BE Hawkes
-
Use of Hawkes processes in a Cramer-Lundberg type model

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Introduction

This project aims to implement Hawkes processes in Python, then to study their self-exciting properties. In order to study this process in a realistic way, we decided to set a particular context.

The main model presented in this analysis to simulate the risk process associated to the insurance company is the Cramer-Lundberg model. The goal is to present the different results, depending on the claims and the parameters that may have the self-exciting process, to our fictive insurance company team.

The first part of this record is a study of the Hawkes processes, to do so we first recalled point process properties. We decided to use the simulation of an homogeneous Poisson process to implement display programs in order to reuse it in the Hawkes simulation part. Afterwards we simulate Hawkes processes and analyse its parameters and properties. In a second part, we recalled the Cramer-Lundberg model and tried to fit in the Hawkes process. The final part consists of a comeback to the realistic context. We will present the results to the company and describe how Hawkes processes could be useful to us. We will also discuss the choices and results we obtained.

1 Study of Hawkes Processes

Before studying a Hawkes process, we compute a Homogeneous Poisson process in order to compare and visualise their differences. This process is well-known, its computation is simple.

1.1 Recall Homogenous Poisson Process:

Definition 1.1 *A point Process on R^+ is a countable subset of R^+, each point representing an arrival time of an event.*

Definition 1.2 *A counting process N_t represents the total number of events that occur by time t.*

Theorem 1 *Knowing the distribution of a counting process N_t is equivalent to knowing the distribution of a point process T_n.*

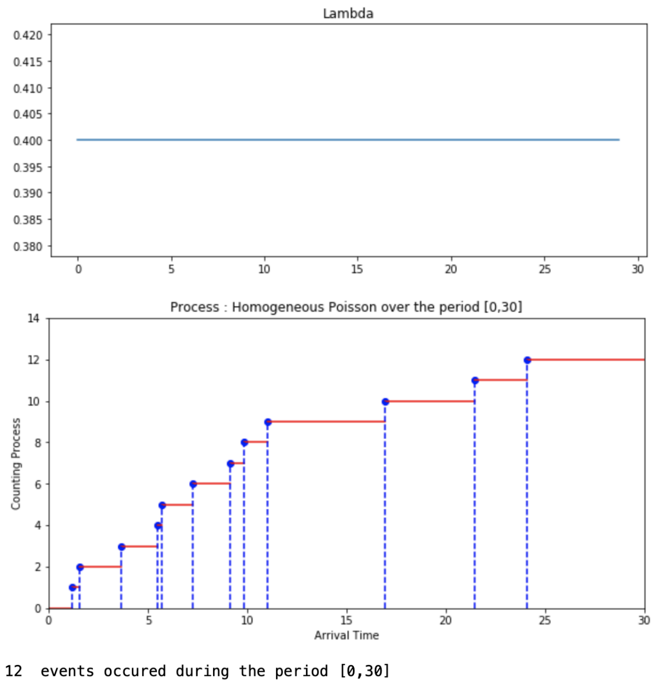


Figure 1: Homogeneous Poisson process ($\mu = 0.4$) on period $T = 30$

As we have seen, the parameter λ has an influence on the number of events that will occur during the period. Nt is an homogeneous Poisson process. Thus, we can compute $E[N_T]$:

$$E[N_T] = \lambda T = 0.4 * 30 = 12$$

This computation will enable to model a Hawkes process. We use the same methods for the following computations.

1.2 Definition and properties of a Hawkes Process:

A Hawkes process is called a self-exciting point process, because the density $\lambda(.)$ depends on time t and the entire past of the process.
In other words, the probability that an event E_2 appears right after another event E_1 is higher than if the event E_1 did not happen.
The density $\lambda(t)$ directly impacts that an event might arise. Indeed, the higher λ is, the higher the occurrence probability of a new event is.
Let $\lambda(t)$ defined as :

$$\lambda(t) = \mu + \int_0^t \alpha e^{-\beta(t-s)} dN(s) = \mu + \sum_{\{k:t_k<t\}} \alpha e^{-\beta(t-t_k)} \tag{1}$$

Hence, we define a Hawkes Process as :
A univariate simple point process $N(t)$ satisfying,
(i) N(0)=0
(ii) The intensity of the point process $\lambda(t)$ is equal to (1)
(iii) The process is regular.

Let H_t be a Hawkes Process, we can compute its expectation :

$$\begin{aligned}
E[H_t] &= E\left[\int_0^t \lambda(s)ds\right] \\
&= \int_0^t E[\lambda(s)]ds \\
&= \int_0^t \mu + \int_0^s \alpha e^{-\beta(s-u)} E[dH(u)] \\
&= \int_0^t \mu + \alpha e^{-\beta s} \int_0^s e^{\beta u} E[\lambda(u)]du
\end{aligned}$$

Denote,

$$f(s) = E[\lambda(s)] = \mu + \alpha e^{-\beta s} \int_0^s e^{\beta u} E[\lambda(u)]du = \mu + \alpha e^{-\beta s} \int_0^s e^{\beta u} f(u)du$$

f is the solution of an Ordinary Differential Equation, whose solution is[1] :

$$f(s) = \frac{\mu}{\alpha - \beta}(\alpha e^{(\alpha-\beta)s} - \beta)$$

Therefore,

$$E[H_t] = \int_0^t f(s)ds = \frac{\mu}{\alpha-\beta}(\frac{\alpha}{\alpha-\beta}(e^{(\alpha-\beta)t} - 1) - \beta t) \tag{2}$$

[1]more details in Appendice

1.3 Hawkes Process simulations

1.3.1 Ogata's Modified Thinning Algorithm

In order to simulate a Hawkes Process, we used a modified Thinning Algorithm developed by Ogata exposed below.

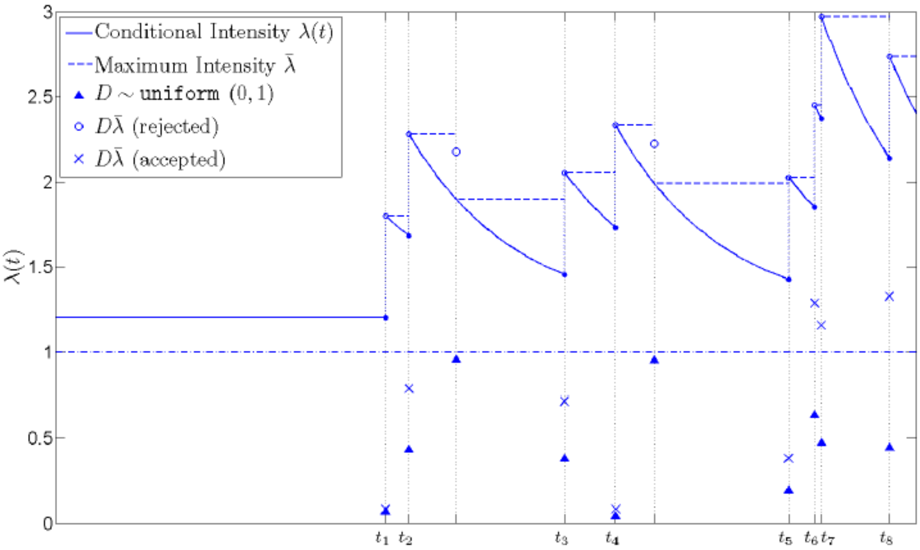
Thinning Modified Algorithm - Simulation of a Hawkes Process on [0, T]

