№6

болели один раз: 181 $\qquad H_0: P_{\text{заболеть}} = B(2)$

болели два раза: 9 $\qquad H_1: \overline{H_0}$

не болели: 10

$$F(x) = \sum_{k=0}^{n} C_n^k p^k (1-p)^{n-k} \theta(x-k)$$

$$p(x) = \sum_{k=0}^{n} C_n^k p^k (1-p)^{n-k} \delta(x-k)$$

$n=2$: $p(x) = \underbrace{\underset{=1}{C_2^0}\, 1 \cdot (1-p)^2}_{\text{не болел ни раза}} + \underbrace{\underset{=2}{C_2^1}\, p(1-p)}_{\text{один раз}} + \underbrace{\underset{=1}{C_2^2}\, p^2 \cdot 1}_{\text{два раза}}$

$L = \prod p_i = ((1-p)^2)^{\overset{\uparrow}{10}} \cdot (2p(1-p))^{\overset{\uparrow}{181}} \cdot (p^2)^{\overset{\uparrow}{9}}$

кол-во не болевших; кол-во болевших 1 раз; –//– 2 раза

$\ln L = 20 \ln(1-p) + 181 \ln(2p) + 181 \cdot \ln(1-p) + 18 \ln p = 201 \ln(1-p) + 199 \ln p + 181 \ln 2$

$$\frac{\partial \ln L}{\partial p} = -\frac{201}{1-p} + \frac{199}{p} = 0 \quad \Leftarrow \quad \downarrow_{\max}$$

$$\searrow \tilde{p} = \frac{199}{400}$$

Проверим, что это max $\left.\frac{\partial^2 \ln L}{\partial p^2}\right|_{p=\tilde{p}} < 0$:

$$\left.\frac{\partial^2 \ln L}{\partial p^2}\right|_{p=\tilde{p}} = -\frac{201}{(1-p)^2} - \left.\frac{199}{p^2}\right|_{p=\tilde{p}} = -\frac{201}{\left(\frac{201}{400}\right)^2} - \frac{199}{\left(\frac{199}{400}\right)^2} = -\frac{400^2}{201} - \frac{400^2}{199} < 0 \Rightarrow \max$$

$$\hat{\Delta} = \sum \frac{(n_i - n\, p_i(\hat{\theta}))^2}{n p_i(\hat{\theta})} = \frac{10 - 200\left(1 - \frac{199}{400}\right)}{200\left(1 - \frac{199}{400}\right)} + \frac{181 - 200\left(2 \cdot \frac{199}{400} \cdot \frac{201}{400}\right)}{200\left(2 \cdot \frac{199}{400} \cdot \frac{201}{400}\right)} + \frac{9 - 200\left(\frac{199}{400}\right)^2}{200\left(\frac{199}{400}\right)^2} \simeq 180$$

$\Delta \sim \chi^2(\underset{m}{3} - 1 - \underset{s}{1}) = \chi^2(1)$

$$\text{p-value} = P(\Delta \geq \hat{\Delta} \mid H_0) = \int_{180}^{+\infty} \underset{\text{pdf } \chi^2(1)}{g(t)}\, dt < 5 \cdot 10^{-3} < 0{,}05$$

$\Rightarrow$ отвергаем H_0

№7

	Заниж.	точн.	Завыш.	
1 партия	25	50	25	$\frac{1}{2}$
2 партия	52	41	7	$\frac{1}{2}$
	$\frac{77}{200}$	$\frac{91}{200}$	$\frac{32}{200}$	

