# Stochastic calculus applied to the Black-Scholes model for the pricing and hedging of options

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March 1, 2021

**Abstract**

**The use of options, as well as other derivatives, has increased sharply over the last decades, becoming a key financial instrument on financial markets. If the well-known Cox-Ross-Rubinstein model has brought a consistent framework for the pricing of financial options, its discrete time structure leads to substantial limitations, specifically when it comes to study the continuity and the dynamics of the derived pricing of the options. Thus, there is a need to develop an extension of that model, continuous in time, which still guarantees the fairness of pricing on financial markets. Our study aims to introduce several mathematical concepts which support the construction of such a theoretical model for the arbitrage pricing of options. To do so, we have studied in detail and implemented various fundamental properties from the stochastic calculus theory, especially inherited from Itô. Then, applying properly these stochastic calculus results to the specific stochastic process of the Bownian motion, we have derived the rigorous framework of the *Black-Scholes* model. This crucial model of financial mathematics allows to derive an arbitrage pricing and hedging process, especially for options, in a continuous-time basis. Our theoretical findings are illustrated through several numerical simulations and computations.**

Martingales | Itô and Stochastic Calculus | Black-Scholes | Pricing and Hedging

## Introduction

This article aims to introduce mathematical concepts and elements which support a theoretical model for continuous-time pricing of financial securities, especially optional products. Specifically, we will focus on structuring various fundamental theorems and results from the stochastic calculus theory in order to derive a consistent extension, continuous in time, of the well-known discrete-time Cox-Ross-Rubinstein model. Through the consideration of crucial theorems of stochastic calculus and important martingale results applied to continuous stochastic processes, we intend to develop the rigorous mathematical framework which leads to the implementation of the *Black-Scholes* model for the pricing and the hedging of financial options. Indeed, we will define and study key martingale properties as well as fundamental results inherited from Itô's calculus, specifically applied to the Brownian motion. This specific approach of stochastic integrability will allow us to derive a continuous and integrable wealth process associated to a portfolio, which constitutes a key step of the studied pricing method. In addition, through the introduction of a specific probability measure, we will bring martingale behavior of prices to light. Finally, the central question of arbitrage will be address throughout the article, and specifically in the construction of the pricing and hedging processes.

Furthermore, numerical simulations and computations will be performed in order to illustrate our theoretical findings. Most of the material on which this study is based can be found in the lecture notes of A. Réveillac (2014).

In a first section, we will start by introducing main characteristics of the Cox-Ross-Rubinstein discrete model, highlighting the key challenging points for a further continuous and integrable extension of this model. A second section we will be dedicated to the definition of important characteristics of continuous stochastic processes and martingales. A specific focus will be done on main properties of the Brownian motion. Then, in the third section we will specifically focus on the fundamental theorems, formulas and results of stochastic calculus inherited from Itô. Indeed, these elements will be keystones in the construction of the Black-Scholes model. Finally, in the forth section, we will apply the previously determined stochastic calculus properties to the implementation of the Black-Scholes model for the pricing and the hedging of options.

Note: Due to length constraint for this article, some theoretical elements of our study do not directly appear in the body of this article, but are provided in the *Appendix* section. We invite the reader to refer to it if necessary.

## 1. Financial framework modelling

In this first section of the article, we will introduce the mathematical modeling of discrete-time financial markets through the *Cox-Ross-Rubinstein (CRR)* model. In addition, considering this specific market framework, we will study the *wealth process* associated to an agent's *portfolio*. We will also address the central notion of *arbitrage*, leading to pricing and hedging processes for financial securities. Finally, we will present the perspective of the continuous-time extension of this first model.

### 1.1. A discrete-time market model: the Cox-Ross-Rubinstein model

We are going to introduce the Cox-Ross-Rubinstein (CRR) model, which allow to derive a probabilistic and discrete-time representation of financial markets. Then, we will properly define the key notion of *information* on the modelled market.

#### 1.1.1. Market structure and dynamics

First of all, we assume that a deterministic and unique interest rate is in force on the market. In addition, we consider this interest rate constant over any period of time, and we make the assumption that it is non-negative[1]. Therefore, let $r \geq 0$, be the interest rate on the market.

Furthermore, we study the market over a specified time interval $[0, T]$ with $T \in \mathbb{R}_+^*$. Then, $[0, T]$ is subdivided in a fixed number $n$ of sub-periods ($n \in \mathbb{N}^*$), such that $t_0 = 0 < ... < t_i < ... < t_n = T$. For instance, let's consider a subdivision family $(t_i)_{i \in [\![0,n]\!]}$ such that $t_i = i\frac{T}{n}$, $\forall i \in [\![0, n]\!]$, and let $\Delta_t = \frac{T}{n}$ denote the time interval between two dates $t_i$ and $t_{i+1}$. Thus, $(t_i)_{i \in [\![0,n]\!]}$ determines deterministic dates at which the market is updated and evaluated. This discrete consideration of time implies that the market changes only at any fixed date $t_i$. That is, for any time period $[t_i, t_{i+1})$, the market's state changes at the date $t_i$, and then it remains stable and unchanged until the next date $t_{i+1}$.

Moreover, we define a simple probabilistic framework which aims to model the intrinsic randomness of the market's evolution and dynamics over time.
Thus, we consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega := \{\mathfrak{T}, \mathfrak{H}\}^n$, and $\mathcal{F} := \mathcal{P}(\Omega)$.
With respect to this probability space, we construct the probability measure $\mathbb{P}$, setting $p \in ]0, 1[$, and such that:

$$\mathbb{P}[\{(\omega_{t_1}, \cdots, \omega_{t_n})\}] \;=\; p^{card(\{i,\, \omega_{t_i} = \mathfrak{T}\})} \cdot (1-p)^{card(\{i,\, \omega_{t_i} = \mathfrak{H}\})}.$$

From this probabilistic framework, we now may construct $n$ specific Bernoulli random variables.
First, let $(d, u) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ such that $0 < d < u$.
Then, we define $(C_{t_i})_{i \in [\![1,n]\!]}$, $n$ particular Bernoulli random variables, ($i.i.d.$), such that:

$$C_{t_i}((\omega_{t_1}, \cdots, \omega_{t_n})) := C_{t_i}(\omega_{t_i}) = \begin{cases} u & \text{if } \omega_{t_i} = \mathfrak{T} \\ d & \text{if } \omega_{t_i} = \mathfrak{H} \end{cases}$$

This set of random variables will allow us to model the random changes in asset prices over periods of time on the market. It is also important to consider that, at any date $t_i$, the realization of the r.v. $C_{t_i}$ is drawn independently from all previous $C_{t_j}$, $j < i$. That is, price changes on the market, at any time $t_i$, are assumed to be independent of all past price changes.

We now introduce the two assets available on the studied market.

- A *riskless asset* $S^0$. This riskless asset is defined by a deterministic sequence of prices $S^0 := (S^0_{t_i})_{i \in [\![0,n]\!]}$, which is directly associated to the interest rate $r$ of the market.
  Indeed, $S^0 := (S^0_{t_i})_{i \in [\![0,n]\!]}$ is defined such that:
  $$\begin{cases} S^0_0 \;=\; 1 \\ S^0_{t_i} \;=\; S^0_{t_{i-1}}(1+r)^{\Delta_t} \;=\; S^0_{t_{i-1}}(1+r)^{\frac{T}{n}}, \;\; \forall\, i \in [\![1, n]\!] \end{cases}$$
  Hence, $\forall\, i \in [\![0, n]\!]$, $S^0_{t_i} \;=\; (1+r)^{t_i}$.

[1] In reality, this interest rate could be negative, but for the simplicity of our study, we take it non-negative.

- A *risky asset* $S$. This risky asset is defined by a random sequence of prices $S := (S_{t_i})_{i\in[\![0,n]\!]}$.
Indeed, $S := (S_{t_i})_{i\in[\![0,n]\!]}$ is defined such that:
$$\begin{cases} S_0 > 0 \\ S_{t_i} = S_{t_{i-1}} C_{t_i}, \quad \forall\, i \in [\![1,n]\!] \end{cases}$$
Hence,
$$S_{t_i}((\omega_{t_1}, \cdots, \omega_{t_i})) = \begin{cases} S_{t_{i-1}}((\omega_{t_1}, \cdots, \omega_{t_{i-1}}))\cdot u \ \text{ if } \ \omega_{t_i} = \mathfrak{T} \\ S_{t_{i-1}}((\omega_{t_1}, \cdots, \omega_{t_{i-1}}))\cdot d \ \text{ if } \ \omega_{t_i} = \mathfrak{H} \end{cases}$$

Since we have defined $(d, u)$ such that $0 < d < u$, at each date $t_i$, the price $S_{t_i}$ of the risky asset represents either a "up" value or a "down" value compared to the previous price $S_{t_{i-1}}$. Such a price process is going to lead us to introduce a formal and mathematical approach of the notion of time-dependent information on the market.

#### 1.1.2. Market information

A *filtration* is a mathematical object related to the concept of information for some *stochastic process*. Thus, in the CRR framework described above, a filtration originates from the random process of risky asset prices. Recall that $(S_{t_i})_{i\in[\![0,n]\!]}$ is defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Hence, we define a filtration deriving from such a stochastic process as follows:

**Definition 1.1.** Filtration.

*A filtration is a family $(\mathcal{F}_{t_i})_{i\in[\![0,n]\!]}$ of sub-$\sigma$-fields of $\mathcal{F}$ such that:*
$$\forall\ (i,j) \in [\![0,n]\!] \times [\![0,n]\!] \ \ s.t.\ 0 \le i \le j, \quad \mathcal{F}_{t_i} \subset \mathcal{F}_{t_j}.$$

From Definition 1.1., one may notice that the filtration $(\mathcal{F}_{t_i})_{i\in[\![0,n]\!]}$ is an increasing family. Thus, at any time $t_i$, $\mathcal{F}_{t_i}$ is a representation of all the information available by time $t_i$ $-$ i.e., all the past information gathered from time 0 to time $t_i$. This interpretation of information is coherent with the increasing family property.

In order to adapt the filtration $(\mathcal{F}_{t_i})_{i\in[\![0,n]\!]}$ more precisely to $(S_{t_i})_{i\in[\![0,n]\!]}$, we assume that, at time $t_0 = 0$, $\mathcal{F}_0 := \{\emptyset, \Omega\}$. This means that $S_0$ is not a random variable and that it takes a deterministic value, $S_0 > 0$. In addition, we set $\mathcal{F}_T = \mathcal{F}$. This implies that, at time $T$, the filtration includes the whole of information possibly available on the market with respect to the defined framework.
Looking more in detail at the informative material of the filtration, we consider that, for any time $t_i$, $\mathcal{F}_{t_i} := \sigma(S_{t_0}, S_{t_1}, ..., S_{t_i})$. That is, by any time $t_i$, all the available information is constituted of all the previous prices taken by the risky asset at each time $t_j$, $0 \le j \le i$.
This consideration of the filtration implies that, for any time $t_i$, $i \in [\![0,n]\!]$, $S_{t_i}$ is $\mathcal{F}_{t_i}-measurable$.

Having introduced the market structure through the CRR model, we will be able to define the important concepts of portfolio and associated wealth process, as well as the key notion of arbitrage, finally leading us to pricing and hedging processes for financial securities.

### 1.2. Portfolios and wealth processes

Having established the CRR market structure, now we introduce the tools used in the modeling of agents' behavior and choices on the market. More precisely, we study the implications of an agent's investment strategy on his capital - or *wealth* - over time. To do so, we will study the notion of *portfolio*. Finally, we will apply the central concept of *discounting* to the previously defined price and wealth processes.

#### 1.2.1. Portfolio, investment strategy and associated wealth process

**Definition 1.2.** Portfolio and Investment strategy.

*A portfolio consists in a couple $(x, \pi) \in \mathbb{R}_+ \times \mathbb{R}^n$, $x$ being an agent's initial capital, and $\pi$ being its investment strategy. We define the investment strategy $\pi$ as a $\mathbb{R}$-valued (discrete-time) stochastic process $\pi := (\pi_{t_1}, ..., \pi_{t_n})$, where each r.v. $\pi_{t_i}$ represents the number of shares of the risky asset an agent has chosen to hold at time $t_{i-1}$, within the scope of his investment over the time period $[t_{i-1}, t_i)$.*

The main principle in the construction of a portfolio $(x, \pi)$ is that the investment choice $\pi_{t_i}$ made by an agent at any time $t_{i-1}$, $i \in [\![1,n]\!]$ derives exclusively from the knowledge of all the available information on the market up until time $t_{i-1}$.

Thus, with respect to the definition of the filtration $(\mathcal{F}_{t_i})_{i\in[\![0,n]\!]}$ on the market, where $\mathcal{F}_{t_i} := \sigma(S_{t_0}, S_{t_1}, ..., S_{t_i})$ at any time $t_i$, the previous principle implies that $\pi_{t_i}$ is $\mathcal{F}_{t_{i-1}}-measurable$, $\forall\ i \in [\![1,n]\!]$. This property constitutes the definition of a *predictable* stochastic process.

The study over time of an agent's wealth, with respect to his portfolio $(x, \pi)$, constitutes an additional point of interest.

**Definition 1.3.** Wealth process associated with a portfolio.

*The wealth process $X^{(x,\pi)}$ associated with the portfolio $(x,\pi)$ is a stochastic process defined such that $X^{(x,\pi)} := (X^{(x,\pi)}_{t_i})_{i\in[\![0,n]\!]}$, with:*

$$\begin{cases} X^{(x,\pi)}_0 &= x \\ \\ X^{(x,\pi)}_{t_i} &= \pi_{t_i} S_{t_i} + (X^{(x,\pi)}_{t_{i-1}} - \pi_{t_i} S_{t_{i-1}})(1+r)^{\Delta_t}, \ \ i \in [\![1,n]\!] \end{cases}$$

We highlight several noteworthy characteristics of such a wealth process.

First, with respect to its construction, this wealth process derives from the assumption that the initial wealth $x$ and the investment strategy $\pi$ entirely and exclusively determine the behavior of the wealth process. That is, we consider neither external in-flows nor external out-flows in this process, and, at any given time $t_i$, the entire capital available $X^{(x,\pi)}_{t_i}$ is completely invested within the market, between the riskless and the risky assets. Such a process is denoted as a *self-financing* portfolio.

At any time $t_{i-1}$, $i \in [\![1,n]\!]$, the agent chooses to hold $\pi_{t_i}$ shares of the risky asset, for an investment cost of $\pi_{t_i} S_{t_{i-1}}$. The remaining amount of capital at time $t_{i-1}$, defined by $(X^{(x,\pi)}_{t_{i-1}} - \pi_{t_i} S_{t_{i-1}})$, is then invested in the riskless asset. Afterwards, the resulting wealth of this investment at the next date $t_i$ consists of the new (random) value $S_{t_i}$ of the risky asset multiplied by $\pi_{t_i}$, the number of shares of this risky asset held by time $t_i$, to which we have to add the amount invested in the riskless asset increased by a (deterministic) factor $(1+r)^{\Delta_t}$, characterizing the compounded interests earned over a single period.

One may note that for any time $t_i$, $X^{(x,\pi)}_{t_i}$ is a function of $\pi_{t_i}$, $S_{t_i}$, $S_{ti-1}$ and $X^{(x,\pi)}_{t_{i-1}}$. Then, considering that $S_{t_i}$ is $\mathcal{F}_{t_i}-measurable$ and that $\pi_{t_i}$, $S_{t_{i-1}}$ are $\mathcal{F}_{t_{i-1}}-measurable$, we can easily show, by mathematical induction, that $X^{(x,\pi)}_{t_i}$ is $\mathcal{F}_{t_i}-measurable$, $\forall\ i \in [\![0,n]\!]$. This property constitutes the definition of an *adapted* stochastic process.

Finally, to end with this set of remarks on the wealth process, we note that $\pi_{t_i}$ r.v. are allowed to be negative. Thus, a negative $\pi_{t_i}$ would imply that, at time $t_{i-1}$, the agent received the amount $\pi_{t_i} S_{t_{i-1}}$ from the sale of $\pi_{t_i}$ borrowed shares of the risky asset. Afterwards, at time $t_i$, the agent has to give the amount $\pi_{t_i} S_{t_i}$ back, which is then denoted by the negative sign of $\pi_{t_i} S_{t_i}$. This kind of specific operation is commonly known as a *short sale*.

#### 1.2.2. Discounted prices and wealth processes

We have introduced the notions of price and wealth processes. Now, an important further consideration lies in the application of the discounting process to these elements.
Indeed, we consider that our observation of the market is established at the date $t_0 = 0$. Thus, it becomes fundamental to discount both price and wealth processes in order to preserve homogeneity in the perception of the notion of value on the market.

Thus, we give the two following definitions.

**Definition 1.4.** Discounted price process.

*The discounted prices process $\tilde{S} := (\tilde{S_{t_i}})_{i\in[\![0,n]\!]}$ is defined such that*

$$\forall\ i \in [\![0,n]\!],\ \ \tilde{S_{t_i}} = \tfrac{S_{t_i}}{(1+r)^{t_i}}.$$

**Definition 1.5.** Discounted wealth process.

*The discounted wealth process $\tilde{X}^{(x,\pi)} := (\tilde{X}^{(x,\pi)}_{t_i})_{i\in[\![0,n]\!]}$ is defined such that*

$$\forall\ i \in [\![0,n]\!],\ \ \tilde{X}^{(x,\pi)}_{t_i} \ = \ \tfrac{X^{(x,\pi)}_{t_i}}{(1+r)^{t_i}}$$

**Remark 1.6.** *For both discounted price and wealth processes, we compute the discount with respect to the riskless asset's price which is intrinsically linked to the interest rate $r$ in force on the market.*

We have introduced all the parameters which define the Cox-Ross-Rubinstein model and other mathematical and financial tools that we will use in the study of pricing and hedging processes. The next subsection will be dedicated to the central notion of *arbitrage*, and how it is closely related to the *fair* pricing of financial securities on the market.

## 1.3. Arbitrage and Asset pricing

In this section, after having properly defined the notion of arbitrage, the main idea will be to study how we may determine a "fair" price for any security on a market.

### 1.3.1. Arbitrage

An *arbitrage* is defined as a financial operation which guarantees an agent will make a non-negative profit, without committing any initial capital, that is, without bearing any risk on the market. In addition, through arbitrage, the agent has a positive probability to earn a positive profit from his arbitrage investment.

**Definition 1.7.** Arbitrage opportunity (AO).

*An Arbitrage Opportunity (AO) is technically defined as a specific portfolio with the following properties :*

- *The arbitrage investment strategy implies no initial capital, it has no initial cost.*
  *i.e.,* $x = 0$.
- *The arbitrage investment strategy is riskless for the agent, it guarantees a certain non-negative profit at the maturity date* $T$ *of the investment.*
  *i.e.,* $\mathbb{P}[X_T^{(0,\pi)} \geq 0] = 1$.
- *The probability for the agent to earn a positive profit from the arbitrage investment strategy is positive.*
  *i.e.,* $\mathbb{P}[X_T^{(0,\pi)} > 0] > 0$.

From Definition 1.7., we note that such an arbitrage portfolio $(0, \pi)$ embodies ineluctably a form of unfairness on the market. The mere existence of such an arbitrage opportunity runs counter to the fundamental "No free lunch with vanishing risk" principle on financial markets. That is, a crucial condition which preserves fairness on a market is that no agent can expect to earn a positive profit from an investment strategy which entails zero initial cost and which does not bear any market risk. From these observations, the *Absence of Arbitrage Opportunity (AAO)* on a market turns out to be an elementary condition in the construction of a "fair" market on which we will later try to define a notion of fair price for securities. Such a market, with no arbitrage opportunity on it, is referred to as a *viable* market.

### 1.3.2. Asset pricing and hedging

The notion of fair price for a security on the market is directly linked to the *viability* of this market (i.e., the fact that the AAO condition is satisfied on this market). Indeed, we will consider that the price of a security is "fair" if it does not lead to any arbitrage opportunity on the global market. We call such a price the *arbitrage price* of a security.

So, an elementary assumption we make is that the studied market is viable[2].
Now, we assume the existence of a third financial security on the market. This new asset is characterized by a random payoff $h$ at the maturity date $T$, and an initial price $\rho_0$ at the date $t_0 = 0$. The random payoff $h$ is considered to be $\mathcal{F}_T-measurable$. Let $\alpha := (\rho_0, h)$ denote this third security.

