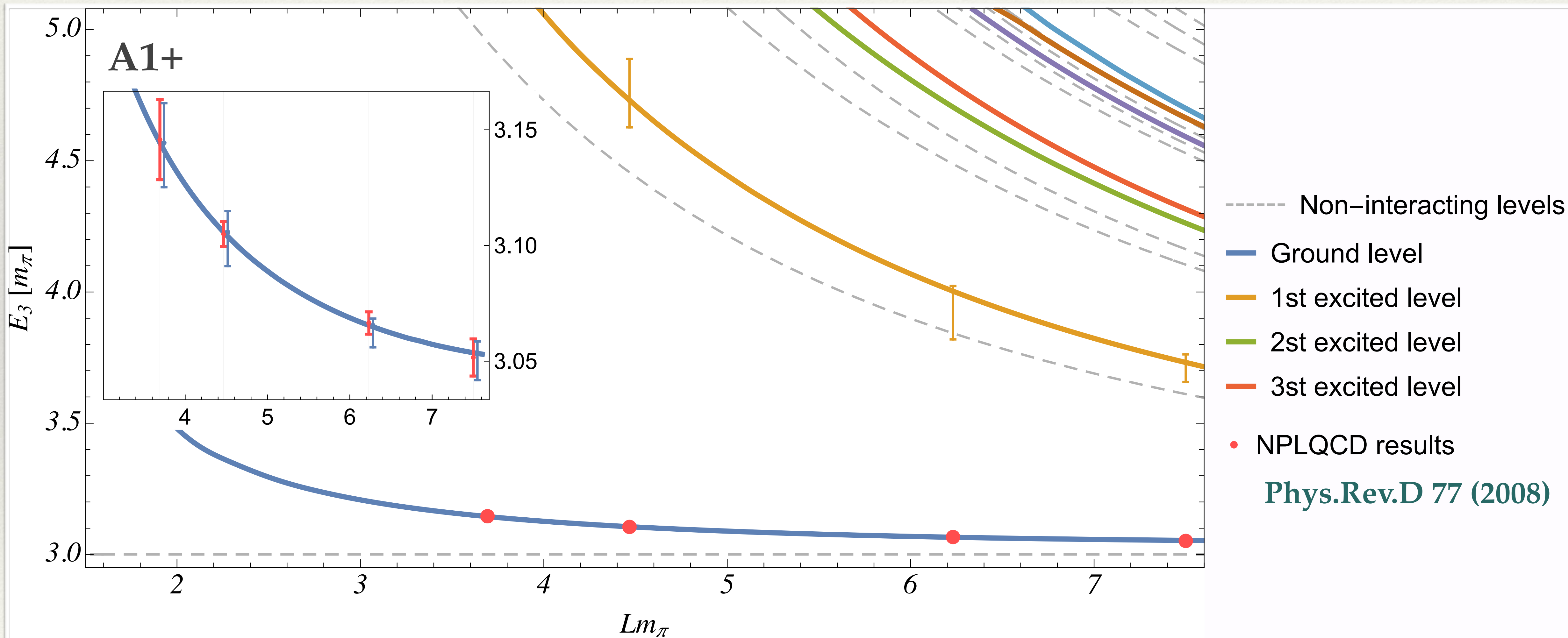
Three positive pions (1)

MM, Döring PRL 122 (2018)

- *3-body spectrum*

$$\left. Det\left[B_0(s) + C_0(s) + 2L^3E_p\cdot\left(K_2^{-1}(s) + \sum_{\mathbf{k}}\frac{1}{2E_{\mathbf{k}}(s-4E_{\mathbf{k}})}\right)\right]_{pq} = 0\right.$$

- *unknown 3-body force* — **ansatz and first fit**: $C_0 \sim id_{pq} \implies m_\pi^2 C_0 = (0.3 \pm 2.3)\cdot 10^{-6}$

