Box-Muller algorithm

The Box-Muller algorithm is used to generate random variables of normal distribution from random variables of uniform distribution.

We will prove that the algorithm below achieves this objective using a very well-known method in probability: the transfer theorem.

Algorithm:

- Generate $U_1, U_2 \sim \mathcal{U}(]0,1])$ independent
- Then $X_1 = \sqrt{-2\ln U_1}\cos(2\pi U_2)$ and $X_2 = \sqrt{-2\ln U_1}\sin(2\pi U_2)$ are independent and their law is $\mathcal{N}(0,1)$

Proof that the algorithm works:

- Let be h a bounded and measurable function. Let's use the transfer theorem.
- $\mathbb{E}\big(h(X_1,X_2)\big) = \int_{\mathbb{R}^2} h(x_1,x_2) f_{(X_1,X_2)}(x_1,x_2)\,dx_1 dx_2$
- $\mathbb{E}(h(F(U_1,U_2))) = \int_{]0,1]^2} h\big(F(u_1,u_2)\big)du_1 du_2$ where $F :]0,1]^2 \to \mathbb{R}^2$ is $F(u_1,u_2) = \left(\sqrt{-2\ln u_1}\cos(2\pi u_2), \sqrt{-2\ln u_1}\sin(2\pi u_2)\right)$
- Jacobian matrix $J = \begin{pmatrix} \frac{-\cos(2\pi u_2)}{u_1\sqrt{-2\ln u_1}} & -2\pi\sqrt{-2\ln u_1}\sin(2\pi u_2) \\ \frac{-\sin(2\pi u_2)}{u_1\sqrt{-2\ln u_1}} & 2\pi\sqrt{-2\ln u_1}\cos(2\pi u_2) \end{pmatrix}$
- $|\det J|^{-1} = \frac{|u_1|}{2\pi} = \frac{1}{2\pi}\exp\left(-\frac{x_1^2+x_2^2}{2}\right)$
- F is a diffeomorphism (F is C^1 and bijective) so we can apply the variable change theorem:

$$\mathbb{E}(h(F(U_1,U_2))) = \int_{]0,1]^2} h\big(F(u_1,u_2)\big)du_1 du_2 = \int_{\mathbb{R}^2} h(x_1,x_2)\frac{1}{2\pi}\exp\left(-\frac{x_1^2+x_2^2}{2}\right)dx_1 dx_2$$

- So $f_{(X_1,X_2)}(x_1,x_2) = \frac{1}{2\pi}\exp\left(-\frac{x_1^2+x_2^2}{2}\right)$, which has separable variables and X_1, X_2 are independent and of law N(0,1).