№3

$\xi \sim p(x) = \begin{cases} \frac{e^{-\frac{x}{\Theta}}}{\Theta}, & x \geq 0 \\ 0, & x < 0 \end{cases}, \Theta > 0 \qquad F(x) = \begin{cases} 1 - e^{-\frac{x}{\Theta}}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

$n = 3 \quad \bar{x}_n = (x_1, \dots x_n)$ - выборка

1) $\tilde{\Theta}_1 = \bar{x}_4$ 2) $\tilde{\Theta}_2 = \frac{x_{min} + x_{max}}{2}$ 3) $\tilde{\Theta}_3 = x_{(2)}$

0.0 $\int_0^\infty x e^{-\lambda x} = -\frac{1}{\lambda} x e^{-\lambda x} \Big|_0^\infty + \frac{1}{\lambda} \int_0^\infty e^{-\lambda x} dx = \frac{1}{\lambda} \cdot \left(-\frac{1}{\lambda}\right) e^{-\lambda x} \Big|_0^\infty = \frac{1}{\lambda^2}$

$\int_0^\infty x^2 e^{-\lambda x} = -\frac{1}{\lambda} x^2 e^{-\lambda x} \Big|_0^\infty + \frac{2}{\lambda} \int_0^\infty x e^{-\lambda x} dx = \frac{2}{\lambda} \cdot \frac{1}{\lambda^2} = \frac{2}{\lambda^3}$

0.1) $M[\xi] = \int_0^\infty x \frac{e^{-\frac{x}{\Theta}}}{\Theta} dx = \frac{1}{\Theta} \int_0^\infty x e^{-\frac{x}{\Theta}} dx = \frac{1}{\Theta} \cdot \Theta^2 = \Theta \quad \lambda = \frac{1}{\Theta}$

$M[\xi^2] = \frac{1}{\Theta} \int_0^\infty x^2 e^{-\frac{x}{\Theta}} dx = \frac{1}{\Theta} \cdot 2 \cdot \Theta^3 = 2\Theta^2 \quad \lambda = \frac{1}{\Theta}$

$D[\xi] = M[\xi^2] - M^2[\xi] = \Theta^2$

а) Несмещенность, определить наиболее эффективную

1) $M[\tilde{\Theta}_1] = M\left[\frac{1}{n} \sum x_i\right] = \frac{1}{n} \sum M[x_i] = \frac{1}{n} \cdot n \cdot M[\xi] = \Theta$ - несмещена

$D[\tilde{\Theta}_1] = D\left[\frac{1}{n} \sum x_i\right] = \frac{1}{n^2} \sum D[x_i] = \frac{1}{n^2} \cdot n \cdot D[\xi] = \frac{\Theta^2}{n}$

2) $\tilde{\Theta}_2 = \frac{x_{min} + x_{max}}{2}$

min: $\xi \sim F(x)$

$\min(\xi_1, \dots \xi_n) \sim 1 - (1 - F(y))^n = \Phi(y)$

$\varphi(y) = \Phi'(y) = n(1 - F(y))^{n-1} p(y)$

$\varphi(y) = n \left(e^{-\frac{x}{\Theta}}\right)^{n-1} \frac{e^{-\frac{x}{\Theta}}}{\Theta} = \frac{n}{\Theta} e^{-\frac{xn}{\Theta}} \Rightarrow M[\xi] = \frac{\Theta}{n}$

max: $M[\xi^2] = \frac{n}{\Theta} \int_0^\infty x^2 e^{-x\frac{n}{\Theta}} dx = \frac{n}{\Theta} \cdot 2 \cdot \frac{\Theta^3}{n^3} = 2\frac{\Theta^2}{n^2}$

$D[\xi] = M[\xi^2] - M^2[\xi] = 2\frac{\Theta^2}{n^2} - \frac{\Theta^2}{n^2} = \frac{\Theta^2}{n^2}$

max: $\xi - F(\xi)$

$\max(\xi_1, \dots \xi_n) \sim (F(z))^n = \Psi(z)$

$\Psi(z) = n\,(F(z))^{n-1} p(z) = n\,(1-e^{-\frac{z}{\Theta}})^{n-1} \frac{1}{\Theta} e^{-\frac{z}{\Theta}}$

