Introduction

Book: Mandatory for exercises
→ Important to retrieve right copy.
→ Lecture Notes vs. Class Notes

Prerequisites: Some recap today
Differential Equations

Exam: 4 hour ; two parts
Documentation must be uploaded
Python part must be .ipynb format.

Tools : Python 3
Jupyter Notebook
↳ VS code
↳ Jupyter Lab
↳ Data Spell (Jetbrains)
↳ Google Colab

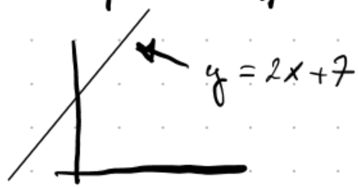
Itslearning not used. Go to
github.com/RBrooksDK/ALI1

Wiseflow : You will receive multiple flows
with assignments. Code is always
O O O O

1.1 Systems of Linear Equations

Linear equations:

$$a_1x_1 + a_2x_2 + \dots a_nx_n = b \quad , \quad a, b \in \mathbb{R}$$

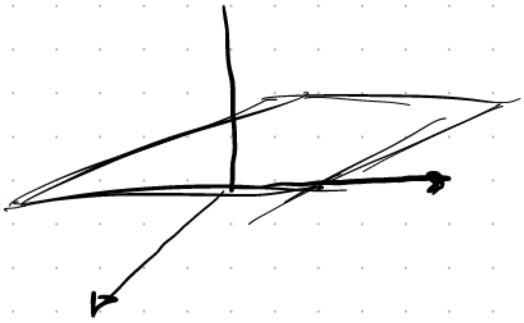


Ex:

$$y = ax+b \rightarrow y = 2x+7$$

Ex: (Plane equation)

$$ax + by + cz = d$$



A system of Linear Equations

A collection of one or more linear equations involving <u>the same</u> variables:

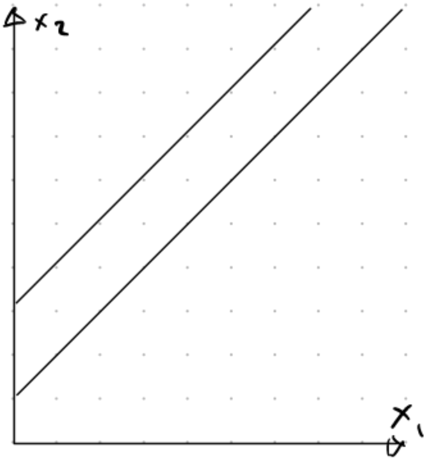
Ex:

$$\begin{aligned} 2x_1 + 3x_2 + x_3 &= 3 \\ 7x_2 - 4x_3 &= 10 \\ x_3 &= 1 \end{aligned}$$

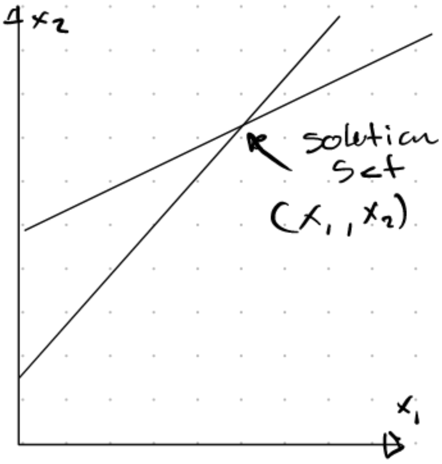
Solution set:

A solution of a linear system is a list of numbers $s_1, s_2, s_3 \dots$ that satisfies the system, i.e. makes the system "true".

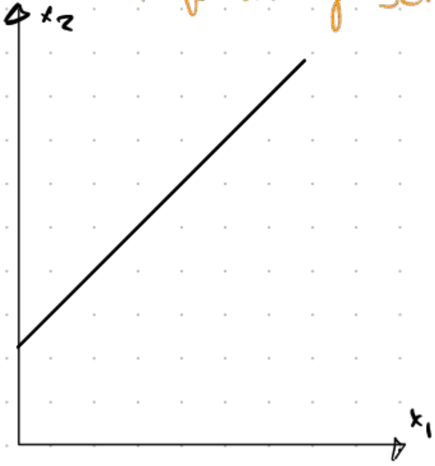
1. No solution



2. One solution



3. Inf. many sol.



1. No solution → Inconsistent
2. Exactly one solution (Unique) } Consistent
3. Infinitely many solutions

Existence question: Does a solution exist?
If yes, is it unique?

