

Two Investors, Two Trees, Two Goods

Maxime Sauzet*

November 5, 2021

[Link to latest version](#)

I characterize the global solution to the portfolio problem of two heterogeneous investors with general preferences, in a two-tree, two-good environment. Investors have recursive preferences and a bias in consumption towards a preferred good. The framework highlights the role of the allocation of wealth across investors for portfolios, asset prices, and risk sharing, an aspect that had received little emphasis in such a setting. The influence of the allocation of wealth grows especially as markets become imperfectly integrated, and as investor heterogeneity rises – be it through a larger bias in consumption, the introduction of labor income, or asymmetries in preferences – to the point where it can match or surpass the impact of fundamentals. The framework lends itself to several applications and extensions, e.g. in international or environmental contexts.

Keywords: Portfolio Choice, Asset Pricing, Wealth Allocation, Heterogeneous Investors, International Financial System, Environmental Fi-

nance. **JEL codes:** E0, F3, F4, G1, Q5.

*Boston University, Questrom School of Business. Email: msauzet@bu.edu. For invaluable support and guidance, I thank my advisor Pierre-Olivier Gourinchas, as well as Nicolae Gârleanu, Martin Lettau, Hélène Rey, and Jón Steinsson. For valuable comments and discussions, I thank Harjoat Bhamra, Markus Brunnermeier, Nicolas Coeurdacier, Xiang Fang, Christian Gollier, Goutham Gopalakrishna (discussant), Tarek Hassan, Chen Lian, Matteo Maggiori, Ulrike Malmendier, Sebastian Merkel, Thomas Mertens, Emi Nakamura, Maurice Obstfeld, Nick Sander, David Sraer, Richard Stanton, Annette Vissing-Jørgensen, Johan Walden, Fernando Zapatero, and seminar participants at the Bank of Canada, Berkeley Economics, Berkeley Haas, Bocconi, Boston University Questrom, Bristol, CERGE-EI, the Foster School of Business, Toulouse, the University of Zurich, and the Wisconsin School of Business. This paper evolved from the first part of my Job Market Paper, previously circulated as “Asset Prices, Global Portfolios, and the International Financial System” (November 2, 2020). All errors are my own.

1. Introduction

Multi-agent multi-good asset pricing models have a number of important applications, from providing a general solution to the international portfolio choice problem, a long-standing open issue, to studying environmental finance topics. Yet, they have been the focus of little emphasis in the literature.

In this paper, I characterize the global solution to the portfolio problem of two heterogeneous investors with general preferences, in a two-tree, two-good environment. One of the main economic messages that emerges from that characterization is that the allocation of wealth across investors matters in a general portfolio choice setting. This finding resonates with an emerging theme in the broader economic literature that has recently emphasized the role of the wealth distribution in determining economic outcomes in macroeconomics (e.g. [Brunnermeier and Sannikov, 2014](#), [Kaplan et al., 2018](#)), finance (e.g. [Gomez, 2017](#), [Lettau et al., 2019](#), [Greenwald et al., 2020](#)), and economics more generally (e.g. [Piketty and Zucman, 2014](#)). In other words, “capital is back” in this setting too: the allocation of wealth across investors has a prime role in driving asset prices, portfolios, and risk sharing, an aspect that had received little emphasis thus far.

To derive this result, I adapt recent advances in multi-agent continuous-time asset pricing models to a two-investor, two-tree, two-good economy in which investors have recursive preferences and a bias in consumption towards their local good. This allows me to overcome two main limitations in the multi-good portfolio choice literature, which has for the most part focused on applications to international finance.¹

First, while a majority of contributions rely on special cases to facilitate the resolution, I allow for general recursive preferences and an arbitrary degree of substitutability across goods. The former matters because (i) recursive preferences are not log so that investors are not myopic and their portfolios feature hedging demands that have a prime role in this context, and (ii) recursive preferences are not constant relative risk aversion (CRRA), which leads the allocation of wealth across investors to become a state variable in its own right that has an important impact beyond

¹Because the application to the international portfolio choice problem has been most prevalent, I sometimes borrow the terminology from this literature when it enhances clarity (e.g. “home bias” and “foreign bias” in equity holdings, for a bias towards the tree that produces the preferred, or least-preferred, good).

current fundamentals.² An arbitrary degree of substitutability across goods ensures, by moving away from the case of unitary elasticity of substitution, that asset returns are not perfectly correlated so that the portfolio choice between them is well-defined.³ Throughout, the generality of the specification allows to study the impact of a number of important dimensions of preferences.

Second, while most contributions have relied on low-order local approximations, I solve the model using a global solution method. This makes it possible to fully trace out the evolution of economic variables with the state of the economy, in sharp contrast to local methods that mostly capture evolutions in a small neighborhood of a specific state.⁴ This innovation is particularly valuable in situations such as here in which economic outcomes turn out to be strongly state-dependent, and in which policy functions can be very non-linear as a result of heterogeneity, or imperfect risk sharing. In addition, because increasing the order of approximation is notoriously cumbersome for the type of local methods that have been used in the literature, most contributions have focused on so-called zero-order (i.e. steady-state) portfolios. Such portfolios, which are constant, are silent on any time variation in investors' positions. Instead, the global method in this paper naturally captures their dynamics, an aspect that is not innocuous: like other outcomes, portfolios are inherently time-varying. For instance, the bias in portfolio holdings towards one of the equity assets that emerges in equilibrium is strongly reinforced as the wealth share of an investor decreases, and the relative portfolio weights of different assets also vary substantially with the relative supply of goods in the economy.

More generally, I augment the framework in a number of dimensions, e.g. by in-

²Specifically, recursive preferences break the link between the elasticity of intertemporal substitution and the inverse of risk aversion. They also help in generating quantitatively more plausible risk premia while maintaining a reasonable risk-free rate. Hedging terms are absent more generally as long as the risk aversion is equal to one.

³The case of unitary elasticity of substitution across goods has received considerable attention in the international portfolio choice literature since the seminal contribution of Cole and Obstfeld (1991). For instance, it is assumed in Pavlova and Rigobon (2007, 2008, 2010), Colacito and Croce (2011, 2013), Maggiori (2017), and Colacito et al. (2018), among others.

⁴Under a set of assumptions, local methods could be used to study an economy further in the state space, cf. for instance Mertens and Judd (2018). However, such methods remain difficult to use in a portfolio choice context due to the portfolio indeterminacy that arises in the corresponding deterministic economy. More generally, defining the state around which to approximate the equilibrium is also non-trivial. The literature has focused on using the symmetric economy as an approximation point, but this might not be a well-defined steady state in particular in the presence of imperfect risk sharing, incomplete markets, and non-stationarity. The global method in this paper circumvents all those difficulties naturally.

troducing labor income as a constant share of output, imperfect financial integration, or asymmetries in preferences, which allow me to analyze the portfolio choice problem in a variety of contexts. This is made possible in part by the fact that throughout, I solve for the *decentralized* equilibrium to the economy so that I am able to study cases in which the standard planner solution (that have been popular in the literature) cannot be used.

