In [1]:

```python
import sympy as sp
import pandas as pd
from sympy import *
init_printing()
from IPython.display import display, Latex
```

Assignment 1

In [29]:

```python
# a)

# a. A is not invertible - TRUE
# b. A is in echelon form - FALSE
# c. Nullity A = 1 - TRUE
# d. Rank A = 3 - TRUE
# e. Nullity A + rank A = 6 - FALSE
# f. The number 0 is an eigenvalue of A - TRUE
# g. A is in reduced echelon form - FALSE
# h. There exists a vector x̄∈ℝ4 such that Ax̄=b̄ is not consistent - FALSE
# i. detA=0 - TRUE
```

In [30]:

```python
# b) 2
```

In [31]:

```python
# c) distinct nonzero eigenvectors, therefore pivots in all columns, hence invertible,
 and they are distinct hence
#     diagonalizable
```

Assignment 2

In [32]:

```
# a) The eigenvalues of A are -1 and 1 where the latter has multiplicity 2.

# Eigenspaces

A = Matrix([
     [1, 3, 1]
   , [0, -1, 0]
   , [0, 0, 1]
])

eig1 , eig2 = A.eigenvects()
print(" A basis for the eigenspace of -1:")
display(eig1[2][0])
print(" A basis for the eigenspace of 1:")
display(eig2[2][0])
```

A basis for the eigenspace of -1:

$$\begin{bmatrix} -\frac{3}{2} \\ 1 \\ 0 \end{bmatrix}$$

A basis for the eigenspace of 1:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

In [33]:

```
# b) Since the dimension of the eigenspace of 1 is 1, and the multiplicity of 1 is 2, A
is not diagonalizable.
```

Assignment 3

In [34]:

```
# a)
A = Matrix([[3, 5, -4], [-3, -2, 4], [6, 1, -8]])
b = Matrix([7, -1, -4])
x, y, z = symbols('x y z')
var = Matrix([x, y, z])
display(Latex("$${}{} = {}$$".format(latex(A), latex(var), latex(b))))
```

$$\begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$$

In [35]:

```python
# b)
print("Augmented matrix:")
A_aug = A.row_join(b)
display(A_aug)

print("Echelon form:")
L, U, perm = A.LUdecomposition()
display(U)
```

Augmented matrix:

$$\begin{bmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{bmatrix}$$

Echelon form:

$$\begin{bmatrix} 3 & 5 & -4 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

From the echelon form we conclude that column three is a non-pivot column, which means z is a free variable.

In [36]:

```python
# c)
A_aug_rref, _ = A_aug.rref()
A_aug_rref
```

Out[36]:

$$\begin{bmatrix} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

In [37]:

```python
z_vec = Matrix([Rational(4,3), 0, 1])
cons = Matrix([A_aug_rref[:,3]])
display(Latex("$$x = {} + z{}$$".format(latex(cons), latex(z_vec))))
```

$$x = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + z \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$$

In [38]:

```python
# d) In part b) we showed that the coefficient matrix A has a non-pivot column.
#    This means that the matrix is non-invertible and hence does not have an inverse.
```

Assignment 4

In [39]:

```
a, b, c, d = symbols('a b c d')
```

In [40]:

```
M = Matrix([[a, 0, d, c], [b, 0, -c, d], [0, c, -b, a], [0, d, a, b]])

# Co-factor expansion on second column
solve(M[0,1]*M.cofactor(0,1)+M[1,1]*M.cofactor(1,1)+
      M[2,1]*M.cofactor(2,1)+M[3,1]*M.cofactor(3,1), a, b, c, d)
```

Out[40]:

$[]$

In [41]:

```
# or
M.rref()
```

Out[41]:

$$\left(\left[\begin{matrix} 1 & 0 & 0 & \frac{c(-ac-bd)-d(ad-bc)}{a(-ac-bd)} \\ 0 & 1 & 0 & \frac{a(-ac-bd)+b(ad-bc)}{c(-ac-bd)} \\ 0 & 0 & 1 & \frac{ad-bc}{-ac-bd} \\ 0 & 0 & 0 & 0 \end{matrix} \right], \quad (0, \quad 1, \quad 2) \right)$$

Since there is a zero on the diagonal, the determinant will always be zero.