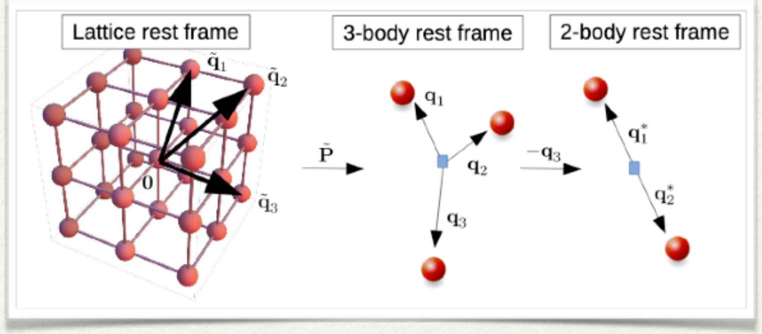


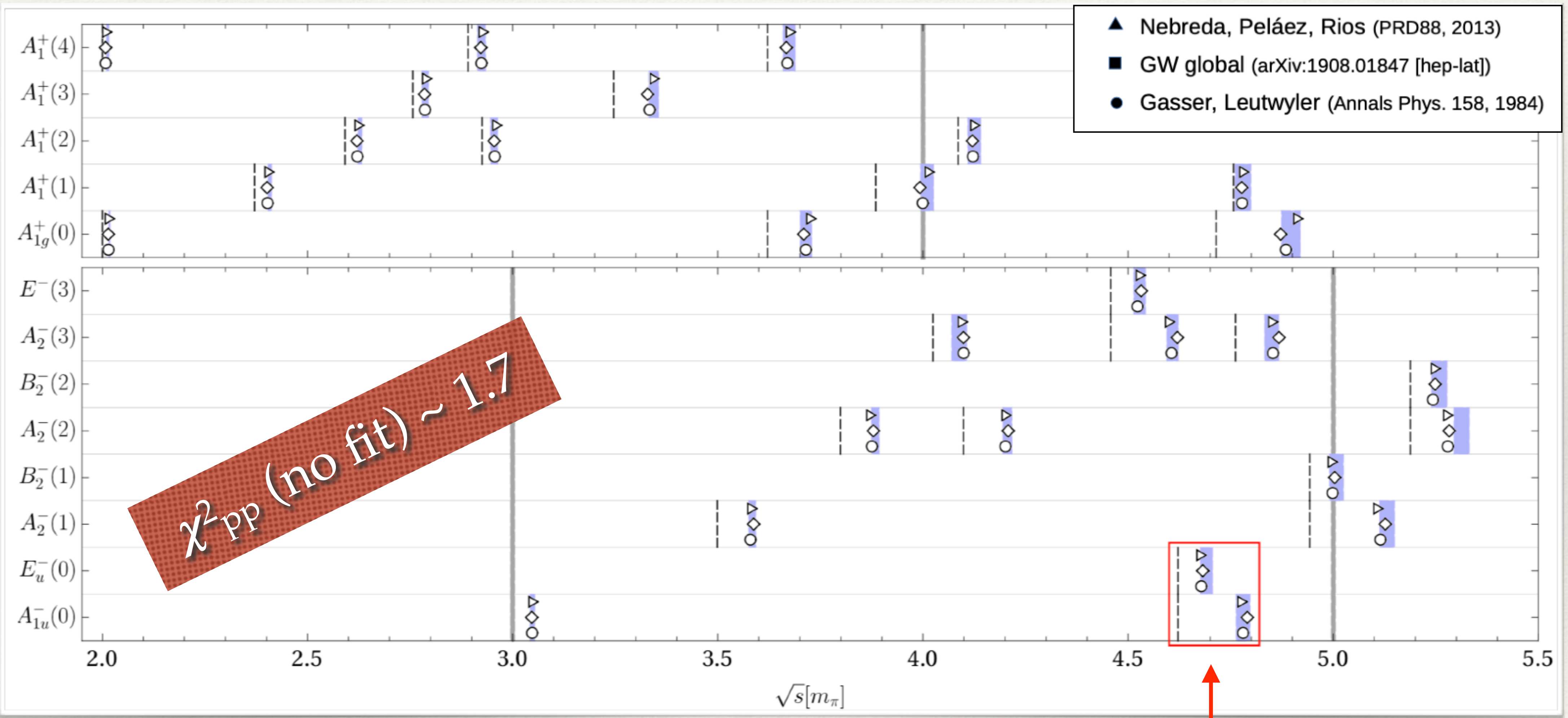
https://maxim-mai.github.io/INT20.pdf

Three positive pions (2)

MM,Culver,Döring,Alexandru(2019) PRD

- *New data* Hörz/Hanlon PRL123 (2019)
