N5

$$p(x) = \begin{cases} \frac{\theta-1}{x^\theta}, & x \geq 1 \\ 0, & x < 1 \end{cases} \qquad \theta > 1 \qquad F(x) = 1 - x^{1-\theta}$$

a) $L(\theta) = \prod p(x_i, \theta) = \prod_{i=1}^{n} \left(\frac{\theta-1}{x_i^\theta}\right) = (\theta-1)^n \left(\prod x_i\right)^{-\theta}$, ecлu вce $x_i \geq 1$ $\Updownarrow$ $x_{min} \geq 1$

$$\ln L(\theta) = n\ln(\theta-1) - \theta \sum_{i=1}^{n} \ln x_i$$

$$\frac{\partial \ln L}{\partial \theta} = \frac{n}{\theta-1} - \sum_{i=1}^{n} \ln x_i = 0 \Rightarrow \hat{\theta} = 1 + \frac{n}{\sum_{i=1}^{n} \ln x_i}$$

$$\left.\frac{\partial^2 \ln L}{\partial \theta^2}\right|_{\theta=\hat\theta} = -\left.\frac{n}{(\theta-1)^2}\right|_{\theta=\hat\theta} = -\left(\sum_{i=1}^{n} \ln x_i\right)^2 \cdot \frac{1}{n} < 0 \Rightarrow \max.$$

б) $\frac{t(\hat\theta) - t(\theta)}{\sigma}\sqrt{n} \rightsquigarrow N(0,1)$ $\qquad \sigma = \sqrt{\nabla^T t(\vec\theta)\, I^{-1}(\theta)\, \nabla t(\vec\theta)}$

$t(\theta) = x_{med}$: $F(x_{med}) = \int_{-\infty}^{x_{med}} \frac{\theta-1}{t^\theta}\, dt = x_{med}^{1-\theta} = \frac{1}{2}$

$\Downarrow$

$x_{med} = 2^{\frac{1}{\theta-1}}$

$$t'(\theta) = 2^{\frac{1}{\theta-1}} \cdot \ln 2 \cdot \frac{-1}{(1-\theta)^2} = -\ln 2 \cdot 2^{\frac{1}{\theta-1}} \cdot \frac{1}{(\theta-1)^2}$$

$$I(\theta) = M\left[\left(\frac{\partial \ln p}{\partial \theta}\right)^2\right]$$

$p = \frac{\theta-1}{x^\theta}$; $\ln p = \ln(\theta-1) - \theta \ln x$; $\frac{\partial \ln p}{\partial \theta} = \frac{1}{\theta-1} - \ln x$

$$I(\theta) = \int_1^{+\infty} \frac{\theta-1}{x^\theta}\left(\frac{1}{\theta-1} - \ln x\right)^2 dx = \frac{1}{(\theta-1)^2}$$

$$\frac{2^{\frac{1}{\hat\theta-1}} - 2^{\frac{1}{\theta-1}}}{\ln 2 \cdot 2^{\frac{1}{\hat\theta-1}} \cdot \frac{1}{(\hat\theta-1)^2} \cdot \underbrace{(\hat\theta-1)}_{\sqrt{I^{-1}}}}\sqrt{n} \rightsquigarrow N(0,1)$$

$$\parallel$$

$$\frac{(\hat\theta-1)\sqrt{n}}{\ln 2}\Big(1 - \underbrace{(x_{med})}_{2^{\frac{1}{\theta-1}}} 2^{-\frac{1}{\hat\theta-1}}\Big) \rightsquigarrow N(0,1)$$

$$u_{\frac{1-\beta}{2}} < \qquad < u_{\frac{1+\beta}{2}}$$

$$\frac{\ln 2}{(\hat\theta-1)\sqrt{n}} u_{\frac{1-\beta}{2}} < 1 - x_{med} \cdot 2^{-\frac{1}{\hat\theta-1}} < \frac{\ln 2}{(\hat\theta-1)\sqrt{n}} u_{\frac{1+\beta}{2}}$$

$$2^{\frac{1}{\hat\theta-1}}\left(1 - \frac{\ln 2}{(\hat\theta-1)\sqrt{n}} u_{\frac{1+\beta}{2}}\right) < x_{med} < 2^{\frac{1}{\hat\theta-1}}\left(1 - \frac{\ln 2}{(\hat\theta-1)\sqrt{n}} u_{\frac{1-\beta}{2}}\right)$$

c) $p(y) = \begin{cases} e^{1-y}, & y \geq 1 \\ 0, & y < 1 \end{cases}$

$\theta \sim \begin{cases} e^{1-\theta}, & \theta \geq 1 \\ 0, & \theta < 1 \end{cases}$

$$P(\theta | \vec{x}_n) = CLP(\theta)$$

$$\downarrow$$

$$\ln p(\theta | \bar{x}_n) = \ln C + \ln L + \ln p(\theta) \to \max$$

$$L = \frac{(\theta-1)^n}{\prod_{i=1}^{n} x_i^{\theta}}, \quad x_i \geq 1$$

$$\ln p(\theta | \bar{x}_n) = \ln c + n\ln(\theta-1) - \theta \sum \ln x_i + 1 - \theta \to \max$$

$$\frac{\partial \ln p(\theta | \bar{x}_n)}{\partial \theta} = \frac{n}{\theta-1} - 1 - \sum \ln x_i = 0 \Rightarrow \frac{n}{\theta-1} = 1 + \sum \ln x_i$$

$$\hat{\theta} = 1 + \frac{n}{1 + \sum \ln x_i}$$

Доверительный интервал

$$P(\theta | \vec{x}_n) = c \cdot e^{1-\theta} \cdot \frac{(\theta-1)^n}{(\prod x_i)^{\theta}}$$

$$\int_1^{+\infty} e^{1-\theta} \frac{(\theta-1)^n \cdot c}{(\prod x_i)^{\theta}} d\theta = 1$$

$$\int_1^{q_1} p(\theta | \vec{x}_n) d\theta = 0{,}025 \to q_1 = 5{,}75$$

$$\int_{q_2}^{+\infty} p(\theta | \vec{x}_n) d\theta = 0{,}025 \to q_2 = 8{,}05$$

d) $\frac{(\hat{\theta} - \theta)}{\sqrt{I^{-1}(\theta)}} \sqrt{n} \rightsquigarrow N(0,1)$

$$\frac{\hat{\theta} - \theta}{\hat{\theta} - 1} \sqrt{n} \rightsquigarrow N(0,1)$$

$$u_{\frac{1-\beta}{2}} < (\hat{\theta} - \theta) \frac{\sqrt{n}}{\hat{\theta} - 1} < u_{\frac{1+\beta}{2}}$$

$$\hat{\theta} - \frac{\hat{\theta}-1}{\sqrt{n}} u_{\frac{1+\beta}{2}} < \theta < \hat{\theta} - \frac{\hat{\theta}-1}{\sqrt{n}} u_{\frac{1-\beta}{2}}, \text{ где } \hat{\theta} = 1 + \frac{n}{\sum_{i=1}^{n} \ln x_i}$$