Compared to the cases that have been the focus of the literature so far – one investor, one good, log or CRRA preferences, perfect risk-sharing, *etc.* –, the economy in this paper differs in a number of ways.⁵

First, I am able to characterize the evolution of the allocation of wealth across investors, and its impact, across the variety of contexts mentioned above. Under perfect risk sharing, the allocation of wealth is not purely monotonically related to fundamentals due to the recursive preferences of investors. This is in sharp contrast to the log case in which the allocation of wealth is constant, or the CRRA case, in which it moves one-for-one with fundamentals and does not play a role of its own. Here, I show that the relationship between allocation of wealth and fundamentals, as captured by the relative supply of goods, is strongly negative, driven by the hedging behavior of investors, even though not purely monotonic. In other words, the share of total wealth held by an investor tends to *decrease* when the relatively supply of her preferred good is *large*. This result is difficult to escape under perfect risk sharing, as it remains valid regardless of the elasticity of substitution across goods and other parameters. As markets become imperfectly integrated, captured here as a tax on dividends, the way the relationship between allocation of wealth and fundamentals evolves is intimately related to how far investors are from having CRRA preferences. For low elasticity of intertemporal substitution, close to the inverse of risk aversion that characterizes CRRA preferences, a small degree of imperfect integration is enough to completely flip the portfolios of investors, which in turn leads to a positive relationship between allocation of wealth and fundamentals. In that case, the loadings of the wealth share on supply shocks become much larger when one of the good becomes dominant, as opposed to the perfect risk sharing case in which those loadings are highest when both goods are broadly in equal supply. When the elasticity of intertemporal substitution is high however, consistent with recent calibrations in

⁵Some of those results are still being integrated to the main text.

the literature, imperfect integration has a much more limited effect quantitatively, so that portfolios are less impacted in their broad direction, and the relationship between allocation of wealth and fundamentals remains strongly negative. This is consistent with both assets being fundamentally important for hedging for both investors in an economy in which the allocation of wealth is less directly related to relative supply, which happens when preferences are further way from CRRA. Those results point to the deep interaction between the different variables in the economy: for instance, the second moments of the allocation of wealth are impacted via portfolios, even though the tax itself only applies to first moments (dividends).

Second, the various dimensions of the economy also impact asset prices. The evolution of risk premia, in particular how they evolve with the allocation of wealth, is influenced for instance by the degree of financial integration and the calibration of preferences, and so are the level and pattern of dividend yields, and how they relate to risk premia. Dividend yields are for the most part driven by discount rates⁶, but the underlying drivers of the latter again depends inherently on the elasticity of intertemporal substitution. When this elasticity is high, i.e. when moving away from CRRA preferences, the riskless interest rate is low on average so that discount rates are driven mostly by risk premia, and are broadly monotonically increasing with the relative supply of the underlying tree. In that case, the relationship between risk premia and dividend yields is reminiscent of the predictability of expected returns observed empirically. When the elasticity is low however, the riskless rate is significantly larger, so that it becomes the main driver of the evolution of discount rates. In that case, the relationship between the dividend yield of an asset and its risk premia is much weaker, and dividend yields actually predict the riskfree interest rate more. Dividend yields are also influenced by the extent of risk sharing and the calibration, and the dividend yield on the market can further be related to the wealth-weighted sum of the consumption to wealth ratios of both investors, corrected for differences in the price level of their baskets of consumption.

Third, in terms of the second moments of returns, their correlation is large even for reasonable degrees of fundamental volatility, and higher on average compared to most cases studied in the literature. This is due to the combination of the two goods, which drive the correlation up through comovements in goods prices, and recursive

⁶This is so even though the capital gain part of discount rates is quantitatively significant in this context, around 2% on average, because capital gains are less state dependent.

preferences. For instance, the correlation increases on average with risk aversion, but also with the elasticity of intertemporal substitution and the degree of consumption bias, while it decreases as goods become better substitutes. Interestingly, comovement and correlation are also strongly state dependent: the changes in correlation with the allocation of wealth can be larger than those with fundamentals, in particular when investors are very heterogeneous, and those patterns are impacted by the various parameters and their interaction. Imperfect financial integration is also a prime driver: when in a case in which it has a large impact (low elasticity), the correlation of returns is *minimized* around the symmetric point of the economy even for low taxes, while it reaches its *maximum* at this point under perfect risk sharing. Taken together, those evolutions point to strongly varying degrees of diversification provided by the assets, both as a function of the state of the economy and of the calibration.

Fourth, the model allows to study the levels and evolutions of portfolios with the state of the economy. Because it is an important aspect of the framework, I come back to it in more details below.

The allocation of wealth impacts the economy in two ways.

Its first role is that of a state variable in its own right, beyond current fundamentals, which captures the average investor. The profile of this average investor varies significantly depending on which investor owns a larger share of total wealth, so that the allocation of wealth directly impacts asset prices, portfolios, and other economic outcomes. Specifically, because an investor has a preference towards a given good, an increase in her wealth share puts upward pressure on the price of that good, so that the returns on the asset that produces it increase.⁷ In turn, the effect on risk premia is reflected on portfolios: as the wealth share of an investor increases, the bias in their equity holdings towards one of the two equity assets that obtains in equilibrium diminishes. Those effects are large, with investors strongly tilting their portfolios when their wealth share is small, but converging towards holding the market portfolio when they dominate total wealth. This stands in sharp contrast for instance to the portfolios that have been the main focus of this literature in an international context, which are constant and computed solely at the point in which both investors own equal wealth.

⁷This is so as long as goods are good enough substitutes, as discussed below.

Where does the bias in portfolio holdings come from? Because preferences are not log, investors tilt their portfolios to hedge against risks in the economy.⁸ To start with fundamentals, the hedging of shocks to relative supply leads both investors to prefer assets whose returns are large when their preferred good is rare, given that their marginal value of wealth is high in such circumstances. When goods are good enough substitutes, the second asset pays more when the relative supply of the first good is small, because this means that the relative supply of the *second* good is large. This leads both investors to bias their equity holdings towards the asset that produces their *least-preferred* good (e.g. a “foreign bias” in an international context), as the returns on that asset are large when their respective marginal value of wealth is high. On the other hand, when goods are poor substitutes, the impact on goods prices of consumer demand is such that an asset pays more when the relative supply of the *other* good is large. This therefore results in a bias in equity holdings towards the asset that produces the preferred good (“home bias”). Due to the fact that relative prices such as the exchange rate are strongly related to relative supply, those findings are consistent in this more general framework with the hedging of real exchange rate risk that has been the focus in the international portfolio choice literature. Importantly, because in that context standard estimates of the elasticity of substitution between goods puts us in the former case, turning the counterfactual “foreign bias” that obtains into a “home bias” in equity holdings like in the data, will rely on the introduction of another plausible channel, imperfect financial integration, that I discuss below.⁹

What about the allocation of wealth? Because it impacts relative prices and asset returns, wealth share risk is also hedged by investors. Under perfect risk sharing, this turns out to reinforce the bias in portfolio holdings towards one of the two equity assets discussed above. This owns to (i) the negative relationship that obtains in equilibrium between wealth share and relative supply, and (ii) the fact that the relative marginal value of wealth of an investor tends to increase with their wealth

⁸Hedging terms are absent more generally as long as the risk aversion is equal to one.

⁹The hedging of real exchange risk has a long history in the international portfolio choice literature.

Cf. [Coeurdacier \(2009\)](#) for a recent take, and [Obstfeld \(2007\)](#) and [Coeurdacier and Rey \(2013\)](#) for surveys. I discuss it in more detail in Section 3.4. [Coeurdacier and Rey \(2013\)](#) also summarize recent empirical findings on the home bias in equity holdings. The impact of the elasticity of substitution is discussed at length throughout Section 3 but modern estimations such as those in [Imbs and Méjean \(2015\)](#) and in the international trade literature put it firmly in the case of goods being good substitutes.

share.¹⁰ (i) emerges regardless of good substitutability because an investor allocates more wealth to the asset – whichever it is – whose returns are large when the physical supply of their preferred good is low. As a result, a shock that tends to improve the relative supply of their preferred good necessarily leads their preferred asset to do poorly, so that their share of wealth decreases. But because in consequence their marginal value of wealth also decreases in the wealth share dimension, an investor values the asset that pays in those conditions even less and therefore overweights its already preferred asset further. In short: the hedging of wealth share risk reinforces the bias in portfolio holdings under perfect risk sharing.

Quantitatively, the impact of the allocation of wealth remains modest in a symmetric baseline under perfect risk sharing with the wealth share evolving in a narrow band around a broad direction given by fundamentals. However, this impact grows tremendously as soon as markets become imperfectly integrated, and as investors become more heterogeneous. In both cases, the role of the allocation of wealth for asset prices, portfolios, and other economic outcomes, can be on par with or surpass that of fundamentals captured by the relative supply.

