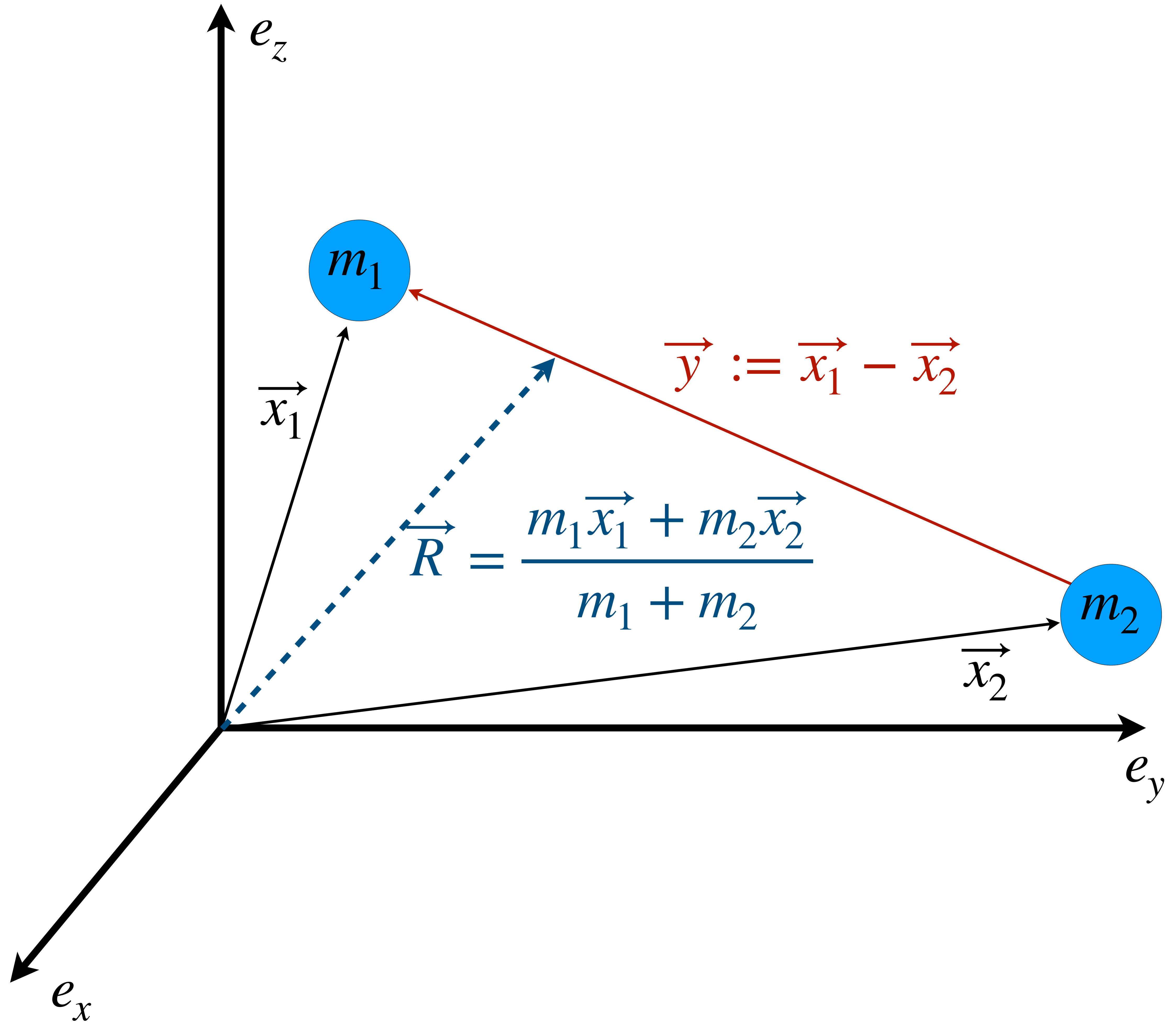
Kepler's laws

Maxim Mai

LECTURE NOTES ONLINE:
maxim-mai.github.io/talks/lecture-01.pdf

Reminder from previous lecture

Setup



Forces & equation of motion

$$\vec{F}_1 = -\vec{F}_2 = \kappa\frac{\vec{y}}{|\vec{y}|^3} \quad\Longrightarrow\quad \mu\ddot{\vec{y}} = \kappa\frac{\vec{y}}{|\vec{y}|^3}$$