  - *Extend to arbitrary irreps & boosts*
  - *Use the same parametric input*





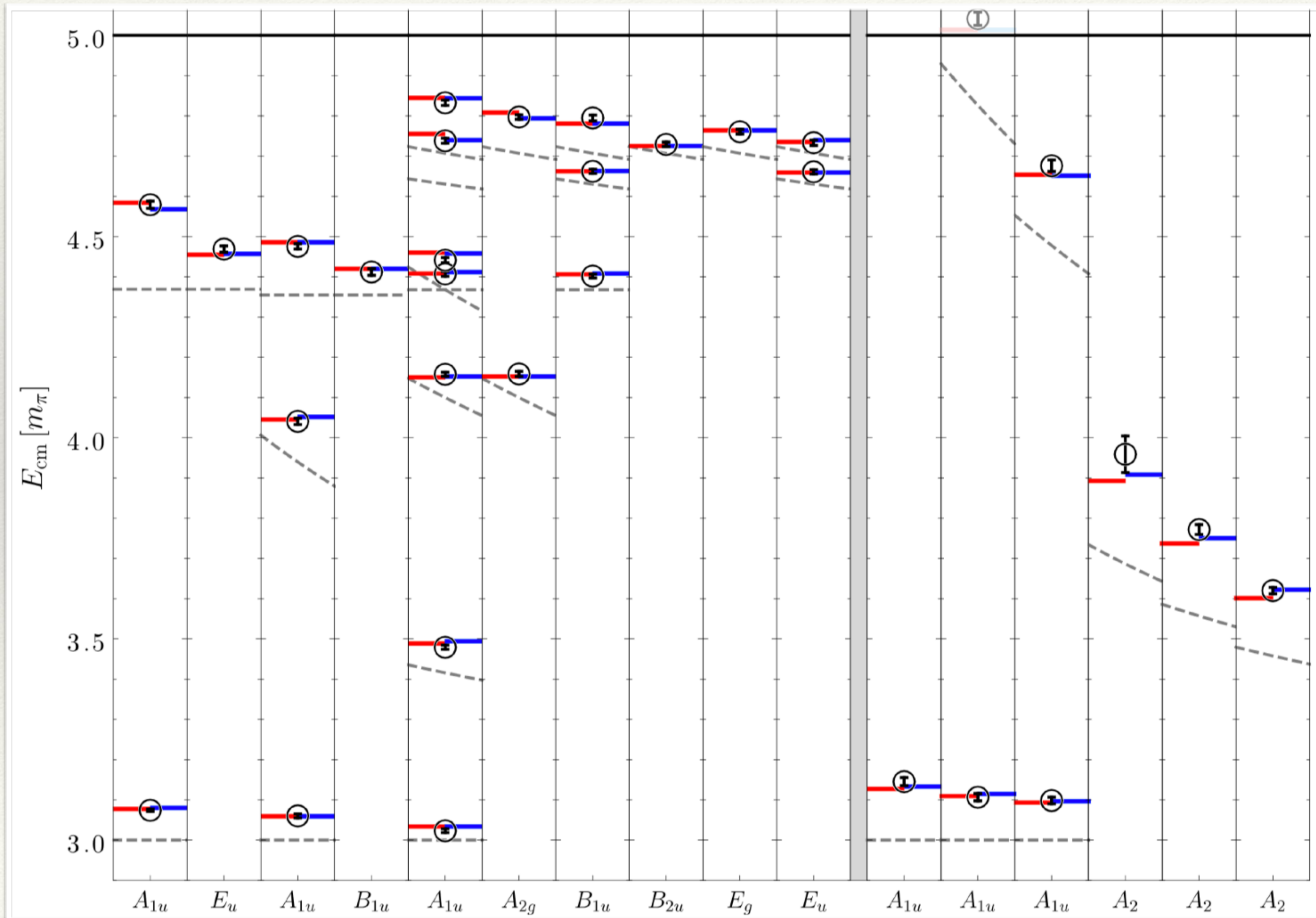
- *Uncertainties dominated by the 2-body input. C~0*
- *Rel. strength between S/D waves fixed dominantly by 3b unitarity*

