The background image shows the Golden Gate Bridge from a low angle, looking towards the Marin Headlands. The bridge's towers and cables are silhouetted against a bright sky. A single sailboat is visible on the water in the foreground.

# Bridging phenomenology and lattice QCD for 2- & 3-hadron systems

<https://maxim-mai.github.io/LBL.pdf>

**Maxim Mai**

# MOTIVATION

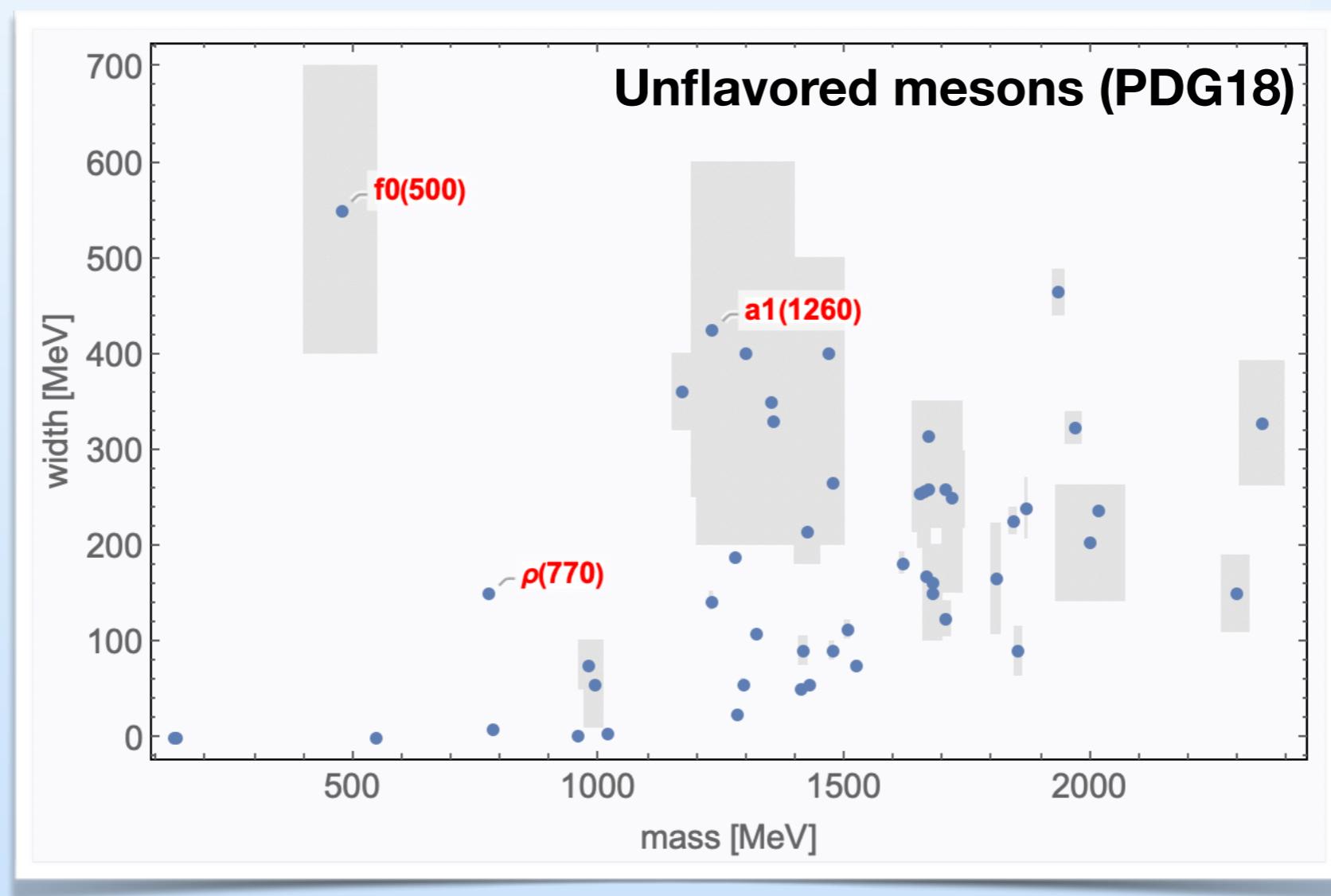
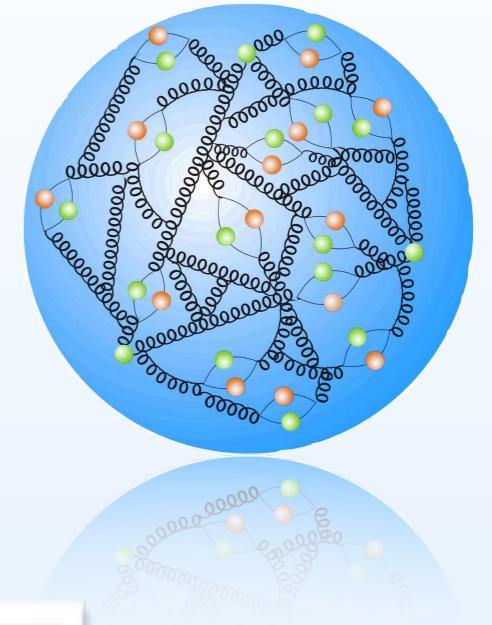


- QCD at low energies:

- Confinement

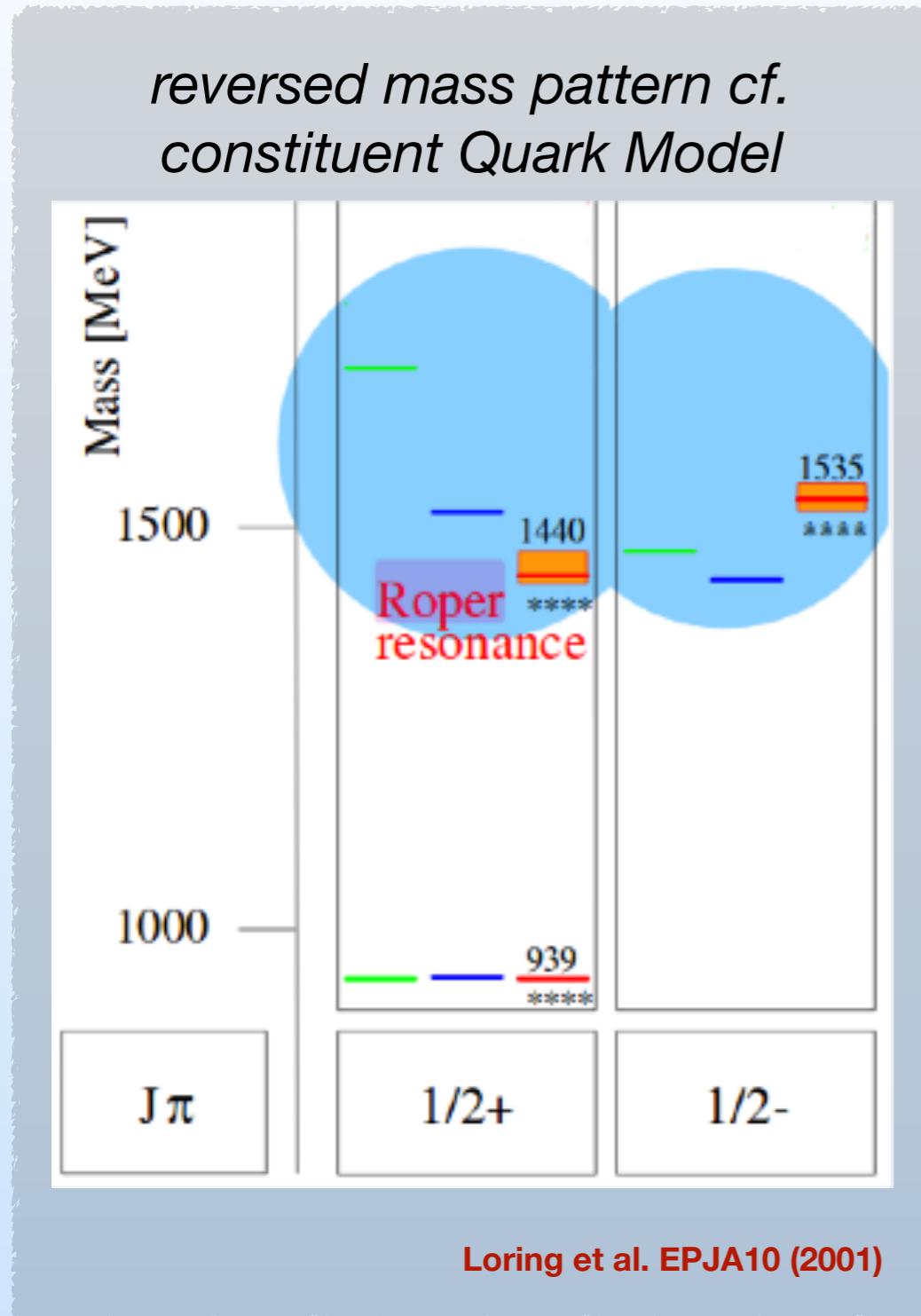
- Mass generation

- Intricate spectrum of excited hadrons



◎ Many unsolved puzzles are related to three-body channels

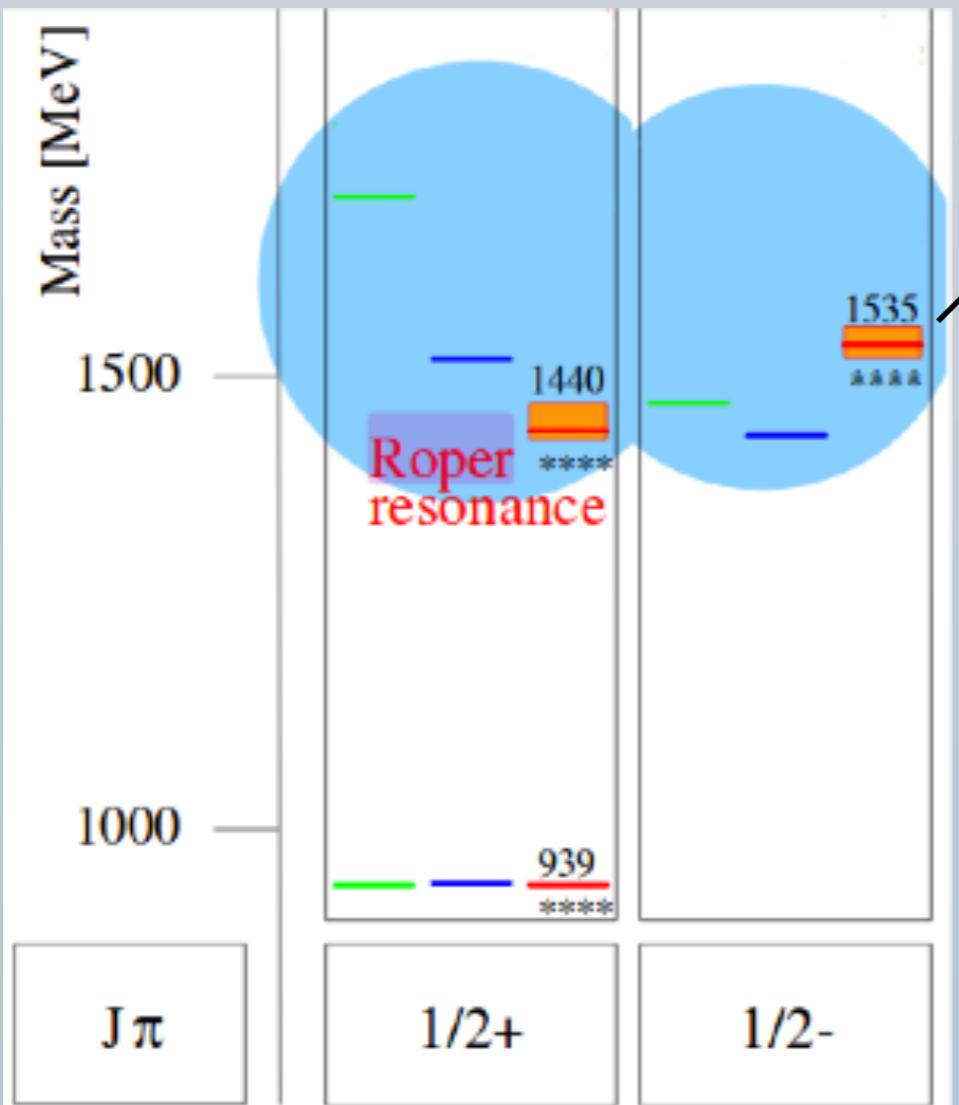
EXAMPLE 1: *Roper-puzzle*



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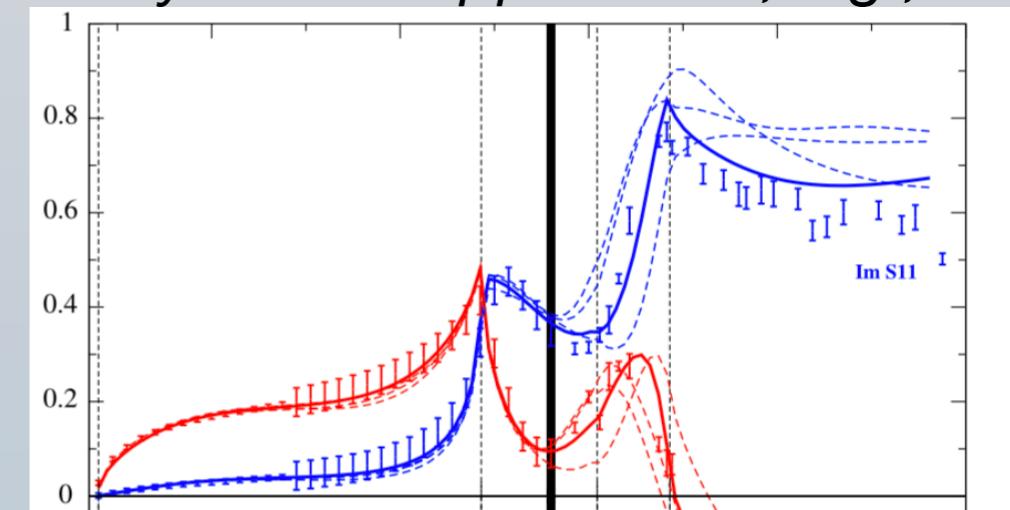
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reversed mass pattern cf.  
constituent Quark Model



Loring et al. EPJA10 (2001)

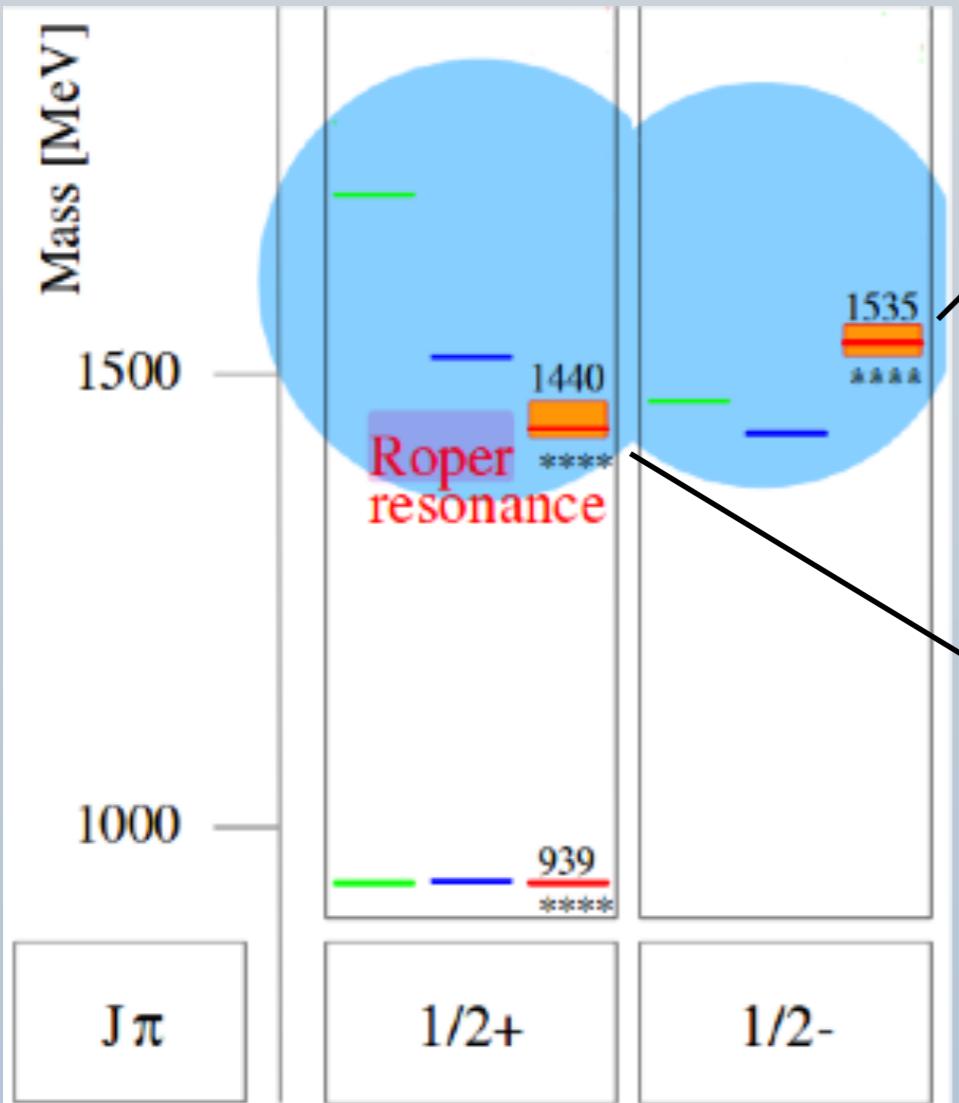
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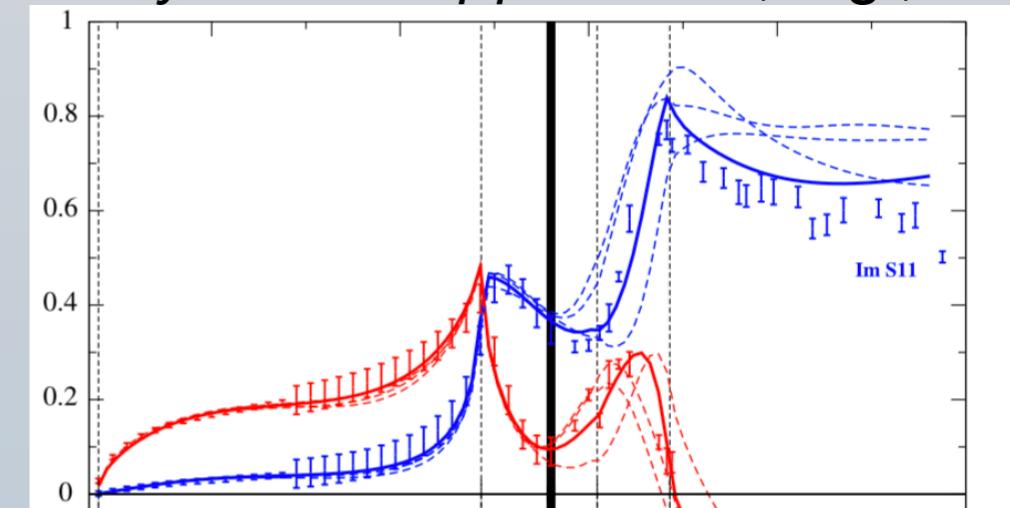
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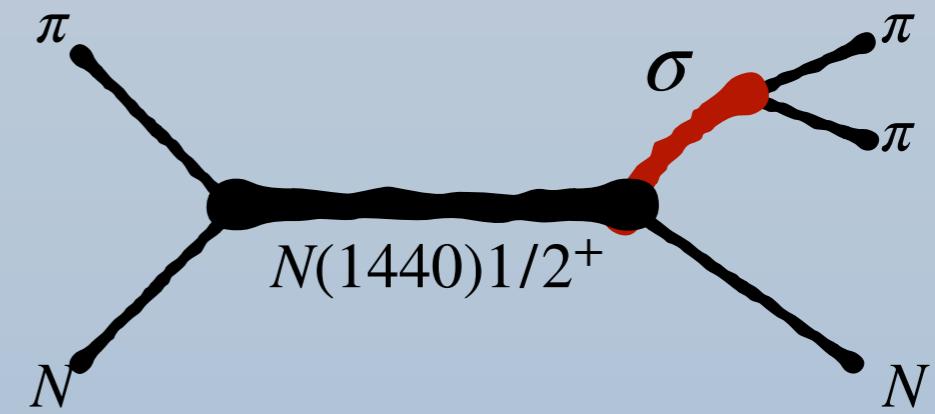
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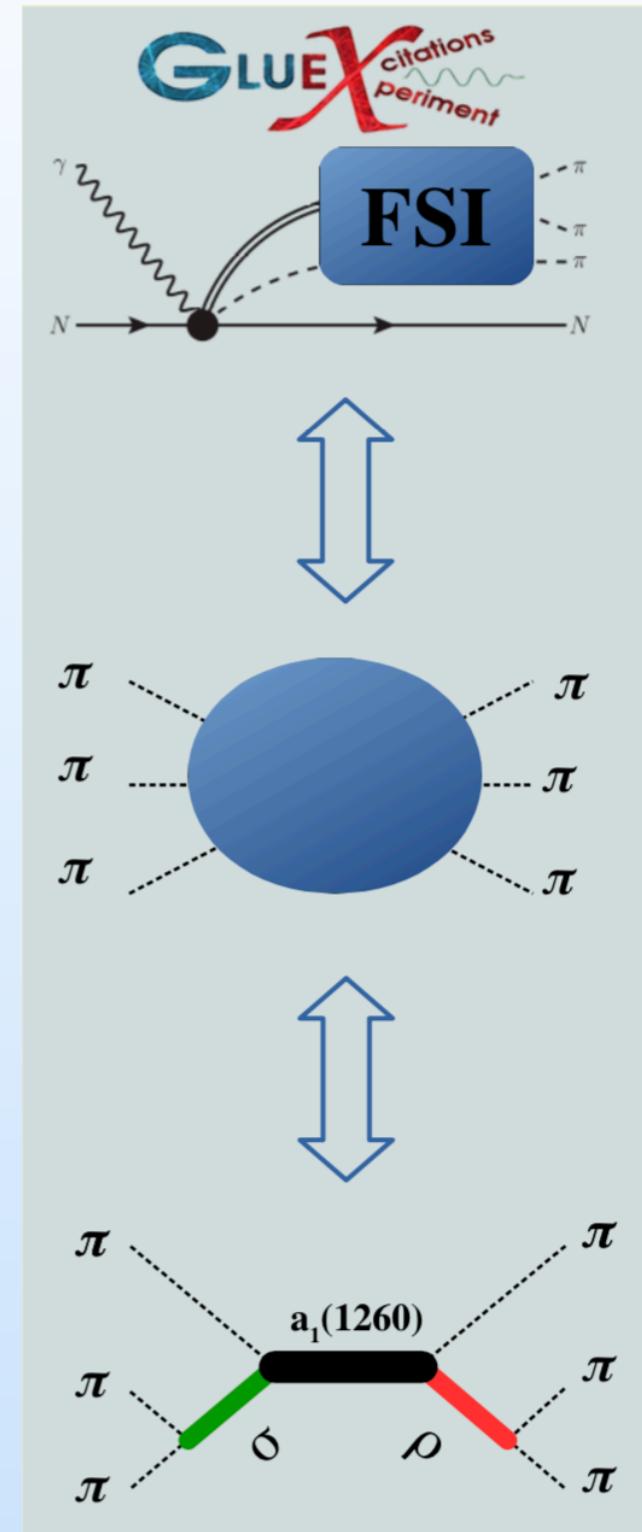
...but  **$N(1440)1/2^+$  has large BR to  $\pi\pi N$**



## ◎ Many unsolved puzzles are related to three-body channels

### EXAMPLE 2: $a_1(1260)$

- does ***not*** decay into  $\pi\pi$  but only  $\pi\pi\pi$  channel
- test channel for the search for ***spin-exotics*** (*GlueX, COMPASS, BESIII*) indicator for the importance of gluonic d.o.f.