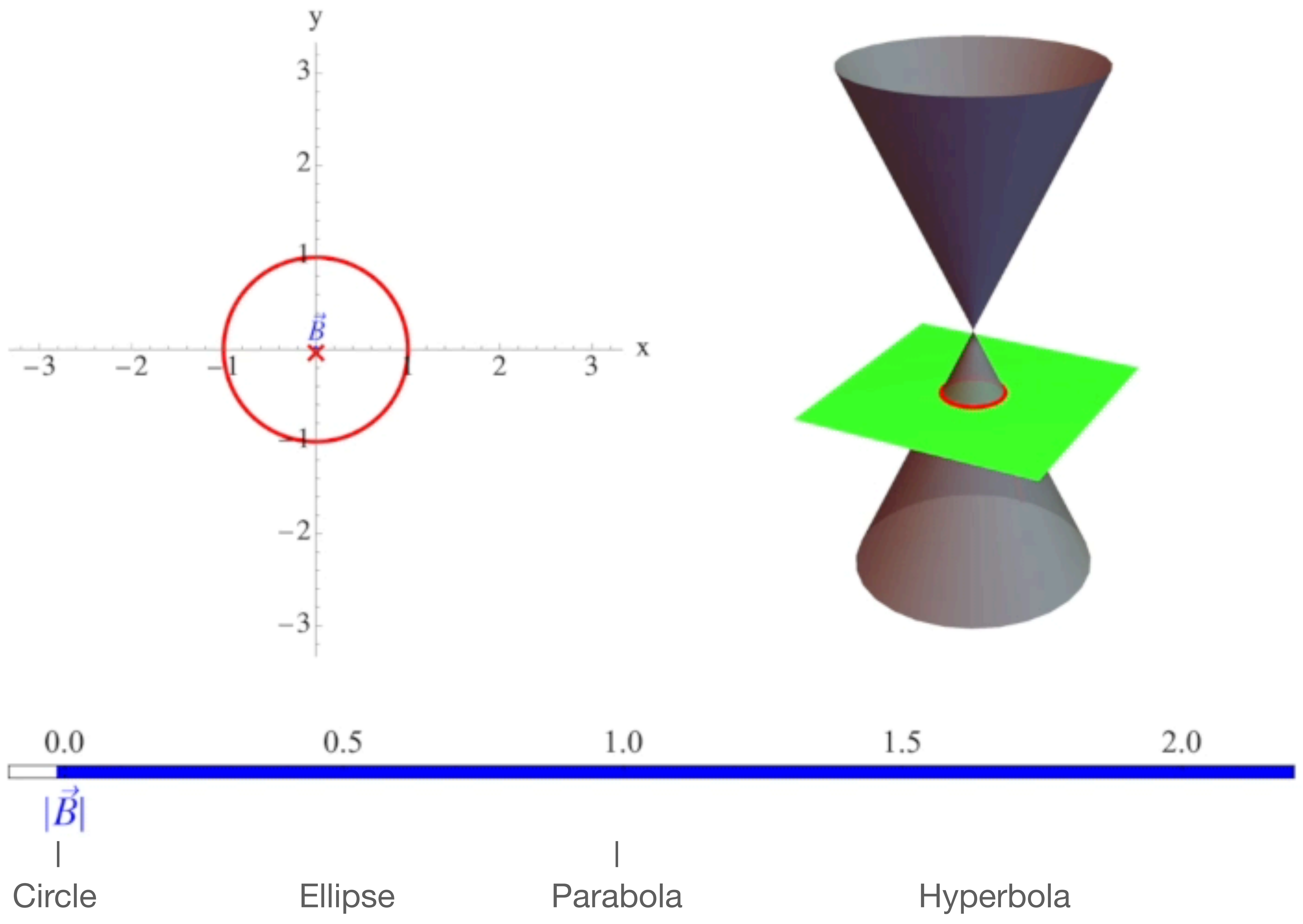
$$\mu = \frac{m_1 m_2}{m_1 + m_2}, \quad \kappa = -Gm_1m_2$$

