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TP4 of Computational Statistics

Exercise 3

Q1. We first study the joint density $\mathbb{P}(X,\mu,\sigma^2,\tau^2;\alpha,\beta,\gamma)$.

$$\mathbb{P}(X,\mu,\sigma^2,\tau^2;\alpha,\beta,\gamma)=\prod_i \mathbb{P}(X_i|\mu,\sigma^2)\,\mathbb{P}(\mu,\sigma^2,\tau^2;\alpha,\beta,\gamma)$$

$$\propto \prod_i \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(X_i-\mu)^2}{2\sigma^2}}\frac{1}{\sigma^{2(1+\alpha)}}\exp\left(-\frac{\beta}{\sigma^2}\right)\frac{1}{\tau^{2(1+\gamma)}}\exp\left(-\frac{\beta}{\tau^2}\right)$$

$$\propto \frac{1}{\sigma^{2(1+\alpha)+N}}\exp\left(-\frac{\beta}{\sigma^2}-\frac{1}{2\sigma^2}\sum_i(X_i-\mu)^2\right)\frac{1}{\tau^{2(1+\gamma)}}\exp\left(-\frac{\beta}{\tau^2}\right)$$

We now study the joint density $\mathbb{P}(Y,X,\mu,\sigma^2,\tau^2;\alpha,\beta,\gamma)$.

$$\mathbb{P}(Y,X,\mu,\sigma^2,\tau^2;\alpha,\beta,\gamma)=\prod_{i,j}\mathbb{P}\left(Y_{ij}|X_i,\tau^2\right)\prod_i \mathbb{P}(X_i|\mu,\sigma^2)\,\mathbb{P}(\mu,\sigma^2,\tau^2;\alpha,\beta,\gamma)$$

$$\propto \prod_{i,j}\frac{1}{\tau\sqrt{2\pi}}e^{-\frac{\left(Y_{ij}-X_i\right)^2}{2\tau^2}}\prod_i\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(X_i-\mu)^2}{2\sigma^2}}\frac{1}{\sigma^{2(1+\alpha)}}\exp\left(-\frac{\beta}{\sigma^2}\right)\frac{1}{\tau^{2(1+\gamma)}}\exp\left(-\frac{\beta}{\tau^2}\right)$$

$$\propto \frac{1}{\sigma^{2(1+\alpha)+N}}\exp\left(-\frac{\beta}{\sigma^2}-\frac{1}{2\sigma^2}\sum_i(X_i-\mu)^2\right)\frac{1}{\tau^{2(1+\gamma)+N+k}}\exp\left(-\frac{\beta}{\tau^2}-\frac{1}{2\tau^2}\sum_{i,j}\left(Y_{ij}-X_i\right)^2\right)$$

Q2. We know that $\sigma_{k+1}^2\sim\mathbb{P}\left(\sigma^2|\tau_k^2,\mu_k,X_k;\alpha,\beta,\gamma\right)=\frac{\mathbb{P}(\sigma^2,\tau_k^2,\mu_k,X_k;\alpha,\beta,\gamma)}{\mathbb{P}(\tau_k^2,\mu_k,X_k;\alpha,\beta,\gamma)}\sim InvGamma\left(1+2\alpha+N,\beta+\frac{1}{2}\sum_i\left(X_i^{(k)}-\mu^{(k)}\right)^2\right)$

$$\tau_{k+1}^2\sim\mathbb{P}\left(\tau^2|\sigma_{k+1}^2,\mu_k,X_k;\alpha,\beta,\gamma\right)=\frac{\mathbb{P}\left(\sigma_{k+1}^2,\tau^2,\mu_k,X_k;\alpha,\beta,\gamma\right)}{\mathbb{P}(\sigma_{k+1}^2,\mu_k,X_k;\alpha,\beta,\gamma)}\sim InvGamma(1+2\gamma,\beta)$$

$$\mu_{k+1}\sim\mathbb{P}\left(\mu|\sigma_{k+1}^2,\tau_{k+1}^2,X_k;\alpha,\beta,\gamma\right)=\frac{\mathbb{P}\left(\sigma_{k+1}^2,\tau_{k+1}^2,\mu,X_k;\alpha,\beta,\gamma\right)}{\mathbb{P}(\sigma_{k+1}^2,\tau_{k+1}^2,X_k;\alpha,\beta,\gamma)}\sim N\left(\frac{1}{N}\sum_{i=1}^N X_i^{(k)},\frac{\sigma_{(k+1)}^2}{N}\right)$$

$$X_{k+1}\sim\mathbb{P}\left(X|\sigma_{k+1}^2,\tau_{k+1}^2,\mu_{k+1};\alpha,\beta,\gamma\right)=\frac{\mathbb{P}\left(\sigma_{k+1}^2,\tau_{k+1}^2,\mu_{k+1},X;\alpha,\beta,\gamma\right)}{\mathbb{P}(\sigma_{k+1}^2,\tau_{k+1}^2,\mu_{k+1};\alpha,\beta,\gamma)}\sim N\left(\mu_{k+1}I_N;\sigma_{k+1}^2I_N\right)$$

Note that the law of μ_{k+1} is a univariate normal distribution and X_{k+1} follows a N-dimensional multivariate normal distribution.

Q3. <u>Block-Gibbs</u>

Now we jointly consider (X,μ)

$$(X,\mu)\sim\exp\left(-\frac{1}{2\sigma^2}\sum_i(X_i-\mu)^2\right)$$

Thus, the challenge is now to sample that as a whole. So let us introduce $Z = X - \mu$ (μ is broadcast in that definition)

Instead of considering (X, μ), let us work on (Z, μ)

$$Z \sim N(0, \sigma^2 I_N)$$

Then, I compute μ by using the mean of Z values. I don't know if that is the best thing to do, but I did not find any easy way to sample the joint variable (X, μ).

Q4.

- Advantage of Block-Gibbs : it can decorrelate variables and make the convergence more rapid
- Disadvantage of Block-Gibbs : the joint distribution is often hard to sample

Q5. See the notebook for implementation (in case N=2).