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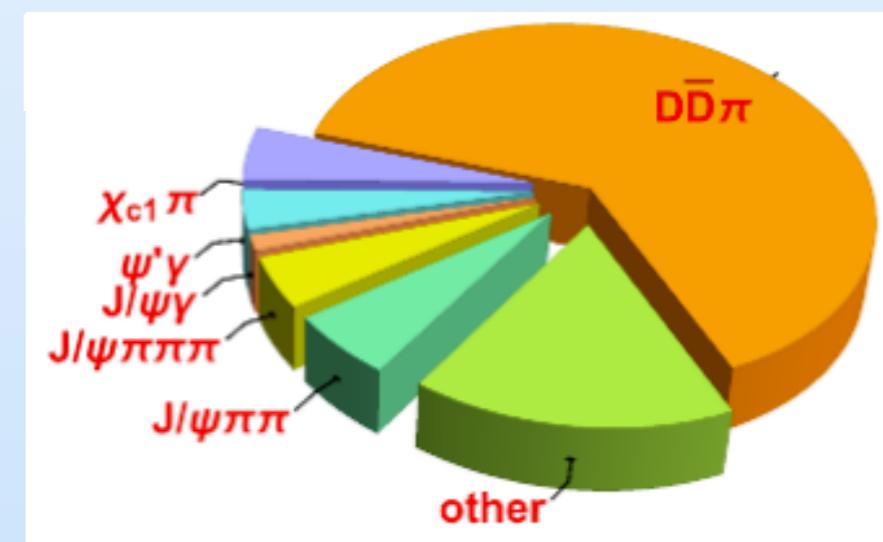


### EXAMPLE 3: $X(3872) \leftrightarrow D\bar{D}\pi$

Belle 2003

- puzzling production mechanism

...similarly for further heavy XYZ exotics



## ● Lattice QCD

- *the only systematic (non-perturbative) approach from first principles*
- *new interesting developments for 3-hadron systems*

Lang et al. JHEP 1404; PRD95 (2017); Woss et al. JHEP 1807; Hoerz/Hanlon PRL 123 (2019); ...

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• *in finite volume*  Quantization Condition

*Well understood for 2 hadrons*

Luescher 1986

*Many new developments and first applications for 3 hadrons*

## ◉ Example: $\pi\pi$ scattering from lattice QCD

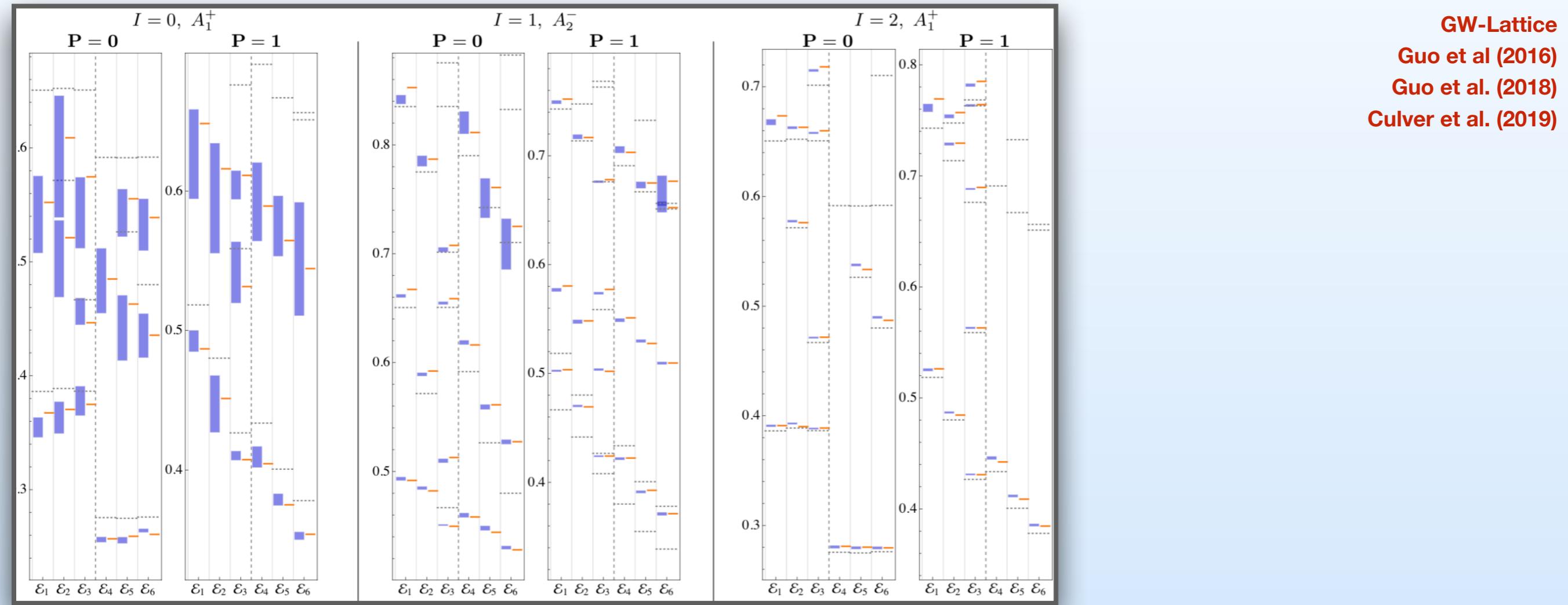
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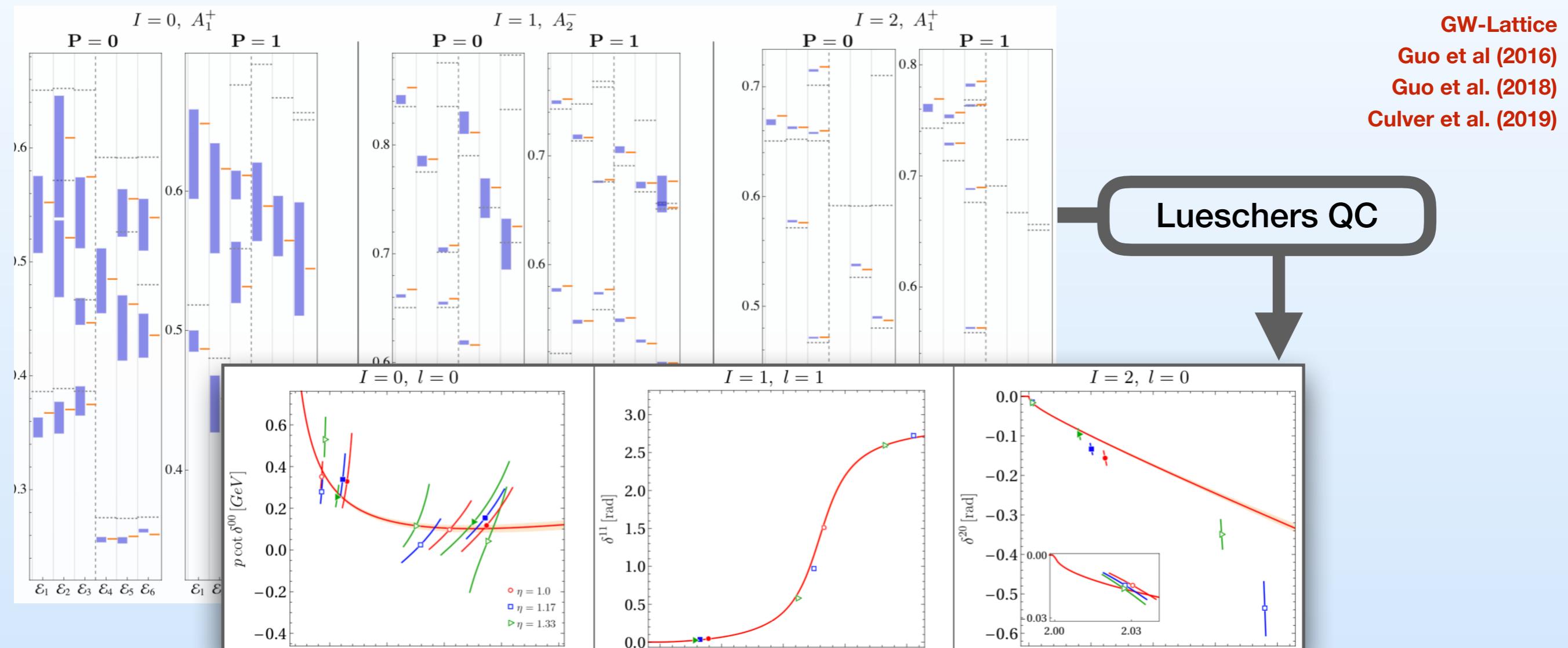
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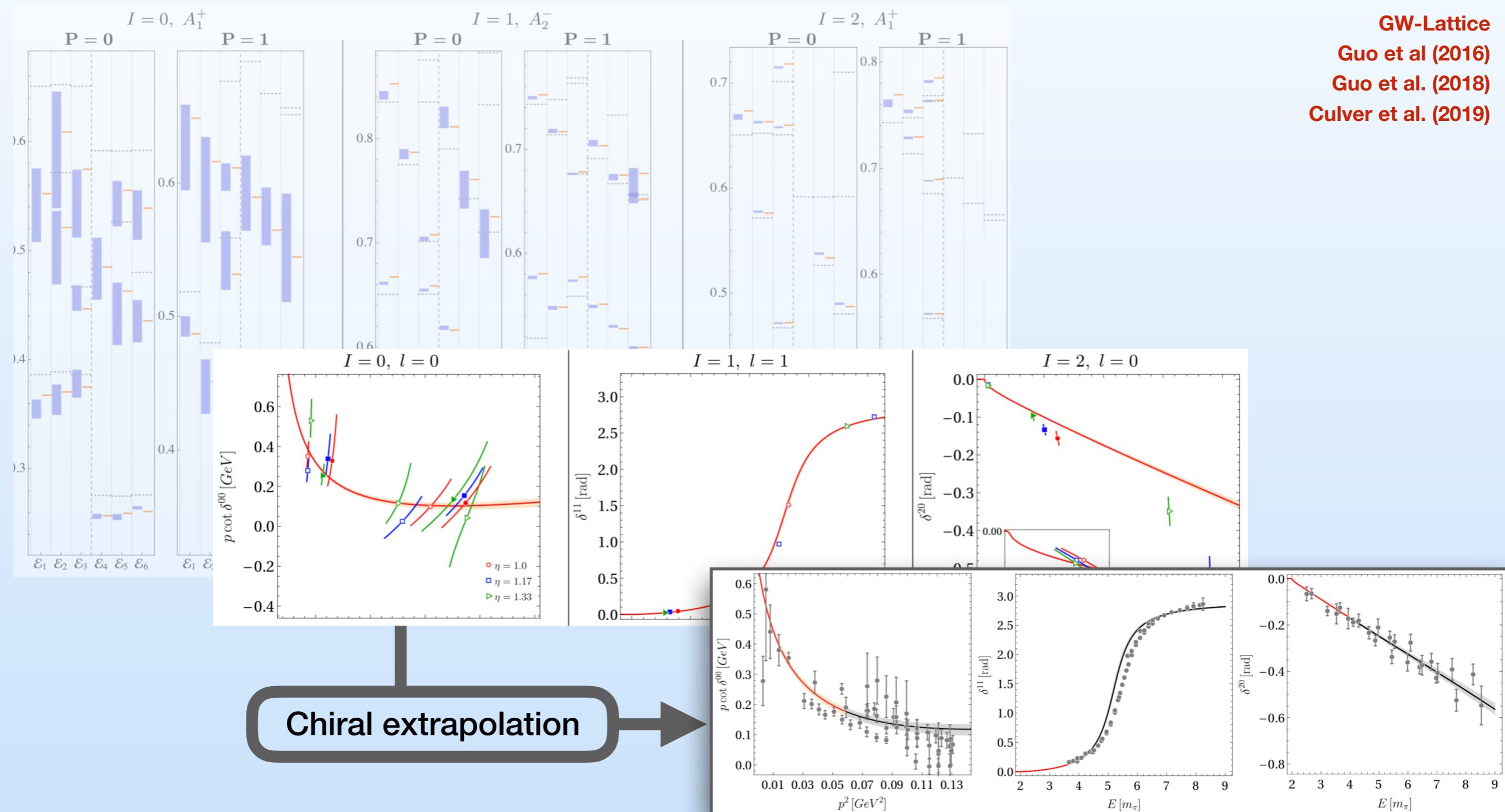
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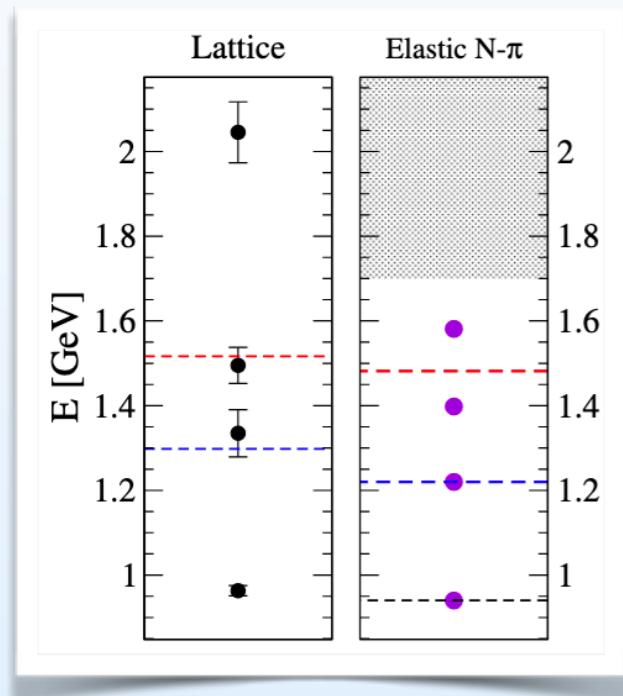
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## **Progress in 3-body lattice calculations:**

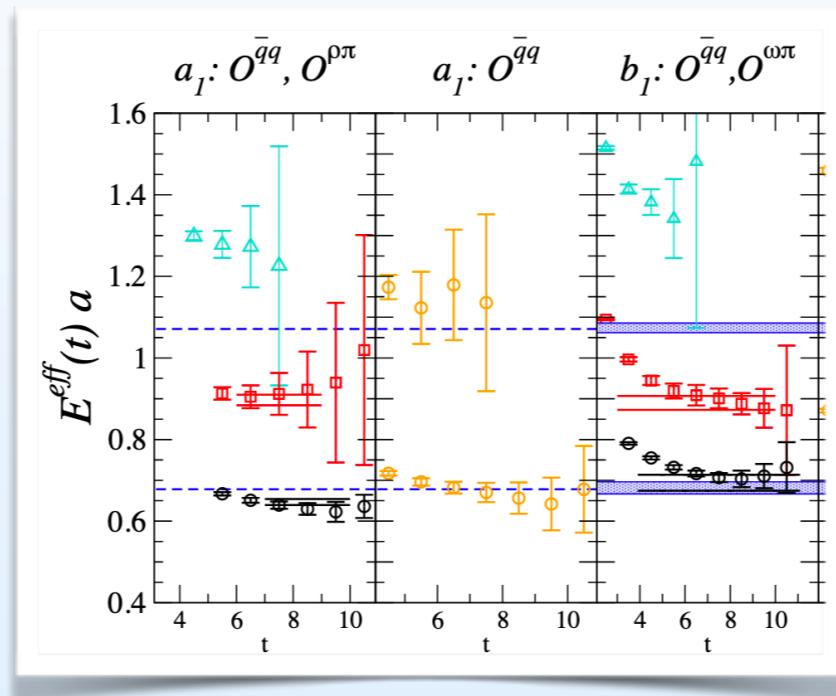
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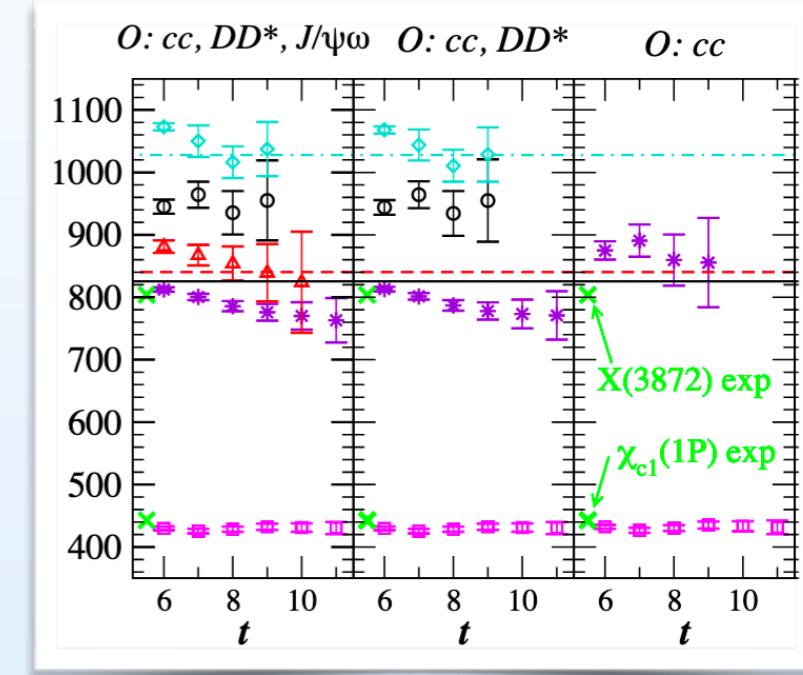
Lang et al. PRD 95(2017)

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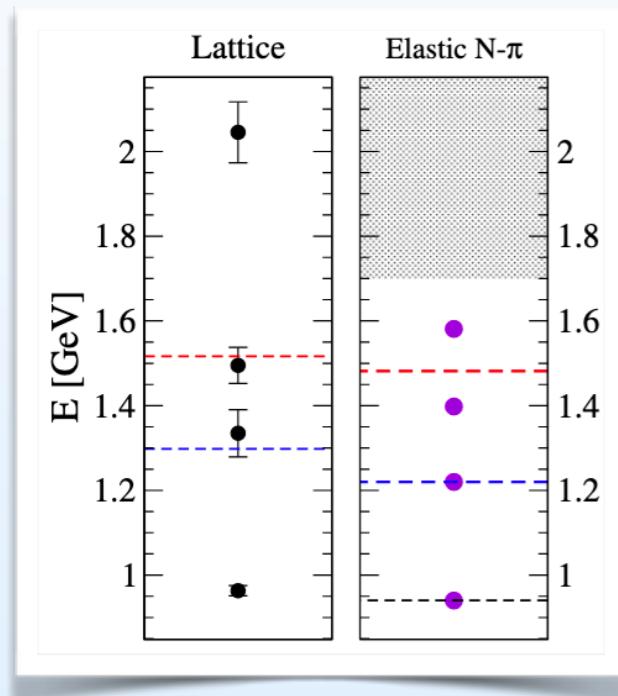
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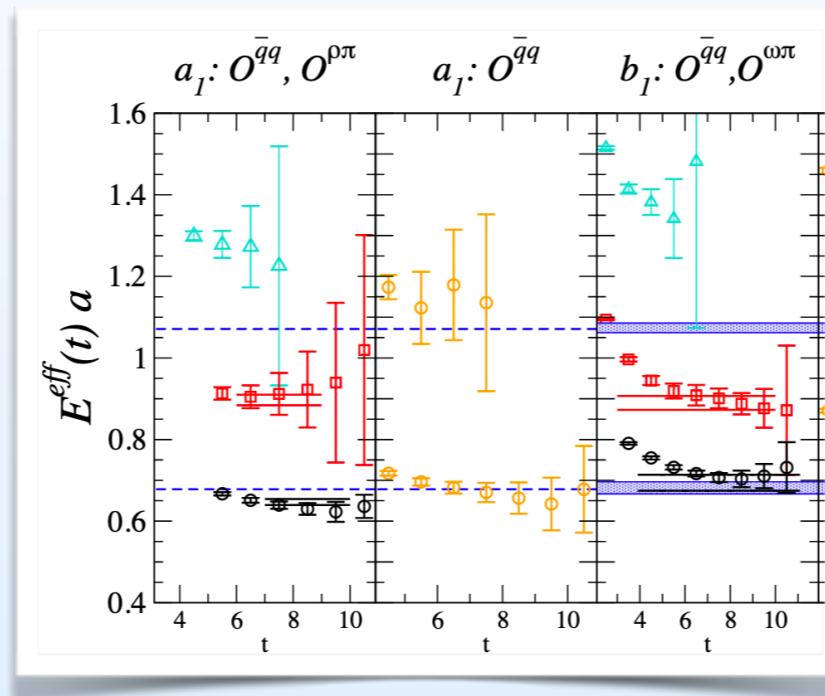
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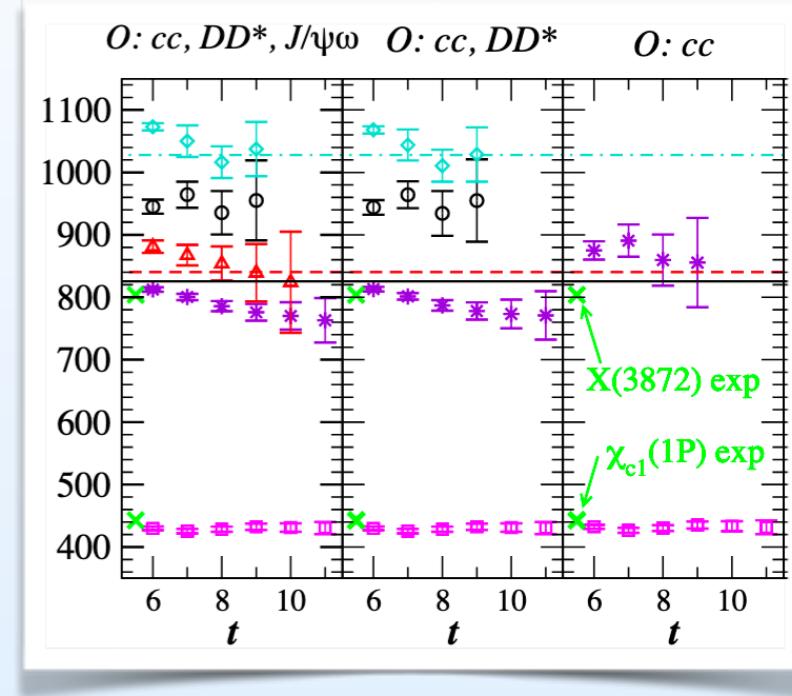
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## ◎ $I=2 \pi\rho$

