№3

$\xi \sim p(x) = \begin{cases} \frac{e^{-\frac{x}{\theta}}}{\theta}, & x \geq 0 \\ 0, & x < 0 \end{cases}, \quad \theta > 0 \qquad F(x) = \begin{cases} 1 - e^{-\frac{x}{\theta}}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

$n=3$ $\bar{X}_n = (X_1, \dots X_n)$ - выборка

1) $\tilde{\theta}_1 = \bar{x}_n$ 2) $\tilde{\theta}_2 = \frac{X_{min} + X_{max}}{2}$ 3) $\tilde{\theta}_3 = X_{(2)}$

о.д) $\int_0^\infty x e^{-\lambda x} = -\frac{1}{\lambda} \underbrace{x e^{-\lambda x} \Big|_0^\infty}_{0} + \frac{1}{\lambda} \int_0^\infty e^{-\lambda x} dx = \frac{1}{\lambda} \cdot \left(-\frac{1}{\lambda}\right) e^{-\lambda x} \Big|_0^\infty = \frac{1}{\lambda^2}$

$\int_0^\infty x^2 e^{-\lambda x} = -\frac{1}{\lambda} \underbrace{x^2 e^{-\lambda x} \Big|_0^\infty}_{0} + \frac{2}{\lambda} \int_0^\infty x e^{-\lambda x} dx = \frac{2}{\lambda} \cdot \frac{1}{\lambda^2} = \frac{2}{\lambda^3}$

о.1) $M[\xi] = \int_0^\infty x \frac{e^{-\frac{x}{\theta}}}{\theta} dx = \frac{1}{\theta} \int_0^\infty x e^{-\frac{x}{\theta}} dx = \frac{1}{\theta} \cdot \theta^2 = \theta$ $\quad \lambda = +\frac{1}{\theta}$

$M[\xi^2] = \frac{1}{\theta} \int_0^\infty x^2 e^{-\frac{x}{\theta}} dx = \frac{1}{\theta} \cdot 2 \cdot \theta^3 = 2\theta^2$ $\quad \lambda = +\frac{1}{\theta}$

$D[\xi] = M[\xi^2] - M^2[\xi] = \theta^2$

а) Несмещенность, определить наиболее эффективную

1) $M[\tilde{\theta}_1] = M\left[\frac{1}{n} \sum x_i\right] = \frac{1}{n} \sum M[x_i] = \frac{1}{n} \cdot n \cdot M[\xi] = \theta$ - несмещена

$D[\tilde{\theta}_1] = D\left[\frac{1}{n} \sum x_i\right] = \frac{1}{n^2} \sum D[x_i] = \frac{1}{n^2} \cdot n \cdot D[\xi] = \frac{\theta^2}{n}$

2) $\tilde{\theta}_2 = \frac{X_{min} + X_{max}}{2}$

min: $\xi \sim F(x)$

$\min(\xi_1, \dots \xi_n) \sim 1 - (1 - F(y))^n = \Phi(y)$

$\varphi(y) = \Phi'(y) = n(1 - F(y))^{n-1} p(y)$

$\Downarrow$

$\varphi(y) = n\left(e^{-\frac{x}{\theta}}\right)^{n-1} \frac{e^{-\frac{x}{\theta}}}{\theta} = \frac{n}{\theta} e^{-\frac{xn}{\theta}} \Rightarrow M[\xi] = \frac{\theta}{n}$

max: $M[\xi^2] = \frac{n}{\theta} \int_0^\infty x^2 e^{-x\frac{n}{\theta}} dx = \frac{n}{\theta} \cdot 2 \cdot \frac{\theta^3}{n^3} = 2 \frac{\theta^2}{n^2}$

$D[\xi] = M[\xi^2] - M^2[\xi] = 2\frac{\theta^2}{n^2} - \frac{\theta^2}{n^2} = \frac{\theta^2}{n^2}$

max: $\xi \sim F(z)$

$\max(\xi_1, \dots \xi_n) \sim (F(z))^n = \Psi(z)$

$\psi(z) = n(F(z))^{n-1} p(z) = \underbrace{n\left(1-e^{-\frac{z}{\theta}}\right)^{n-1} \frac{1}{\theta} e^{-\frac{z}{\theta}}}$

$M[\xi] = \int_0^\infty x \frac{n}{\theta}\left(1-e^{-\frac{x}{\theta}}\right)^{n-1} e^{-\frac{x}{\theta}} dx$

$n=3$:

$\lambda = \frac{1}{\theta} \qquad \lambda = \frac{2}{\theta} \qquad \lambda = \frac{3}{\theta}$