$M[\xi] = \int_0^\infty x \frac{n}{\Theta} (1-e^{-\frac{x}{\Theta}})^{n-1} e^{-\frac{x}{\Theta}} dx$

$n=3$:

$M[\bar{x}_{(3)}] = \frac{3}{\Theta} \int_0^\infty x (1-e^{-\frac{x}{\Theta}})^2 e^{-\frac{x}{\Theta}} dx = \frac{3}{\Theta} \left[\int_0^\infty x e^{-\frac{x}{\Theta}} dx - 2 \int_0^\infty x e^{-\frac{2x}{\Theta}} dx + \int_0^\infty x e^{-\frac{3x}{\Theta}} dx \right] =$

($\lambda = \frac{1}{\Theta}$, $\lambda = \frac{2}{\Theta}$, $\lambda = \frac{3}{\Theta}$)

$= \frac{3}{\Theta} \left[\Theta^2 - 2\frac{\Theta^2}{4} + \frac{\Theta^2}{9} \right] = \underline{\frac{11}{6}\Theta}$

$M[\xi^2] = \int_0^\infty x^2 \frac{n}{\Theta} (1-e^{-\frac{x}{\Theta}})^{n-1} e^{-\frac{x}{\Theta}} dx$

$n=3$:

$M[\xi^2] = \frac{3}{\Theta} \int_0^\infty x^2 (1-e^{-\frac{x}{\Theta}})^2 e^{-\frac{x}{\Theta}} dx = \frac{3}{\Theta} \left[\int_0^\infty x^2 e^{-\frac{x}{\Theta}} dx - 2 \int_0^\infty x^2 e^{-\frac{2x}{\Theta}} dx + \int_0^\infty x^2 e^{-\frac{3x}{\Theta}} dx \right] =$

($\lambda = \frac{1}{\Theta}$, $\lambda = \frac{2}{\Theta}$, $\lambda = \frac{3}{\Theta}$)

$= \frac{3}{\Theta} \left[2\Theta^3 - 4\frac{\Theta^3}{8} + 2\frac{\Theta^3}{27} \right] = \left(\frac{9}{2} + \frac{6}{27}\right)\Theta^2$

$D[\bar{x}_{(3)}] = M[\xi^2] - M^2[\xi] = \left(\frac{9}{2} + \frac{6}{27}\right)\Theta^2 - \frac{121}{36}\Theta^2 = \underline{\frac{49}{36}\Theta^2}$

$\tilde{\Theta}_2 = \frac{x_{min} + x_{max}}{2}$:

$M[\tilde{\Theta}_2] = \frac{1}{2}\left(M[\bar{x}_{(1)}] + M[\bar{x}_{(3)}]\right) = \frac{1}{2}\left(\frac{\Theta}{3} + \frac{11}{6}\Theta\right) = \frac{13}{12}\Theta$ – смещенная

$\Downarrow$

$\tilde{\Theta}_2' = \frac{12}{13} \frac{x_{min} + x_{max}}{2}$ – несмещенная

$D[\tilde{\Theta}_2'] = \frac{6^2}{12^2}\left[D[\bar{x}_{(1)}] + D[\bar{x}_{(3)}] + 2\,cov(\bar{x}_{(1)}, \bar{x}_{(3)})\right]$

$cov(\bar{x}_{(1)}, \bar{x}_{(3)}) = M[\bar{x}_{(1)} \bar{x}_{(3)}] - M[\bar{x}_{(1)}] \cdot M[\bar{x}_{(3)}]$

$K(y,z) = \begin{cases} y > z, & F^n(z) \\ y \le z, & F^n(z) - (F(z) - F(y))^n \end{cases}$ $\quad \varkappa(y,z) = n(n-1)p(y)p(z)(F(y)-F(z))^{n-2}$ $(z \ge y)$

