1 Introduction to Ordinary Differential Equations

Definition. An equation containing the derivatives of one or more dependent variables, with respect to one or more independent variables, is said to be a **differential equation (DE)**. For the equation

$$\frac{d^2x}{dt^2} + 16x = 0,$$

the unknown function or dependent variable is x and the independent variable is t.

Definition. If an equation contains only ordinary derivatives of one or more dependent variables with respect to a single independent variable it is said to be an **ordinary differential equation (ODE)**. For example,

$$\frac{dy}{dx} + 5y = e^x, \quad \frac{d^2y}{dx^2} + 6y = 0, \quad \text{and} \quad \frac{dx}{dt} + \frac{dy}{dt} = 2x + y.$$

are ordinary differential equations.

The ordinary derivatives may be written by using either the **Leibniz notation** dy/dx, d^2y/dx^2, d^3y/dx^3, ...or the **prime notation** y', y'', y''', ...By using the prime notation, the two first equations above can be written as

$$y' + 5y = e^x, \quad y'' - y' + 6y = 0.$$

Observe that the prime notation is used to denote only the first three derivatives. Other derivatives are $y^{(4)}$, $y^{(5)}$, ..., $y^{(n)}$ where $y^{(n)} = d^ny/dx^n$.

In symbols we can express an nth-order ordinary differential equation in one dependent variable by the general form,

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0, \tag{1}$$

where F is a real-valued function of $n + 2$ variables: $x, y, y', y'', \ldots, y^{(n)}$.

Classification of ODE: Ordinary differential equations are classified by **order** and **linearity**.

The **order** of a differential equation is the order of the highest derivative in the equation. In the first example above, the first and the third equations are of the first order and the second equation is of the second order.

An nth-order ordinary differential equation (1) is said to be **linear** if F is linear in $y, y', y'', \ldots, y^{(n)}$. This means that an n^{th} ODE is linear if (1) can be written as

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

For a first and second order ODE, we have

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x), \quad a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{x} + a_0(x)y = g(x)$$

Observe that

1. The dependent variable y and its derivatives $y', y'', \ldots, y^{(n)}$ are of the first degree, that is, the power of each term involving y is 1.

2. The coefficients $a_0, a_1, a_2, \ldots, a_n$ of $y, y', y'', \ldots, y^{(n)}$ depend at most on the dependent variable x.

$$(1-y)y' + 2y = e^x, \qquad \text{nonlinear}$$
$$\frac{d^2y}{dx^2} + \sin y = 0, \qquad \text{nonlinear}$$
$$y'' - y' + 2y = x, \qquad \text{linear}$$

Definition. Any function y, defined on an interval I and possessing at least n derivatives that are continuous on I, which when substituted into an nth-order ordinary differential equation reduces the equation to an identity, is said to be a **solution** of the equation on the interval.

Example. The functions $y = c_1 \cos 4x$ and $y = c_2 \sin 4x$, where c_1 and c_2 are arbitrary constants or parameter, are both solutions of the linear differential equation

$$y'' + 16y = 0.$$

Solution. For $y = c_1 \cos 4x$, the first two derivatives are $y' = -4c_1 \sin 4x$ and $y'' = -16c_1 \cos 4x$. Substituting y'' and y then gives

$$y'' + 16y = -16c_1 \cos 4x + 16c_1 \cos 4x = 0$$

Similarly, $y = c_2 \sin 4x$, we have $y'' = -16c_2 \sin 4x$, so that

$$y'' + 16y = -16c_2 \sin 4x + 16c_2 \sin 4x = 0$$

Finally, it is straightforward to verify that the linear combination of solutions, or the two-parameter family $y = c_1 \cos 4x + c_2 \sin 4x$, is also a solution of the differential equation.

Exercises

1. State the order of the given ODE, and whether it is linear or nonlinear. If it is linear, is it homogeneous or nonhomogeneous?

   (a) $\frac{dy}{dx} = 5y$

   (b) $\frac{d^2y}{dx^2} + x = y$

   (c) $y\frac{dy}{dx} = x$

   (d) $y''' + xy' = x \sin x$

   (e) $y'' + x \sin xy' = y$

   (f) $y'' + 4y' - 3y = 2y^2$

   (g) $\frac{d^3y}{dt^3} + t\frac{dy}{dt} + t^2y = t^3$

   (h) $\cos x\frac{dx}{dt} + x \sin t = 0$

   (i) $y^{(4)} + e^x y'' = x^3 y'$

   (j) $x^2 y'' + e^x y' = \frac{1}{y}$

2. Verify that $y = \cos x$ and $y = \sin x$ are solutions of the ODE $y'' + y = 0$. Are any of the following functions solutions?