Three positive pions (3)

- *Newer data is available*

  Culver, MM, Brett, Alexandru, Döring (2019) PRD

  - *very dense spectrum from elongated boxes*
  - *different pion masses (chiral extrapolations?)*



predictions from MM/Döring(2018)

lattice calculation

χ^2_{pp} (no fit) ~ 2

C=0 still works fine

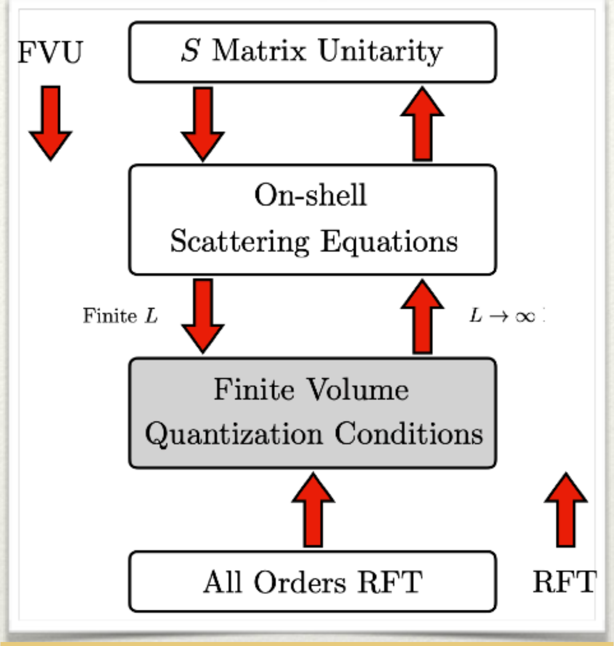
https://maxim-mai.github.io/INT20.pdf

Alternatives to “TRUF”

- *Alternative way for counting* ***3-body on-shell configurations*** *(relativistic cases)*

  - *Diagrammatic approach — “RFT”* **Hansen,Sharpe(2014)(2015)** **Briceño,Hansen,Sharpe(2017)(2018)**

  - *TOPT diagrammatic approach* **Blanton/Sharpe(2020)**



FROM TALK of A. Jackura

https://maxim-mai.github.io/INT20.pdf

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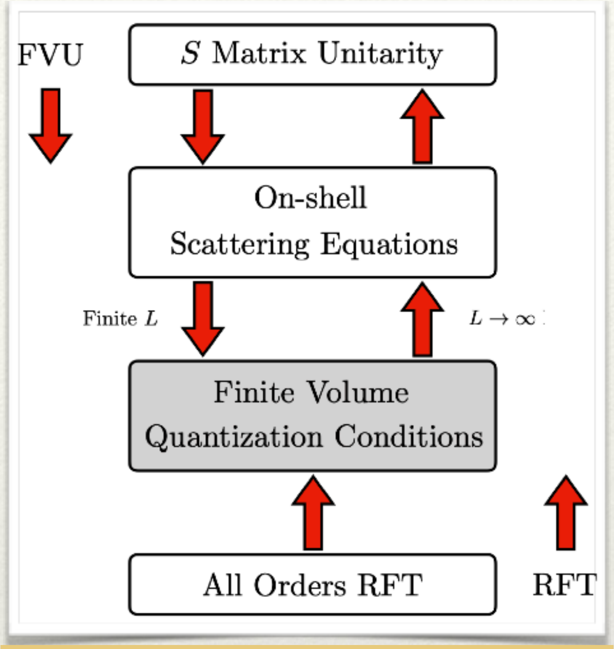


FROM TALK of A. Jackura

- *Equivalent building blocks of RFT / FVU*

... up to details: cutoffs, LHC of sub-channel,...