Introducing imperfectly integrated markets in this economy is particularly relevant because for most applications of the framework (international context, environmental context, *etc.*), investing in some assets comes with a number of frictions – be they legal, technical, informational, or otherwise. I capture those frictions in a parsimonious way as a tax on “foreign” dividends, i.e. a tax on the dividends of the tree that produces the least-preferred good, generalizing [Bhamra et al. \(2014\)](#).¹¹ The formulation allows me to study the effect of a range of financial integration degrees without having to take a specific stance on the source of the underlying imperfections.

By making the asset that pays the least-preferred good (“foreign asset”) less at-

¹⁰(ii) comes from the fact that as the wealth share of an investor increases, the impact of that investor on relative prices grows and the price of its preferred good increases, which makes them relatively worse-off. Intuitively, this is also consistent with this investor growing more dominant in total wealth so that diversifying risks with the other, increasingly small, investor is more difficult.

¹¹Cf. also the seminal contribution of [Basak and Gallmeyer \(2003\)](#), who study a dynamic asset pricing model with asymmetric dividend taxation and a unique risky asset in a one-investor one-good setting. As shown in [Gărleanu et al. \(2020\)](#), models with investment taxes constitute an equivalent, but substantially simpler, way to capture a rich set of impediments to financial trade. As such, the tax is meant to capture not only actual differential tax treatments or transaction costs for investing in different assets, but more generally any friction that prevent investors from freely participating in all markets equally.

tractive, due to the direct required payment of the tax as well as a modest general equilibrium effect, imperfect financial integration can rapidly overcome the “foreign bias” in equity holdings that obtains in the baseline and deliver a “home bias” in equity holdings in line with empirical observations e.g. in an international context. When this happens, the impact of the allocation of wealth is also strongly reinforced, consistent with the fact that risk sharing becomes imperfect so that insuring against risks in the economy becomes more difficult for investors.¹² The allocation of wealth has a larger direct effect as a state variable, but the impact is also visible in terms of hedging demands: insuring against shocks to their wealth share becomes as important a driver of investors’ portfolios as the hedging of fundamentals. In addition, the hedging of wealth share risk now contributes to obtaining a “*home bias*” in equity holdings, in contrast to the baseline in which it reinforced the “*foreign bias*” coming from fundamentals. This occurs because of the overpowering effect of imperfect financial integration on risk premia, which makes the asset producing the preferred good more attractive and therefore on average more prevalent in the portfolio of an investor. This in turn yields a switch in the equilibrium relationship between wealth share and relative supply: a shock that increases the relative supply of a good also leads the asset that produces it to do well, and therefore the wealth share of the investor that prefers that good to now increase. As a result, the hedging of wealth risk flips sign and contributes positively to obtaining a “*home bias*” in equity holdings.¹³

Importantly, the calibration of preferences has a substantial effect on the ultimate potency of imperfect financial integration, with the elasticity of intertemporal substitution taking center stage. When this elasticity is low, modest taxes on the order of $\tau = 7$ to 10% are enough to deliver a “*home bias*” in equity holdings e.g. like the one in the data in an international context, qualitatively throughout the state space, and quantitatively in at least some regions of it.¹⁴ When the elasticity is high however, and even though the same mechanisms are at play, the effects are much more muted

¹²Imperfect risk sharing arises because the tax makes the opportunity sets of the two investors different so that their stochastic discount factors are no longer perfectly correlated. Another consequence is that the standard planner solution that has been popular in the literature can no longer be used.

¹³This switch also has long-term consequences in terms of which investor survives in the long run. In addition, the dispersion of the wealth share in equilibrium increases with imperfect risk sharing so that the quantitative effect of the wealth share is larger.

¹⁴E.g. the home bias can be made consistent with empirical measures reported in [Coeurdacier and Rey \(2013\)](#) around the symmetric point in the state space. Importantly, portfolios remain inherently state-dependent.

and for reasonable taxes, a “foreign bias” in equity holdings remains.¹⁵ This additional novel result arises because the dividend yields on the two equity assets, which are the ultimate driver of the effect of the tax on returns, are significantly smaller in magnitude in this case. Economically, this happens in part because as ψ increases, variations in the wealth share beyond the broad direction given by fundamentals are larger (this is true even under perfect risk sharing), so that both state variables have a clear distinct role and two equity assets are required to hedge against changes in both of them. It also reflects the fact that the elasticity of intertemporal substitution has a large impact on the extent of trading in the risk-free bond, which was unused under perfect risk sharing but becomes important with imperfect integration, as well as on the diversification benefits provided by the two equity assets that vary a lot both with parameters and with the state of the economy.

Taken together, those results confirm but qualify the findings in [Bhamra et al. \(2014\)](#) for an international context in this general setting with non-log preferences and home bias in consumption: imperfectly integrated markets *can* deliver portfolios consistent with the data *provided that* the elasticity of intertemporal substitution is moderate. From the perspective of international applications like that mentioned in Section 5.1, a realistic home bias in equity holdings can therefore be generated by combining a moderate elasticity of intertemporal substitution with modest taxes on foreign dividends.

The heterogeneity of investors is another factor that has a sizable effect on the equilibrium. This is visible even in a symmetric baseline calibration: as the degree of bias in consumption increases, the fundamental level of heterogeneity between investors also increases. As a result, the quantitative impact of the allocation of wealth across those – now more different – investors grows. For instance, the hedging of wealth share risk becomes once again on par with that of fundamentals. The same observation is true when introducing labor income, which can strongly reinforces the bias in portfolio holdings.¹⁶ Heterogeneity in the form of asymmetric preferences is also especially potent, in particular in terms of its effect on risk premia, and is

¹⁵Generating a home bias consistent with empirical observations requires implausible taxes as high as $\tau = 75\%$ or 90% .

¹⁶In the spirit of [Baxter and Jermann \(1997\)](#), labor income tends to lead to a “foreign bias” in equity holdings in this setting because it is modeled as a constant share of the output of each tree. More general specification such as a time-varying share in the spirit of [Coeurdacier and Gourinchas \(2016\)](#) or idiosyncratic labor income risk as in [Kaplan et al. \(2018\)](#) are interesting avenues for further exploration.

explored in detail in the applications mentioned below.

In summary, the impact of the wealth share grows markedly with the degree of imperfect financial integration, and the degree of investor heterogeneity, to the point where it can come to be on par with or surpass the effect of fundamentals on portfolios, asset prices, and other economic outcomes. This reiterates the main message: capital is back in this economy too!¹⁷ On a more theoretical note, the results emphasize both the strong state-dependence of most economic variables in this environment, and the vital impact of the calibration of preferences. This makes the novel framework presented in this paper, which is based on a global solution method and allows for general recursive preferences including asymmetries, particularly suited to study this economy.

Because of its generality, this “222” framework in this paper represents a versatile building block towards several applications and extensions. Those are explored in ongoing work, and I only provide a brief overview.

A first and prominent one, to which I have referred throughout, is international finance. The framework indeed allows to characterize the global solution to the international portfolio problem in full generality, which had been a long-standing open issue. This makes it possible to reassess various results in this literature under a unified framework. Beyond that aspect, the ability of the framework to handle truly general preferences, including asymmetries, also allows it to reproduce a number of stylized facts about the structure and dynamics of the international financial system, and in particular the role of the United States, and of asset returns in this context. I discuss this application in detail in [Sauzet \(2021a\)](#).