$n=200$

H_0: номер партии и размер детали независимы

$$\Delta = \sum_{i,j} \frac{n_{ij} - nP_i q_j}{nP_i q_j} \sim \chi^2((\underset{3}{k}-1)(\underset{2}{m}-1))$$

$$\hat{\Delta} = \frac{\left(25 - 200\cdot\frac{77}{200}\cdot\frac{1}{2}\right)^2}{200\cdot\frac{77}{200}\cdot\frac{1}{2}} + \frac{\left(50 - 200\cdot\frac{91}{200}\cdot\frac{1}{2}\right)^2}{200\cdot\frac{91}{200}\cdot\frac{1}{2}} + \frac{\left(25 - 200\cdot\frac{32}{200}\cdot\frac{1}{2}\right)^2}{200\cdot\frac{32}{200}\cdot\frac{1}{2}} +$$

$$+ \frac{\left(52 - 200\cdot\frac{77}{200}\cdot\frac{1}{2}\right)^2}{200\cdot\frac{77}{200}\cdot\frac{1}{2}} + \frac{\left(41 - 200\cdot\frac{91}{200}\cdot\frac{1}{2}\right)^2}{200\cdot\frac{91}{200}\cdot\frac{1}{2}} + \frac{\left(7 - 200\cdot\frac{32}{200}\cdot\frac{1}{2}\right)^2}{200\cdot\frac{32}{200}\cdot\frac{1}{2}} = 20,3$$

$\Delta \sim \chi^2(2)$

$$\text{p-value} = P(\Delta \geq \hat{\Delta} \mid H_0) = \int_{20,3}^{+\infty} \underset{\text{pdf } \chi^2(2)}{g(t)}\,dt = 3,67\cdot 10^{-5} < 0,05 \Rightarrow \text{отвергаем } H_0$$

№8

	2	3	4	5	n
1 поток	33	43	80	144	300
2 поток	39	35	72	154	300

H_0: потоки однородны

H_1: $\overline{H_0}$

$P(\text{„2"}) = \frac{72}{600}$ $\quad P(\text{„3"}) = \frac{78}{600}$ $\quad P(\text{„4"}) = \frac{152}{600}$ $\quad P(\text{„5"}) = \frac{298}{600}$

$$\tilde{\Delta}_1 = \frac{\left(33 - 300\cdot\frac{72}{600}\right)^2}{300\cdot\frac{72}{600}} + \frac{\left(43 - 300\cdot\frac{78}{600}\right)^2}{300\cdot\frac{78}{600}} + \frac{\left(80 - 300\cdot\frac{152}{600}\right)^2}{300\cdot\frac{152}{600}} + \frac{\left(144 - 300\cdot\frac{298}{600}\right)^2}{300\cdot\frac{298}{600}} = 1,05$$

$$\hat{\Delta}_2 = \frac{\left(39 - 300\cdot\frac{72}{600}\right)^2}{300\cdot\frac{72}{600}} + \frac{\left(35 - 300\cdot\frac{78}{600}\right)^2}{300\cdot\frac{72}{600}} + \frac{\left(72 - 300\cdot\frac{152}{600}\right)^2}{300\cdot\frac{152}{600}} + \frac{\left(154 - 300\cdot\frac{298}{600}\right)^2}{300\cdot\frac{298}{600}} = 1,03$$

$\hat{\Delta} = \hat{\Delta}_1 + \hat{\Delta}_2 = 2,08$ $\qquad \Delta \sim \chi^2((m-1)(k-1)) = \chi^2(3)$

$$\text{p-value} = P(\Delta \geq \hat{\Delta} \mid H_0) = \int_{2,08}^{+\infty} \underset{\text{pdf}(\chi^2(3))}{g(t)}\,dt = 0,57 > \alpha = 0,05 \Rightarrow \text{нет оснований отвергнуть } H_0$$

№9

m	0	1	2	3	4	5	6	7	8	9
n	5	8	6	12	14	18	11	6	13	7

$n = 100$

$-\infty$ 0,5 1,5 2,5 3,5 4,5 5,5 6,5 7,5 8,5 $+\infty$

а) $H_0: \xi \sim R(0,9)$

$H_1: \bar{H}_0$

$$\tilde{\Delta} = \frac{(5-100\cdot\frac{1}{10})^2}{100\cdot\frac{1}{10}} + \frac{(8-10)^2}{10} + \frac{(6-10)^2}{10} + \frac{(12-10)^2}{10} + \frac{(14-10)^2}{10} + \frac{(18-10)^2}{10} + \frac{(11-10)^2}{10} + \frac{(6-10)^2}{10} + \frac{(13-10)^2}{10} + \frac{(7-10)^2}{10} = 16{,}4$$