Input: μ, α, β

Initialise $\mathcal{T} = \varnothing, s = 0, n = 0$;
while $s < T$ **do**
  Set $\bar{\lambda} = \lambda(s^+) = \mu + \sum_{\tau \in \mathcal{T}} \alpha e^{-\beta(s-\tau)}$;
  Generate $u \sim$ `uniform(0,1)`;
  Let $w = -\ln u / \bar{\lambda}$; `// so that` $w \sim$ `exponential(`$\bar{\lambda}$`)`
  Set $s = s + w$; `// so that` s `is the next candidate point`
  Generate $D \sim$ `uniform(0,1)`;
  if $D\bar{\lambda} \leq \lambda(s) = \mu + \sum_{\tau \in \mathcal{T}} \alpha e^{-\beta(s-\tau)}$ **then**
    $n = n + 1$; `// updating the number of points accepted`
    $t_n = s$; `// naming it` t_n
    $\mathcal{T} = \mathcal{T} \cap \{t_n\}$; `// adding` t_n `to the ordered set` $\mathcal{T}$
  end if
end while
if $t_n \leq T$ **then**
  return $\{t_n\}_{k=1,2,\ldots,n}$
else
  return $\{t_n\}_{k=1,2,\ldots,n-1}$
end if

- **Explanation of the Algorithm :**

This algorithm seems very difficult at first reading but it is not that much complicated. Before using the algorithm, we need to compute $\lambda(t)$, the Hawkes conditional intensity. A graphic display will allow us to understand the simulation process.



Explanation of the Thinning Modified Algorithm

Compute: $\lambda(t)$ the Hawkes Process intensity with parameters μ, α, β

Start from time t=0

Set $\bar{\lambda}$ the constant which will help us to accept a candidate point

Generate an exponential variable w of parameter $\bar{\lambda}$ to simulate the inter-arrival time.

Go to the next candidate point computed with w

Generate $D \sim$ `uniform(0,1)`;

if $D\bar{\lambda} \leq \lambda(t)$ evaluated at the candidate point **then**

  Accept the candidate point

end if

Restart: this simulation process up to the moment when a candidate point is bigger than the period T

return The set of accepted candidate points

After implementing this algorithm in Python, we were able to simulate Hawkes processes and tune its parameters.

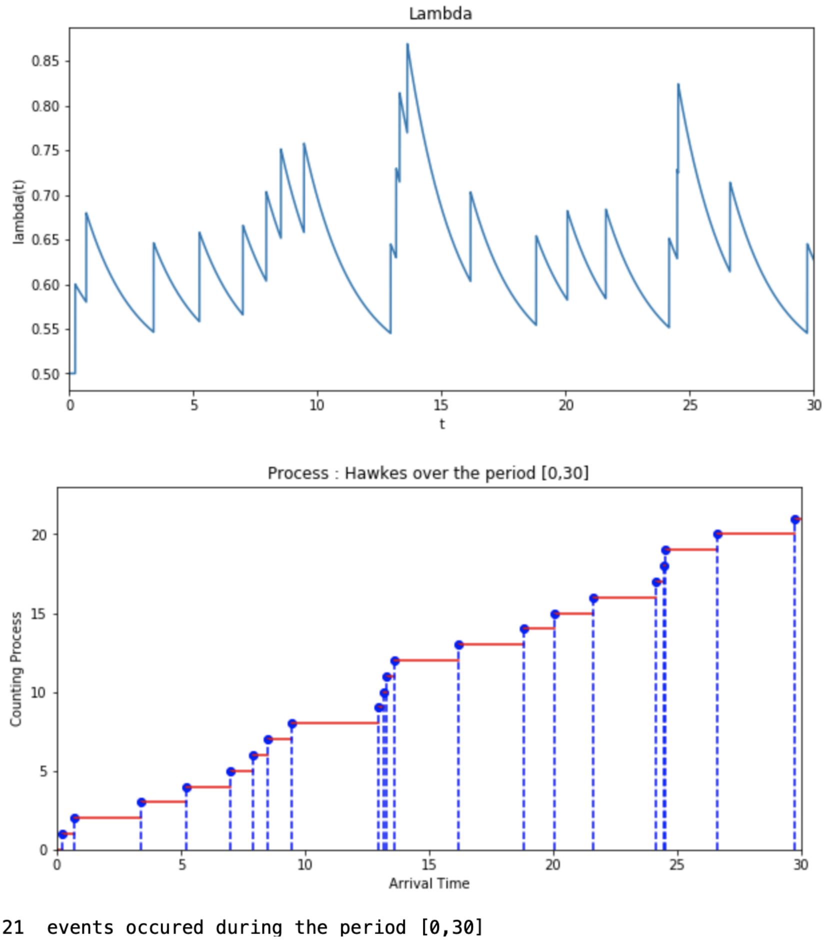


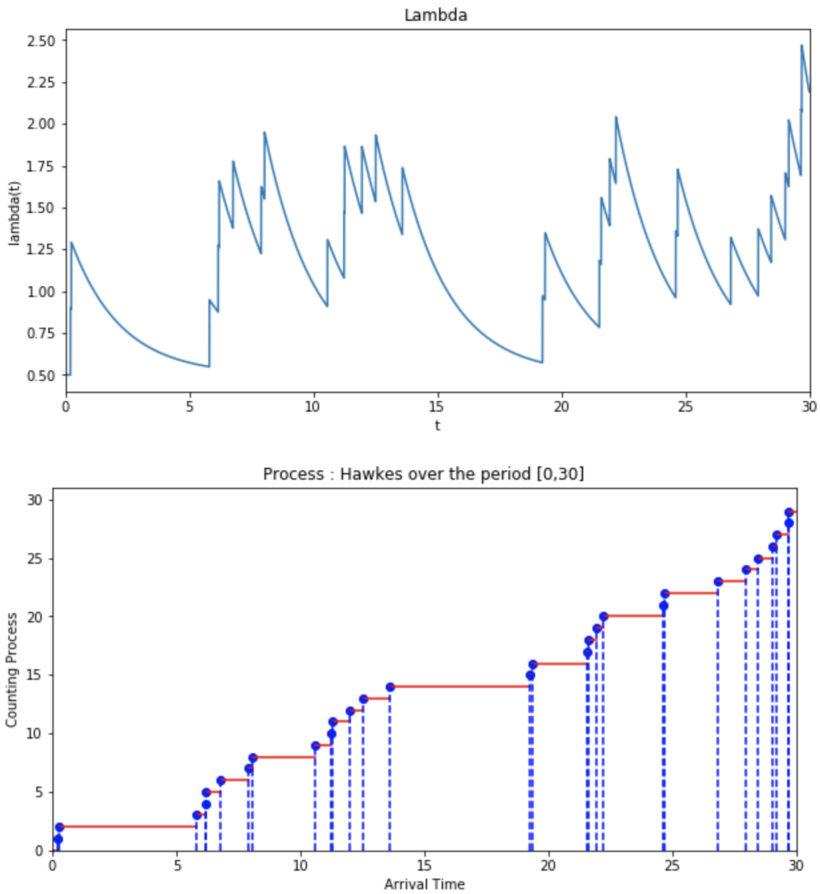
Figure 2: Hawkes process ($\mu = 0.5$, $\alpha = 0.1$ and $\beta = 0.5$) over the period $T = 30$