... later“RFT” application show $K_{3,df} \neq 0$ (3b term)

Blanton, Romero-López, Sharpe (2019)
Fischer et al (2020)

$$C_0 = -K_2^{-1} K_{3,df} K_2^{-1} \left[1 + \frac{1}{2E_p L^3} K_{3,df}\ K_2^{-1} \right]^{-1}$$

... Unitarity is blind to the structure of 3-body contact term (general parametrization)

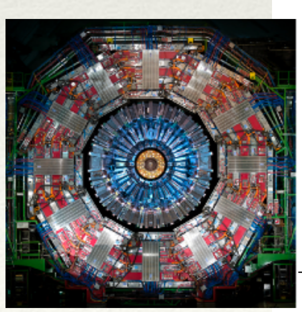
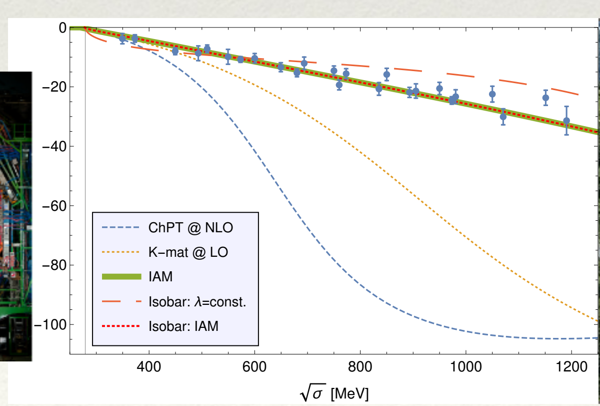
... depending on interaction and needs, RFT of TRUF are more advantageous

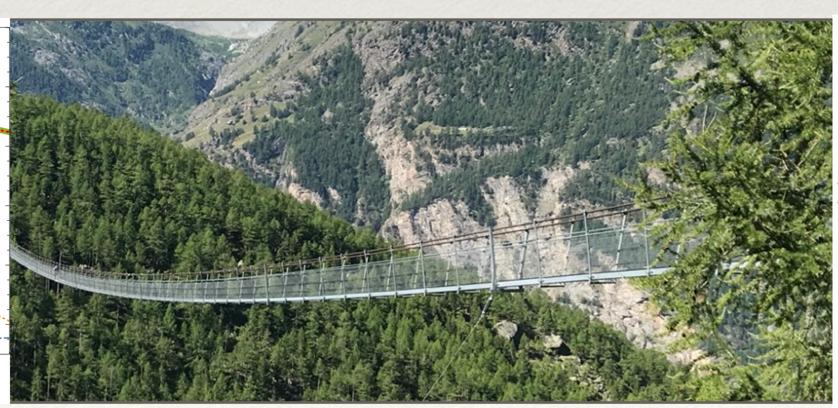
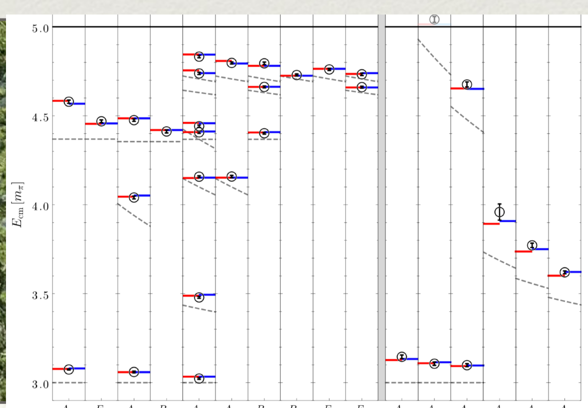
SUMMARY

- ***2-body spectroscopy from Lattice is highly evolved:***
  - *new high-precision information on properties of resonances*
  - *complementary insights to experimental studies*

  e.g. m_π dependence, extreme quantum numbers,…

- ***3-body spectroscopy from Lattice is maturing quickly***
  - *many theoretical tools have been developed*
  - *first data-driven studies are performed (details/statistics become important)*
  - *2 relativistic approaches → complementary insights*