Conservation laws

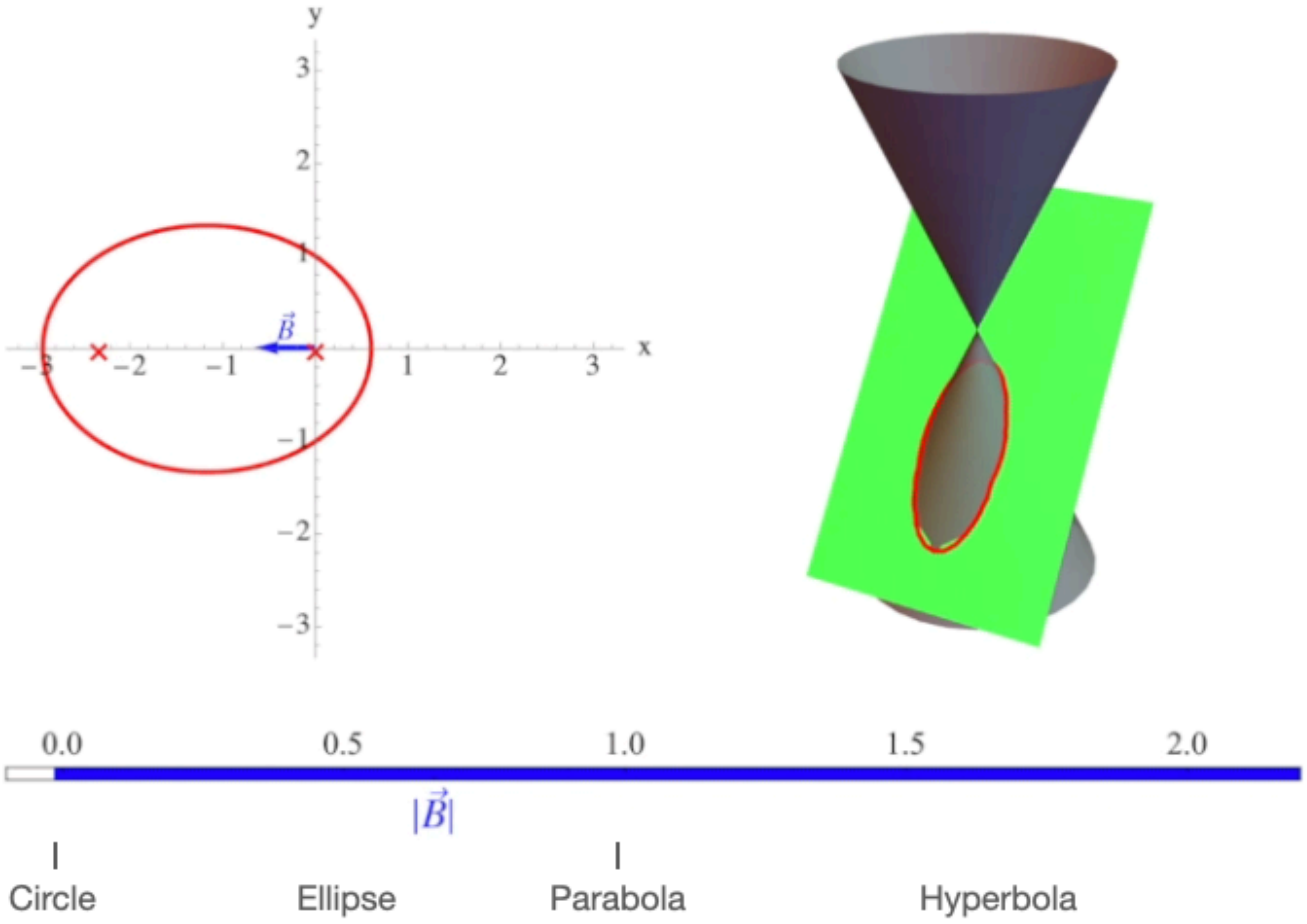
$\vec{P} := (m_1 + m_2)\dot{\vec{R}}$	total momentum
$\vec{R}' := \vec{R} - \dot{\vec{R}}t$	center of mass motion
$\vec{L}' := \mu\vec{y} \times \dot{\vec{y}}$	relative angular momentum
$H' := \dfrac{\mu}{2}\lvert\dot{\vec{y}}\rvert^2$	inner energy
$\vec{B} := \dfrac{1}{\kappa}\dot{\vec{y}} \times \vec{L}' + \dfrac{\vec{y}}{\lvert\vec{y}\rvert}$	Runge – Lenz vector

