№4

$\xi \sim R(\theta, 2\theta)$ $\qquad p(x,\theta) = \frac{1}{\theta}\ (\theta, 2\theta)$ $\qquad F(x) = \begin{cases} 0, & x < \theta \\ \frac{x-\theta}{\theta}, & \theta \le x < 2\theta \\ 1, & x \ge 2\theta \end{cases}$

а) $\overline{X}_n$

$$M\xi = \int_{\theta}^{2\theta} x \frac{1}{\theta} dx = \frac{x^2}{2\theta}\Big|_{\theta}^{2\theta} = \frac{4\theta^2 - \theta^2}{2\theta} = \frac{3\theta}{2}$$

$$M\xi^2 = \int_{\theta}^{2\theta} x^2 \frac{1}{\theta} dx = \frac{x^3}{3\theta}\Big|_{\theta}^{2\theta} = \frac{8\theta^3 - \theta^3}{3\theta} = \frac{\cancel{8}\,7\theta^2}{3}$$

$$D\xi = M\xi^2 - (M\xi)^2 = \frac{7\theta^2}{3} - \frac{9\theta^2}{4} = \frac{28\theta^2 - 27\theta^2}{12} = \frac{1\theta^2}{12}$$

ОММ

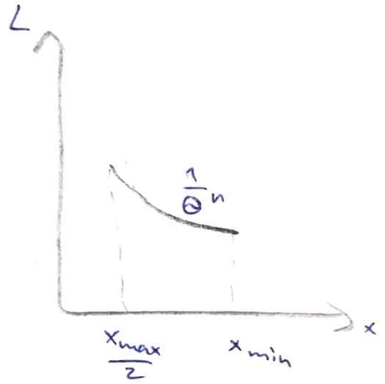
$$\alpha_1 = \int_{\theta}^{2\theta} x \cdot \frac{1}{\theta} dx = \frac{3}{2}\theta = M\xi = \overline{X} \Rightarrow \underline{\tilde{\theta}_1 = \frac{2}{3}\overline{X}}$$

ОМП

$$L(\theta) = \begin{cases} \frac{1}{\theta^n}, & \text{если все } x_i \in [\theta, 2\theta] \\ 0, & \text{иначе} \end{cases} = \begin{cases} \frac{1}{\theta^n}, & \theta \le x_{min} \le x_{max} \le 2\theta \\ 0, & \text{иначе} \end{cases} =$$

$$= \begin{cases} \frac{1}{\theta^n}, & \frac{x_{max}}{2} \le \theta \le x_{min} \\ 0 \end{cases}$$

L - max, при $\hat{\theta}_2 = \frac{x_{max}}{2}$ $\Leftarrow$



б) 1) $\tilde{\theta}_1 = \frac{2\bar{x}_n}{3}$ - ОММ

$$M[\tilde{\theta}_1] = M\left[\frac{2\hat{x}_n}{3}\right] = \frac{2}{3} M\left[\frac{1}{n}\sum x_i\right] = \frac{2}{3}\cdot M[\xi] = \frac{2}{3}\cdot\frac{3}{2}\theta = \theta$$ - несмещена

$$D[\hat{\theta}_1] = \frac{4}{9} D[\bar{x}_n] = \frac{4}{9} D\left[\frac{1}{n}\sum x_i\right] = \frac{4}{9n} D\xi = \frac{4}{9n}\cdot\frac{1}{12}\theta^2 = \frac{\theta^2}{27n} \xrightarrow[n\to\infty]{} 0$$

состоятельна

2) $\tilde{\theta}_2 = \frac{x_{max}}{2}$ - ОМП

$$M[\hat{\theta}_2] = \frac{1}{2} M[x_{max}] = \frac{1}{2}\int_{\theta}^{2\theta} x\, n\, (x-\theta)^{n-1}\cdot\frac{1}{\theta^n} dx = \frac{1}{2}\frac{n}{\theta^n}\int_{\theta}^{2\theta} x(x-\theta)^{n-1} dx =$$

$$= \frac{1}{2} n\theta \int_1^2 x(x-1)^{n-1} dx = \frac{1}{2} n\theta \int_0^1 y^{n-1}(y+1) dy = \frac{2n+1}{2n+2}\theta$$ - смещена

$$\hat{\theta}_2' = \frac{2n+2}{2n+1}\hat{\theta}_2 = \frac{n+1}{2n+1} x_{max}$$ - несмещ.