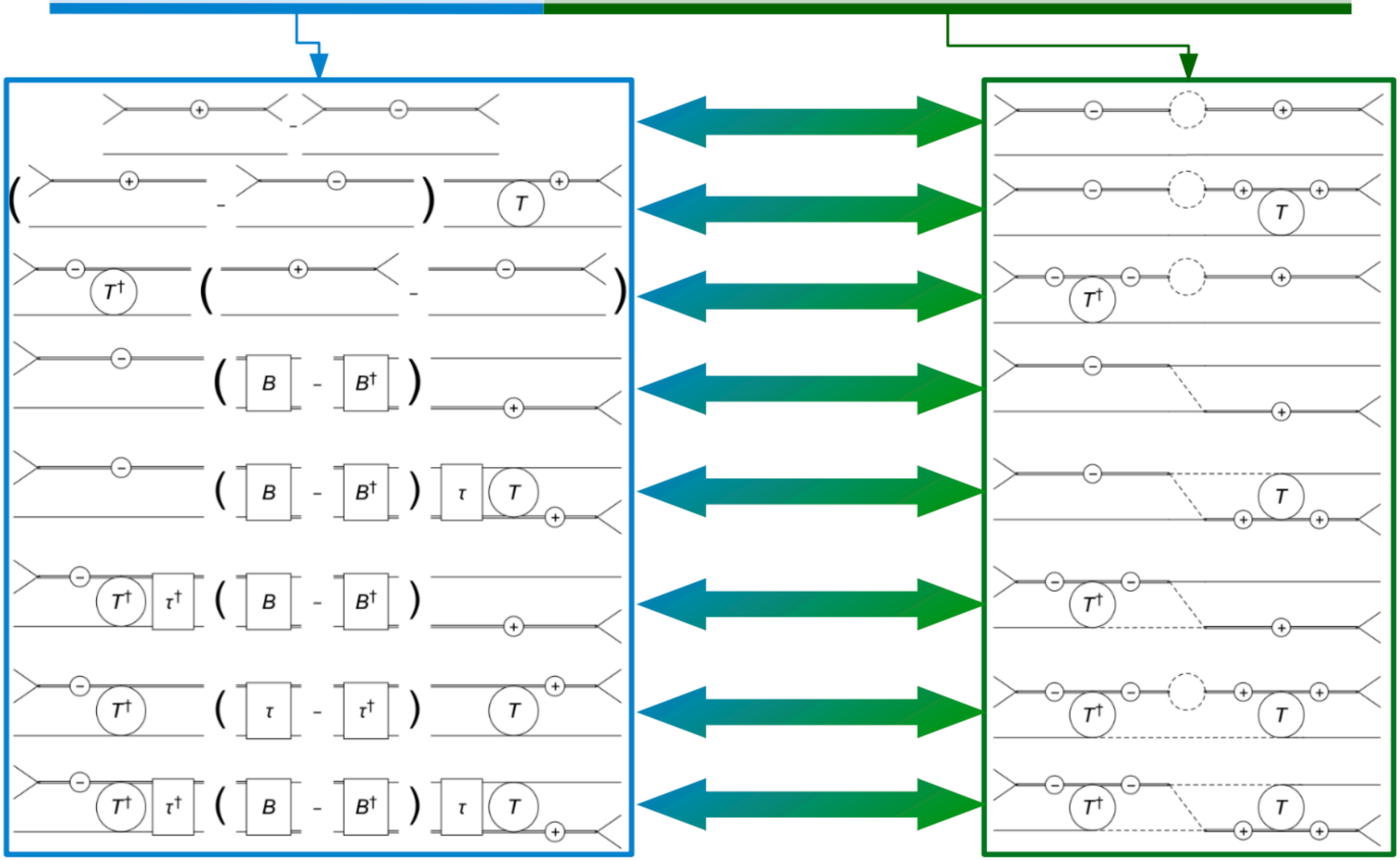
QCD

https://maxim-mai.github.io/INT20.pdf

SPARES

3-body Unitarity

$$\langle q_1,q_2,q_3|(\hat{T}-\hat{T}^\dagger)|p_1,p_2,p_3\rangle \;=\; i\int_P \langle q_1,q_2,q_3|\hat{T}^\dagger|k_1,k_2,k_3\rangle\langle k_1,k_2,k_3|\hat{T}|p_1,p_2,p_3\rangle$$

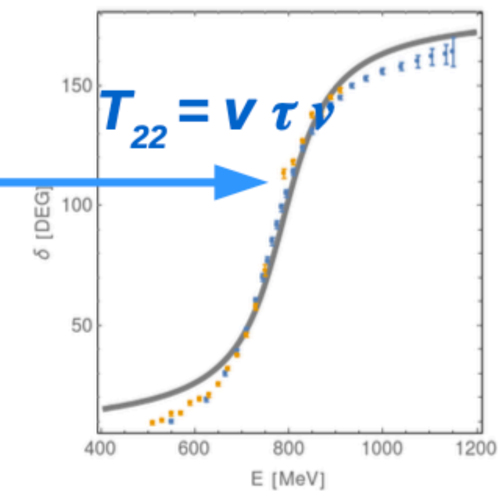


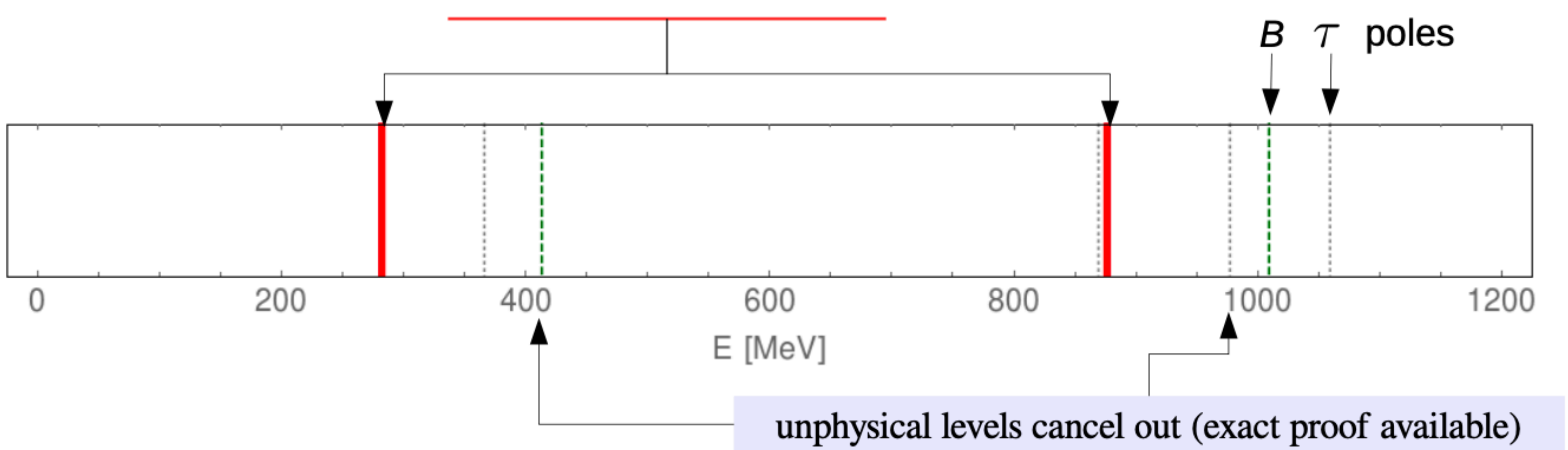
https://maxim-mai.github.io/INT20.pdf

SPARES

[M. Mai, MD, EPJA 2017 [arXiv: 1709.08222]]