These graphics are the computations of the intensity $\lambda(t)$ and its associated counting process N(t) for an Hawkes process with exponential decay that has parameters $\mu = 0.5$, $\alpha = 0.1$ and $\beta = 0.5$ over the period $T = 30$.
The intensity starts μ from and keeps at the base intensity $\mu = 0.5$ until the occurrence of the first point. After each occurrence, the intensity jumps with a size of α and then immediately starts decaying at a rate determined by $\beta = 0.5$. $\lambda(t)$ is minimized by $\mu = 0.5$. The associated counting process is a step function that jump by 1 at each occurrence.

Let us study the influence of the parameters α and β.

1.3.2 Influence of the parameter α

Now, β and μ are fixed (both are equals to 0.5).



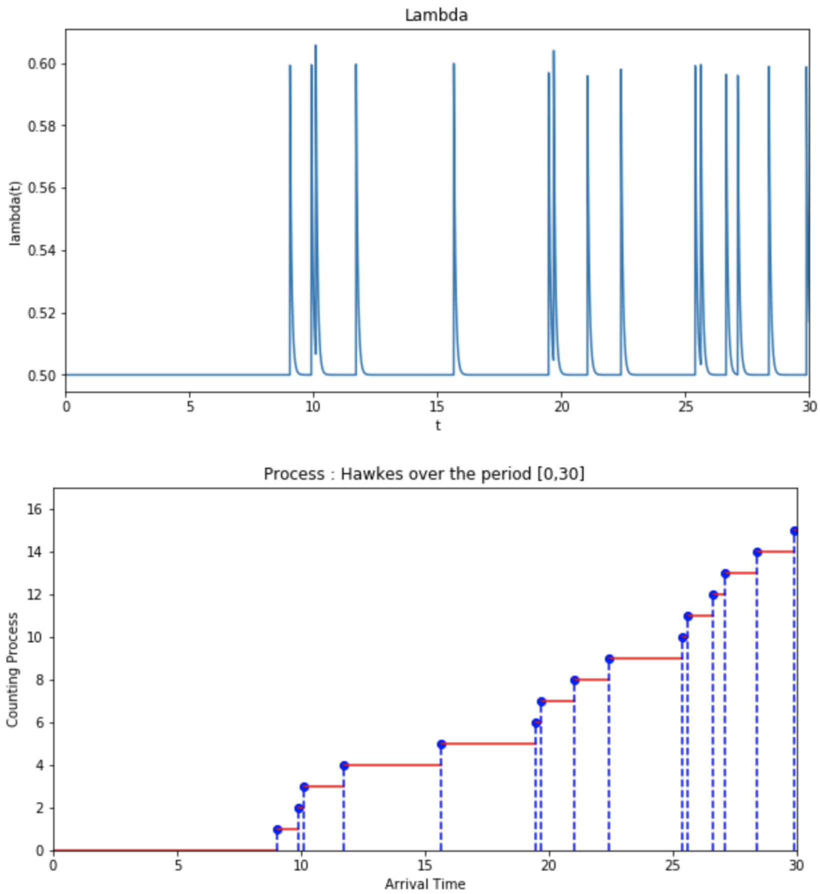

```
29  events occured during the period [0,30]
```

Figure 3: Hawkes process ($\mu = 0.5$, $\alpha = 0.4$ and $\beta = 0.5$) over the period $T = 30$

α is increasing compared to figure 2, $\alpha = 0.4$.
α is a "scale parameter". In fact, if α is equal to 4, the intensity jumps of 4.

1.3.3 Influence of the parameter β

Now, α and μ are fixed (equals to 0.1 and 0.5 respectively).




```
15  events occured during the period [0,30]
```

Figure 4: Hawkes process ($\mu = 0.5$, $\alpha = 0.1$ and $\beta = 15$) on period $T = 30$

Let us take a greater value for β than in figure 2, $\beta = 15$.
We can easily see that $\lambda(t)$ is minimised by μ.
If β increases, the process tends to loose its memory property. In fact, when an event E_1 occurs, the higher β is, the quicker the intensity decreases after an event. Thus it de-energises the process. The occurrence probability of an event E_2 right after E_1 decreases, since it depends on $\lambda(t)$ β is the "memory parameter".

2 Cramer-Lundberg Model

The Cramer-Lundberg model is used for modelling the risk process associated to the wealth of an insurance company. It is defined as :

$$R_t = u + ct - \sum_{i=1}^{N_t} Y_i \tag{3}$$

• u represents the premium (first transfer done by the client).
• c represents the amount paid every period by the client to subscribe to the insurance.
• N_t is a counting process that is therefore random. We use the point process to simulate the random arrival times. It can be a Poisson or a Hawkes process for example.
• Y_i represent the claim i's size.

In this part we will firstly show the properties of the Cramer-Lundberg model with N_t an Homogeneous Poisson Process and then we will use some of these results on R_t with the Hawkes model as it is difficult to compute them.

2.1 Cramer-Lundberg Model with Homogeneous Poisson Process

Knowing that the claim's sizes and the counting process N are independent, we can therefore compute the expectation of R_t.

$$\begin{aligned}
\mathbb{E}[R_t] &= \mathbb{E}\left[u + ct - \sum_{i=1}^{N_t} Y_i\right] \\
&= u + ct - \mathbb{E}\left[\sum_{i=1}^{N_t} Y_i\right] \\
&= u + ct - \mathbb{E}[N_t]\mathbb{E}[Y_1] \\
&= u + ct - \lambda t \mathbb{E}[Y_1]
\end{aligned} \tag{4}$$

We can see that the expectation will depend on the distribution the Y_i will have.