$$D[\hat{\theta}_2] = \frac{1}{4} D[x_{max}] = \ldots = \frac{n\theta^2}{4(n+2)(n+1)^2}$$

$$D[\hat{\theta}_2'] = \left(\frac{2n+2}{2n+1}\right)^2 \frac{n\theta^2}{4(n+2)(n+1)^2} = \frac{n\theta^2}{(n+2)(2n+1)^2} \xrightarrow[n\to\infty]{} 0$$ - состоятельна

3) $\tilde{\Theta}_3 = \frac{1}{5}(X_{min} + 2X_{max})$

$$M[\tilde{\Theta}_3] = \frac{1}{5} M[X_{min}] + \frac{2}{5} M[X_{max}] = \frac{1}{5}\left(n\Theta\left(\frac{2}{n} - \frac{1}{n+1}\right)\right) + \frac{2}{5}\left(n\Theta\left(\frac{1}{n+1} + \frac{1}{n}\right)\right) =$$

$$= \Theta \frac{5n+4}{5n+5} \text{ – смещена} \Rightarrow \tilde{\Theta}_3' = \frac{5(n+1)}{5n+4}\tilde{\Theta}_3 = \frac{n+1}{5n+4}(X_{min} + 2X_{max})$$

$$D[\tilde{\Theta}_3] = D\left[\frac{X_{min}}{5}\right] + D\left[\frac{2 \cdot X_{max}}{5}\right] + 2\operatorname{cov} X_{(1)} X_{(n)}$$

$$D[X_{min}] = D[X_{max}] = \frac{n\Theta^2}{(n+2)(n+1)^2}$$

$$\operatorname{cov}(X_{(1)} X_{(n)}) = M[X_{(1)} X_{(n)}] - M X_{(1)} M X_{(n)}$$

$$M[X_{(1)} X_{(n)}] = \iint uv\, f_{1n}(u,v)\,du\,dv$$

$$f_{1n} = \begin{cases} n(n-1) f(x) f(y) (F(y) - F(x))^{n-2}, & x, y \in [\Theta, 2\Theta] \\ 0, & \text{иначе} \end{cases} = \begin{cases} n(n-1) \frac{1}{\Theta^n} (y-x)^{n-2}, & x, y \in [\Theta, 2\Theta] \\ 0, & \text{иначе} \end{cases}$$

$$M X_{(1)} X_{(n)} = n(n-1)\frac{1}{\Theta^n} \int_\Theta^{2\Theta}\int_u^{2\Theta} uv(v-u)^{n-2}\,du\,dv = \frac{n(n-1)}{\Theta^n}\int_\Theta^{2\Theta} u\left(\int_u^{2\Theta} v(v-u)^{n-2}\,dv\right)du =$$

$$= \frac{n(n-1)}{\Theta^n}\int_\Theta^{2\Theta} u\left(\frac{(2\Theta-u)^n}{n} + \frac{u(2\Theta-u)^{n-1}}{n-1}\right)du = \Theta^2\left(2 + \frac{1}{n+2}\right)$$

$$\operatorname{cov}(X_{(1)}, X_{(n)}) = \Theta^2\left(2 + \frac{1}{n+2}\right) - \left(n\Theta\left(\frac{1}{n+1} + \frac{1}{n}\right)\right)\left(n\Theta\left(\frac{2}{n} - \frac{1}{n+1}\right)\right) = \Theta^2\left(\frac{1}{n+2} - \frac{n}{n+1} + \frac{n^2}{(n+1)^2}\right) =$$

$$= \frac{\Theta^2(2n-1)}{(n+2)(n+1)^2}$$

$$D[\Theta_3] = \frac{1}{25}\frac{n\Theta^2}{(n+2)(n+1)^2} + \frac{4}{25}\frac{n\Theta^2}{(n+2)(n+1)^2} + 2 \cdot \frac{2}{25}\frac{(2n-1)\Theta^2}{(n+2)(n+1)^2} =$$

$$= \frac{(13n-4)\Theta^2}{25(n+2)(n+1)^2}$$

$$D[\tilde{\Theta}_3'] = \left(\frac{5n+5}{5n+4}\right)^2 D[\tilde{\Theta}_3] = \frac{(13n-4)\Theta^2}{(n+2)(5n+4)^2} \xrightarrow[n\to\infty]{} 0 \text{ – состоятельна}$$

c) $\tilde{\Theta}_1 = \frac{2\alpha_1}{3}$: $D[\tilde{\Theta}_1] = \frac{1}{27}\frac{\Theta^2}{n}$