- First numerical demonstration of three-body finite volume formalism
- 3 particles in finite volume: ***m=138 MeV, L=3 fm***
- one S-wave isobar → two unknowns:
  - – vertex(Isobar→2 stable particles)
  - – subtraction constant (~mass)
- **Project to $\Gamma = A^{1+}$**
  - **→ prediction of 3body energy-eigenlevels (C=0)**





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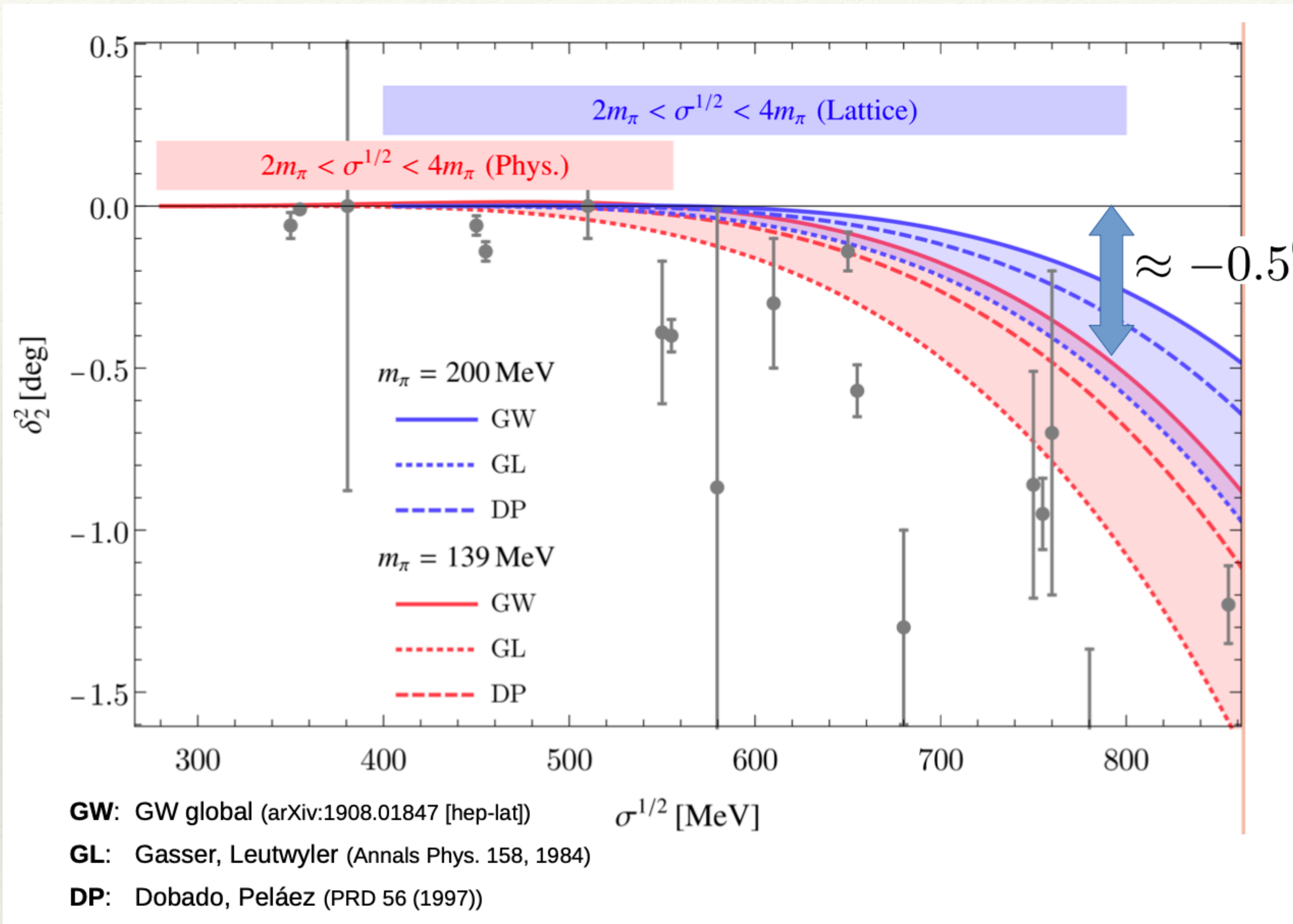
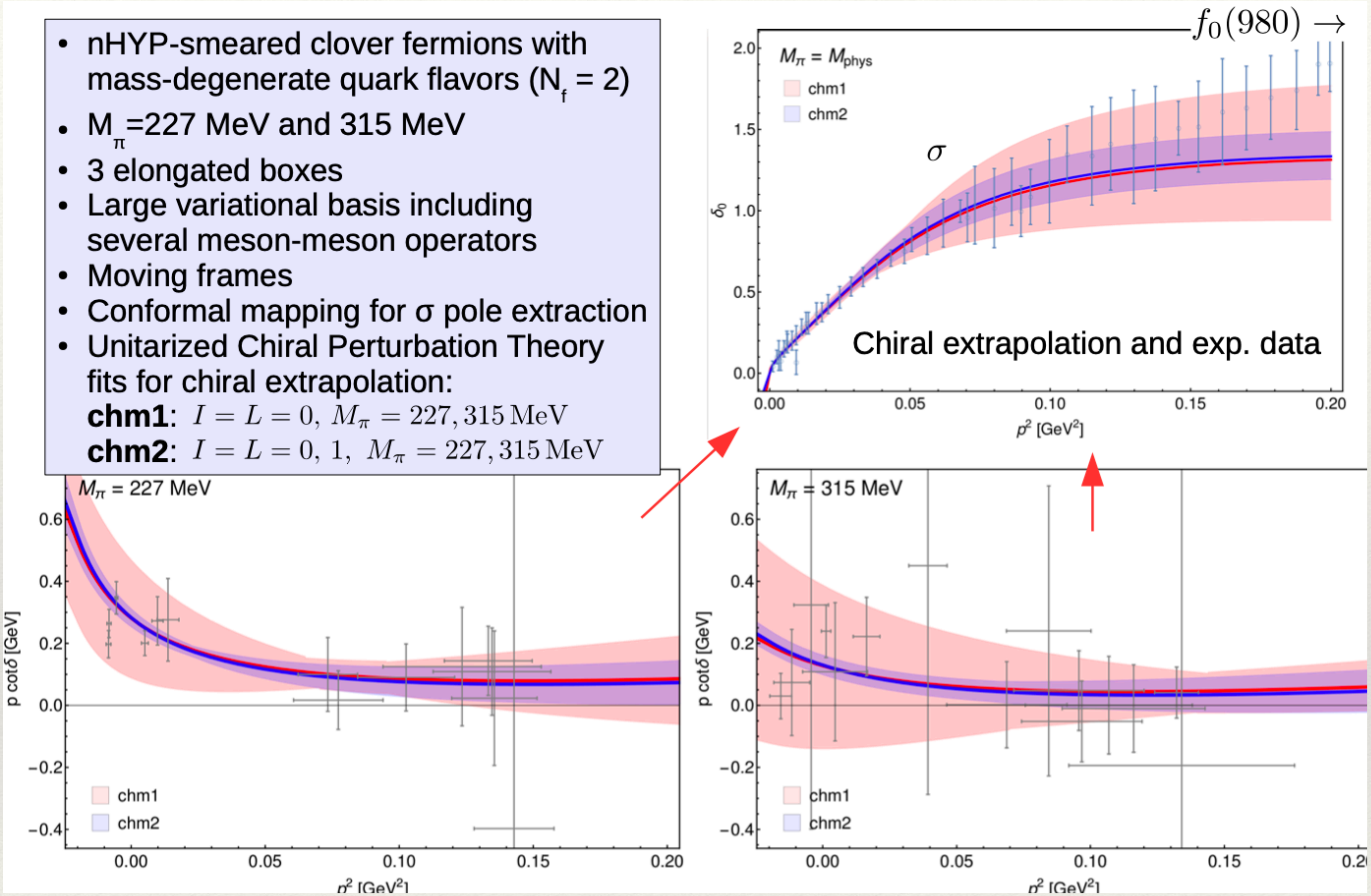