2.1.1 Graphic Representation of Rt with a Poisson Process :

In order to display Rt, we chose the following set of parameters :
- $u = 15$
- $c = 2$

We suppose in that first case that Y_i have a special Bernoulli distribution and are independent and identically distributed, so that :
P$(Y_i = 1) = p = 0.3$ and P$(Y_i = 5) = 1 - p = 0.7$

Therefore,

$$E[Y_1] = p + 5(1-p) = 3.8 \tag{5}$$

We fixed the Period T = 30 to observe the process

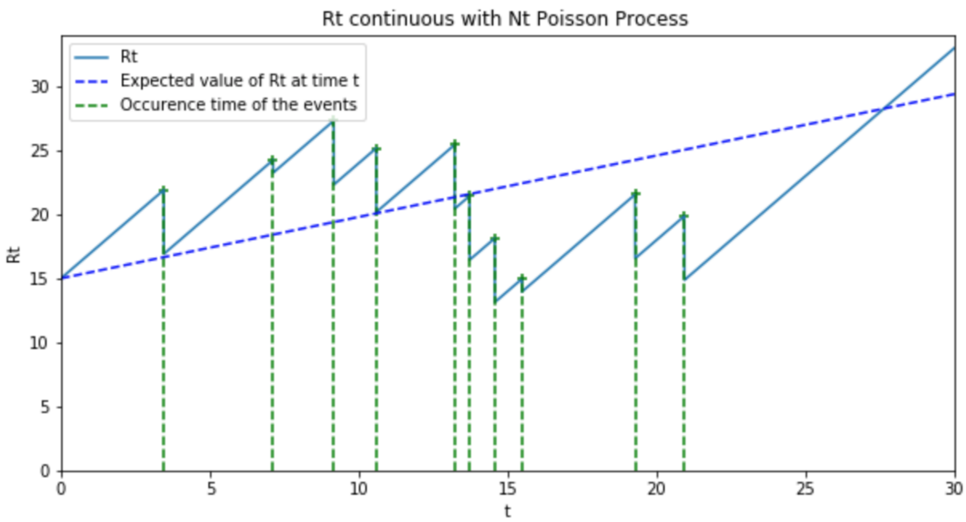


Figure 5: $\lambda = 0.4, u = 15, c = 2, Y_i Bernoulli$

We can compute $E[R_T] = 29.4$ thanks to the previous formula.
In order to show the influence of the rate λ we will now take $\lambda = 0.8$.

- **Case of study : $\lambda = 0.8$**

We can compute the expected value of R_T : $E[R_T] = -16.2$
We expect to see a Ruin Time happening on the simulation of the process Rt.
If we want a positive expected value,we can slightly increase $c \leftarrow 3$.
And with $c = 3$: $E[R_T] = 13.8$
Let us now display these two cases to see the difference between theory and reality.

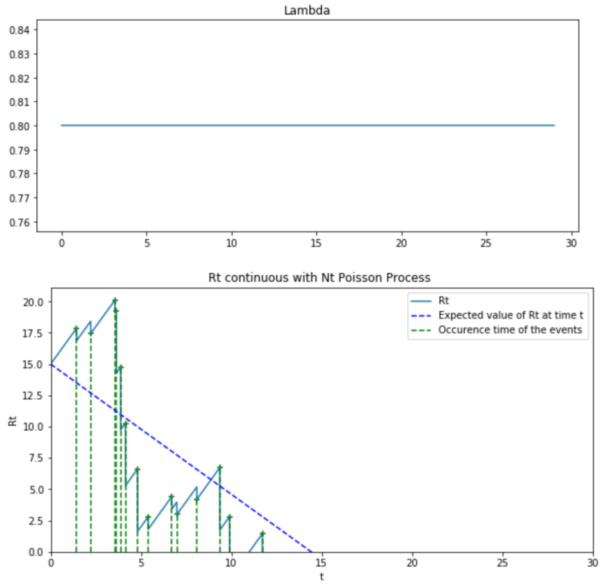


Figure 6: c = 2

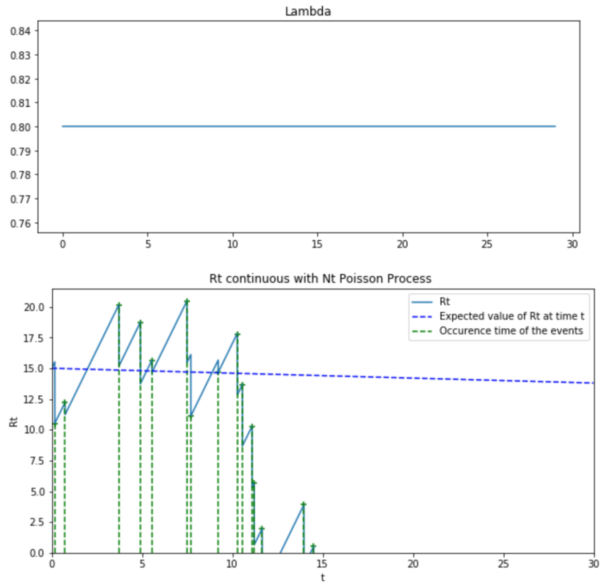


Figure 7: c = 3

As expected, a ruin time occurred on the first simulation (Figure 6). The expected value of R_T being negative, it is not represented on the graph when it goes below 0.

The really interesting phenomenon on the figure 7 is that even if theoretically we expected a positive R_T, the reality is different : around day 14, many events with big claims occurred, leading R_t to a ruin time.

It shows us that even if theoretically we expect some results, the randomness of Poisson Process can lead to a totally different perspective.

We will also show this point in part 2.2 and find a way to minimise it.

The question is now : **How to choose well u and c for Rt ?**

2.1.2 Net Profit Condition with u and c in the small claim case:

Let us remind some definitions and properties around the ruin time for a Poisson Process:

- Recall the **Safety Loading Coefficient :**
  $r = \frac{c}{\lambda E} - 1 = (\rho - 1)\frac{c}{\lambda}$; with $\rho = \frac{\lambda E}{c}$ and $E = E[Y_1]$

- Ruin Term : $\tau(u) = inf(t > 0, R_t < 0)$

ie : the first time time for which R_t goes below 0.

- Probability of Ruin : $\Psi(u) = P(\tau(u) < \infty)$

ie : the probability that there is a finite Ruin time

- Survival Probability : $\theta(u) = 1 - \Psi(u)$

Our goal is then to have $\theta(\infty) = 1$; ie : $\Psi(\infty) = 0$. **In other words, we don't want any ruin time**

• **Cramer-Lundberg Coefficient : Small Claim Case**
If there exists K such that : $E[e^{K(Y_1-cW_1)}] = 1$
Then **K** is the lundberg coefficient, with Y_1 claim , W_1 Inter-Arrival Time

Lemma 2.1 *If $\rho < 1$ then K exists and is unique*

Theorem 2.1 *If $\rho < 1$: $\Psi(u) \leq e^{-Ku}$*

So, the condition to ensure that $\Psi(\infty) = 0$ is to have $\rho < 1$
We will call this condition the **Net Profit Condition**.