$\Rightarrow$11 independent conserved quantities

Conic section



Conic section



Reality check #1

Orbit of a star (S2) around the center of Milky Way

- 20y of data by VLT and UCLA-Keck

Data: www.mpe.mpg.de/369216/The_Orbit_of_S2

Reality check #1

Orbit of a star (S2) around the center of Milky Way

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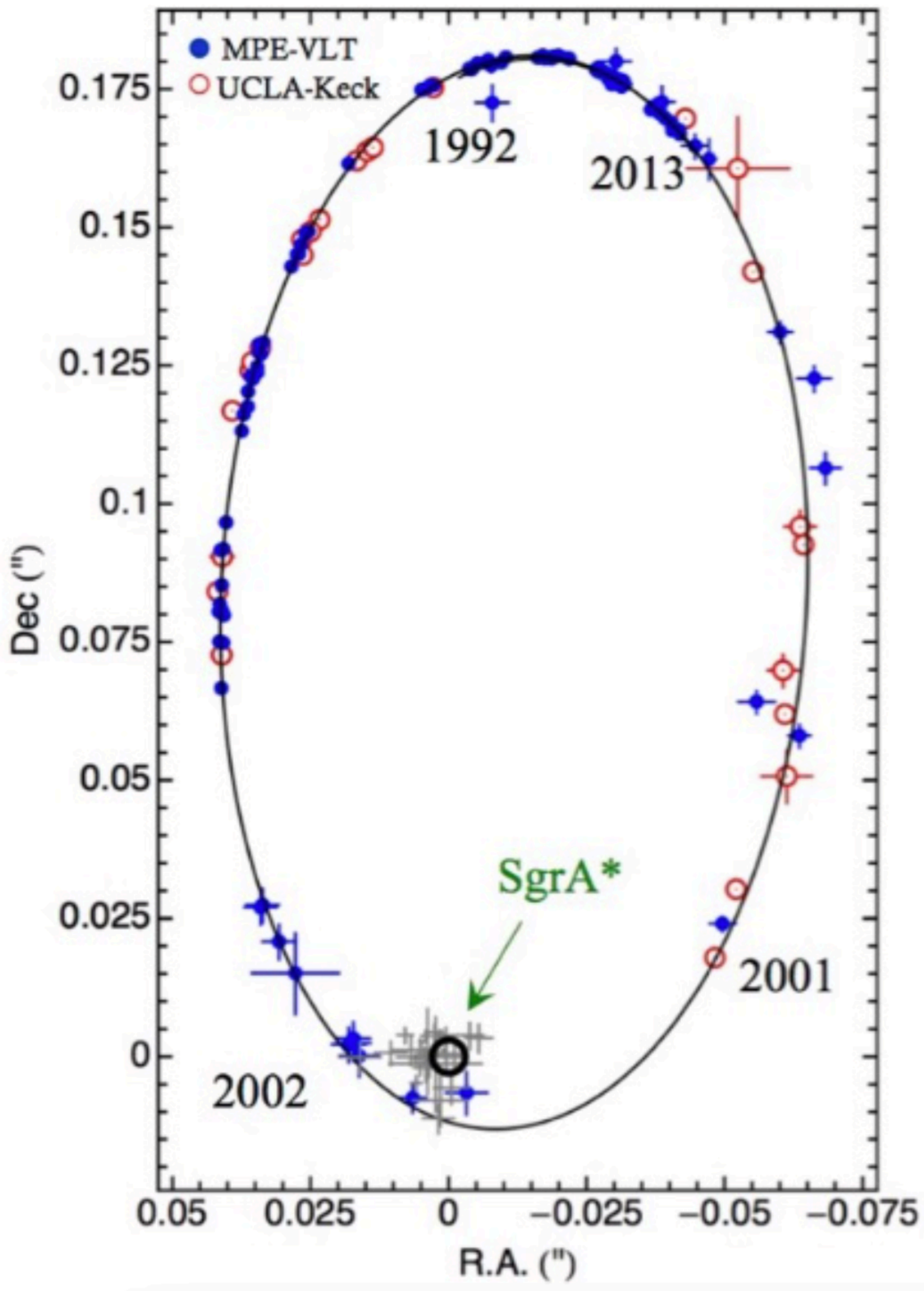
⇒ **perfectly elliptic orbit:**

T=15.2y, I=46Deg, e=0.87, a=0.119''