Woss et al. JHEP 1807 [Hadspec]

## ◎ $I=3 \pi\pi\pi$ systems

Hörz/Hanlon PRL123 (2019)

## ◎ $\pi\omega/\pi\phi$ systems

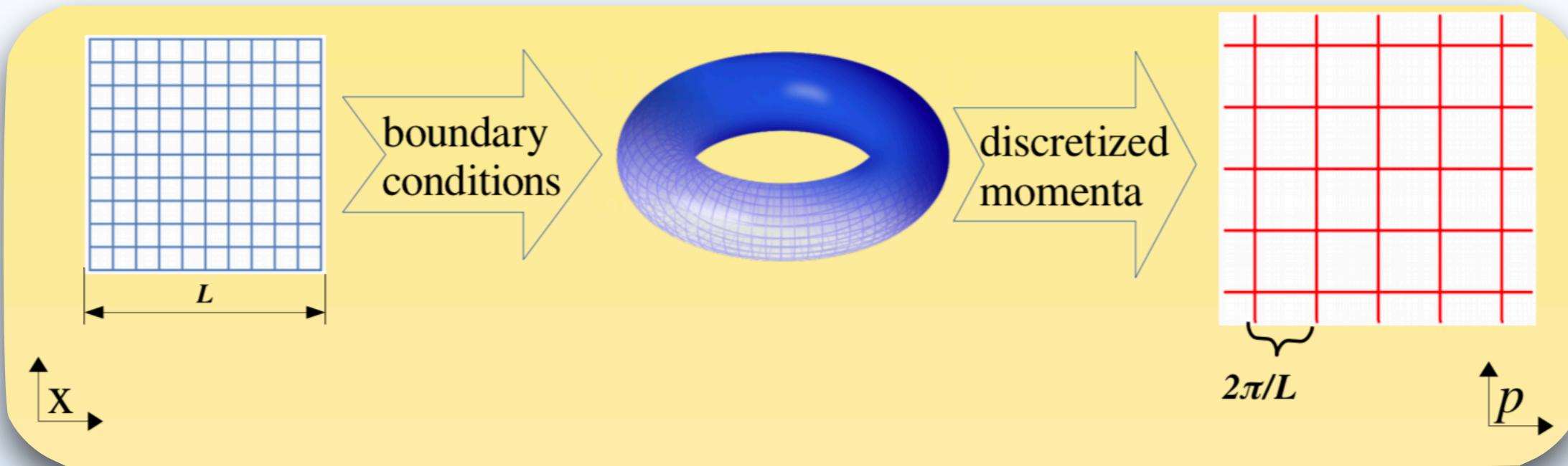
Woss et al (2019) arXiv:1904.04136

... more to come

# QUANTIZATION CONDITION

# FINITE VOLUME SPECTRUM

## ● Numerical calculations of QCD Greens functions in finite volume

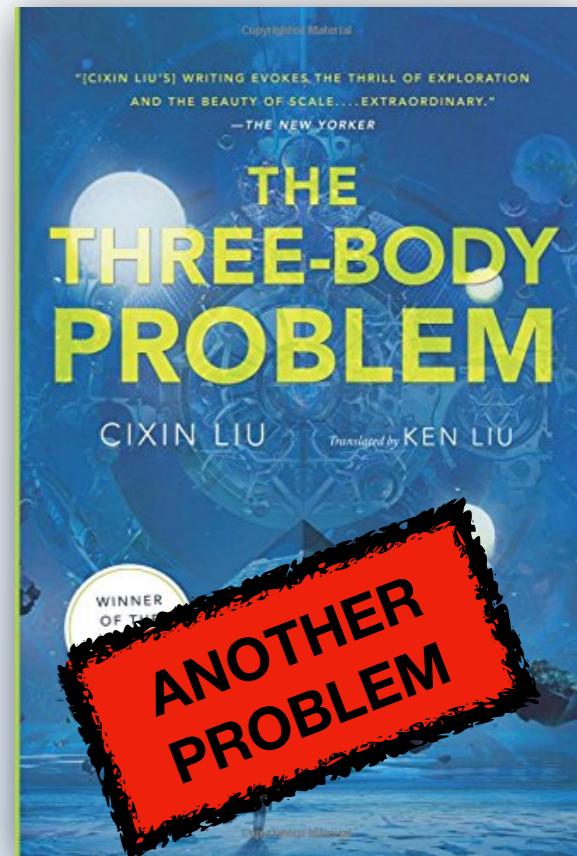


- momenta are discretized
- interaction spectrum real-valued and discrete

⇒ Quantization condition is required

# STATE OF THE ART

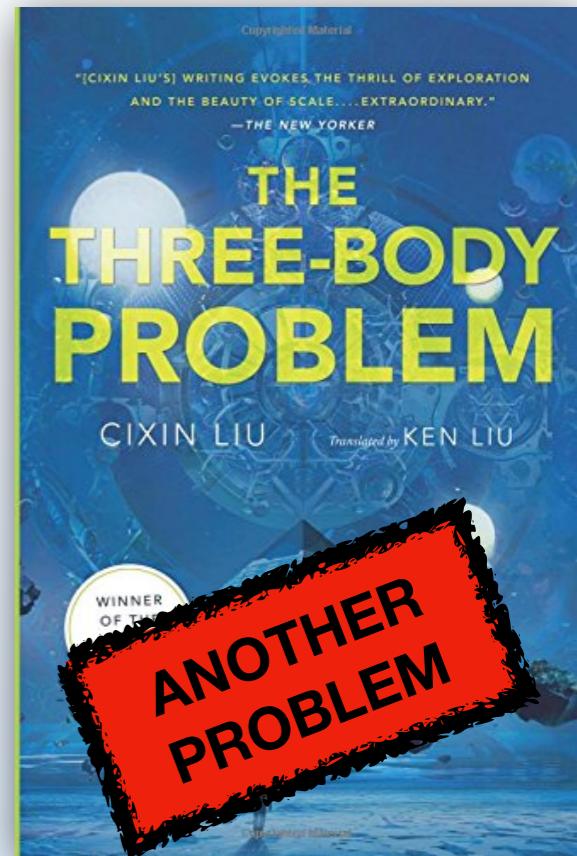
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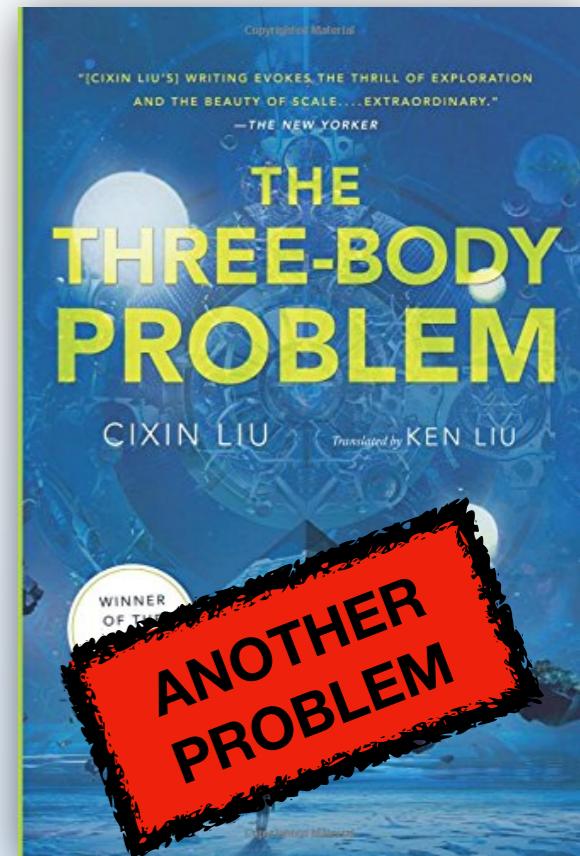
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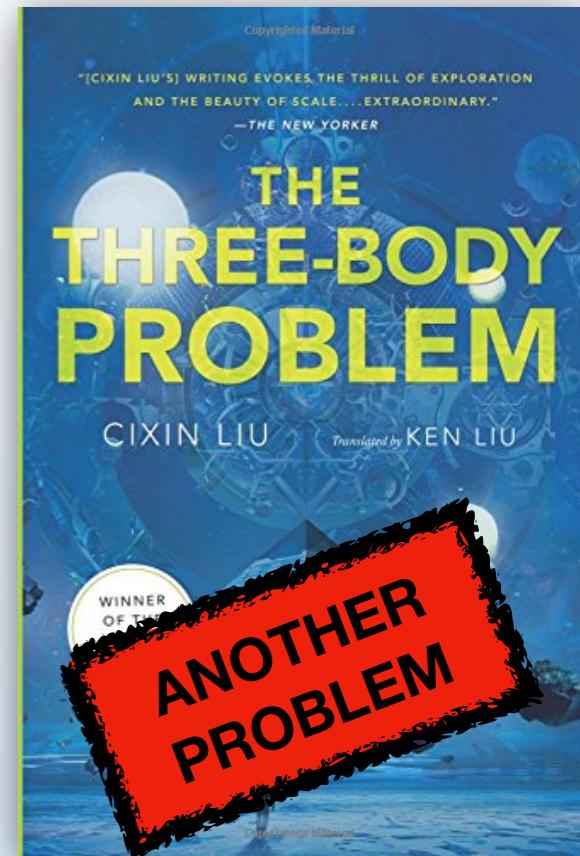
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- analysis of 2019 Hoerz/Hanlon ( $\pi^+\pi^+\pi^+$ ) spectrum (boosted systems/...) MM, Alexandru, Culver, Doering [1909.05749] Blanton, Romero-López, Sharpe [1909.02973]

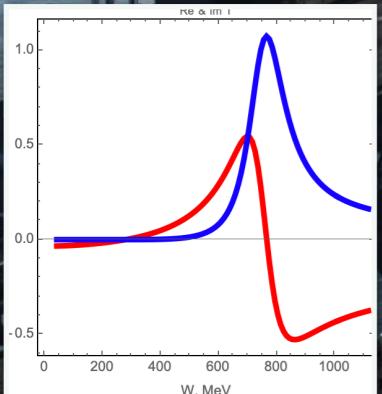
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*Unitarity*

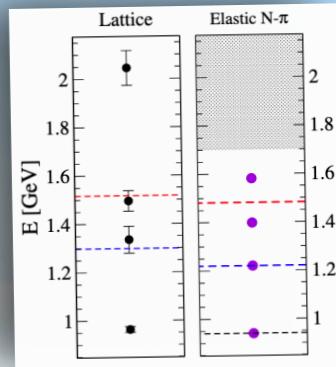
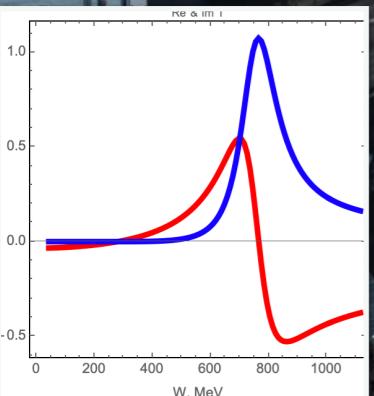


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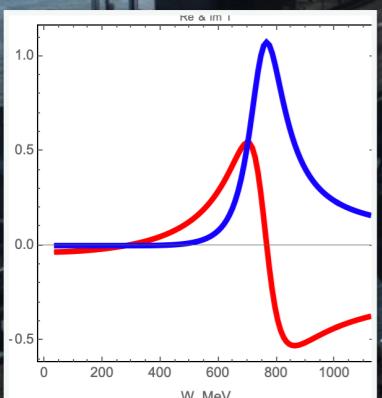
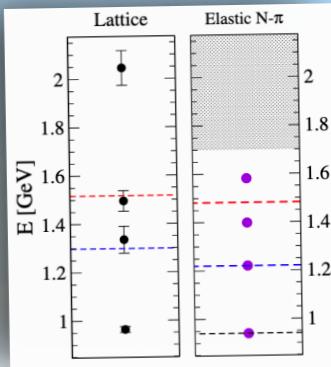


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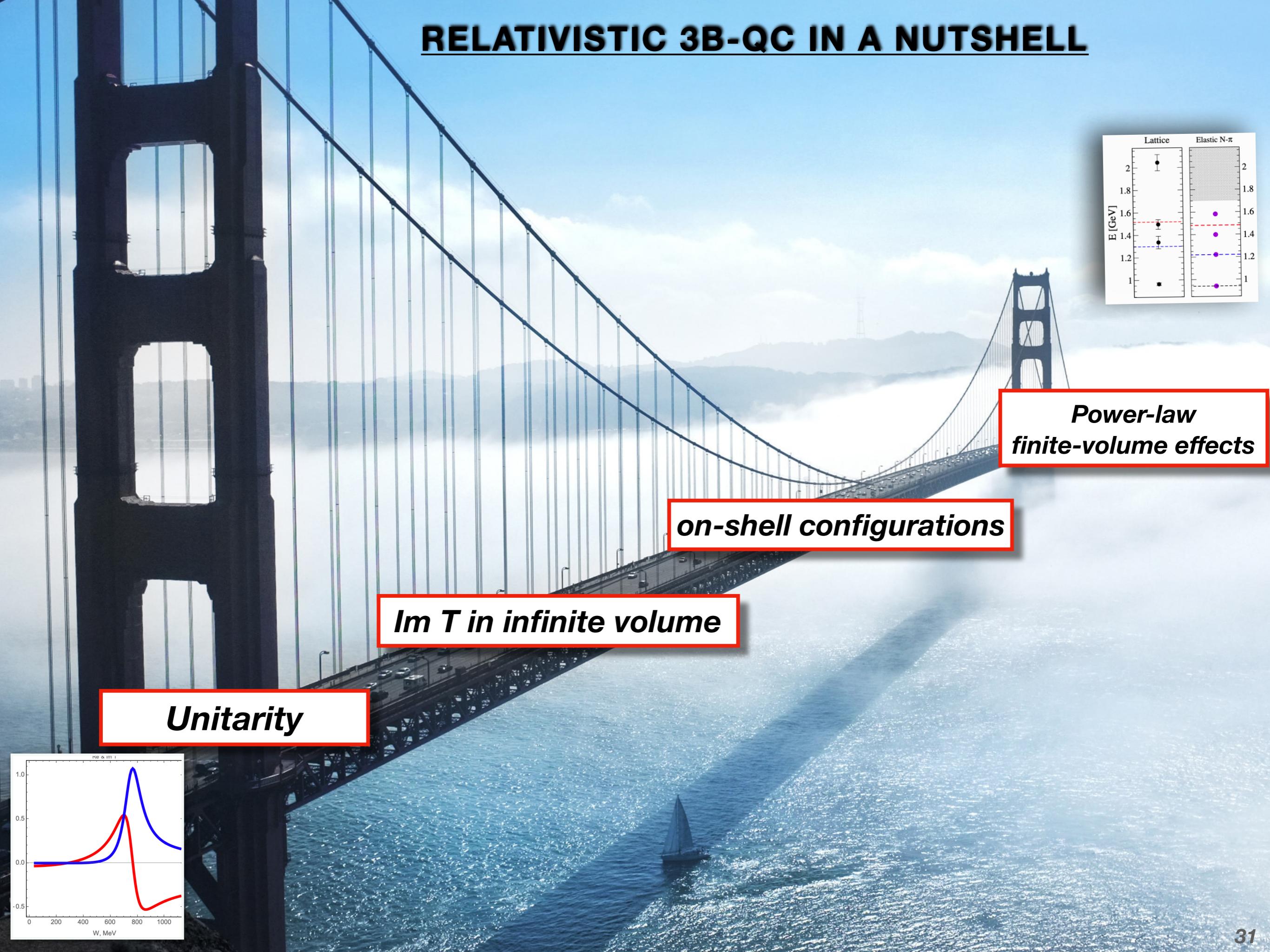
***Im T in infinite volume***



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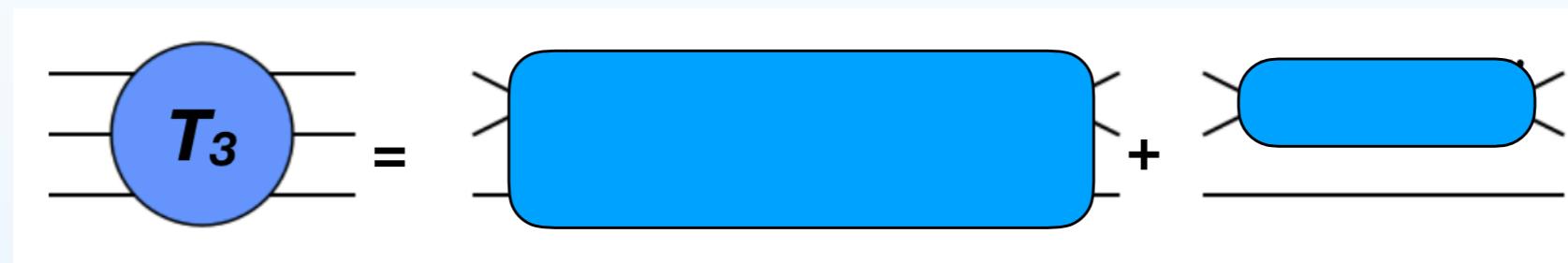
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MM, Hu, Doring, Pilloni, Szczepaniak Eur.Phys.J. A53 (2017)

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- Connected part

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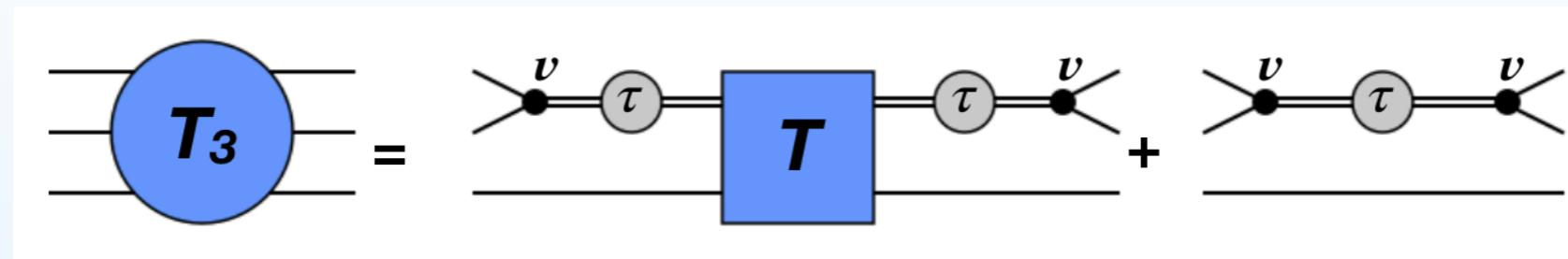
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→ cut-free-function