$\tilde{\Theta}_2' = \frac{n+1}{2n+1} X_{max}$: $D[\tilde{\Theta}_2'] = \frac{n\Theta^2}{(n+2)(2n+1)^2}$ – наиболее эффективна при $n \geq 3$

$\tilde{\Theta}_3' = \frac{n+1}{5n+4}(X_{min} + 2 \cdot X_{max})$: $D[\tilde{\Theta}_3'] = \frac{(13n-4)\Theta^2}{(n+2)(5n+4)^2}$

d) $X_i \sim U_{\theta,2\theta}$

$$y_i = \frac{x_i}{\theta} - 1 \sim U_{0,1}$$

$$y_{(n)} = \max(y_1, \dots y_n) = \frac{\max(x_1, \dots x_n)}{\theta} - 1 = \frac{X_{(n)}}{\theta} - 1 = G(x, \theta)$$

$$F_{y_{(n)}}(y) = \begin{cases} 0, & y < 0 \\ y^n, & y \in [0;1] \\ 1, & y > 1 \end{cases} \qquad p(y) = n y^{n-1} [0,1]$$

$$P_\theta(g_1 < G(x,\theta) < g_2) = P_\theta\Big(\underbrace{g_1}_{= q_{\frac{1-\beta}{2}}} < \frac{X_{(n)}}{\theta} - 1 < \underbrace{g_2}_{= q_{\frac{1+\beta}{2}}}\Big)$$

$$\Downarrow$$

$$P_\theta\left(\frac{X_{(n)}}{g_2+1} < \theta < \frac{X_{(n)}}{g_1+1}\right) \geq \beta$$

$$g_1: P_\theta(0 < y_{(n)} < g_1) = F_{y_{(n)}}(g_1) - F_{y_{(n)}}(0) = g_1^n - 0 = \frac{1-\beta}{2}$$

$$\Downarrow$$

$$g_1 = \sqrt[n]{\frac{1-\beta}{2}}$$

$$g_2: n \int_0^{q_{\frac{1+\beta}{2}}} y^{n-1} dy = \frac{1+\beta}{2} \Rightarrow y^n \Big|_0^{q_{\frac{1+\beta}{2}}} = \frac{1+\beta}{2}$$

$$\Downarrow$$

$$\left(q_{\frac{1+\beta}{2}}\right)^n = \frac{1+\beta}{2} \Rightarrow g_2 = q_{\frac{1+\beta}{2}} = \sqrt[n]{\frac{1+\beta}{2}}$$

$$\Downarrow$$

$$P_\theta\left(\frac{X_{(n)}}{\sqrt[n]{\frac{1+\beta}{2}} + 1} < \theta < \frac{X_{(n)}}{\sqrt[n]{\frac{1-\beta}{2}} + 1}\right) \geq \beta$$

e) uz OMM: $\tilde{\theta} = \frac{2}{3}\bar{X}$

$$\frac{f(\tilde{\alpha}) - f(\alpha)}{\sigma} \sqrt{n} \rightsquigarrow N(0,1) \qquad \sigma = \sqrt{\nabla^T f(\alpha)\, K\, \nabla f(\alpha)}$$

$$\alpha = (\alpha_1, \dots \alpha_m)$$
$$\tilde{\alpha} = (\tilde{\alpha}_1, \dots \tilde{\alpha}_m)$$
$$K_{ij} = \alpha_{i+j} - \alpha_i \cdot \alpha_j$$

$$f(\alpha) = \frac{2\alpha_1}{3} = \theta$$

$$f'(\alpha) = \frac{2}{3} \qquad K_{11} = \alpha_2 - \alpha_1^2$$

$$\sigma = \sqrt{\frac{2}{3}(\alpha_2 - \alpha_1^2)\frac{2}{3}} = \sqrt{\frac{4}{9}(\alpha_2 - \alpha_1^2)}$$

$$\frac{\hat{\theta} - \theta}{\sqrt{\frac{4}{9}(\tilde{\alpha}_2 - \tilde{\alpha}_1^2)}} \sqrt{n} \rightsquigarrow N(0,1) \longrightarrow -1{,}96 < \frac{\tilde{\theta} - \theta}{\sqrt{\frac{4}{9}(\tilde{\alpha}_2 - \tilde{\alpha}_1^2)}} < 1{,}96$$

$$-1{,}96 \cdot \frac{2}{3}\sqrt{\tilde{\alpha}_2 - \tilde{\alpha}_1^2} + \tilde{\theta} < \theta < 1{,}96 \cdot \frac{2}{3}\sqrt{\tilde{\alpha}_2 - \tilde{\alpha}_1^2} + \tilde{\theta}$$