The Matrix:

Consider the system:

$$\begin{aligned} 2x_1 + 3x_2 + x_3 &= 3 \\ 7x_2 - 4x_3 &= 10 \\ x_3 &= 1 \end{aligned}$$

We can "code" this system into two types of matrices:

Coefficient Matrix

$$\begin{array}{ccc} x_1 & x_2 & x_3 \end{array}$$
$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & 7 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

Augmented Matrix

$$\begin{array}{cccc} x_1 & x_2 & x_3 & b \text{ or } y \end{array}$$
$$\text{3 rows}\left\{\underbrace{\left[\begin{array}{ccc|c} 2 & 3 & 1 & 3 \\ 0 & 7 & -4 & 10 \\ 0 & 0 & 1 & 1 \end{array}\right]}_{\text{4 columns}}\right.$$

3x4
n x m (sometimes m x n!)

Solving a system:

We can solve a system by working on the augmented matrix. The objective is to get all ones on the diagonal of the coefficient ~~part~~ parts and then we will have the solution on the augmented part (the last column)

Ex:

$$\begin{array}{cccc} x_1 & x_2 & x_3 & b \end{array}$$

$$\begin{bmatrix} 2 & 3 & 1 & 3 \\ 0 & 7 & -4 & 10 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{r_1 \to r_1 - r_3} \begin{bmatrix} 2 & 3 & 0 & 2 \\ 0 & 7 & -4 & 10 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{r_2 \to r_2 + 4r_3} \begin{bmatrix} 2 & 3 & 0 & 2 \\ 0 & 7 & 0 & 14 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{r_2 \to \frac{1}{7} r_2} \begin{bmatrix} 2 & 3 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{r_1 \to r_1 - 3r_2} \begin{bmatrix} 2 & 0 & 0 & -4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{r_1 \to \frac{1}{2} r_1} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & b \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Before we went from equations to matrix. Let us now go from matrix to equations:

$$\begin{cases} 1x_1 + 0x_2 + 0x_3 = -2 \\ 0x_1 + 1x_2 + 0x_3 = 2 \\ 0x_1 + 0x_2 + 1x_3 = 1 \end{cases}$$

$\Downarrow$

$$x_1 = -2$$
$$x_2 = 2 \quad \longrightarrow \quad (x_1, x_2, x_3) = (-2, 2, 1)$$
$$x_3 = 1$$

Elementary Row Operations:

1) Replacement (one row by self + multiple of another)
2) swap (swap two rows)
3) Scaling (multiply all entries in a row with a non-zero constant)

Consistency and Matrices:

1. If a system has no solution (i.e. is inconsistent), then the matrix will also have an inconsistency when reduced:

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{aligned} x_1 &= 2 \\ x_2 &= 3 \\ 0x_1 + 0x_2 + 0x_3 &= 1 \end{aligned}$$

2. If a system has a unique solution, the reduced matrix will be "nice" like in our example, i.e. only ones on the diagonal on the coefficient part and real numbers in the last column.

3. If a system has infitely many solutions, the reduced matrix will have a row of all zeros.

Exercises:

ⓐ Determine if consistent

$$x_2 + 4x_3 = 2$$
$$x_1 - 3x_2 + 2x_3 = 6$$
$$x_1 - 2x_2 + 6x_3 = 9$$

ⓑ Give solution

$$x_1 + 2x_2 + 3x_3 = 4$$
$$3x_1 + 6x_2 + 9x_3 = 12$$

ⓒ Find h and k s.t. system is consistent:

$$2x_1 - x_2 = h$$
$$-6x_1 + 3x_2 = k$$

1.2. Row Reduction and Echelon Forms

Echelon Form:

i) All nonzero rows are above all rows of zeros

ii) Each leading non-zero entry is to the right of the above leading non-zero entry

iii) All entries in a column below a leading non-zero entry are zero

$$\begin{bmatrix} 2 & 3 & 1 & 3 \\ 0 & 7 & -4 & 10 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 7 & 4 & 9 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Reduced Echelon Form:

iv) the leading non-zero entry in each row is 1

v) Each leading 1 is the only non-zero entry in its column.

