ALI Reexam 2019 Solution

In [2]:

```
import sympy as sp
from scipy import *
from sympy import *
init_printing()
from IPython.display import display, Latex, HTML, Math
import numpy as np
import pandas as pd
from latex2sympy import lat2py
```

Assignment 1 ¶

a.

B is not in echelon form since the upper right position is populated by a zero-entry and there are non-zero entries below it.

C is not in echelon form since there is a row of zeros above a row of non-zeros.

D is not in echelon form since the upper right position is populated by a zero-entry and there are non-zero entries below it.

b.

Since is triangular, thee eigenvalues are on the diagonal and that means -1 with multiplicity of 2; 1 and 2 with multiplicity of 1.

c.

Since the matrix is triangular the eigenvalues are given on the diagonal. And since all these values are distinct it follows that the matrix is diagonalizable.

d.

Since the matrix is symmetric, the sum of the diagonal equals to the sum of eigenvalues which means that $\lambda_3 = 6$.

Assignment 2

In [2]:

```
A = Matrix([[2,2],[3,4]])
B = Matrix([[1,2],[-1,2]])
C = Matrix([[4,-2],[1,0]])
```

In [3]:

```
#a)

# Using the determinant
display(Math(r'\text{Det}(A) =' + latex(A.det())))

# Using identity matrix and row reduction
display(Math(r'B^{-1} =' + latex(B.row_join(eye(2)).rref()[0][:, 2:])))

# Using nullspace by showing it is empty
display(Math(r'\text{nullspace}(C) = ' + latex(C.nullspace())))
```

$$\text{Det}(A) = 2$$

$$B^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

$$\text{nullspace}(C) = []$$

In [12]:

```
# b)
display(Math(r'AX=B:'))
X1 = (A**-1)*B
display(Math(r'X =' + latex(X1)))

display(Math(r'A^{2}X+B=0:'))
X2 = -(A**-2)*B
display(Math(r'X =' + latex(X2)))

display(Math(r'AXB=C:'))
X3 = (A**-1)*C*B**-1
display(Math(r'X =' + latex(X3)))

display(Math(r'AX +BX=C:'))
X4 = ((A+B)**-1)*C
display(Math(r'X =' + latex(X4)))

display(Math(r'ACX=0:'))
X5 = Matrix(zeros(2,2))
display(Math(r'X =' + latex(X5)))
```

$AX = B:$

$$X = \begin{bmatrix} 3 & 2 \\ -\frac{5}{2} & -1 \end{bmatrix}$$

$A^2X + B = 0:$

$$X = \begin{bmatrix} -\frac{17}{2} & -5 \\ 7 & 4 \end{bmatrix}$$

$AXB = C:$

$$X = \begin{bmatrix} \frac{5}{2} & -\frac{9}{2} \\ -\frac{7}{4} & \frac{13}{4} \end{bmatrix}$$

$AX + BX = C:$

$$X = \begin{bmatrix} 2 & -\frac{6}{5} \\ -\frac{1}{2} & \frac{2}{5} \end{bmatrix}$$

$ACX = 0:$

$$X = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Assignment 3

In [16]:

```
# a)

A = Matrix([[1,-1,3,5],[-1,-3,1,-1],[2,6,-2,2]])

display(Latex('Basis for nullspace:'))
display(A.nullspace())

display(Latex('Basis for colspace:'))
display(A.columnspace())

display(Latex('Basis for rowspace:'))
display(A.rowspace())
```

Basis for nullspace:

$$\left[\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right]$$

Basis for colspace:

$$\left[\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 6 \end{bmatrix} \right]$$

Basis for rowspace:

$$\left[\begin{bmatrix} 1 & -1 & 3 & 5 \end{bmatrix}, \begin{bmatrix} 0 & -4 & 4 & 4 \end{bmatrix} \right]$$

In [17]:

```
# b)

A.rref()
```

Out[17]:

$$\left(\begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, (0, 1) \right)$$

In [ ]:

```
# Since A has a free variable (in fact it has two), there are an
# infinite number of solutions.
```

In [19]:

```
# c) One such b will lie in the columnspace of A:
display(Math(r'\mathbf{b} =' + latex(A[:,0] + A[:,1])))
display(Latex('Check for inconsistencies:'))
display(A.row_join(A[:,0] + A[:,1]).rref()[0])
```

$$\mathbf{b} = \begin{bmatrix} 0 \\ -4 \\ 8 \end{bmatrix}$$

Check for inconsistencies:

$$\begin{bmatrix} 1 & 0 & 2 & 4 & 1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Assignment 4

In [3]:

```
a, b = symbols('a b')
A = Matrix([[-6,a,-1,3],[2,0,3,0],[4,5,6,0],[8,b,1,-4]])
A.det()
```

