Stock portfolio optimization subject to risk constraints with simple management

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Abstract

Any individual can invest capital in the financial markets, many people are afraid to do so because of their ignorance of the markets. While everybody wants to invest his capital optimally, a portfolio can be managed in many different ways as its structure, composed of financial assets, is very unpredictable since it relies on financial markets. Our goal is to study the fundamental aspects of portfolio management by presenting different notions of risk and expected return, so that an individual can understand the basics notions of a portfolio through a simple problem. To do so, we define the notion of portfolio and we apply optimization methods with constraints to our problem in order to find the best strategy. These solutions have allowed us to determine the portfolio that has the best expected return with a given risk and also the one that minimizes the risk with a given return. Although the randomness of financial markets still frightens people, this application will be useful for people who want to invest in financial markets and understand how they should do it. This paper is built on mathematical modeling based on past data, the next step of this research is to use temporal series to approximate the incoming values of the stocks and then apply the strategy described in this article.

Introduction

We have recently experienced a financial crash due to the decline in activity and investors' fear of Covid-19's effects on the economy. After a dramatic decline, capital markets are now slowly rebounding. During the previous decades, increased access to the Internet has also made investing capital in financial markets easier.

It is therefore the right moment to invest and create an asset portfolio, but there are still many individuals who do not know how to invest, what a portfolio is and how to optimize its profitability with minimal risk.

A portfolio is a grouping of financial assets such as stocks, bonds, commodities or currencies. Its structure is quite simple: it is composed of an initial capital brought by the individual and the proportions allocated to each asset the individual invests in. In this paper we will only focus on investing in stocks.

If an agent wants to invest in the financial markets, he first needs to define his objectives. They can be an expected return related to a risk taken, the time of return, etc. Then he can decide on his investment strategy.

The objectives of the present paper are the following: first we want to mathematically define a portfolio and explain how it works over on a simple period, with a precise strategy . Second we want to find the optimum investment strategy for a set of assets, on a fixed period based on different risk measures. To do so, we will work on historical data from 2010 to 2018 (in which we have different indicators for each day of the period) on a set of five American stocks. We will not take into account the dividends that could be distributed by the company we invest in, even if for the majority of individual investors this is one of the main reasons to invest in financial assets.

Therefore, in a first part we will focus on defining mathematically, the return and wealth of a portfolio π. We will then model our first optimisation problem with a simple investment strategy on a period and find an optimum portfolio for the given set of assets, knowing that at the same time, there may have been assets with higher returns on the market. We will introduce the term Value at Risk, a risk measure, which will help us to quantify the risk associated with a portfolio. Then we will show that several important hypotheses made in the literature are not correct. Throughout this paper, we will provide graphs obtained with numerical simulations to confirm theoretical results and explain the best strategies on how investments should be made.

1 Problem and notations

In this first part we will describe the mathematical notations and formulas that we will use throughout this article to describe the financial problem, using I.B. Tahar Lecture notes [3].

- **Market Model:**

The market is composed of d risky assets, denoted $S^i, i = 1, ..., d$, the risky assets and S^0 the riskless asset.
There are plenty of ways of investing among these assets. In order to simplify comprehension and equations, we suppose that assets are perfectly divisible and that there are no transaction or order costs.
Let P_t^i denote the price of the asset i at time t.

- **Notion of profitability:**

First we will suppose that agents can only access the market at time $t = 0$ and $t = T$, $(T > 0)$.
Hence, at time $t = 0$, an investor creates a portfolio composed of risky and riskless assets and awaits time $t = T$ to obtain his return.

Therefore, the profitability R^i of an asset i becomes:

$$R^i = \frac{P_T^i}{P_0^i} - 1, \quad \forall i = 1...d \tag{1}$$

As R^i is a random variable[1], we will compute $m^i = E[R^i]$, the expected profitability of the asset i.

Let $R = (R^1, ..., R^d)'$ be the vector of assets' profitability and denote: $M = E[R] = (m^1, .., m^d)'$ its expectation. Let us denote $\sigma^{ij} = cov(R^i, R^j)$, then the covariance Matrix is: $\Sigma = (\sigma^{ij})_{i=1..d}$

- **Investment Strategy:**

We suppose that the agent has w_0 initial wealth and that he wants to split it between one riskless asset and different risky assets. Let π^i be the proportion of wealth invested in the risky asset i.

Then $\pi = (\pi^1, .., \pi^d)$ is **the investment strategy**. Indeed, since the agent invests all his money:

$$\pi^0 = 1 - \sum_{i=1}^{d} \pi^i$$

Where π^0 represents the proportion allocated to the riskless asset.
We will therefore consider the following strategy:
Investing our whole capital at time $t = 0$ in risky

[1] We already know the value of the riskless asset S^0 at time T, hence, R_T^0 is a constant deterministic variable

assets and studying the return at time $t = T$, where T may be one day, two months, one year, etc. This first simple investment strategy will allow us to work on simpler problems and have a better idea of the concepts. If $T = 8$ years for example, this is a long-term investment strategy an individual would make to invest their capital without needing to examine markets everyday.

Finding the best investment strategy over a fixed period will then allow us to extend the program to many other periods to replicate an individual behaviour. Let's say for example every month, the individual invests more capital, buys and sells financial positions.

- **Portfolio:**

Hence, a portfolio is characterized by w the money invested and π the way it is split among the different assets. We call it a two funds investment. For example,

$$\pi = [0.1, 0.5, 0.4] \quad w = 100$$

is a portfolio composed of three assets.
An important notion to take into consideration is the **Short Sale Limit**, which is the ability to go into debt on our assets. Meaning that the agent can sell an asset that he does not own and buy it afterwards. In this paper, we fix the maximum proportion short sale limit at 2.
Let A denote the set of all admissible portfolios, in our case, $A = \left\{\pi \in [-2, 3]^d, \sum_{i=0}^{d} \pi^i = 1\right\}$

- **Portfolio Value and Profitability:**

As T is fixed and known, and the agent does not buy or sell assets during this period, the wealth W of the portfolio π at time T with the initial capital w is equal to:

$$W_T^{w,\pi} = w(1 - \sum_{i=1}^{d} \pi^i)(1 + R^0) + w(\sum_{i=1}^{d} \pi^i(1 + R^i)) \quad (2)$$

We can use vectors to do the same operations: with $\mathbf{1} = (1, .., 1)' \in R^d$,
$\pi = (\pi^1, .., \pi^d) \in R^d$ and $R = (R^1, ..., R^d)'$:

$$\begin{aligned} W_T^{w,\pi} &= w(1 - \pi'\mathbf{1})(1 + R^0) + w\pi'(1 + R)) \\ &= w(1 + R^0 + \pi'(\mathbf{1} + R - \mathbf{1} - R^0\mathbf{1}) \\ &= w(1 + R^0 + \pi'(R - R^0\mathbf{1})) \end{aligned} \quad (3)$$

$W_T^{w,\pi}$ is a random variable and we can compute its expectation : $E[W_T^{w,\pi}] = w(1 + R^0 + \pi'(E[R] - R^0\mathbf{1}))$ which we can rewrite as:

$$E[W_T^{w,\pi}] = w(1 + R^0 + \pi'(M - R^0\mathbf{1})) \quad (4)$$

And its Variance: $Var(W_T^{w,\pi}) = w^2 Var(\pi' R)$

$$Var(W_T^{w,\pi}) = w^2 \pi'\Sigma\pi \quad (5)$$

Let's compute its return:

$$\begin{aligned} R^\pi = \frac{W_T^{w,\pi}}{W_0^{w,\pi}} - 1 &= \frac{w(1 + R^0 + \pi'(M - R^0\mathbf{1}))}{w} \\ &= (1 + R^0 + \pi'(R - R^0\mathbf{1}) \end{aligned} \quad (6)$$

Remark: the return does not depend on the initial wealth w.
As R^π is a random variable we can also compute its expectations and its variance:

$$E[R^\pi] = 1 + R^0 + \pi'(M - R^0\mathbf{1})$$
$$Var(R^\pi) = \pi'\Sigma\pi$$

In this paper, we will consider investing only in risky assets. Therefore,

$$E[R^\pi] = \pi' M \quad (7)$$
$$Var(R^\pi) = \pi'\Sigma\pi \quad (8)$$

Now that we have defined the expected return and the variance of a portfolio, we want to find an optimal one considering these two terms. An optimal portfolio can be defined as the portfolio that maximizes the return with a given variance or minimizes the variance with a given return. In fact, the variance of a portfolio represents somehow the risk. The higher the variance is, the more the portfolio can have an obtained return different than the expected one.

2 Optimisation of a portfolio

2.1 Maximization of the expected return with a given variance

The agent aims to find the optimal portfolio π among the set of all possible portfolios A. This agent wants to maximize the expected return $E[R^\pi]$ and minimize the risk $Var(R^\pi)$. Where $R^\pi = \pi' R$, is the unknown return of the portfolio π.

2.1.1 Definition of an efficient frontier

Let us introduce the notion of **efficient portfolio**.
Let π^1 be a portfolio of A.
If $\forall \pi^2 \in A$ such that $Var(R^{\pi^1}) = Var(R^{\pi^2})$,

$$E[R^{\pi^1}] \geq E[R^{\pi^2}]$$

and if $\forall \pi^3 \in A$ such that $E[R^{\pi^1}] = E[R^{\pi^3}]$

$$Var(R^{\pi^1}) \leq Var(R^{\pi^3})$$

then, π^1 is said to be an efficient portfolio. It is not dominated by any other portfolio of A.