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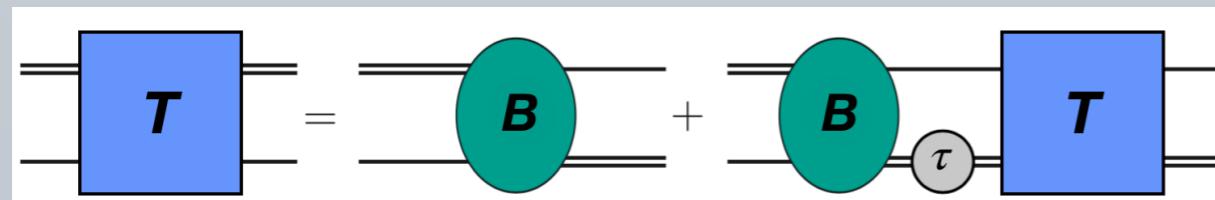
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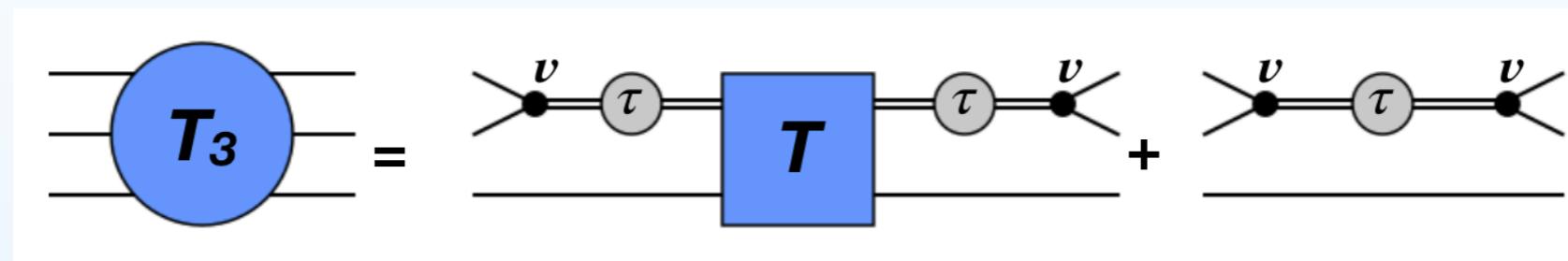
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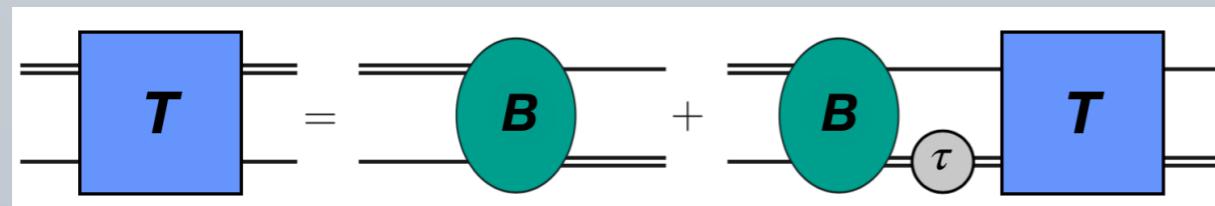
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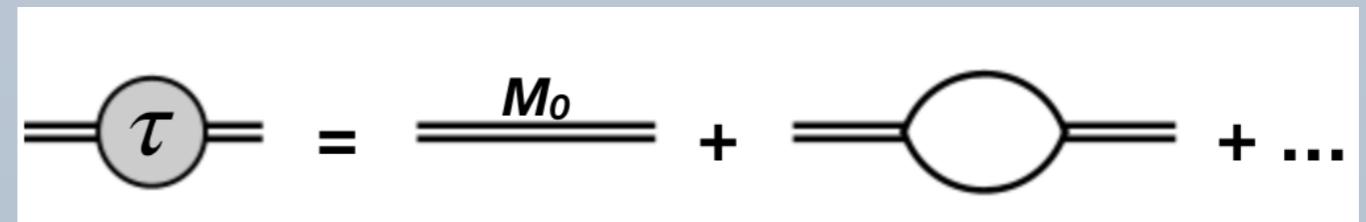
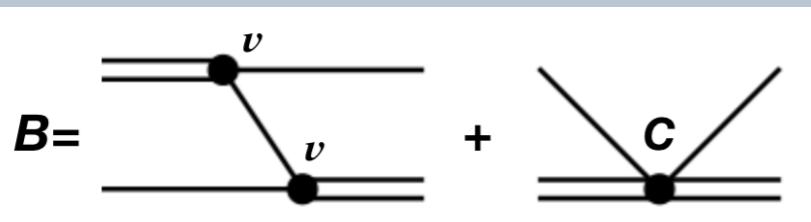
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## $B$ & $\tau$ from 3-body unitarity

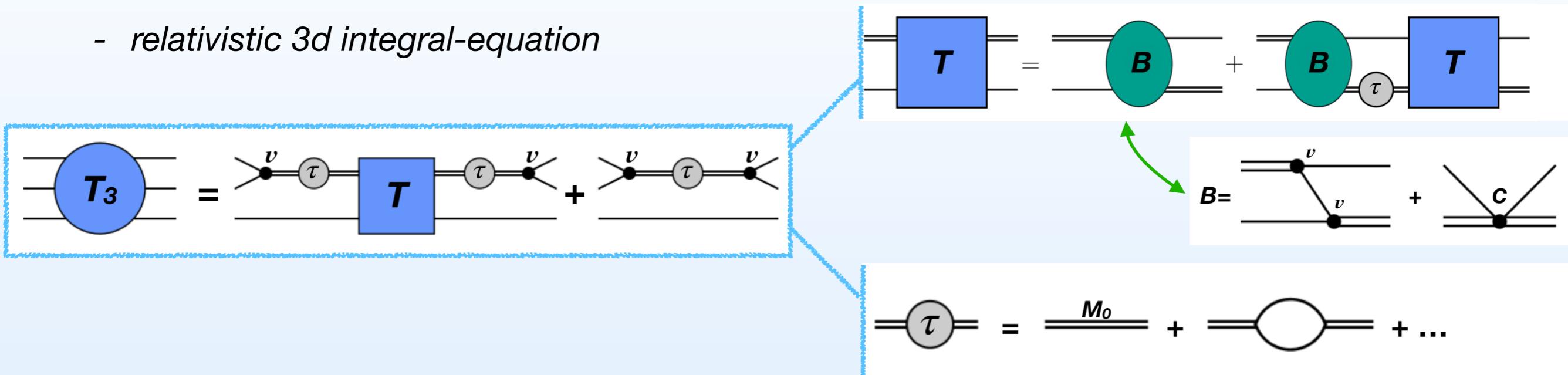


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## Result:

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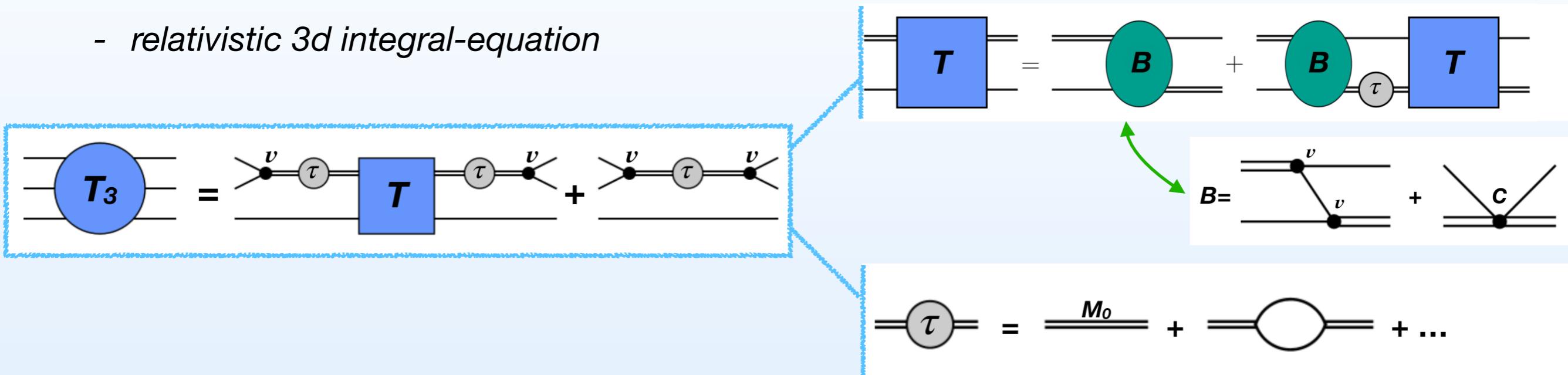


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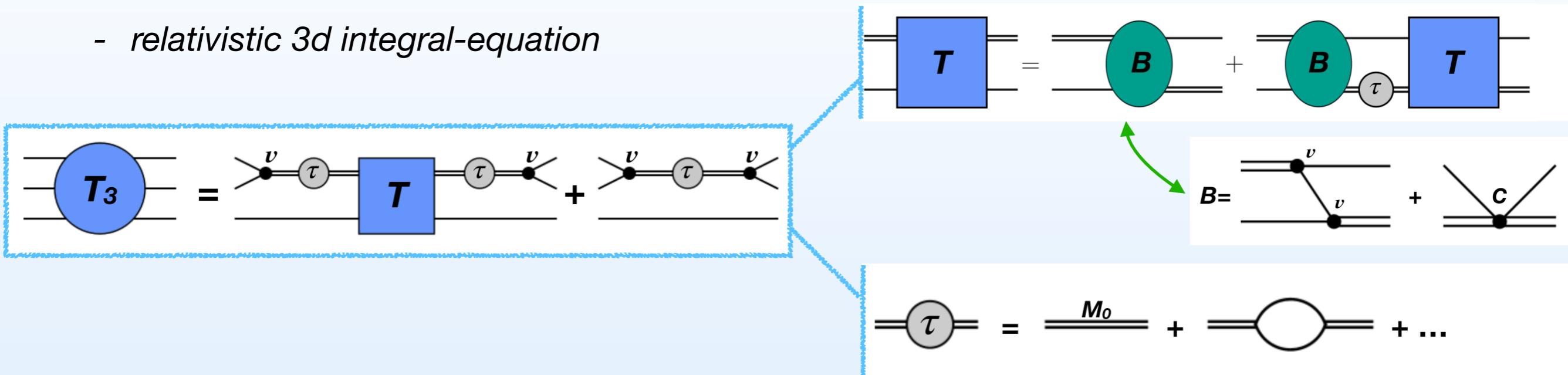
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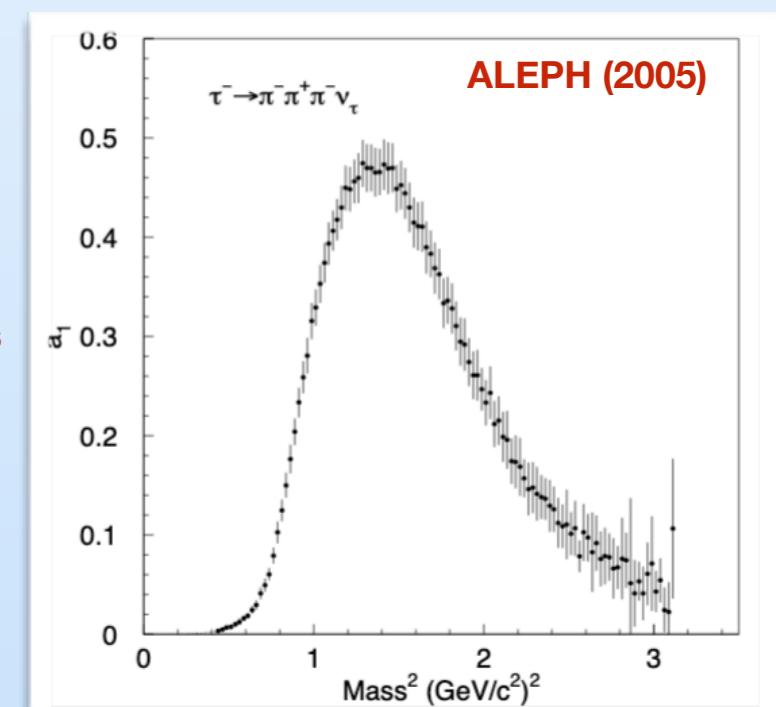


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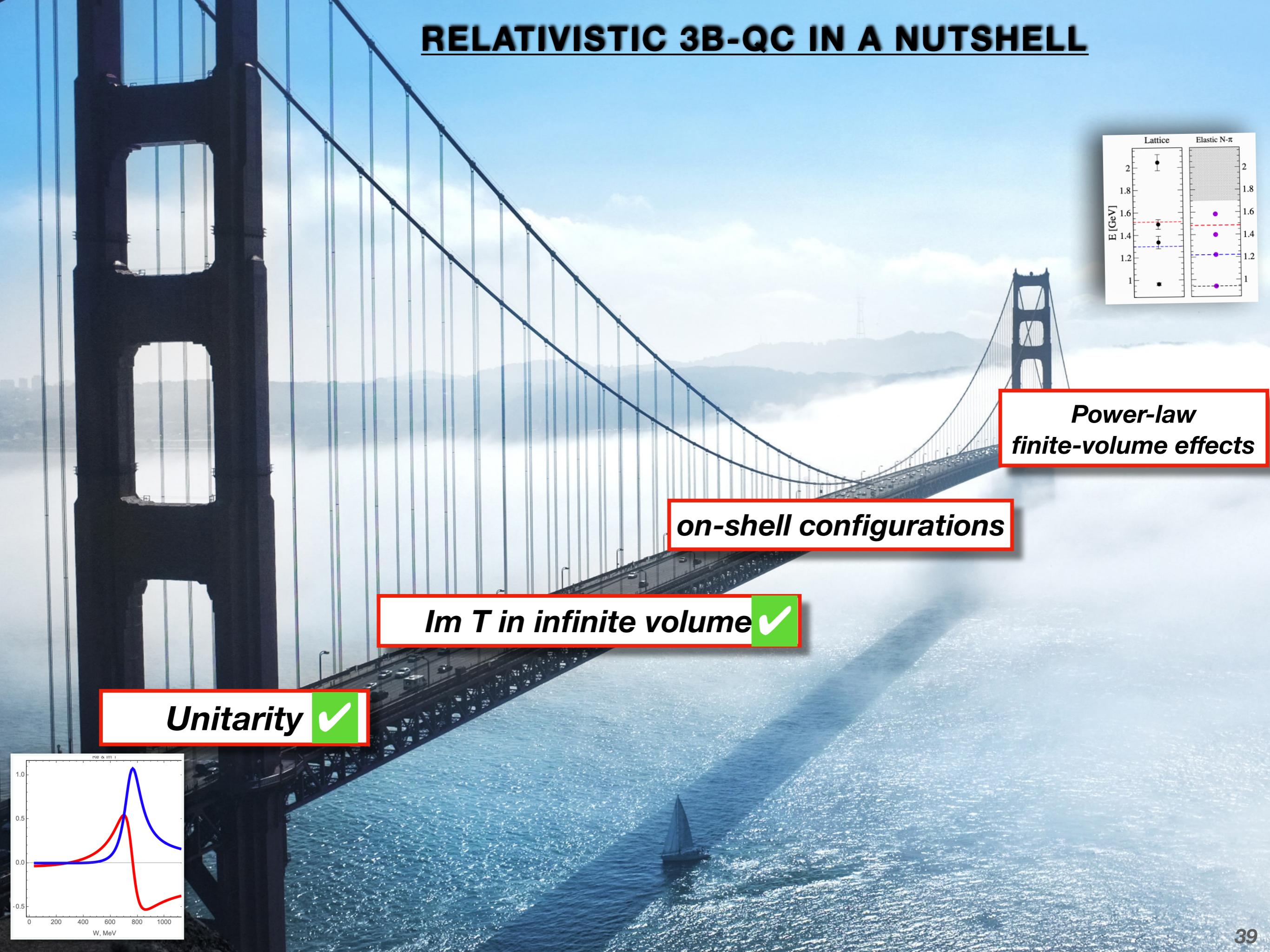
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- application to experimental data:  
 $\tau$  decay data in the  $a_1(1260)$  channel

D. Sadasivan, MM, Doring [GWU] ... work in progress



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- In “finite volume world” momenta are discretized -> replace integrals by sums
    - ⇒ scattering amplitude is **real-valued** and **singular** matrix equation
    - ⇒ singularities, iff total energy ( **$E^*$** ) is such that **3 pions are on-shell**
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MM Doring (2017)

$$[E_L]_{pq} = \delta_{pq} 2L^3 \sqrt{m_\pi^2 + \mathbf{p}^2}$$

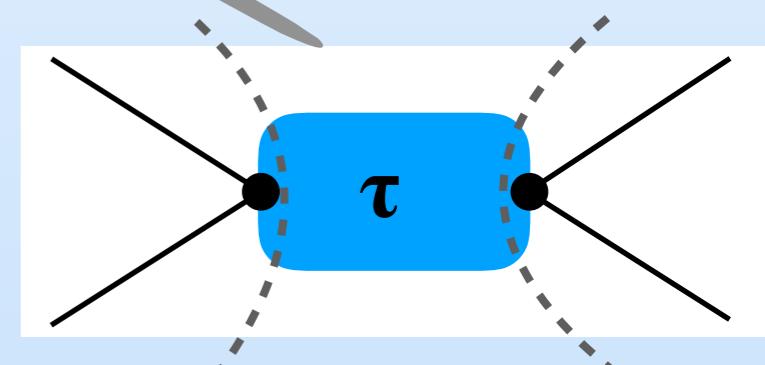
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~ 2-body partial wave

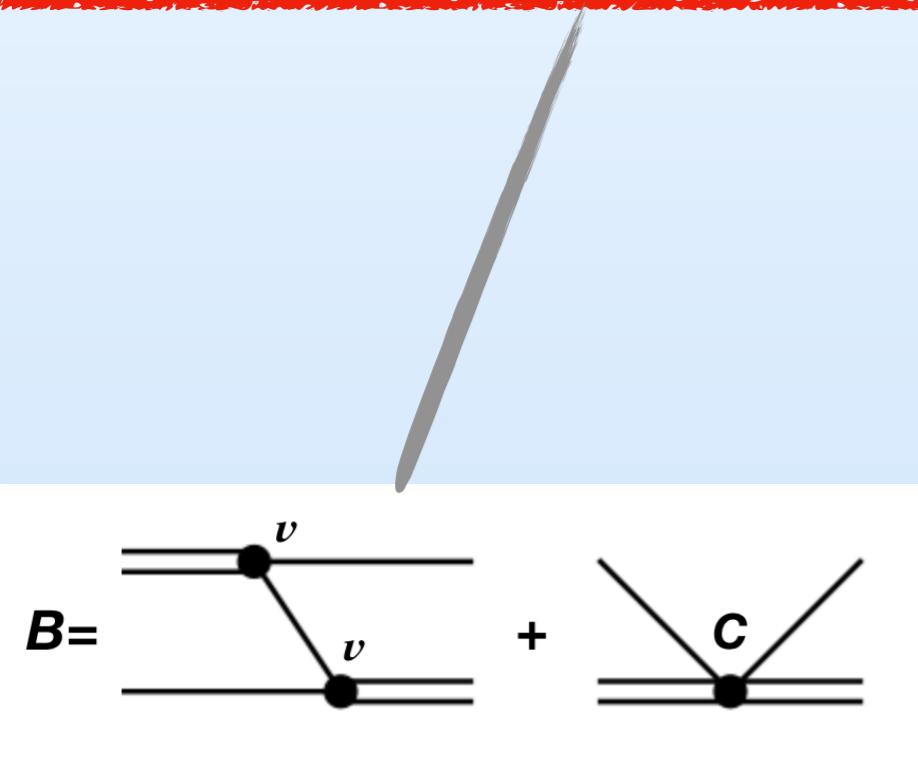
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MM Doring (2017)



- In “finite volume world” momenta are discretized -> replace integrals by sums
  - ⇒ scattering amplitude is **real-valued** and **singular matrix equation**
  - ⇒ singularities, iff total energy ( **$E^*$** ) is such that **3 pions are on-shell**

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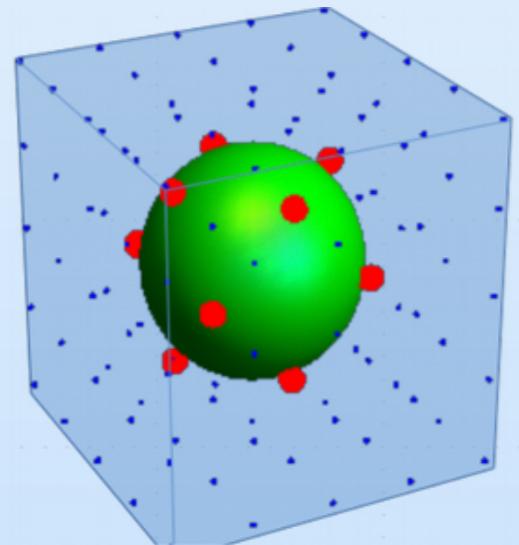
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  - on the level of  $B$  in [...]