Let's illustrate with 3 results of simulations the role of ρ: We keep $\lambda = 0.4$, $u = 15$, and the claims are still Bernoulli.

Case 1: $\rho = 1$

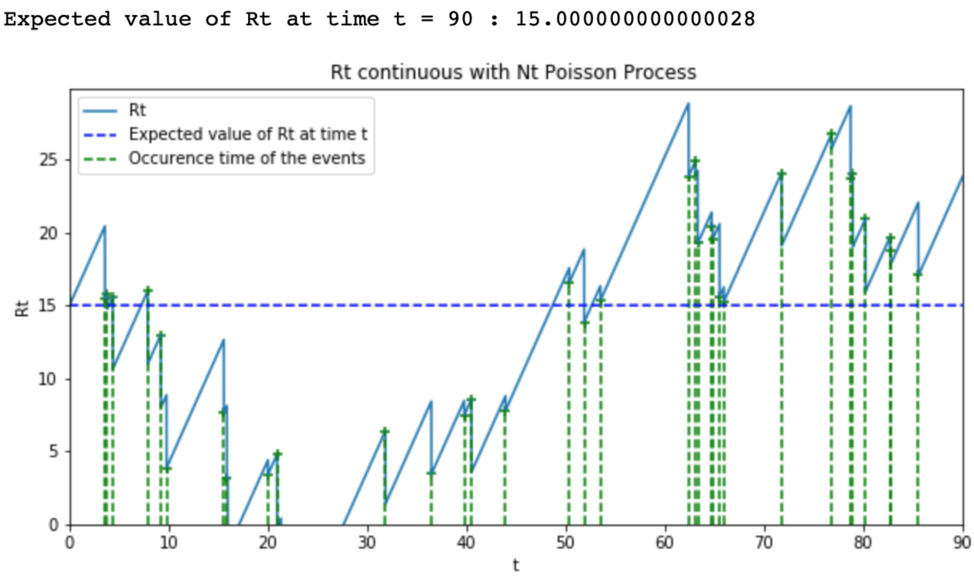


Figure 8: $\rho = 1$

We can see that with $\rho = 1 \Leftrightarrow c = \lambda E = 1.52$, even if the expectation of Rt is positive by time T, there is a ruin time before.

Case 2: $\rho > 1$

```
Expected value of Rt at time t = 90 : -74.99999999999997
```

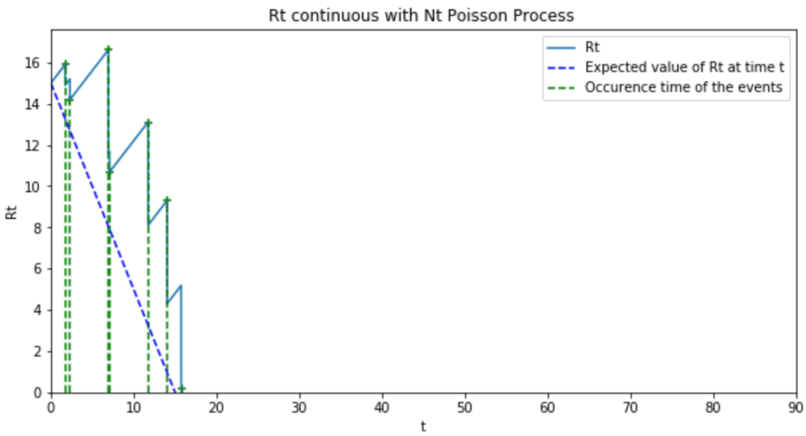


Figure 9: $\rho > 1$

$\rho > 1 \Leftrightarrow c < \lambda E$ is the worst case. We chose u=15 and c = 0.52 for the simulation displayed above (figure 9).

Case 3: $\rho < 1$

```
Expected value of Rt at time t = 90 : 105.00000000000003
```

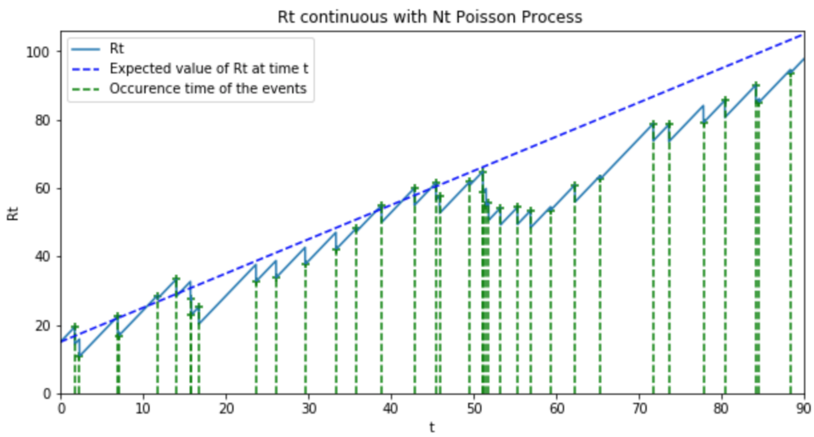


Figure 10: $\rho < 1$

First the expectation is the highest and then, we can see that there is no ruin time. We chose u=15 and c = 3.52 for the simulation displayed on the figure 10.
Now that we have seen the right way to choose u and c for Rt with a Poisson Process, we will try to adapt it to Hawkes Processes.

2.2 Cramer-Lundberg Model with Hawkes Process

Let us first compute the theoretical expectation of Rt :
Let H_t be a Hawkes Process and $\forall i, H_t \amalg Y_i$ then :

$$\begin{aligned}\mathbb{E}[R_t] &= \mathbb{E}\left[u + ct - \sum_{i=1}^{H_t} Y_i\right] \\ &= u + ct - \mathbb{E}[H_t]\mathbb{E}[Y_1] \\ &= u + ct - \frac{\mu}{\alpha - \beta}(\frac{\alpha}{\alpha - \beta}(e^{(\alpha-\beta)t} - 1) - \beta t)\mathbb{E}[Y_1] \end{aligned} \tag{6}$$

The claims Y_i still have the special Bernoulli distribution.
Let us observe Rt with Hawkes Processes. We have a seen a **Net Profit Condition** for Homogeneous Poisson Processes, which was : $\frac{\lambda E}{c} < 1$.

As $\lambda(t)$ is non constant for Hawkes processes, we can not easily find a Net Profit Condition.

So we will use the Poisson NPC with different approximation of λ in order to find the right u and c for the models.