Now that we have introduced an efficient portfolio, we can define the **efficient frontier**, the set of the couples

$$\Big\{ (\mathrm{E}[\mathrm{R}^\pi], Var(R^\pi)) \text{ , with } \pi \text{ an efficient portfolio } \Big\}$$

In the same way, we introduce the notion of **inefficient portfolio** which this time has the minimum expected return for a given variance and the **inefficient frontier**:

$$\Big\{ (\mathrm{E}[\mathrm{R}^\pi], Var(R^\pi)) \text{ , with } \pi \text{ an inefficient portfolio } \Big\}$$

In order to determine the efficient frontier we can solve the two following equivalent problems:

$$(P_{ff\sigma}) \text{: } \begin{cases} sup_{\pi \in A} E[R^\pi] \\ s.t. \ var(R^\pi) = \sigma^2 \end{cases}$$

$$(P_{ffm}) \text{: } \begin{cases} inf_{\pi \in A} \ var(R^\pi) \\ s.t. \ E[R^\pi] = m \end{cases}$$

where σ^2 and m are given.

Thus, an agent can find his optimal portfolio depending on his preferences. In fact, he could prefer to minimize risk by fixing the variance and then maximizing the profitability by solving the first problem. Otherwise, he could expect a profit and minimize the variance with the second problem.

After introducing these two problems, we are now going to solve the first one. We will discuss the equivalence between both of them in section 2.1.4.

2.1.2 Maximum expected return solution of the optimization problem

In this optimisation problem, we assume that the agent cannot invest in the riskless asset

Finding a portfolio on the efficient frontier means determining the best portfolio for the given assets.
In order to determine the efficient frontier we will solve $(P_{ff\sigma})$ which is, with no riskless asset, equivalent to the following convex problem:

$$(P_\sigma) \Leftrightarrow \begin{cases} sup_{\pi \in A} \pi' M = -inf_{\pi \in A} - \pi' M \\ s.t. \\ \pi' \Sigma \pi = \sigma^2 \\ \pi' \mathbf{1} = 1 \end{cases}$$

The constraint $\pi'\mathbf{1} = 1$ represents the fact that the agent invests all his money in the risky assets.
First, let us introduce the reals:

$$a = \mathbf{1}' \Sigma^{-1} \mathbf{1} = \left\| \mathbf{1} \right\|^2_{\Sigma^{-1}}$$

$$b = \mathbf{1}' \Sigma^{-1} M =< \mathbf{1}, M >_{\Sigma^{-1}}$$

Constraints:

$$X = \Big\{ \pi \in R^d, \ \pi' \Sigma \pi = \sigma^2 \text{ and } \pi' \mathbf{1} - 1 = 0 \Big\}$$

X is closed and bounded and the objective function $-\pi' M$ is continuous, therefore according the Weierstrass theorem, the objective function has a global minimum and maximum on X.

If such a portfolio π exists in X then we can deduce a condition on σ^2 for which the optimisation problem has a solution:

$$a\sigma^2 = \|\mathbf{1}\|^2_{\Sigma^{-1}} \|\Sigma \pi\|^2_{\Sigma^{-1}} \geq (< \mathbf{1}, \Sigma \pi >)^2 = 1$$
$$\Rightarrow \sigma^2 \geq \tfrac{1}{a}$$

We know now that σ^2 has to be greater than $\frac{1}{a}$. Let us solve the optimisation problem P_σ.

Constraints qualification:

We denote $h_1(\pi)$ and $h_2(\pi)$ the constraint of the problem P_σ.

$$h_1(\pi) = \pi' \Sigma \pi - \sigma^2 \quad \Rightarrow \quad \nabla h_1(\pi) = 2 \Sigma \pi$$

$$h_2(\pi) = \pi' \mathbf{1} - 1 \quad \Rightarrow \quad \nabla h_2(\pi) = \mathbf{1} = \begin{pmatrix} 1 \\ . \\ . \\ 1 \end{pmatrix}$$

Lagrangian function:

$$\mathcal{L}(\pi, \lambda, \mu) = -\pi' M + \lambda(\pi' \Sigma \pi - \sigma^2) + \mu(\pi' \mathbf{1} - 1)$$

Karush–Kuhn–Tucker conditions:

$$\begin{cases} \nabla \mathcal{L}(\pi, \lambda, \mu) = 0_{R^d} \\ h_1(\pi) = 0_{R^d} \\ h_2(\pi) = 0_{R^d} \end{cases}$$

$$\Leftrightarrow \begin{cases} -M + 2\lambda\Sigma\pi + \mu\mathbf{1} = 0_{R^d} \\ \pi'\Sigma\pi - \sigma^2 = 0_{R^d} \\ h_2(\pi) = 0_{R^d} \end{cases}$$

$$\Leftrightarrow \begin{cases} \pi^\star = \frac{1}{2\lambda}\Sigma^{-1}(M - \mu\mathbf{1}) \\ \lambda = \pm\frac{1}{2\lambda}\left\|M - \mu\mathbf{1}\right\|_{\Sigma^{-1}} \\ \mu = \frac{1}{a}(b - 2\lambda) \end{cases}$$

Using the second and the third equations, we obtain:

$$\lambda^\pm = \pm\frac{1}{2}\frac{\left\|M - \frac{b}{a}\mathbf{1}\right\|_{\Sigma^{-1}}}{\sqrt{\sigma^2 - \frac{1}{a}}}$$

and then

$$\mu^\mp = \frac{1}{a}(b \mp \frac{\left\|M - \frac{b}{a}\mathbf{1}\right\|_{\Sigma^{-1}}}{\sqrt{\sigma^2 - \frac{1}{a}}})$$

Second order conditions:

Let us compute the Hessian matrix of the Lagrangian:

$$H_{\mathcal{L}}(\pi, \lambda, \mu) = 2\lambda\Sigma$$

This matrix is definite positive if and only if $\lambda > 0$, then we choose λ^+ which is positive to find the minimum of our problem. Thus, the optimal portfolio π which maximizes the profitability with a fixed variance $\sigma^2 \geq \frac{1}{a}$ is:

$$\pi^+(\sigma) = \pi^{\lambda^+} = \frac{1}{a}\Sigma^{-1}\mathbf{1} + \sqrt{\sigma^2 - \frac{1}{a}}\Sigma^{-1}\frac{M - \frac{b}{a}\mathbf{1}}{\left\|M - \frac{b}{a}\mathbf{1}\right\|_{\Sigma^{-1}}} \tag{9}$$

Finally, the expected profitability can be written as:

$$\overline{m}(\sigma) = \pi^{+\prime}M = \frac{b}{a} + \sqrt{\sigma^2 - \frac{1}{a}}\|M - \frac{b}{a}\mathbf{1}\|_{\Sigma^{-1}} \tag{10}$$

Solving this problem also allow us to find the minimum profitability reached with $\pi^-(\sigma)$ by using the others Lagrange multipliers (λ^-, μ^+). The portfolio $\pi^-(\sigma)$ is on the inefficient frontier. The minimum expected profitability is:

$$\underline{m}(\sigma) = \pi^{-\prime}M = \frac{b}{a} - \sqrt{\sigma^2 - \frac{1}{a}}\|M - \frac{b}{a}\mathbf{1}\|_{\Sigma^{-1}}$$

2.1.3 Efficient frontier visualisation

Once the problem (P_σ) is solved, we compute the minimum and maximum expected profitability for a given σ. Then we can create a graphic representation of all possible (admissible) portfolios and show at the same time the efficient and inefficient frontier. This area can be defined as:

$$\left\{(m, \sigma),\ \sigma^2 \geq \frac{1}{a},\ \underline{m}(\sigma) \leq m \leq \overline{m}(\sigma)\right\}$$

In order to show what this area looks like, we downloaded historical data for five stocks: Apple, Amazon, Microsoft, Walmart, and Nasdaq. We assume that we can invest in the Nasdaq index as an asset. The data goes from year 2010 to 2018.

We need to compute the vector M (composed of all expected returns) as well as the associated covariance matrix. To do so, we compute the daily returns for each asset and substitute them into a dataset as shown in Figure 1.

	AAPL	AMZN	MSFT	WMT	NASD
0	0.271751	-1.724775	1.077727	0.911794	0.610615
1	-0.102521	0.944322	0.356561	-0.739510	0.062409
2	-1.590631	-1.745918	-0.356214	0.130841	-0.286859
3	-0.552539	-1.522608	-0.587653	-0.223386	0.085287
4	0.798865	2.267161	1.254950	-0.187157	1.087579

Figure 1: Extract from the dataset composed of the daily returns

One row of the dataset represents one day and each column one asset. The figures we see in Figure 1 are the returns of the asset for different days i.
We estimate the expected return of the asset i denoted $E[R_i]$ by taking the mean of all daily returns of the asset i throughout all the daily values.
Then we compute the reals a and b described in the previous part and use the equations we have seen previously. Thus we obtain the graph shown in Figure 2.

We can easily see the the maximum, as well as the minimum expected profitability for a given σ. Moreover, we observe that the higher σ we choose, the higher the maximum expected profitability is; this result is logical.
All the portfolios will be in the grey area.