Out[3]:

$$10$$

In [70]:

```
# a)
display(Math("Det(A) = {}".format(
    A[1,0]*A.cofactor(1,0)+A[1,2]*A.cofactor(1,2))))
```

$$Det(A) = 10$$

In [67]:

```
# b)
# Matrix B can be obtained from A via elementary row operations: One swap, one scaling
# and one replacement, the latter not having an effect on the determinant of the
# resulting matrix. One swap changes sign and the scale is 2. That means:
display(Math("Det(B) = -20"))
```

$$Det(B) = -20$$

Assignment 5

In [3]:

```
A = Matrix([[2,2,1],[2,-1,4],[0,1,-1]])
b = Matrix([[1],[1],[1]])
x1 = Matrix([[1],[1],[2]])
x2 = Matrix([[1],[1],[0]])
```

In [6]:

```
A.row_join(b).echelon_form()
```

Out[6]:

$$\begin{bmatrix} 2 & 2 & 1 & 1 \\ 0 & -6 & 6 & 0 \\ 0 & 0 & 0 & -6 \end{bmatrix}$$

In [30]:

```
# a)

display(A.row_join(b).rref()[0])
display(Latex('The system is inconsistent since the final row contains 0 = 1. Hence' +
        ' the matrix has no solution'))
```

$$\begin{bmatrix} 1 & 0 & \frac{3}{2} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The system is inconsistent since the final row contains 0 = 1. Hence the matrix has no solution

In [35]:

```
# b)
display(Math(r'||\mathbf{b}-A\mathbf{x}_{1}|| = ' +
             latex(round(float((b-A*x1).norm()), 2))))
display(Math(r'||\mathbf{b}-A\mathbf{x}_{2}|| = ' +
             latex(round(float((b-A*x2).norm()), 2))))
```

$$||\mathbf{b} - A\mathbf{x}_1|| = 9.64$$

$$||\mathbf{b} - A\mathbf{x}_2|| = 3.0$$

In [ ]:

```
# We conclude that x2 is the best solution of the two candidates.
```

Assignment 6

In [36]:

```
x = np.array([0, 2, 5, 6])
y = np.array([1, 6, 17, 19])
```

In [40]:

```
# a) Design matrix for y1

X1 = Matrix([ones(len(x), 1)]).row_join(Matrix(x**2))
XtX = X1.T*X1
Xty = X1.T*Matrix(y)
Mat, _ = XtX.row_join(Xty).rref()
B1 = Mat[:,-1]

display(Math(r'\text{Design Matrix for } y_1 =' + latex(X1)))
display(Math(r'\text{Observation Matrix for } y_1 =' + latex(Matrix(y))))
```

$$\text{Design Matrix for } y_1 = \begin{bmatrix} 1 & 0 \\ 1 & 4 \\ 1 & 25 \\ 1 & 36 \end{bmatrix}$$

$$\text{Observation Matrix for } y_1 = \begin{bmatrix} 1 \\ 6 \\ 17 \\ 19 \end{bmatrix}$$

In [41]:

```
# a) Design matrix for y2

X2 = Matrix(x).row_join(Matrix(x**2))
XtX = X2.T*X2
Xty = X2.T*Matrix(y)
Mat, _ = XtX.row_join(Xty).rref()
B2 = Mat[:,-1]

display(Math(r'\text{Design Matrix for } y_2 =' + latex(X2)))
display(Math(r'\text{Observation Matrix for } y_2 =' + latex(Matrix(y))))
```

$$\text{Design Matrix for } y_2 = \begin{bmatrix} 0 & 0 \\ 2 & 4 \\ 5 & 25 \\ 6 & 36 \end{bmatrix}$$

$$\text{Observation Matrix for } y_2 = \begin{bmatrix} 1 \\ 6 \\ 17 \\ 19 \end{bmatrix}$$

In [42]:

```
# a) Design matrix for y3

X3 = Matrix([ones(len(x), 1)]).row_join(Matrix(x)).row_join(Matrix(x**2))
XtX = X3.T*X3
Xty = X3.T*Matrix(y)
Mat, _ = XtX.row_join(Xty).rref()
B3 = Mat[:,-1]

display(Math(r'\text{Design Matrix for } y_3 =' + latex(X3)))
display(Math(r'\text{Observation Matrix for } y_3 =' + latex(Matrix(y))))
```

$$\text{Design Matrix for } y_3 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 4 \\ 1 & 5 & 25 \\ 1 & 6 & 36 \end{bmatrix}$$

$$\text{Observation Matrix for } y_3 = \begin{bmatrix} 1 \\ 6 \\ 17 \\ 19 \end{bmatrix}$$