Use of an approximation of λ to use Poisson NPC on this model:

We have :

$$\lim_{t\to+\infty} E[\lambda(t)] = \frac{-\beta\mu}{\alpha - \beta} = \lambda_B$$

Therefore we can use this λ_B which rely on the parameters of the Hawkes process to get a best approximation of c, thanks to the Poisson NPC:

$$\lambda_B E < c \Leftrightarrow c > \frac{-\beta\mu}{\alpha - \beta}E \tag{7}$$

In order to try this new value of c we arbitrarily choose c $= \frac{-\beta\mu}{\alpha-\beta}E + k, \quad k \in \mathbb{R}_+^\star$.
We will allow ourselves to observe the results with different value of this constant k. **We will display results with k=2** and afterwards draw a tabular to explain the impact of k.

Using numerical simulations is great, but as we study random process, there is a huge randomness from one simulation to another, even if we use the same parameters..

We come up with a good idea to ignore this random part : for one set of parameters $(u, c, \mu, \alpha, \beta)$ we do a big number of simulations and then we have a look at the percentage of Ruin Time in those simulations. This method has a big numeric cost but give pertinent results

- **Case 1 : Few events occurring**

In order to simulate a few events occurring we use $(\mu, \alpha, \beta) = (0.1, 0.5, 1)$, with such parameter we can expect $E[H_t] = 72.8$ events occurring over the period T$= 365$ days.
The condition (7), k=2 and the values of the set of parameters give us : $c = 2.76$. This value is smaller than the previous one which was equal to 3.52.
With such parameters we have 2% of ruin time occurring instead of 0% previously.

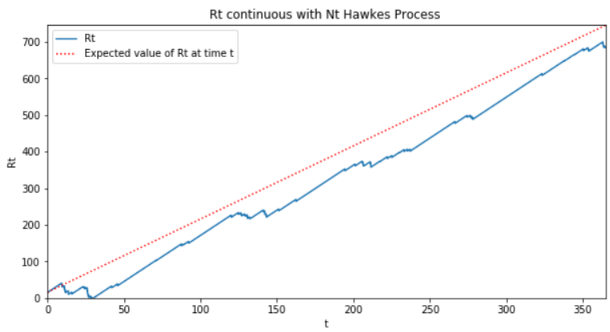


Figure 11: Ruin Time

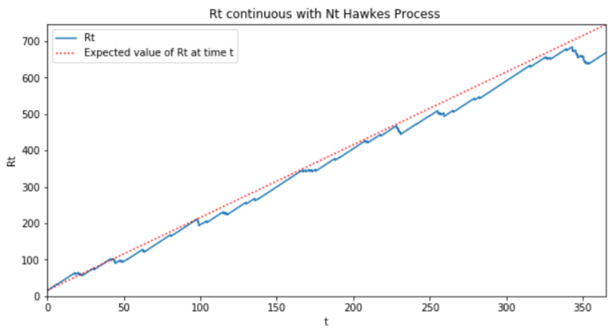


Figure 12: No Ruin Time

As we can see, even if the amount the client has to pay each month is smaller that before, we only have 2% of Ruin time if there is a small number of events occurring. Thus, this condition fixing c seems more fare than the previous one.

- **Case 2 : More events occurring**

In order to simulate that more events occur, we use $(\mu, \alpha, \beta) = (0.5, 1, 2)$, with such parameter we can expect $E[H_t] = 364.5$ events occurring over the period T$= 365$ days.
The condition (7), k=2 and the values of the set of parameters give us : $c = 5.8$. This value is bigger than the previous one which was equal to 3.52.
With such parameters we have 20% of ruin time occurring. If we had kept $c = 3.52$ we would have had : 45% of ruin time.

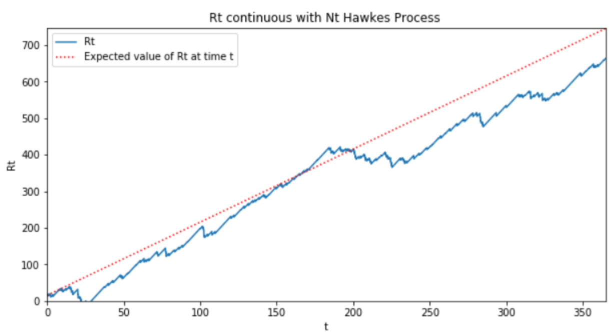


Figure 13: Ruin Time

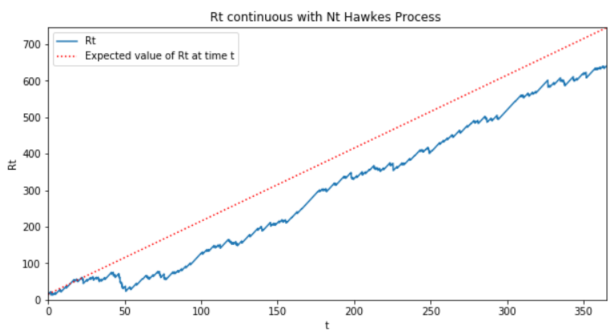


Figure 14: No Ruin Time

Here, c is bigger than before, and thus there are way less ruin times than before. It gives the insurer a better condition to fix the amount c.

- **Case 3 : Lot of events occurring**

In order to simulate that a lot of events are occurring, we use $(\mu, \alpha, \beta) = (1, 0.5, 2)$, with such parameter we can expect $E[H_t] = 486.44$ events occurring over the period T= 365 days. The condition (7), k=2 and the values of the set of parameters give us : $c = 7.06$. This value is bigger than the previous one which was equal to 3.52.
With such parameters we have 24% of ruin time occurring. If we had kept $c = 3.52$ we would have had : 85 % of ruin time.

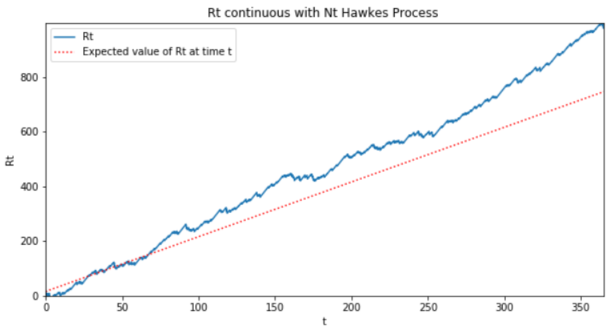


Figure 15: Ruin Time

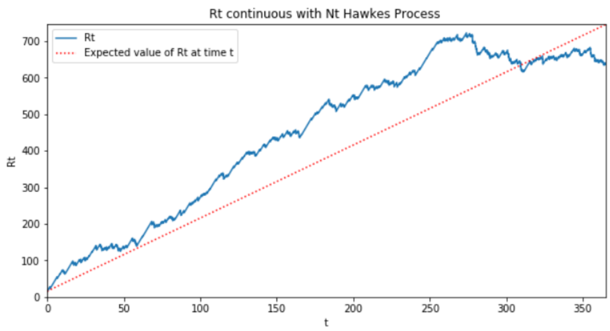


Figure 16: No Ruin Time

Here again, c is bigger than before, and thus there are way less ruin times than before. It gives the insurer a better condition to fix the amount c.