The framework can also be used to study a completely different set of questions, in the domain of environmental finance.¹⁸ Indeed, following [Guesnerie \(2004\)](#), [Hoel and Sterner \(2007\)](#), [Sterner and Persson \(2008\)](#), [Gollier \(2010\)](#), [Traeger \(2011\)](#), [Barro and Misra \(2016\)](#), and [Gollier \(2019\)](#), the two goods can be taken to represent aggregate economic capital (physical capital, labor, scientific knowledge, *etc.*) on one

¹⁷These results are also reminiscent of recent findings in the price impact literature, in which quantities, represented here by the portfolios held by each investor and captured in aggregate by the wealth share, strongly impact asset prices and risk compensations. Contributions in this spirit include [Kouri \(1982\)](#), [Jeanne and Rose \(2002\)](#), [Hau and Rey \(2006\)](#), and more recently [Gabaix and Maggiori \(2015\)](#), [Camanho et al. \(2018\)](#), [Gabaix and Koijen \(2020\)](#), and [Koijen and Yogo \(2020\)](#).

¹⁸I am grateful to Christian Gollier for this suggestion.

hand, and various ecosystem services that are generated by natural capital on the other. Importantly, because my framework embeds not only two goods and two trees, but also *two investors* with possibly heterogeneous preferences towards the goods, it can allow to study not only how relative prices can be crucial for the pricing of the ecological services provided by natural assets, as has been discussed in the literature, but also how this pricing interacts with the allocation of wealth across investors. This can for instance make it possible to connect environmental issues to those of economic inequality in which one group of investors is holding an increasingly larger share of total wealth. One can also study the impact of having investors with different preferences towards environmental goods, which is likely relevant in practice. I explore this application in ongoing work ([Sauzet, 2021b](#)).

The model could be applied to a number of other topics (e.g. sectors of the economy, in the spirit of [Menzly et al., 2004](#), [Santos and Veronesi, 2006](#)), but more generally, it is also a well-suited building block for many potential extensions. For instance, one can study a generalization of this economy with N investors, M trees, and L goods (“NML” model, explored in ongoing work, [Sauzet, 2021e](#)). By having more than two trees, one can explore limiting their trading to a subset of investors so as to introduce a natural source of market incompleteness that can fundamentally impact the equilibrium. Another natural extensions would be to consider a production economy in which investors can also directly influence the supply of their preferred good. Additional promising avenues are related to the introduction in this setting of financial intermediaries of the type that has been discussed in the recent intermediary asset pricing literature e.g. in [Danielsson et al. \(2012\)](#), [He and Krishnamurthy \(2013\)](#), [Adrian and Shin \(2014\)](#), or [Adrian and Boyarchenko \(2015\)](#). Illustrations are briefly discussed in Section 5.3 and Appendix E, for instance with the inclusion of a global asset manager ([Sauzet, 2021d](#)). From the perspective of extensions, solving for the decentralized equilibrium of this economy like I do in this paper will prove particularly valuable: the framework is readily set to tackle a wide range of market structures beyond imperfect risk sharing. In addition, the implementation of those extensions will likely require higher-dimensional methods such as the “projection methods via neural networks” being developed in [Sauzet \(2021c\)](#). I leave all these promising avenues for future research.

Related literature

This paper contributes to two main strands of literature.

First, I contribute to the literature on multi-agent asset pricing models, which has a long and distinguished history since the seminal contributions of [Dumas \(1989, 1992\)](#), [Wang \(1996\)](#), [Basak and Cuoco \(1998\)](#), [Chan and Kogan \(2002\)](#), and more recently [Brunnermeier and Pedersen \(2009\)](#), [Weinbaum \(2009\)](#), [Bhamra and Uppal \(2009, 2014\)](#), [Brunnermeier and Sannikov \(2014\)](#), [Gârleanu and Pedersen \(2011\)](#), [Chabakauri \(2013\)](#), [Gârleanu and Panageas \(2015\)](#), [Drechsler et al. \(2018\)](#). This literature is also related to the modern literature on heterogeneous agents in closed-economy macroeconomics such as [Kaplan et al. \(2018\)](#). To those contributions, I bring two goods, two assets, two countries, as well as a bias in consumption. The bias in consumption is particularly important because it introduces a fundamental level of heterogeneity between investors even absent asymmetries, and is responsible for most mechanisms in the economy including the rise of a substantial bias in portfolio holdings through hedging demands, the shape and comovement of risk premia, and a well-defined exchange rate. As such, this is one of the main differences with the international model of [Brunnermeier and Sannikov \(2015, 2019\)](#). Having two assets also fundamentally relates my paper to contributions with multiple securities but one agent e.g. [Cochrane et al. \(2008\)](#), [Martin \(2013\)](#).

Most related to my contribution are those of [Pavlova and Rigobon \(2007, 2008, 2010\)](#) and [Stathopoulos \(2017\)](#), inspired in part by [Zapatero \(1995\)](#), who study a pure exchange economy similar to mine, but in which preferences are log and the elasticity of intertemporal substitution across goods is equal to one. The combination of those assumptions leads the allocation of wealth to be constant, equity assets to be perfectly correlated in the absence of demand shocks, and hedging demands to be absent due to myopic portfolios. All three are important dimensions that arise in my framework once I allow for general recursive preferences and an arbitrary elasticity of substitution between goods. I therefore see my contribution has the natural continuation of this earlier research effort.

Breaking those limitations does not come without a cost however, and solving the model requires a whole new set of methods compared to those papers. In particular, the resolution of my framework is based on global projection methods, as presented

in Judd (1992, 1998), the NBER Summer SI Lecture by Fernández-Villaverde and Christiano (2011), or Parra-Alvarez (2018), and as applied to multi-agent models for instance in Drechsler et al. (2018), Fang (2019), or Kargar (2019). The approximation is based on Chebyshev polynomials and orthogonal collocation, although in ongoing work, I am also developing a natural extension based on neural networks (Sauzet, 2021c, cf. Section 5.3).¹⁹

In addition, I also introduce asymmetries in preferences, labor income in the form of a constant share of output as in Baxter and Jermann (1997), and most importantly, imperfect financial integration. The latter is captured in a parsimonious way as an asymmetric tax on dividends by generalizing Bhamra et al. (2014) to a non-log environment that also features home bias, and following the seminal contribution of Basak and Gallmeyer (2003) who study a dynamic asset pricing model with asymmetric dividend taxation and a unique risky asset in a one-country one-good setting. Compared to Bhamra et al. (2014), the introduction of general preferences makes a significant difference: imperfect risk sharing has a large impact *provided that* the elasticity of intertemporal substitution is modest, a novel insight. In addition, I use a global solution instead of relying on local approximations, and am able to study the effect on the exchange rate and of hedging terms. Theoretically, the use of a tax to capture a wide range of frictions is related to the work of Gârleanu et al. (2020), who show that models with investment taxes constitute an equivalent, but substantially simpler, way to capture a rich set of impediments to financial trade.

Other related papers include Cass and Pavlova (2004), Brandt et al. (2006), Martin (2011), and Maggiori (2017) that I discuss below, as well as Fang (2019) who focuses on a small open economy in which the rest of the world is taken as exogenous and in which investors do not have consumption biases. On the theoretical front, my paper is also related to contributions introducing recursive preferences in continuous-time e.g. Duffie and Epstein (1992), and contributions focusing on the existence and uniqueness of equilibria in the presence of multiple agents, and possibly multiple goods and incomplete markets e.g. Polemarchakis (1988), Geanakoplos and Polemarchakis (1986), Geanakoplos and Mas-Colell (1989), Geanakoplos (1990), Duffie et al. (1994), Berrada et al. (2007), Anderson and Raimondo (2008), Hugonnier et al. (2012), Ehling

¹⁹I solve for the decentralized economy throughout, but the method of Dumas et al. (2000), based on a planner, could also be used in cases in which risk sharing is perfect.

and Heyerdahl-Larsen (2015).