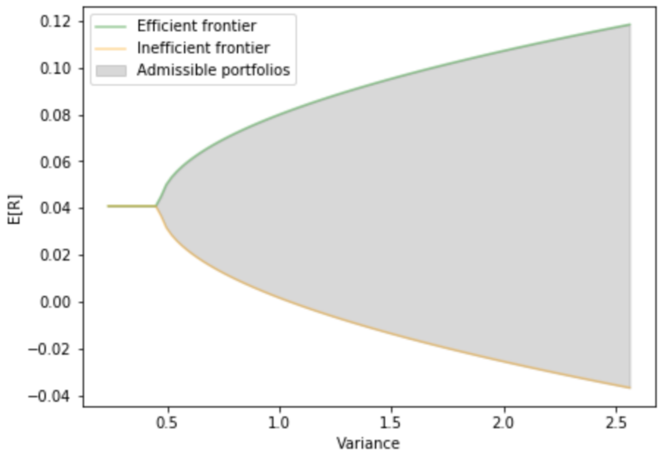

Figure 2: Graphic display of the frontiers

2.1.4 Equivalence between $P_{ff\sigma}$ and P_{ffm}

This optimization problem can also be solved by fixing an expected return and then minimizing its variance: in other words, an agent desiring a return. Thus we will try to minimise the risk for such a return.

$$(P_{ffm}) : \begin{cases} \inf_{\pi \in A} \ var(R^\pi) \\ s.t. \ E[R^\pi] = m \end{cases}$$

We use the same assumptions as those made in part 3.2. The reals a and b are still equal to:

$$a = \mathbf{1}'\Sigma^{-1}\mathbf{1} = \left\|\mathbf{1}\right\|^2_{\Sigma^{-1}}$$

$$b = \mathbf{1}'\Sigma^{-1}M =< \mathbf{1}, M >_{\Sigma^{-1}}$$

Constraints:

The constraints are the same as those given previously.

$$X = \left\{\pi \in R^d, \ \pi' M = m \ and \ \pi'\mathbf{1} - 1 = 0\right\}$$

where X is closed and bounded and the objective function $var(R^\pi)$ is continuous, then according to the Weierstrass theorem, $var(R^\pi)$ has a global minimum and maximum on X.

Using the same method as before, we obtain the following results:

Karush–Kuhn–Tucker conditions:

$$\begin{cases} 2\Sigma\pi + \lambda M + \mu\mathbf{1} = 0 \\ \pi' M - m = 0 \\ \pi'\mathbf{1} - 1 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \pi^\star = -\frac{1}{2}\Sigma^-1(\lambda^* M - \mu^*\mathbf{1}) \\ \lambda^\star = \frac{-2+2m\frac{a}{b}}{b-\frac{a}{b}||M||^2_{\Sigma-1}} \\ \mu^\star = -2\frac{(1+\frac{a}{b}m)\|M\|_{\Sigma-1}}{b^2-a\|M\|_{\Sigma-1}} \end{cases}$$

The portfolio $\pi^\star$ minimizes the variance σ^2 with a given m. Then, we show the equivalence between the problems $P_{ff\sigma}$ and P_{ffm}. To do so, we replace m by $m(\sigma)$ (computed in the previous part) and we obtain:

$$\pi^\star(\sigma) = \frac{1}{a}\Sigma^{-1}\mathbf{1} + \sqrt{\sigma^2 - \frac{1}{a}}\Sigma^{-1}\frac{M - \frac{b}{a}\mathbf{1}}{\left\|M - \frac{b}{a}\mathbf{1}\right\|_{\Sigma^{-1}}}$$

equal to the previous portfolio $\pi^+(\sigma)$.

Hence, the two problems have the same solution $\pi^\star$ that is unique.

Numerical simulation of the equivalence:

Algorithm

- **Choose** an arbitrary variance σ_1^2
- **Using** $P_{ff\sigma}$, compute the maximum expected return $m(\sigma_1)$ (portfolio with variance σ_1^2)
- **Using** P_{ffm}, compute the lowest variance σ_2^2 (portfolio with an expected return of $m(\sigma_1)$)

if $\sigma_1 = \sigma_2$ **then**
    The two optimum portfolios are the same.
    The problems are equivalent.
else
    $P_{ff\sigma}$ and P_{ffm} are not equivalent
end

Using this algorithm, we numerically found that $\sigma_1 = \sigma_2$ which confirmed the mathematical analysis that we made before.

This first optimisation method was very simple as the risk was only represented by the variance, but in the next part, we will introduce risk measures to quantify the risk. This application will be useful for the agent who wants to invest as it will allow him to reduce the risk of losing his initial capital.

2.2 Maximization of the expected return with Risk measures as a risk constraint

2.2.1 Value at Risk definition

We have previously seen that the risk associated with a portfolio can be represented by the variance of its return.

In this section we will focus on a new function which represents the risk: **the Value at Risk**. It is an homogeneous risk measure. We will write it $V@R_\lambda$ as it relies on a confidence level 1-λ.

- **Explanation:**

In general, the $V@R_\lambda$ associated with a portfolio is the Maximum loss value that we can expect with a confidence 1-λ.

In our case, we work with returns, therefore, the $V@R_\lambda$ associated with a portfolio corresponds to the opposite of **the minimum return** we could have with a confidence 1-λ.

- **Mathematical explanation:**

Let X be a monetary random variable. The Value at Risk of X is a monetary measure of the risk of the financial position X.
According to *Mr.Réveillac*'s lecture [2], it is defined as:

$$V@R_\lambda(X) = -q_X^+(\lambda) = q_{-X}^-(1-\lambda) \tag{11}$$

With,

$$q_X^+(\lambda) = \sup_{z \in R}\{P[X < z] \leq \lambda\}$$
$$q_X^-(\lambda) = \inf_{z \in R}\{P[X \leq z] \geq \lambda\}$$

In our case, the monetary random variable is R^π and as we have seen in part 1, R^π is the return of a portfolio π over a period T.
We do not know the distribution function or the Cumulative Distribution Function (CDF) of R^π. Therefore, we will use the following strategy to obtain an approximation of R^π's CDF:

Let R_j^π be the return of the portfolio π for the day j. Then $\Re^\pi = (R_1^\pi, ..., R_T^\pi)$ is the vector of each day's return for the portfolio π.

We consider each R_j^π as a realisation of one R^π, and we use it to compute the distribution of R^π, so that we can calculate the quantile:

$$V@R_\lambda(R^\pi) = -q_{R^\pi}^+(\lambda) = q_{-R^\pi}^-(1-\lambda) \tag{12}$$

With,

$$q_{R^\pi}^+(\lambda) = \sup_{z \in R}\{P[R^\pi < z] \leq \lambda\}, \quad \text{the } \lambda \text{ quantile}$$

Example:
If we have $V@R_\lambda(R^\pi) = 5$ it means that the minimum expected return is equal to -5. Thus, we need to save $w * (\frac{5}{100})$ in order not to be bankrupt with a confidence level of $1-\lambda$.
However, if we have $V@R_\lambda(R^\pi) = -2$ it means that the minimum expected return is equal to 2, which means we can spend $w*(\frac{2}{100})$ without being bankrupt with a confidence level of $1-\lambda$.

2.2.2 Evolution of the V@R in terms of the confidence level

In this section, we try to determine which values take a risk measure like the V@R. Knowing the approximate size of the V@R can be very useful to determine how much money we should save in order not to be bankrupt.

To represent how the Value at Risk evolves in terms of the confidence level we simulate four random portfolios in a market of four assets (Apple, Amazon, Microsoft and Walmart). Then we apply the V@R to each of these portfolios for different confidence levels $1-\lambda$ as shown in Figure 3:

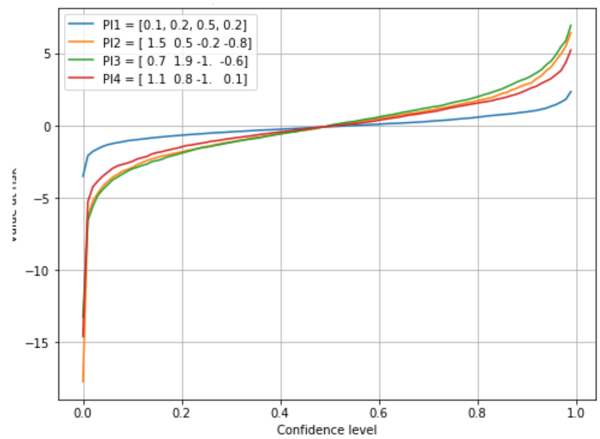

Figure 3: Evolution of $V@R_\lambda$ for different portfolios

This figure shows that the strategy $PI1$ (blue line) is less risky than the strategy $PI2$ (orange line) because the V@R is always lower for a confidence level between 80-100% for example. All of these lines are coherent, if we want to be sure (high confidence level) not to be bankrupt we need to save a positive amount of money. Moreover if we take a very low confidence level, we could even spend money because the V@R is negative.

2.2.3 Tail Value at Risk definition

The Value at Risk is a good risk measure, but it has limitations as it does not adequately take into account the distribution behaviour. It just gives a quantile. In order to solve this limitation we introduce the Tail Value at Risk defined as:

$$TV@R_\lambda(R^\pi) = E[R^\pi | R^\pi \leq V@R_\lambda(R^\pi)] \qquad (13)$$

We can see its evolution in regards of on the confidence level $1-\lambda$ for the same four random portfolios with the same assets as shown in Figure 4:

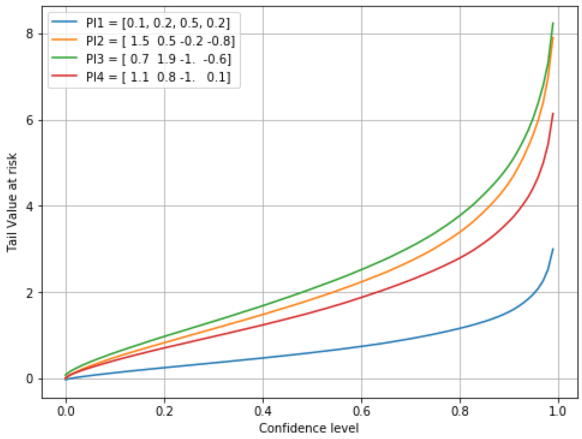

Figure 4: Evolution of the $TVaR_\lambda$ for different portfolios

However, this risk measure adds more constraints as it is bigger than the previous one; it has the same behaviour among the set of portfolios. We will therefore only work with the V@R.