$M[\bar{X}_{(3)}] = \frac{3}{\theta}\int_0^\infty x\left(1-e^{-\frac{x}{\theta}}\right)^2 e^{-\frac{x}{\theta}} dx = \frac{3}{\theta}\left[\int_0^\infty x e^{-\frac{x}{\theta}}dx - 2\int_0^\infty x e^{-\frac{2x}{\theta}}dx + \int_0^\infty x e^{-\frac{3x}{\theta}}dx\right] =$

$= \frac{3}{\theta}\left[\theta^2 - 2\frac{\theta^2}{4} + \frac{\theta^2}{9}\right] = \underline{\frac{11}{6}\theta}$

$M[\xi^2] = \int_0^\infty x^2 \frac{n}{\theta}\left(1-e^{-\frac{x}{\theta}}\right)^{n-1} e^{-\frac{x}{\theta}} dx$

$n=3$:

$\lambda = \frac{1}{\theta} \qquad \lambda = \frac{2}{\theta} \qquad \lambda = \frac{3}{\theta}$

$M[\xi^2] = \frac{3}{\theta}\int_0^\infty x^2\left(1-e^{-\frac{x}{\theta}}\right)^2 e^{-\frac{x}{\theta}} dx = \frac{3}{\theta}\left[\int_0^\infty x^2 e^{-\frac{x}{\theta}}dx - 2\int_0^\infty x^2 e^{-\frac{2x}{\theta}}dx + \int_0^\infty x^2 e^{-\frac{3x}{\theta}}dx\right] =$

$= \frac{3}{\theta}\left[2\theta^3 - 4\frac{\theta^3}{8} + 2\frac{\theta^3}{27}\right] = \left(\frac{9}{2} + \frac{6}{27}\right)\theta^2$

$D[\bar{X}_{(3)}] = M[\xi^2] - M^2[\xi] = \left(\frac{9}{2} + \frac{6}{27}\right)\theta^2 - \frac{121}{36}\theta^2 = \underline{\frac{49}{36}\theta^2}$

$\tilde{\theta}_2 = \frac{X_{min} + X_{max}}{2}$:

$M[\tilde{\theta}_2] = \frac{1}{2}\left(M[\bar{X}_{(1)}] + M[\bar{X}_{(3)}]\right) = \frac{1}{2}\left(\frac{\theta}{3} + \frac{11}{6}\theta\right) = \frac{13}{12}\theta$ – смещенная

$\Downarrow$

$\tilde{\theta}_2' = \frac{12}{13}\,\frac{X_{min} + X_{max}}{2}$ – несмещенная

$D[\tilde{\theta}_2'] = \frac{6^2}{13^2}\left[D[\bar{X}_{(1)}] + D[\bar{X}_{(3)}] + 2\,\mathrm{cov}(\cancel{\tilde{\theta}_2}\,\bar{X}_{(1)}, \bar{X}_{(3)})\right]$

$\mathrm{cov}(\bar{X}_{(1)}, \bar{X}_{(3)}) = M[\bar{X}_{(1)}\bar{X}_{(3)}] - M[\bar{X}_{(1)}] \cdot M[\bar{X}_{(3)}]$

$K(y,z) = \begin{cases} y > z, & F^n(z) \\ y \le z, & F^n(z) - (F(z) - F(y))^n \end{cases}$ $\quad \varkappa(y,z) = n(n-1)p(y)p(z)(F(y) - F(z))^{n-2}_{(z \ge y)}$

$\varkappa(y,z) = n(n-1)\frac{e^{-\frac{y}{\theta}}}{\theta}\frac{e^{-\frac{z}{\theta}}}{\theta}\left(1 - e^{-\frac{y}{\theta}} - 1 + e^{-\frac{z}{\theta}}\right)^{n-2} = \frac{n(n-1)}{\theta^2} y e^{-\frac{y}{\theta}} e^{-\frac{z}{\theta}}\left(e^{-\frac{z}{\theta}} - e^{-\frac{y}{\theta}}\right)^{n-2}$

$M[\bar{X}_{(1)}\bar{X}_{(3)}] = \iint yz\,\varkappa(y,z)\,dy\,dz$

$n=3$: 2

$M[\bar{X}_{(1)}\bar{X}_{(3)}] = \frac{6}{\theta^2}\iint_{\substack{y \ge 0 \\ z \ge y}} yz e^{-\frac{y}{\theta}} e^{-\frac{z}{\theta}}\left(e^{-\frac{z}{\theta}} - e^{-\frac{y}{\theta}}\right)dy\,dz$ (дальше не получается)