A crucial concept intrinsically linked to arbitrage pricing is the notion of *replication*.

**Definition 1.8.** Replicating portfolio.

*A portfolio* $(\bar{x}, \bar{\pi})$ *is said to be a replicating portfolio of* $\alpha$ *- equivalently, we say that* $(\bar{x}, \bar{\pi})$ *replicates* $\alpha$ *- if and only if:*

$$X_T^{(\bar{x},\bar{\pi})} = h\,, \quad \mathbb{P} - a.s.$$

[2] Concretely, this assumption lies on underlying assumptions on the probabilistic structure of the CRR market, but it is not the point of this article to study in detail these underlying assumptions which ensure that the AAO condition is in force on modelled market.

**Proposition 1.9.** Arbitrage pricing.

*Assuming that such a $\alpha$-replicating portfolio $(\bar{x}, \bar{\pi})$ exists, i.e. $\exists\ (\bar{x}, \bar{\pi})$ s.t. $X_T^{(\bar{x},\bar{\pi})} = h$, $\mathbb{P}-a.s.$, then, the AAO condition holds on the $(S^0, S, \alpha)$ market if and only if:*

$$\rho_0 = \bar{x}.$$

*Thus, $\rho_0 = \bar{x}$ is called the arbitrage price of the security $\alpha$, and $\bar{\pi}$ is the hedging strategy.*

*Proof.*

The previous result directly derives from the well-known *Law of one price*, which states that two assets with identical payoffs must have the same price.

If we assume $\bar{x} \neq \rho_0$, let's say $\bar{x} < \rho_0$[3], then the strategy consisting in selling the asset $\alpha$ and adopting a long position on the $(\bar{x}, \bar{\pi})$[4] at $t_0$ would lead to the following final payoff at the maturity date $T$:

$$\rho_0 - \bar{x} - h + X_T^{(\bar{x},\bar{\pi})} = (\rho_0 - \bar{x}) > 0.$$

Thus, the described strategy is an arbitrage opportunity since it involves a zero initial cost, it even assure the agent to realize an instantaneous profit at date $t = 0$.

□

To sum up, the CRR model allowed us to determine a consistent pricing and hedging process. However, this discrete-time model faces clear limitations, especially in the perspective of an accurate consideration of the intrinsic randomness of the prices in the pricing process. Hence, we intend to think about a continuous-time extension of the CRR model.

### 1.4. Extension of the discrete-time model toward time continuity

We have introduced and described a discrete-time market model, consistent with the AAO condition. Now, we aim to extend this first discrete model to the pricing and hedging of financial securities in a continuous-time basis.

First, we define a formal discretization of the studied time interval $[0, T]$, similar to the one previously defined in the CRR model. Indeed, for $n \in \mathbb{N}^*$, we consider a subdivision family $(t_i^n)_{i \in [\![0, m_n]\!]}$ such that $t_0^n = 0 < ... < t_i^n < ... < t_{m_n}^n = T$. In order to tend to a continuous-time modelling, we study the limit behavior of our model when we subdivide the time interval $[0, T]$ in infinitely small subdivisions. That is, we consider the subdivision $(t_i^n)_{i \in [\![0, m_n]\!]}$ such that its step $|\rho_n| = \sup_{1 \leq i \leq m_n} |t_i^n - t_{i-1}^n| \underset{n \to +\infty}{\longrightarrow} 0$.

Similarly to the CRR model, with this new time discretization, we consider that the studied market is updated at each time $t_i^n \in (t_i^n)_{i \in [\![0, m_n]\!]}$. Indeed, at each time $t_i^n$, the investor sets a new investment strategy $\pi_{i+1}^n$, given the observation of the updated stock quotation $S_{t_i^n}$. Hence, we define a portfolio $(x, \pi^n)$ with $\pi^n = (\pi_1^n, ..., \pi_{m_n}^n)$.

Thereafter, in order to simplify the mathematical expressions of our model, we will consider a zero interest rate on the market, i.e., $r = 0$. Thus, within the scope of this model extension toward continuous-time modelling, the final wealth associated to a portfolio $(x, \pi^n)$ can be expressed as follows:

$$X_T^{x,\pi} = \lim_{n \to +\infty} X_T^{(x,\pi^n)} = \lim_{n \to +\infty} x + \sum_{i=1}^{m_n} \pi_i^n (S_{t_i^n} - S_{t_{i-1}^n}).$$

In another way, the limit behavior of the wealth process can be studied as

$$X_T^{(x,\pi)} = x + \int_0^T \pi_t dS_t. \qquad [1]$$

However, even if the previous modelling appears to give a satisfying limit behavior of our first market model in a continuous-time basis, the randomness of both $S$ and $\pi$ processes will lead us to study in more detail continuous stochastic processes in order to derive a proper definition of integrability properties for Eq.(1). These considerations of stochastic calculus will finally lead us to the definition of Itô integral and Itô's formula.

[3] The reverse case $\bar{x} > \rho_0$ would lead to a strictly similar proof.

[4] A long position on $(\bar{x}, \bar{\pi})$ means that $\bar{x}$ are invested at $t_0$ following the $\bar{\pi}$ investment strategy, eventually leading to a final payoff of $X_T^{(\bar{x},\bar{\pi})}$ at the maturity date $T$.

## 2. Fundamental notions of continuous-time stochastic processes and martingales

In the perspective of our study of a continuous-time approach for the pricing and hedging of options, continuous stochastic processes will play a central role, especially the Brownian motion. Thus, in this section, we will recall basic definitions about continuous stochastic processes as well as key properties of martingales. Finally, we will focus on the Brownian motion and its regularity, showing that its integration can not be carried out using a standard *Lebesgues-Stieltjes* integral. This will lead us to further considerations of Itô's calculus.

### 2.1. Continuous stochastic processes: Generalities

#### 2.1.1. A first definition

In order to introduce this section, we start by giving a proper definition of continuous stochastic processes. We still consider a time interval of study $[0,T]$, $T \in \mathbb{R}_+^*$ and a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

**Definition 2.1.** (Continuous) stochastic process.

*A stochastic process* $X := (X_t)_{t\in[0,T]}$ *is defined as a collection of (*$\mathbb{R}$*-valued) time-indexed random variables such that:*

$$\begin{array}{rcl} X \,:\, [0,T]\times\Omega & \to & \mathbb{R} \\ (t,\omega) & \mapsto & X_t(\omega) \end{array}$$

*Defined in this way, the stochastic process* $X$ *is* $\mathcal{B}([0,T])\otimes\mathcal{F}-measurable$*.*

*The application* $t \mapsto X_t(\omega)$ *is called the trajectory of* $X$*.*

*In particular, in this article, when we mention continuous stochastic processes, we refer to stochastic processes with continuous trajectory.*

#### 2.1.2. The notion of filtration

As already mentioned in our first presentation of the CRR model, the notion of *filtration* appears to be fundamental in the perspective of our study. We recall the definition of a filtration associated with a (continuous) stochastic process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

**Definition 2.2.** Filtration.

*A filtration is a family* $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$ *of sub-tribes of* $\mathcal{F}$ *such that:*

$$\begin{array}{rl} & \mathcal{F}_t \;\subset\; \mathcal{F}\,, \quad \forall\, t\in[0,T] \\ \textit{and} & \mathcal{F}_s \;\subset\; \mathcal{F}_t\,, \quad 0\le s\le t\le T. \end{array}$$

*In particular, the natural filtration* $\mathbb{F}^X := (\mathcal{F}_t^X)_{t\in[0,T]}$ *associated with a stochastic process* $X$ *is defined as:*

$$\mathcal{F}_t^X := \sigma\{X_s,\ s\le t\}\,, \quad t\in[0,T].$$

**Remark 2.3.** *Throughout this article, we will always consider the natural filtration associated with the studied stochastic processes, and we will also consider that* $\mathcal{F}_T = \mathcal{F}$*.*
*Therefore, these considerations of the natural filtration imply that,* $\forall\, \mathcal{F}_t^X,\ t\in[0,T],\ \mathcal{F}_t^X$ *includes all the available information on the stochastic process* $X$ *up to date* $t$*. We also say that* $\mathcal{F}_t^X$ *represents the observation of* $X$ *up to date* $t$*.*

Furthermore, we give two other important definitions about the relationship between stochastic processes and their associated filtration. We consider the previously defined subdivision family $(t_i^n)_{i\in[\![0,m_n]\!]}$ of the time interval $[0,T]$.

**Definition 2.4.** Adapted process.

*Let $X$ be a stochastic process and $\mathbb{F} := (\mathcal{F}_{t_i})_{i\in[\![0,m_n]\!]}$ a filtration.*

*$X$ is said to be adapted (according to the filtration $\mathbb{F}$) if:*

$$X_{t_i} \text{ is } \mathcal{F}_{t_i}\text{−measurable}\,, \ \ \forall\, t_i\,,\ i \in [\![0, m_n]\!].$$

*In particular, by definition, any stochastic process $X$ is $\mathbb{F}^X$−adapted.*

**Definition 2.5.** Predictable process.

*Let $X$ be a stochastic process and $\mathbb{F} := (\mathcal{F}_{t_i})_{i\in[\![0,m_n]\!]}$ a filtration.*

*$X$ is said to be predictable (according to the filtration $\mathbb{F}$) if:*

$$X_{t_i} \text{ is } \mathcal{F}_{t_{i-1}}\text{−measurable}\,, \ \ \forall\, t_i\,,\ i \in [\![1, m_n]\!]$$

#### 2.1.3. Finite variation processes

A final point of interest about continuous stochastic processes lies in their regularity. Indeed, we will observe later in our study that stochastic processes may have very different behaviours according to their regularity. Then, we define here a specific set of regular stochastic processes, the so-called *finite variation processes.*

**Definition 2.6.** Finite variation process.

*For $n\in\mathbb{N}^*$, let $(t_i^n)_{i\in[\![0,m_n]\!]}$ be a subdivision family of the time interval $[0,T]$, such that $t_0^n = 0 < ... < t_i^n < ... < t_{m_n}^n = T$.*
*Let $A := (A_{t_i})_{i\in[\![0,m_n]\!]}$ be a stochastic process. $A$ is said to be a finite variation process if and only if:*

$$\sup_{(t_i^n)_{i\in[\![0,m_n]\!]}} \sum_{i=1}^{m_n} |A_{t_i^n} - A_{t_{i-1}^n}| \,<\, +\infty.$$

*In addition, considering a subdivision $(t_i^n)_{i\in[\![0,m_n]\!]}$ of $[0,T]$ such that its step $|\rho_n| = \sup\limits_{1\leq i\leq m_n} |t_i^n - t_{i-1}^n| \underset{n\to+\infty}{\longrightarrow} 0$,*
*$A$ is said to be a finite variation process if and only if:*

$$\lim_{n\to+\infty} \sum_{i=1}^{m_n} |A_{t_i^n} - A_{t_{i-1}^n}|^2 \,=\, 0.$$

Thus, we have defined the basics of (continuous) stochastic processes, and we are now going to introduce in more detail martingale objects and their key properties.

### 2.2. Martingales: definitions and properties

#### 2.2.1. Continuous martingales

**Definition 2.7.** Martingale.

*Let $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$ be a general filtration.*
*Let $M$ be a (continuous) stochastic process.*
*$M$ is a martingale if it satisfies the following properties:*

1. *$M$ is $\mathbb{F}$−adapted,*
2. *$\mathbb{E}\,[|M_t|] \leq +\infty\,, \ \forall\, t \in [0, T]\,,$ i.e., $M$ is intregrable,*
3. *$\mathbb{E}\,[M_t|\mathcal{F}_s] = M_s\,, \ \forall\, 0 \leq s \leq t \leq T.$*

Having properly define a martingale, we are now going to introduce a fundamental theorem from the study of martingales. This crucial result, known as the *Doob-Meyer decomposition*, will be particularly useful later in our study, to derive some important properties of the Brownian motion.

**Theorem 2.8.** Doob-Meyer decomposition.

*Let $M$ be continuous martingale. Then, it exists a unique continuous increasing predictable process, denoted $\langle M, M\rangle$ such that $M^2 - \langle M, M\rangle$ is a continuous martingale.*

*Furthermore, for any subdivision family $(t_i^n)_{i\in[\![0,m_n]\!]}$ of $[0,T]$ such that $t_0^n = 0 < ... < t_i^n < ... < t_{m_n}^n = T$, and such that its step $|\rho_n| \underset{n\to+\infty}{\longrightarrow} 0$, we have:*

$$\langle M, M\rangle_T \ = \lim_{n\to+\infty} \sum_{i=1}^{m_n} |M_{t_i^n} - M_{t_{i-1}^n}|^2$$

**Remark 2.9.** *According to the previous theorem, the defined process $\langle M, M\rangle$ is predictable and continuous. Therefore, it follows that $\langle M, M\rangle$ is adapted.*

Another consideration of interest on (continuous) martingales consists in the study of the *covariation* of two (continuous) martingales.

**Definition 2.10.** Covariation of martingales.

*Let $M$ and $N$ be two continuous martingales.*
*Then, the covariation of $M$ and $N$, denoted $\langle M, N\rangle$, is the following process:*

$$\langle M, N\rangle \ := \ \frac{1}{2}(\langle M+N, M+N\rangle - \langle M, M\rangle - \langle N, N\rangle).$$

**Proposition 2.11.** Covariation of martingales: Properties.

*Let M and N be two continuous martingales.*
*Then,*

1. *$\langle M, N\rangle$ is a continuous finite variation process.*
2. *$\langle M, N\rangle$ is the unique predictable finite variation process such that $MN$ is a martingale.*
3. *For any subdivision family $(t_i^n)_{i\in[\![0,m_n]\!]}$ of $[0,T]$ such that $t_0^n = 0 < ... < t_i^n < ... < t_{m_n}^n = T$, and such that its step $|\rho_n| \underset{n\to+\infty}{\longrightarrow} 0$, we have:*

$$\langle M, N\rangle_T \ = \lim_{n\to+\infty} \sum_{i=1}^{m_n} \Big(M_{t_i^n} - M_{t_{i-1}^n}\Big)\Big(N_{t_i^n} - N_{t_{i-1}^n}\Big)$$

Thus, we have presented the main characteristics of martingales as well as important associated theorems and properties. We are now going to focus on semimartingales, a specific stochastic object which will be central in our study of the Black-Scholes model.

### 2.2.2. Continuous semimartingales

**Definition 2.12.** Continuous Semimartingale

*An adapted process $X := (X_t)_{t\in[0,T]}$ is a continuous semimartingale if it exists a continuous martingale $M$ and a continuous adapted finite variation process $A$ such that:*

$$X = M + A.$$

*In addition, such a decomposition of $X$ is unique.*

Semimartingales will constitute objects of interest in the construction of Itô's formula which is a keystone of the Black-Scholes model. Thus, we are going to introduce and detail the main properties associated with the use of semimartingales within the scope of stochastic calculus. In particular, we are going to highlight noteworthy results around the covariation of semimartingales.

**Proposition 2.13.** Covariation of semimartingales.

*Let $X$ and $\tilde{X}$ be two continuous semimartingales.*
*Then, for any subdivision family $(t_i^n)_{i\in[\![0,m_n]\!]}$ of $[0,T]$ such that $t_0^n = 0 < ... < t_i^n < ... < t_{m_n}^n = T$, and such that its step $|\rho_n| \underset{n\to+\infty}{\longrightarrow} 0$, we have:*

$$\langle X, \tilde{X}\rangle_T \;=\; \lim_{n\to+\infty} \sum_{i=1}^{m_n} \Big(X_{t_i^n} - X_{t_{i-1}^n}\Big)\Big(\tilde{X}_{t_i^n} - \tilde{X}_{t_{i-1}^n}\Big)$$

*where the process $\langle X, \tilde{X}\rangle$ is the covariation of $X$ and $\tilde{X}$ in the sense of Definition 2.10.*

As a following, from Proposition 2.13., we may derive the following properties for covariation of semimartingales calculation.

**Proposition 2.14.** Covariation of semimartingales: Properties.

*Let $X := M + A$ and $\tilde{X} := \tilde{M} + \tilde{A}$ be two continuous semimartingales, where $M$ and $\tilde{M}$ are two continuous martingales, and $A$ and $\tilde{A}$ are two continuous predictable finite variation processes.*
*Then, we have the following properties:*

1. $\langle X, \tilde{X}\rangle \;=\; \langle X, \tilde{M}\rangle \;=\; \langle M, \tilde{X}\rangle \;=\; \langle M, \tilde{M}\rangle$
2. $\langle X, \tilde{A}\rangle \;=\; \langle M, \tilde{A}\rangle \;=\; \langle A, \tilde{A}\rangle \;=\; 0$

*Proof.*

From Proposition 2.13.,

$$\begin{aligned}
\langle X, \tilde{X}\rangle_T \;&=\; \lim_{n\to+\infty} \sum_{i=1}^{m_n} \Big(X_{t_i^n} - X_{t_{i-1}^n}\Big)\Big(\tilde{X}_{t_i^n} - \tilde{X}_{t_{i-1}^n}\Big) \\
&=\; \lim_{n\to+\infty} \sum_{i=1}^{m_n} \Big((M_{t_i^n} + A_{t_i^n}) - (M_{t_{i-1}^n} + A_{t_{i-1}^n})\Big)\Big((\tilde{M}_{t_i^n} + \tilde{A}_{t_i^n}) - (\tilde{M}_{t_{i-1}^n} + \tilde{A}_{t_{i-1}^n})\Big) \\
&=\; \lim_{n\to+\infty} \sum_{i=1}^{m_n} \Big((M_{t_i^n} - M_{t_{i-1}^n}) + (A_{t_i^n} - A_{t_{i-1}^n})\Big)\Big((\tilde{M}_{t_i^n} - \tilde{M}_{t_{i-1}^n}) + (\tilde{A}_{t_i^n} - \tilde{A}_{t_{i-1}^n})\Big) \\
&=\; \lim_{n\to+\infty} \sum_{i=1}^{m_n} \Big(M_{t_i^n} - M_{t_{i-1}^n}\Big)\Big(\tilde{M}_{t_i^n} - \tilde{M}_{t_{i-1}^n}\Big) \;+\; \lim_{n\to+\infty} \sum_{i=1}^{m_n} \Big(M_{t_i^n} - M_{t_{i-1}^n}\Big)\Big(\tilde{A}_{t_i^n} - \tilde{A}_{t_{i-1}^n}\Big) \\
&+\; \lim_{n\to+\infty} \sum_{i=1}^{m_n} \Big(A_{t_i^n} - A_{t_{i-1}^n}\Big)\Big(\tilde{M}_{t_i^n} - \tilde{M}_{t_{i-1}^n}\Big) \;+\; \lim_{n\to+\infty} \sum_{i=1}^{m_n} \Big(A_{t_i^n} - A_{t_{i-1}^n}\Big)\Big(\tilde{A}_{t_i^n} - \tilde{A}_{t_{i-1}^n}\Big)
\end{aligned}$$

where

$$\lim_{n\to+\infty} \sum_{i=1}^{m_n} \Big(M_{t_i^n} - M_{t_{i-1}^n}\Big)\Big(\tilde{M}_{t_i^n} - \tilde{M}_{t_{i-1}^n}\Big) \;=\; \langle M, \tilde{M}\rangle_T \;,$$

$$\sum_{i=1}^{m_n} \Big(M_{t_i^n} - M_{t_{i-1}^n}\Big)\Big(\tilde{A}_{t_i^n} - \tilde{A}_{t_{i-1}^n}\Big) \;\leq\; \underbrace{\sup_{i\in[\![1,m_n]\!]} (M_{t_i^n} - M_{t_{i-1}^n})}_{\underset{n\to+\infty}{\longrightarrow} 0} \underbrace{\sum_{i=1}^{m_n} \Big(\tilde{A}_{t_i^n} - \tilde{A}_{t_{i-1}^n}\Big)}_{\underset{n\to+\infty}{\longrightarrow} \int_0^T d\tilde{A}_t < +\infty} \;\underset{n\to+\infty}{\longrightarrow}\; 0\,,$$

$$\sum_{i=1}^{m_n} \Big(A_{t_i^n} - A_{t_{i-1}^n}\Big)\Big(\tilde{M}_{t_i^n} - \tilde{M}_{t_{i-1}^n}\Big) \;\leq\; \underbrace{\sup_{i\in[\![1,m_n]\!]} (\tilde{M}_{t_i^n} - \tilde{M}_{t_{i-1}^n})}_{\underset{n\to+\infty}{\longrightarrow} 0} \underbrace{\sum_{i=1}^{m_n} \Big(A_{t_i^n} - A_{t_{i-1}^n}\Big)}_{\underset{n\to+\infty}{\longrightarrow} \int_0^T dA_t < +\infty} \;\underset{n\to+\infty}{\longrightarrow}\; 0\,,$$

$$\sum_{i=1}^{m_n} \Big(A_{t_i^n} - A_{t_{i-1}^n}\Big)\Big(\tilde{A}_{t_i^n} - \tilde{A}_{t_{i-1}^n}\Big) \;\leq\; \underbrace{\sup_{i\in[\![1,m_n]\!]} (\tilde{A}_{t_i^n} - \tilde{A}_{t_{i-1}^n})}_{\underset{n\to+\infty}{\longrightarrow} 0} \underbrace{\sum_{i=1}^{m_n} \Big(A_{t_i^n} - A_{t_{i-1}^n}\Big)}_{\underset{n\to+\infty}{\longrightarrow} \int_0^T dA_t < +\infty} \;\underset{n\to+\infty}{\longrightarrow}\; 0.$$

$\square$

To sum up, we have shown that the covariation of two semimartingales is equal to the covariation of their respective martingale component, and that the covariation of a semimartingale and a finite variation process is equal to zero.