Assignment 5

In [43]:

```
# a)
a = symbols('a')
A = Matrix([
      [a, 3, -1, 0, -2]
    , [0, 2, -4, -1, -6]
    , [-2, -6, 2, 3, 9]
    , [3, 7, -3, 8, -7]
    , [3, 5, 5, 2, 7]
])
var = solve(A.det()-24,a)
display(Latex("$$a = {}$$".format(var[0])))
```

$a = 1$

In [44]:

```
# b)
A = Matrix([
     [1, 3, -1, 0, -2]
   , [0, 2, -4, -1, -6]
   , [-2, -6, 2, 3, 9]
   , [3, 7, -3, 8, -7]
   , [3, 5, 5, 2, 7]
])

A.rref()
```

Out[44]:

$$\left(\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad (0, \quad 1, \quad 2, \quad 3, \quad 4) \right)$$

Since we have pivots in all columns, the columns of A necessarily form a basis for $\mathbb{R}^5$

In [45]:

```
# c)
A.T*A
```

Out[45]:

$$\begin{bmatrix} 23 & 51 & 1 & 24 & -20 \\ 51 & 123 & -19 & 46 & -86 \\ 1 & -19 & 55 & -4 & 100 \\ 24 & 46 & -4 & 78 & -9 \\ -20 & -86 & 100 & -9 & 219 \end{bmatrix}$$

Had A been an orthogonal basis, the off-diagonals would have been 0's. To figure out which columns are orthogonal, we need to find the inner product between them. The dot products can be read in the above matrix which is called the Gramian Matrix. The dot product between columns i and j, is exactly the entry in entry i, j (and/or j, i). So the dot product between column 1 and 2 is 51. Since none of the off-diagonals are zero, none of the columns are orthogonal.

In [46]:

```
# Gram Schmidt
u1 = A[:,0]
u2 = A[:,1]-A[:,1].project(u1)
u3 = A[:,2]-A[:,2].project(u1)-A[:,2].project(u2)
u4 = A[:,3]-A[:,3].project(u1)-A[:,3].project(u2)-A[:,3].project(u3)
u5 = A[:,4]-A[:,4].project(u1)-A[:,4].project(u2)-A[:,4].project(u3)-A[:,4].project(u4)
U = u1.row_join(u2).row_join(u3).row_join(u4).row_join(u5)
U.T*U
```

Out[46]:

$$\begin{bmatrix} 23 & 0 & 0 & 0 & 0 \\ 0 & \frac{228}{23} & 0 & 0 & 0 \\ 0 & 0 & \frac{544}{57} & 0 & 0 \\ 0 & 0 & 0 & \frac{63}{17} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{14} \end{bmatrix}$$

Since the off-diagonals are all 0's, we conclude that U is an orthogonal basis.

Assignment 6

In [47]:

```
import sympy as sp
from sympy import *
init_printing()
from IPython.display import display, Latex
```

In [48]:

```
from scipy.io import loadmat
Data = loadmat("ALI2018_6.mat")
Data.keys()
```

Out[48]:

```
dict_keys(['__header__', '__version__', '__globals__', 'CPU', 'Order_lines
_per_day'])
```

In [49]:

```
x = Data['Order_lines_per_day']
y = Data['CPU']
x = x.astype(float)
```

In [50]:

```
# a) Design matrix for linear model

X1 = Matrix([ones(len(x),1)]).row_join(Matrix(x))
XtX = X1.T*X1
Xty = X1.T*Matrix(y)
Mat, _ = XtX.row_join(Xty).rref()
B1 = Mat[:,-1]
display(Latex("$$y_1(x) = {}+{}x$$".format(round(B1[0],2), round(B1[1], 4))))
```

$$y_1(x) = 4.63 + 0.0019x$$

In [51]:

```
# a) Design matrix for quadratic model

X2 = Matrix([ones(len(x),1)]).row_join(Matrix(x)).row_join(Matrix(x**2))
XtX = X2.T*X2
Xty = X2.T*Matrix(y)
Mat, _ = XtX.row_join(Xty).rref()
B2 = Mat[:,-1]
display(Latex("$$y_2(x) = {}+{}x{}x^2$$"
              .format(round(B2[0],2), round(B2[1], 4), round(B2[2], 4))))
```

$$y_2(x) = 1.51 + 0.0043x - 0.0x^2$$

In [52]:

```
# We see that the coefficient for the quadratic part is very small. It's actual value i
s
B2[2]
```

Out[52]:

$$-1.59605429178864 \cdot 10^{-7}$$

In [53]:

```
# b)

display(Latex(
    "The error of the linear model is {}, and the error of the quadratic is model {}."
    " We conclude that the quadratic model has the best fit for this specific data."
    .format(round((y-X1*B1).norm(), 2), round((y-X2*B2).norm(), 2))))
```

The error of the linear model is 42.7, and the error of the quadratic is model 37.87. We conclude that the quadratic model has the best fit for this specific data.

In [54]:

```
# c)

display(Latex("$$y_2(20000) = {}$$".format(round(B2[0]+B2[1]*20000+B2[2]*20000**2, 2
))))
```

$$y_2(20000) = 23.16$$

Assignment 7

In [99]:

```
# This assignment had no (real) solution. Will be explained in class.
```