$\Delta \sim \chi^2(1-9)$

$$\text{p-value} = P(\Delta \geq \tilde{\Delta} \mid H_0) = \int\limits_{16,4}^{+\infty} \underbrace{g(t)}_{\text{pdf } \chi^2} dt \approx 0{,}06 > \alpha \Rightarrow \text{нет оснований отвергнуть } H_0$$

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$$\tilde{\Delta} = \sqrt{n}\, \max\left(|\hat{F}(x_i-0) - F(x_i)|, |F(x_i+0) - F(x_i)|\right) \approx \underline{1{,}6}$$

посчитали в блокноте на github

$$K(x) = P(\Delta < x) = 1 + 2\sum_{k=1}^{\infty}(-1)^k e^{-2k^2x^2} \quad (0;+\infty)$$

$$\text{p-value} = P(\Delta \geq \tilde{\Delta} \mid K) = 1 - P(\tilde{\Delta}) = 1 - \left(1 + 2\sum_{k=1}^{\infty}(-1)^k e^{-2k^2}\right) = \underline{0{,}012} < 0{,}05 \text{ – отвергаем } H_0$$

б) $\xi \sim N(a, \sigma^2)$

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-a)^2}{2\sigma^2}}$$

$$p_1 = \int\limits_{-\infty}^{0,5} p(x)dx \quad p_2 = \int\limits_{0,5}^{1,5} p(x)dx \ldots \quad p_{10} = \int\limits_{8,5}^{+\infty} p(x)dx$$

$$L = p_1^5 \cdot p_2^8 \cdot \ldots \cdot p_{10}^7 \to \max$$

численно получили $a = 4{,}78$, $\sigma = 2{,}7$

$$\tilde{\Delta} = \sum_{i=1}^{n} \frac{m_i - np_i(\hat{a},\hat{\sigma})}{np_i(\hat{a},\hat{\sigma})} = 9{,}82$$

$$\Delta \sim \chi^2(\underset{m}{10} - \underset{\hat{a}\hat{\sigma}}{2} - 1) = \chi^2(7) \qquad P(\Delta \geq \tilde{\Delta} \mid H_0) = \int\limits_{9,82}^{+\infty} \underbrace{g(t)}_{\text{pdf}(\chi^2(7))} dt \approx 0{,}2 > \alpha$$

$\Rightarrow$ нет оснований отвергнуть H_0

Критерий Колмогорова

p-value $= 0{,}55 > \alpha \Rightarrow$ нет оснований отвергнуть H_0

№10

$H_0: p_0(x) = \begin{cases} 1, & x \in (0,1) \\ 0, & x \notin (0,1) \end{cases}$

$H_1: p_1(x) = \begin{cases} \dfrac{e^{1-x}}{e-1}, & x \in (0,1) \\ 0, & x \notin (0,1) \end{cases}$

а) $n = 1$ α

$$l = \frac{L_1}{L_0} = \frac{p_1}{p_0} = \frac{e}{e-1} e^{-x} \geq C$$

$$\Downarrow$$

$$e^{-x} \geq B$$

$$\Downarrow$$

$G: x \leq A$ – критическая область

$$P(x \leq A | H_0) = \alpha$$

$$\int_0^A \underset{p_0}{1}\, dx = \alpha \Rightarrow A = \alpha \Rightarrow G: x \leq \alpha$$

$$\alpha_1 = P(H_1 | H_0) = \alpha$$

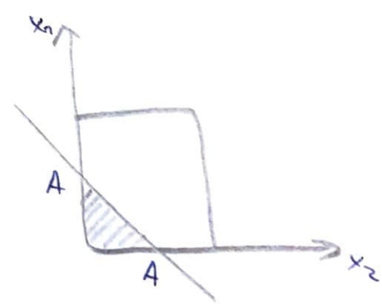
$$W = P(x \leq A | H_1) = \int_0^\alpha \frac{e}{e-1} e^{-x} dx = \frac{e}{e-1} e^{-x} \Big|_\alpha^0 = \frac{e}{e-1}(1 - e^{-\alpha})$$