In [46]:

```
# b)

display(Latex("$$y_1(t) = {}+{}t^2$$"
              .format(round(B1[0], 2), round(B1[1], 4))))

display(Latex("$$y_2(t) = {}t+{}t^2$$"
              .format(round(B2[0], 2), round(B2[1], 4))))

display(Latex("$$y_2(t) = {}+{}t+{}t^2$$"
              .format(round(B3[0], 2), round(B3[1], 4), round(B3[2], 4))))
```

$$y_1(t) = 2.74 + 0.4930t^2$$

$$y_2(t) = 3.24t + 0.0015t^2$$

$$y_2(t) = 0.78 + 2.7641t + 0.0607t^2$$

In [50]:

```
# c)

display(Latex(
    "The error of $y_1$ = {}"
    .format(round(float((Matrix(y)-X1*B1).norm()), 2))))

display(Latex(
    "The error of $y_2$ = {}"
    .format(round(float((Matrix(y)-X2*B2).norm()), 2))))

display(Latex(
    "The error of $y_3$ = {}"
    .format(round(float((Matrix(y)-X3*B3).norm()), 2))))

display(Latex("$y_3$ has the smallest error and is the best fitted model."))
```

The error of y_1 = 3.26

The error of y_2 = 1.44

The error of y_3 = 1.2

y_3 has the smallest error and is the best fitted model

Assignment 7

In [4]:

```
A = Matrix([[9,3,3],[-3,9,3]])
AtA = A.T*A
vecs = AtA.eigenvects()
vecs
```

Out[4]:

$$\left[\left(0,\ 1,\ \left[\left[\begin{matrix}-\frac{1}{5}\\-\frac{2}{5}\\1\end{matrix}\right]\right]\right),\ \left(90,\ 1,\ \left[\left[\begin{matrix}-2\\1\\0\end{matrix}\right]\right]\right),\ \left(108,\ 1,\ \left[\left[\begin{matrix}1\\2\\1\end{matrix}\right]\right]\right)\right]$$

In [6]:

```
# a)
s1 = sqrt(vecs[2][0])
s2 = sqrt(vecs[1][0])

v1 = vecs[2][2][0].normalized()
v2 = vecs[1][2][0].normalized()
v3 = vecs[0][2][0].normalized()

u1 = (s1**-1)*A*v1
u2 = (s2**-1)*A*v2

U = u1.row_join(u2)
V = v1.row_join(v2).row_join(v3)
Vt = V.T
S = diag(s1, s2).row_join(zeros(2,1))

display(Math('U \Sigma V^T = {}{}{}'.format(latex(U), latex(S), latex(Vt))))
display(Latex("Test:"))
display(U*S*Vt)
display(V)
```

$$U\Sigma V^T = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 6\sqrt{3} & 0 & 0 \\ 0 & 3\sqrt{10} & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{3} & \frac{\sqrt{6}}{6} \\ -\frac{2\sqrt{5}}{5} & \frac{\sqrt{5}}{5} & 0 \\ -\frac{\sqrt{30}}{30} & -\frac{\sqrt{30}}{15} & \frac{\sqrt{30}}{6} \end{bmatrix}$$

Test:

$$\begin{bmatrix} 9 & 3 & 3 \\ -3 & 9 & 3 \end{bmatrix}$$

$$\begin{bmatrix} \frac{\sqrt{6}}{6} & -\frac{2\sqrt{5}}{5} & -\frac{\sqrt{30}}{30} \\ \frac{\sqrt{6}}{3} & \frac{\sqrt{5}}{5} & -\frac{\sqrt{30}}{15} \\ \frac{\sqrt{6}}{6} & 0 & \frac{\sqrt{30}}{6} \end{bmatrix}$$

In [68]:

```
# b)
# If the columns of U are eigenvectors of AAt, the following must be the case
display(Math('AA^T \cdot u_1 = \lambda_1 u_1'))
display(Math('AA^T \cdot u_2 = \lambda_2 u_2'))
```

$$AA^T \cdot u_1 = \lambda_1 u_1$$

$$AA^T \cdot u_2 = \lambda_2 u_2$$

In [69]:

```
display(Math(r'AA^T \cdot u_1 =' + latex(AAt) + latex(u1) + '=' + latex(AAt*u1)))
display(Math(r'AA^T \cdot u_2 =' + latex(AAt) + latex(u2) + '=' + latex(AAt*u2)))
display(Latex("That means"))
display(Math(r'AA^T \cdot u_1 =' + latex(108) + latex(u1)))
display(Math(r'AA^T \cdot u_2 =' + latex(90) + latex(u2)))
display(Latex("And we get"))
display(Math('\lambda_1 =' + latex(108)))
display(Math('\lambda_2 =' + latex(90)))
```

$$AA^T \cdot u_1 = \begin{bmatrix} 99 & 9 \\ 9 & 99 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} 54\sqrt{2} \\ 54\sqrt{2} \end{bmatrix}$$

$$AA^T \cdot u_2 = \begin{bmatrix} 99 & 9 \\ 9 & 99 \end{bmatrix} \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} -45\sqrt{2} \\ 45\sqrt{2} \end{bmatrix}$$

That means

$$AA^T \cdot u_1 = 108 \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

$$AA^T \cdot u_2 = 90 \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

And we get

$$\lambda_1 = 108$$

$$\lambda_2 = 90$$