Second, a large part of multi-agent multi-good asset pricing has been studied in an international context, and I therefore contribute to the literature on the international portfolio problem. Specifically, the advances presented above allow me to characterize the general and global solution to the international portfolio choice problem, a long-standing issue in this literature since the seminar contributions of Stulz (1983), Dumas (1989, 1992), Cole and Obstfeld (1991), Zapatero (1995), Baxter and Jermann (1997), Baxter et al. (1998), Obstfeld and Rogoff (2001), Obstfeld (2004), among many others. Obstfeld (2007) and Coeurdacier and Rey (2013) provide surveys.

To a large part of the more recent literature on the topic, such as Corsetti et al. (2008), Tille and van Wincoop (2010), Coeurdacier (2009), Devereux and Sutherland (2011), Evans and Hnatkovska (2012), Coeurdacier and Rey (2013), Coeurdacier and Gourinchas (2016), I bring (i) a solution that is global and does not rely on approximations. This allows to complete the picture and trace out the evolution of economic outcomes as we move away from the point of approximation (typically the symmetric point), which proves important in this context where variables are strongly state-dependent and potentially non-linear. I also bring (ii) general preferences, which allow to move away from special cases and study all situations under a unified framework (cf. also the discussion above of Pavlova and Rigobon, 2007, 2008, 2010, Stathopoulos, 2017). A limited number of contributions have relied on global methods in similar settings e.g. Kubler and Schmedders (2003) (one country), Stepanchuk and Tsyrennikov (2015) (one good), Rabitsch et al. (2015), and Coeurdacier et al. (2020) (one good). To those, I bring (iii) continuous-time methods, which make it possible to study portfolio drivers, in particular hedging demands, asset prices and their conditional first and second moments, as well as the determinants of wealth and state variable dynamics, in ways that are inaccessible in a discrete-time formulation and therefore make continuous-time the natural tool of choice to study this type of questions. Finally (iv), to all, in addition to labor income as in Baxter and Jermann (1997) and asymmetries in preferences, I bring imperfect financial integration, which is an important topic in international finance but had not been studied thus far in a general international portfolio choice context.²⁰

²⁰More general specification of labor such as a time-varying share in the spirit of Coeurdacier and Gourinchas (2016) or idiosyncratic labor income risk as in Kaplan et al. (2018) are interesting

My contribution is also related to those of [Colacito and Croce \(2011, 2013\)](#), and [Colacito et al. \(2018\)](#), who introduce recursive preferences in an international context. Compared to those, output does not feature long-run risk dynamics. Instead, I bring in an arbitrary elasticity of substitution across goods, which makes the two equity assets no longer perfectly correlated so that the portfolio choice is no longer indeterminate in my context. More generally, I bring (i), (ii), (iii) and (iv) above to that economy. [Dou and Verdelhan \(2015\)](#) deserves particular mention as well: the authors solve an international portfolio problem globally, with general preferences and endowments, portfolio constraints, and incomplete markets. However, their focus on the volatility of international capital flows is different. In addition, partly because their framework is cast in discrete time, they do not focus on describing the underlying determinant of portfolios, such as hedging demands, which are an important part of my contribution.

Finally, the first application in this paper, to international finance, is in the spirit of [Gourinchas and Rey \(2007a,b\)](#), [Caballero et al. \(2008\)](#), [Gourinchas et al. \(2017\)](#), and [Maggiori \(2017\)](#) that I bring to the general international portfolio choice context of my framework. The second application, to environmental finance, is most related to [Guesnerie \(2004\)](#), [Hoel and Sterner \(2007\)](#), [Sterner and Persson \(2008\)](#), [Gollier \(2010\)](#), [Traeger \(2011\)](#), [Barro and Misra \(2016\)](#), and [Gollier \(2019\)](#). Cf. [Sauzet \(2021a,b\)](#) for details.

The paper is organized as follows. Section 2 describes the set-up of the economy, and introduces the two state variables that drive economic mechanisms: the wealth share of an investor, and the relative supply of the two goods, i.e. fundamentals. Section 3 characterizes the solution to the model both theoretically, and by presenting the resulting equilibrium variables. It discusses in particular the role of the wealth share and how it grows as markets become less perfectly integrated, and agents become more heterogeneous. Section 5 briefly describes two applications of the framework – to modeling the international financial system, and to environmental finance – as well as possible extensions. Section 6 concludes. Additional material is provided in Appendix.

avenues for further exploration.

2. The Economy

This section presents the theoretical setup. I introduce a pure-exchange economy à la Lucas (1978) with two groups of investors, two trees, and two goods. Each tree produces a given differentiated good, and $j \in \{1, 2\}$ denotes both the tree and the corresponding good. Each group of investors, denoted $i \in \{A, B\}$, consists of a representative investor with recursive preferences and whose consumption is biased towards a given (“local”) good: $j = 1$ for A , $j = 2$ for B .²¹ I show that the equilibrium can be characterized as a function of two state variables: the wealth share of the first investor, x_t , and the relative supply of the two goods, y_t . The former captures the allocation of wealth between the two groups, and therefore the identity of the average investor in the economy, while the latter captures fundamentals. The setup is summarized in Figure F.1 in Appendix. Appendix A gathers additional results that are omitted in the main text.

Time is continuous and the horizon is infinite, $t \in [0, \infty)$. Uncertainty is represented by a probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ supporting a two-dimensional Brownian motion $\vec{Z} \equiv (Z_1, Z_2)^T \in \mathbb{R}^2$. The filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, \infty)}$ is the usual augmentation of the filtration generated by the Brownian motions, and $\mathcal{F} \equiv \mathcal{F}_\infty$.

2.1. Endowments, prices, assets

Each tree produces a differentiated good, and its output follows a geometric Brownian motion

$$\frac{dY_{j,t}}{Y_{j,t}} = \mu_{Y_j} dt + \sigma_{Y_j}^T d\vec{Z}_t, \quad j \in \{1, 2\}$$

The price of the goods are $p_{1,t}, p_{2,t}$. The “terms of trade” is $q_t \equiv p_{2,t}/p_{1,t}$, defined so that an increase in q_t corresponds to a worsening of the terms of trade for good 1. The “real exchange rate” is $\mathcal{E}_t \equiv P_t^B/P_t^A$, defined so that an increase in \mathcal{E}_t corresponds to a depreciation for investor A . P_t^A, P_t^B are the prices of the consumption baskets

²¹I sometimes refer to them as the “local” or “home” good, and “foreign” good, of each agent, borrowing the international terminology when it helps clarity.

for each investor discussed below. All prices are defined with respect to a global numéraire taken to be a CES-basket with weight a on good 1.²²

Both trees are traded as equity assets, with returns given by

$$dR_{j,t} = \frac{dQ_{j,t}}{Q_{j,t}} + \frac{p_{j,t}Y_{j,t}}{Q_{j,t}}dt = \frac{d(p_{j,t}Y_{j,t}/F_{j,t})}{p_{j,t}Y_{j,t}/F_{j,t}} + F_{j,t}dt \equiv \mu_{R_{j,t}}dt + \sigma_{R_{j,t}}^T d\vec{Z}_t, \quad j \in \{1, 2\} \quad (1)$$

where $Q_{j,t}$ are the equity prices, and $F_{j,t} \equiv p_{j,t}Y_{j,t}/Q_{j,t}$ are the dividend yields, for both assets. Drifts $\mu_{R_{j,t}}$, which measure conditional expected returns, and diffusion terms $\sigma_{R_{j,t}}$, which measure the loadings on the shocks and therefore the conditional volatilities, are obtained from Itô's Lemma and given in Appendix A.2.

The supply of each equity asset is normalized to unity, and there also exists a bond in net zero supply, which is locally riskless in units of numéraire. Its price is B_t , and the corresponding instantaneous interest rate is r_t , so that $dB_t/B_t = r_t dt$.