$$\alpha_2 = 1 - W = 1 - \frac{e}{e-1}(1 - e^{-\alpha})$$

б) $n = 2$

$$l = \frac{L_1}{L_0} = \frac{\left(\frac{e}{e-1}\right)^2 \cdot e^{-x_1} \cdot e^{-x_2}}{1 \cdot 1} \geq C \Rightarrow e^{-x_1 - x_2} \geq B$$

$G: x_1 + x_2 \leq A$



$$P(x_1 + x_2 \leq A | H_0) = \frac{A^2}{2} = \alpha \Rightarrow A = \sqrt{2\alpha}$$

$$\Downarrow$$

$$G: x_1 + x_2 \leq \sqrt{2\alpha}$$

$$\alpha_1 = P(H_1 | H_0) = \alpha$$

$$W = P(x_1 + x_2 \leq A | H_1) = \int_0^A dx_1 \int_0^{A - x_1} \left(\frac{e}{e-1}\right)^2 e^{-x_1} e^{-x_2} dx_2 = \frac{e^2}{(e-1)^2}\left(1 - e^{-\sqrt{2\alpha}} - \sqrt{2\alpha}\, e^{-\sqrt{2\alpha}}\right)$$

$$\alpha_2 = 1 - W$$

c) n, α

$$L = \frac{L_1}{L_0} = \prod_{i=1}^{n} p_1(x_i) \geq C \qquad P(L \geq C \mid H_0) = \alpha$$

$$\ln L = \sum_{i=1}^{n} \ln p_1(x_i)$$

$$\eta = \ln p_1(\xi) = \ln \frac{e^{1-\xi}}{e-1} = \ln \frac{e}{e-1} - \xi$$

ЦПТ

$$\frac{\sum \eta_i - n M\{\eta_i\}}{\sqrt{n D\{\eta_i\}}} \rightsquigarrow N(0,1)$$

$$H_0: \; M\{\eta_i\} = M\left[\ln \frac{e}{e-1} - x_i\right] = \ln\frac{e}{e-1} - \overset{M[x_i]}{\overset{\shortparallel}{\frac{1}{2}}}$$

$$D\{\eta_i\} = D\left\{\ln \frac{e}{e-1} - x_i\right\} = D[x_i] \overset{H_0}{\overset{d}{=}} \frac{1}{12}$$

$$P(\ln L \geq \ln C \mid H_0) = P\left(\underbrace{\frac{\ln L - n\left(\ln\frac{e}{e-1} - \frac{1}{2}\right)}{\sqrt{n \cdot \frac{1}{12}}}}_{\sim N(0,1)} \geq \frac{\ln C - n\left(\ln \frac{e}{e-1} - \frac{1}{2}\right)}{\sqrt{n \cdot \frac{1}{12}}}\right) = \alpha$$

$$\Downarrow$$

$$\frac{\ln C - n\left(\ln\frac{e}{e-1} - \frac{1}{2}\right)}{\sqrt{n \cdot \frac{1}{12}}} = u_{1-\alpha} \Rightarrow \ln C = n\left(\ln\frac{e}{e-1} - \frac{1}{2}\right) + u_{1-\alpha}\sqrt{\frac{n}{12}}$$

$G: \ln L \geq \ln C$

$$\ln L = \sum_{i=1}^{n} \eta_i = \sum\left(\ln\frac{e}{e-1} - x_i\right) = n \cdot \ln\frac{e}{e-1} - n\bar{x} \geq \underbrace{\sqrt{\frac{n}{12}}\, u_{1-\alpha} + n \ln \frac{e}{e-1} - \frac{n}{2}}_{\ln C}$$

$$\Leftarrow$$

$$G: \; \bar{x} \leq \frac{1}{2} - \frac{u_{1-\alpha}}{\sqrt{12n}}$$

$\alpha_1 = \alpha$

$\alpha_2 = 1 - W$

$$W = P\left(\bar{x} \leq \frac{1}{2} - \frac{u_{1-\alpha}}{\sqrt{12n}} \;\Big|\; H_1\right)$$