Remark : Ruin Times always happens before the 50th day. A first thought was to increase the premium u but as a matter of fact, it did not change drastically the percentage of Ruin time.

We can first display a tabular of the evolution of c we would choose depending on k according to the amount of events.

(μ, α, β)	k=0.5	k=1	k=1.5	k=2	k=2.5	k=3	Event amount
$(0.1, 1, 2)$	c = 1.26	c = 1.76	c = 2.26	c = 2.76	c = 3.26	c = 3.76	Small
$(0.5, 1, 2)$	c = 4.3	c = 4.8	c = 5.3	c = 5.8	c = 6.3	c = 6.8	Medium
$(1, 0.5, 2)$	c = 5.56	c = 6.06	c = 6.56	c = 7.06	c = 7.56	c = 8.06	Big

Let PR be the proportion of Ruin Times, we have the corresponding tabular with the proportion of Ruin Times :

(μ, α, β)	k=0.5	k=1	k=1.5	k=2	k=2.5	k=3	Event amount
$(0.1, 1, 2)$	PR=26%	PR=10%	PR=6%	PR=2%	PR=1%	PR=0%	Small
$(0.5, 1, 2)$	PR=67%	PR=49%	PR=35%	PR=20%	PR=14%	PR=11%	Medium
$(1, 0.5, 2)$	PR=56%	PR=34%	PR=26%	PR=20%	PR=11%	PR=7%	Big

Hence, this condition seems better as it take into account the parameters (μ, α, β) but picking a $c > \frac{-\beta\mu}{\alpha-\beta}E$ does not always lead to a situation with no ruin time; we need to choose it carefully depending on the quantity of events we expect.

3 A more realistic Model to use for our insurance company

The idea of the project is to present and explain the results as if this project has been led by engineers and has to be presented to the team. In order to put us back in this **context** we modelled larger claims. Indeed, since the Hawkes process has a **self-excitation property** we chose to be an **home insurance company**. The Hawkes process could easily model some natural disasters such as **flood** or **earthquakes**.

Our company is specialised in high seismic risk area. In order to fit as best as we can this realistic context, the parameters of the Hawkes process are fixed to have rare occurence but an high self-excitation property. Hence, we choose : $\alpha = 30, \beta = 32, \mu = 0.1$. We observe over the period $T = 1000$ months.

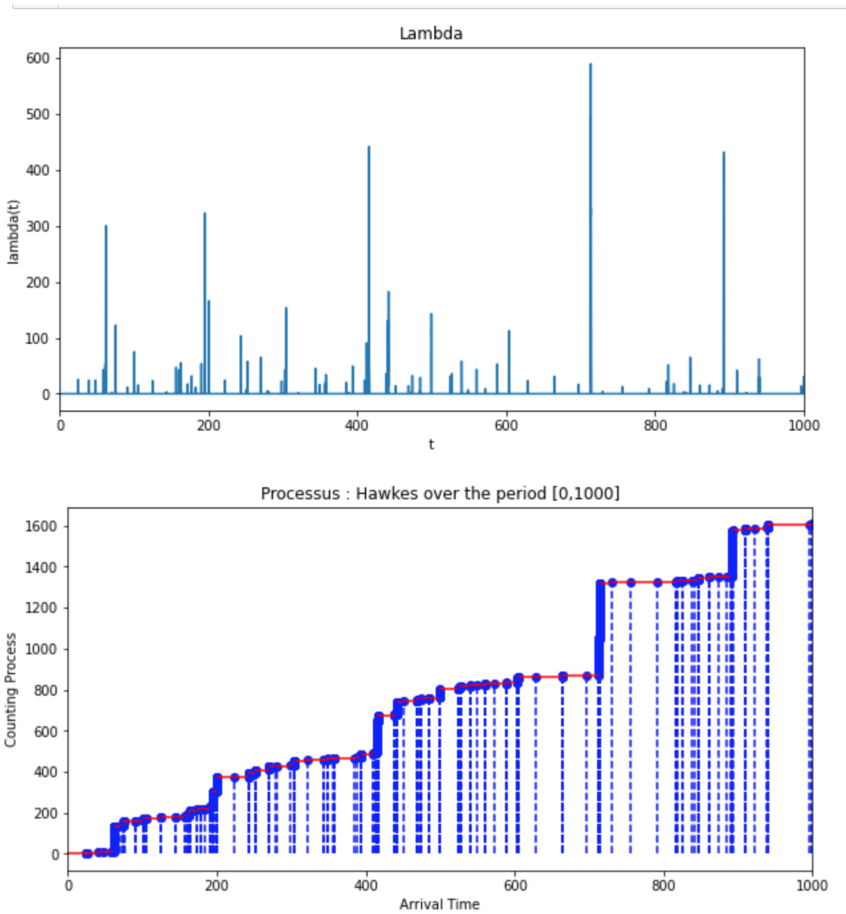


Figure 17: Hawkes process ($\mu = 0.1$, $\alpha = 30$ and $\beta = 32$) over the period $T = 30$

We can see on Figure 17 that if an event occur at time t the probability of having new events close to time t is high.

Each claim Y_i is modelled with a Pareto distribution of parameter $\alpha = 1.01$ (cf Figure 18).

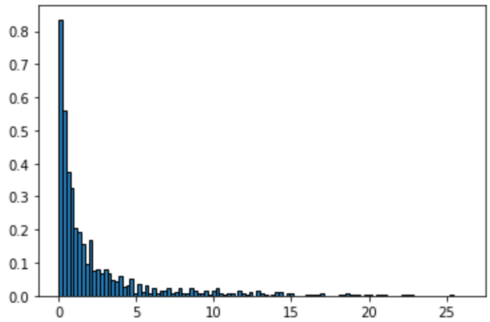


Figure 18: Repartition with N=1000 Pareto simulations

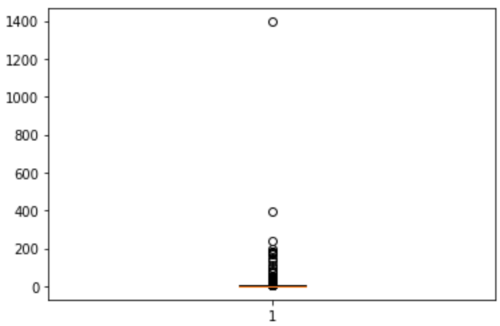


Figure 19: Boxplot

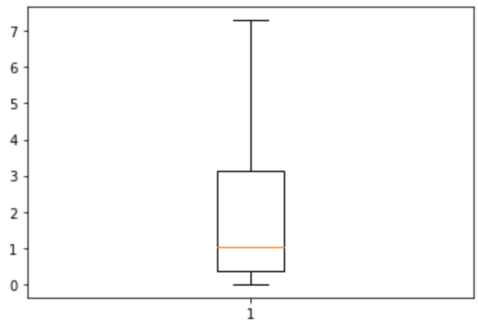


Figure 20: Boxplot without outliers

A Pareto distribution allow a huge dispersion between the different claims. Indeed, the boxplots show us that 75% of the claims are lower than 3 but some claims extend up to 1400.