Having introduce general definitions, properties and theorems about continuous processes and martingales, we are going to define and study specifically the Brownian motion, a specific continuous stochastic process which will be at the heart of the Black-Scholes model.

### 2.3. A specific continuous stochastic process: the Brownian motion

**Definition 2.15.** Brownian motion.

*Let* $W := (W_t)_{t\in[0,T]}$ *be a stochastic process adapted to a filtration* $\mathbb{F}$.

*Then,* $W$ *is a Brownian motion if:*

- $W_0 = 0\,,\quad \mathbb{P}-a.s.$
- $\mathcal{L}(W_t) \,=\, \mathcal{N}(0,t)\,,\ \ \forall\, t \in [0,T]$
- $\forall\, n \in \mathbb{N}^*,\ \forall\, t_i \in (t_i)_{i\in[\![0,n]\!]}$ *s.t.* $0 = t_0 < t_1 < ... < t_n \leq T$,
  *the increments* $W_{t_1} - W_{t_0}\,,\, W_{t_2} - W_{t_1}\,, ...\,,\, W_{t_n} - W_{t_{n-1}}$ *are independents random variables.*

**Proposition 2.16.** Brownian motion: Properties.

*Let* $W$ *be a Brownian motion.*

1. $W$ *is a Gaussian process with covariance function* $\mathbb{E}[W_s W_t] = s \wedge t\,,\ \ \forall\,(s,t) \in [0,T]^2$
2. $W$ *has stationary increments*
   *i.e.,* $W_t - W_s \sim \mathcal{N}(0, t-s) \,=\, \mathcal{L}(W_{t-s})\,,\ \forall\,(s,t) \in [0,T]^2\,,\ t-s \in [0,T]$
3. $W$ *is a self-similar process*
   *i.e.,* $\forall\, a > 0,\ \mathcal{L}(W_{at}) = \mathcal{L}(a^{1/2} W_t)\,,\ \forall\, t \in [0,T]$
4. $W$ *admits a continuous path modification* $\tilde{W}$.
5. $W$ *is a* $\mathbb{F}^W-$*martingale,* $\mathbb{F}^W$ *being its associated natural filtration.*

According to the described properties, we may simulate the iterative construction of a Brownian motion path in Python.[5]

Furthermore, as already mentioned, the Brownian motion will have a central role in our study of price processes within the scope of the Black-Scholes framework. Thus, we are going to introduce two important results characterizing in more detail the nature of such a stochastic process.

**Theorem 2.17.** Lévy.

*A continuous martingale* $M$ *is a Brownian motion if and only if:*

$$\langle M, M\rangle_t \,=\, t\,,\ \ \forall\, t \in [0,T]$$

The previous result represent a noteworthy characterization of the Brownian motion.
Now, we are going to derive an important result on the regularity of the Brownian motion.

[5] See Appendix I.

**Proposition 2.18.**

*Let $W := (W_t)_{t\in[0,T]}$ be a Brownian motion.*
*Then, $W$ is not a finite variation process.*
*i.e.,*

$$\mathbb{P}\left[(W_s)_{s\in[0,t]} \textit{ is a finite variation process } \right] = 0.$$

The proof of Proposition 2.18. is given in Appendix II[6].

Placing the consideration of the Brownian motion within the scope of the Black-Scholes model, the main idea is that studied price process is materialized by a Brownian motion. Thus, in order to derive a continuous wealth process associated with a portfolio, the integrability of such a process turns out to be a central point of interest. What the previous result tells us is that, due to the non-regularity of the Brownian motion, it is not possible to define an integral of the following form: "$\int_0^T H_t(\omega)dW_t(\omega)$ , $\omega \in \Omega$" using Lebesgue-Stieltjes approach.

Thus, in the next section, we will intend to introduce concepts of stochastic calculus in order to define a proper framework around the Itô's integral which will allow us to characterize a continuous wealth process. Indeed, we will see that the stochastic calculus tools introduced by Itô constitute a keystone of the Black-Scholes model we will study in the last section of this article.

[6] See Appendix II.

# 3. Stochastic integral and Itô's calculus

Having defined some key elements such as the Brownian motion or semimartingales that will be central in the construction of the Black-Scholes model, we now have to study the theoretical framework that will allow us to properly process these stochastic objects within the scope of the implementation of the Black-Scholes model. Thus, in this section, we will introduce specific objects such as stochastic integrals which will then lead us to the study of fundamental theorems and results from *Itô's calculus*.

## 3.1. First definitions

In this entire section, any martingale $M$ we will consider is supposed to be continuous.

First, we start by defining a specific space of continuous martingales $M$ on which we will perform stochastic calculus. We also define specific processes that we will use in the constriction of Itô's integral.

**Definition 3.1.** $\mathbb{H}_2$ space.

*Let* $\mathbb{H}_2$ *be a space of martingales* $M := (M_t)_{t\in[0,T]}$ *such that* $\mathbb{E}\left[|M_T|^2\right] < +\infty$.

Thereafter, we assume that any martingale $M$ we consider belongs to $\mathbb{H}_2$.

Further, we define a specific class of stochastic processes we will consider in the studied stochastic calculus framework. Such processes will be called *simple processes*.

**Definition 3.2.** Simple processes.

*A stochastic process* $H := (H_t)_{t\in[0,T]}$ *is said to be a simple process if it is of the form:*

$$H_t \;=\; \sum_{i=1}^{n} \lambda_i \mathbb{1}_{]t_{i-1},t_i]}(t)\;,\; t\in[0,T]$$

*where* $(t_i)_{i\in[\![0,n]\!]}$ *is a subdivision family of* $[0,T]$ *such that* $t_0 = 0 < ... < t_i < ... < t_n = T$, *with* $n \in \mathbb{N}^*$,
*and* $\forall\, i \in [\![1,n]\!]$, $\lambda_i \in L^\infty(\Omega, \mathcal{F}_{t_{i-1}}, \mathbb{P})$.
*i.e.,* $\lambda_i$ *is bounded and* $\mathcal{F}_{t_{i-1}}$*−measurable* $\forall\, i \in [\![1,n]\!]$.

*Let* $\mathcal{S}$ *denote the set of simple processes.*

**Remark 3.3.** *In the perspective of the integration of a wealth process, as described in sub-section 1.4., we may notice that such a definition of simple processes is appropriate to our consideration of an investment strategy* $\pi^n$*, which is predictable by construction.*

Such simple processes will be used in the definition of Itô's integral we will study later. Before that, we are going to present a preliminary result known as *Itô's isometry*.

## 3.2. Itô's isometry

Having defined simple processes, their stochastic nature leads us to introduce a specific definition for their integrability.

**Proposition 3.4.** Stochastic integral of simple processes

*Let* $I$ *be a linear operator,*

$$\begin{aligned} I\,:\,\mathcal{S} \quad &\longrightarrow \quad L^2(\Omega,\mathcal{F}_T,\mathbb{R}) \\ H \quad &\mapsto \quad I(H) = \sum_{i=1}^{n} \lambda_i (M_{t_i} - M_{t_{i-1}}). \end{aligned}$$

**Theorem 3.5.** Itô's Isometry

*The previous operator* $I(H)$ *is an isometry in the way that:*

$$\forall\, H \in \mathcal{S}\,,\quad \mathbb{E}[(I(H))^2] \;=\; \mathbb{E}\left[\int_0^T (H_t)^2 d\left\langle M,M\right\rangle_t\right]. \qquad [2]$$

*Proof.*

Let $(t_i)_{i\in[\![0,n]\!]}$ be a subdivision family of $[0,T]$ such that $t_0 = 0 < ... < t_i < ... < t_n = T$ and let $H \in \mathcal{S}$. Then,

$$\begin{aligned}
\mathbb{E}[(I(H))^2] &= \mathbb{E}\left[\left(\sum_{i=1}^{n}\lambda_i(M_{t_i} - M_{t_{i-1}})\right)^2\right] \\
&= \mathbb{E}\left[\sum_{i=1}^{n}(\lambda_i)^2(M_{t_i} - M_{t_{i-1}})^2 \ + \ 2\sum_{1\le i<j\le n}\lambda_i\lambda_j(M_{t_i} - M_{t_{i-1}})(M_{t_j} - M_{t_{j-1}})\right] \\
&= \sum_{i=1}^{n}\mathbb{E}\left[(\lambda_i)^2(M_{t_i} - M_{t_{i-1}})^2\right] \ + \ 2\sum_{1\le i<j\le n}\mathbb{E}\left[\lambda_i\lambda_j(M_{t_i} - M_{t_{i-1}})(M_{t_j} - M_{t_{j-1}})\right].
\end{aligned}$$

By definition, $\forall\, i \in [\![1,n]\!]\,,\ \lambda_i$ is $\mathcal{F}_{t_{i-1}}-$*measurable* and $i\ <\ j \Longrightarrow \lambda_i,\ \lambda_j,\ (M_{t_i} - M_{t_{i-1}})$ are $\mathcal{F}_{t_{j-1}}-$*measurable*. Therefore,

$$\begin{aligned}
\mathbb{E}\left[\lambda_i\lambda_j(M_{t_i} - M_{t_{i-1}})(M_{t_j} - M_{t_{j-1}})\right] &= \mathbb{E}\left[\mathbb{E}\left[\lambda_i\lambda_j(M_{t_i} - M_{t_{i-1}})(M_{t_j} - M_{t_{j-1}})\,|\,\mathcal{F}_{t_{j-1}}\right]\right] \\
&= \mathbb{E}\left[\lambda_i\lambda_j(M_{t_i} - M_{t_{i-1}})\mathbb{E}\left[(M_{t_j} - M_{t_{j-1}})\,|\,\mathcal{F}_{t_{j-1}}\right]\right] \\
&= \mathbb{E}\left[\lambda_i\lambda_j(M_{t_i} - M_{t_{i-1}})\underbrace{\left(\mathbb{E}\left[M_{t_j}\,|\,\mathcal{F}_{t_{j-1}}\right] - M_{t_{j-1}}\right)}_{=0}\right] \\
&= 0.
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathbb{E}[(I(H))^2] &= \sum_{i=1}^{n}\mathbb{E}\left[(\lambda_i)^2(M_{t_i} - M_{t_{i-1}})^2\right] \\
&= \sum_{i=1}^{n}\mathbb{E}\left[(\lambda_i)^2\mathbb{E}\left[(M_{t_i} - M_{t_{i-1}})^2\,|\,\mathcal{F}_{t_{i-1}}\right]\right] \\
&= \sum_{i=1}^{n}\mathbb{E}\left[(\lambda_i)^2\mathbb{E}\left[M_{t_i}^2 - M_{t_{i-1}}^2\,|\,\mathcal{F}_{t_{i-1}}\right]\right] \\
&= \sum_{i=1}^{n}\mathbb{E}\left[(\lambda_i)^2\mathbb{E}\left[(M_{t_i}^2 - \langle M,M\rangle_{t_i}) - (M_{t_{i-1}}^2 - \langle M,M\rangle_{t_{i-1}}) + \langle M,M\rangle_{t_i} - \langle M,M\rangle_{t_{i-1}}\,|\,\mathcal{F}_{t_{i-1}}\right]\right].
\end{aligned}$$

From the Doob-Meyer decomposition (Theorem 2.8.), we know that the process $M^2 - \langle M,M\rangle$ is a martingale, so we deduce that:

$$\mathbb{E}\left[(M_{t_i}^2 - \langle M,M\rangle_{t_i}) - (M_{t_{i-1}}^2 - \langle M,M\rangle_{t_{i-1}})\,|\,\mathcal{F}_{t_{i-1}}\right] \ = \ 0\,,\ \ \forall\, i \in [\![1,n]\!].$$

Hence,

$$\begin{aligned}
\mathbb{E}[(I(H))^2] &= \sum_{i=1}^{n}\mathbb{E}\left[(\lambda_i)^2\mathbb{E}\left[\langle M,M\rangle_{t_i} - \langle M,M\rangle_{t_{i-1}}\,|\,\mathcal{F}_{t_{i-1}}\right]\right] \\
&= \mathbb{E}\left[\sum_{i=1}^{n}(\lambda_i)^2(\langle M,M\rangle_{t_i} - \langle M,M\rangle_{t_{i-1}})\right] \\
&= \mathbb{E}\left[\int_0^T(\lambda_t)^2 d\,\langle M,M\rangle_t\right].
\end{aligned}$$

□

This isometry property, coupled with the convergence of a specific Cauchy sequence in a more general space of martingales will allow us to generalize this notion of stochastic integral, leading us to the definition of well-known Itô's integral.

### 3.3. Itô's integral

Within the scope of the construction of Itô's integral, we first define a specific space of stochastic processes related to $\mathbb{H}_2-$martingales.

**Definition 3.6.** $L^2(M)$.

*Denote $L^2(M)$ the set of predictable processes $H$ such that:*

$$\|H\|^2_{L^2(M)} \;=\; \mathbb{E}\left[\int_0^T (H_t)^2 d\left\langle M, M\right\rangle_t\right] < +\infty$$

**Lemma 3.7.**

*The space $\mathcal{S}$ is dense in $L^2(M)$.*

Now we have all the necessary theoretical elements to define and construct Itô's integral, for which we are going to develop some elements of proof of its existence and uniqueness.

**Definition 3.8.** Itô's integral.

*Let $H \in L^2(M)$ and $M \in \mathbb{H}_2$.*
*Then, Itô's integral, denoted $I(H)$, is defined as:*

$$I(H) := \int_0^T H_t dM_t.$$

*$I(H) \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ is defined according to the following proposition:*

$$\exists\; (H^n)_n \text{ an approximate sequence of } H \text{ such that } \lim_{n\to+\infty} \mathbb{E}\left[(I(H^n) - I(H))^2\right] \;=\; 0.$$

*Proof.*

From Lemma 3.7., $\forall$ H $\in L^2(M)$, $\exists\; (H^n)_n \subseteq \mathcal{S}$ such that $(H^n)_n \xrightarrow[n\to+\infty]{\|\cdot\|_{L^2(M)}} H$,

$$i.e., \quad \lim_{n\to+\infty} \|H^n - H\|^2_{L^2(M)} \;=\; 0 \quad\iff\quad \lim_{n\to+\infty} \mathbb{E}\left[\int_0^T (H^n_t - H_t)^2 d\left\langle M, M\right\rangle_t\right] \;=\; 0.$$

In addition, since $(H^n)_n \subseteq \mathcal{S}$, from Itô's isometry (Theorem 3.5.), it follows that

$$\|H^n\|^2_{L^2(M)} \;=\; \mathbb{E}\left[\int_0^T (H^n_t)^2 d\left\langle M, M\right\rangle_t\right] \;=\; \mathbb{E}\left[(I(H))^2\right].$$

Thereby, we deduce that $(I(H^n))_n$ is a Cauchy sequence in $L^2(M)$, which proves the existence of $I(H) \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ defined such that

$$\lim_{n\to+\infty} \mathbb{E}\left[(I(H^n) - I(H))^2\right] \;=\; 0.$$

□

**Remark 3.9.** *The object $I(H) := \int_0^T H_t dM_t$ we have just defined is an integral fully probabilistic, it is a random variable which can be evaluated as $\omega \mapsto \left(\int_0^T H_t dM_t\right)(\omega)$.*

We have defined Itô's integral which will play an important role in the derivation of Itô's formula. Itô's formula will be central in the construction of the Black-Scholes model. Before to define Itô's formula, we are going to introduce some specific features as well as important properties of Itô's integral.

**Proposition 3.10.**

*Let $H$ belong to $L^2(M)$.*
*Then,*

$$\mathbb{E}[I(H)] \;=\; 0 \qquad \text{and} \qquad \mathbb{E}[(I(H))^2] \;=\; \mathbb{E}\left[\int_0^T (H_t)^2 d\left\langle M, M\right\rangle_t\right]$$

*Proof.*

Let $(H^n)_n$ be a sequence of elements of $\mathcal{S}$, converging to $H$ in $L^2(M)$.
$\forall\, K \in \mathcal{S}\,,\; \mathbb{E}[I(K)] \,=\, 0.$
Indeed,

$$\begin{aligned}
\mathbb{E}[I(K)] &= \mathbb{E}\left[\sum_{i=1}^{n} \lambda_i (M_{t_i} - M_{t_{i-1}})\right] \\
&= \sum_{i=1}^{n} \mathbb{E}\left[\mathbb{E}\left[\lambda_i (M_{t_i} - M_{t_{i-1}})\,|\,\mathcal{F}_{t_{i-1}}\right]\right] \\
&= 0.
\end{aligned}$$

Then,

$$\begin{aligned}
\mathbb{E}[I(H)] &= \mathbb{E}[I(H) - I(H^n)] \\
&\leq \mathbb{E}[(I(H) - I(H^n))^2]^{\frac{1}{2}} \quad (by\;\; Cauchy-Schwarz\;\; inequality) \\
where\;\; \mathbb{E}[(I(H) - I(H^n))^2]^{\frac{1}{2}} &= \|H - H^n\|_{L^2(M)} \underset{n \to +\infty}{\longrightarrow} 0
\end{aligned}$$

For the second-order moment,

$$\begin{aligned}
\mathbb{E}[(I(H))^2] &= \mathbb{E}[(I(H) - I(H^n) + I(H^n))^2] \\
&= \underbrace{\mathbb{E}[(I(H) - I(H^n))^2]}_{\underset{n\to+\infty}{\longrightarrow} 0} + \underbrace{2\,\mathbb{E}[(I(H) - I(H^n))(I(H^n))]}_{\substack{\underset{n\to+\infty}{\longrightarrow} 0 \\ (by\;\; Cauchy-Schwarz\;\; inequality)}} + \underbrace{\mathbb{E}[(I(H^n))^2]}_{= \|H^n\|^2_{L^2(M)} \underset{n\to+\infty}{\longrightarrow} \|H\|^2_{L^2(M)}}
\end{aligned}$$

Hence,

$$\mathbb{E}[(I(H))^2] \;=\; \|H\|^2_{L^2(M)} \;=\; \mathbb{E}\left[\int_0^T (H_t)^2 d\left\langle M, M\right\rangle_t\right].$$

□

From the previous proposition, we may derive an result of interest about the nature of Itô's integral.