Doring, Hammer, MM, ... PRD97(2018)



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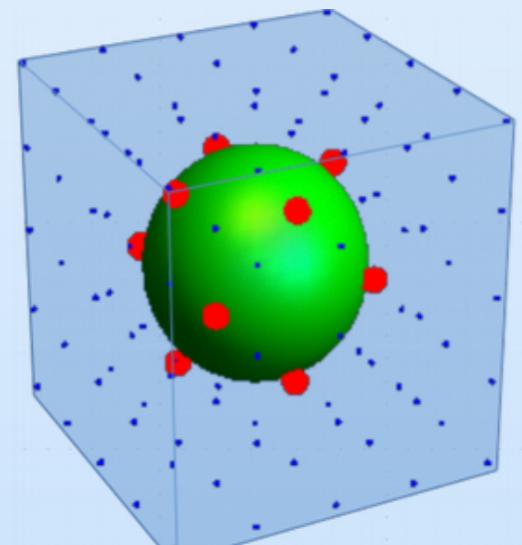
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- Or project final expression to irreps
  - on the level of  $\langle \dots \rangle$

Doring, Hammer, MM, ... PRD97(2018)

MM et al. (2019)



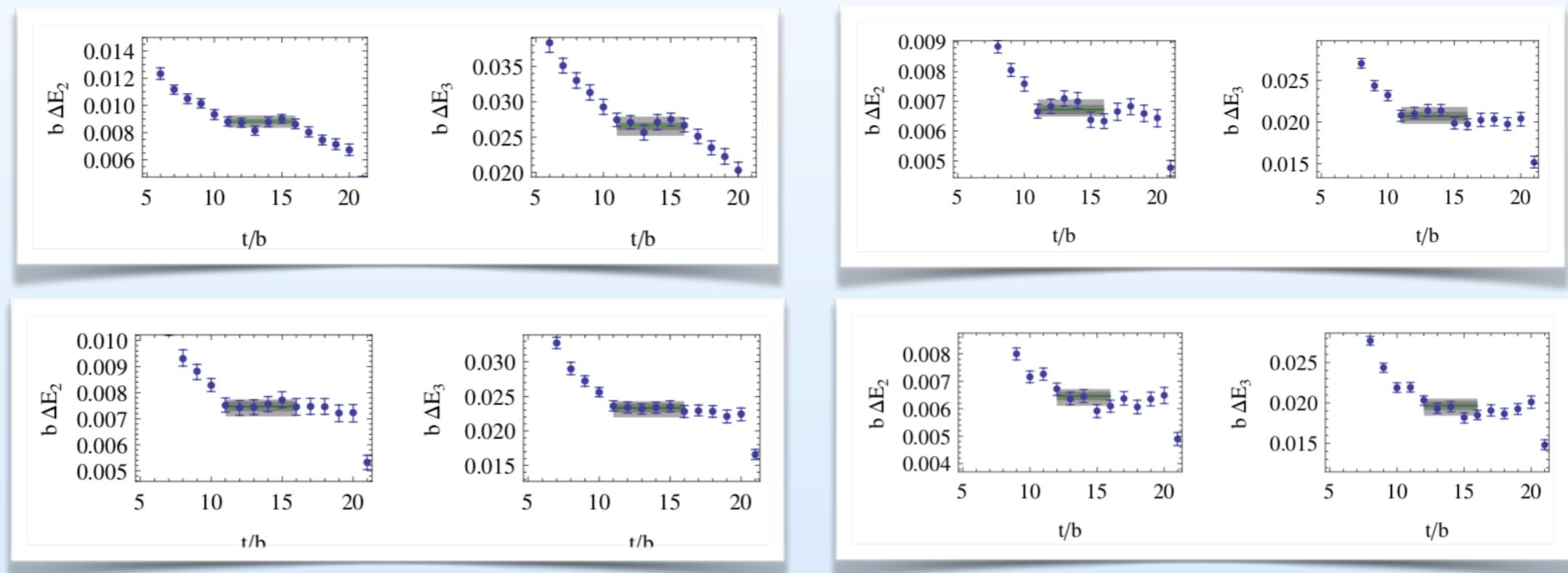
# APPLICATIONS

# 1. PHYSICAL APPLICATION

MM, Doring PRL 122 (2019) arXiv: 1807.04746

## ● Interesting system to study: $\pi^+\pi^+\pi^+$

- Results availability → **Ground level for  $2\pi^+$  &  $3\pi^+$  @  $L=2.5$  fm**



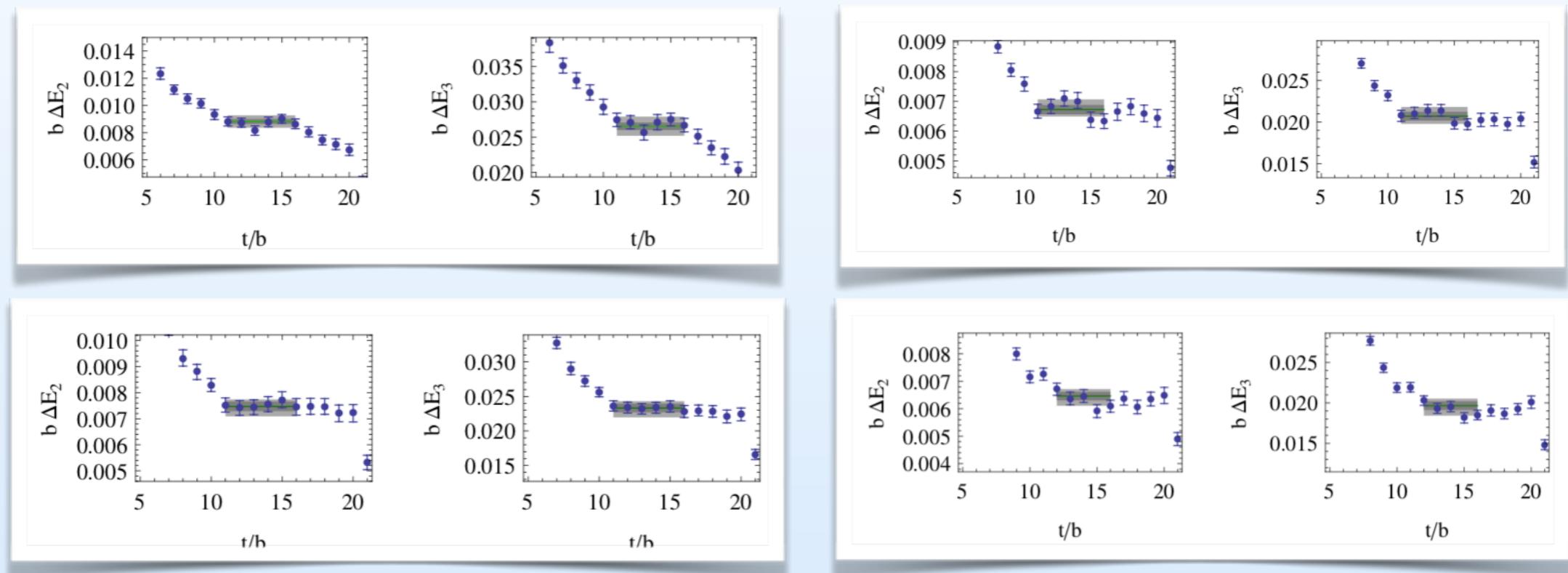
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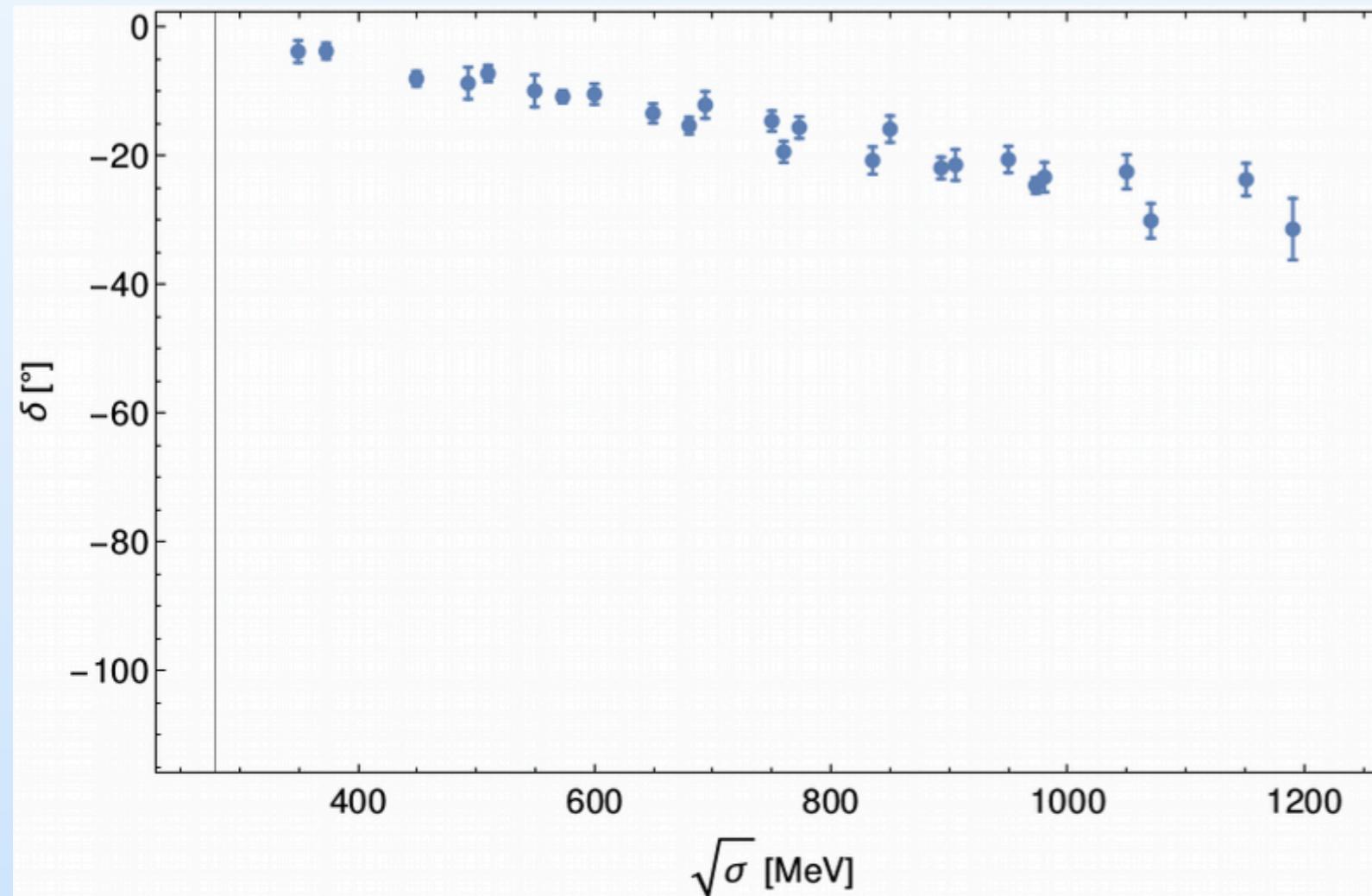
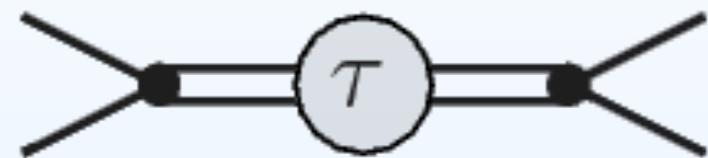
- Repulsive channel → **Does the “isobar” picture hold?**
- $m_\pi = 291 / 352 / 491 / 591$  MeV → **Chiral extrapolation in 3body system?**

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## ◎ 2-body sub-channel:

- One-channel problem –  $\pi^+\pi^+$  system in S-wave
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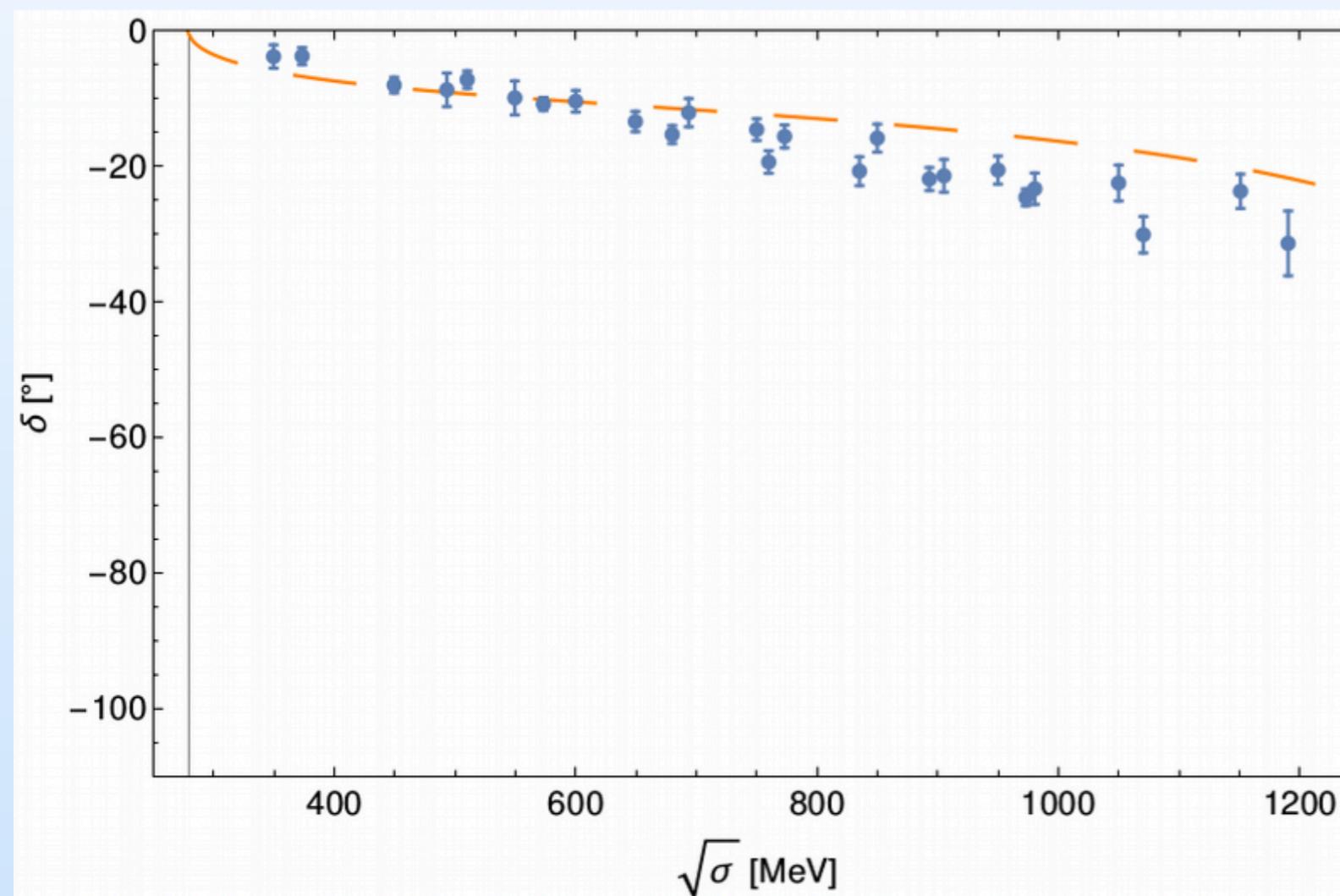
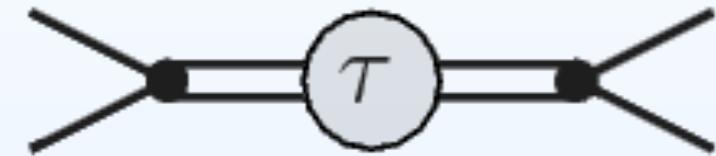