⇒ **conformation of a Black Hole (SgrA*) scenario**



Data: www.mpe.mpg.de/369216/The_Orbit_of_S2

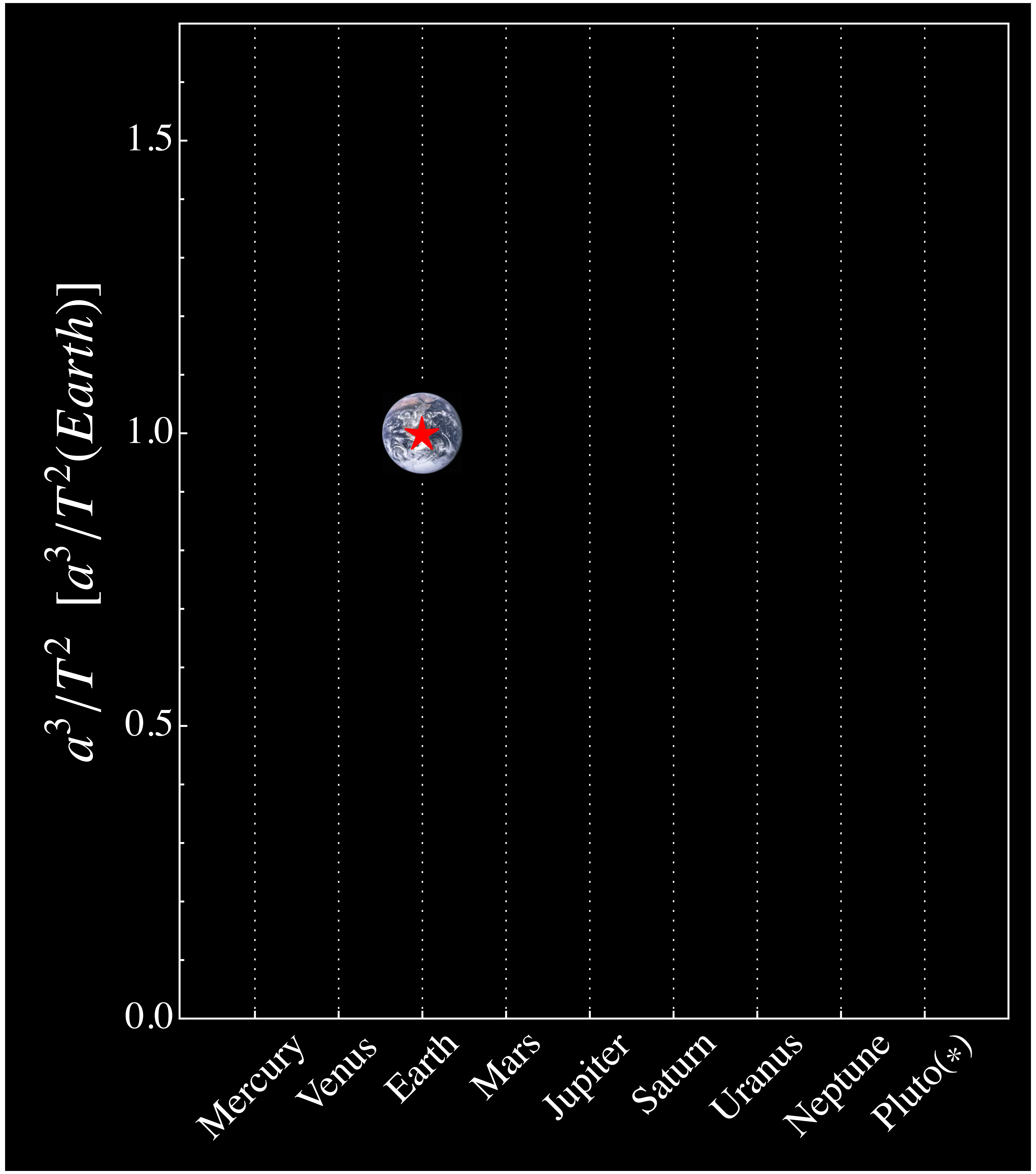
Reality check #2

3rd Kepler's law in solar system:

- 8 planets orbiting Sun ($M_{planet} << M_{sun}$)