**Theorem 3.11.**

*Let $M$ belong to $\mathbb{H}_2$ and $H$ belong to $L^2(M)$.*
*Then, the process $X := (X_t)_{t\in[0,T]}$ defined such that $X_t \,=\, \int_0^t H_s dM_s$ is a continuous martingale, with $\mathbb{E}[\sup\limits_{t\in[0,T]} |X_t|^2] < +\infty$*

*Proof.*

From the definition of Itô's integral $I(H)$,
$\forall\, t \in [0,T]\,,\; \mathbb{E}\left[|X_t|^2\right] \;=\; \mathbb{E}\left[\int_0^t (H_s)^2 d\left\langle M, M\right\rangle_s\right] \;\leq\; \mathbb{E}\left[\int_0^T (H_s)^2 d\left\langle M, M\right\rangle_s\right] \;=\; \mathbb{E}\left[|I(H)|^2\right] \;<\; +\infty.$
By Cauchy-Schwarz inequality, it follows that $\forall\, t \in [0,T]\,,\; \mathbb{E}\left[|X_t|\right] \;<\; +\infty.$

Also,

$$\begin{aligned}\mathbb{E}\left[(X_t - X_s)\,|\,\mathcal{F}_s\right] &= \mathbb{E}\left[\int_s^t H_u dM_u \,|\, \mathcal{F}_s\right] \\ &= \mathbb{E}\left[I(H.\mathbb{I}_{[s,t]})\,|\,\mathcal{F}_s\right] \\ &= 0.\end{aligned}$$

Then, from the two previous results, we deduce that $X := (\int_0^t H_s dM_s)_{t\in[0,T]}$ is a continuous martingale.
Finally, from Doob's inequality[7] and considering $p = 2$, we have:

$$\mathbb{E}[\sup_{t\in[0,T]} |X_t|^2] \;\leq\; 4\,\mathbb{E}[|X_T|^2] \;<\; +\infty.$$

□

We introduce a last noteworthy result that will be useful later in our approach of Itô's formula.

**Proposition 3.12.** *Let $M$ and $N$ be two martingales belonging to $\mathbb{H}_2$, and $(H, K) \in L^2(M) \times L^2(M)$. Then,*

$$\left\langle \int_0^{\cdot} H_t dM_t, N \right\rangle = \int_0^{\cdot} H_t d\left\langle M, N\right\rangle_t$$

*In particular,*

$$\left\langle \int_0^{\cdot} H_t dM_t, \int_0^{\cdot} K_t dNt \right\rangle = \int_0^{\cdot} H_t K_t d\left\langle M, N\right\rangle_t$$

All these considerations about Itô's integral allow us to make a consistent parallelism between the discrete-time and continuous-time approaches for the pricing of options, in line with the continuous-time model extension presented in sub-section 1.4. of this article. Indeed, we might define the investment strategie $\pi$ as an element of $L^2(S)$, $S$ being a martingale characterising the price process. Thus, we would have a sequence $(\pi_n)_n$, $\pi_n \in \mathcal{S}$ , $\forall\, n$, which would approximate $\pi \in L^2(S)$, and whose consideration over a subdivision $(t_i^n)_{i\in[\![0,n]\!]}$ of $[0, T]$ would correspond to the notion of investment strategy associated to a portfolio as defined in Definition 1.2. Therefore, the stochastic integral of $\pi$ against $S$ would constitute a consistent construction of the notion of wealth process, continuous in time, as the limit of the described discrete model.
As a final remark, we notice that the martingale property of the (discounted) price process and of the wealth process we observed in the discrete model is still preserved in this continuous-time framework.

### 3.4. Itô's Formula

In this section we will introduce and study one of the major tool of the stochastic calculus theory: Itô's formula. Further, this formula will help us to compute both specific derivatives and integrals of price and wealth processes within the scope of the Black-Scholes model.

**Definition 3.13.**

*Let $X = M + A$ be a semimartingale with $M \in \mathbb{H}_2$.*
*For any $H \in L^2(M)$ such that $\sup_{t\in[0,T]} |H_t| \;<\; +\infty$, we have:*

$$\int_0^T H_t dX_t \;=\; \int_0^T H_t dMt \;+ \int_0^T H_t dA_t$$

*Furthermore, defining $Y_t = \int_0^t H_s dX_s$, we have equivalently:*

$$dY_t = H_t dM_t + H_t dA_t$$

[7] See Appendix III.

**Proposition 3.14.** *Let $X$ be a continuous semimartingale, and $H$ be a continuous predictable process. Then, for any subdivision family $(t_i^n)_{i\in[\![0,m_n]\!]}$ of $[0,T]$ such that $t_0^n = 0 < ... < t_i^n < ... < t_{m_n}^n = T$, whose step $|\rho_n| \underset{n\to+\infty}{\longrightarrow} 0$, we have:*

$$\int_0^T H_t dX_t \overset{\mathbb{P}}{=} \lim_{n\to+\infty} \sum_{i=1}^{m_n} H_{t_{i-1}^n}(X_{t_i^n} - X_{t_{i-1}^n}).$$

**Theorem 3.15.** Itô's formula.
*Let $F : \mathbb{R} \to \mathbb{R}$, $F \in \mathcal{C}^2(\mathbb{R})$.*
*Let $X$ be a continuous semimartingale.*
*Then, we may define $(F(X_t))_{t\in[0,T]}$ such that:*

$$F(X_t) \ = \ F(X_0) + \int_0^t F'(X_s)dX_s + \frac{1}{2}\int_0^t F''(X_s)d\left\langle X,X\right\rangle_s \ , \ \forall \ t \in [0,T] \qquad [3]$$

*where integrals are stochastic integrals in the sense of Itô's integral (Definition 3.8.).*
*Defined in this way, the process $(F(X_t))_{t\in[0,T]}$ is a semimartingale.*
*Eq.(3) can also be written as:*

$$dF(X_t) \ = \ F'(X_t)dX_t + \frac{1}{2}F''(X_t)d\left\langle X,X\right\rangle_t \ , \ \forall \ t \in [0,T]. \qquad [4]$$

*Proof.*
We only intend to give some elements of proof here, to get the main intuition behind this Itô's formula.
For any subdivision family $(t_i^n)_{i\in[\![0,m_n]\!]}$ of $[0,T]$ such that $t_0^n = 0 < ... < t_i^n < ... < t_{m_n}^n = T$, whose step $|\rho_n| \underset{n\to+\infty}{\longrightarrow} 0$, Taylor formula gives us:

$$\begin{aligned} F(X_T) \ - \ F(X_0) \ &= \ \sum_{i=1}^{m_n} F(X_{t_i^n} - F(X_{t_{i-1}^n}) \\ &= \ \sum_{i=1}^{m_n} F'(X_{t_{i-1}^n})(X_{t_i^n} - X_{t_{i-1}^n}) + \frac{1}{2}\sum_{i=1}^{m_n} F''(X_{\theta_i^n})(X_{t_i^n} - X_{t_{i-1}^n})^2 \\ &= \ \sum_{i=1}^{m_n} F'(X_{t_{i-1}^n})(X_{t_i^n} - X_{t_{i-1}^n}) + \frac{1}{2}\sum_{i=1}^{m_n} F''(X_{t_{i-1}^n})(X_{t_i^n} - X_{t_{i-1}^n})^2 \\ &+ \ \frac{1}{2}\sum_{i=1}^{m_n}(F''(X_{\theta_i^n}) - F''(X_{t_{i-1}^n}))(X_{t_i^n} - X_{t_{i-1}^n})^2 \end{aligned}$$

where $\theta_i^n$ is a random variable taking its values in $[t_{i-1}^n, t_i^n]$.
Then, from Proposition 3.14., we deduce that:

$$\lim_{n\to+\infty} \sum_{i=1}^{m_n} F'(X_{t_{i-1}^n})(X_{t_i^n} - X_{t_{i-1}^n}) \overset{\mathbb{P}}{=} \int_0^T F'(X_t)dX_t \ , \ \ \mathbb{P}-a.s.$$

Further developments would allow us to show that:

$$\lim_{n\to+\infty} \left( \frac{1}{2}\sum_{i=1}^{m_n} F''(X_{t_{i-1}^n})(X_{t_i^n} - X_{t_{i-1}^n})^2 + \frac{1}{2}\sum_{i=1}^{m_n}(F''(X_{\theta_i^n}) - F''(X_{t_{i-1}^n}))(X_{t_i^n} - X_{t_{i-1}^n})^2 \right) \overset{\mathbb{P}}{=} \frac{1}{2}\int_0^T F''(X_t)d\left\langle X,X\right\rangle_t \ , \ \ \mathbb{P}-a.s.$$

$\square$

**Remark 3.16.** *Recalling the result given by the fundamental theorem of calculus for $\mathcal{C}^2$ functions, i.e., $F(X_t) \ = \ F(X_0) + \int_0^t F'(X_s)dX_s$, we observe that Itô's formula applied to processes with non-regular trajectories adds a corrective term $\frac{1}{2}\int_0^t F''(X_s)d\left\langle X,X\right\rangle_s$.*

A generalization of Itô's formula is given in Appendix IV.

Furthermore, an importannt result associated to Itô's formula lies in the extension of the integration by parts principle to this stochastic framework.

**Theorem 3.17.** Integration by parts formula.

*Let $X$ and $Y$ be two continuous semimartingales.*
*Then the process $XY$ is a semimartingale, and:*

$$X_tY_t \; = \; X_0Y_0 \, + \int_0^t X_s dY_s \, + \int_0^t Y_s dX_s \, + \, \langle X, Y \rangle_t \;, \; \forall \; t \in [0, T]. \qquad [5]$$

*Eq.(5) can also be written as:*

$$d(X_tY_t) \; = \; X_t dY_t \, + \, Y_t dX_t \, + \, d \, \langle X, Y \rangle_t \;, \; \forall \; t \in [0, T]. \qquad [6]$$

We have defined the main objects and results of Itô's calculus. We are now going to introduce some more elements of stochastic calculus, more specifically related to the Black-Scholes framework we will study later.

### 3.5. Martingale representation theorem

Let $W := (W_t)_{t \in [0,T]}$ be a Brownian motion and its natural filtration $\mathbb{F}^W := (\mathcal{F}_t^W)_{t \in [0,T]}$.

From Itô's integral (Definition 3.8.), for a given process $H \in L^2(W)$, the stochastic integral $(\int_0^t H_s dW_s)_{t \in [0,T]}$ is a martingale.

The martingale representation theorem states that, for a Brownian motion, the reciprocal is valid, i.e., if $(\int_0^t H_s dM_s)_{t \in [0,T]}$ is a martingale, then the process $M$ associated to $H$ is a Brownian motion ($M \equiv W$a Bronian motion) and $H \in L^2(W)$.

**Theorem 3.18.** Martingale representation theorem.

*Let $M$ be a martingale belonging to $\mathbb{H}_2$.*
*Then, it exists a unique predictable process $H \in L^2(M)$ such that:*

$$M_t \; = \; \mathbb{E}[M_T] \, + \int_0^t H_s dW_s \, , \; \forall \; t \in [0, T]$$

**Remark 3.19.** *The previous theorem may be stated in a more general way, for martingales not necessarily in $\mathbb{H}_2$. But in this article we are only interested in martingales $M$ such that $\mathbb{E}\left[|M_T|^2\right] < +\infty$.*

Theorem 3.18. represent a important result specific to the Brownian motion. It illustrates the central place that the Brownian motion will have within the scope of our study of the Black-Scholes model. Indeed, the martingale representation theorem indicates that any martingale adapted to the natural filtration of a Brownian motion may be expressed as a stochastic integral with respect to the Brownian motion.

After having defined the main objects of interest from Itô's calculus, we are going to introduce new notions from the stochastic calculus. Indeed, first, we are going to define the *stochastic exponential* before to present an important theorem associated to its use. Then, we will study *stochastic differential equations*.

### 3.6. Stochastic exponential

**Definition 3.20.** Stochastic exponential.

*Let $M$ be a continuous martingale.*
*Then, the process $X := (X_t)_{t \in [0,T]}$ defined such that:*

$$X_t \; = \; \exp(M_t - \frac{1}{2} \langle M, M \rangle_t)$$

*is called the stochastic exponential of $M$.*
*Such a process is commonly denoted by $\mathcal{E}(M)$.*

We are going to introduce some important properties associated to stochastic exponential processes.

**Proposition 3.21.**

*Let $M$ be a continous martingale, and $X := \mathcal{E}(M)$.*
*Then, we have the following properties:*

1. $dX_t \;=\; X_t dM_t$.
2. *$X$ is a continuous martingale.*[8]
3. $\mathbb{E}[X_T] \;\leq\; 1$.

We are going to give elements of proof of the previous properties through an example of stochastic exponential applied to the Brownian motion.

Thus, let $W := (W_t)_{t\in[0,T]}$ be a Brownian motion.
$W$ being a $\mathbb{F}^W$-martingale (Property 5., Proposition 2.16.), we may define its stochastic exponential $\mathcal{E}(W)$ as:

$$X_t \;:=\; \mathcal{E}(W)_t \;=\; \exp(W_t - \frac{1}{2}\left\langle W,W\right\rangle_t) \;=\; \exp(W_t - \frac{1}{2}t)\,,\; \forall\, t \in [0,T].$$

Let $Y := (Y_t)_{t\in[0,T]}$ be the following process: $Y_t \;=\; W_t - \frac{1}{2}t\,,\; \forall\, t \in [0,T]$. $Y$ is a semimartingale.
Then, considering the function:

$$\begin{array}{rccl} F \;: & \mathbb{R} & \to & \mathbb{R}_+ \\ & x & \mapsto & F(x) = \exp(x) \end{array}$$

and applying Itô's formula to $Y$, we obtain:

$$F(Y_t) \;=\; F(Y_0) \,+\, \int_0^t F'(Y_s)dY_s \,+\, \frac{1}{2}\int_0^t F''(Y_s)d\left\langle Y,Y\right\rangle_s \,,\; \forall\, t \,\in [0,T]. \tag{7}$$

First, by definition of processes $X$ and $Y$, we have $X_t \;=\; F(Y_t)\,,\; \forall\, t \,\in [0,T]$.

In addition, since we consider $F(.) \;=\; \exp(.)$, we have $F(Y_t) \;=\; F'(Y_t) \;=\; F''(Y_t)$.

Finally, $dY_t \;=\; d(W_t - \frac{1}{2}\left\langle W,W\right\rangle_t) \;=\; dW_t - \frac{1}{2}dt$,
and $d\left\langle Y,Y\right\rangle_t \;=\; d\left\langle W,W\right\rangle_t \;=\; dt\,,\; \forall\, t \,\in [0,T]$ (from Proposition 2.14. and Theorem 2.17.).

Then, rewriting Eq.(7), we have:

$$\begin{aligned} F(Y_t) \;&=\; F(W_0 - \frac{1}{2}.0) \,+\, \int_0^t F(Y_s)dY_s \,-\, \frac{1}{2}\int_0^t F(Y_s)ds \,+\, \frac{1}{2}\int_0^t F(Y_s)\underbrace{d\left\langle Y,Y\right\rangle_s}_{=ds},\;\; \forall\, t \,\in [0,T]. \\ \Longleftrightarrow \qquad X_t \;&=\; 1 + \int_0^t X_s dW_s\,,\;\; \forall\, t \,\in [0,T]. \end{aligned}$$

**Remark 3.22.** *In particular, the previous development demonstrates the Property 1. of Proposition 3.21.*

We have defined the stochastic exponential object as well as some properties related to its use. Now, we are going to introduce an important theorem which will allow us to define a probability measure adapted to our study of the price process within the scope of Black-Scholes model. Indeed, the following theorem aims to describe the changes in stochastic processes induced by a change in the probability measure. It is widely use as it allows to transform studied processes from the historic probability measure (describing the probability that an asset's price takes a specific value in the future) to the risk-neutral measure which is central when carrying out the pricing of a derivative.

[8] The martingale property for such stochastic exponential processes is a delicate question to be addressed. Indeed, it implies a notion of locality for martingales, which is not developed in this article. However, we may note that sufficient conditions exist to ensure that $X := \mathcal{E}(M)$ is a "real" martingale, such as *Novikov's criterion.*

**Theorem 3.23.** Girsanov.

*Let $W := (W_t)_{t\in[0,T]}$ be a Brownian motion.*
*Let $\theta = (\theta_t)_{t\in[0,T]}$ be a predictable process (according to the Brownian filtration), and assumed to be bounded.*
*We set:*

$$W_t^\theta \;=\; W_t + \int_0^t \theta_s ds\,,\;\;\forall\, t\in[0,T].$$

*Then, the stochastic exponential $\mathcal{E}(-\int_0^{\cdot}\theta_s dW_s)$ is a (real) martingale, and $W^\theta$ is a $\mathbb{P}^\theta$-Brownian motion, with $\mathbb{P}^\theta$ defined such that:*

$$\frac{d\mathbb{P}^\theta}{d\mathbb{P}} \;=\; \mathcal{E}(-\int_0^{\cdot}\theta_s dW_s).$$

*In addition, $\mathbb{P}^\theta \sim \mathbb{P}$.*

*Proof.*

The main idea of this proof is to determine a specific process $X^\theta$ such that $X^\theta := \frac{d\mathbb{P}^\theta}{d\mathbb{P}}$, and such that the process $W^\theta$ would be a $\mathbb{P}^\theta$-martingale (and therefore a $\mathbb{P}^\theta$-Brownian motion by nature).

In a similar way as for our first approach of stochastic exponential of a Brownian motion, we intend to determine a solution for $X^\theta$, of the form of a stochastic exponential and satisfying:

$$X_t^\theta \;=\; 1 - \int_0^t \theta_s X_s^\theta dW_s \quad\Longleftrightarrow\quad dX_t^\theta \;=\; -\,\theta_t X_t^\theta dW_t\,,\quad \forall\, t\,\in[0,T].$$

Let $M := (M_t)_{t\in[0,T]}$ be a continuous martingale defined such that:

$$M_t \;=\; -\int_0^t \theta_s dW_s\,,\quad \forall\, t\,\in[0,T].$$

Then, setting $X^\theta := \mathcal{E}(M)$, we have:

$$\begin{aligned}
\forall\, t\,\in[0,T]\,,\quad X_t^\theta \;&=\; \exp\left(-\int_0^t \theta_s dW_s \,-\, \frac{1}{2}\left\langle -\int_0^{\cdot}\theta_s dW_s\,, -\int_0^{\cdot}\theta_s dW_s\right\rangle_t\right)\\
&=\; \exp\left(-\int_0^t \theta_s dW_s \,-\, \frac{1}{2}\int_0^t (\theta_s)^2 d\left\langle W,W\right\rangle_s\right)\\
&=\; \exp\left(-\int_0^t \theta_s dW_s \,-\, \frac{1}{2}\int_0^t (\theta_s)^2 ds\right)
\end{aligned}$$

Now we define $Y := (Y_t)_{t\in[0,T]}$ such that:

$$Y_t \;=\; -\int_0^t \theta_s dW_s \,-\, \frac{1}{2}\int_0^t (\theta_s)^2 ds\,,\quad \forall\, t\,\in[0,T].$$

We notice that $Y$ is a semimartingale, with a martingale component $\big(-\int_0^{\cdot}\theta_s dW_s\big)$ and a finite variation component $\big(-\frac{1}{2}\int_0^{\cdot}(\theta_s)^2 ds\big)$.
Then, considering again the function

$$\begin{aligned}
F\,:\,\mathbb{R}\;&\to\;\mathbb{R}_+\\
x\;&\mapsto\;F(x) = \exp(x)\,,
\end{aligned}$$

we may express $X^\theta$ as $X_t^\theta \;=\; F(Y_t)\,,\;\forall\, t\,\in[0,T]$.
In addition, applying Itô's formula, we obtain:

$$dF(Y_t) \;=\; F'(Y_t)dY_t \,+\, \frac{1}{2}F''(Y_t)d\left\langle Y,Y\right\rangle_t\,,\;\forall\, t\,\in[0,T]$$

where

$$
\begin{aligned}
F(Y_t) \;&=\; F'(Y_t) \;=\; F''(Y_t) \;=\; X_t^\theta\,,\\
dY_t \;&=\; d\left(-\int_0^t \theta_s dW_s \,-\, \frac{1}{2}\int_0^t (\theta_s)^2 ds\right) \;=\; -\,\theta_t dW_t - \frac{1}{2}(\theta_t)^2 dt\,,\\
d\left\langle Y, Y\right\rangle_t \;&=\; d\left\langle -\int_0^\cdot \theta_s dW_s\,,\, -\int_0^\cdot \theta_s dW_s\right\rangle_t \;=\; d\left(\int_0^t (\theta_s)^2 d\left\langle W, W\right\rangle_s\right) \;=\; d\left(\int_0^t (\theta_s)^2 ds\right) \;=\; (\theta_t)^2 dt.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
dX_t^\theta \;&=\; X_t^\theta(-\,\theta_t dW_t - \frac{1}{2}(\theta_t)^2 dt) \,+\, \frac{1}{2}X_t^\theta(\theta_t)^2 dt\\
&=\; -\,X_t^\theta\theta_t dW_t.
\end{aligned}
$$

We recognize the characteristic form of the solution sought for the process $X^\theta$, and we deduce that:

$$
X^\theta \;:=\; \mathcal{E}\left(-\int_0^\cdot \theta_s dW_s\right) \;:=\; 1 \,-\, \int_0^\cdot \theta_s X_s^\theta dW_s.
$$

Thus, for the rest of the proof, we consider the process $X^\theta \;:=\; \mathcal{E}\left(-\int_0^\cdot \theta_s dW_s\right)$.

