Applied Linear Algebra Re-Exam

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Please state all answers in the "ALI-Reexam.ipynb". If you have handwritten answers (that are scanned), please state In paper or similar so the examiner knows where to find the answer. Also, please include all ipynb files when you hand in.

Assignment 1 (10%)

The questions in this assignment require little or no calculations. **The points for this assignment are given entirely for your reasons**.

a. Identify the matrices that are not in echelon form and explain why they are not in echelon form.

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -4 & 1 \\ 2 & 0 & 0 \\ 1 & -3 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 2 & -7 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

b. Determine the eigenvalues and their multiplicity of A below and explain why it is possible to answer this question without any further calculations.

$$A = \begin{bmatrix} -1 & 0 & 2 & 4 \\ 0 & 1 & 5 & 7 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

c. Explain why the following matrix contains sufficient information to determine whether it is diagonalizable.

$$\begin{bmatrix} 1 & 2 & -2 & 0 & 1 & -3 & 3 \\ 0 & 2 & 1 & -2 & 3 & 0 & 2 \\ 0 & 0 & 3 & -1 & 9 & 11 & 2 \\ 0 & 0 & 0 & 4 & 7 & -1 & 3 \\ 0 & 0 & 0 & 0 & 5 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 7 \end{bmatrix}$$

d. The following matrix has three eigenvectors

$$\begin{bmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{bmatrix}$$

It is know that $\lambda_1 = 8$ and $\lambda_2 = 3$. Explain how this information enables you to easily find λ_3, and then find λ_3.

Assignment 2 (20%)

Let three matrices be given by:

$$A = \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 4 & -2 \\ 1 & 0 \end{bmatrix}$$

a. Show that A is invertible by using the determinant, show that B is invertible by finding its inverse using the identity matrix and row reduction, and show that C is invertible by finding the null space of C.

b. Find solutions to all of the following matrix equations, if they exist

$$AX = B, \quad A^2X + B = 0, \quad AXB = C, \quad AX + BX = C, \quad ACX = 0$$

Assignment 3 (15%)

A matrix A is given by

$$A = \begin{bmatrix} 1 & -1 & 3 & 5 \\ -1 & -3 & 1 & -1 \\ 2 & 6 & -2 & 2 \end{bmatrix}$$

a. Determine bases for the null space, column space and row space of A.

b. Find the number of solutions to the homogenous equation $A\mathbf{x} = \mathbf{0}$.

c. Find a vector $\mathbf{b}$ such that $A\mathbf{x} = \mathbf{b}$ can be solved.

Assignment 4 (10%)

a. By co-factor expanding on the second row, show that the determinant of the following matrix is 10

$$A = \begin{bmatrix} -6 & a & -1 & 3 \\ 2 & 0 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 8 & b & 1 & -4 \end{bmatrix}$$

b. Explain how you can use $\det A = 10$ to determine the determinant of B, and then find $\det B$.

$$B = \begin{bmatrix} -6 & a & -1 & 3 \\ 4 & 5 & 6 & 0 \\ 4 & 0 & 6 & 0 \\ 12 & b & 7 & -4 \end{bmatrix}$$

Assignment 5 (10%)

Consider the matrix $A\mathbf{x} = \mathbf{b}$ where A and $\mathbf{b}$ are given by

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 2 & -1 & 4 \\ 0 & 1 & -1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

a. The equation $A\mathbf{x} = \mathbf{b}$ can not be solved. Show why.

Two guesses of approximate solutions to $A\mathbf{x} = \mathbf{b}$ are

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

b. Determine which of the two proposed solutions is the best one in the least squares sense.

Assignment 6 (20%)

Assume that the following corresponding values of time t and output y for a system has been measured

t	y
0	1
2	6
5	17
6	19

It is assumed that the system can be approximated by a model of the form $y_1(t) = \beta_0 + \beta_2 t^2$ or of the form $y_2(t) = \gamma_1 t + \gamma_2 t^2$ or of the form $y_3(t) = \delta_0 + \delta_1 t + \delta_2 t^2$

a. Determine the design matrix and observation matrix of all three models.

b. Determine the parameters of all three models

c. State, with substantiation, which of the three models is the best fit for the measured data

Assignment 7 (15%)

Consider the following matrix:

$$\begin{bmatrix} 9 & 3 & 3 \\ -3 & 9 & 3 \end{bmatrix}$$

a. Do a full singular value decomposition of matrix A. This includes the following steps:

   1) Finding the eigenvalues of either $A^T A$
   2) Determining the corresponding eigenvectors
   3) Finding the columns of V and from this deriving the columns of U
   4) Setting up $A = U\Sigma V^T$
   5) Testing that $A = U\Sigma V^T$

b. Show that the columns of U are eigenvectors of AA^T and determine the corresponding eigenvalues.