2.2. Preferences

Each investor has recursive preferences over consumption à la Duffie and Epstein (1992). This is in contrast to a large part of the literature that focuses on log or constant relative risk aversion (CRRA) utility. The former has the drawback that investors are myopic so that state variables are not hedged, and have therefore a limited impact on portfolios, asset prices, and other quantities of interest.²³ Contrary to the CRRA case, recursive preferences also allow to disentangle the risk aversion and elasticity of intertemporal substitution (EIS) of each investor. This is important to get closer to empirical moments, but also to be able to study the specific role of

²²Specifically, I normalize $[ap_{1,t}^{1-\theta} + (1-a)p_{2,t}^{1-\theta}]^{1/(1-\theta)}$ to unity.

²³Hedging terms are absent more generally as long as the risk aversion is equal to one.

the EIS, which will be crucial. Preferences are given, for $i \in \{A, B\}$ by

$$V_t^i = \max_{\{C_{1,u}^i, C_{2,u}^i, w_{1,u}^i, w_{2,u}^i\}_{u=t}^\infty} \mathbb{E}_t \left[\int_t^\infty f^i(C_u^i, V_u^i) du \right] \quad (2)$$

$$f^i(C, V) \equiv \left(\frac{1 - \gamma^i}{1 - 1/\psi^i} \right) V \left[\left(\frac{C}{[(1 - \gamma^i)V]^{1/(1-\gamma^i)}} \right)^{1-1/\psi^i} - \rho^i \right]$$

where γ^i is the coefficient of relative risk aversion, $\psi^i \neq 1/\gamma^i$ the elasticity of intertemporal substitution, and ρ^i is the discount rate.

The consumption basket of each investor is composed of the two goods, which are combined according to an aggregator with constant elasticity of substitution θ , and bias in consumption α^i

$$C_t^i = \left[\alpha^{i\frac{1}{\theta}} C_{1,t}^{i\frac{\theta-1}{\theta}} + (1 - \alpha^i)^{\frac{1}{\theta}} C_{2,t}^{i\frac{\theta-1}{\theta}} \right]^{\frac{\theta}{\theta-1}} \quad (3)$$

The two goods could for instance be the one produced respectively in the domestic and foreign country in an international context (application of Section 5.1 and [Sauzet, 2021a](#)), or aggregate economic capital on one hand, and various ecosystem services generated by natural capital on the other in an environmental context (application of Section 5.2 and [Sauzet, 2021b](#)). I sometimes adopt the terminology of the international context, and refer to the preferred good, or the tree that produces it, as “local” or “home”, and the least-preferred good, or the tree that produces it, as “foreign”. For instance, good 1 and tree 1 are “local” or “home” for investor A . This is purely when it helps to be clear or concise, and should not obscure the fact that framework is general and can be applied to other contexts.

The resolution method allows for each parameter to differ across investors, although most of the exposition focuses on a baseline symmetric calibration in which all parameters are equal except for the bias in consumption which is symmetric: $\alpha_A = \alpha, \alpha_B = 1 - \alpha$ ([Assumption 1](#)). In what follows, I drop the i superscript for parameters, unless needed for clarity.

Two characteristics of the consumption baskets are noteworthy.

First, the elasticity of intertemporal substitution θ is in general not equal to unity. Due to its specificity, the case with $\theta = 1$, in which C_t^i collapse to Cobb-Douglas aggregators, has received considerable attention in the literature since the seminal contribution of Cole and Obstfeld (1991).²⁴ In this case and under some conditions, the Pareto optimal equilibrium that would obtain under complete markets can in fact be attained under financial autarky. This is so because under this specification, the relative price of goods moves just enough to offset changes in their relative supply so that investors are perfectly insured against shocks in the economy. As a result, trade in asset is not required to reach perfect risk sharing. Another consequence is that the payoffs of the two equity assets are perfectly correlated, so that the portfolio choice of international investors is indeterminate.²⁵ Further, an economy with unit elasticity of substitution across goods satisfy the conditions of the no-trade theorem in Berrada et al. (2007), so that there is no trade in equilibrium, resulting in no realistic capital flows and no nontrivial portfolio rebalancing. Taken together, those reasons make this case clearly peculiar, and I instead focus on the general environment in which $\theta \neq 1$, which has received less attention.

Second, the bias in consumption, captured by parameter $\alpha > \frac{1}{2}$, turns out to be a core driver of economic outcomes in the model.²⁶ It is indeed responsible for the differing patterns of asset returns and, by introducing hedging motives, is also a prime determinant of portfolios. Because it is symmetric in the baseline calibration, the bias in consumption leads to a natural and fundamental degree of heterogeneity between investors, even in the absence of other differences. This is one of the reasons why the allocation of wealth is neither constant nor purely monotonically related to the relative supply of goods even under perfect risk sharing. In addition, this heterogeneity makes the hedging motives different across investors, and is therefore responsible for part of the differential tilt in their portfolios that ultimately explains their individual bias towards holding more of a given asset. How those hedging motives interact with the

²⁴For instance, $\theta = 1$ in Pavlova and Rigobon (2007, 2008, 2010), Colacito et al. (2018), Maggiori (2017), or Colacito et al. (2018), among others.

²⁵I discuss this case in more detail throughout Section 3. Interestingly, another consequence of the equity assets being perfectly correlated is that markets are technically dynamically incomplete when the investors can only trade the two equity assets and a bond, as discussed in Ehling and Heyerdahl-Larsen (2015). Despite this fact, investors are perfectly insured via changes in the relative price of goods.

²⁶This bias is realistic and well-established for instance in Application 1, in which each investor represents a country. It is then dubbed the “home bias” in consumption. Cf. Section 5.1.

impact of the allocation of wealth is an important dimension in this environment. Lastly, without consumption bias, both investors would consume identical baskets, so that their relative price would be constant and equal, i.e. the real exchange rate would be constant and equal to unity. This would therefore prevent the analysis of any phenomenon involving the real exchange rate, which is key quantity in an international or environmental context.

Investors allocate a share $w_{j,t}^i$ of their wealth to each equity asset, earning an expected risk premia $\mu_{R_j,t} - r_t$ on each, and the rest $(1 - w_{1,t}^i - w_{2,t}^i)$ to the bond. They use the proceeds to purchase their desired baskets of consumption $c_t^i \equiv C_t^i/W_t^i$, at price P_t^i . In other words, they choose their consumption and portfolios to maximize (2) subject to the following budget constraint

$$\begin{aligned} \frac{dW_t^i}{W_t^i} = & (r_t + w_{1,t}^i (\mu_{R_1,t} - r_t) + w_{2,t}^i (\mu_{R_2,t} - r_t) - P_t^i c_t^i) dt \\ & + (w_{1,t}^i \sigma_{R_1,t} + w_{2,t}^i \sigma_{R_2,t})^T d\vec{Z}_t \end{aligned} \quad (4)$$

The impact on budget constraints of the introduction of imperfect financial integration and labor income of the form considered in this paper is discussed in Section 2.4. Finally, to complete the definition of the optimization problem, investors are subject to a standard transversality condition, and W_0^i is given. Note also that $W_t^i \geq 0$.

2.3. Equilibrium and state variables

The definition of the equilibrium is standard: (1) investors solve their optimization problems by taking aggregate stochastic processes as given, and (2) goods and equity markets clear. It is shown in Appendix A.3. The bond market clears by Walras's law, which gives rise to the following useful relationship: $W_t^A + W_t^B = Q_{1,t} + Q_{2,t}$. In words, world wealth has to be held in the form of the two equity assets in aggregate.

Stationary recursive Markovian equilibrium Most importantly, the equilibrium can be recast as a stationary recursive Markovian equilibrium in which all variables of interest are expressed as a function of a pair of state variables $X_t \equiv (x_t, y_t)'$, whose

dynamics are also solely a function of X_t . x_t is the wealth share of investor A , and y_t is the relative supply of the good 1.²⁷ Both are defined below.