Now, returning to the process $W^\theta$, we are going to study its behavior with respect to $X^\theta$.

By the integration by parts formula (Theorem 3.17.), we have:

$$
d(W_t^\theta X_t^\theta) \;=\; W_t^\theta dX_t^\theta \,+\, X_t^\theta dW^\theta \,+\, d\left\langle W^\theta, X^\theta\right\rangle_t\,,\;\; \forall\, t \in [0,T].
$$

where

$$
\begin{aligned}
dX_t^\theta \;&=\; X_t^\theta\theta_t dW_t\,,\\
dW_t^\theta \;&=\; d\left(W_t \,+\, \int_0^t \theta_s ds\right) \;=\; dW_t + \theta_t dt\,,\\
d\left\langle W^\theta, X^\theta\right\rangle_t \;&=\; d\left\langle W \,+\, \int_0^\cdot \theta_s ds\,,\, 1 \,-\, \int_0^\cdot \theta_s X_s^\theta dW_s\right\rangle_t\\
&=\; \underbrace{d\left\langle W, 1\right\rangle_t}_{=\,0} \,-\, d\left\langle W, \int_0^\cdot \theta_s X_s^\theta dW_s\right\rangle_t \,+\, \underbrace{d\left\langle\int_0^\cdot \theta_s ds, 1\right\rangle_t}_{=\,0} \,-\, \underbrace{d\left\langle\int_0^\cdot \theta_s ds, \int_0^\cdot \theta_s X_s^\theta dW_s\right\rangle_t}_{=\,0}\\
&=\; -\,d\left(\int_0^t \theta_s X_s^\theta d\left\langle W, W\right\rangle_s\right)\\
&=\; -\,d\left(\int_0^t \theta_s X_s^\theta ds\right)\\
&=\; -\,\theta_t X_t^\theta dt.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
d(W_t^\theta X_t^\theta) \;&=\; W_t^\theta(X_t^\theta\theta_t dW_t) \,+\, X_t^\theta(dW_t + \theta_t dt) \,-\, \theta_t X_t^\theta dt\\
&=\; X_t^\theta(W_t^\theta\theta_t + 1)dW_t.
\end{aligned}
$$

So, from the martingale representation theorem (Theorem 3.18.), we deduce that the process $W^\theta X^\theta$ is a $\mathbb{P}$-martingale.

Finally, defining $\frac{d\mathbb{P}^\theta}{d\mathbb{P}} \;=\; X^\theta$, we introduce a new probability measure $\mathbb{P}^\theta$. With this new probability measure $\mathbb{P}^\theta$, we observe that, for any stochastic process $Z := (Z_t)_{t\in[0,T]}$ defined on $\mathbb{P}$,

$$
\mathbb{E}^{\mathbb{P}^\theta}[Z_t] \;=\; \mathbb{E}^{\mathbb{P}}[Z_t X_t^\theta]\,,\;\; \forall\, t \in [0,T].
$$

Given the fact that $W^\theta X^\theta$ is a $\mathbb{P}$-martingale, it follows that $W^\theta$ is a $\mathbb{P}^\theta$-martingale.

Indeed,

$$
\forall\, 0 \leq s \leq t \leq T\,,\quad \mathbb{E}^{\mathbb{P}^\theta}\left[W_t^\theta \,|\, \mathcal{F}_s\right] \;=\; \mathbb{E}^{\mathbb{P}}\left[\frac{W_t^\theta X_t^\theta}{X_s^\theta}\;\Big|\,\mathcal{F}_s\right] \;=\; \frac{W_s^\theta X_s^\theta}{X_s^\theta} \;=\; W_s^\theta.
$$

$\square$

This theorem will be very usefull to demonstrate that it exists a unique equivalent martingale measure within the Black-Scholes framework, and it will also allow us to study the risky asset's dynamics in terms of this risk-neutral measure.

Before to focus on the direct construction and implementation of the Black-Scholes model, we are going to end this section dedicated to the fundamentals of stochastic calculus by studying a resolution approach for *stochastic differential equations.*

### 3.7. Stochastic differential equations

Here, we are going to develop the resolution of a stochastic differential equation, considering a crucial example taken the Black-Scholes model. Indeed, we are going to derive the solution associated to the stochastic differential equation which governs the risky asset's price process dynamics in the Black-Scholes model.

First, we place our development on a $(\Omega, \mathcal{F}, \mathbb{P})$ probability space, where a Brownian motion $W := (W_t)_{t\in[0,T]}$ is defined, with its natural filtration $\mathbb{F}^W$.

Let's consider a price process $S := (S_t)_{t\in[0,T]}$ defined such that:

$$\frac{dS_t}{S_t} \;=\; \mu dt \,+\, \sigma dW_t \;, \quad \forall\, t \in [0,T] \;, \quad (\mu,\sigma) \in \mathbb{R} \times \mathbb{R}_+^*. \tag{8}$$

Eq.(8) describes price dynamics and, defined in this way, it appears that $\mu$ characterizes a trend parameter when $\sigma$ embodies a volatility parameter.

In another way, the price dynamics is better illustrated in the following form of Eq.(8):

$$\begin{aligned} dS_t \;&=\; \mu S_t dt \,+\, \sigma S_t dW_t \\ \Longleftrightarrow \qquad S_t \;&=\; S_0 \,+\, \mu\int_0^t S_s ds \,+\, \sigma\int_0^t S_s dW_s. \end{aligned} \tag{9}$$

**Remark 3.24.** *Such a definition of* $S := (S_t)_{t\in[0,T]}$ *constitutes a linear equation.*

Given that $S$ represents a price process, it is supposed to be positive, and we make the assumption that $S_t > 0\,,\ \forall\, t \in [0,T]$.

In addition, defined in this way, $S$ is a (continuous) semimartingale. Thus, we may apply Itô's formula to it, with

$$\begin{aligned} F \,:\, \mathbb{R} \;&\to\; \mathbb{R}_+ \\ x \;&\mapsto\; F(x) \,=\, \log(x) \end{aligned}$$

Doing so, we obtain:

$$dF(S_t) \;=\; F'(S_t)dS_t \,+\, \frac{1}{2}F''(S_t)d\left\langle S,S\right\rangle_t \,,\; \forall\, t\, \in [0,T]$$

where

$$\begin{aligned} F'(S_t) \;&=\; \frac{1}{S_t}\,, \\ F''(S_t) \;&=\; -\frac{1}{{S_t}^2}\,, \\ d\left\langle S,S\right\rangle_t \;&=\; d\left\langle S_0 \,+\, \mu\int_0^\cdot S_s ds \,+\, \sigma\int_0^\cdot S_s dW_s \,,\, S_0 \,+\, \mu\int_0^\cdot S_s ds \,+\, \sigma\int_0^\cdot S_s dW_s \right\rangle_t \\ &=\; d\left\langle \int_0^\cdot \sigma S_s dW_s \,,\, \int_0^\cdot \sigma S_s dW_s \right\rangle_t \\ &=\; d\left(\int_0^t (\sigma S_s)^2 d\left\langle W,W\right\rangle_s\right) \\ &=\; d\left(\int_0^t (\sigma S_s)^2 ds\right) \\ &=\; \sigma^2 {S_t}^2 dt \end{aligned}$$

Hence,

$$
\begin{aligned}
d\log(S_t) &= \frac{dS_t}{S_t} + \frac{1}{2}\left(-\frac{1}{{S_t}^2}\right)\sigma^2 {S_t}^2 dt \\
&= \frac{dS_t}{S_t} - \frac{\sigma^2}{2}dt \\
&= \frac{1}{S_t}(\mu S_t dt + \sigma S_t dW_t) - \frac{\sigma^2}{2}dt \qquad (from\ \ Eq.\ (9)) \\
&= \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dW_t.
\end{aligned}
$$

It follows that,

$$
\log(S_t) = \log(S_0) + \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t.
$$

And finally we deduce

$$
S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right), \quad \forall\, t \in [0,T].
$$

Further considerations about stochastic differential equations would allow us to prove that this solution for $S := (S_t)_{t\in[0,T]}$ is unique, but we will not develop such theoretical elements in this article.

Finally, we may propose a generalized solution form for stochastic differential equations applied to such linear equations.

Indeed, considering predictable and bounded processes $\mu := (\mu_t)_{t\in[0,T]}$ and $\sigma := (\sigma_t)_{t\in[0,T]}$, equations of the type:

$$
\begin{cases}
dX_t = \mu_t X_t dt + \sigma_t X_t dW_t\,, \ \ \forall\, t \in [0,T] \\
X_0 = x_0
\end{cases}
$$

admits a unique solution of the form:

$$
X_t = x_0 \exp\left(\int_0^t \left(\mu_s - \frac{{\sigma_s}^2}{2}\right) ds + \int_0^t \sigma_s dW_s\right), \quad \forall\, t \in [0,T].
$$

In the next section, we will finally combine all the elements, theorems and properties from the stochastic calculus theory that we have studied so far in order to construct the Black-Scholes model, especifically for the pricing and hedging of options.

## 4. Black-Scholes model

We have introduced and studied all the relevant elements from stochastic calculus theory that will allow us to construct properly the Black-Scholes model, as a continuous-time market model for the pricing and hedging of financial options. As already mentioned, we are more precisely going to construct this Black-Scholes model as the continuous-time extension of the CRR model introduced in the first section of this article.

### 4.1. Model description

#### 4.1.1. Assets and their price dynamics

First, we define the same market framework as in the discrete-time CRR model. Thus, the studied market is composed of two assets:

- A riskless asset $S^0 := (S^0_t)_{t\in[0,T]}$, associated with a fixed interest rate $r \geq 0$.
- A risky asset $S := (S_t)_{t\in[0,T]}$.

A point of interest in the scope of the Black-Scholes model is the characterization of the respective dynamics of these two assets' price processes.

On one hand, being intrinsically linked to the interest rate $r \geq 0$ in force on the market, riskless asset's price dynamics is governed by the following ordinary differential equation:

$$\frac{dS^0_t}{S^0_t} \;=\; r dt\,, \quad \forall\, t \in [0,T].$$

Therefore, riskless asset's price process $S^0 := (S^0_t)_{t\in[0,T]}$ is defined such that:

$$S^0_t \;=\; S^0_0 e^{rt}\,, \quad \forall\, t \in [0,T].$$

Setting $S^0_0 = 1$ as in the CRR model, we have $S^0_t = e^{rt}\,,\ \forall\, t \in [0,T]$.

On the other hand, we define the risky asset's price dynamics as a linear combination of a trend component and a volatility component associated with a Brownian motion. Thus, making the assumption of positive prices for the risky asset, we may define its price dynamics under the following logarithmic form:

$$\frac{dS_t}{S_t} \;=\; \mu dt + \sigma dW_t \quad \iff \quad dS_t \;=\; \mu S_t dt + \sigma S_t dW_t\ , \quad \forall\, t \in [0,T]\,, \quad (\mu,\sigma) \in \mathbb{R}\times\mathbb{R}^*_+.$$

where

$\mu \in \mathbb{R}$ characterizes a trend parameter,

$\sigma \in \mathbb{R}$ embodies a volatility parameter,

$\mu$ and $\sigma$ are assumed to be constant,

$W := (W_t)_{t\in[0,T]}$ is a Brownian motion defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with its associated natural filtration $\mathbb{F}^W$.

As already developed in section 3.7. of this article, we may derive the following explicit expression for the risky asset's price process:

$$S_t \;=\; S_0 \exp\left(\sigma W_t \,+\, \left(\mu \,-\, \frac{\sigma^2}{2}\right) t\right)\,, \quad \forall\, t \in [0,T].$$

#### 4.1.2. Portfolio and associated wealth process

On the presented market, we consider that an agent is endowed with an initial capital $x$, and that he (continuously) invests a portion $\pi_t$ of his capital on the risky asset, at each date $t \in [0,T]$. Thus, we are going to define the central notions of *portfolio* and associated *wealth process* within the scope of The Black-Scholes model.

**Definition 4.1.** Portfolio and Investment strategy.

*A portfolio consists in a couple $(x,\pi)$, $x$ being an agent's initial capital, and $\pi$ being its investment strategy, as defined in the scope of the CRR model.*
*We define $(x,\pi)$ as $x \geq 0$ and $\pi := (\pi_t)_{t\in[0,T]}$ being a predictable process such that:*

$$\int_0^T (\pi_t)^2 dt \; < \; +\infty \,, \mathbb{P} - a.s.$$

**Definition 4.2.** Wealth process associated with a portfolio.

*The wealth process $X^{(x,\pi)}$ associated with the portfolio $(x,\pi)$ is a stochastic process defined such that:*

$$\begin{aligned} X_t^{(x,\pi)} &= x + \int_0^t \pi_s dS_s + \int_0^t \frac{(X_s^{(x,\pi)} - \pi_s S_s)}{S_s^0} dS_s^0 \\ &= x + \int_0^t \pi_s dS_s + \int_0^t (X_s^{(x,\pi)} - \pi_s S_s) \underbrace{rds}_{=\frac{dS_s^0}{S_s^0}} \,, \;\; \forall \, t \in [0,T]. \end{aligned}$$

*We may also express the dynamics of such a wealth process under the following infinitesimal form:*

$$\begin{aligned} dX_t^{(x,\pi)} &= \pi_t dS_t + (X_t^{(x,\pi)} - \pi_t S_t) \frac{dS_t^0}{S_t^0} \\ &= \pi_t dS_t + (X_t^{(x,\pi)} - \pi_t S_t) r dt. \end{aligned}$$

**Remark 4.3.** *At each date $t$, the capital not invested in the risky asset is then invested in the riskless asset. Hence, at each date $t$, $(X_t^{(x,\pi)} - \pi_t S_t)$ represents the amount of capital invested in $S^0$, and we have to divide this amount by the price $S_t^0$ in order to get the quantity of riskless asset detained by the agent at date $t$.*

However, as in the CRR model, the main point of interest for the construction of the pricing and hedging process lies in the consideration of *discounted* price and wealth processes.

Thus, discounting the price process $S$ with respect to the value of the riskless asset's value at each date $t$, we obtain the following discounted price process $\tilde{S} := (\tilde{S}_t)_{t\in[0,T]}$:

$$\tilde{S}_t \; = \; S_t e^{-rt} \,, \;\; \forall \, t \in [0,T].$$

Therefore, we may derive the following dynamics for the discounted price process $\tilde{S}$:

We set $X_t = S_t$ and $Y_t = e^{-rt}$.
We notice that $Y \; s.t. \; Y_t = e^{-rt} = 1 + \int_0^t -re^{-rs} ds$ is a finite variation process.
Then, from the Integration by parts formula (Theorem 3.17.), we have:

$$\begin{aligned} d\tilde{S}_t &= d(X_t Y_t) \; = \; X_t dY_t + Y_t dX_t + \underbrace{d\langle X,Y\rangle_t}_{=0} \\ &= -rS_t e^{-rt} dt + e^{-rt} dS_t \\ &= -rS_t e^{-rt} dt + e^{-rt}(\mu S_t dt + \sigma S_t dW_t) \\ &= (\mu - r)\tilde{S}_t dt + \sigma \tilde{S}_t dW_t \end{aligned} \qquad [10]$$

Therefore, we may express the discounted price process $\tilde{S}$ under the form:

$$\tilde{S}_t \; = \; \tilde{S}_0 \exp\left( \sigma W_t + \left( \mu - r - \frac{\sigma^2}{2} \right) t \right) \,, \;\; \forall \, t \in [0,T].$$

Now, we aim to determine the expression of the discounted wealth process. Thus, we are going to demonstrate the following proposition.

**Proposition 4.4.** Discounted wealth process.

*Let $(x, \pi)$ be a portfolio. Then, the discounted wealth process $\tilde{X}^{(x,\pi)} := (\tilde{X}_t^{(x,\pi)})_{t\in[0,T]}$ is expressed as:*

$$\tilde{X}_t^{(x,\pi)} \;=\; X_t^{(x,\pi)} e^{-rt} \;=\; x + \int_0^t \pi_s d\tilde{S}_s\,, \quad \forall\, t \in [0,T].$$

*Proof.*

We set $X_t = X_t^{(x,\pi)}$ and $Y_t = e^{-rt}$.
Again, we notice that $Y$ *s.t.* $Y_t = e^{-rt} = 1 + \int_0^t -re^{-rs}ds$ is a finite variation process.
Then, from the Integration by parts formula (Theorem 3.17.), we have:

$$\begin{aligned}
d\tilde{X}_t^{(x,\pi)} \;&=\; d(X_tY_t) \;=\; X_t dY_t \,+\, Y_t dX_t \,+\, \underbrace{d\left\langle X, Y\right\rangle_t}_{=0} \\
&=\; -rX_t^{(x,\pi)} e^{-rt} \,+\, e^{-rt} dX_t^{(x,\pi)} \\
&=\; -rX_t^{(x,\pi)} e^{-rt} \,+\, e^{-rt} \left(\pi_t dS_t + (X_t^{(x,\pi)} - \pi_t dS_t) r d_t\right) \\
&=\; \pi_t (dS_t - rS_t dt) e^{-rt}
\end{aligned}$$

Recalling that we have derived $d\tilde{S}_t \;=\; (dS_t - rS_t dt)e^{-rt}$ (Eq. (10)), we deduce that:

$$d\tilde{X}_t^{(x,\pi)} \;=\; \pi_t d\tilde{S}_t$$

which finally gives us

$$\tilde{X}_t^{(x,\pi)} \;=\; X_t^{(x,\pi)} e^{-rt} \;=\; x + \int_0^t \pi_s d\tilde{S}_s\,, \quad \forall\, t \in [0,T].$$

□

#### 4.1.3. Equivalent martingale measure

A last point of interest in the description of the Black-Scholes market model is related to the notion of *arbitrage*. We have already defined this notion of arbitrage (Definition 1.7.), specifying that *Absence of Arbitrage Opportunity (AAO)* is a crucial condition that has to be in force on the market. We are going to see that this AAO condition is directly linked to the notion of *Equivalent Martingale Measure (EMM)* on the probability space $(\Omega, \mathcal{F})$.

**Definition 4.5.** Equivalent martingale measure.

*An equivalent martingale measure (EMM) on $(\Omega, \mathcal{F})$ is a probability measure $\mathbb{Q}$ such that $\mathbb{Q} \sim \mathbb{P}$, and such that $\tilde{S}$ is a $\mathbb{Q}$-martingale.*

These considerations on equivalent martingale measure lead to a main theorem in financial mathematics, known as the first fundamental theorem of asset pricing.

**Theorem 4.6.** First fundamental theorem of asset pricing.

*The Absence of Arbitrage Opportunity condition is in force on the market if and only if an Equivalent Martingale Measure (EMM) exists on $(\Omega, \mathcal{F})$.*

Within the scope of the Black-Scholes model, we may demonstrate that it exists indeed a equivalent martingale measure, and that this EMM is unique on $(\Omega, \mathcal{F})$.

**Proposition 4.7.** *It exists a unique equivalent martingale measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$, equivalent to $\mathbb{P}$, defined such that:*

$$\frac{d\mathbb{Q}}{d\mathbb{P}} \;=\; \exp\left(-\frac{\mu - r}{\sigma} W_T \,-\, \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 T\right). \qquad [11]$$

*Proof.*

First, we aim to demonstrate that the probability measure $\mathbb{Q}$ defined above is indeed an EMM.