$\varkappa(y,z) = n(n-1)\frac{e^{-\frac{y}{\Theta}}}{\Theta}\frac{e^{-\frac{z}{\Theta}}}{\Theta}\left(1 - e^{-\frac{y}{\Theta}} - 1 + e^{-\frac{z}{\Theta}}\right)^{n-2} = \frac{n(n-1)}{\Theta^2} y e^{-\frac{y}{\Theta}} e^{-\frac{z}{\Theta}} \left(e^{-\frac{z}{\Theta}} - e^{-\frac{y}{\Theta}}\right)^{n-2}$

$M[\bar{x}_{(1)}\bar{x}_{(3)}] = \iint y z\, \varkappa(y,z)\, dy\, dz$

$n=3$:

$M[\bar{x}_{(1)}\bar{x}_{(3)}] = \frac{6}{\Theta^2} \iint_{\substack{z \ge 0 \\ y \ge z}} y z e^{-\frac{y}{\Theta}} e^{-\frac{z}{\Theta}} \left(e^{-\frac{z}{\Theta}} - e^{-\frac{y}{\Theta}}\right) dy\, dz$ (дальше не получается)

3) $\tilde{\Theta}_3 = \bar{x}_{(2)}$

$\bar{x}_{(k)} \sim n C_{n-1}^{k-1} \frac{e}{\Theta} \left(1-e^{-\frac{x}{\Theta}}\right)^{k-1} \left(e^{-\frac{x}{\Theta}}\right)^{n-k}$

$n=3 \quad k=2$:

$\bar{x}_{(2)} \sim 3 C_2^1 \frac{e}{\Theta} \left(1-e^{-\frac{x}{\Theta}}\right) e^{-\frac{x}{\Theta}} = \frac{6}{\Theta} \left(1-e^{-\frac{x}{\Theta}}\right) e^{-\frac{2x}{\Theta}}$

$M\left[\bar{x}_{(2)}\right] = \frac{6}{\Theta} \int_0^\infty x \left(1-e^{-\frac{x}{\Theta}}\right) e^{-\frac{2x}{\Theta}} dx = \frac{6}{\Theta} \left[\int_0^{\infty, z=\frac{\Theta}{2}} x e^{-\frac{2x}{\Theta}} dx - \int_0^{\infty, z=\frac{\Theta}{3}} x e^{-\frac{3x}{\Theta}} dx \right] =$

$= \frac{6}{\Theta} \left[\frac{\Theta^2}{4} - \frac{\Theta^2}{9} \right] = \frac{5}{6}\Theta$ - смещенная

$\tilde{\Theta}_3' = \frac{5}{6} \bar{x}_{(2)}$ - несмещенная

$M\left[\bar{x}_{(2)}^2\right] = \frac{6}{\Theta} \int_0^\infty x^2 \left(1-e^{-\frac{x}{\Theta}}\right) e^{-\frac{2x}{\Theta}} dx = \frac{6}{\Theta} \left[\int_0^{\infty, z=\frac{\Theta}{2}} x^2 e^{-\frac{2x}{\Theta}} dx - \int_0^{\infty, z=\frac{\Theta}{3}} x^2 e^{-\frac{3x}{\Theta}} dx \right] =$

$= \frac{6}{\Theta} \left[2\frac{\Theta^3}{8} - 2\frac{\Theta^3}{27} \right] = \frac{19}{18}\Theta^2$

$D\left[\bar{x}_{(2)}\right] = M\left[\bar{x}_{(2)}^2\right] - M^2\left[\bar{x}_{(2)}\right] = \left\{\frac{19}{18} - \frac{25}{36}\right\}\Theta^2 = \frac{\Theta^2}{3}$