Each matrix is row equivalent to one and only one matrix in reduced echelon form

Pivots

A leading non-zero entry in echelon form is called a **pivot** and its column a pivot column:

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot columns → Basic variables

non-pivot cols → free variables

Ex:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 6 & 7 & 8 & 9 \\ 0 & 0 & 0 & 7 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

x_1, x_2, x_4 are Basic

x_3 and x_5 are free

If a system is consistent:

a) Unique → no free variables

b) at least one free variable

Exercise:

ⓐ i) Find reduced echelon

ii) State solution

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

ⓑ Choose h and k s.t.

$$x_1 - 3x_2 = 1$$

$$2x_1 + hx_2 = k$$

$$\begin{bmatrix} 1 & -3 & 1 \\ 0 & h+6 & k-2 \end{bmatrix}$$

has

i) no solution — $k \neq, h = -6$

ii) Unique sol. — $h \neq -6$

iii) Inf. sol. — $k = 2, h = -6$

1.3. Vector Equations

A matrix with only one column is called a column vector.

$$\bar{u} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \quad \bar{v} = \begin{bmatrix} 5 \\ 3 \end{bmatrix} \qquad \bar{u}, \bar{v} \in \mathbb{R}^2$$

$$\bar{u} \neq \bar{v}$$

Same rules apply for vectors as for numbers (see p. 27)

Linear Combinations:

Given a set of vectors $\bar{v}_1, \bar{v}_2 \ldots \bar{v}_p \in \mathbb{R}^n$ and scalars $c_1, c_2 \ldots c_p \in \mathbb{R}$, the vector $\bar{y}$ given by:

$$\bar{y} = c_1 \cdot \bar{v}_1 + c_2 \cdot \bar{v}_2 + \ldots + c_p \cdot \bar{v}_p$$

is called a linear combination of $\bar{v}_1, \bar{v}_2 \ldots \bar{v}_p$ with weights $c_1, c_2 \ldots c_p$.

EX: $\bar{V}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \bar{V}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\bar{V}_1 + \bar{V}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\bar{V}_1 + 2V_2 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$-2\bar{V}_1 - 3V_2 = \begin{bmatrix} -2 \\ -4 \end{bmatrix} + \begin{bmatrix} -3 \\ 3 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$$

$$2\bar{V}_1 + 3V_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

Often we want to know if another vector $\bar{b}$ can be formed as a lin. comb. of some other vectors $\bar{a}_1, \bar{a}_2 \dots \bar{a}_n$.

A vector equation:

$$x_1 \cdot \bar{a}_1 + x_2 \cdot \bar{a}_2 + \dots x_n \cdot \bar{a}_n = \bar{b}$$

has the same solution set as the linear system whose augmented matrix is

$$[\bar{a}_1 \quad \bar{a}_2 \quad \bar{a}_3 \quad \dots \quad \bar{a}_n \quad \bar{b}]$$

More specifically,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \ddots & & \\ a_{m1} & a_{m2} & a_{m3} & & a_{mn} & b_m \end{bmatrix} \quad m \times n$$

$$\begin{matrix} \uparrow & \uparrow & \uparrow & & \uparrow & \uparrow \\ \bar{a}_1 & \bar{a}_2 & \bar{a}_3 & & \bar{a}_n & \bar{b}_n \end{matrix}$$

Ex:

$$\bar{a}_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \quad \bar{a}_2 = \begin{bmatrix} 0 \\ 8 \\ -2 \end{bmatrix}, \quad \bar{a}_3 = \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix}, \quad \bar{b} = \begin{bmatrix} 10 \\ 3 \\ 7 \end{bmatrix}$$

Vector Equation:

$$x_1 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 8 \\ -2 \end{bmatrix} + x_3 \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \\ 7 \end{bmatrix}$$

Augmented Matrix

$$\begin{bmatrix} 2 & 0 & 6 & 10 \\ -1 & 8 & 5 & 3 \\ 1 & -2 & 1 & 7 \end{bmatrix}$$

Linear System of Equations

$$\begin{aligned} 2x_1 + 0x_2 + 6x_3 &= 10 \\ -x_1 + 8x_2 + 5x_3 &= 3 \\ x_1 - 2x_2 + x_3 &= 7 \end{aligned}$$

Solution:

$$\begin{bmatrix} 2 & 0 & 6 & 10 \\ -1 & 8 & 5 & 3 \\ 1 & -2 & 1 & 7 \end{bmatrix} \overset{r_1 \leftrightarrow r_3}{\sim} \begin{bmatrix} 1 & -2 & 1 & 7 \\ -1 & 8 & 5 & 3 \\ 2 & 0 & 6 & 10 \end{bmatrix}$$

$$\overset{r_2 \to r_2 + r_1}{\sim} \begin{bmatrix} 1 & -2 & 1 & 7 \\ 0 & 6 & 6 & 10 \\ 2 & 0 & 6 & 10 \end{bmatrix}$$

$$r_3 \to r_3 - 2r_1 \begin{bmatrix} 1 & -2 & 1 & 7 \\ 0 & 6 & 6 & 10 \\ 0 & 4 & 4 & -4 \end{bmatrix}$$

$$r_3 \to r_3 - \frac{4}{6} r_2 \begin{bmatrix} 1 & -2 & 1 & 7 \\ 0 & 6 & 6 & 10 \\ 0 & 0 & 0 & -4 - \frac{40}{6} \end{bmatrix}$$

Inconsistent, so

$\bar{b}$ is not lin. comb of $\bar{a}_1, \bar{a}_2, \bar{a}_3$.