The characterization of the solution as a system of coupled algebraic and second-order partial differential equations is the focus of Section 3. For now, let us discuss the intuition behind both state variables. Note that an additional variable, which is not a state variable *per se* but is useful throughout, is z_t , the ratio of the equity price on asset 1 to world wealth. It captures the weight of asset 1 in the market portfolio, and it can be shown that

$$z_t \equiv \frac{Q_{1,t}}{Q_{1,t} + Q_{2,t}} = \left(1 + \left(\frac{F_{1,t}}{F_{2,t}} \right) q_t \left(\frac{1 - y_t}{y_t} \right) \right)^{-1} \quad (5)$$

Wealth share The wealth share of investor A is a measure of the average investor in the economy. It is defined as

$$x_t \equiv \frac{W_t^A}{W_t^A + W_t^B} \quad (6)$$

Importantly, the wealth share is not constant, even under perfect risk sharing. This is due to the fact that preferences are not log, contrary to a large subset of the literature, and to the presence of the bias in consumption. In addition, the wealth share is not solely a monotonic function of current fundamentals, so that it is required as an additional state variable. This comes from the combination of heterogeneity, introduced if nothing else by the bias in consumption, and recursive preferences. The intuition is that the wealth share captures Negishi weights, which are time-varying in this case, as discussed among others in [Dumas et al. \(2000\)](#), [Anderson \(2005\)](#), or [Colacito and Croce \(2011, 2013\)](#). One of the advantages of characterizing the solution directly as a function of the wealth share is that the method remains valid even in cases in which risk sharing is imperfect, markets are incomplete, and the characterization of a solution using the Pareto weights chosen by a fictitious planner is no longer necessarily possible.

²⁷Formally, this is shown using a guess and verify approach like e.g. in [Gârleanu and Panageas \(2015\)](#). The variables of interest are: $\{c_{1,t}^A, c_{2,t}^A, c_{1,t}^B, c_{2,t}^B, w_{1,t}^A, w_{2,t}^A, w_{1,t}^B, w_{2,t}^B, \mu_{R1,t}, \mu_{R2,t}, r_t, F_{1,t}, F_{2,t}, p_{1,t}, p_{2,t}, P_t^A, P_t^B, q_t, \mathcal{E}_t\}$.

Relative supply The relative supply of good 1 captures the effect of current fundamentals and is defined as

$$y_t \equiv \frac{Y_{1,t}}{Y_{1,t} + Y_{2,t}} \quad (7)$$

This variable has been the focus in one form or another of a large part of the portfolio choice literature in an international context.²⁸ As I discuss in Section 3.4, it is for instance closely related to the impact of real exchange rate hedging, as emphasized e.g. in Coeurdacier (2009), and Coeurdacier and Rey (2013), although the mapping is not one-to-one. An appeal of my framework is to analyze the effect of this variable in a context with more general preferences, various specifications, as well as globally throughout the state space, instead of having to rely on local approximation methods around a particular point like as been common in that literature. In addition, the interaction of the hedging of y_t with the impact of the wealth share x_t constitutes an important new dimension. I discuss those elements in detail in Section 3.

Note that because $W_t^i \geq 0$ and $Y_{j,t} \geq 0$, x_t and y_t are both evolving in the bounded interval $[0, 1]$. This has the advantage that solving for unknown functions on a bounded domain is numerically more stable. Conceptually, as x_t gets closer to either of the boundaries, the economy converges (continuously) to a natural one-investor environment. As y_t gets closer to either of the boundaries, the economy converges to a one-good one-equity asset economy, but this has consequences in terms of marginal values of wealth as the investors still want to consume both goods.

Throughout, I focus on the solution to the decentralized, i.e. Radner, equilibrium instead of relying on the social planner's problem. When markets are complete and risk sharing is perfect, both solutions must coincide. In Appendix D.1, I show that this is indeed the case for instance under symmetric CRRA preferences in which the elasticity of intertemporal substitution is inversely related to the risk aversion.²⁹ Solving for the planner solution can be extended to recursive preferences in a Markovian

²⁸Note that the ratio involves *quantities* of the two different goods. This poses no particular theoretical issue and is used because it simplifies the characterization of the equilibrium. This definition is a monotonic transformation of $Y_{2,t}/Y_{1,t}$: $y_t \equiv (1 + Y_{2,t}/Y_{1,t})^{-1}$, which ensures that the state variable evolves in the bounded interval $[0, 1]$. $Y_{2,t}/Y_{1,t}$ has the clear interpretation of the output of good 1 per unit of good 2. An economic intuition is that one compares the economy to the symmetric point in which relative prices are $q_t = \mathcal{E}_t = 1$.

²⁹I also check the solution with Monte-Carlo simulations.

setting, following [Dumas et al. \(2000\)](#). However, I stick to the study of the decentralized economy because part of the appeal of the framework is that it remains valid even in cases in which the usual planner solution can no longer necessarily be used, such as with imperfect financial integration or even incomplete markets. This will also prove useful as the framework is extended in several directions, some of which presented in Section 5.3. An additional benefit is to put the solution closer to observables, which could prove interesting from the perspective of bringing the model to the data.

The existence and uniqueness of the equilibrium should be guaranteed, for instance following the work of [Duffie and Epstein \(1992\)](#), who use partial differential equation techniques to prove them in a infinite-horizon Markov diffusion setting with stochastic differential utility, or [Chabakauri \(2013\)](#) and [Bhamra and Uppal \(2014\)](#), who do so constructively for economies with heterogeneous agents and incomplete/complete markets, respectively. Both are also shown in situations with potentially dynamically complete markets³⁰ using a planner solution in [Anderson and Raimondo \(2008\)](#), and under complete markets with a full set of Arrow-Debreu securities in [Hugonnier et al. \(2012\)](#). As has been known since the seminal example of [Hart \(1975\)](#) however, the introduction of multiple goods could complicate the matter, for instance because markets can become dynamically incomplete even if the number of assets should technically be sufficient to span risks. Those multiple-good contexts are discussed e.g. in [Berrada et al. \(2007\)](#), and [Ehling and Heyerdahl-Larsen \(2015\)](#), again for the most part through the lens of the Pareto efficient allocation obtained from a social planner. Overall, equilibrium existence and uniqueness in the context of this paper with multiple goods, imperfect risk sharing or even incomplete markets, and a decentralized Radner solution, could therefore be analyzed further from a theoretical perspective, and represent an interesting avenue for further research.

2.4. Additions

Together with preferences that are general and potentially heterogeneous beyond the bias in consumption, the framework accommodates two important additions: imperfect financial integration, and labor income.

³⁰A securities market is potentially dynamically complete if the number of securities with non-colinear payoffs is equal to one plus the number of risk factors (Brownian motions) to be spanned.

Market structure and imperfect financial integration In the environment described so far, markets are potentially dynamically complete in the sense of [Anderson and Raimondo \(2008\)](#), i.e. the number of securities is at least one more than the number of independent sources of uncertainty and they can therefore span all risk. Even though the introduction of multiple goods could actually render markets dynamically incomplete, the assumption that the elasticity of substitution across goods θ is different from one, i.e. that the aggregator is not Cobb-Douglas, limits that possibility in practice ([Berrada et al., 2007](#); [Ehling and Heyerdahl-Larsen, 2015](#)). This is because as θ differs from one, the payoffs of the two equity assets are not perfectly correlated so that they can indeed span both sources of uncertainty and the portfolio choice between them is well-defined. In short, in this setup, risk sharing is perfect, markets are complete in the usual sense, and the decentralized equilibrium is Pareto efficient and corresponds to the planner's problem.

An aspect that is important in practice however, is that markets are likely to be imperfectly integrated, e.g. internationally. This can come from a number of frictions – informational, legal, technical –, with the result that the risk sharing between investors is likely to be imperfect. This aspect is particularly relevant in this context because as investors have a more difficult time sharing risks with one another, the allocation of wealth among them, which is captured by x_t and is an important new dimension in this paper, is likely to have a more significant impact on economic outcomes.