   (a) $\sin x - \cos x$ (b) $\sin(x + 3)$ (c) $\sin 2x$

   Justify your answer.

3. Verify that $y = e^x$ and $y = e^{-x}$ are solutions to the ODE $y'' - y = 0$. Are any of the following functions solutions?

(a) $\cosh x = \frac{1}{2}(e^x + e^{-x})$ (b) $\cos x$ (c) x^e

Justify your answer.

4. $y_1 = \cos(kx)$ is a solution of $y'' + k^2 y = 0$. Guess and verify another solution y_2 that is not a multiple of y_1. Then find a solution that satisfies $y(\pi/k) = 3$ and $y'(\pi/3) = 3$.

5. $y_1 = e^{kx}$ is a solution of $y'' - k^2 y = 0$. Guess and verify another solution y_2 that is not a multiple of y_1. Then find a solution that satisfies $y(1) = 3$ and $y'(1) = 2$.

6. Using exercise 2, find a solution of $y'' + y = 0$ that satisfies $y(\pi/2) = 2y(0)$ and $y(\pi/3) = 3$.

7. Find two values of r such that $y = e^{rx}$ is a solution of $y'' - y' - 2y = 0$. Then find a solution of the equation that satisfies $y(0) = 1$, $y'(0) = 2$.

8. Verify that $y = x$ is a solution of $y'' + y = x$, and find a solution y of this ODE that satisfies $y(\pi) = 1$ and $y'(\pi) = 0$.

9. Verify that $y = -e$ is a solution of $y'' - y = e$, and find a solution y of this ODE that satisfies $y(1) = 0$ and $y'(1) = 1$.

2 First order differential equation

2.1 Separable equation

Definition. A separable equation is one of the form

$$\frac{dy}{dx} = g(x)h(y), \tag{2}$$

where the derivative dy/dx is a product of a function of x alone times a function of y alone, rather than a more general function of the two variables x and y.

$$\frac{dy}{dx} = y^2 x e^{3x+4y}, \qquad \text{separable}$$
$$\frac{dy}{dx} = y + \sin x, \qquad \text{non-separable}$$

An equation of the form (2) can be written as

$$p(y)\frac{dy}{dx} = g(x),$$

where $p(y) = 1/h(y)$ and whose solution is

$$\int p(y)\,dy = \int g(x)\,dx, \quad \text{i.e.} \quad H(y) = G(x) + C,$$

where $H(y)$ and $G(x)$ are antiderivatives of $p(y)$ and $g(x)$, respectively and C is the constant of integration.

Example. Solve $(1 + x)\,dy = y\,dx = 0$.

Solution. This equation can be written as $\dfrac{dy}{dx} = \dfrac{dx}{1+x}$. Integrating on both sides, we get

$$\int \frac{dy}{dx} = \int \frac{dx}{1+x}, \quad \text{so} \quad \ln|y| = \ln|1+x| + \ln C, \quad \text{that is,} \quad y = C(1+x)$$

Example. Solve the initial-value problem

$$\frac{dy}{dx} = x^2 y^3, \qquad y(1) = 3$$

Solution. Separating this ODE gives $\dfrac{dy}{y^3} = x^2\,dx$. Thus,

$$\int \frac{dy}{y^3} = \int x^2\,dx, \quad \text{so} \quad \frac{-1}{2y^2} = \frac{x^3}{3} + C.$$

Since $y = 3$ when $x + 1$, we have $\dfrac{-1}{18} = \dfrac{1}{3} + C$ and $C = \dfrac{-7}{18}$. Substituting this value into the above solution and solving for y, we obtain

$$y(x) = \frac{3}{\sqrt{7 - 6x^3}}.$$

This solution is valid for $x < \left(\dfrac{7}{6}\right)^{1/3}$.

2.2 First-Order Linear Equations

A first-order linear equation is an equation of the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

When $g(x) = 0$, it is said to be **homogeneous**; otherwise it is **nonhomogeneous**. Dividing both sides by the leading coefficients $a_1(x)$, we obtain a **standard form** of a linear equation

$$\frac{dy}{dx} + P(x)y = f(x) \tag{3}$$

The solution is a sum of two solutions $y = y_c + y_p$, where y_c is a solution to the associated homogeneous equation

$$\frac{dy}{x} + P(x)y = 0$$

and y_p is a particular solution of the nonhomogeneous equation (3).
The homogeneous equation is separable and solving for y gives

$$y_c = Ce^{-\int P(x)\,dx}$$

The particular solution y_p can be found using the method of variation of parameters described below.