Let $M := (M_t)_{t\in[0,T]}$ be the process defined such that:

$$M_t \;=\; \exp\left(-\frac{\mu-r}{\sigma}W_t \,-\, \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 t\right)\,, \quad \forall\, t \in [0,T].$$

Thus, $M := \mathcal{E}\left(-\frac{\mu-r}{\sigma}W\right)$ and, in particular, $M_T = \frac{d\mathbb{Q}}{d\mathbb{P}}$.

With $r \geq 0$, $\mu \in \mathbb{R}$, and $\sigma \geq 0$ constant, it is clear that the process $-\frac{\mu-r}{\sigma}W$ is a martingale. Hence, as a stochastic exponential, we deduce that $M$ is a martingale (Property 2., Proposition 3.21.).

In addition, $\frac{\mu-r}{\sigma}$ being constant, it is clear that the process $\left(\left(\frac{\mu-r}{\sigma}\right)_t\right)_{t\in[0,T]}$ is bounded and predictable.

Thus we may rewrite $M$ as $M := \mathcal{E}\left(-\int_0^{\cdot} \frac{\mu-r}{\sigma}dW_t\right)$.

From Girsanov's theorem (Theorem 3.23.), we deduce that $W^{\mathbb{Q}} := (W_t^{\mathbb{Q}})_{t\in[0,T]}$ defined such that:

$$\begin{aligned} W_t^{\mathbb{Q}} \;&=\; W_t \,+\, \int_0^t \frac{\mu-r}{\sigma}ds \\ &=\; W_t \,+\, \frac{\mu-r}{\sigma}t \end{aligned}$$

is a $\mathbb{Q}$-Brownian motion.

Now, we study the dynamics of $\tilde{S}$ under the $\mathbb{Q}$ probability measure.

$$\begin{aligned} d\tilde{S}_t \;&=\; e^{-rt}(-rS_t dt \,+\, dS_t) \\ &=\; e^{-rt}(-rS_t dt \,+\, \mu S_t dt \,+\, \sigma S_t dW_t) \\ &=\; e^{-rt}S_t(\sigma dW_t \,+\, (\mu-r)dt) \\ &=\; \sigma \tilde{S}_t dW_t^{\mathbb{Q}} \end{aligned} \qquad [12]$$

Therefore, from the martingale representation theorem (Theorem 3.18.), we deduce that $\tilde{S}$ is a $\mathbb{Q}$-martingale, and then that $\mathbb{Q}$ is indeed an equivalent martingale measure.

Now we intend to show that $\mathbb{Q}$ is the unique EMM on $(\Omega, \mathcal{F})$.

To do so, we start by assuming that it exists another EMM $\hat{\mathbb{Q}}$, and we define the process $L := (L_t)_{t\in[0,T]}$ such that:

$$\begin{cases} L_T \;=\; \frac{d\hat{\mathbb{Q}}}{d\mathbb{P}} \\ \\ L_t \;=\; \mathbb{E}\left[L_T \,|\, \mathcal{F}_t\right]\,, \;\; \forall\, t \in [0,T]. \end{cases}$$

We also suppose that $\mathbb{E}\left[|L_T|^2\right] \;<\; +\infty$.

Defined in this way, $L$ is a martingale. Then, from the martingale representation theorem (Theorem 3.18.), it exists a process $Z \in L^2(W)$ such that:

$$L_t \;=\; L_0 \,+\, \int_0^t Z_s dW_s\,, \;\; \forall\, t \in [0,T].$$

By definition, since $\hat{\mathbb{Q}}$ is an EMM, equivalent to $\mathbb{P}$, it follows that $\tilde{S}L$ is a $\mathbb{P}$-martingale.

From the Integration by parts formula (Theorem 3.17.), we derive that:

$$\begin{aligned} d(\tilde{S}_t L_t) \;&=\; \tilde{S}_t dL_t \,+\, L_t d\tilde{S}_t \,+\, d\left\langle \tilde{S}, L\right\rangle_t \\ &=\; \tilde{S}_t Z_t dW_t \,+\, L_t(\sigma\tilde{S}_t dW_t \,+\, (\mu-r)\tilde{S}_t dt) \,+\, \sigma Z_t \tilde{S}_t dt \\ &=\; (Z_t + \sigma L_t)\tilde{S}_t dW_t \,+\, (L_t(\mu-r) + \sigma Z_t)\tilde{S}_t dt \end{aligned}$$

To be verified, the $\mathbb{P}$-martingale property for $\tilde{S}L$ implies that the term in $dt$ has to be equal to zero.
Therefore, we deduce that:

$$\begin{aligned} & (L_t(\mu-r) + \sigma Z_t) \;=\; 0 \\ \Longleftrightarrow \qquad & Z_t \;=\; -\frac{\mu-r}{\sigma}L_t. \end{aligned}$$

Thus, we obtain:

$$dL_t \ = \ -\frac{\mu - r}{\sigma} L_t dW_t. \qquad [13]$$

Recalling that $M := \mathcal{E}\left(-\frac{\mu-r}{\sigma}W\right) \ = \ \frac{d\mathbb{Q}}{d\mathbb{P}}$, from Property 1., Proposition 3.21., we have that:

$$dM_t \ = \ -\frac{\mu - r}{\sigma} M_t dW_t. \qquad [14]$$

Therefore, we notice that Eq.(13) and Eq.(14) are strictly equivalent, which shows that $\hat{\mathbb{Q}} = \mathbb{Q}$.

□

**Remark 4.8.** *In the previous consideration about EMM, we have notice that the term* $\frac{\mu-r}{\sigma}$ *appeared to be recurrent. In fact, this term* $\frac{\mu-r}{\sigma}$ *is a crucial ratio in financial theory, known as the Sharpe ratio. It is also called the market price of risk.*

**Remark 4.9.** *From the definition of that EMM* $\mathbb{Q}$ *and the introduction of the* $\mathbb{Q}$*-Brownian motion* $W^{\mathbb{Q}}$*, risky asset's price dynamics can be expressed as* $\frac{dS_t}{S_t} \ = \ rdt + \sigma dW^{\mathbb{Q}}$*. This dynamics structure is really remarkable in the sense that it reflects the fact that, under* $\mathbb{Q}$*, the returns associated with the risky assets are characterized by a trend parameter which is the riskless interest rate, plus a volatility parameter following a Brownian motion behavior. That is, under* $\mathbb{Q}$*, the expected return of the risky asset is exactly the riskless interest rate, which defines the measure* $\mathbb{Q}$ *as a risk-neutral measure.*

We have just proven the existence as well as the uniqueness of an EMM on $(\Omega, \mathcal{F})$. Then, by the first fundamental theorem of asset pricing (Theorem 4.6.), it follows the the AAO condition is in force on the Black-Scholes market.

We introduce a last definition of interest related to this notion of equivalent martingale measure.

**Definition 4.10.** Admissible Portfolio
*A portfolio* $(x, \pi)$ *is said to be admissible if its associated discounted wealth process* $\tilde{X}_t^{(x,\pi)}$ *is a* $\mathbb{Q}$*-martingale.*

From Definition 4.9., we may easily show that it does not exist any arbitrage opportunity (AO) on the market.

Indeed, considering the defined EMM $\mathbb{Q}$, and assuming that it exists an admissible portfolio $(0, \pi)$ which is an AO, then we have:

$$\begin{aligned} X_0^{(0,\pi)} \ &= \ \tilde{X}_0^{(0,\pi)} \ = \ 0 \\ \textit{and} \quad \mathbb{E}^{\mathbb{Q}}[\tilde{X}_T^{(0,\pi)}] \ &= \ \tilde{X}_0^{(0,\pi)} \ = \ 0. \end{aligned}$$

Therefore, the admissible portfolio $(0, \pi)$ cannot be an arbitrage opportunity.

**Proposition 4.11.**

*The absance of arbitrage opportunity (AAO) condition is in force on the Black-Scholes market,*
*i.e., no any admissible portfolio can be an arbitrage opportunity.*

### 4.2. Pricing and hedging of options in the Black-Scholes model

In this part, we are going to focus on the main aim for which we have studied this Black-Scholes model: the pricing and the hedging of financial options.
First, we are going to expose the arbitrage pricing principle for a general financial security, and then we will derive the Black-Scholes formula specifically dedicated to the pricing and hedging of options.

#### 4.2.1. Arbitrage pricing

Let's consider a financial security whose payoff at maturity (date $T$) is characterized by a random variable $V$. We assume that $V$ is of integrable square with respect to $\mathbb{Q}$ (i.e., $\mathbb{E}^{\mathbb{Q}}\left[|V|^2\right] \ < \ +\infty$).

The consideration of the EMM $\mathbb{Q}$ will allow us to determine an arbitrage pricing process as well as an hedging process for the financial security characterized by the payoff $V$ at maturity.

**Theorem 4.12.** Arbitrage pricing and hedging.

*Let $V \in L^2(\Omega, \mathbb{Q})$ characterize the random payoff of a given security at maturity (date $T$). Then, it exists a unique admissible portfolio $(x, \pi)$ such that:*

$$\tilde{X}_T^{(0,\pi)} = V \,, \quad \mathbb{P} - a.s.$$

*In addition, the price process $P := (p_t)_{t\in[0,T]}$ is given by $p_t \;=\; \mathbb{E}^{\mathbb{Q}}\left[V e^{-r(T-t)} \,|\, \mathcal{F}_t\right] \,, \;\; \forall\, t \in [0,T]$.*

In particular, $p_T \;=\; V$, and $p_0 \;=\; x \;=\; \mathbb{E}^{\mathbb{Q}}\left[V e^{-rT}\right]$ is called the arbitrage price of the studied security. The associated process $\pi$ is called its hedging strategy.

*Proof.*

The price process $P := (p_t)_{t\in[0,T]}$ defined as $p_t \;=\; \mathbb{E}^{\mathbb{Q}}\left[V e^{-r(T-t)} \,|\, \mathcal{F}_t\right] \,, \quad \forall\; t \;\in\; [0,T]$ is a $\mathbb{Q}$-martingale, and $P \in L^2(\Omega, \mathbb{Q})$.

From the definition of $\mathbb{Q}$, we set $L := \frac{d\mathbb{Q}}{d\mathbb{P}} \;=\; \mathcal{E}\left(-\frac{\mu - r}{\sigma} W\right)$ such that:

$$L_t \;=\; \exp\left(-\frac{\mu - r}{\sigma} W_t \;-\; \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 t\right) \;, \;\; \forall\, t \in [0,T].$$

$P := (p_t)_{t\in[0,T]}$ being a $\mathbb{Q}$-martingale, it follows that the process $M := PL$ is a $\mathbb{P}$-martingale.

Hence, from the martingale representation theorem (Theorem 3.18.), it exists a predictable process $H \in L^2(W)$ such that:

$$M_t \;\;=\;\; M_0 \;+\; \int_0^t H_s dW_s \,, \;\; \forall\, t \in [0,T].$$

Now, we are going to determine the dynamics of $(p_t)_{t\in[0,T]}$ implementing Itô's formula.

First, we derive:

$$\begin{aligned}
d(L_t{}^{-1}) \;&=\; d\left(\exp\left(\frac{\mu - r}{\sigma} W_t \;+\; \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 t\right)\right) \\
&=\; L_t{}^{-1}\left(\frac{\mu - r}{\sigma} dW_t \;+\; \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 dt\right) \;+\; \frac{1}{2} L_t{}^{-1}\left(\frac{\mu - r}{\sigma}\right)^2 dt \\
&=\; \frac{\mu - r}{\sigma} L_t{}^{-1} dW_t \;+\; \left(\frac{\mu - r}{\sigma}\right)^2 L_t{}^{-1} dt.
\end{aligned}$$

We also have $dM_t \;=\; H_t dW_t$.

As a following, by definition, it appears that $p_t \;=\; M_t L_t{}^{-1}$, and we derive:

$$\begin{aligned}
dp_t \;&=\; d(M_t L_t{}^{-1}) \;=\; M_t dL_t{}^{-1} \;+\; L_t{}^{-1} dM_t \;+\; d\left\langle M, L^{-1}\right\rangle_t \\
&=\; M_t\left(\frac{\mu - r}{\sigma} L_t{}^{-1} dW_t \;+\; \left(\frac{\mu - r}{\sigma}\right)^2 L_t{}^{-1} dt\right) \;+\; L_t{}^{-1} H_t dW_t \;+\; \frac{\mu - r}{\sigma} H_t L_t{}^{-1} dt \\
&=\; \frac{\mu - r}{\sigma} p_t dW_t \;+\; \left(\frac{\mu - r}{\sigma}\right)^2 p_t dt \;+\; L_t{}^{-1} H_t dW_t \;+\; \frac{\mu - r}{\sigma} H_t L_t{}^{-1} dt \\
&=\; \left(\frac{\mu - r}{\sigma} p_t \;+\; H_t L_t{}^{-1}\right)\left(dW_t \;+\; \frac{\mu - r}{\sigma} dt\right).
\end{aligned}$$

With $\left(dW_t \;+\; \frac{\mu - r}{\sigma} dt\right)$, we recognize the infinitesimal form of the $\mathbb{Q}$-Brownian motion $W^{\mathbb{Q}}$ introduced in the proof of Proposition 4.7.

Hence,

$$dp_t \;=\; \left(\frac{\mu - r}{\sigma} p_t \,+\, \frac{H_t}{L_t}\right) dW_t^{\mathbb{Q}}$$

$$= \; \frac{\frac{\mu-r}{\sigma} p_t + \frac{H_t}{L_t}}{\sigma \tilde{S}_t} d\tilde{S}_t. \qquad \textit{(from Eq.(12))}$$

Thus, defining $\pi_t := \frac{\frac{\mu-r}{\sigma} p_t + \frac{H_t}{L_t}}{\sigma \tilde{S}_t}$, $dp_t$ is expressed under the form:

$$dp_t \;=\; \pi_t d\tilde{S}_t \;=\; d\tilde{X}_t^{(x,\pi)}.$$

We have demonstrated the existence of an admissible portfolio $(x, \pi)$ such that $\tilde{X}_T^{(x,\pi)} \;=\; V$ , $\mathbb{P} - a.s.$, with $x \;=\; p_0 \;=\; \mathbb{E}^{\mathbb{Q}}\left[V e^{-rT}\right]$.

In addition, the uniqueness of that portfolio $(x, \pi)$ may be proven assuming that it exists two distinct admissible portfolios $(x, \pi)$ and $(\hat{x}, \hat{\pi})$ which replicates $V$, and showing that $(x, \pi)$ is necessarily equal to $(\hat{x}, \hat{\pi})$.

For instance, given that $(x, \pi)$ and $(\hat{x}, \hat{\pi})$ are admissible, $\tilde{X}^{(x,\pi)}$ and $\tilde{X}^{(\hat{x},\hat{\pi})}$ are both $\mathbb{Q}$-martingales. Then, since $\tilde{X}^{(x,\pi)}$ and $\tilde{X}^{(\hat{x},\hat{\pi})}$ have the same terminal value ($(x, \pi)$ and $(\hat{x}, \hat{\pi})$ being replicating portfolios of $V$), it follows necessarily that $x = \hat{x}$. $\square$

#### 4.2.2. Black-Scholes formula for the pricing and hedging of options

We have developed the arbitrage pricing principle in the Black-Scholes model and we are going to derive the well-known Black-Scholes formula specifically applied to the pricing and hedging of European options[9]. In this article, we are going to demonstrate the Black-Scholes formula for the pricing and hedging of European Call options. The formula for European Put options can be derived in a strictly similar way.

**Theorem 4.13.** Black-Scholes formula

*Let $V := F(S_T)$ be the payoff of a European Call option at maturity date $T$, and with strike price $K$.*
*In particular,*

$$\begin{array}{rcl} F \,:\, \mathbb{R} & \rightarrow & \mathbb{R}_+ \\ x & \mapsto & F(x) \;=\; (x - K)_+ \end{array}$$

*Then, the arbitrage price process $P := (p_t)_{t \in [0,T]}$ is given by $\phi_C(t, S_t)$, with:*

$$\phi_C(t, x) \;=\; x N(d_1(x)) \,-\, K e^{-r(T-t)} N(d_2(x))$$

*where*

$N(d) \;=\; \int_{-\infty}^{d} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$ *is the cumulative distribution function of the standard normal distribution* $\mathcal{N}(0, 1)$,

$d_1(x) \;=\; \frac{log(\frac{x}{K}) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$,

$d_2(x) \;=\; d_1(x) - \sigma\sqrt{T-t}$.

*Thus, $p_0 \;=\; \phi_C(0, S_0)$ is the arbitrage price of the European Call option.*
*In addition, the process $\pi := (\pi_t)_{t \in [0,T]}$ defined such that:*

$$\pi_t \;=\; \frac{\partial F}{\partial x}(t, S_t)\,, \quad \forall\, t \in [0, T]$$

*is the hedging strategy of that European Call option, i.e., $(p_0, \pi)$ replicates $V$.*

[9] Definitions and considerations about options given in Appendix V.

*Proof.*

Let $V$ denote the random payoff of a European Call option at maturity date $T$.
$V$ can be expressed as $V = F(S_T) = (S_T - K)_+$.

From Theorem 4.11., the arbitrage price process $P := (p_t)_{t\in[0,T]}$ of the European Call option is defined by:

$$\begin{aligned} p_t &= \mathbb{E}^{\mathbb{Q}}\left[V e^{-r(T-t)} \,|\, \mathcal{F}_t\right] \\ &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[F(S_T) \,|\, \mathcal{F}_t\right] \end{aligned}$$

In order to develop further the pricing process, we decompose the structure of $S_T$, the price of the underlying at maturity. A noteworthy characteristic of $S$ is that it is a Markov process.
From the expression of $S := (S_t)_{t\in[0,T]}$, we notice that $S_T$ may be also expressed as:

$$\begin{aligned} S_T &= S_0 + \int_0^T \sigma S_s dW_s + \int_0^T \mu S_s ds \\ &= S_t + \int_t^T \sigma S_s dW_s + \int_t^T \mu S_s ds \end{aligned}$$

Equivalently,

$$\begin{aligned} S_T &= S_0 \exp\left(\sigma W_T + \left(\mu - \frac{\sigma^2}{2}\right)T\right) \\ &= S_t \exp\left(\sigma W_{T-t} + \left(\mu - \frac{\sigma^2}{2}\right)(T-t)\right) \end{aligned}$$

From these considerations, defining $S_T^t = \exp\left(\sigma W_{T-t} + \left(\mu - \frac{\sigma^2}{2}\right)(T-t)\right)$, we can express the pricing process as:

$$\begin{aligned} p_t &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[F(S_T) \,|\, \mathcal{F}_t\right] \\ &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[F(S_t S_T^t)\right] \qquad \textit{by Markov's property of } S. \end{aligned}$$

As a following, we may define the function $\phi$ such that:

$$\phi(t,x) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[F(xS_T^t)\right],$$

and we have $p_t = \phi(t, S_t)$.
From the general expression of $\phi$, we have:

$$\begin{aligned} \phi(t,x) &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[F(xS_T^t)\right] \\ &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[(xS_T^t - K)_+\right]. \end{aligned}$$

By construction,

$$\begin{aligned} S_T^t &= \exp\left(\sigma W_{T-t} + \left(\mu - \frac{\sigma^2}{2}\right)(T-t)\right) \\ &= \exp\left(\sigma W_{T-t}^{\mathbb{Q}} + \left(r - \frac{\sigma^2}{2}\right)(T-t)\right) \end{aligned} \qquad [15]$$

since $\frac{dS_t}{S_t} = \mu dt + \sigma dW_t \iff \frac{dS_t}{S_t} = r dt + \sigma dW_t^{\mathbb{Q}}$, by definition of the $\mathbb{Q}$-Brownian motion $W^{\mathbb{Q}}$ given previously.
We focus on the implications of this new expression of $S_T^t$ under $\mathbb{Q}$.

$$\begin{aligned} \phi(t,x) &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[(xS_T^t - K)_+\right] \\ &= e^{-r(T-t)}\left(x\mathbb{E}^{\mathbb{Q}}\left[\left(S_T^t - \frac{K}{x}\right)_+\right]\right) \\ &= e^{-r(T-t)}\left(x\mathbb{E}^{\mathbb{Q}}\left[\left(S_T^t - \frac{K}{x}\right)\mathbb{I}\left\{S_T^t - \frac{K}{x} \geq 0\right\}\right]\right) \\ &= \underbrace{x\, e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[S_T^t \mathbb{I}\left\{S_T^t - \frac{K}{x} \geq 0\right\}\right]}_{A} - Ke^{-r(T-t)}\underbrace{\mathbb{Q}\left[S_T^t - \frac{K}{x} \geq 0\right]}_{B}. \end{aligned} \qquad [16]$$

Under $\mathbb{Q}$, Eq.(15) can be expressed as:

$$S_T^t \;=\; \exp\left(\sigma\sqrt{T-t}\,Z \,+\, \left(r \,-\, \frac{\sigma^2}{2}\right)(T-t)\right) \quad \text{where} \quad Z \sim \mathcal{N}(0,1).$$

From this expression of $S_T^t$,

$$\begin{aligned}
S_T^t \,-\, \frac{K}{x} \,\geq\, 0 \quad &\iff \quad \exp\left(\sigma\sqrt{T-t}\,Z \,+\, \left(r \,-\, \frac{\sigma^2}{2}\right)(T-t)\right) \,-\, \frac{K}{x} \,\geq\, 0 \\
&\iff \quad Z \,\geq\, \frac{\log(\frac{K}{x}) \,-\, \left(r \,-\, \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \\
&\iff \quad Z \,\leq\, \frac{\log(\frac{x}{K}) \,+\, \left(r \,-\, \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \quad \text{since} \quad Z \equiv -Z.
\end{aligned}$$

Let $d_2(x)$ denote the quantity $\frac{\log(\frac{x}{K}) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$.