With such parameter, we can compute its expectation :

$$\mathbb{E}[Y_i] = \frac{\alpha}{\alpha - 1} = 10.1 \tag{8}$$

Due to this dispersion it is relevant to compute a conditional expectation depending on a fixed level l, let's say that l is the **Upper Quartile**. Then we can compute :

$$\mathbb{E}[Y|Y > l] = \frac{\mathbb{E}[Y 1_{Y>l}]}{\mathbb{P}(Y > l)} \simeq 250$$

In order to estimate this conditional expectation we use Monte-Carlo theory on a sample of size $N = 1000000$.

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Now we would like to know how much customers have to pay each month to have less than 15% probability of ruin time. So, we would like to know c.

We fixed u=1000 :

c	ruin time rate(%)
1	100.0
5	96.0
10	75.0
20	46.0
30	30.0
40	20.0
50	18.0
60	15.0
70	15.0
80	14.0
90	11.0
100	8.0

Figure 21

To estimate this ruin time rate on Figure 21 we computed R_t 100 times for each c.

In order to reach a satisfactory ruin time rate ($\leq$ 15%) we found that c have to be higher than 60 (cf Figure 21)

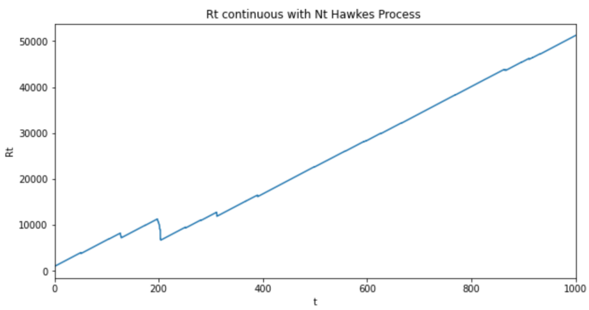

Figure 22: R_t without ruin time

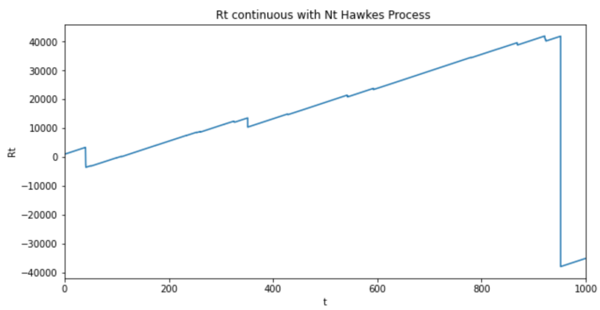

Figure 23: R_t with ruin time

On figure 22 and 23 we compute Rt with Hawkes Processes with the final parameters : u=1000, c=60, $\alpha = 30, \beta = 32, \mu = 0.1$.

On figure 22 no big claims occur, hence there is no ruin time contrary to figure 23 where a big claim occur at time $t \simeq 950$. This event is responsible of ruin time. So, an event can have a significant impact and it's totally random.

Conclusion

This project enabled to understand Hawkes processes and its self-exciting property. We used a modified thinning algorithm to model Hawkes processes. Thanks to this project, we have been able to understand the role of each parameters of the process. Afterwards, we tried to fit in the context of a company as a home insurance company. This part was the hardest one, since had to choose each value of all parameters by ourselves. After tuning all these parameters we obtained some satisfactory results. However these results have to be analysed with caution.

Indeed, the model used during the study is the Cramer-Lundberg model. One of its assumption is the independent and identically distributed property of the claims. Another assumption is the Independence of the claims to the counting process. These assumptions are not verified here, indeed when a earthquake occurs, it seems clear that there will be a dependence between claims as many houses in an area would be destroyed and therefore the claims be in the same magnitude. Moreover, the claims are modelled thanks to a Pareto law in order to model some large claims in short times like earthquakes could be modelled. The lack of knowledge concerning these phenomena did not allow us to model it properly. Such results have been achieved thanks to many computations due to a high variance of the process from one simulation to another, indeed, Hawkes process has a too large variability which provides it an "unpredictable" character. In order to fit in a satisfactory context and avoid this variance, we had to run a big amount of simulations for a set of parameter, trying to reach less than 15% of ruin times. The use of such a level in actuarial field does not seem appropriate in this context.

Hawkes process could suit other studies such as modelling terrorism attacks. The unpredictable character of such attacks could be modelled thanks to Hawkes process[2]. Indeed, in 2015 series of attacks happened in Paris. These attacks were coordinated and happened in a very short time. This phenomenon suits the self-exciting property of Hawkes process.

[2]Resource online at (clickon the following URL) : Modelisation du risque terroriste par les processus de Hawkes

Appendices

$$f(s) = \mu + \alpha e^{-\beta s} \int_0^s e^{\beta u} f(u) du$$

Hence,

$$\begin{aligned} f'(s) &= -\alpha\beta e^{-\beta s} \int_0^s e^{\beta u} f(u) du + \alpha e^{-\beta s} e^{\beta s} f(s) \\ &= -\alpha\beta e^{-\beta s} \int_0^s e^{\beta u} f(u) du + \alpha f(s) \\ &= -\beta[f(s) - \mu] + \alpha f(s) \\ &= f(s)(\alpha - \beta) + \mu\beta \end{aligned}$$

So we have :

$$\begin{cases} f'(s) = f(s)(\alpha - \beta) + \mu\beta \\ f(0) = \mu \end{cases} \tag{9}$$

Let us introduce

$$h(t) = e^{-(\alpha-\beta)t} f(t) \tag{10}$$

Thus,

$$\begin{aligned} h'(t) &= -(\alpha - \beta)e^{-(\alpha-\beta)t} f(t) + e^{-(\alpha-\beta)t} f'(t) \\ &= -(\alpha - \beta)h(t) + e^{-(\alpha-\beta)t}(f(t)(\alpha - \beta) + \beta\mu) \\ &= \beta\mu e^{-(\alpha-\beta)t} \end{aligned}$$

By definition,

$$h(t) = h(0) + \int_0^t h'(s)ds = \mu + \int_0^t \beta\mu e^{-(\alpha-\beta)s} ds = \mu - \frac{\mu\beta}{\alpha - \beta}(e^{-(\alpha-\beta)t} - 1) \tag{11}$$

From (8) we have :

$$f(t) = e^{(\alpha-\beta)t} h(t)$$

We can therefore compute $f(t)$:

$$f(t) = \frac{\mu}{\alpha - \beta}(\alpha e^{(\alpha-\beta)t} - \beta) \tag{12}$$