Variation of parameters

The variation of parameter method is a method used to find a particular solution y_p. The idea is to find u so that $y_p = uy_1$ where $y_1 = e^{-\int P(x)\,dx}$ solves the corresponding homogeneous equation. Substituting y_p in (3) gives

$$u\frac{dy_1}{dx} + y_1\frac{du}{dx} + P(x)uy_1 = f(x) \quad \text{or} \quad y_1\frac{du}{dx} + u\left(\frac{dy_1}{dx} + P(x)y_1\right) = f(x)$$

This gives

$$y_1\frac{du}{dx} = f(x)$$

since $\dfrac{dy_1}{dx} + P(x)uy_1 = 0$ as y_1 is the homogeneous solution. We therefore obtain

$$u = \int \frac{f(x)}{y_1}\,dx = \int e^{\int P(x)\,dx} f(x)\,dx$$

Hence

$$y_p = uy_1 = e^{-\int P(x)\,dx}\int e^{\int P(x)\,dx} f(x)\,dx$$

and

$$y = Ce^{-\int P(x)\,dx} + e^{-\int P(x)\,dx}\int e^{\int P(x)\,dx} f(x)\,dx$$

The function $e^{\int P(x)\,dx}$ is called an **integrating factor** of the given differential equations because, if we multiply both sides of the equation (3) by $e^{\int P(x)\,dx}$, the left side becomes the derivative of $e^{\int P(x)\,dx}y(x)$

Example. Solve $\dfrac{dy}{dx} + xy = x^3$

Solution. Here $P(x) = x$ and $f(x) = x^3$. We have $e^{-\int P(x)\,dx} = e^{-\int x\,dx} = e^{-x^2/2}$ The homogeneous solution is

$$y_c = Ce^{-x^2/2}$$

and a particular solution is

$$y_p = e^{-x^2/2}\int e^{x^2/2}x^3\,dx$$

Let us integrate by parts

$$\begin{aligned}\int x^3 e^{x^2/2}\,dx &= [\text{Let} \quad U = x^2, \quad dV = xe^{x^2/2} \quad \text{Then} \quad dU = 2x\,dx, \quad V = e^{x^2/2}] \\ &= x^2e^{x^2/2} - 2\int xe^{x^2/2}\,dx = x^2e^{x^2/2} - 2e^{x^2/2}\end{aligned}$$

So

$$y_p = e^{-x^2/2}(x^2e^{x^2/2} - 2e^{x^2/2}) = x^2 - 2$$

and

$$y = Ce^{-x^2/2} + x^2 - 2.$$

2.3 Examples of applications

2.3.1 Growth and Decay

The initial-value problem

$$\frac{dx}{dt} = kx, \qquad x(t_0) = x_0,$$

where k is a constant of proportionality, serves as a model for diverse phenomena involving either growth or decay.
Example: Bacteria growth. A culture initially has P_0 number of bacteria. At $t = 1h$ the number of bacteria is measured to be $\frac{3}{2}P_0$. If the rate of growth is proportional to the number of bacteria $P(t)$ present at time t, determine the time necessary for the number of bacteria to triple.
Solution. The differential equation to solve is

$$\frac{dP}{dt} = kP, \qquad P(0) = P_0, \qquad P(1) = \frac{3}{2}P_0,$$

Solving this equation, we obtain

$$P(t) = Ce^{-kt}.$$

At $t = 0$, $P_0 = C$ so that $P(t) = P_0 e^{kt}$.
At $t = 1$, $\frac{3}{2}P_0 = P_0 e^k$, or $e^k = \frac{3}{2}$. From this equation, we get $k = \ln\frac{3}{2} = 0.4055$. So,

$$P(t) = P_0 e^{0.4055t}$$

To find the time at which the number of bacterias has tripled, we solve

$$3P_0 = P_0 e^{0.4055t}$$

for t. It follows that

$$0.4055t = \ln 3 \to t = \frac{\ln 3}{0.4055} \approx 2.71h$$

2.3.2 Newton's second law of motion

According to Newton's empirical law of cooling/warming, the rate at which the temperature of a body changes is proportional to the difference between the temperature of the body and the temperature of the surrounding medium, the so-called ambient temperature. If $T(t)$ represents the temperature of a body at time t, T_m the temperature of the surrounding medium, and dT/dt the rate at which the temperature of the body changes, then Newton's law of cooling/warming translates into the mathematical statement

$$\frac{dT}{dt} = k(T - T_m),$$

where k is a constant of proportionality. In either case, cooling or warming, if T_m is a constant, it tends to reason that $k < 0$.
Example. When a cake is removed from an oven, its temperature is measured at $300°F$. Three minutes later its temperature is $200°F$. How long will it take for the cake to cool off to a room temperature of $70°F$?
Solution. Observe that $T_m = 70$. We must solve