3) $\underline{\tilde{\Theta}_3 = \bar{X}_{(2)}}$

$$\bar{X}_{(k)} \sim n\, C_{n-1}^{k-1} \frac{e^{-\frac{x}{\Theta}}}{\Theta} \left(1-e^{-\frac{x}{\Theta}}\right)^{k-1} \left(e^{-\frac{x}{\Theta}}\right)^{n-k}$$

$k=2$:

$$\bar{X}_{(2)} \sim n\, C_{n-1}^{1} \frac{e^{-\frac{x}{\Theta}}}{\Theta} \left(1-e^{-\frac{x}{\Theta}}\right)^{1} \left(e^{-\frac{x}{\Theta}}\right)^{n-2} = \frac{n(n-1)}{\Theta}\left(1-e^{-\frac{x}{\Theta}}\right)\left(e^{-\frac{x}{\Theta}}\right)^{n-1}$$

$$M[\bar{X}_{(2)}] = \frac{n(n-1)}{\Theta}\int_0^\infty x\left(1-e^{-\frac{x}{\Theta}}\right)e^{-\frac{x}{\Theta}(n-1)}dx = \frac{n(n-1)}{\Theta}\left(\int_0^\infty x e^{-\frac{x}{\Theta}(n-1)}dx + \int_0^\infty x e^{-\frac{x}{\Theta}n}dx\right) =$$

$$= \frac{n(n-1)}{\Theta}\left(\frac{\Theta^2}{(n-1)^2} - \frac{\Theta^2}{n^2}\right) = \frac{n\Theta}{n-1} - \frac{(n-1)\Theta}{n} = \frac{2n-1}{n(n-1)}\Theta \text{ - смещенная}$$

$$\Downarrow$$

$$\underbrace{\tilde{\Theta}_3' = \frac{n(n-1)}{2n-1}\bar{X}_{(2)}}\text{ - несмещенная}$$

$$M[\bar{X}_{(2)}^2] = \frac{n(n-1)}{\Theta}\int_0^\infty x^2\left(1-e^{-\frac{x}{\Theta}}\right)e^{-\frac{x}{\Theta}(n-1)}dx = \frac{n(n-1)}{\Theta}\left(\int_0^\infty x^2 e^{-\frac{x}{\Theta}(n-1)}dx - \int_0^\infty x^2 e^{-\frac{x}{\Theta}n}dx\right) =$$

$$= \frac{n(n-1)}{\Theta}\left(2\frac{\Theta^3}{(n-1)^3} - 2\frac{\Theta^3}{n^3}\right) = 2\Theta^2\frac{3n^2-3n+1}{n^2(n-1)^2}$$

$$D[\bar{X}_{(2)}] = M[\bar{X}_{(2)}^2] - M^2[\bar{X}_{(2)}] = 2\Theta^2\frac{3n^2-3n+1}{n^2(n-1)^2} - \frac{(2n-1)^2}{n^2(n-1)^2}\Theta^2 = \frac{2n^2-2n+1}{n^2(n-1)^2}\Theta^2$$

Для несмещенной: $D[\tilde{\Theta}_3'] = \frac{n^2(n-1)^2}{(2n-1)^2}D[\tilde{\Theta}_3] = \underline{\frac{2n^2-2n+1}{(2n-1)^2}\Theta^2}$

4) Сравнение эффективностей:

$D[\tilde{\Theta}_1] = \frac{\Theta^2}{n}$ $\qquad D[\tilde{\Theta}_1] < D[\tilde{\Theta}_3']$

$D[\tilde{\Theta}_3'] = \frac{2n^2-2n+1}{(2n-1)^2}\Theta^2$ $\qquad \frac{1}{n} < \frac{2n^2-2n+1}{(2n-1)^2}$

$D[\tilde{\Theta}_2]$ - ?

5) Н-во Крамера-Рао: $D[\tilde{\Theta}] \geq \frac{1}{nI(\Theta)}$

$$I(\Theta) = M\left[\left(\frac{\partial \ln p}{\partial \Theta}\right)^2\right]$$

$p = \frac{e^{-\frac{x}{\Theta}}}{\Theta}$ $\quad \ln p = -\frac{x}{\Theta} - \ln\Theta$ $\quad \frac{\partial \ln p}{\partial \Theta} = \frac{x}{\Theta^2} - \frac{1}{\Theta} \Rightarrow M\left[\left(\frac{\partial \ln p}{\partial \Theta}\right)^2\right] = M\left[\frac{x^2}{\Theta^4} - \frac{2x}{\Theta^3} + \frac{1}{\Theta^2}\right] =$

$$= \frac{1}{\Theta^4}M[x^2] - \frac{2}{\Theta^3}M[x] + \frac{1}{\Theta^2}$$

$I(\Theta) = \frac{2}{\Theta^2} - \frac{2}{\Theta^2} + \frac{1}{\Theta^2} = \frac{1}{\Theta^2} \Rightarrow \underline{D[\tilde{\Theta}] \geq \frac{\Theta^2}{n}}$