Теорема Фишера

$$\frac{\bar{x} - M[\xi]}{\sqrt{D[\xi]}}\sqrt{n} \rightsquigarrow N(0,1)$$

H_1: справедлива

$$\Downarrow$$

$$M\{\xi\} = M\left[\frac{e}{e-1} e^{-x}\right] = \int_0^1 \frac{e}{e-1} x e^{-x} dx = \frac{e-2}{e-1}$$

$$M\{\xi^2\} = \int_0^1 \frac{e}{e-1} x^2 e^{-x} dx = \frac{2e-5}{e-1}$$

$$D\{\xi\} = \frac{e^2 - 3e + 1}{e-1}.$$

$$W = P\left(\overbrace{\frac{\bar{x} - M[\xi]}{\sqrt{D[\xi]}}\sqrt{n}}^{\rightsquigarrow N(0,1)} \leq \frac{\frac{1}{2} - \frac{u_{1-\alpha}}{\sqrt{12n}} - M[\xi]}{\sqrt{D[\xi]}}\sqrt{n}\right) =$$

$$= \int_{-\infty}^{\frac{\frac{1}{2} - \frac{u_{1-\alpha}}{\sqrt{12n}} - M[\xi]}{\sqrt{D[\xi]}}} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx$$

д) $G: x_{min} < C$

$H_0: \quad F_0(x) = \begin{cases} 0, & x<0 \\ x, & x\in[0;1] \\ 1, & x>1 \end{cases} \qquad F_{min}(x) = 1-(1-F(x))^n$

$P(x_{min} \le C \mid H_0) = F_{min}(c) = \alpha$

$1-(1-c)^n = \alpha \Rightarrow c = 1-\sqrt[n]{1-\alpha} \qquad \alpha_1 = \alpha$

$W = P(x_{min} \le c \mid H_1)$

$H_1: F(x) = \int\limits_0^x \frac{e}{e-1} e^{-t} dt = \frac{e}{e-1}(1-e^{-x}) \; (0,1)$

$W = F_{min}(c) = 1-\left(1-\frac{e}{e-1}(1-e^{-c})\right)^n = 1-\left(1-\frac{e}{e-1}\left(1-e^{\sqrt[n]{1-\alpha}-1}\right)\right)^n$

$\alpha_2 = 1-W = \left(1-\frac{e}{e-1}\left(1-e^{\sqrt[n]{1-\alpha}-1}\right)\right)^n$

№11

$H_0:$ „1": $\frac{1}{4}$; „2": $\frac{1}{4}$; „3": $\frac{1}{6}$; „4": $\frac{1}{3}$ $\qquad p_0(x) = \frac{1}{3}\delta(x-4) + \frac{1}{6}\delta(x-3) + \frac{1}{4}\delta(x-2) + \frac{1}{4}\delta(x-$

$H_1:$ „1": $\frac{1}{4}$; „2": $\frac{1}{4}$; „3": $\frac{1}{4}$; „4": $\frac{1}{4}$ $\qquad p_1(x) = \frac{1}{4}\delta(x-4) + \frac{1}{4}\delta(x-3) + \frac{1}{4}\delta(x-2) + \frac{1}{4}\delta(x-$

$n=2$

$L_1 = (p_1(x))^2 \qquad L_0 = (p_0(x))^2$

$l = \frac{L_1}{L_0}$

$G: P(l \ge c \mid H_0) = \alpha = 0{,}2$

$H_0:$

2-ой бросок \ 1-ый бросок	1	2	3	4
1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{24}$	$\frac{1}{12}$
2	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{24}$	$\frac{1}{12}$
3	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{36}$	$\frac{1}{18}$
4	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{9}$