Span $\{\bar{v}\}$:

If $\bar{v}_1, \dots \bar{v}_p$ are in $\mathbb{R}^n$, then the set of all lin. comb. of $\bar{v}_1 \dots \bar{v}_p$ is denoted $\text{span}\{\bar{v}_1 \dots \bar{v}_p\}$ and is called the subset of $\mathbb{R}^n$ spanned by $\bar{v}_1, \dots \bar{v}_p$.

Ex: $\bar{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$, $\bar{a}_2 = \begin{bmatrix} 5 \\ -13 \\ -3 \end{bmatrix}$, $\bar{b} = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}$

$\text{span}\{\bar{a}_1, \bar{a}_2\}$ is a plane through the origin in $\mathbb{R}^3$. Is $\bar{b}$ in that plane?

$$\begin{bmatrix} 1 & 5 & -3 \\ -2 & -13 & 8 \\ 3 & -3 & 1 \end{bmatrix} \underset{r_3 \to r_3 - 3r_1}{\overset{r_2 \to r_2 + 2r_1}{\sim}} \begin{bmatrix} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & -18 & 10 \end{bmatrix}$$

$$\overset{r_3 \to r_3 - 6r_2}{\sim} \begin{bmatrix} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$ Inconsist!

1.4. Matrix Equation

A column vector $\bar{x}$ and a Matrix A can be combined as the product of a Matrix and a vector:

$$A\bar{x} = [\bar{a}_1 \ \bar{a}_2 \cdots a_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \underbrace{x_1\bar{a}_1 + x_2\bar{a}_2 + \dots + x_n\bar{a}_n}_{\text{Vector Equation}}$$

If A is an $m \times n$ matrix and if $\bar{b} \in \mathbb{R}^m$ the matrix equation $A\bar{x} = \bar{b}$ has the same solution as the corresponding vector equation.

Ex:

Linear eq

$$x_1 + x_2 + x_3 = 6$$
$$2x_1 + x_2 + 3x_3 = 11$$
$$x_1 + 2x_2 + x_3 = 8$$

Vector eq

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \\ 8 \end{bmatrix}$$

Matrix Eq.

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \\ 8 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 2 & 1 & 3 & 11 \\ 1 & 2 & 1 & 8 \end{bmatrix} \underset{r_3 \to r_3 - r_1}{\overset{r_2 \to r_2 - 2r_1}{\sim}} \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

$$\overset{r_3 \to r_3 - r_2}{\sim} \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\overset{r_2 \to r_2 - r_3}{\sim} \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$r_1 \to r_1 - r_3 \quad \begin{bmatrix} 1 & 1 & 0 & 5 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$r_2 \to -1 \cdot r_2 \quad \begin{bmatrix} 1 & 1 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$r_1 \to r_1 - r_2 \quad \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

so $A\bar{x} = \bar{b}$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \\ 8 \end{bmatrix}$$

$$3\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \\ 8 \end{bmatrix}$$

Please Note:

If A is a $m \times n$ matrix, then all of the following are equivalent

a) For each $\bar{b}$ in $\mathbb{R}^m$, $A\bar{x} = \bar{b}$ has a solution

b) Each $\bar{b}$ in $\mathbb{R}^m$ is a lin. comb. of the columns of A.

c) The columns of A span $\mathbb{R}^m$

d) A has a pivot in every row.

1.5. Solution sets of Linear Systems

A linear system is said to be homogeneous if $A\bar{x}=0$

We call $\bar{x}=\bar{0}$ the trivial solution

↳ looking for non-trivial.

$A\bar{x}=\bar{0}$ has a non-trivial solution iff. the equation has at least one free variable.