In the next part, we will see that under Gaussian return assumption, the Value at Risk of a portfolio's return can be expressed simply.

2.2.4 Optimisation with Gaussian Return Hypothesis

If the $R^\pi \sim N(\mu, \sigma^2)$, we have:

$$V@R_\lambda(R^\pi) = -\mu + \sigma\Phi^{-1}(1-\lambda) \qquad (14)$$

where Φ is the CDF of the $N(0,1)$.
Therefore we can solve analytically the problem with constraints which is:

$$(P_{52}) : \begin{cases} \sup_{\pi \in A} E[R^\pi] \\ s.t. \\ V@R_\lambda(R^\pi) \leq v_0 \\ \pi'\mathbf{1} = 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} \sup_{\pi \in A} \pi' M \\ s.t. \\ -\pi' M + \pi'\Sigma\pi \times \Phi^{-1}(1-\lambda) \leq v_0 \\ \pi'\mathbf{1} = 1 \end{cases}$$

Let us denote $\Phi^{-1} = \Phi^{-1}(1-\lambda)$
The optimum portfolio $\pi^\star$ is:

$$\pi^\star = \frac{\Sigma^{-1}(M(\varphi+1) - \nu\mathbf{1}}{2\varphi\Phi^{-1}}$$

With φ and ν the Lagrange multipliers:

$$\varphi = \frac{b + \nu a}{2\Phi^{-1}}$$

$$\nu = \frac{b + \frac{2ab\gamma}{(2\Phi^{-1}-b)^2} \pm \sqrt{\Delta}}{2(a - \frac{a^2\gamma}{(2\Phi^{-1}-b)^2})}$$

With γ and Δ equal to:

$$\gamma = 4v_0\Phi^{-1} + ||M||$$

$$\Delta = b^2 - a||M|| + \frac{\gamma a(8b^2 + a||M||)}{(2\Phi^{-1} - b)^2}$$

These results are found thanks to a heavy computation of optimization methods. This is why we used the reals γ and Δ to simplify reading and also did not replace the value of ν in φ.

In the case where the Gaussian hypothesis is not verified, we can derive such an expression of the value at risk and therefore need to use another strategy.

2.2.5 Optimisation Problem without Gaussian Return Hypothesis

In this part we cannot derive the Value at Risk function as we cannot find an explicit formula for the quantile. Therefore we will use a numerical stochastic optimisation approach to solve the problem. We can now try to find the optimum portfolio which satisfies the following equations:

$$(P_5) : \begin{cases} \sup_{\pi \in A} E[R^\pi] \\ s.t.\ \pi'\mathbf{1} = 1 \\ V@R_\lambda(R^\pi) \leq v_0 \end{cases}$$

Indeed, as the V@R models the risk, we do not have to take into account σ in our optimisation problem.

- **Numerical Solving of the problem**
  Algorithm:

In short, this algorithm is designed to find the global minimum of a function, adding a part of randomness in order to avoid local minimums.

Let $C(\pi)$ be the cost function of the optimisation problem, $C(\pi) = -E[R^{\pi}]$. Our goal is to minimise this function under both constraints. But we do not want to try all the admissible portfolios to find the solution. To find the minimum of this function, we will go from one random neighbour π to another, if some cost conditions are respected. Two neighbours are two portfolios "mathematically close" and we have to define what relation two portfolios neighbours.

The best way to describe how we choose neighbours is through an example: imagine that we are in a four dimension case (i.e. we invest in four risky assets), here are two neighbours π_1 and π_2:

$$\pi_1 = (0.2, 0.1, 0.3, 0.4)$$
$$\pi_2 = (0.2, 0.2, 0.3, 0.3)$$

Only two values change, the difference is equal to a discretization step (here 0.1). One component increases by 0.1 and the other decreases by the same value.

Moreover, we verify that the Value at Risk of the random portfolio is lower than v_0. With these conditions respected, π_1 and π_2 are neighbours portfolios.

Metropolis Hasting Algorithm

Initialization

- Create randomly π, the starting portfolio
- Compute C(π)
- $k = 0$

Main

while $k < iter_{max}$ **do**
- **Compute** π_{next} a random neighbour of π
- **Compute** $C(\pi_{next})$
- **Generate** U following $\mathcal{U}_{[0,1]}$
- **if** $U < min(1, e^{\frac{C(\pi)-C(\pi_{next})}{h/ln(k)}})$
  - $\pi \leftarrow \pi_{next}$
- $k+=1$

Return π_k

The difficult part in this algorithm (Inspired from C.Breton's paper [1]) is to fix the parameter h corresponding theoretically to the highest value that we would have to surpass to escape from a local minimum.

After presenting the different methods, we to define many numerical parameters in order to simulate portfolios on a graph. We will use these graphs to represent the optimum portfolio for each method.

3 Simulation Methods

Let us now explain how we conduct the simulations that will be displayed in the results part.

3.1 Simulation parameters

In order to compute different portfolio simulations, we first have to define several parameters such as the number of assets. Thus, the simulations will be conducted with the following parameters:

- A dimension
- A discretization step
- A short sale limit

The dimension will be defined by the number of assets we want to study.
Next, we need a discretization step that will allow us to adjust the number of portfolios we want to generate. To illustrate, the following Figure 5 and 6 are extracts of portfolios sets (in dimension 2) discretized with a different step:

```
   ⋮
[0.4 0.6]
[0.5 0.5]
[0.6 0.4]
[0.7 0.3]
   ⋮
```

Figure 5: Dicretizations with step of 0.1

```
    ⋮
[0.39 0.61]
[0.4  0.6 ]
[0.41 0.59]
[0.42 0.58]
    ⋮
```

Figure 6: Dicretizations with step of 0.01

Finally, we need to set a short sales limit. This means we accept short sale but with a limit. For example, if we set this limit to 1, it means that the agent cannot go into debt for more than 100% of their initial wealth. The introduction of this limit leads us to consider another limit for the proportion of wealth invested in an asset.
We could simulate a portfolio with a large proportion for one asset and very small one for the other which would be equal to the short sale limit.

Such a portfolio would have a huge variance as we would invest a lot of money in only one asset and go into debt on all the others which is not relevant to consider in our simulations.
We can also visualize different levels of risk on the same graph.

3.2 Simulation of Risk Levels with the V@R

In order to visualize the Value at Risk (V@R) of different portfolios, we decided to define four risk levels. The risk level is based on the V@R value.

We first estimated the V@R distribution of the expected return on a given market (d risky assets). We denote $(q_{0,25}, q_{0,5}, q_{0,75})$ the related quantiles.

Let $\pi \in R^d$ be a portfolio of the specific market we consider and R^π its return. Then the risk level of π is defined as:

$$\begin{cases} Risk1 & \text{if} \quad V@R(R^\pi) < q_{0.25}; \\ Risk2 & \text{if} \quad V@R(R^\pi) \in [q_{0.25}, q_{0.5}[; \\ Risk3 & \text{if} \quad V@R(R^\pi) \in [q_{0.5}, q_{0.75}[; \\ Risk4 & \text{if} \quad V@R(R^\pi) \geq q_{0.75}; \end{cases}$$

After assigning four colors to the four risk levels, we could display the portfolios as before but with four different colors. This basic display will allow the user to see with a simple risk level which portfolio he could choose to invest in.

4 Results

4.1 Graphic representation of portfolios

In this section we will compute the different methods explained before. We are also going to show the best portfolios depending on the risk chosen.

4.1.1 Portfolio composed of two risky assets

We take historical data from Apple and Microsoft. We first compute the expected return of each asset by taking the mean of all daily returns as well as the covariance matrix. Then, as before, we display the efficient and inefficient frontiers for these two specific assets.

- **Classic Representation:**

We simulate many portfolios with a short sale limit equal to 2 and a discretization step of 0.1 in order to visualize them on the graph as shown in Figure 7.

Each cross represents one portfolio with a variance and an expected return. We notice that all the

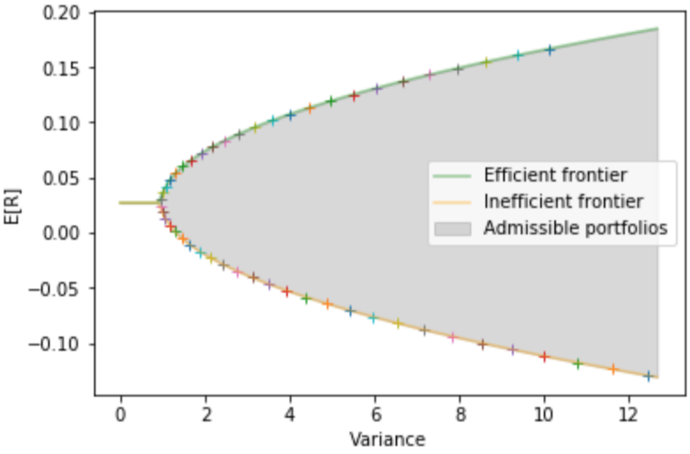

Figure 7: Portfolio composed of two assets

crosses are either on the efficient frontier or on the inefficient frontier, none appear within the admissible portfolios area. We can observe the same phenomenon with different assets.
In fact, with a dimension 2 we can easily express the return in terms of the variance and we obtain the same equations as the efficient and inefficient frontiers.