$$\frac{dT}{dt} = k(T - 70), \quad T(0) = 300,$$

and determine the value of k so that $T(3) = 200$.
The equation can be written as a separable equation

$$\frac{dT}{T-70} = k\,dt,$$

whose solution is

$$\ln|T-70| = kt + C_1, \quad \text{or} \quad T(t) = 70 + C_2 e^{kt}.$$

When $T(0) = 0$, we observe that $C_2 = 230$. Therefore

$$T(t) = 70 + 230e^{kt}.$$

Finally $T(3) = 200$ leads to

$$200 = 70 + 230e^{3k} \to e^{3k} = \frac{13}{23} \quad \text{or} \quad k = \frac{1}{3}\ln\frac{13}{23} = -0.19018.$$

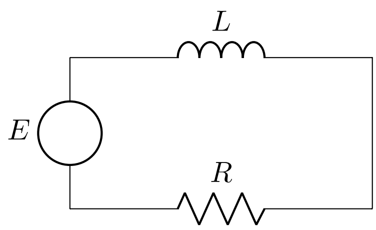
Thus,

$$T(t) = 70 + 230e^{-0.19018t}.$$

2.3.3 Series of circuit

For a series circuit containing only a resistor and an inductor, Kirchhoff's second law states that the sum of the voltage across the inductor $L(di/dt)$ and the voltage drop across the resistor iR is the same as the impressed voltage $E(t)$ on the circuit. See Figure below

LR series circuit

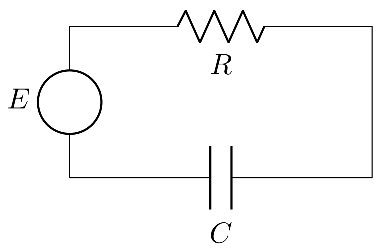


Thus, we obtain the linear differential equation of the current $i(t)$,

$$L\frac{di}{dt} + Ri = E(t)$$

where L and R are constants known as the inductance and resistance, respectively. The current $i(t)$ is also called the **response** of the system.
The voltage drop across a capacitor with capacitance C is given by $q(t)/C$, where q is the charge on the capacitor. Hence, for the series of circuit shown in the Figure below,

RC series circuit



Kirchhoff's second law gives

$$Ri + \frac{1}{C}q = E(t).$$

But the current i and the charge q are related by $i = dq/dt$, so that the equation becomes,

$$R\frac{dq}{dt} + \frac{1}{C}q = E(t).$$

Example. A 12-volt battery is connected to a series circuit in which the inductance is $\frac{1}{2}$ henry and the resistance is 10 ohms. Determine the current i if the initial current is zero.
Solution. Since $L = \frac{1}{2}$, $R = 10$ and $E = 12$, we must solve

$$\frac{1}{2}\frac{di}{dt} + 10i = 12, \qquad i(0) = 0.$$

Solving this equation, we find $i(t) = \frac{6}{5} + Ce^{-20t}$. Using the initial condition $i(0) = 0$, we find $C = -\frac{6}{5}$. Therefore,

$$i(t) = \frac{6}{5} - \frac{6}{5}e^{-20t}$$

Exercises

1. Solve the following separable equations

   (a) $\frac{dy}{dx} = \frac{y}{2x}$ (b) $\frac{dy}{dx} = \frac{3y-1}{x}$ (c) $\frac{dy}{dx} = \frac{x^2}{y^2}$ (d) $\frac{dy}{dx} = x^2y^2$ (e) $\frac{dx}{dt} = e^x \sin t$

2. Solve the following linear equations

   (a) $\frac{dy}{dx} - \frac{2y}{x} = x^2$ (b) $\frac{dy}{dx} + \frac{2y}{x} = \frac{1}{x^2}$ (c) $\frac{dy}{dx} + y = x$ (d) $\frac{dy}{dx}2e^x y = e^x = x^2$

3. Solve the following initial-value problems

   (a) $\begin{cases} \frac{dy}{dt} + 10y = 1 \\ y(1/10) = 2/10 \end{cases}$

   (b) $\begin{cases} \frac{dy}{dt} + 3x^2y = x^2 \\ y(0) = 1 \end{cases}$

4. An RL circuit has an electromagnetic force ($E(t)$) given (in volts) by $3\sin 2t$, a resistance (R) of 10 ohms, an inductance (L) of 0.5 henry, and an initial current (i) of 6 amperes. Find the current in the circuit at any time t?