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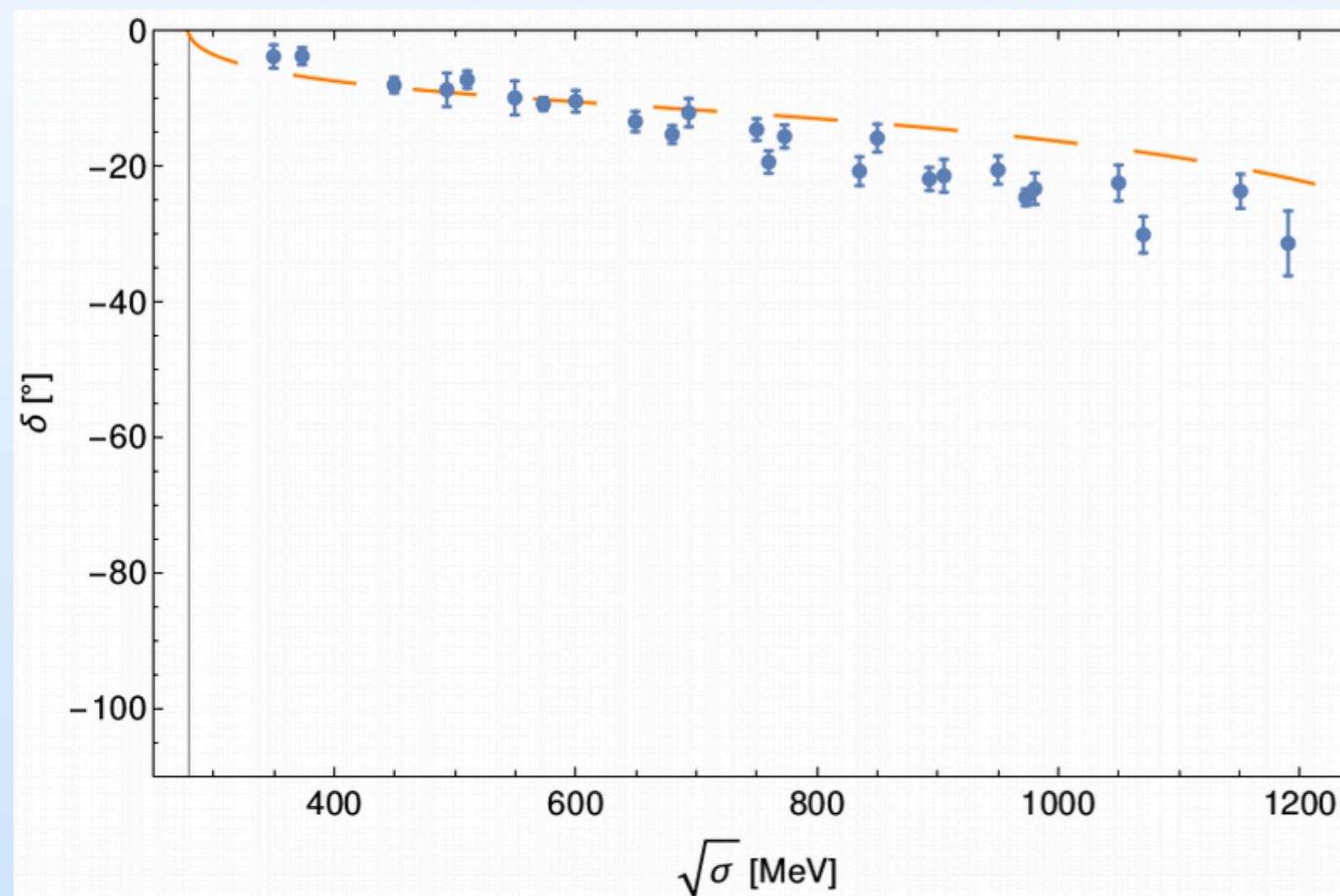
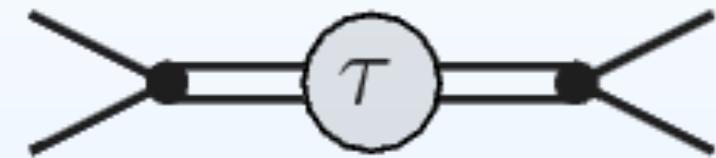
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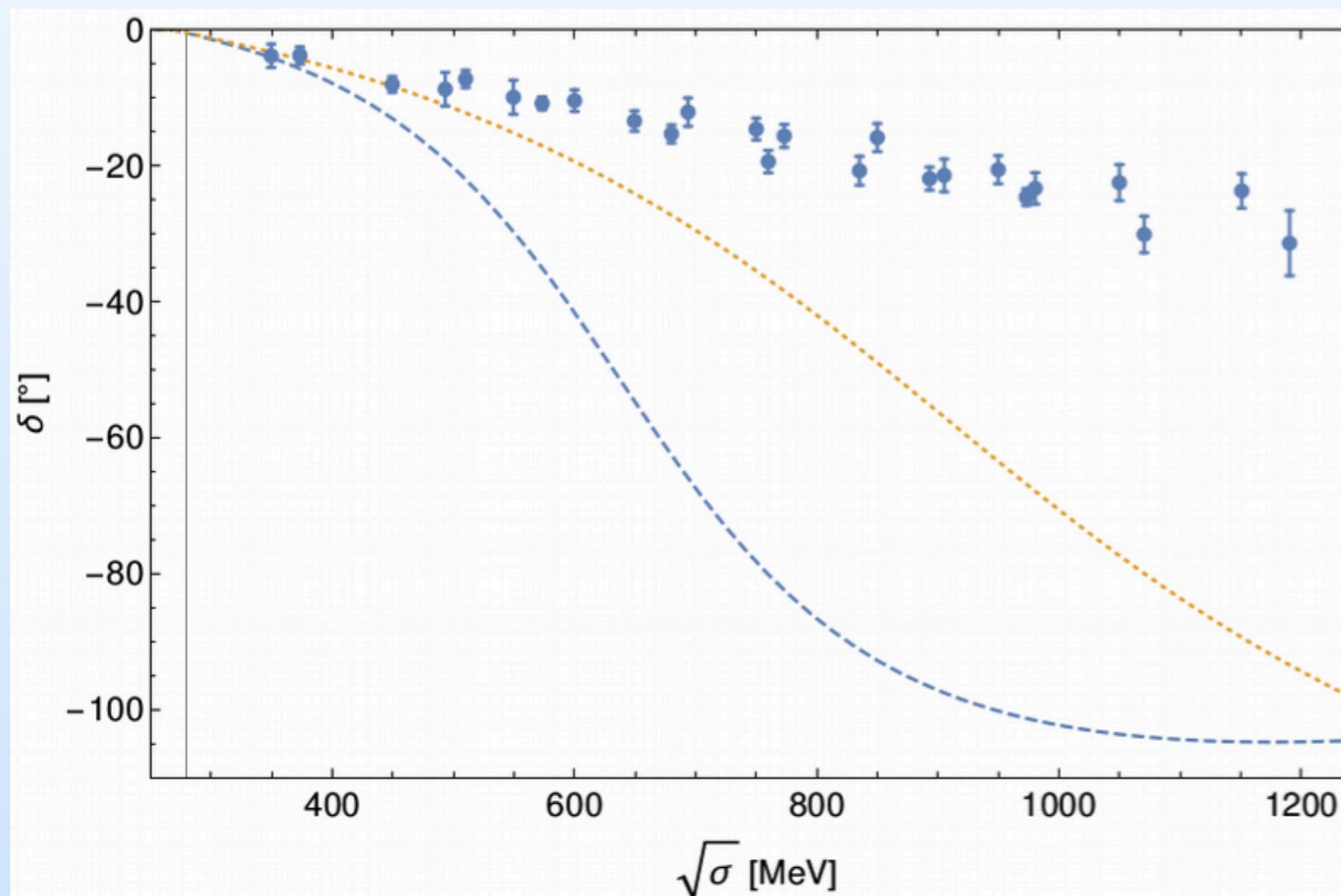
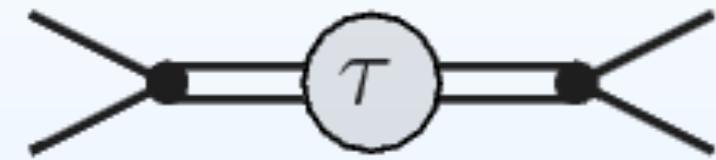
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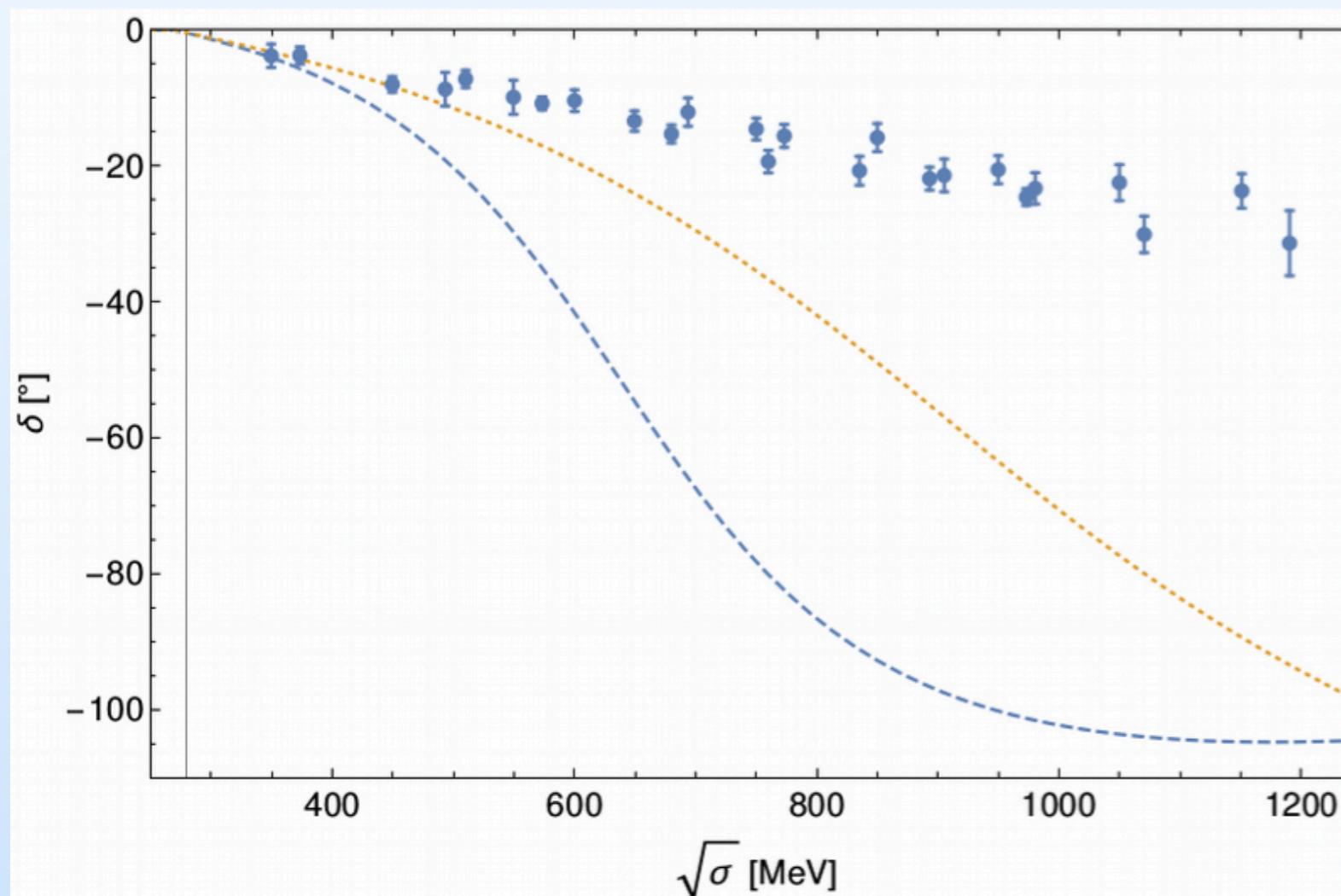
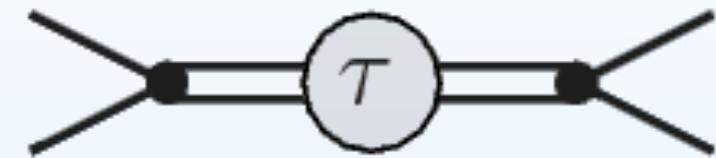
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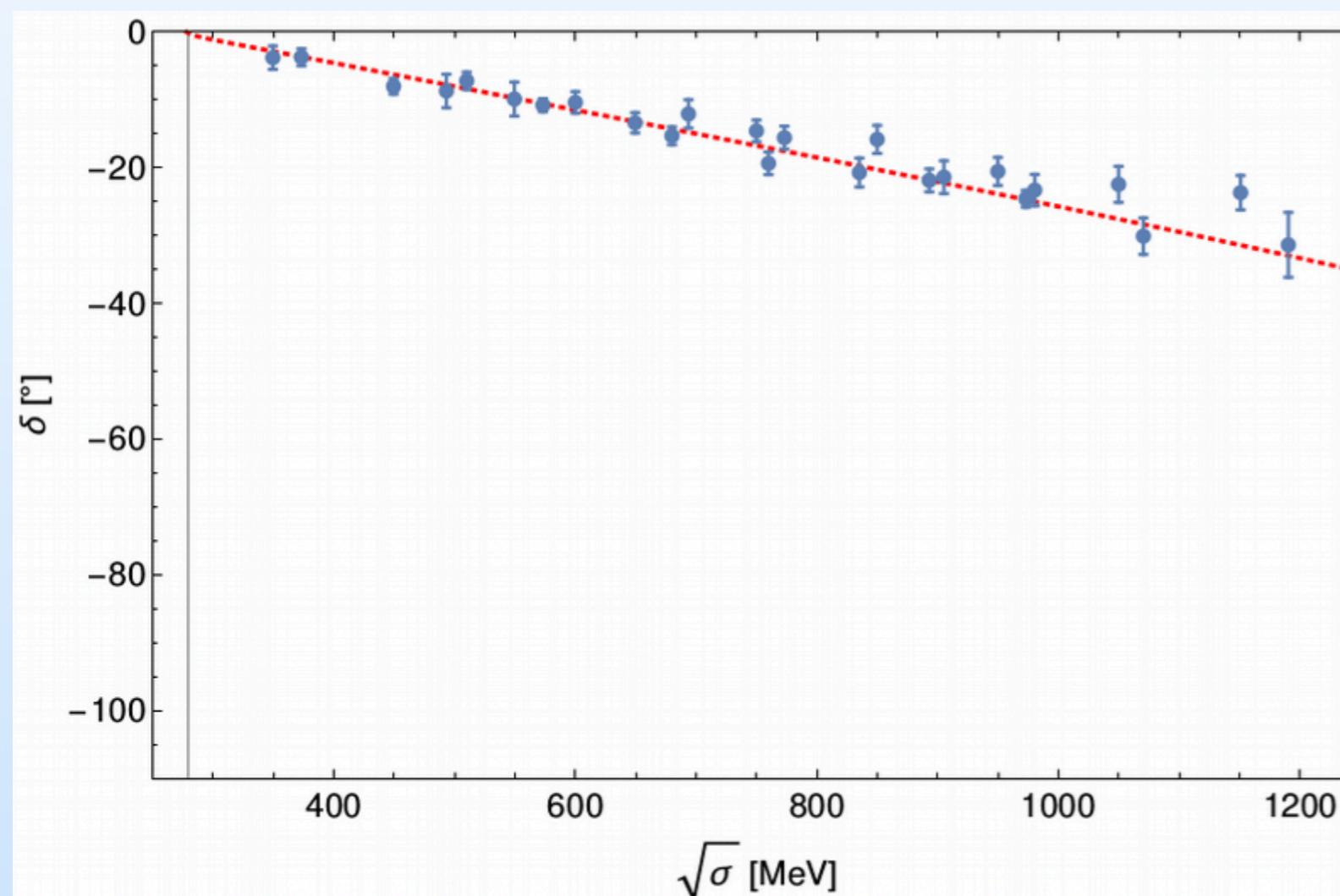
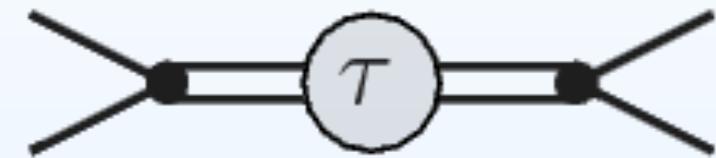
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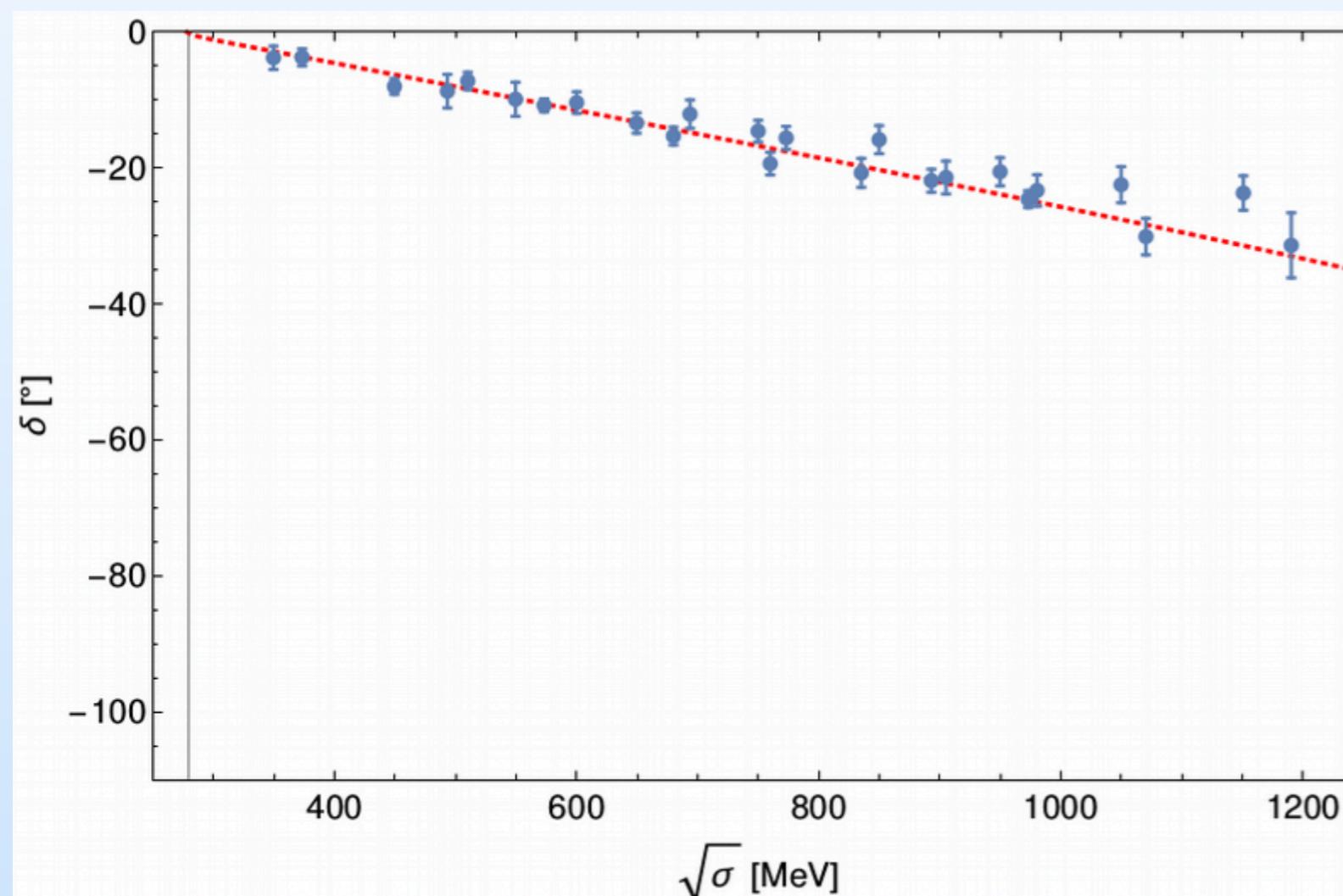
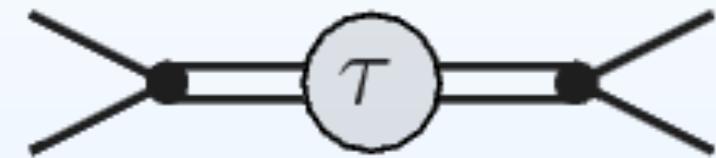
Truong(1988)

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 ✓

Truong(1988)

- correct  $\sigma$  &  $m_\pi$  behavior  
- parameters known

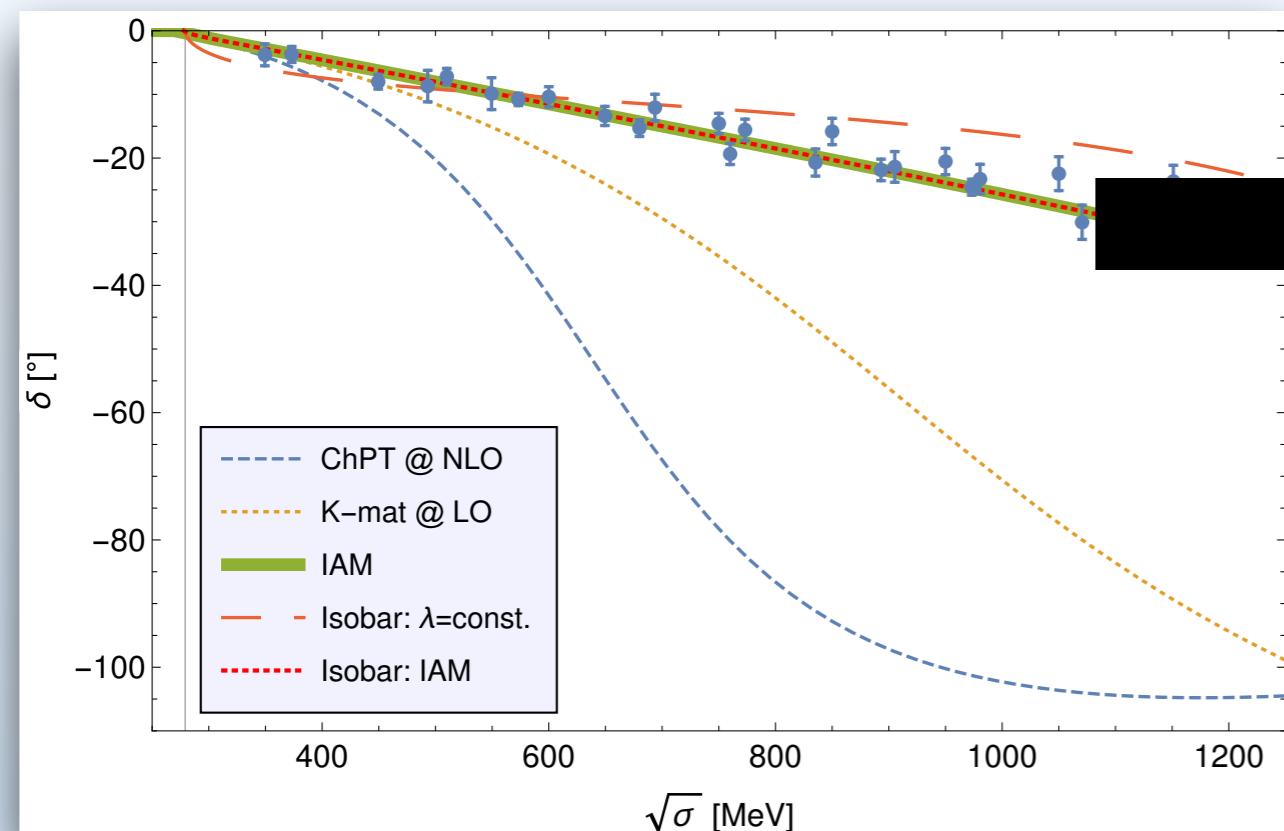
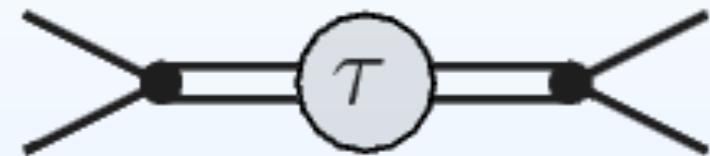
Gasser/Leutwyler(1984)

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MM, Doring PRL 122 (2019) arXiv: 1807.04746

## ◎ 2-body sub-channel:

- One-channel problem –  $\pi^+\pi^+$  system in S-wave
- IAM parametrized scattering amplitude



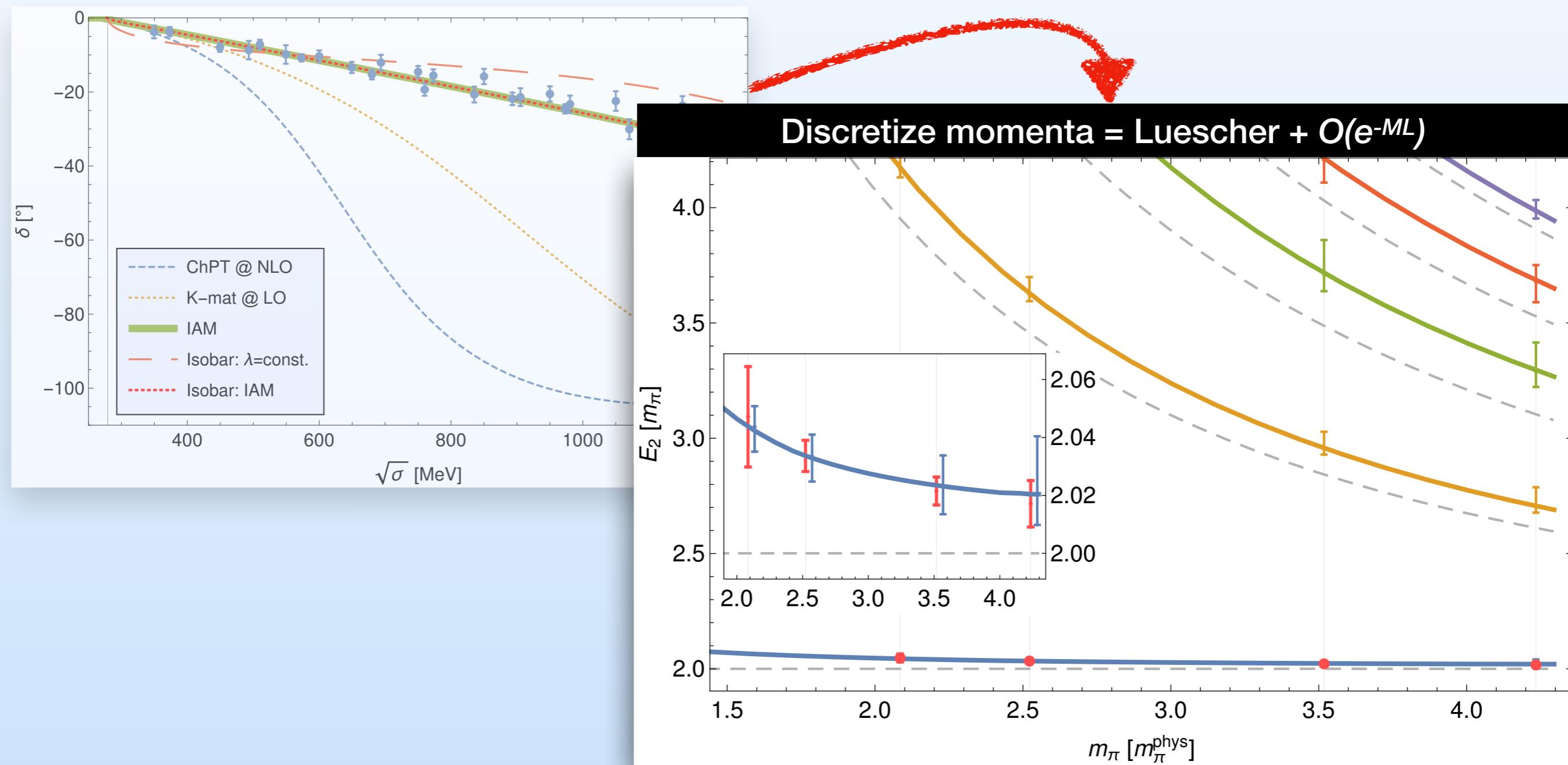
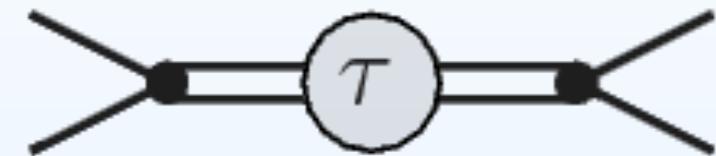
Discretize momenta = Luescher +  $O(e^{-ML})$

# PHYSICAL APPLICATION

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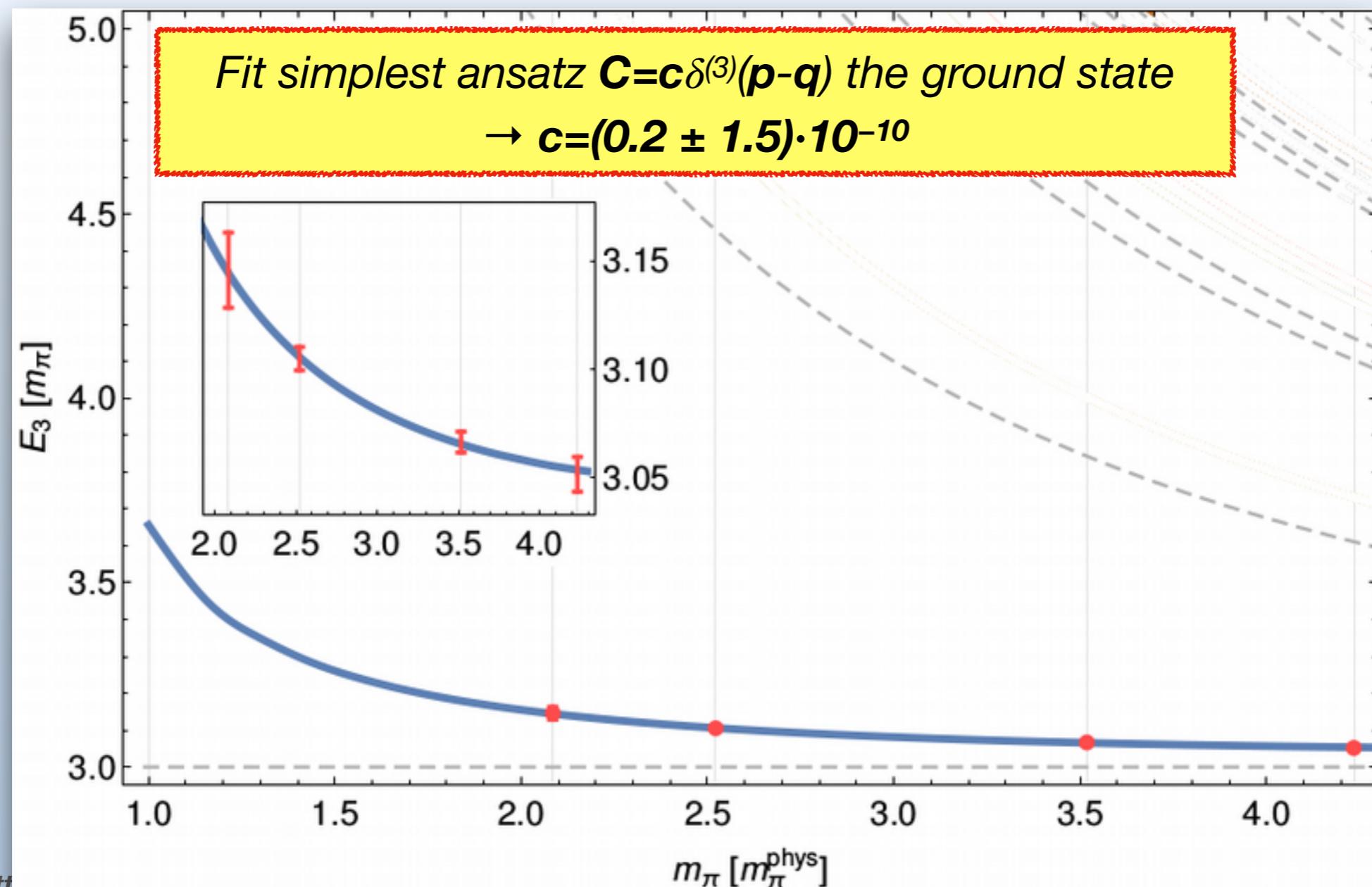
# PHYSICAL APPLICATION