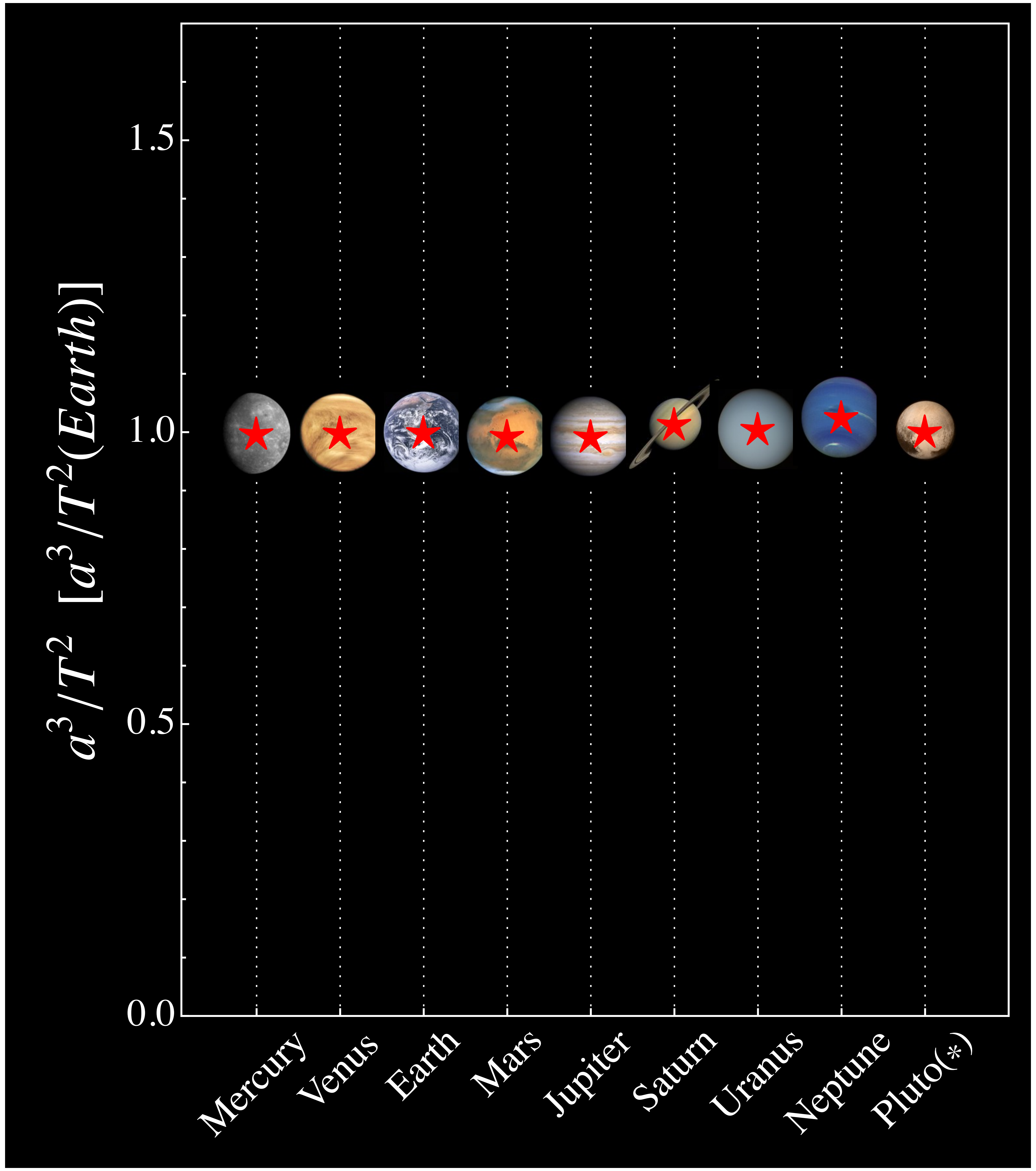
$$\frac{a^3}{T^2} = \frac{|\kappa|}{4\pi^2\mu} \approx \frac{GM_{\rm sun}}{4\pi^2} = \text{const}$$