- **Representation with levels of V@R:**

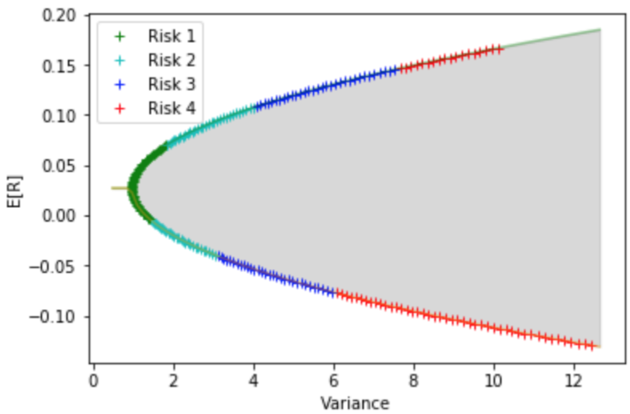

Figure 8: Portfolio composed of two assets

We observe in Figure 8 that the more the variance increases, the higher the risk level is. This is logical because a large variance represents a risk more important. On our graph, the V@R is continuous on the efficient and inefficient frontier. Moreover, we can link an interval of variance to a risk level.

Next, we simulate different portfolios with three assets in order to see crosses inside the admissible portfolios area.

4.1.2 Portfolio Composed of three risky assets

We include Amazon's data from the same period and run the same simulation. We investigate how relevant the short sale is by completing another simulation without authorizing it (short sale limit equal to zero). We now have three assets: Microsoft, Apple and Amazon.

- **Classic Representation:**

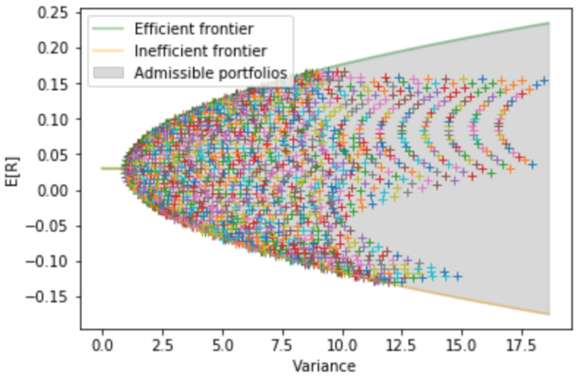

Figure 9: Short Sale limit = 1

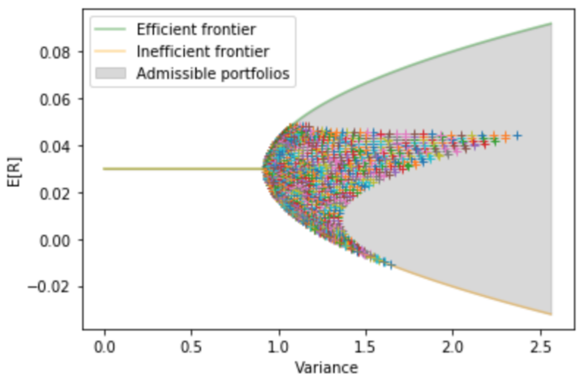

Figure 10: Short Sale limit = 0

As shown in Figures 9 and 10, we observe different portfolios distributed in the admissible portfolios area and not only on the frontiers. Moreover, the set of portfolios is spread over almost all the admissible area. We note that the expected return is very limited when the authorized short sale is low. For example, no portfolio with a variance equal to 2 reaches the efficient frontier whereas before there was one portfolio which could. After many simulations we concluded that the more we increase the short sale limit, the higher the return we can expect. This can also be shown with the result of the optimization problem. Thus, we can expect many different returns but with a high associated variance.

- **Representation with levels of V@R :**

We decide to investigate the case with a short sale limit equal to one.

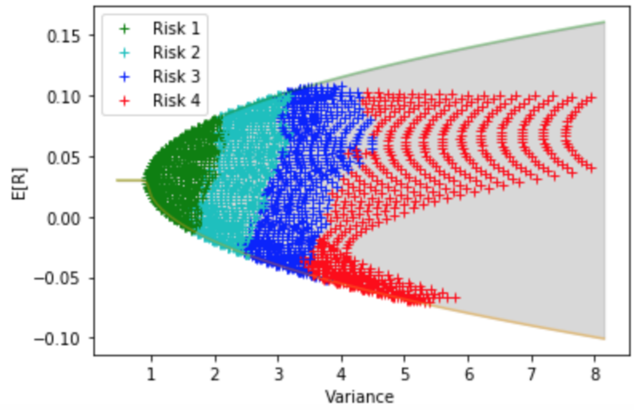

Figure 11: Risk levels for the [Microsoft, Apple, Amazon] portfolio

As shown in Figure 11, we simulated all the possible portfolios and took quantiles to define the risks, therefore, the number of crosses for each color is the same. Regarding the dispersal, the red crosses (risk level 4) correspond to the extreme values of the V@R. In fact these values are very different as shown by how the red crosses are spread over a large area on the graph.

Once again, the V@R is continuous over all the admissible area. We can observe different "slices" of colors. Because these slices (vertical) can almost be associated with variance interval, we conclude that there is an equivalence relation between the variance and the V@R.

In fact with these 3 assets, we find a portfolio with risk level 3 (blue crosses) that has a greater expected return than any other portfolio of risk level 4 (red crosses). In fact the maximum expected return with risk level 3 is 0.107 compared to 0.102 for risk level 4. This is unexpected because we would assume that by increasing the risk level we could expect an higher return, but this is not true in our case because of the short sale limit. Moreover, the maximum expected return for risk level 2 is almost the same as risk level 3 (0.101 compared to 0.107).

Therefore, without applying any optimisation algorithm, we consider that the best strategy, displayed in Figure 12, with the second level of risk is the following portfolio (black point):

This strategy means that we should have gone into debt for 90% of our initial wealth in Apple. This would allow us to invest 170% of our wealth in Microsoft. The remaining 20% is invested in Amazon.

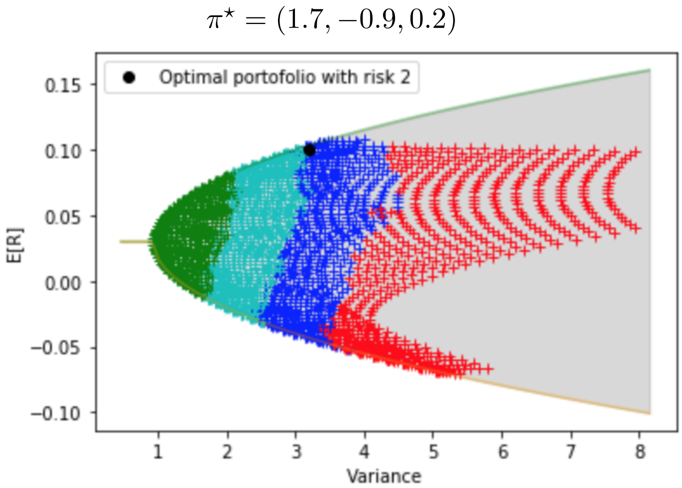

Figure 12: Optimal portfolio for a given risk level

4.1.3 Portfolio composed of five risky assets

Let us now consider the graph with five risky assets. We add Walmart and the Nasdaq index to our dataset which is now composed of : Apple, Microsoft, Amazon, Walmart and the Nasdaq index.

- **Classic Representation:**

Again, we can determine two limits depending on the authorized short sale, as shown in Figures 13 and 14, it is therefore relevant to authorise a short sale limit of at least one, considering the results of the simulations:

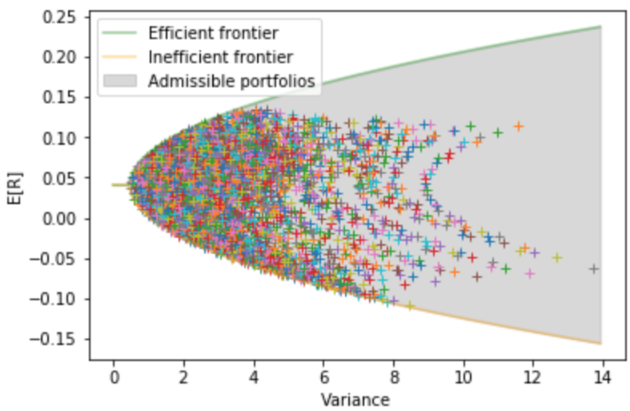

Figure 13: Short Sale = 1

- **Representation with levels of V@R:**

The simulation shown in Figure 15 has been done with a short sale limit equal to 1.

We observe the same behaviour as previously, except that the borders between the different risks is less distinct.