$\underline{D[\tilde{\Theta}_1] = \frac{\Theta^2}{n} \text{ - эффективная}}$ $\left(D[\tilde{\Theta}_1] < D[\tilde{\Theta}_2']\right)$ не посчитана

№2

1) Нахождение значений моментов и коэф. ассиметрии

$$p(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$\gamma = \frac{\mu_3}{\sigma_3} = \frac{\mu_3}{\mu_2^{3/2}}$$

$$\alpha_1 = \int_0^\infty x e^{-x} dx = -x e^{-x}\Big|_0^\infty + \int_0^\infty e^{-x} dx = -e^{-x}\Big|_0^\infty = 1$$

$$\mu_2 = \int_0^\infty (x-1)^2 e^{-x} dx = \int_0^\infty \overset{\lambda=1}{x^2 e^{-x}} dx - 2\int_0^\infty \overset{\lambda=1}{x e^{-x}} dx + \int_0^\infty e^{-x} dx =$$

$$= 2 - 2 + e^{-x}\Big|_0^\infty = 1$$

$$\mu_3 = \int_0^\infty (x-1)^3 e^{-x} dx = \int_0^\infty x^3 e^{-x} dx - 3\underbrace{\int_0^\infty x^2 e^{-x} dx}_{2} + 3\underbrace{\int_0^\infty x e^{-x} dx}_{1} - \underbrace{\int_0^\infty e^{-x} dx}_{1} =$$

$$= -x^3 e^{-x}\Big|_0^\infty + 3\underbrace{\int_0^\infty x^2 e^{-x} dx}_{2} - 6 + 3 - 1 = 2$$

$$\gamma = \frac{2}{1^{3/2}} = 2$$

2) Нахождение плотности распределения среднего арифметического элементов выборки

Классическое ЦПТ

$$\frac{\tilde{\alpha}_1 - \alpha_1}{\tilde{\sigma}}\sqrt{n} \sim N(0,1) \qquad p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$\tilde{\alpha}_1 = \frac{1}{n}\sum_{i=1}^n x_i \qquad \tilde{\alpha}_2 = \frac{1}{n}\sum_{i=1}^n x_i^2$$

$$\tilde{\sigma} = \sqrt{\tilde{\alpha}_2 - \tilde{\alpha}_1^2}$$

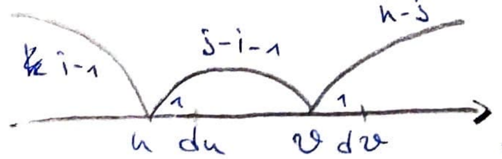
$$\tilde{\alpha}_1 - \alpha_1 \sim \frac{\tilde{\sigma}}{\sqrt{n}} N(0,1)$$

$$\tilde{\alpha}_1 - \alpha_1 \sim N\left(0, \frac{\tilde{\sigma}^2}{n}\right) \quad |\cdot(-1) - N\left(0, \frac{\tilde{\sigma}^2}{n}\right) \text{ — симметрично}$$

$$\alpha_1 \sim N\left(\tilde{\alpha}_1, \frac{\tilde{\sigma}^2}{n}\right)$$

$$p(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases} \qquad F(x) = \begin{cases} 1 - e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

3) Плотность совместного распр. i-го и j-го элементов вариационного ряда



$$P(X_{(i)} \in (u, u+du), X_{(j)} \in (v, v+dv)) =$$

$$= n \overset{p(u)}{P(u \leq \xi < u+du)} (n-1) \overset{p(v)}{P(v \leq \xi < v+dv)} \cdot C_{n-2}^{i-1} [\overset{F(u)}{P(-\infty < \xi < u)}]^{i-1} \cdot$$

$$\cdot C_{n-i-1}^{j-i-1} [\overset{F(v)-F(u)}{P(u \leq \xi < v)}]^{j-i-1} \cdot 1 \cdot [\overset{1-F(v)}{P(v \leq \xi < \infty)}]^{n-j} =$$

$$= n(n-1) C_{n-i-1}^{j-i-1} p(u) p(v) F^{i-1}(u) (F(v) - F(u))^{j-i-1} (1 - F(v))^{n-j}$$