MM, Doring PRL 122 (2019) arXiv: 1807.04746

## ◎ 3-body spectrum

- genuine 3-body force ***unknown***
- momenta dependent function

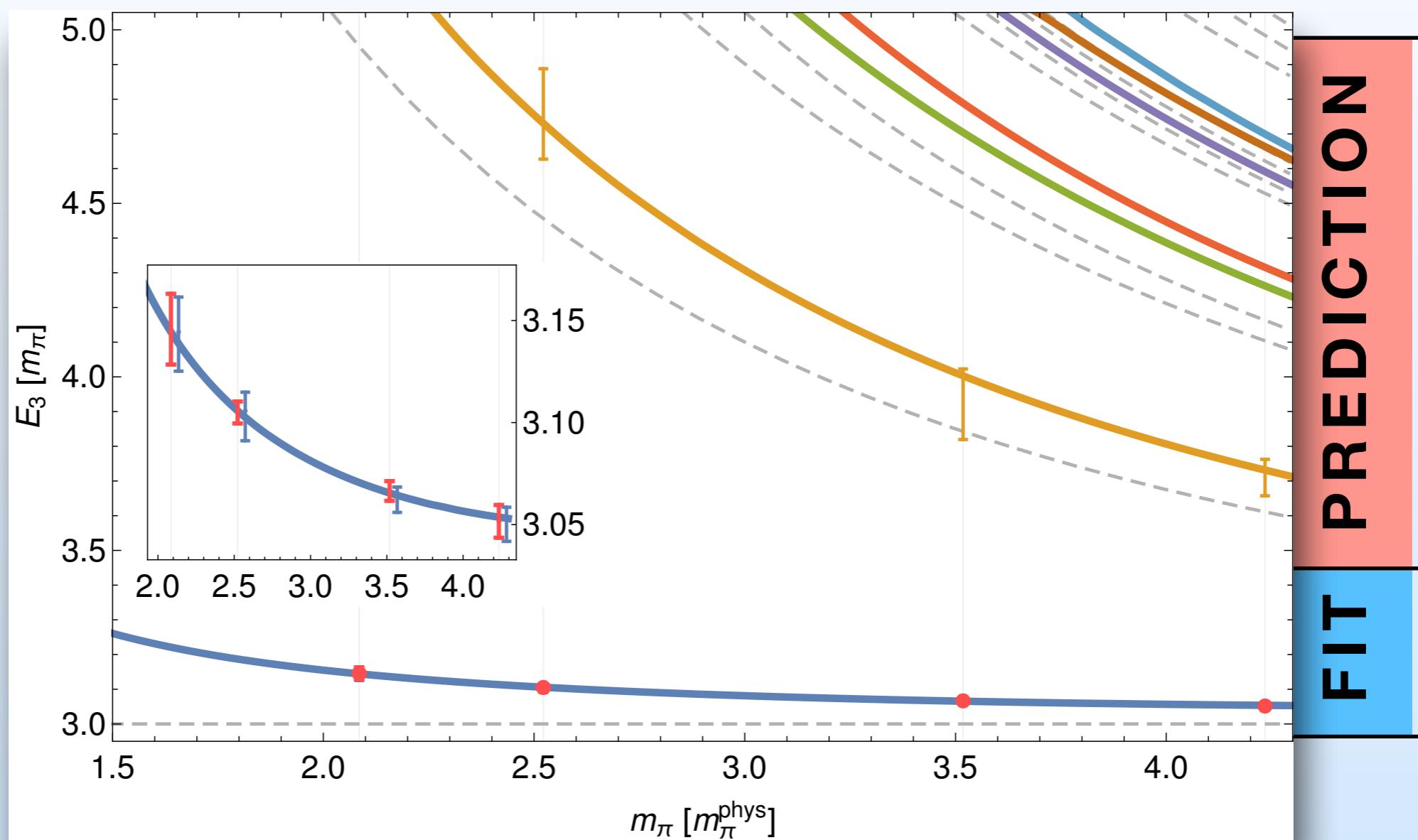
$$\left\langle v(p_1, p_2) \left[ B(E^*) + E_L \tau_{LP}^{-1}(E^*) \right]_{p_3, q_3}^{-1} v(q_1, q_2) \right\rangle_{p_i, q_j} = \infty$$



# PHYSICAL APPLICATION

MM, Doring PRL 122 (2019) arXiv: 1807.04746

## ◎ 3-body spectrum – predictions – excited level spectrum



1. 1-to-1 correspondence of interacting and non-interacting levels
2. Energy levels are shifted block-wise noted also in Briceno et al. PRD98(2018)
3. Corresponding poles are simple

## 2. PHYSICAL APPLICATION

- New data is now available for higher boosts and irreps

Hoerz Hanlon PRL 123 (2019)

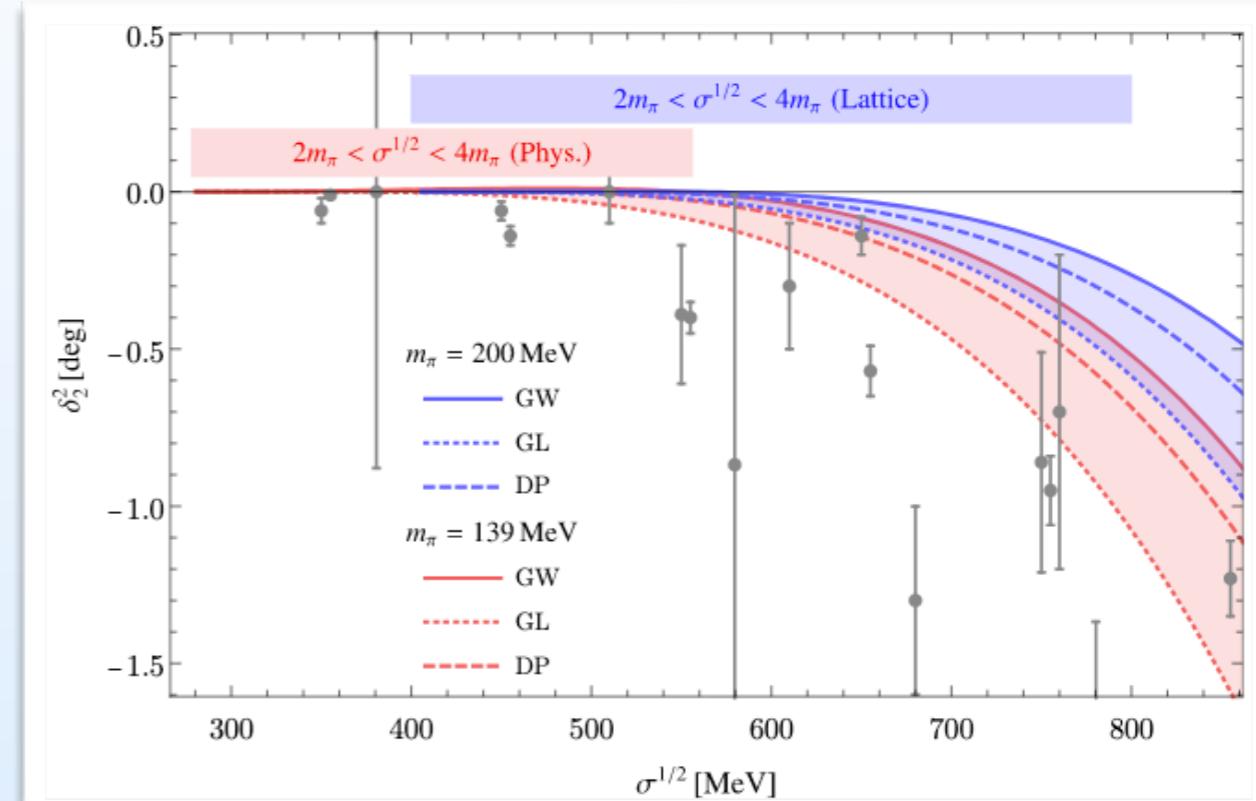
- *excited level spectrum*
- **$m_\pi \sim 200 \text{ MeV}$ ,  $m_\pi L \sim 4.16$**
- *multiple boosts and irreps*

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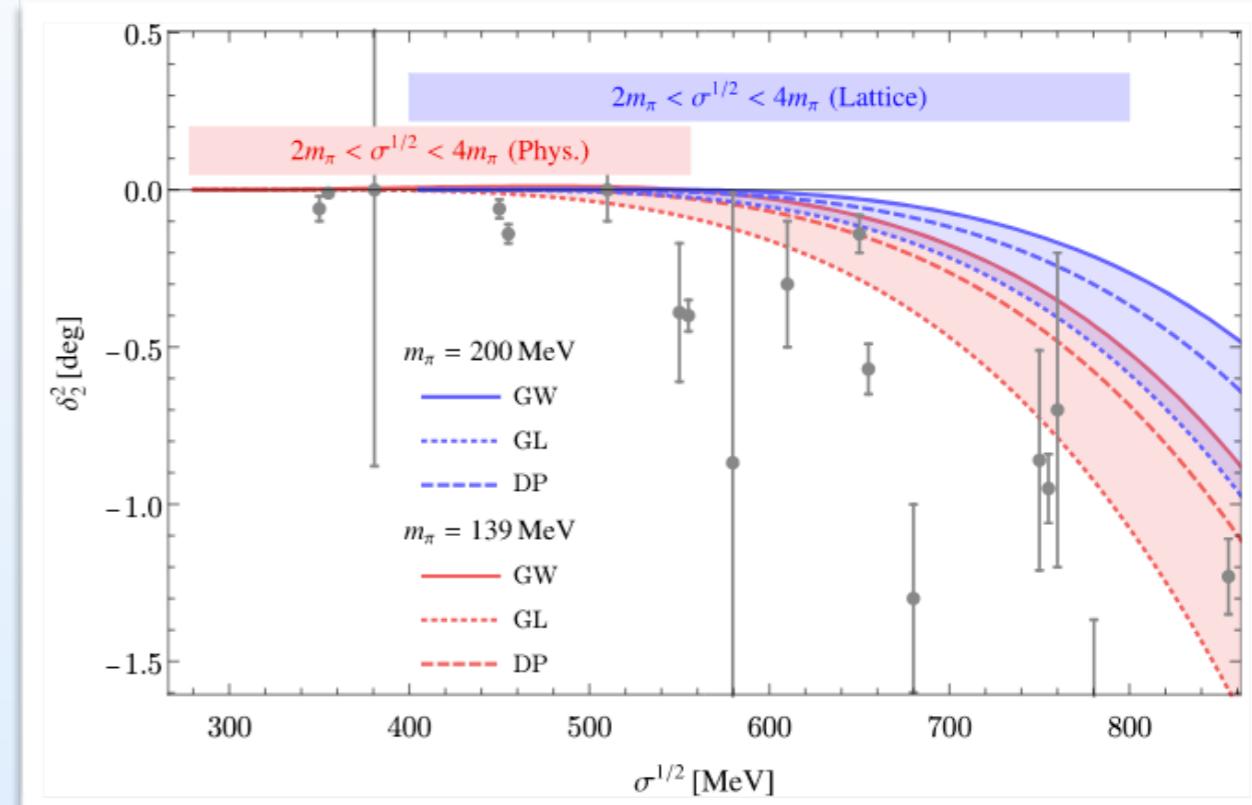


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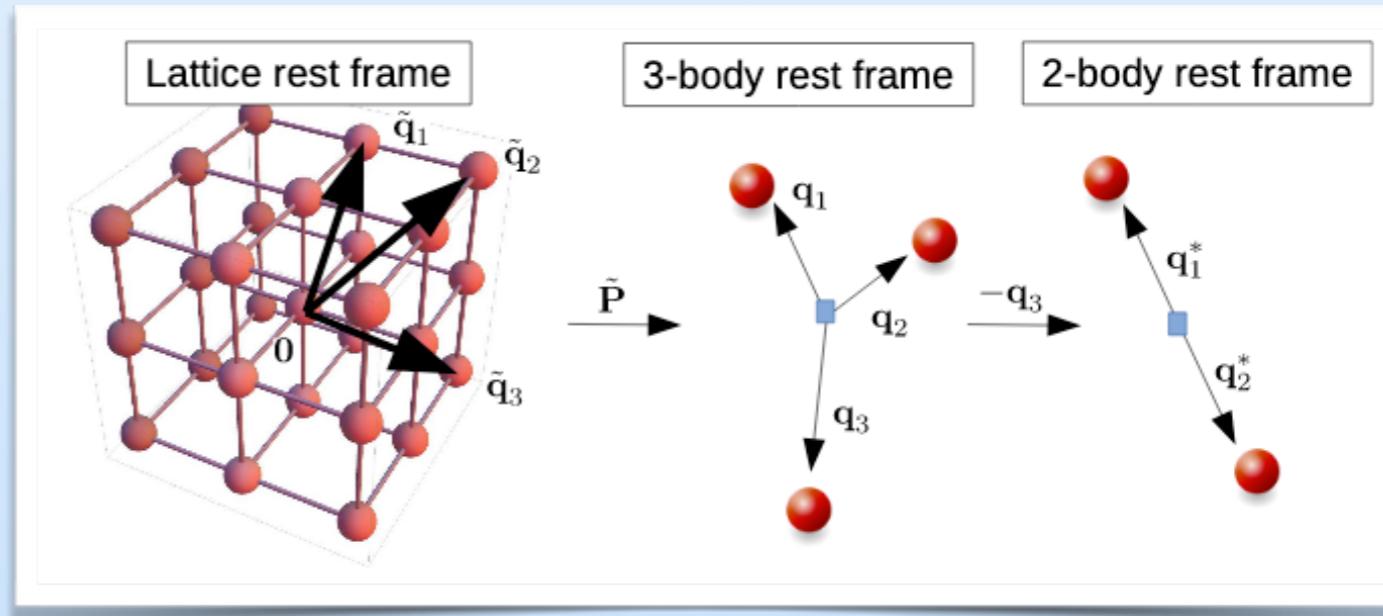
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- Generalize 3bQC for arbitrary boosts and irreps

MM, Alexandru, Culver, Doering [1909.05749]



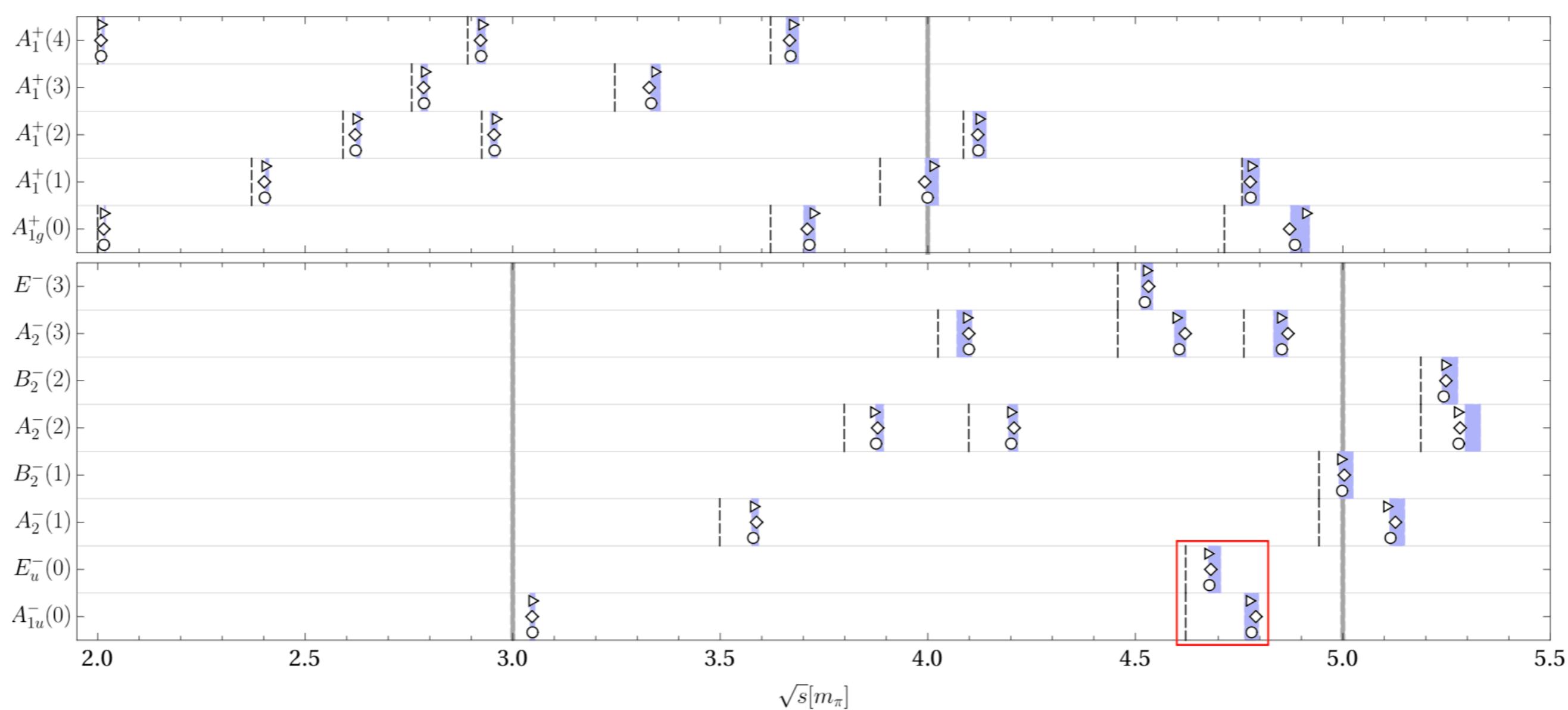
## 2. PHYSICAL APPLICATION

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Hoerz Hanlon PRL 123 (2019)

- compare with the 3b-QC predictions
- systematic uncertainties due to different 2-body input:  $\diamond$  vs  $\circ$  vs  $\triangleright$

MM, Alexandru, Culver, Doering [1909.05749]



# SUMMARY

- ★ *Relativistic 3b scattering amplitude via 2-b. sub-channel amplitudes*
- ★ *3d integral equation* EPJA53 (2017)
- ★ *Applications in progress ...*

- ★ *Discretization of 3b T-matrix leads to relativistic 3bQC* EPJA53 (2017)
- ★ *Projection to irreps of Oh -> reduce dimensionality* PRD97 (2018)
- ★ *NPLQCD(2009) results are analyzed, excited level spectrum predicted* PRL 122 (2019)
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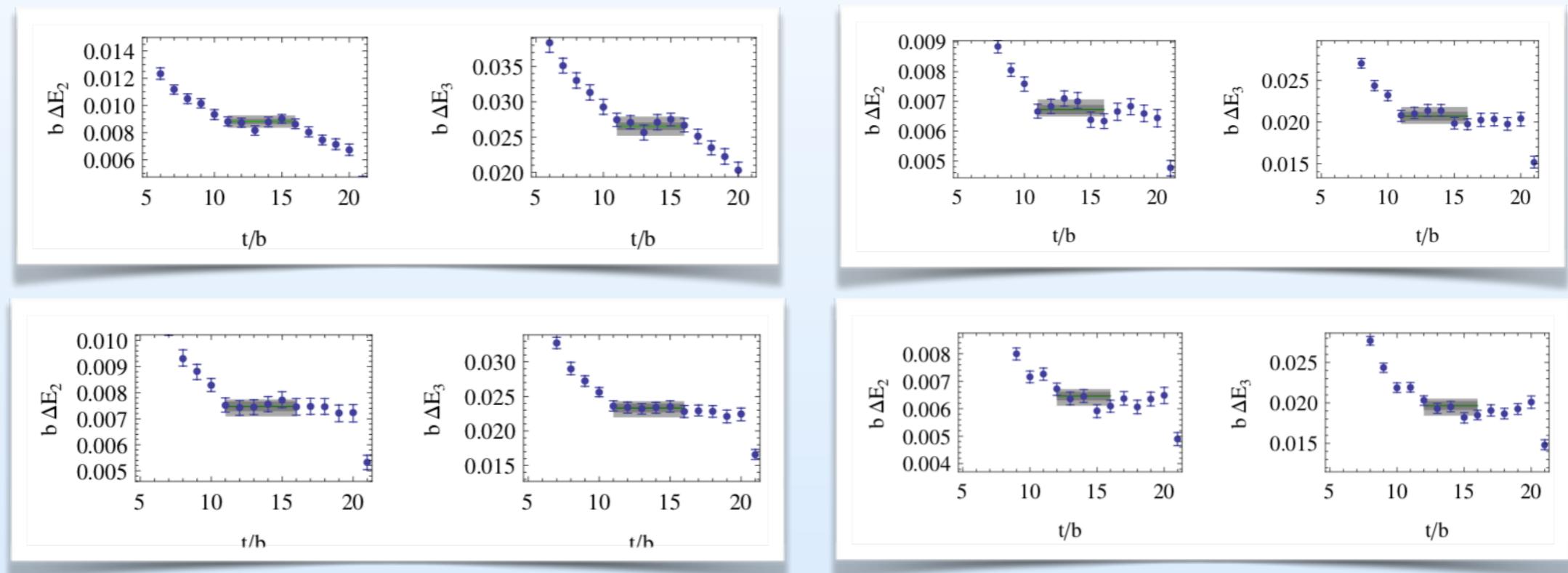
# **SPARES**

# PHYSICAL APPLICATION

MM, Doring PRL 122 (2019)