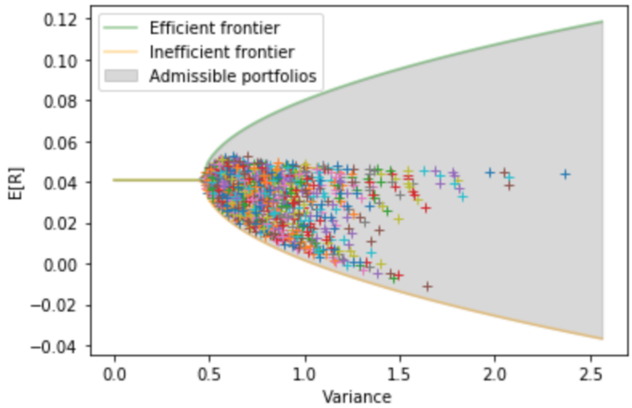

Figure 14: Short Sale = 0

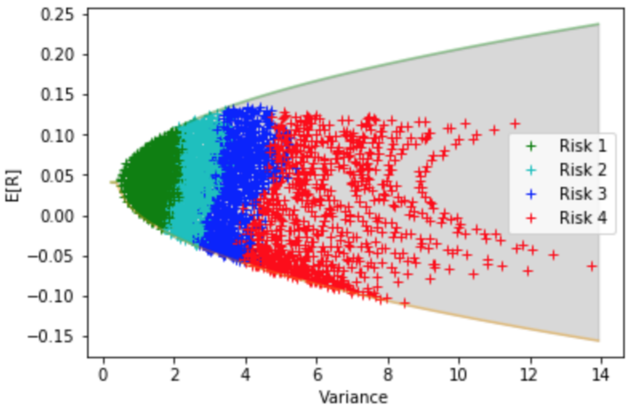

Figure 15: Risk levels for the Apple, Microsoft, Amazon, Walmart and the Nasdaq index portfolio

Conclusion:

V@R is a coherent risk measure compared to the variance. The more the variance increases, the more the risk defined by the V@R increases.

Since the borders on the graphs are almost linear, in our case a fixed V@R corresponds to a line of the expected return in terms of the variance. We will study this further in part 4.4

We have also seen that because of a short sale limit, the expected return is bounded. Therefore, the higher risk is not reliable and it is less profitable than lower risks.
We now want to find the best portfolio, among all those displayed above, for the different risks level measured.

4.2 Best portfolio for a given variance as a risk constraint

Let us recall the optimization problem we want to solve:

$$(P_{ff\sigma}): \begin{cases} sup_{\pi \in A} E[R^\pi] \\ s.t.\ var(R^\pi) = \sigma^2 \end{cases}$$

We have seen graphs depicting portfolio behaviors. We can easily locate a portfolio that satisfies a given variance and an expected return. Thanks to the efficient frontier we have a global overview of all the expected returns available on the market with such assets. For a given variance, most of the portfolios on the efficient frontier require short sales.
In our simulation, if for a given variance there is not any efficient portfolio displayed, it means that we need to increase the short sale limit to obtain such a portfolio.
However, this process considerably increases the variance, in other words the risk.
We can display the optimal portfolio on the graph using the optimization results seen in part 2.1.2.

Optimum Portfolio with a given variance:

If we want to invest in Microsoft, Apple and Amazon, with a maximum variance of 2 then, the best portfolio (displayed with a black point) is:

$$\pi^\star = (1.195, -0.565, 0.370)$$

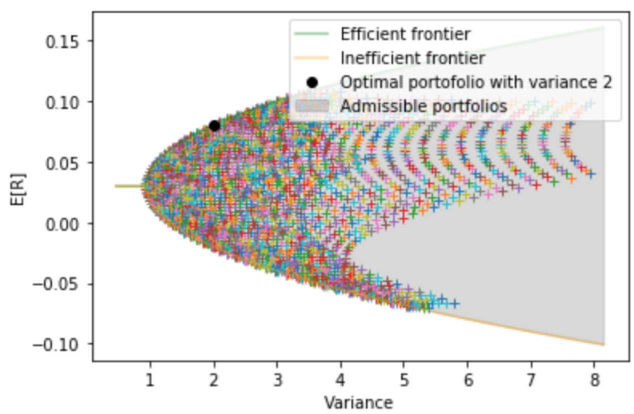

Figure 16: Optimum portfolio with Microsoft, Apple and Amazon assets

The graph shown in Figure 16 is coherent with the theory, the black point is on the efficient frontier and has a variance equal to 2.
The best portfolio for this set of assets uses short sales. We could have guessed this behaviour with the two previous figures because we can see in Figure 14 that there is no portfolio with variance equal to 2 on the efficient frontier.
The proportions allocated to the assets are very precise, in reality we cannot make such investments if the initial wealth w is too small. In fact, 119.5% of the original wealth does not necessarily match an integer number of stocks.

Conclusion:

To conclude, short sale allows us to reach the efficient frontier with a given variance while also allowing us to take more risk (increased variance) in order to obtain this specific high expected return. In other words, we consider the short sale as a risk constraint. We have seen the optimisation solution with the variance as a risk constraint. Let us now explain what happens when with the risk measures as constraints of the optimisation problem to find the most efficient portfolio.

4.3 Best Portfolio with the V@R Constraints

4.3.1 Gaussian Return Hypothesis

An important hypothesis made in the resolution of this problem is the Gaussian return. We test this hypothesis and prove that it is not accurate.

Hypothesis 1: simple Returns $R_1, .., R_T$ are independent and identically distributed, with $R_j \sim \mathcal{N}(\mu, \sigma)$

In order to test this hypothesis, we downloaded historical data from Apple and ploted it in Figure 17:

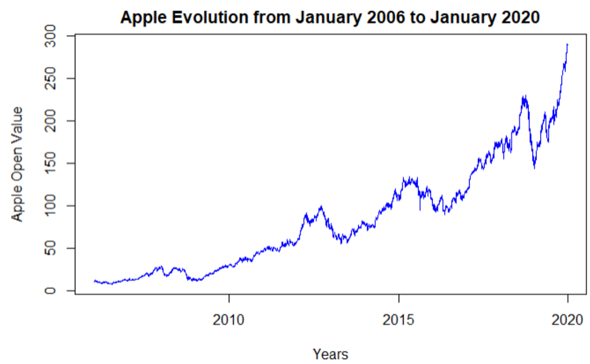

Figure 17: Evolution of Apple's Stock Value

The data consists of open values which correspond to the value of the asset when the market opens. Close Values correspond to the value of the index when the market closes. Therefore we can input the Profitability for a period of one day i defined as:

$$R_j = \frac{CloseValue_j - OpenValue_j}{OpenValue_j} \tag{15}$$

We plot every daily return in Figure 18, and then compute a histogram in Figure 19 to visualize the distribution.
It is a centered distribution but it does resemble a Gaussian one.
Using Q-Q plot, and examining the tail of the curve

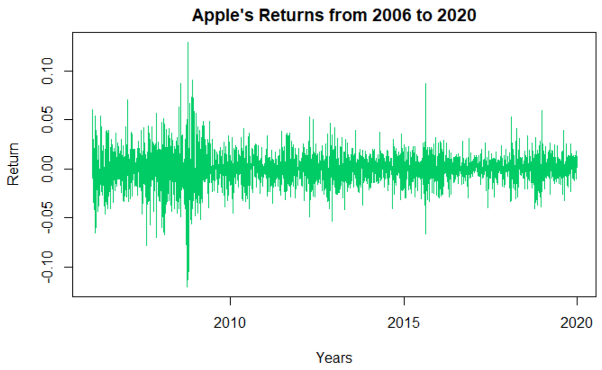

Figure 18: Evolution of Apple's Stock Returns

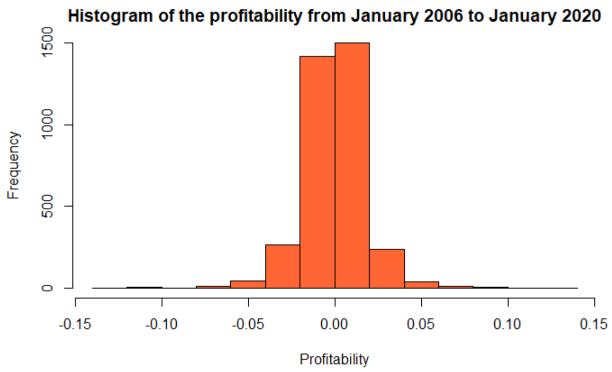

Figure 19: Distribution of Apple's Stock Returns

(as shown in Figure 20), we assume that it is not a Gaussian Distribution:

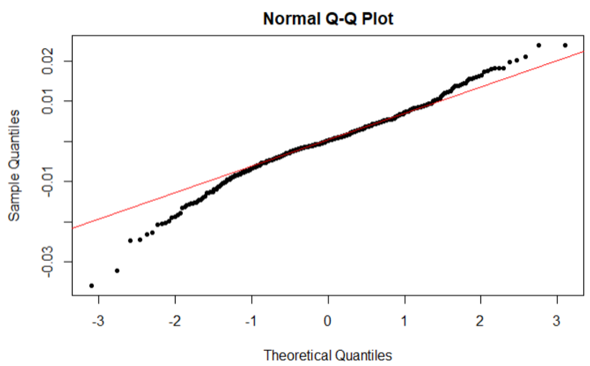

Figure 20: Distribution's tail study with QQ-plot

We also perform a Gaussian test using lillie and shapiro functions in R for Kolmogorov-Smirnov and Shapiro-Wilk Test to confirm this:

```
Lilliefors (Kolmogorov-Smirnov) normality test

data:  Renta_Apple
D = 0.06318, p-value < 2.2e-16


Shapiro-Wilk normality test

data:  Renta_Apple
W = 0.9484, p-value < 2.2e-16
```

We see that the p-values are weak, meaning that we reject the hypothesis with a level of confidence of almost 100%. Even if we take into consideration another period (trying to avoid the 2008 Financial Crisis), the results are quite similar.