From this observation, we derive expressions of terms $A$ and $B$ in Eq.(16).

Let's start by the term $B$.

$$\begin{aligned}
B \;&=\; \mathbb{Q}\left[S_T^t - \frac{K}{x} \,\geq\, 0\right] \\
&=\; \mathbb{Q}\left[Z \,\leq\, d_2(x)\right] \quad \text{where} \quad Z \sim \mathcal{N}(0,1) \\
&=\; \int_{-\infty}^{d_2(x)} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\,dz \;:=\; N(d_2(x)).
\end{aligned}$$

Now we determine the term $A$.

$$\begin{aligned}
A \;&=\; e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[S_T^t \mathbb{1}\left\{S_T^t - \frac{K}{x} \geq 0\right\}\right] \\
&=\; e^{-r(T-t)} \int_{-\infty}^{d_2(x)} e^{\sigma\sqrt{T-t}\,Z + \left(r - \frac{\sigma^2}{2}\right)(T-t)} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\,dz \\
&=\; \int_{-\infty}^{d_2(x)} \frac{1}{\sqrt{2\pi}} e^{-\frac{\left(z^2 - 2\sigma\sqrt{T-t}z + \sigma^2(T-t)\right)}{2}}\,dz \\
&=\; \int_{-\infty}^{d_2(x)} \frac{1}{\sqrt{2\pi}} e^{-\frac{\left(z \,-\, \sigma\sqrt{T-t}\right)^2}{2}}\,dz \\
&=\; \int_{-\infty}^{d_2(x)+\sigma\sqrt{T-t}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\,dz.
\end{aligned}$$

Finally, setting $d_1(x) \;=\; d_2(x) + \sigma\sqrt{T-t} \;=\; \frac{\log(\frac{x}{K}) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$, and using the notation $N(d) \;:=\; \int_{-\infty}^{d} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\,dz$ for the cumulative distribution function of the standard normal distribution $\mathcal{N}(0,1)$, we have shown that:

$$p_t \;=\; \phi(t, S_t) \;=\; S_t N(d_1(S_t)) \,-\, K e^{-r(T-t)} N(d_2(S_t))\,, \quad \forall\, t \in [0,T].$$

**Remark 4.14.** *For European Put options, a similar approach, but considering a payoff at maturity* $V := (K - S_T)_+$ *leads to the following pricing formula:*

$$p_t \;=\; \phi_P(t, S_t)\,, \quad \forall\, t \in [0,T].$$

$$\textit{where} \qquad \phi_P(t,x) \;=\; K e^{-r(T-t)} N(-d_2(x)) \,-\, x N(-d_1(x)).$$

□

### 4.3. Partial differential equations for the pricing of securities

In this last theoretical part of this article, we are going to introduce another pricing approach in the Black-Scholes model, using partial differential equations (PDE). Indeed, we are going to demonstrate that the pricing process for a security is solution of a specific partial differential equation.

First, we consider a financial security whose random payoff at maturity date $T$ is characterized by a function of the underlying's price at maturity, i.e., $V(S_T)$, with

$$\begin{array}{rccl} V : & \mathbb{R}_+ & \to & \mathbb{R} \\ & x & \mapsto & V(x) \end{array}$$

and $\mathbb{E}\left[|V(S_T)|^2\right] < +\infty$.

By arbitrage pricing theorem (Theorem 4.11.), the pricing process $P := (p_t)_{t\in[0,T]}$ for such a security is given by:

$$p_t = \mathbb{E}^{\mathbb{Q}}\left[V(S_T)e^{-r(T-t)} \,|\, \mathcal{F}_t\right]$$

By definition of the EMM $\mathbb{Q}$, defining the process $L := (L_t)_{t\in[0,T]}$ such that:

$$L_t = \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \,|\, \mathcal{F}_t\right] = \exp\left(-\frac{\mu-r}{\sigma}W_t - \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 t\right), \quad \forall\, t \in [0,T],$$

the following equation is verified:

$$p_t = \mathbb{E}^{\mathbb{Q}}\left[V(S_T)e^{-r(T-t)} \,|\, \mathcal{F}_t\right] = \frac{\mathbb{E}\left[V(S_T)e^{-r(T-t)}L_T \,|\, \mathcal{F}_t\right]}{L_t}.$$

It follows that

$$p_t = \mathbb{E}\left[V(S_T)e^{-\frac{\mu-r}{\sigma}W_{T-t} - \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2(T-t) - r(T-t)} \,|\, \mathcal{F}_t\right].$$

Finally, introducing the following notations:

$$\begin{aligned} S_T^{t,x} &= x\exp\left(\sigma W_{T-t} + \left(\mu - \frac{\sigma^2}{2}\right)(T-t)\right) \\ \textit{and} \quad F(t,x) &= \mathbb{E}\left[V(S_T^{t,x})e^{-\frac{\mu-r}{\sigma}W_{T-t} - \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2(T-t) - r(T-t)} \,|\, \mathcal{F}_t\right], \end{aligned}$$

Markov's property for the process $S$ leads to the following result:

$$p_t = F(t, S_t).$$

Having defined such a price process, we note that, in order to preserve the $AAO$ condition on the market, the discounted price process $(p_t e^{-rt})_{t\in[0,T]}$ has to be a $\mathbb{Q}$-martingale, $\mathbb{Q}$ being the unique EMM on $(\Omega, \mathcal{F})$.

Studying the infinitesimal form of $(p_t e^{-rt})_{t\in[0,T]}$, we obtain that:

$$\begin{aligned} d(p_t e^{-rt}) &= d(F(t,S_t)e^{-rt}) = F(t,S_t)d(e^{-rt}) + e^{-rt}d(F(t,S_t)) + \underbrace{d\left\langle F(.,S_.), e^{-r.}\right\rangle_t}_{=0} \\ &= -re^{-rt}F(t,S_t)dt \\ &+ e^{-rt}\left[\frac{\partial F}{\partial t}(t,S_t)dt + \frac{\partial F}{\partial x}(t,S_t)dS_t + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(t,S_t)d\left\langle S,S\right\rangle_t\right] \quad \textit{by generalized Itô's formula} \end{aligned}$$

where

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

$$d\left\langle S,S\right\rangle_t = (\sigma S_t)^2 d\left\langle W,W\right\rangle_t = (\sigma S_t)^2 dt.$$

Hence,

$$\begin{aligned}
d(p_t e^{-rt}) &= -re^{-rt}F(t,S_t)dt \\
&+ e^{-rt}\left[\frac{\partial F}{\partial t}(t,S_t)dt + \frac{\partial F}{\partial x}(t,S_t)(\mu S_t dt + \sigma S_t dW_t) + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(t,S_t)(\sigma S_t)^2 dt\right] \\
&= -re^{-rt}F(t,S_t)dt \\
&+ e^{-rt}\left[\frac{\partial F}{\partial t}(t,S_t)dt + \frac{\partial F}{\partial x}(t,S_t)(\sigma S_t dW_t^{\mathbb{Q}} + rS_t dt) + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(t,S_t)(\sigma S_t)^2 dt\right] \\
&= e^{-rt}\frac{\partial F}{\partial x}(t,S_t)\sigma S_t dW_t^{\mathbb{Q}} \\
&+ e^{-rt}\left[-rF(t,S_t) + \frac{\partial F}{\partial t}(t,S_t) + r\frac{\partial F}{\partial x}(t,S_t)S_t + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(t,S_t)(\sigma S_t)^2\right] dt
\end{aligned}$$

Then, the $\mathbb{Q}$-martingale property for $(p_t e^{-rt})_{t\in[0,T]}$ implies that the term in $dt$ has to be equal to zero.

Therefore, we deduce that

$$-rF(t,S_t) + \frac{\partial F}{\partial t}(t,S_t) + r\frac{\partial F}{\partial x}(t,S_t)S_t + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(t,S_t)(\sigma S_t)^2 = 0.$$

Thus, we have shown that the price process of a security, characterized by the function $F$, is solution of the following partial differential equation:

$$-rF(t,x) + \frac{\partial F}{\partial t}(t,x) + r\frac{\partial F}{\partial x}(t,x)\,x + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(t,x)(\sigma\, x)^2 = 0. \ , \quad (t,x) \in [0,T]\times\mathbb{R}_+ \qquad [17]$$

with the terminal condition $F(T,x) = V(x)$ where $V$ represents the payoff function of the studied security.

In particular, considering the case of a European Call option, the terminal condition is $F(T,x) = (x-K)_+$, and the function $F$ solution of Eq.(17) is the one derived in Theorem 4.12.

### 4.4. Numerical simulations

In this last section, we will focus on numerical simulations around various elements discussed in the development of our study. In particular, we will elaborate simulations around price processes in order to derive pricing and hedging processes, numerically computed using theoretical formulas introduced previously. In addition, we will implement a Monte-Carlo approach to verify the proper convergence of the Monte-Carlo simulated arbitrage price (discounted expected payoff at maturity) to the theoretical price given by the Black-Scholes closed formula.

However, before to concretely run simulations, we intend to control the number of simulations implied by our Monte-Carlo approach, in order to avoid too big (and useless) numerical cost.

**Theorem 4.15.** Strong Law of Large Numbers.
*Let $(X_i)_{i=1\dots n}$ i.i.d, then :*

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{i=1}^{n} X_i \;=\; \mathbb{E}[X], \quad \mathbb{P}-a.s$$

**Theorem 4.16.** Central Limit Theorem
*Let $(X_i)_{i=1\dots n}$ i.i.d, then :*

$$\lim_{n\to+\infty} \sqrt{n}\frac{\frac{1}{n}\sum_{i=1}^{n} X_i - \mathbb{E}[X]}{\sigma} \xrightarrow[n\to+\infty]{\mathcal{L}} \mathcal{N}(0,1) \qquad where \quad \sigma \;=\; \sqrt{Var(X)}$$

The two previous theorem ensure us that if we want a $10^{-k}$ precision in a result approximated by Monte-Carlo simulations, then we have to carry out at least $n = 10^{2k}$ simulations.

We estimate in our framework that a $10^{-2}$ approximation is reasonable. So we set $n\_iter = 10000$ as our number of simulations for our Monte-Carlo approach.

Then, we need to set our parameters for our numerical simulations.

- **Stock (underlying) price at t = 0:** $S_0 = 100$
- **Strike price:** $K = 100$
- **Maturity in years:** $T = 1$
- **Risk-free rate:** $r = 0.05$
- **Implicit volatility:** $\sigma = 0.2$

#### 4.4.1. Risky asset's price simulations

The idea is to simulate a path of the price S over the time interval $[0, T] = [0, 1]$, using the formula derived in section 3.7.
First we subdivide $[0, 1]$ in $n = 1000$ sub-periods.
We simulate a Brownian motion path $(W_{t_i})_{i\in[0,n]}$ over $[0, 1]$, using the method described in Appendix 1.

Then, for each time $(t_i)_{i\in[0,n]}$, $t_0 = 0, ..., t_n = 1$, we compute $S_{t_i}$ as $S_{t_i} = S_0.e^{\sigma W_{t_i}+(r-\frac{\sigma^2}{2})t_i}$ (price formula under EMM $\mathbb{Q}$).

In order to illustrate the randomness inherent to the price process $S$, we have respectively simulated 5 and 50 different stock price paths, and thus we obtain the following random paths for $(S_t)_{t\in[0,1]}$:

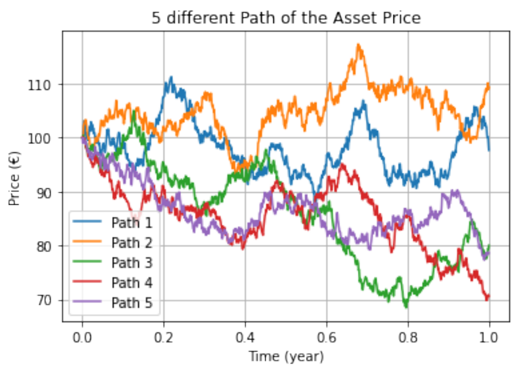


**Fig. 1.** 5 Paths

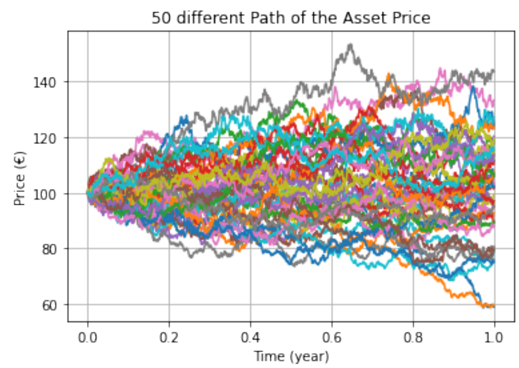


**Fig. 2.** 50 Paths

A last noteworthy element that we can simulate about stock prices is their "mean behavior". Indeed, implementing Monte-Carlo simulations, we obtain the following results:

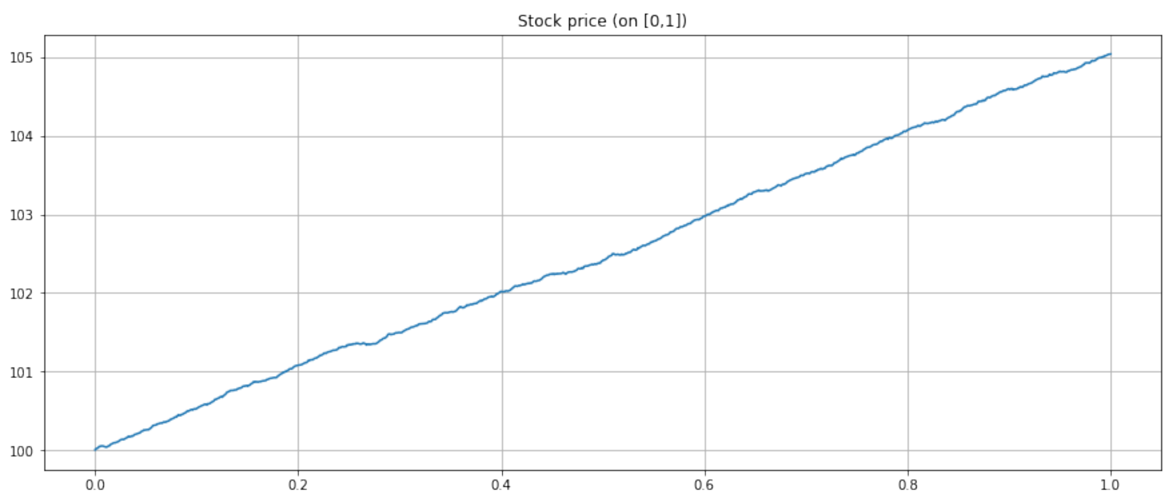


**Fig. 3.** Mean behavior of stock prices

This obtained result is really remarkable, and refers to Remark 4.9. of this article. Indeed, the previous graph reflects the structure of stock prices under the EMM $\mathbb{Q}$. The trend parameter equal to the riskless interest rate is clearly illustrated. We observe that, in average, prices tend to reach (linearly) a level of \$105 after one year (from a starting point of \$100). It is exactly in line with the defined (annual) riskless interest rate of 5% on the market. Thus, we observe that the expected return on risky asset is equal to the riskless rate, which characterizes the risk neutrality under $\mathbb{Q}$.

### 4.4.2. European Call option pricing simulations

As already mentioned, we have also intended to simulate the arbitrage price of an European Call option, using the arbitrage pricing principle developed in Theorem 4.12. Indeed, we simulate $n\_iter$ payoff at maturity date for a European Call option, and we determine a proxy for the (discounted) expectation of these payoffs taking their empirical mean. An ideal simulation of the entire price process $(p_{t_i})_{i\in[0,n]}$ would consist in $n\_iter$ Monte-Carlo simulations for each time step $(t_i)_{i\in[0,n]}$ to determine $p_{t_i}$ in the same way, but this is too heavy computationally speaking.
Therefore, we will only use Monte-Carlo simulation to price our option at time $t = 0$ and check if it corresponds with $\phi_C(0, S_0)$ from the closed Black-Scholes formula (Theorem 4.13.). The next figure illustrate our results:

```
Option type: European
Option: Call
Initial spot price: $100
Strike price: $100
Time interval: [0,1]

Option price from Black-Scholes formula: $10.451

Option price from Monte-Carlo simulations: $10.449
```

**Fig. 4.** Arbitrage pricing Monte-Carlo simulation (t=0)

We can see that our Monte-Carlo reproduce well the theoretical formula. For simpler and cheaper numerical cost, we will now only use the theoretical formula for further simulations.

### 4.4.3. Hedging strategy and European Call replication simulation

Through this last simulation, the aim is to simulate the investment strategy which hedges the risk associated to a short position in a given European Call option. In other words, we aim to compute numerically the strategy of hedging ratios $(\pi_{t_i})_{i\in[1,n]}$ (also called *Delta hedging*) which allow to replicate that Call option.

From Theorem 4.13., such an hedging strategy is given by $\pi_t = \frac{\partial \phi_C}{\partial x}(t, S_t)$, we we have to derive the explicit expression of the function $(t,x) \mapsto \frac{\partial \phi_C}{\partial x}(t,x)$ to be able to properly implement numerical computations of $(\pi_{t_i})_{i\in[1,n]}$.