$2m_\pi < \sigma^{1/2} < 4m_\pi$ (Lattice)
$2m_\pi < \sigma^{1/2} < 4m_\pi$ (Phys.)
$\approx -0.5^{0}$
δ_2^2 [deg]
m_π = 200 MeV
GW
GL
DP
m_π = 139 MeV
GW
GL
DP
$\sigma^{1/2}$ [MeV]
GW: GW global (arXiv:1908.01847 [hep-lat])
GL: Gasser, Leutwyler (Annals Phys. 158, 1984)
DP: Dobado, Peláez (PRD 56 (1997))

SPARES



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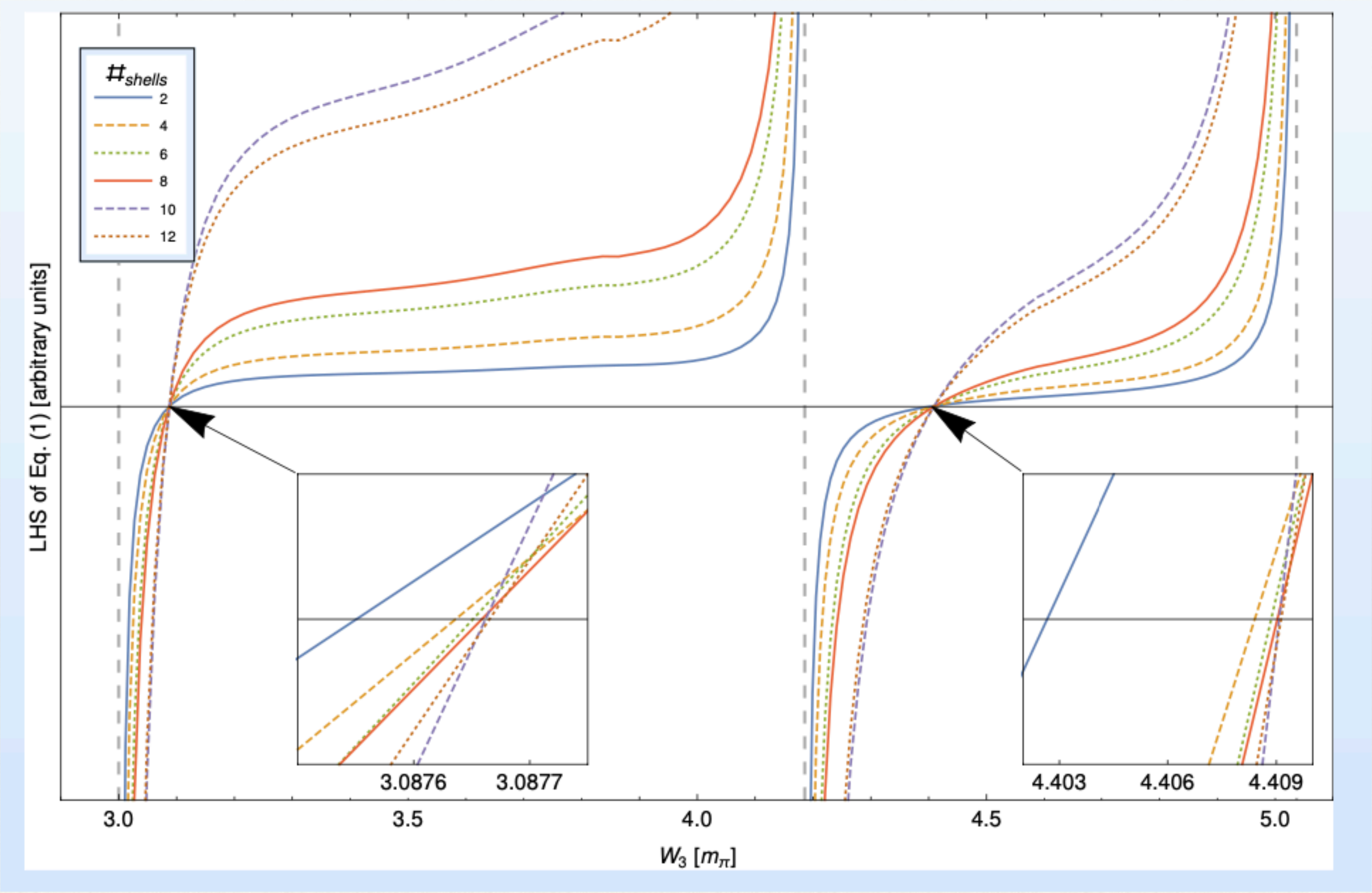

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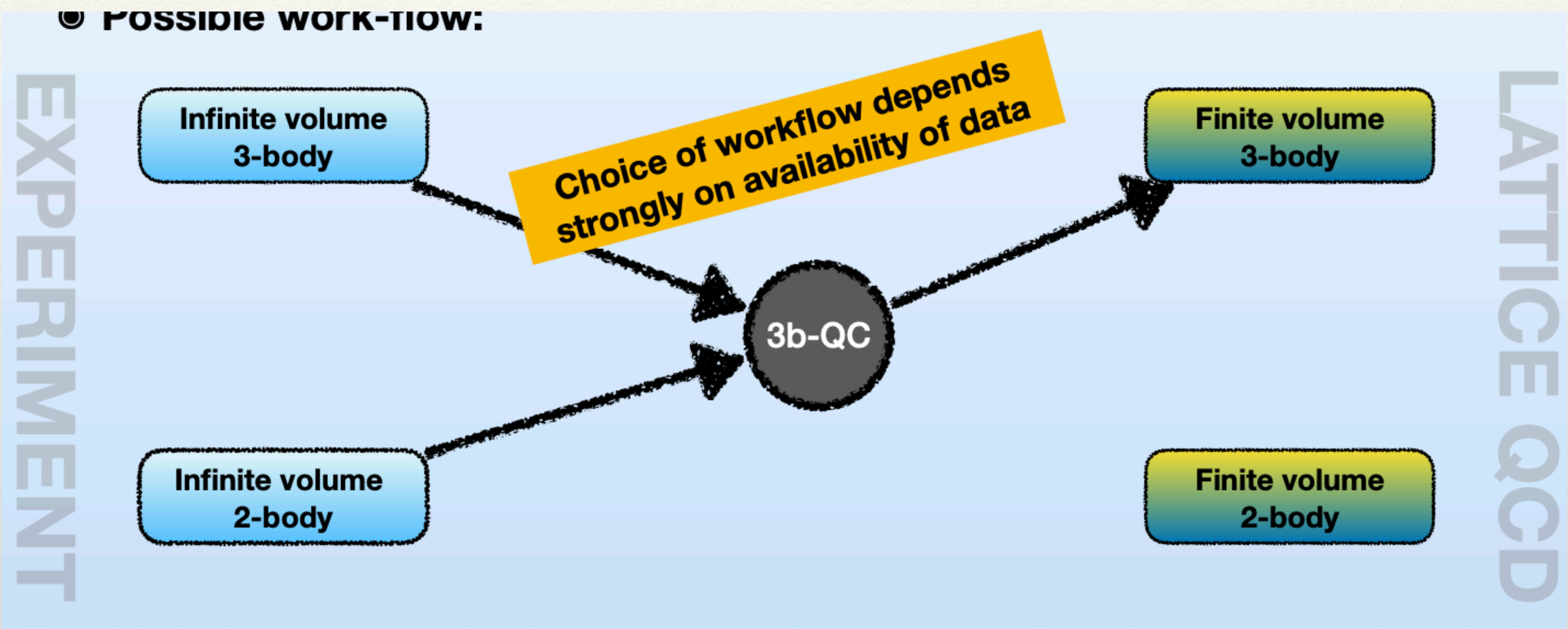
Possible work-flow:
EXPERIMENT
LATTICE QCD
Infinite volume
3-body
Infinite volume
2-body
Choice of workflow depends
strongly on availability of data
3b-QC
Finite volume
3-body
Finite volume
2-body

SPARES

SPARES

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