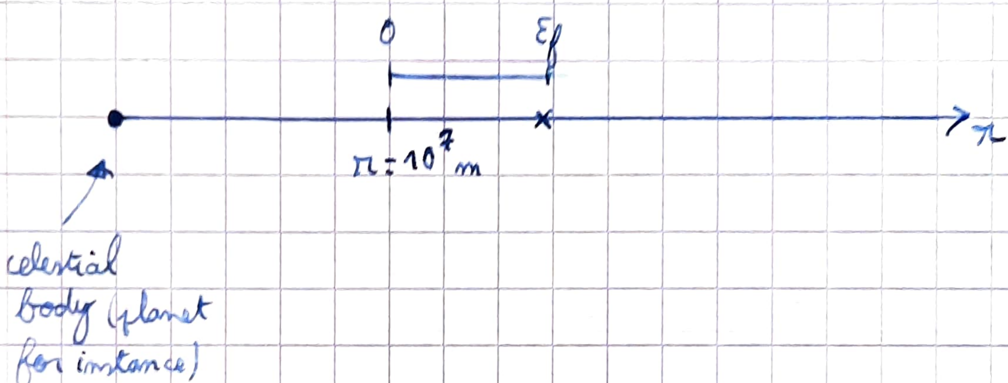
– Optimal control of rocket flow –

Let's denote by $D(t) \in [0, D_{max}]$ the flow.

We assume we want to go from r to $r + \varepsilon_f$ with $\frac{\varepsilon_f}{r} \ll 1$

For instance we could say that $r = 10\,000$ km
and $\varepsilon_f = 100$ km



The rocket is subjected to the following forces:

- $\vec{F}_{grav} = -\frac{\alpha}{r^2}\vec{e_r}$ gravity exerted by the planet
- $\vec{F}_t = \gamma D \vec{e_r}$ rocket thrust

We assume that:
$$\varepsilon(0) = 0$$
$$\varepsilon(T) = 10^5 \text{ m} = 100 \text{ km}$$
$$\dot{\varepsilon}(0) = 0$$

$$\|\vec{F}_{gar}\|(\varepsilon) = \frac{\alpha}{(r+\varepsilon)^2} = \frac{\alpha}{r^2}\left(1+\frac{\varepsilon}{r}\right)^{-2} = \frac{\alpha}{r^2}\left(1 - \frac{2\varepsilon}{r} + o\left(\frac{\varepsilon}{r}\right)\right)$$

So the state at time $t \in [0, T]$ is:

$$X(t) = \begin{pmatrix} \varepsilon(t) \\ \dot{\varepsilon}(t) \end{pmatrix} \text{ s.t. } \ddot{\varepsilon} = -\frac{\alpha}{r^2}\left(1 - \frac{2\varepsilon}{r}\right) + \gamma D$$

Moreover, we want to minimize the fuel, so

$$\text{Min} \int_0^T D^2(t)\, dt$$

$$\text{s.t.} \quad \dot{X} = \begin{pmatrix} \dot{\varepsilon} \\ -\frac{\alpha}{r^2}\left(1 - \frac{2\varepsilon}{r}\right) + \gamma D \end{pmatrix}$$

$$\varepsilon(0) = 0, \quad \varepsilon(T) = 10^5, \quad \dot{\varepsilon}(0) = 0$$

__Costate equations :__

The pre-Hamiltonian is

$$\mathcal{H} = \beta D^2 + p_\varepsilon \dot{\varepsilon} + p_{\dot{\varepsilon}} \left(-\frac{\alpha}{r^2}\left(1 - \frac{2\varepsilon}{r}\right) + \gamma D\right)$$

The Initial-Final Lagrangian is

$$\mathcal{L}^{IF} = \psi_1 (1 \ \ 0) X_0 + \psi_2 (0 \ \ 1) X_0 + \psi_3 (1 \ \ 0) X_f$$

Costate dynamics are :

$$-\dot{p}_\varepsilon(t) = \frac{2\alpha}{r^3} p_{\dot{\varepsilon}}(t) \qquad -p_\varepsilon(0) = \psi_1 \qquad p_\varepsilon(T) = \psi_3$$

$$-\dot{p}_{\dot{\varepsilon}}(t) = p_\varepsilon(t) \qquad -p_{\dot{\varepsilon}}(0) = \psi_2$$

__Necessary optimality conditions :__

$\beta \in \{0; 1\}$

$\beta + |4| > 0$

So we have : $\exists A, B \in \mathbb{R}$ s.t.

$$\bar{p}_\varepsilon(t) = A e^{t\sqrt{C}} + B e^{-t\sqrt{C}} \qquad \text{with } C = \frac{2\alpha}{\pi^3}$$

Then we use the Hamiltonian inequality :

$$\mathcal{H}'_D(t)(D - \bar{D}) \geq 0 \quad \forall D \in [0, D_{max}]$$

So $\quad (2\beta\bar{D} + \gamma\bar{p}_\varepsilon, D - \bar{D}) \geq 0 \quad \forall D \in [0, D_{max}]$

If $\beta = 0$,

$$D(t) = \mathbb{1}_{\{A e^{t\sqrt{C}} + B e^{-t\sqrt{C}} \leq 0\}}(t)\, D_{max}$$

If $\beta = 1$,

$$D(t) = \underbrace{P_{[0, D_{max}]}}_{\substack{\text{projection on} \\ [0, D_{max}]}} \left(-\frac{\gamma}{2} \bar{p}_\varepsilon(t) \right)$$

<u>Remark 1</u>: I have dealt with this problem only in 1D and only with $\varepsilon \ll 1$, otherwise costate equations and optimality conditions become much more complex!

<u>Remark 2</u>: Theory and theorems come from Lecture Notes for SOD311 by Frédéric Bonnans!