**Proposition 4.17.**

*Within the scope of Theorem 4.13., the hedging strategy* $(\pi_t)_{t\in[0,T^-]}$ *is given by:*

$$\pi_t = \frac{\partial \phi_C}{\partial x}(t, S_t) \ = \ N(d_1(S_t))\,, \quad \forall\, t \in [0,T].$$

*Proof.*

From Theorem 4.13.,

$$\begin{aligned}
\phi_C(t,x) \ &= \ x\,N(d_1(x)) \ - \ Ke^{-r(T-t)}N(d_2(x)) \\
&= \ x\,N(d_1(x)) \ - \ Ke^{-r(T-t)}N(d_1(x) - \sigma\sqrt{T-t})
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{\partial \phi_C}{\partial x}(t,x) \ &= \ N(d_1(x)) \ + \ x\,N'(d_1(x)) \ - \ Ke^{-r(T-t)}N'(d_1(x) - \sigma\sqrt{T-t}) \\
&= \ N(d_1(x)) \ + \ \underbrace{x\,d_1'(x)N'(d_1(x))}_{A} \ - \ \underbrace{Ke^{-r(T-t)}d_1'(x)N'(d_1(x) - \sigma\sqrt{T-t})}_{B}
\end{aligned}$$

where

$$N'(d(x)) \ = \ \tfrac{1}{\sqrt{2\pi}}e^{-\frac{(d(x))^2}{2}}$$

$$d_1'(x) \ = \ \tfrac{1}{\sigma\sqrt{T-t}}\tfrac{1}{K}\tfrac{1}{\frac{x}{K}} \ = \ \tfrac{1}{x\sigma\sqrt{T-t}},$$

Therefore, we evaluate the terms $A$ and $B$:

$$\begin{aligned}
A \ &= \ x\,d_1'(x)N'(d_1(x)) \\
&= \ x\,\frac{1}{x\sigma\sqrt{T-t}})\frac{1}{\sqrt{2\pi}}e^{-\frac{(d_1(x))^2}{2}} \\
&= \ \frac{1}{\sigma\sqrt{(2\pi)(T-t)}}e^{-\frac{(d_1(x))^2}{2}}
\end{aligned}$$

$$\begin{aligned}
B \ &= \ \frac{1}{\sigma\sqrt{(2\pi)(T-t)}}\frac{K}{x}e^{-r(T-t)}e^{-\frac{(d_1(x)-\sigma\sqrt{T-t})^2}{2}} \\
&= \ \frac{1}{\sigma\sqrt{(2\pi)(T-t)}}\frac{K}{x}e^{-r(T-t)}e^{-\frac{(d_1(x))^2-2\sigma\sqrt{T-t}d_1(x)+\sigma^2(T-t)}{2}} \\
&= \ \frac{1}{\sigma\sqrt{(2\pi)(T-t)}}\frac{K}{x}e^{-\frac{(d_1(x))^2}{2}}e^{\log(\frac{x}{K})+(r+\frac{\sigma^2}{2})(T-t)-\frac{\sigma^2(T-t)}{2}-r(T-t)} \\
&= \ \frac{1}{\sigma\sqrt{(2\pi)(T-t)}}\frac{K}{x}e^{-\frac{(d_1(x))^2}{2}}e^{\log(\frac{x}{K})} \\
&= \ \frac{1}{\sigma\sqrt{(2\pi)(T-t)}}e^{-\frac{(d_1(x))^2}{2}}
\end{aligned}$$

We have shown that $A \ = \ B$, and finally we conclude that $\frac{\partial \phi_C}{\partial x}(t,x) \ = \ N(d_1(x))$.

□

From Proposition 4.17., we are able to numerically compute the hedging strategy $(\pi_{t_i})_{i\in[1,n]}$ such that $\pi_{t_i} = N(d_1(S_{t_i}))$.

As a following, we can construct the replication process $(X_{t_i}^{(p_0,\pi)})_{i\in[0,n]}$ of the priced Call option such that $X_{t_0}^{(p_0,\pi)} = p_0$ and $X_{t_i}^{(p_0,\pi)} = \pi_{t_{i-1}} S_{t_i} + (X_{t_{i-1}}^{(p_0,\pi)} - \pi_{t_{i-1}} S_{t_{i-1}})e^{r(t_i - t_{i-1})}$.

Doing so, we obtain the following simulations:

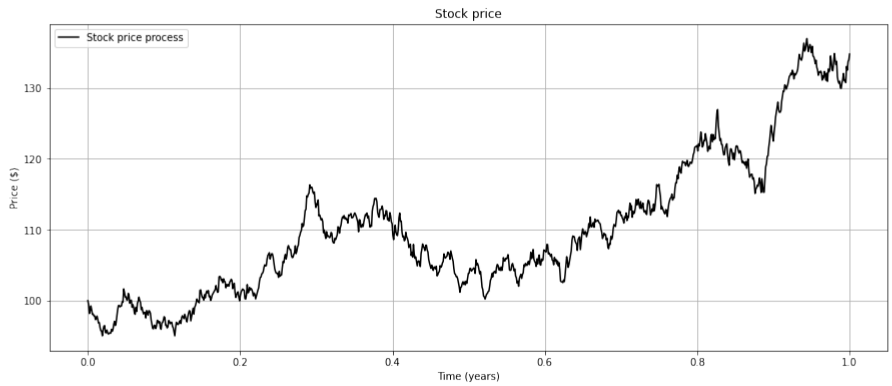


**Fig. 5.** Stock price path simulation

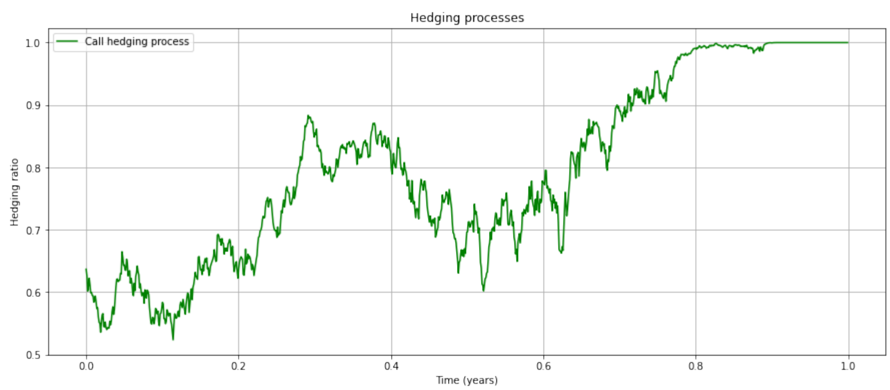


**Fig. 6.** Associated hedging process


**Fig. 7.** European Call replication

We observe that the computed hedging process leads to an exact replication of the European Call price process.

## Conclusion

The main purpose of this article was to derive a theoretical model for continuous-time pricing of financial options.

We recalled the well-known Cox-Ross-Rubinstein Model, supporting a theoretical model of discrete-time markets for the pricing and hedging, and tried to extend it towards a continuous-time modelling framework. As developed in this article, the extension of this pricing model towards time continuity requires theoretical and technical elements from stochastic calculus theory.

Thus, a specific study was carried out on continuous stochastic processes, and main properties of martingales and Brownian motions have been introduced. Then, complex notions of stochastic calculus and specifically Itô's calculus have been developed, as they play a key role in the construction and implementation of the Black-Scholes model.

With all these considerations, we could finally derive the Black-Scholes model, applying all the theoretical elements introduced previously. Thus, defining a new probabilistic measure which sets our study in a risk neutral context, we elaborated the principle of arbitrage pricing in the defined framework. In particular, we derived the famous Black-Scholes formula for pricing and hedging of European Call options.

We finally carried out numerical simulations around various elements of the Black-Scholes framework, in order to illustrate abstract concepts and to support the theoretical findings. Through Monte-Carlo simulatons, we have highlighted the equivalence between the arbitrage pricing approach and the theoretical results derived from the Black-Scholes formula. In addition, we have illustrated the notion of hedging and replication simulating the price process and the hedging process for a European Call option.

## Appendix

### I. Brownian motion path simulation

For easiest visualisation, the time interval $[0,T]$ on which the process is studied is fixed to $[0,1]$.
In addition, we consider a subdivision family $(t_i)_{i\in[\![0,N]\!]}$ of $[0,1]$ such that $t_i = \frac{i}{N}$ , $\forall\, i \in [\![0,N]\!]$. That is, the step of the considered subdivision of $[0,1]$ is constant and equal to $\frac{1}{N}$.

| **Simulation of a Brownian motion path** |
|---|
| **Initialization** |
| - Choose the number of subdivisions $N$ of the interval $[0,1]$<br>- Set $W_0 = 0$<br>- Set $t = 1$ |
| **Main** |
| **While** $t < N$ **do**<br>    **Generate** $D$ a centered Gaussian variable with variance $\frac{1}{N}$<br>    Compute $W_t = W_{t-1} + D$<br>    $W \mathrel{+}= [W_t]$<br>    $t \mathrel{+}= 1$ |
| **Return** $W$ |

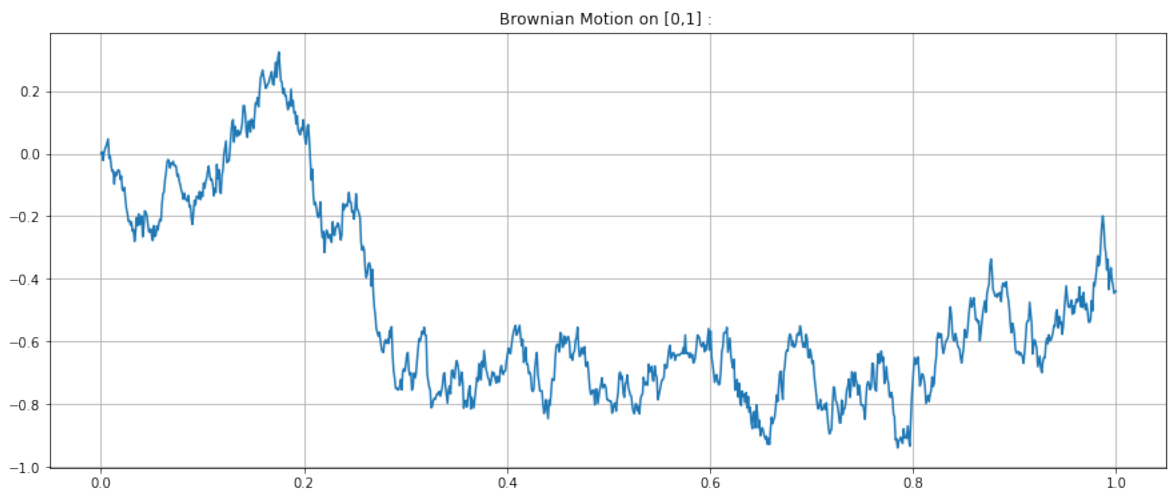


**Fig. 8.** Brownian motion path simulation

### II. Proposition 2.18.: Proof

*Proof.*

$\forall\, t\in[0,T]$ and $\forall\, n\in\mathbb{N}^*$, let $(t_i^n)_{i\in[\![0,m_n]\!]}$ be a subdivision of $[0,t]$, such that $t_0^n = 0 < ... < t_i^n < ... < t_{m_n}^n = t$.
Also, $(t_i^n)_{i\in[\![0,m_n]\!]}$ is defined such that its step $|\rho_n| = \underset{1\leq i\leq m_n}{sup}\ |t_i^n - t_{i-1}^n| \underset{n\to+\infty}{\longrightarrow} 0$.
Showing that the Brownian motion is not a finite variation process is equivalent to showing that

$$\forall\, t\in[0,T],\ \lim_{n\to+\infty} \mathbb{E}\left[\left|\sum_{i=1}^{m_n}(W_{t_i^n} - W_{t_{i-1}^n})^2 - t\right|^2\right] = 0. \qquad [18]$$

In order to simplify the writing of the proof, we are going to restrict the scope of the study of the previous equation to the case $t = T$. In addition, we define $(t_i^n)_{i\in[\![0,m_n]\!]}$ such that $t_i^n = \frac{iT}{m_n}$, $\forall\, i\in[\![0,m_n]\!]$. Thus $m_n$ is the number of subdivisions of $[0,T]$ and the step of $(t_i^n)_{i\in[\![0,m_n]\!]}$ is $|\rho_n| = \frac{1}{m_n}$. Hence, by defining $m_n$ such that $m_n \underset{n\to+\infty}{\longrightarrow} +\infty$, we make $(t_i^n)_{i\in[\![0,m_n]\!]}$ verify $|\rho_n| \underset{n\to+\infty}{\longrightarrow} 0$.

Therefore, we have:

$$
\begin{aligned}
\mathbb{E}\left[\left|\sum_{i=1}^{m_n}(W_{t_i^n}-W_{t_{i-1}^n})^2-T\right|^2\right] &= \mathbb{E}\left[\left(\sum_{i=1}^{m_n}(W_{t_i^n}-W_{t_{i-1}^n})^2\right)^2 + 2T\sum_{i=1}^{m_n}(W_{t_i^n}-W_{t_{i-1}^n})^2 + T^2\right] \\
&= \underbrace{\mathbb{E}\left[\left(\sum_{i=1}^{m_n}(W_{t_i^n}-W_{t_{i-1}^n})^2\right)^2\right]}_{A} + 2T\,\underbrace{\mathbb{E}\left[\sum_{i=1}^{m_n}(W_{t_i^n}-W_{t_{i-1}^n})^2\right]}_{B} + T^2
\end{aligned}
$$

We recall some relevant properties and results about the Brownian motion that will allow us to derive an explicit form of terms $A$ and $B$.

- Independence and stationarity of the increments.
- $\forall\, i \in [\![0,m_n]\!],\ (W_{t_i^n}-W_{t_{i-1}^n}) \sim \mathcal{N}(0, t_i^n - t_{i-1}^n) \ \Rightarrow\ \mathbb{E}\left[(W_{t_i^n}-W_{t_{i-1}^n})^2\right] = t_i^n - t_{i-1}^n = \frac{1}{m_n}.$
- $\mathbb{E}\left[(W_{t_i^n}-W_{t_{i-1}^n})^4\right] = 3Var(W_{t_i^n}-W_{t_{i-1}^n})^2 = 3(t_i^n - t_{i-1}^n)^2,$
  as the $4^{th}-$ order moment of a r.v. following a centered normal distribution $\mathcal{N}(0, t_i^n - t_{i-1}^n)$.

Thus, we evaluate both terms $A$ and $B$.

$$
\begin{aligned}
A &= \mathbb{E}\left[\left(\sum_{i=1}^{m_n}(W_{t_i^n}-W_{t_{i-1}^n})^2\right)^2\right] \\
&= \mathbb{E}\left[\sum_{i=1}^{m_n}(W_{t_i^n}-W_{t_{i-1}^n})^2 \times \sum_{j=1}^{m_n}(W_{t_j^n}-W_{t_{j-1}^n})^2\right] \\
&= \mathbb{E}\left[\sum_{i=1}^{m_n}(W_{t_i^n}-W_{t_{i-1}^n})^4 + 2\sum_{1\le i<j\le m_n}(W_{t_i^n}-W_{t_{i-1}^n})^2(W_{t_j^n}-W_{t_{j-1}^n})^2\right] \\
&= \sum_{i=1}^{m_n}\mathbb{E}\left[(W_{t_i^n}-W_{t_{i-1}^n})^4\right] + 2\sum_{j=2}^{m_n}\mathbb{E}\left[(W_{t_j^n}-W_{t_{j-1}^n})^2\right]\sum_{i=1}^{j-1}\mathbb{E}\left[(W_{t_i^n}-W_{t_{i-1}^n})^2\right] \\
&= \sum_{i=1}^{m_n}3Var(W_{t_i^n}-W_{t_{i-1}^n})^2 + 2\sum_{j=2}^{m_n}Var(W_{t_j^n}-W_{t_{j-1}^n})\sum_{i=1}^{j-1}Var(W_{t_i^n}-W_{t_{i-1}^n}) \\
&= \sum_{i=1}^{m_n}3(t_i^n-t_{i-1}^n)^2 + 2\sum_{j=2}^{m_n}(t_j^n-t_{j-1}^n)\sum_{i=1}^{j-1}(t_i^n-t_{i-1}^n) \\
&= \frac{3T^2}{m_n^2}m_n + \frac{2T^2}{m_n^2}\sum_{j=2}^{m_n}(j-1) \\
&= \frac{3T^2}{m_n^2}m_n + \frac{2T^2}{m_n^2}\sum_{j=1}^{m_n-1} j \\
&= \frac{3T^2}{m_n} + \frac{2T^2}{m_n^2}\cdot\frac{(m_n-1)m_n}{2} \\
&= \frac{3T^2}{m_n} + T^2 - \frac{T^2}{m_n}
\end{aligned}
$$

$$\begin{aligned} B &= \mathbb{E}\left[\sum_{i=1}^{m_n}(W_{t_i^n}-W_{t_{i-1}^n})^2\right] = \sum_{i=1}^{m_n}\mathbb{E}\left[(W_{t_i^n}-W_{t_{i-1}^n})^2\right] \\ &= \sum_{i=1}^{m_n} Var\left(W_{t_i^n}-W_{t_{i-1}^n}\right) = \sum_{i=1}^{m_n}(t_i^n - t_{i-1}^n) \\ &= T \end{aligned}$$

Finally,

$$\mathbb{E}\left[\left|\sum_{i=1}^{m_n}(W_{t_i^n}-W_{t_{i-1}^n})^2 - T^2\right|^2\right] = \frac{3T^2}{m_n} + T^2 - \frac{T^2}{m_n} - 2T^2 + T^2 = \frac{2T^2}{m_n}$$

And therefore :

$$\lim_{n\to+\infty}\mathbb{E}\left[\left|\sum_{i=1}^{m_n}(W_{t_i^n}-W_{t_{i-1}^n})^2 - T\right|^2\right] = 0$$

□

### III. Doob's inequality

Let $M$ be a (continuous) martingale and let $p > 1$.
Then,

$$\mathbb{E}[\sup_{t\in[0,T]}|M_t|^p] \leq \left(\frac{p}{p-1}\right)^p . \mathbb{E}[|M_T|^p]$$

### IV. Generalized Itô's formula

**Theorem 4.18.** Generalised Itô's formula
*Let $F : [0,T]\times\mathbb{R}\to\mathbb{R}$, $F\in\mathcal{C}^{1,2}([0,T]\times\mathbb{R})$.*
*Let $X$ be a continuous semimartingale.*
*Then, $(F(X_t))_{t\in[0,T]}$ is a continuous semimartingale, and:*

$$F(t,X_t) = F(0,X_0) + \int_0^t \frac{\partial F}{\partial t}(s,X_s)ds + \int_0^t \frac{\partial F}{\partial x}(s,X_s)dX_s + \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial x^2}(s,X_s)d\langle X,X\rangle_s \ , \ \forall\, t\in[0,T].$$

### V. European Call and Put options

**Definition 4.19.** Financial option.

*An option is a financial security which entitles its owner to the right (but not the obligation) to either buy or sell (depending on the nature of the option) a predetermined quantity of a given* underlying asset*, at a fixed date $T$ in the future (the* maturity*), and at a predetermined price $K$ (called the* strike*).*
*All these predetermined terms of the contract are defined at the signature of the contract between the two agents involved in the trade.*

An option is a particular type of *derivative*. Indeed, the value of an option derives directly from the value of the *underlying asset* associated to the option.

Furthermore, an option is said to be of *European-type* if it only can be exercised at the maturity date $T$. This is the main difference with *American* options, which can be exercised at any time $t$ belonging to the time-interval $[\![0, T]\!]$. Specifically, the random variable which represents the value of a European option at the maturity $T$ is $\mathcal{F}_T-measurable$.

**Definition 4.20.** European Call option*.*

*A European Call option entitles its owner to the right (but not the obligation) to buy a predetermined quantity of a given* underlying asset*, at a fixed date $T$ in the future (the* maturity*), and at a predetermined price $K$ (called the* strike*).*
*Thus, a Call option on an underlying $S$ and with a strike price $K$ is characterized, at the maturity date $T$, by a payoff function equal to* $(S_T - K)_+ := \max(S_T - K, 0)$*.*

Thus, taking out a European Call option on the underlying $S$ with strike $K$, the Call option holder has, at maturity date $T$, the right to either buy or not the underlying asset at the price $K$. Let $S_T$ denote the price of the underlying asset at date $T$. The Call option leads to a positive payoff if the price of the underlying asset at time $T$ is greater than the strike price $K$ – i.e., if $S_t > K$. On the contrary, if $S_T \leq K$, the option is not exercised and the Call option is worth zero at its maturity. All in all, a European Call option turns out to be profitable if and only if its final payoff at maturity is greater than the compounded value (at time $T$) of the price paid for the Call at the signature of the contract.

**Definition 4.21.** European Put option*.*

*A European Put option entitles its owner to the right (but not the obligation) to sell a predetermined quantity of a given* underlying asset*, at a fixed date $T$ in the future (the* maturity*), and at a predetermined price $K$ (called the* strike*).*
*Thus, a Put option on an underlying $S$ and with a strike price $K$ is characterized, at the maturity date $T$, by a payoff function equal to* $(K - S_T)_+ := \max(K - S_T, 0)$*.*

Thus, taking out a European Put option on the underlying $S$ with strike $K$, the Put option holder has, at maturity date $T$, the right to either sell or not the underlying asset at the price $K$. The Put option leads to a positive payoff if the price of the underlying asset at time $T$ is lower than the strike price $K$ – i.e., if $S_t < K$. On the contrary, if $S_T \geq K$, the option is not exercised and the Put option is worth zero at its maturity. All in all, a European Put option turns out to be profitable if and only if its final payoff at maturity is greater than the the compounded value (at time $T$) of the price paid for the Put at the signature of the contract.

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