<u>Ex:</u>

$$\begin{aligned} 2x_1 + x_2 - 3x_3 &= 0 \\ x_1 - x_2 + x_3 &= 0 \\ -2x_1 + 5x_2 - 7x_3 &= 0 \end{aligned} \to \begin{bmatrix} 2 & 1 & -3 & 0 \\ 1 & -1 & 1 & 0 \\ -2 & 5 & -7 & 0 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & -\frac{2}{3} & 0 \\ 0 & 1 & -5/3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 - 2/3\,x_3 &= 0 \\ x_2 - 5/3\,x_3 &= 0 \end{aligned} \implies \begin{aligned} x_1 &= 2/3\,x_3 \\ x_2 &= 5/3\,x_3 \end{aligned}$$

We can write this as:

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2/3\,x_3 \\ 5/3\,x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 2/3 \\ 5/3 \\ 1 \end{bmatrix}$$

Parametric vector Form

Ex: $x_1 - 2x_2 - 5x_3 = 0$

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_2 + 5x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$$

Ex (non-homogeneous)

$$\begin{aligned} 2x_1 + x_2 - 3x_3 &= 2 \\ x_1 - x_2 + x_3 &= 1 \\ -2x_1 + 5x_2 - 7x_3 &= -2 \end{aligned} \rightarrow \begin{bmatrix} 2 & 1 & -3 & 2 \\ 1 & -1 & 1 & 1 \\ -2 & 5 & -7 & -2 \end{bmatrix} \sim$$

$$\begin{aligned} x_1 - 2/3\, x_3 &= 1 \\ x_2 - 5/3\, x_3 &= 0 \end{aligned} \Rightarrow \begin{aligned} x_1 &= 1 + \tfrac{2}{3} x_3 \\ x_2 &= \tfrac{5}{3} x_3 \end{aligned} \qquad \begin{bmatrix} 1 & 0 & -2/3 & 1 \\ 0 & 1 & -5/3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 + 2/3\, x_3 \\ 5/3\, x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2/3 \\ 5/3 \\ 1 \end{bmatrix}$$

Ex: $\left.\begin{aligned} x_1 - 2x_2 - 5x_3 &= 3 \\ x_1 &= 3 + 2x_2 + 5x_3 \end{aligned}\right\} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 + 2x_2 + 5x_3 \\ x_2 \\ x_3 \end{bmatrix}$

$$= \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$$

Exercise:

(a)
$$x_1 + 4x_2 - 5x_3 = 0$$
$$2x_1 - x_2 + 8x_3 = 9$$
$$\to \begin{bmatrix} 1 & 4 & -5 & 0 \\ 2 & -1 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & -1 \end{bmatrix}$$

$$\bar{x} = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$

(b)
$$\begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ RREF of augmented

$$\bar{x} = \begin{bmatrix} 5 \\ -3 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

1.7. Linear Dependence

If the vector equation

$$x_1 \bar{v}_1 + x_2 \bar{v}_2 + \dots x_p \cdot \bar{v}_p = 0$$

has only the trivial solution, then the vectors are lin. independent.

Also two vectors are independent if one is not a multiple of the other.

In general a vector is independent of a set of vectors if it is NOT a lin. comb. of the set.

ex: $\bar{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \bar{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \bar{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & 4 & 2 \\ 2 & 5 & 1 \\ 3 & 6 & 0 \end{bmatrix} \overset{RREF}{\sim} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

V_1, V_2, V_3 are not independent.

V_3 is a lin comb. of $\bar{v}_1$ and $\bar{v}_2$.

$$-2 \cdot \bar{v}_1 + \bar{v}_2 = \bar{v}_3$$

Ex:

$\bar{v}_1 = \begin{bmatrix} ? \\ 1 \\ 2 \end{bmatrix}, \bar{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \bar{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Important Theorems:

a) If a set $S = \{\bar{v}_1, \dots \bar{v}_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is lin. dep.

b) A set of two vectors in $\mathbb{R}^n$ is lin. independent iff. neither is a multiple of the other

c) Any set $\{\bar{v}_1 \dots \bar{v}_p\}$ in $\mathbb{R}^n$ is lin. dep. if $p > n$, i.e. more columns than rows.

d) $S = \{\bar{v}_1, \dots \bar{v}_p\}$ is lin. dep. iff. at least one vector in S is a linear comb. of the others, assuming $\bar{v}_1 \neq \bar{0}$. So $\bar{v}_j$ $(1 < j \leq p)$ is a linear comb. of the preceeding vectors $\bar{v}_1, \dots \bar{v}_{j-1}$

Exercise:

$$\bar{u} = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}, \quad \bar{v} = \begin{bmatrix} -6 \\ 1 \\ 7 \end{bmatrix}, \quad \bar{w} = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix}, \quad \bar{z} = \begin{bmatrix} 3 \\ 7 \\ -5 \end{bmatrix}$$

① Is any pair lin. dep.?

② Is $\{\bar{u}, \bar{v}, \bar{w}, \bar{z}\}$ lin. dep.? (c)

$$3\bar{u} + \bar{v} = \bar{z}$$