The histogram shown in Figure 21 has a better "Gaussian shape" but the return is still non-Gaussian:

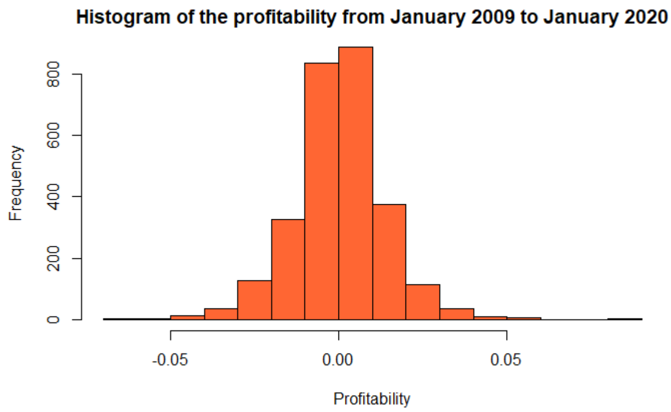

Figure 21: Distribution of Apple's Stock returns on another period

```
Lilliefors (Kolmogorov-Smirnov) normality test

data:  Renta_Apple_ap2008
D = 0.0425, p-value = 1.604e-12


Shapiro-Wilk normality test

data:  Renta_Apple_ap2008
W = 0.98076, p-value < 2.2e-16
```

Applying log function on (1+Return) leads to the same results: $log(1 + R)$ does not have a Gaussian Distribution.We tried the same test with different assets on different periods (taking account of the 2008 Financial Crisis) and obtained the same type of result. Hence we can affirm that profitability does not follow Gaussian Distribution and that the first hypothesis is incorrect. Therefore, we will use the **Metropolis-Hastings Algorithm** to solve the problem defined in part 2.2.5.

4.3.2 Best portfolio with V@R constraint using Metropolis-Hastings Algorithm

Recall the problem to optimise the return of a portfolio subject to a V@R constraint.

$$(P_5) : \begin{cases} \sup_{\pi \in A} E[R^\pi] \\ s.t.\ \pi' \mathbf{1} = 1 \\ V@R_\lambda(R^\pi) \leq v_0 \end{cases}$$

In order to verify that the algorithm works, we study the problem on a small dimension case 3, meaning that we try to find the best portfolio with the value at risk as a constraint for 3 assets.
We compute the empirical solution of the problem, determining the portfolio that minimizes the cost function by trying all the combinations. The cost function that we want to minimize is $-E[R^\pi]$, in other words, we want to find the portfolio with the highest return subject to the constraints. Then we launch the algorithm and compare both solutions.
Figure 22 explains several phenomena:

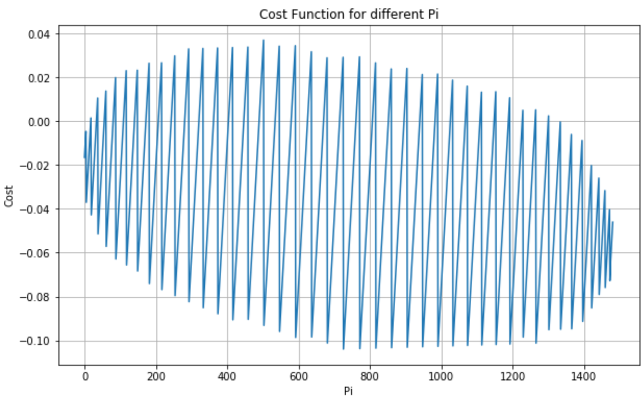

Figure 22: Evolution of the cost function for all portfolios, in a 3 dimensional case, with a short sale limit of 1 and a discretization step of 0.05

The x axis represents the i^{th} portfolio simulated and the y axis the corresponding cost-function. The cost function reaches a peak and then suddenly goes drops. This phenomenon comes from our way of generating all the admissible portfolios as shown in Figure 23.

	AMZN	AAPL	MSFT
35	-1.00	1.75	0.25
36	-1.00	1.80	0.20
37	-1.00	1.85	0.15
38	-1.00	1.90	0.10
39	-1.00	1.95	0.05
40	-1.00	2.00	0.00
41	-0.95	-0.05	2.00
42	-0.95	-0.00	1.95
43	-0.95	0.05	1.90
44	-0.95	0.10	1.85

Figure 23: Extract of the different pi generated

We explain the variation in the cost function as there is a huge difference between portfolio 40 and portfolio 41.

In a small dimension case, for a small discretization step small, the numerical cost of the empirical minimum research is acceptable. However, any dimension greater than 3 leads to increased numeric cost in order to find an empirical solution and thus a very long running time.
If we kept the same discretization step and short sale limit and studied the empirical solution in the 4 dimension case, there would be more than 100,000 combinations of portfolios to test.
The aim of the Metropolis-Hasting Algorithm is to reduce the numerical cost. Indeed, in less than 10,000 iterations, we can find a satisfactory solution regardless of the dimension. While generating random neighbours, these jumps in the cost function can also occur, however, we avoid them thanks to the parameter h of the Metropolis-Hastings algorithm.
We fix the value at risk constraint to 2, meaning that the investor does not want to lose more than 2% of his initial wealth invested. Here are the results for the set of assets Microsoft, Apple and Amazon. With 4,500 iterations we find the right solution:

```
Empirical Solution : [ 1.8  -0.95  0.15]
Algorithm solution : [ 1.8  -0.95  0.15]
Maximum jump encoutered during the Algorithm : 0.002960
```

It is relevant to note that we had to fix a parameter h to find relevant solutions. This parameter depends on the discretization step. Once it is fixed in a small dimension case, it will also work in a larger dimension case:

```
Empirical Solution : [ 0.1 -1.  -0.1  2. ]
Algorithm solution : [ 0.1 -1.   0.   1.9]
Maximum jump encoutered during the Algorithm : 0.00318
```

The results displayed above are computed with the same h as previously. It takes 30 seconds to find the empirical solution and less than 3 seconds to find the algorithm solution.
There is little difference between the best solution and the solution found by the algorithm and we can reduce this difference by increasing the number of iterations of the Metropolis-Hasting algorithm or by fixing a better h. Indeed, as explained in Figure 24, the algorithm reaches very quickly the neighbourhood of the solution:

The blue line represents the value returned by the

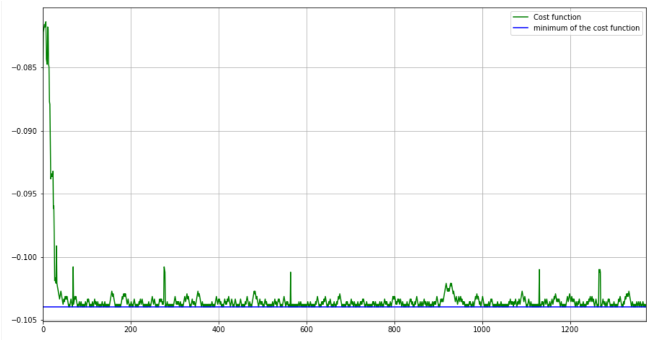

Figure 24: Evolution of the cost function during the algorithm

algorithm and the green one represents the cost function. We can see in Figure 24 that we reach very quickly the minimum of the cost function (and therefore the maximum return).
The cost function keeps bouncing because of the randomness introduced in the algorithm to avoid local minima.

Now that we have found a solution, let us display it through Figure 25 with the other portfolios to compare them. With Microsoft, Apple and Amazon we found that the best portfolio to invest in with the constraint of value at risk was $[1.8, -0.95, 0.15]$:

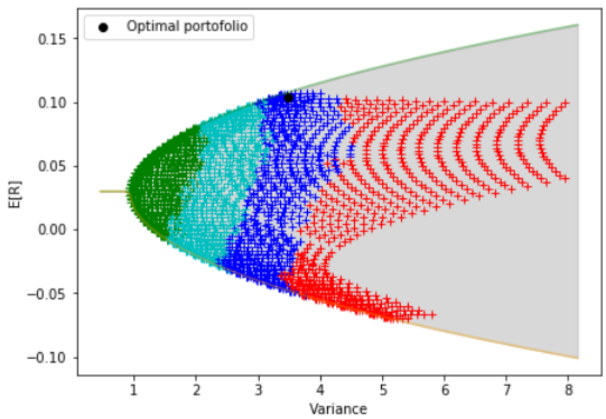

Figure 25: Optimal Portfolio found with problem P_5

This optimal portfolio corresponds to going into debt for 95% of our wealth on Apple, investing 180% in Microsoft and 15% in Amazon. We compute the V@R of the optimal portfolio $[1.8, -0.95, 0.15]$ and we find 1.9969, a value very close to the limit we fixed which was 2.

However, the fact that the parameter h has to be determined when one simulation parameter changes makes this algorithm non-viable if we want to extend it to a wider range of assets as we would have to estimate h every time.

We can numerically find a link between the variance and the V@R for a given set of assets.
Therefore, for a given Value at Risk, we can find an optimum portfolio thanks to the first optimization problem.

4.4 Best portfolio with a given V@R

4.4.1 Method to find a link between the variance and the V@R

In this section we establish a link between the V@R and the variance in order to find the optimal portfolio with a given $V@R_\lambda$:

$$(P_l) : \begin{cases} \sup_{\pi \in A} E[R^\pi] \\ s.t.\ \pi'\mathbf{1} = 1 \\ V@R_\lambda(R^\pi) = v_0 \end{cases}$$

We want to find the optimum variance $\sigma^{\star 2}$ associated with a given $V@R_\lambda$ so that we can apply the results of the first optimization problem. Thus we will be able to determine the optimum portfolio.

This is very useful because the V@R is much easier to interpret. In fact, for example if we take a V@R equal to 2, the optimal portfolio ensures (with a certain confidence level) that we do not lose more than 2% of our initial wealth.

As we have seen before, Figure 25 shows us the V@R levels.
There are many portfolios with the same V@R on a same line. In fact a portfolio color changes when its V@R reaches a certain level, which means that every portfolio around the cyan/blue border has the same V@R. This border is a line, an equation of the expected return in terms of the variance. Therefore, we conclude that there is a link between V@R and variance.
We want to implement the following process: the agent gives a V@R and then we find the optimum portfolio.
The steps to do so are described below.

- **First step: Estimate of the given V@R line**

Firs we estimate the line on the graph that corresponds to all the portfolios with a V@R equal to the one given:

$$E[R^\pi] = c \times \sigma^2 + d \quad , \quad (c, d) \in R$$

such that every portfolio that satisfies this equation has a V@R equal to the one given.

- **Second step: Optimal variance equivalent to the given V@R**

After estimating this line equation, we find the variance $\sigma^\star$ for which the estimated line and the efficient frontier are equal. We use the equation expressed in the first optimization problem:

$$c \times \sigma^{\star 2} + d = \frac{b}{a} + \sqrt{\sigma^{\star 2} - \frac{1}{a}} \|M - \frac{b}{a}\mathbf{1}\|_{\Sigma^{-1}} \quad (16)$$

The optimal portfolio for the given V@R will graphically be at the intersection of the line we estimate and the efficient frontier. The variance of the intersection point is $\sigma^\star$.

- **Third step: Optimal portfolio**

Then we express the optimal portfolio using the first optimization problem result with the variance $\sigma^\star$ we found in the second step:

$$\pi^\star(\sigma^\star) = \frac{1}{a}\Sigma^{-1}\mathbf{1} + \sqrt{\sigma^{\star 2} - \frac{1}{a}}\Sigma^{-1}\frac{M - \frac{b}{a}\mathbf{1}}{\|M - \frac{b}{a}\mathbf{1}\|_{\Sigma^{-1}}}$$

4.4.2 Simulation

We take three assets : Microsoft, Apple and Amazon. In order to test our algorithm, we fix the target V@R equal to 2 and a confidence level of $1 - \lambda = 90\%$.

- **Estimate of the given V@R line**

To do so, we first simulate some portfolios with a V@R close to the one given. Then we define two categories: portfolio with a V@R lower than the given V@R (red color) and the others (blue color), as shown in Figure 26:

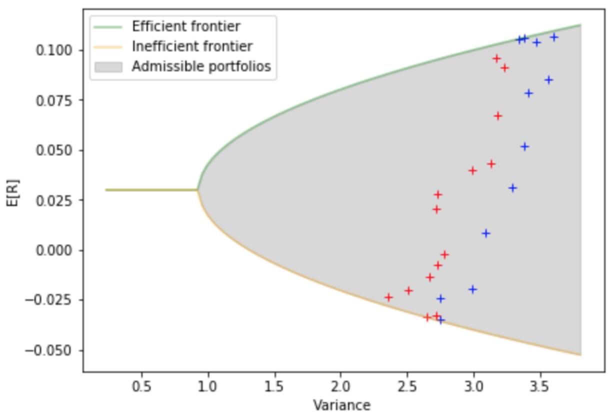

Figure 26: Some portfolios around the given V@R

Creating two categories allows us to estimate the desired line by performing a logistic regression. For any portfolio π, this regression allows us to obtain the coefficient (α, β, γ) such that:

$$\alpha + \beta \times \sigma_\pi^2 + \gamma \times E[R^\pi] = \ln\left(\frac{P[cross = red \mid \pi]}{1 - P[cross = red \mid \pi]}\right)$$

As we want to classify the two categories equally, we replace $P[cross = red \mid \pi]$ by $0,5$. This leads us to:

$$\alpha + \beta \times \sigma_\pi^2 + \gamma \times E[R^\pi] = 0$$

$$\Leftrightarrow E[R^\pi] = -\frac{\beta}{\gamma} \times \sigma_\pi^2 - \frac{\alpha}{\gamma}$$

Now that we have the equation of this specific line, let us plot it on the graph displayed in Figure 27:

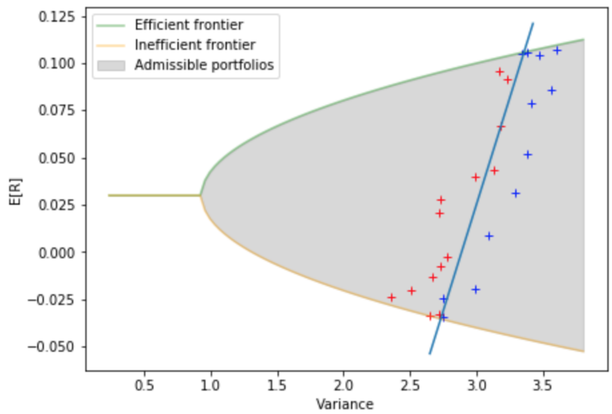

Figure 27: Line on which every portfolio has the V@R equal to the one given

To conclude,

$$c = -\frac{\beta}{\gamma} \quad \text{and} \quad d = -\frac{\alpha}{\gamma}$$

- **Optimal variance equivalent to the given V@R**

In order to solve the previous equation (16) we use the Python function *fsolve* from library *scipy*. Thus we obtain the value of $\sigma^{\star 2}$ which is equal in this case to 3.36.

- **Optimal portfolio**

Now that we have computed the optimal variance $\sigma^{\star 2}$, we compute the optimal portfolio (with a given V@R equal to 2 for example) and visualize it in Figure 28:

$\pi^{\star} = (1.51, -1.00, 0.49)$

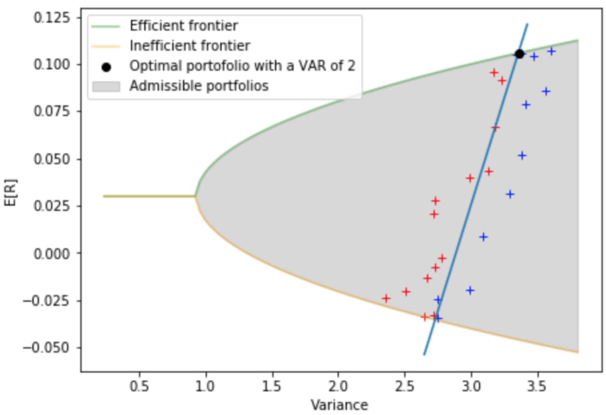

Figure 28: Optimum portfolio with a given V@R equals to 2

- **Result control**

Now that we have the optimum portfolio, we compute its V@R in order to verify if our result is true. We obtain a V@R very close to the desired V@R:

$$V@R_{0.1}\left(R^{\pi^{\star}}\right) = 2.037$$

Conclusion

We have mathematically and graphically seen different portfolio behaviours. In order to increase the expected return of a portfolio, short sales were necessary even if this implies an increasing variance. For realistic reasons, we chose not to allow short sales greater than 200% of the initial wealth. The efficient frontiers allow us to find an optimum portfolio for any given variance; some of them were not attainable in our simulations as they required too many short sales. We used risk measure as the V@R and the TVaR to quantify the risk, as they are more precise. The aim was then to find an optimum portfolio for a given risk. Under the Gaussian assumptions of the individual returns, we derive theoretical results, solving this problem mathematically is not relevant as the Gaussian hypothesis is false. Implementing a stochastic optimisation algorithm enables us to find an optimum portfolio under the V@R constraint. However, it has a weakness, the parameter h has to be determined when one parameter changes.

We finally come up with the idea of finding a numerical link between the variance and the V@R after seeing the results of the graphs. This link enables us to find the optimum portfolio very quickly for a given V@R.

Therefore we implemented a means of finding the optimum investment strategy for an agent with a passive portfolio management on a given set of assets. The next steps of this research are threefold. First, modelling the investment considering riskless assets and finding optimum portfolios. This paper was an introduction to stock portfolio management with a simple investment strategy, a more sophisticated strategy should be elaborated and furthermore, including other assets. Third, using temporal series to have an idea of the future behaviour of the stocks and applying the optimisation techniques found in the paper to this data will allow us to find an optimum portfolio.

However, even if we find the best strategy and apply it to the predicted data, there is a high probability that the market will not evolve as predicted. Indeed, the rational or irrational behaviour of the agents on the market cannot always be predicted, nor sanitary crises like Covid-19 or the financial crisis such as in 2008.

Nomenclature

$\mathbf{1} = (1, ..., 1)' \in R^d$
$a = \mathbf{1}'\Sigma^{-1}\mathbf{1} = ||1||^2_{\Sigma^{-1}}$
$b = \mathbf{1}'\Sigma^{-1}M =< \mathbf{1}, M >_{\Sigma^{-1}}$
π^i proportion of wealth invested in asset i
$\pi = (\pi^1, ..., \pi^d)$ a portfolio
Σ covariance matrix of the assets' return
A All admissible portfolios
$m^i = E[R^i]$ expected return of the asset i
$M = E[R]$ vector of assets expected return
$m = E[R^\pi] = \pi' M$ expected value of π's return
P^i price of the asset i (random)
R^i return of the asset i (random)
$R = (R^1, ..., R^d)$ vector of assets return (random)
R^π return of the portfolio π
S^i asset i
$V@R_\lambda(X)$ Value at Risk (confidence level $1-\lambda$) of the random variable X
$W_T^{w,\pi}$ value of the portolio π at time T with the initial wealth w invested

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