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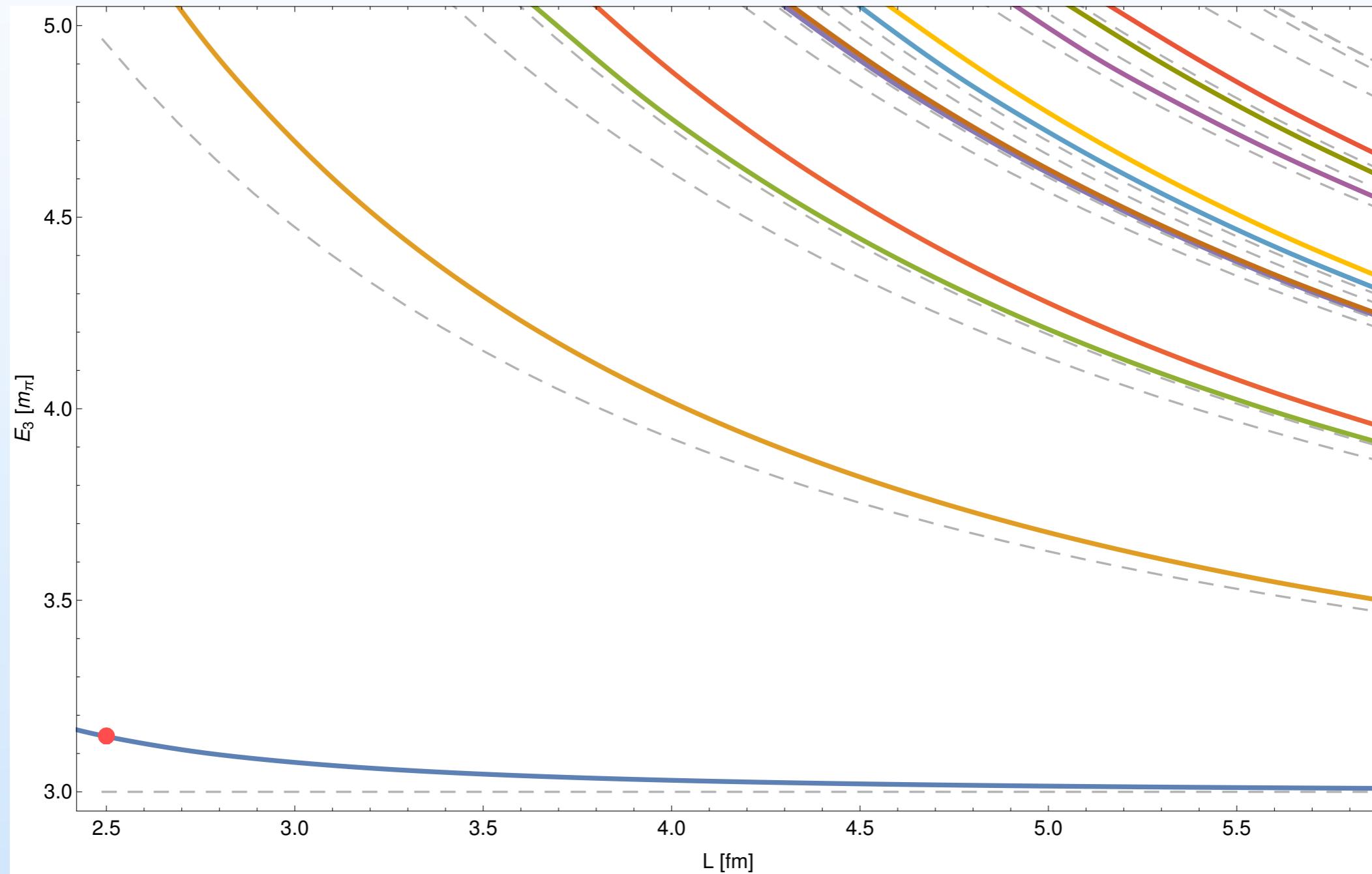
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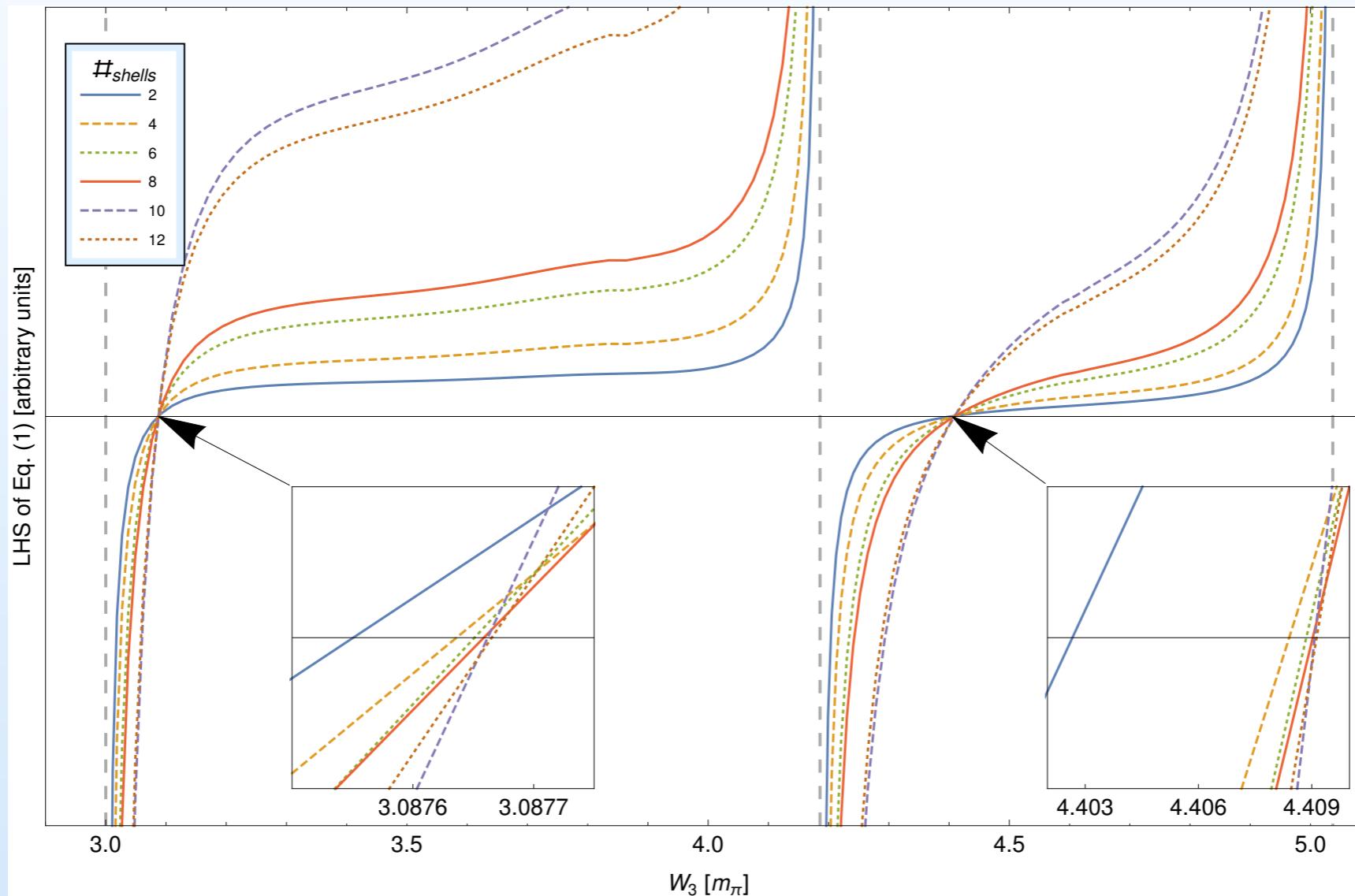
Beane et al. [NPLQCD] PRL100 (2008) Detmold et al. [NPLQCD] PRD78 (2008)

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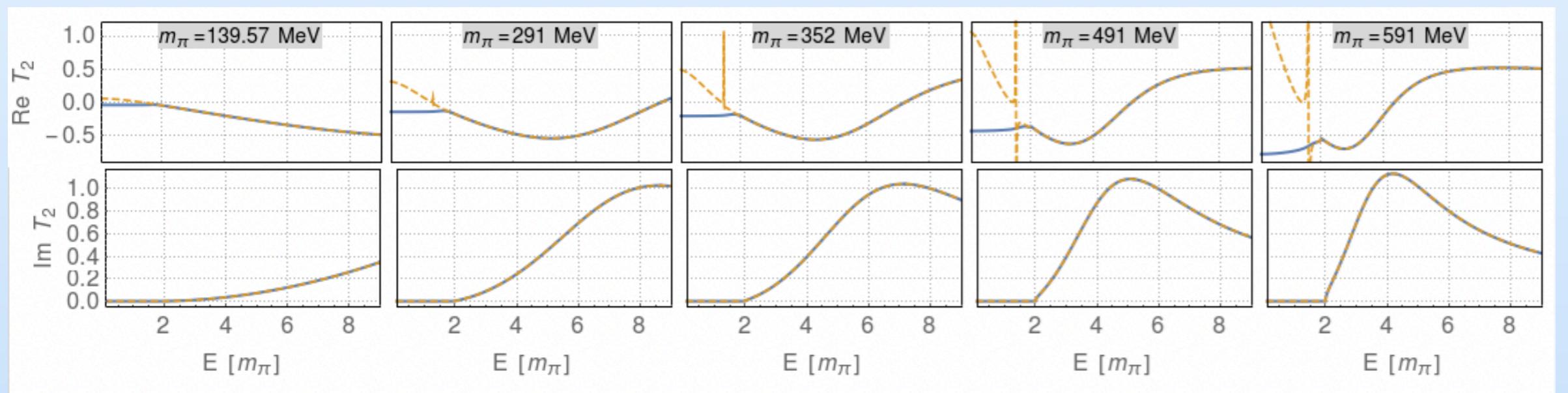
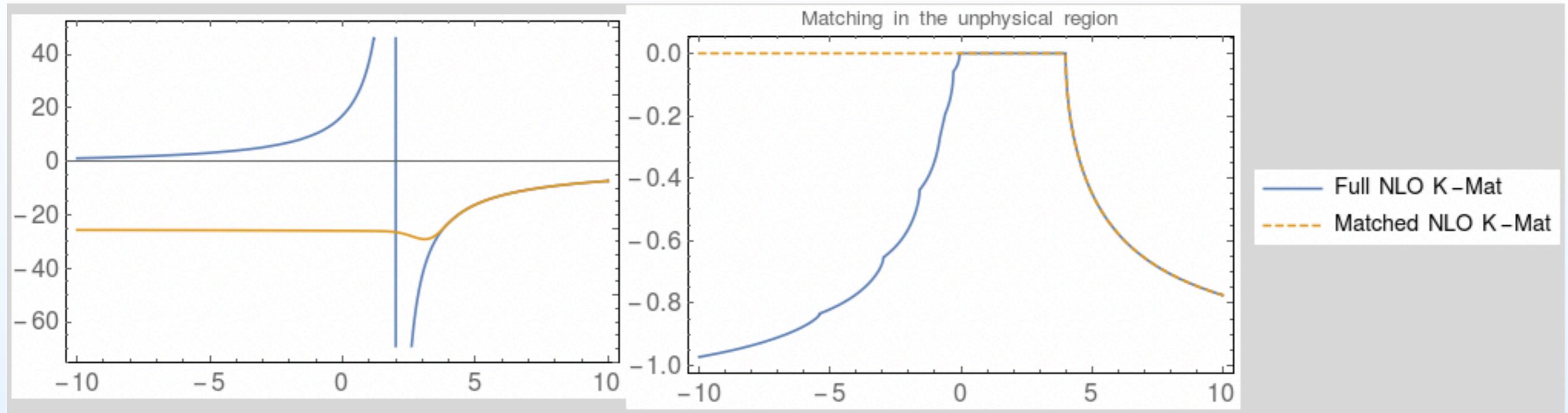
# **L-DEPENDENCE (M=291 MEV)**



# **“SHELL-CUTOFF” DEPENDENCE**



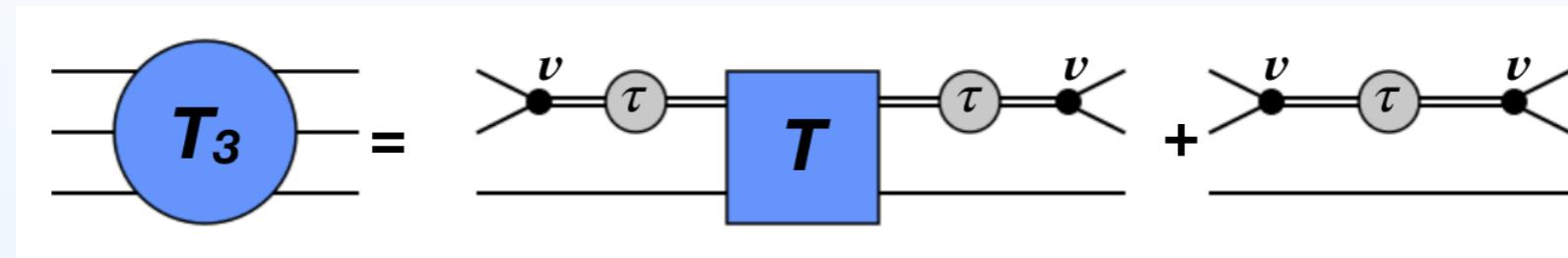
$m_\pi$ [MeV]	139.57	291	352	491	591
$E_2^1$ [ $m_\pi$ ]	$2.1228^{+0.0068}_{-0.0069}$	$2.0437^{+0.0071}_{-0.0086}$	$2.0334^{+0.0076}_{-0.0086}$	$2.0233^{+0.0105}_{-0.0098}$	$2.0204^{+0.0200}_{-0.0106}$
Refs. [24, 25]	—	<b>2.0471(27)(65)</b>	<b>2.0336(22)(22)</b>	<b>2.0215(16)(13)</b>	<b>2.0171(16)(19)</b>
$E_2^2$ [ $m_\pi$ ]	—	—	$3.6245^{+0.0746}_{-0.0299}$	$2.9556^{+0.0728}_{-0.0263}$	$2.7045^{+0.0827}_{-0.0271}$
$E_2^3$ [ $m_\pi$ ]	—	—	—	$3.7114^{+0.1482}_{-0.0737}$	$3.2911^{+0.1241}_{-0.0688}$
$E_2^4$ [ $m_\pi$ ]	—	—	—	—	$3.6802^{+0.0707}_{-0.0902}$
$E_2^5$ [ $m_\pi$ ]	—	—	—	—	$3.9829^{+0.0500}_{-0.0299}$
$E_3^1$ [ $m_\pi$ ]	$3.6564^{+0.1014}_{-0.0847}$	$*3.1444^{+0.0171}_{-0.0192}$	$*3.1058^{+0.0091}_{-0.0147}$	$*3.0655^{+0.0029}_{-0.0095}$	$*3.0537^{+0.0048}_{-0.0119}$
Refs. [24, 25]	—	<b>3.1458(49)(125)</b>	<b>3.1050(27)(27)</b>	<b>3.0665(26)(22)</b>	<b>3.0516(27)(53)</b>
$E_3^2$ [ $m_\pi$ ]	—	—	$4.7301^{+0.1577}_{-0.1027}$	$4.0031^{+0.0196}_{-0.1836}$	$3.7315^{+0.0309}_{-0.0742}$
$E_3^3$ [ $m_\pi$ ]	—	—	—	$4.7043^{+0.0126}_{-0.5923}$	$4.2621^{+0.0001}_{-0.1739}$
$E_3^4$ [ $m_\pi$ ]	—	—	—	$4.7890^{+0.0506}_{-0.1722}$	$4.3155^{+0.0837}_{-0.1341}$
$E_3^5$ [ $m_\pi$ ]	—	—	—	—	$4.5913^{+0.0001}_{-0.1995}$
$E_3^6$ [ $m_\pi$ ]	—	—	—	—	$4.6634^{+0.0001}_{-0.1070}$
$E_3^7$ [ $m_\pi$ ]	—	—	—	—	$4.6995^{+0.0001}_{-0.0661}$



# UNITARY ISOBAR INFINITE VOLUME AMPLITUDE

MM, Hu, Doring, Pilloni, Szczepaniak Eur.Phys.J. A53 (2017)

## 1) $T$ is a sum of a connected and disconnected parts

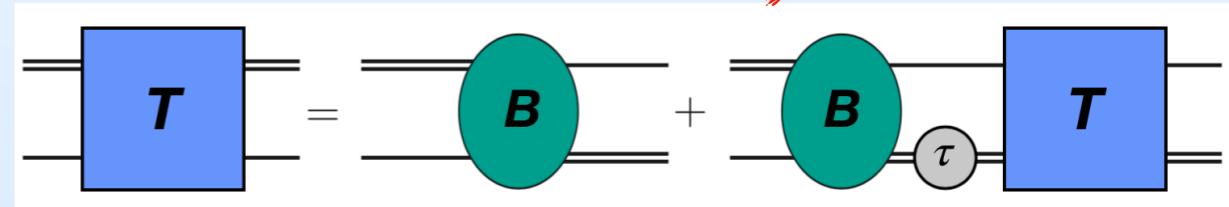


## 2) Disconnected part

- spectator + tower of functions  $\tau(M_{\text{Inv}})$  with correct right-hand-singularities
- coupling to asymptotic states: cut-free-function  $v(q,p)$

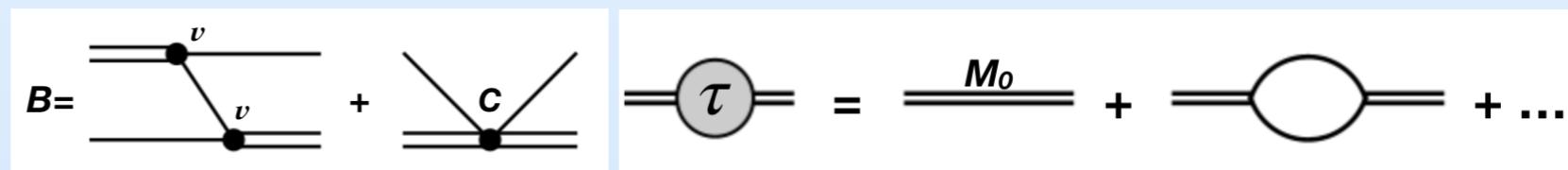
## 3) Connected part

- general 4d BSE-like equation w.r.t kernel  $B$



## 4) 3-body unitarity

- constrains  $B, \tau$



RESULT:

relativistic 3d integral-equation –  $v, C, M_0$  to be fixed from data

useful for phenomenological applications

Sadasivan, MM, Doering... in progress

- recent study of analytic properties

Jackura et al. [JPAC] Eur.Phys.J. C79 (2019)

# 3-BODY QUANTIZATION CONDITION

**Power-law finite-volume effects**

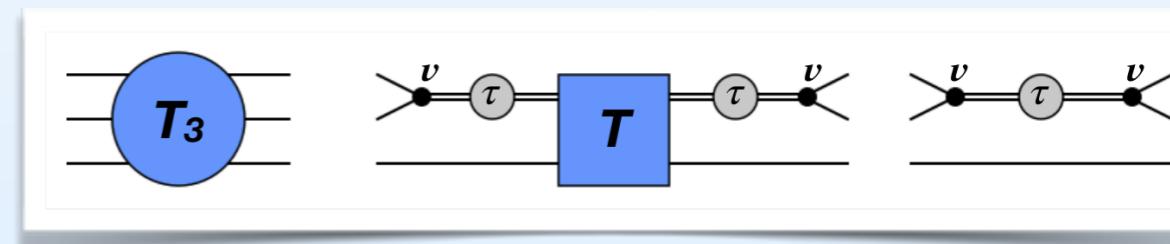
**$\text{Im } T$  in infinite volume**

**on-shell configurations in  $T$**

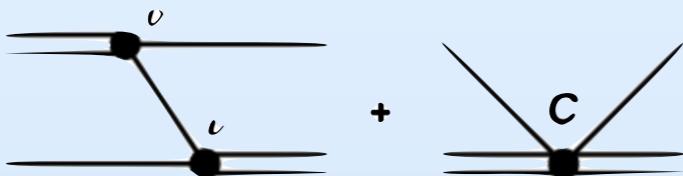
**Determined by Unitarity**

## ◎ Replace *integrals* by *sums*

- Energy eigenvalues in a box:  $\{ E^* \mid T(E^*) = \infty \}$

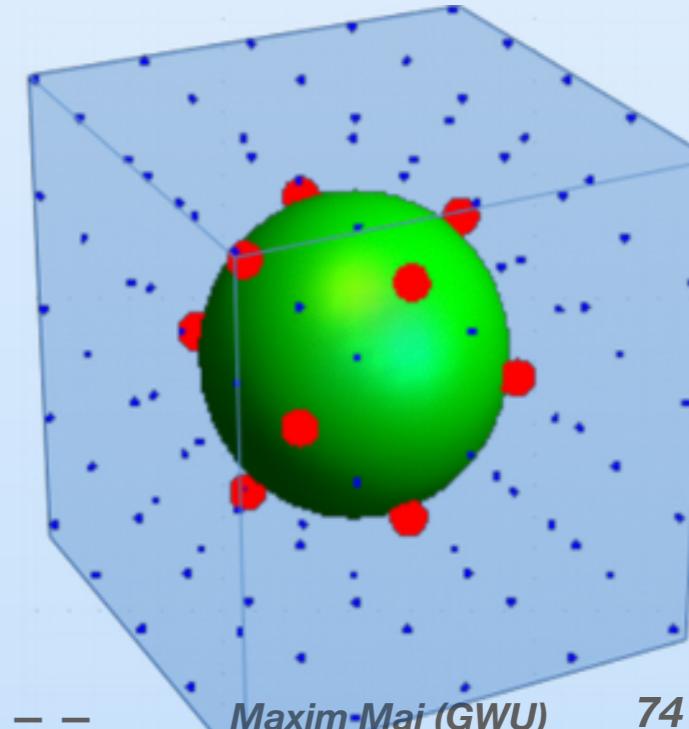


## ◎ $B$ is NOT regular



- projection to irreps is required
- similar to **partial wave projection**: introduce shells

Doring, Hammer, MM, PRD97(2018)



# 3-BODY QUANTIZATION CONDITION

- ◎ Final result on shells, projected to irreps

$$\text{Det} \left( B_{uu'}^{\Gamma ss'}(W^2) + \frac{2E_s L^3}{\vartheta(s)} \tau_s(W^2) \delta_{ss'} \delta_{uu'} \right) = 0$$

$W$  – total energy  $s^{(\prime)}$  - shell index  $u^{(\prime)}$  - basis index  $\theta$  – multiplicity  
 $L$  – lattice volume  $E_s$  – 1p. energy

- ◎ Possible work-flow:

