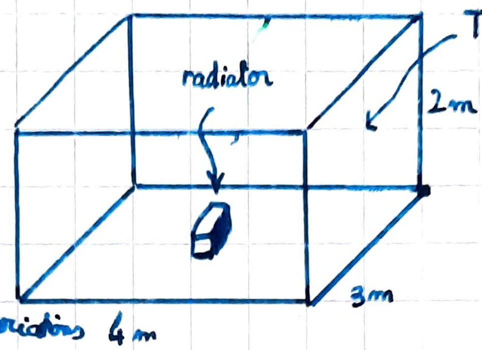
- Optimal control of radiator power -

Wooden hut heated by a radiator :

$T_{ext}(t) = 290K + \underbrace{10K\cos\left(\frac{2\pi t}{86400}\right)}_{\text{daily temperature variations}}$



$T = T(t)$ uniform in the room

(not to scale)

4 walls and ceiling are made of 5 cm-thick wood with a thermal conductivity of 0,15 $W.m^{-1}.K^{-1}$

Assumption : the floor is a thermal insulator

Total thermal resistance : $R = \dfrac{0{,}05\,m}{0{,}15\,W.m^{-1}.K^{-1}} = 0{,}3\ m^2.K.W^{-1}$

Thermal capacity of the hut : $C = V C_v = 3{,}1.10^4\ JK^{-1}$

volume of the hut

air volumetric heat capacity $1{,}3.10^3\ JK^{-1}m^{-3}$

Power of the radiator : $P(t)$

System = {hut}

First principle applied to this System :

$$\underbrace{H(t+dt) - H(t)}_{C(T(t+dt)-T(t))} = \delta Q = \Big(P(t) - \underbrace{\phi^{\text{walls, ceiling}}_{\text{outgoing}}(t)}_{\frac{S}{R}(T(t) - T_{ext}(t))}\Big)\,dt$$

wall + ceiling area $S = 40\ m^2$

Fourier's phenomenological law

Therefore:

$$\dot{T}(t) = \frac{1}{C} P(t) - \frac{S}{CR}\left(T(t) - T_{ext}(t)\right)$$

$$\dot{T}(t) = \alpha P(t) - \beta\left(T(t) - T_{ext}(t)\right)$$

with: $\alpha = \frac{1}{C} = 3{,}2 \cdot 10^{-5}\ K \cdot J^{-1}$

$\beta = \frac{S}{CR} = 4 \cdot 10^{-3}\ s^{-1}$

Goals: ① indoor temperature $T_g(t) \simeq 293\,K$ (g: goal)

② minimize $P(t)$

Data/constraint: ① when we arrive (at $t=0$)
$T(0) = T_{ext}(0)$
② we stay 3 days:
$t \in [0; \underbrace{259\,200}_{t_f}]\ s$

Therefore we want to solve:

$$\underset{T \in W^{1,\infty}(0,t_f,\mathbb{R})}{\mathrm{Min}} \underbrace{\int_0^{t_f} \ell(t, P(t), T(t))\,dt + \varphi(T(t_f))}_{\text{general form of cost function } J(P(t),T(t))}$$

Sobolev:
$W^{1,\infty}(0,t_f,\mathbb{R})$
$= \{ y \in L^\infty(0,t_f,\mathbb{R}) \ \underbrace{\text{such that}}_{\text{s.t.}}\ \dot{y} \in L^\infty(0,t_f,\mathbb{R}) \}$

with $\begin{cases} \ell & \text{uniformly quasi } C^r \\ \varphi & C^r \end{cases}$

$\dot{y}$: derivative in the sense of distributions (cf L. Schwartz) and satisfying $\dot{T}(t) = f(t, \overbrace{P(t)}^{\text{control}}, \overbrace{T(t)}^{\text{state}})$ for a.a. $t \in [0, t_f]$

(equality in $L^\infty(0, t_f, \mathbb{R})$)

with $\begin{cases} t_f > 0 \\ \text{initial condition } T(0) = T_{ext}(0) = 280K \\ \text{dynamics : } f : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \begin{cases} \text{Lipschitz w.r.t. } T \\ \text{uniformly quasi } C^r \end{cases} \end{cases}$ $(r \geq 1)$

In optimal control theory, deriving the following Lagrangian

lower-case ; lagrange multiplier

$$J(P,T) + \underbrace{\int_0^{t_f} \langle p(t), f(t, P(t), T(t)) - \dot{T}(t) \rangle_{\mathbb{R}', \mathbb{R}} \, dt}_{\text{dual notation (not very useful here because the state space is } \mathbb{R})} + \langle q, T^0 - T(0) \rangle_{\mathbb{R}', \mathbb{R}}$$

gives us the <u>costate dynamics equations</u>

$$\begin{cases} -\dot{\bar{p}}(t) = \nabla_T H(t, \bar{P}(t), \bar{T}(t), \bar{p}(t)) \text{ for a.a. } t \in [0, t_f] \\ \bar{p}(t_f) = \nabla \varphi(\bar{T}(t_f)) \end{cases}$$

where $H(t, P, T, p) = \ell(t, P, T) + \langle p, f(t, P, T) \rangle_{\mathbb{R}', \mathbb{R}}$

is called « pre-hamiltonian »

From the derivative of $J(P,t)$ we can obtain <u>necessary optimality conditions</u>:

$$\bar{P} \text{ local minimum of } \underbrace{J(P, T[P, \underset{\text{fixed}}{T^0}])}_{\substack{\text{reduced cost} \\ \text{mapping}}} \Rightarrow \underbrace{\nabla_P H(t, \bar{P}(t), \bar{T}(t), \bar{p}(t))}_{\text{called } \nabla_P \bar{H}(t)} = 0 \text{ for a.a } t \in [0, t_f]$$

$$\bar{T}^0 \text{ local minimum of } J(\underbrace{P, T[P}_{\text{fixed}}, T^0]) \Rightarrow \bar{p}(0) = 0$$

PROOF

We remind $J(P,T)$:

$$J(P,T) = \int_0^{t_f} \ell(t, P(t), T(t))\,dt + \varphi(T(t_f))$$

Its derivative is

$$J'(P,T)(\tilde{P},\tilde{T}) = \int_0^{t_f} \ell'(t, P(t), T(t))(\tilde{P}(t), \tilde{T}(t))\,dt$$
$$+ \langle \nabla\varphi(T(t_f)), \tilde{T}(t_f) \rangle_{\mathbb{R}', \mathbb{R}}$$

Why? Because we have:

$$J(P+\tilde{P}, T+\tilde{T}) := J(P,T) + J'(P,T)(\tilde{P},\tilde{T}) + o((\tilde{P},\tilde{T}))$$
$$\parallel$$
$$\int_0^{t_f} \ell(t, P(t)+\tilde{P}(t), T(t)+\tilde{T}(t))\,dt + \varphi(T(t_f)+\tilde{T}(t_f))$$
$$\parallel$$
$$J(P,T) + \int_0^{t_f} \ell'(t, P(t), T(t))(\tilde{P}(t), \tilde{T}(t))\,dt$$
$$+ \underbrace{\varphi'(T(t_f))(\tilde{T}(t_f))}_{\text{Jacobian notation}} + o((\tilde{P},\tilde{T}))$$

$$\underbrace{\langle \nabla\varphi(T(t_f)), \tilde{T}(t_f) \rangle_{\mathbb{R}', \mathbb{R}}}_{\text{dual notation with gradient}}$$

because it's a costate equation

But

$$\langle \underbrace{\nabla \varphi(T(t_f))}_{= p(t_f)}, \tilde{T}(t_f) \rangle_{\mathbb{R}', \mathbb{R}} = \langle p(0), \tilde{T}(0) \rangle_{\mathbb{R}', \mathbb{R}} + \int_0^{t_f} \frac{d}{dt} \langle p(t), \tilde{T}(t) \rangle_{\mathbb{R}', \mathbb{R}} \, dt$$

$$= \langle p(0), \tilde{T}(0) \rangle_{\mathbb{R}', \mathbb{R}} + \int_0^{t_f} \langle \dot{p}(t), \tilde{T}(t) \rangle_{\mathbb{R}', \mathbb{R}} + \langle p(t), \dot{\tilde{T}}(t) \rangle_{\mathbb{R}', \mathbb{R}} \, dt$$

1st costate dynamics

Linearized equation

$$= \langle p(0), \tilde{T}(0) \rangle_{\mathbb{R}', \mathbb{R}} + \int_0^{t_f} \langle -\nabla_T \bar{H}(t), \tilde{T}(t) \rangle_{\mathbb{R}', \mathbb{R}} + \langle p(t), f'(t, P, T)(\tilde{P}, \tilde{T}) \rangle \, dt$$

$$= \langle p(0), \tilde{T}(0) \rangle_{\mathbb{R}', \mathbb{R}} + \int_0^{t_f} \langle -\nabla_T \bar{\ell}(t) - \bar{f}'^{T}_{T}(t)\, p(t), \tilde{T}(t) \rangle_{\mathbb{R}', \mathbb{R}} \, dt + \int_0^{t_f} \langle p(t), \bar{f}'_P(t, P, T)\tilde{P} + \bar{f}'_T(t, P, T)\tilde{T} \rangle_{\mathbb{R}', \mathbb{R}} \, dt$$

Therefore:

$$J'_R(P, T^0)(\tilde{P}, \tilde{T}^0) = \langle p(0), \tilde{T}^0 \rangle_{\mathbb{R}', \mathbb{R}} + \int_0^{t_f} \underbrace{\left(\bar{\ell}'_P(t)\tilde{P}(t) + p(t)^T \bar{f}'_P \tilde{P}\right)}_{\langle \nabla_P \bar{H}(t), \tilde{P}(t) \rangle_{\mathbb{R}', \mathbb{R}}} dt$$

which concludes.

Let's apply this to

$$J(P,T) = \int_0^{t_f} \ell(t, P(t), T(t))\,dt + \varphi(T(t_f))$$

with $\varphi = 0 \in C^{\infty}$

and $\ell(t,P,T) = \frac{1}{2}\left(|P|^2 + |T - T_g|^2\right)$ uniformly quasi C^{∞}

to minimize the consumption of power

goal

Costate dynamics:

lower-case

$$-\dot{\bar{p}}(t) = \bar{T}(t) - \bar{T}_g(t) + \beta\,\bar{p}(t)$$

lower-case

lower case $\rightarrow$ $\cancel{\bar{p}(T) = 0}$

Necessary optimality condition:

$$\bar{P}(t) + \alpha\,\bar{p}(t) = 0$$

uppercase

lower case

And of course, $\bar{T}$ must satisfy the differential equation:

$$\dot{\bar{T}}(t) = \alpha\,\bar{P}(t) - \beta\left(T(t) - T_{ext}(t)\right)$$

Therefore, with $Y(t) := \begin{pmatrix} T(t) \\ p(t) \end{pmatrix}$ we obtain:

$$\dot{Y}(t) = \underbrace{\begin{pmatrix} -\beta & -\alpha^2 \\ -1 & \beta \end{pmatrix}}_{A} Y(t) + \underbrace{\begin{pmatrix} \beta\,T_{ext}(t) \\ T_g \end{pmatrix}}_{B(t)}$$

This can be solved analytically, but it's very time-consuming!

Here is the beginning of the analytical resolution:

there exists a vector $\lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \in \mathbb{R}^2$ s.t.

$$Y(t) = e^{At} \left(C + \int_0^t e^{-As} B(s)\, ds \right)$$

From Cayley-Hamilton theorem we know that

$$e^{At} = c_0(t)\, I_2 + c_1(t)\, A$$

with

$$\left| \begin{aligned} C_0(t) &= \frac{e^{\xi_1 t} - e^{\xi_2 t}}{\xi_1 - \xi_2} \\ c_1(t) &= \frac{e^{\xi_1 t}\xi_2 - e^{\xi_2 t}\xi_1}{\xi_2 - \xi_1} \end{aligned} \right.$$

where ξ_1, ξ_2 are the eigenvalues of $A = \begin{pmatrix} -\beta & -\alpha^2 \\ -1 & \beta \end{pmatrix}$

We stop here the calculations, because we use a numerical solution with Runge Kutta. See the file .ipynb

On that file we discover a very strange phenomenon: high frequency oscillations of $p(t)$ and $T(t)$.
Another "strange" phenomenon: $P(t)$ is negative for some values of t. This shows we should add constraints:

③ $0 \leqslant P(t) \leqslant P_{max}$

④ $\ell(t, P, T) = \frac{1}{2}\left(|P|^2 + \underbrace{|\dot{P}|^2}_{\text{to avoid high frequency oscillations}} + |T - T_g|^2 \right)$