$H_1:$ все значения „ $\frac{1}{16}$

Тогда при разных c:

c	$\frac{9}{16}$	$\frac{3}{4}$	1	$\frac{9}{8}$	$\frac{3}{2}$	$\frac{9}{4}$
α_1	1	$\frac{8}{9}$	$\frac{5}{9}$	$\frac{11}{36}$	$\frac{7}{36}$	$\frac{1}{36}$
W	1	$\frac{15}{16}$	$\frac{11}{16}$	$\frac{7}{16}$	$\frac{5}{16}$	$\frac{1}{16}$

$\alpha_1 > 0{,}2$ $\qquad$ $\alpha_1 < 0{,}2$

определяется суммой событий при H_0

—‖— H_1

выбираем $W \to max$

$W = \frac{5}{16}, \; c = \frac{3}{2}$

крит. область: выпадение хотя бы одной „3"

$l = \frac{L_1}{L_0}$:

2-ой бросок \ 1-ый бросок	1	2	3	4
1	1	1	$\frac{3}{2}$	$\frac{3}{4}$
2	1	1	$\frac{3}{2}$	$\frac{3}{4}$
3	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{9}{4}$	$\frac{9}{8}$
4	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{9}{8}$	$\frac{9}{16}$

№12

$n=3$ $N(a, \sigma^2)$: $-1,11;\ -6,10;\ 2,42$

$H_0: a=0$ $H_1:$ $a>0$
$a<0$
$a\neq 0$

$\bar{x} = -1,596$

$S^2 = \frac{1}{n-1}\sum (x_i - \bar{x})^2 = \frac{1}{2}\left[(-1,11-\bar{x})^2 + (-6,1-\bar{x})^2 + (2,42-\bar{x})^2\right] \approx 18,325$

теорема Фишера

$\frac{\bar{x}-\theta_1}{S}\sqrt{n} \sim t(n-1)$

$\frac{-1,596-\theta_1}{\sqrt{18,325}}\sqrt{3} \sim t(2)$

$\hat{\Delta} = \frac{-1,596-0}{4,281}\sqrt{3} = -0,646$ p-value $= p(|\Delta| \geq |\hat{\Delta}| \mid H_0) =$

$= 2\int_{0,646}^{+\infty} g(t)\,dt = 0,585 > 0,05$ – нет оснований отвергнуть H_0

$g(t)$ – pdf $t(2)$

$H_1: a>0$ – отвергаем

$H_2: a<0$: p-value $= P(\Delta \leq -|\hat{\Delta}|) > 0,25 > 0,05$ – нет оснований отвергнуть H_0

№13

x_n, y_m – независимые случайные выборки из $N(a, \sigma_x^2)$ и $N(b, \sigma_y^2)$, $\sigma_x^2 = 2$, $\sigma_y^2 = 1$

$x = \{-1,11;\ -6,10;\ 2,42\} \to \hat{a} = -1,596$

$y = \{-2,29;\ -2,91\} \to \hat{b} = -2,6$ $\Rightarrow$ отвергаем $a<b$

$\hat{\Delta} = \frac{\bar{x}-\bar{y}}{\sqrt{\frac{\sigma_x^2}{n_x} + \frac{\sigma_y^2}{n_y}}} \to N(0,1)$ $\hat{\Delta} = \frac{-1,596+2,6}{\sqrt{\frac{2}{3}+\frac{1}{2}}} = 0,92$ $H_0: a=b$

1) $H_1: a>b$ p-value $= P(\Delta \geq |\hat{\Delta}|) = \int_{0,92}^{+\infty} g(t)\,dt = 0,17 > 0,05$ $\Rightarrow$ нет оснований отвергнуть H_0

$g(t)$ – pdf $N(0,1)$

3) $H_2: a\neq b$ p-value $= P(|\Delta| \geq |\hat{\Delta}|) = 2\int_{0,92}^{+\infty} g(t)\,dt = 0,34 > 0,05$

$\Downarrow$

нет оснований отвергнуть H_0