- normalize a^3/T^2 to Earth's values



Data: https://nssdc.gsfc.nasa.gov/planetary/factsheet/

Reality check #2



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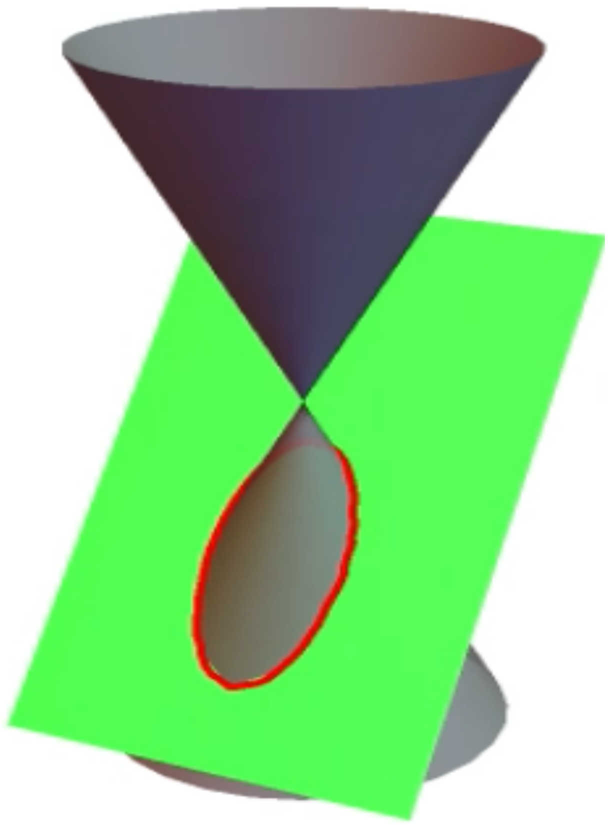
Summary

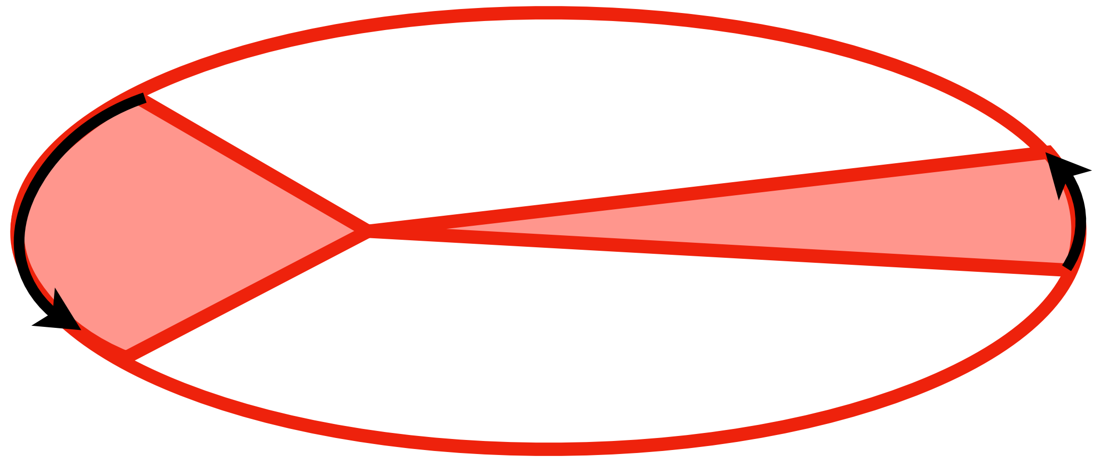
Kepler's laws (two-body with central forces)

1. Motion is one-dimensional >> conic sections



$$|\vec{y}| = \frac{|\vec{L}'|^2}{\kappa\mu} \frac{1}{|\vec{B}|\cos\varphi - 1}$$

2. Area per time is constant



$$\frac{dA}{dt} = \frac{|\vec{L}'|}{2\mu}$$

3. Ratio of cubes of major axis to squares of rotation period is constant

$$\frac{a^3}{T^2} = \frac{|\kappa|}{4\pi^2\mu}$$

KEPLER's LAWS

Introduction : (say)

- Mockup lecture by Maxim Mai as part of Application for a tenure track position.
- all materials are online

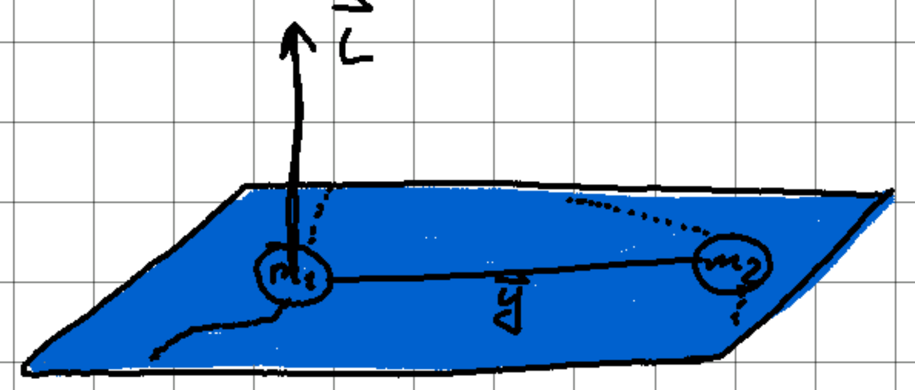
- What will we learn :
  - universal laws of two-body motion with central force (planetary motion)
  - historically an important test of Newtonian dynamics.
  - Kepler was also active in Graz.

