Applied Linear Algebra

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Please state all answers in the "ALI_exam.ipynb"-file. If you have handwritten answers (that are scanned), please state "In paper" or similar so the examiner knows where to find the answer. Also, please include all ipynb files when you hand in.

Assignment 1 (15%)

The questions in this assignment require little or no calculations. **The points for this assignment are given entirely for your reasons**.

a. The following matrix has three eigenvectors

$$\begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

It is know that $\lambda_1 = -1$ and that it has multiplicity 2. Explain how this information enables you to easily find the last eigenvalue and state its value.

b. Identify the matrices that are in reduced echelon form, are in echelon form, and are not in any echelon form and state your reasons for identifying them as such.

$$A = \begin{bmatrix} 1 & 0 & 0 & -10 \\ 0 & 1 & 0 & -10 \\ 0 & 0 & 0 & 0 \end{bmatrix} B = \begin{bmatrix} -8 & -4 & -8 & -9 & -8 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} C = \begin{bmatrix} 1 & 1 & -4 \\ 1 & 0 & -10 \end{bmatrix} D = \begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

c. If A is *any* m by n matrix with $m > n$, explain why AA^T is necessarily singular. *Hint*: In your answer, consider the rank of AA^T.

d. The determinant of A is 0. Explain how you can deduce this simply by observing the matrix.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

e. Below you see a matrix A and its echelon form. Explain how you from this information can deduce that dim Col $A = 2$ and dim Nul $A = 3$.

$$A = \begin{bmatrix} 2 & 4 & -5 & 2 & -3 \\ 3 & 6 & -8 & 3 & -5 \\ 0 & 0 & 9 & 0 & 9 \\ -3 & -6 & -7 & -3 & -10 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -5 & 1 & -4 \\ 0 & 0 & 5 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

f. Give an example of each of the following or explain why no such example can exist:

- An inconsistent linear system in three variables, with a coefficient matrix of rank two.
- A consistent linear system with three equations and two unknowns, with a coefficient matrix of rank one.
- A consistent linear system with three equations and two unknowns, with a coefficient matrix of rank larger than one
- A linear system of two equations in three unknowns, with an invertible coefficient matrix.
- A linear system in three variables, whose geometrical interpretation is three planes intersecting in a line.

Assignment 2 (15%)

a. Let the matrix A be given by

$$A = \begin{bmatrix} 3-2q & 1 \\ 4 & 3+2q \end{bmatrix},$$

where q is a scalar. Calculate q so that

$$A^2 = \begin{bmatrix} 29 & 6 \\ 24 & 125 \end{bmatrix}$$

b. Let B be an invertible $n \times n$ matrix. Reduce the expression $B^2 B^T B B^{-1} \left(B^{-1}\right)^T B \left(B^{-1}\right)^2$ as much as possible and account for the rules used in each step of the reduction.

c. Let $A = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 9 \\ -3 & k \end{bmatrix}$. What value(s) og k, if any, will make $AB = BA$?

Assignment 3 (10%)

a. Do the three lines $x_1 + 2x_2 = 5$, $3x_1 - 2x_2 = 1$ and $2x_1 + 4x_2 = 10$ have a common point of intersection? If yes, find the point; if no, explain the reason.

b. Let the *augmented* matrix of a system of linear equations be

$$A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 2 & \alpha & 2\alpha \\ 1 & \alpha & 2 & -2 \end{bmatrix}$$

Find the value(s) of α for which the system of linear equations has (i) three basic variables; (ii) two basic variables and one free variable.

Assignment 4 (10%)

a. Calculate the following determinant (using suitable properties of the determinant to simplify the calculation):

$$\begin{vmatrix} x & y & z & 1 \\ 1 & -2 & 3 & 1 \\ 2 & -3 & 1 & 1 \\ 4 & -6 & 3 & 1 \end{vmatrix}$$

b. Let A and B be 4×4 square matrices such that $\det(A) = 3$ and $\det(B) = -2$. Compute $\det(2A)$, $\det\left(A^3\right)$, $\det\left(A^{-1}\right)$, $\det\left(A^2B^3\right)$ and $\det\left(A^3B^{-2}\right)$

Assignment 5 (10%)

Diagonalize the following matrices, if possible. If it is not possible, supply a substantiated explanation of why this is the case.

$$A = \begin{bmatrix} 3 & 2 & -2 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} -5 & 2 & -1 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Assignment 6 (25%)

To measure the takeoff performance of an airplane, the horizontal position of the plane was measured every second, from $t = 0$ to $t = 12$. The positions (in metres) were: 0, 26.4, 89.7, 186.0, 314.1, 477.3, 666.0, 883.5, 1.141.2, 1.413.3, 1.715.1, 1.715.1, and 2.427.6.

It is assumed that the system can be approximated by a model of the form $y_1(t) = \beta_0+\beta_1 t+\beta_2 t^2+\beta_3 t^3$ or of the form $y_2(t) = \gamma_1 t + \gamma_2 t^2 + \gamma_3 t^3$ or of the form $y_3(t) = \delta_0 + \delta_1 t + \delta_2 t^2$

a. Determine the design matrix and observation matrix of all three models.

b. Determine the parameters of all three models

c. State, with substantiation, which of the three models is the best fit for the measured data

d. Use the best fitted model to estimate the velocity of the plane when $t = 4.5$ seconds.

Assignment 7 (15%)

Consider the following matrix:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

a. Do a full singular value decomposition of matrix A. This includes the following steps:

   1) Finding the eigenvalues of either $A^T A$ or AA^T
   2) Determining the corresponding eigenvectors
   3) Finding the columns of V and from this deriving the columns of U or vice versa depending on which method you used in (1).
   4) Setting up $A = U\Sigma V^T$
   5) Testing that $A = U\Sigma V^T$

b. Find orthonormal bases for the four fundamental subspaces: Col A; Nul A^T; Col A^T; Nul A associated with A (It is actually possible to derive these from the SVD).