№2

1) Нахождение значений моментов и коэф. ассиметрии

$$p(x)=\begin{cases} e^{-x}, & x\geq 0 \\ 0, & x<0 \end{cases}$$

$$\gamma=\frac{\mu_3}{\sigma_3}=\frac{\mu_3}{\mu_2^{3/2}}$$

$$\alpha_1=\int_0^\infty xe^{-x}dx=-xe^{-x}\Big|_0^\infty+\int_0^\infty e^{-x}dx=-e^{-x}\Big|_0^\infty=1$$

$$\mu_2=\int_0^\infty (x-1)^2e^{-x}dx=\underbrace{\int_0^\infty x^2e^{-x}dx}_{2=1}-2\underbrace{\int_0^\infty xe^{-x}dx}_{A=1}+\int_0^\infty e^{-x}dx =$$

$$=2-2+e^{-x}\Big|_0^\infty=1$$

$$\mu_3=\int_0^\infty (x-1)^3e^{-x}dx=\int_0^\infty x^3e^{-x}dx-3\underbrace{\int_0^\infty x^2e^{-x}dx}_{2}+3\underbrace{\int_0^\infty xe^{-x}dx}_{1}-\underbrace{\int_0^\infty e^{-x}dx}_{1}=$$

$$=-x^3e^{-x}\Big|_0^\infty+3\underbrace{\int_0^\infty x^2e^{-x}dx}_{2}-6+3-1=2$$

$$\gamma=\frac{2}{1^{3/2}}=2$$

2) Нахождение плотности распределения среднего арифметического элементов выборки

Классическое ЦПТ

$$\frac{\tilde\alpha_1-\alpha_1}{\tilde\sigma}\sqrt{n}\sim N(0,1) \qquad p(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$\tilde\alpha_1=\frac{1}{n}\sum_{i=1}^n x_i \qquad \tilde\alpha_2=\frac{1}{n}\sum_{i=1}^n x_i^2$$

$$\tilde\sigma=\sqrt{\tilde\alpha_2-\tilde\alpha_1^2}$$

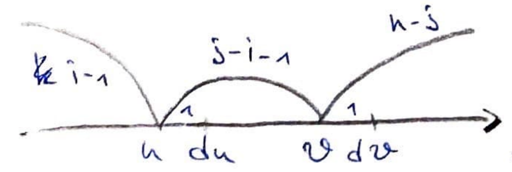
$$\tilde\alpha_1-\alpha_1\sim\frac{\tilde\sigma}{\sqrt{n}}N(0,1)$$

$$\tilde\alpha_1-\alpha_1\sim N\left(0,\frac{\tilde\sigma^2}{n}\right) \quad \cdot(-1)-N\left(0,\frac{\tilde\sigma^2}{n}\right)\text{ — симметрично}$$

$$\alpha_1\sim N\left(\tilde\alpha_1,\frac{\tilde\sigma^2}{n}\right)$$

$$p(x)=\begin{cases} e^{-x}, & x\geq 0 \\ 0, & x<0 \end{cases} \qquad F(x)=\begin{cases} 1-e^{-x}, & x\geq 0 \\ 0, & x<0 \end{cases}$$

3) Плотность совместного распр. i-го и j-го элементов вариационного ряда



$$P(\overline{X}_{(i)}\in(u,u+du),\overline{X}_{(j)}\in(v,v+dv))=$$

$$=nP(\underbrace{u\leq\xi<u+du}_{p(u)})(n-1)P(\underbrace{v\leq\xi<v+dv}_{p(v)})\cdot C_{n-2}^{i-1}[\underbrace{P(-\infty<\xi<u)}_{F(u)}]^{i-1}\cdot$$

$$\cdot C_{n-i-1}^{j-i-1}[\underbrace{P(u\leq\xi<v)}_{F(v)-F(u)}]^{j-i-1}\cdot 1\cdot[\underbrace{P(v\leq\xi<\infty)}_{1-F(v)}]^{n-j}=$$

$$=n(n-1)C_{n-i-1}^{j-i-1}p(u)p(v)F^{i-1}(u)(F(v)-F(u))^{j-i-1}(1-F(v))^{n-j}$$