- We begin with by repeating results of last lecture. $\Longrightarrow$ SLIDE 2
  - setup
  - relative coordinate are useful
  - constraints on motion $\Longrightarrow$ 11

- Can we learn something about the motion ? (Blackboard)

1) $(\vec{y}, \vec{L}) = 0$ & $(\dot{\vec{y}}, \vec{L}) = 0$

$\Rightarrow$ relative motion takes place in a plane



0's Kepler law

2) $(\vec{B}, \vec{L}) = 0$ R-L vector also lies in the plane

Both $\vec{B}$ & $\vec{y}$ lie in the plane ...

3) $$(\vec{B}\cdot\vec{y}) = \frac{1}{\kappa}\left(\vec{y}, \dot{\vec{y}}\times\vec{L}\right) + \frac{(\vec{y},\vec{y})}{|\vec{y}|}$$

$(\vec{a}, \vec{b}\times\vec{c}) = (\vec{c}, \vec{a}\times\vec{b}) = \ldots$

$$= 1/\kappa\left(\vec{L}, \vec{y}\times\dot{\vec{y}}\right) + |\vec{y}|$$

$$= \frac{1}{\kappa\mu}|\vec{L}|^2 + |\vec{y}|$$ $\rightarrow$ examine degrees of freedom

$$= \cos\varphi\cdot|\vec{B}||\vec{y}| \quad \text{for } \varphi : \measuredangle(\vec{B},\vec{y})$$

$$\Longrightarrow \boxed{|\vec{y}| = \frac{|\vec{L}|^2/(\kappa\mu)}{|\vec{B}|\cos\varphi - 1}}$$

1st Kepler law

- motion is 1-dimensional (as expected)
- $|\vec{y}|(\varphi)$ $\rightarrow$ circle for $B = 0$ $(\kappa < 0)$
  $\rightarrow$ in general conic section

$\Longrightarrow$ SLIDE 3

- We know now the geometrical form of motion, but what is with the <u>time evolution</u>?

<u>Consider</u> dA infinitesimal area drawn by $\vec{y}$

$$dA = 1/2 \left| \vec{y} \times d\vec{y} \right| = 1/2 \left| \vec{y} \times \frac{d\vec{y}}{d\varphi} \, d\varphi \right|$$

$$= \frac{1}{2} \left| \vec{y} \times \frac{d\vec{y}}{d\varphi} \right| \cdot d\varphi$$

$\Rightarrow$ per time increment dt:

$$\frac{dA}{dt} = \frac{d\varphi}{dt} \cdot \frac{1}{2} \left| \vec{y} \times \frac{d\vec{y}}{d\varphi} \right|$$ looks like $|\vec{L}'|$

$$\vec{L}' = \mu \, \vec{y} \times \dot{\vec{y}} = \mu \frac{d\varphi}{dt} \, \vec{y} \times \frac{d\vec{y}}{d\varphi}$$

$$\Rightarrow \boxed{\frac{dA}{dt} = \frac{1}{2\mu} |\vec{L}'| = \text{const}}$$

<u>2nd Kepler's law</u>

e.g. ellipse

- Full area (ellipse)

$$A = \int_0^T \frac{dA}{dt} \cdot dt = \frac{1}{2\mu} |\vec{L}'| \cdot T \qquad \text{(2KL)}$$

$$A = \pi a \cdot b = \pi \cdot \left(a \frac{|\vec{L}'|^2}{\mu K} \right)^{3/2} \qquad \text{(geometry)}$$

$$\Rightarrow \boxed{\frac{a^3}{T^2} = \frac{K}{4\pi^2 \mu}}$$

<u>3rd Kepler's law</u>

HOW UNIVERSAL ARE THESE LAWS $\longrightarrow$ $\boxed{\text{SLIDES 4+}}$