Fuller's optimal control problem

Minimize $$J(u) = \int_0^T x^2(t)\,dt$$

where $$\begin{cases} \ddot{x} = u \\ x(0) = 0 \\ \dot{x}(0) = 1 \\ u(t) \in [-1,1] \end{cases}$$

Hamiltonian $H(x, \dot{x}, u, \lambda, t) = x^2(t) + \lambda_1(t)\dot{x}(t) + \lambda_2(t)u(t)$

$$\left.\begin{aligned} \dot{\lambda}_1(t) &= -\frac{\partial H}{\partial x} = -2x(t) \\ \dot{\lambda}_2(t) &= -\frac{\partial H}{\partial \dot{x}} = -\lambda_1(t) \end{aligned}\right\} \text{co-states}$$

We are trying to solve

$$\min_{\substack{u \\ u \in [-1,1]}} H(x, \dot{x}, u, \lambda, t)$$

We see that $u^*(t) = -\operatorname{sgn}(\lambda_2^*(t)) \in \{-1, 0, 1\}$

This evolution of the control is called "chattering":