To study this friction, I introduce imperfect financial integration in a parsimonious way as a tax on dividends, adapting [Bhamra et al. \(2014\)](#) to a non-log two-good context with bias in consumption.³¹ The assumption allows me to study the effect of a range of financial integration degrees without having to take a specific stance on the source of the underlying imperfections. This tax is meant to encompass the wide array of frictions mentioned above – be they legal, technical, informational, or otherwise – that prevent investors from freely participating in certain financial markets. As shown in [Gârleanu et al. \(2020\)](#), models with investment taxes constitute

³¹Cf. also the seminal contribution of [Basak and Gallmeyer \(2003\)](#), who study a dynamic asset pricing model with asymmetric dividend taxation and a unique risky asset in a one-country one-good setting. The friction could also affect the diffusion of asset returns, which could be adapted to capture more specifically effects about information and uncertainty in the spirit of [Gehriger \(1993\)](#). I leave this exploration for future research.

an equivalent, but substantially simpler, way to capture a rich set of impediments to financial trade. Note that the spanning condition above is still verified in that the number of securities is still one more than the number of independent sources of uncertainty. As a result, investors still individually face markets that are dynamically complete. However, the opportunity sets that they face are now different due to the tax that differentially affects the assets for each of them, so that the equilibrium need not be Pareto efficient and the usual planner solution that has been popular for instance in the international finance literature cannot be used.³² Relatedly, the stochastic discount factors of the two investors are no longer perfectly correlated, and risk sharing is therefore imperfect. The latter is the phenomenon of interest here, e.g. with respect to the broader international finance and international portfolio choice literature. From a more general perspective, recall that I solve for the decentralized equilibrium of this economy, so that the framework is readily set to tackle a wide range of market structures including incomplete market settings. This will prove useful when tackling a number of promising extensions of the framework.³³

In practice, each investor pays a tax τ^i on the dividends of the tree producing their least preferred good. For instance, investor A only receives a dividend $(1 - \tau^A)p_{2,t}Y_{2,t}$ per share of the second equity asset (of which she holds $w_{2,t}^A W_t^A / Q_{2,t}$) because she pays $\tau^A p_{2,t} Y_{2,t}$ as a tax. As a result, the risk premium on asset 2 faced by investor A and therefore appearing in her budget constraint becomes $\mu_{R_{2,t}} - r_t - \tau^A F_{2,t}$, while the risk premium on asset 1 faced by investor B and appearing in his budget constraint becomes $\mu_{R_{1,t}} - r_t - \tau^B F_{1,t}$. This highlights the role of dividend yields in driving the effect of the tax, a point that is important in practice as discussed in Section 4. The amount of tax collected from one investor is rebated lump-sum to the other investor, so as not to distort decisions further. The exact details of this rebate do not make material difference, as discussed in [Bhamra et al. \(2014\)](#). In terms of budget

³²Cf. [Basak and Gallmeyer \(2003\)](#) for details. In a simpler context, e.g. with log preferences, one good and no home bias, one could potentially use a weaker notion of a social planner to solve the equilibrium by introducing time-varying Pareto weights, à la [Cuoco and He \(1994\)](#) or [Basak and Cuoco \(1998\)](#).

³³As a stark example, I consider the case in which market integration is so limited that investors can only trade the equity asset producing their preferred good, and a bond. In that case, the spanning condition is no longer satisfied and markets are incomplete. I omit those results in the interest of space, but they are available upon request. Another way to naturally include incomplete markets could be for the tax on dividends to be time-varying, or to introduce idiosyncratic labor income as in [Kaplan et al. \(2018\)](#), or capital risk as in [Brunnermeier and Sannikov \(2014, 2015\)](#). Assessing the impact of those extensions is an exciting avenue for future research.

constraints, the domestic investor receives an additional $w_{1,t}^B(1-x_t)\tau^BF_{1,t}/x_t$ per unit of wealth each infinitesimal period, while the foreign investor receives $w_{2,t}x_t\tau^AF_{2,t}/(1-x_t)$.

Labor Income Another aspect that has been analyzed in the literature and that can have a large impact on portfolios is labor income. Although I only touch upon it briefly in Section 4, this is also captured in the framework as a constant share of the output of each tree in the spirit of [Baxter and Jermann \(1997\)](#). Specifically, a share δ_j of the output of each tree is paid as labor income to the investor that prefers its good, while the remainder $1 - \delta_j$ is paid as dividends. In turn, this means that the dividend yields of the equity assets become $F_{1,t} \equiv (1 - \delta_1)p_{1,t}Y_{1,t}/Q_{1,t}$ and $F_{2,t} \equiv (1 - \delta_2)p_{2,t}Y_{2,t}/Q_{2,t}$, while the budget constraints have an additional term, $\delta_1F_{1,t}z_t/((1 - \delta_1)x_t)$ and $\delta_2F_{2,t}(1 - z_t)/((1 - \delta_2)(1 - x_t))$, for investors A and B respectively.³⁴

A more general specification of labor income could be an interesting extension, and is left for future research. It could for instance take the form of a time-varying share of output, as in [Coeurdacier and Gourinchas \(2016\)](#), and could naturally give rise to incomplete markets, or more realistic hedging terms in portfolios. The discussion in Section 4 and Appendix A.7 provides additional details.

2.5. Computation of the equilibrium

Section 3, which follows, characterize all variables of interest as a function of the state variables, $X_t = (x_t, y_t)'$, and a set of unknown functions $\mathcal{G} \equiv \{J_t^A, J_t^B, F_{1,t}, F_{2,t}, q_t, w_{1,t}^A, w_{2,t}^A\}$.³⁵ Due to the stationary recursive Markovian structure of the equilibrium, those unknown functions are themselves solely functions of X_t , and are determined by a set of coupled algebraic and second-order partial differential equations. Before describing those in the next section, let me say a brief word about the numerical approach.

³⁴ z_t , the ratio of the home equity price to the total wealth in the economy, is updated accordingly:

$$z_t \equiv Q_{1,t}/(Q_{1,t} + Q_{2,t}) = (1 + ((1 - \delta_2)/(1 - \delta_1))(F_{1,t}/F_{2,t})q_t(1 - y_t)/y_t)^{-1}.$$

³⁵ J_t^A, J_t^B are introduced in Section 3.2 and capture (an increasing monotonic transformation of) the marginal values of wealth of each investor. In addition, as a point of notation, for any function g , g_t simply denotes $g(X_t)$, not the time-derivative of g (which is zero because the model is stationary due to the infinite horizon).

Each of the unknown function $g : [0, 1]^2 \rightarrow \mathcal{D}^g \subseteq \mathbb{R}$ in \mathcal{G} is approximated using projection methods based on Chebyshev polynomials and orthogonal collocation. Details are provided in Appendix C. Let us simply discuss a few characteristics of the approach.

First, as with many continuous-time approaches, projection methods provide a global solution throughout the state space. This is in sharp contrast to a large subset of the international portfolio choice literature that has historically focused on local approximations in neighborhoods of specific points.³⁶ The use of a global solution approach instead makes it possible to study economic outcomes throughout the state space, which is particularly relevant in contexts in which variables e.g. portfolios are strongly state-dependent such as here. In addition, the use of a global solution method is important in cases in which the evolution of the variables of interest are very non-linear throughout the state space as can be the case in this context when investors become more risk averse, and even more importantly when markets become less perfectly integrated or investors become more heterogeneous. Such a method is also particularly adapted when there is no particularly well-suited point around which to perform a local approximation, such as a steady state. This is the case in my framework due to the specification of outputs as geometric Brownian motions but more importantly is also typically true in international contexts with incomplete markets. Finally, a global method will prove crucial when different types of constraints on the investor are introduced, a natural element that I plan to include in future research as discussed in Section 5.3.

Second, projection methods are also well-suited to contexts with multiple state variables in which other approaches like finite-difference methods become rapidly computationally too costly.³⁷ More generally, the addition of new state variables, as will

³⁶A typical point around which the local approximation is performed in the literature is the deterministic or risky steady state if it is well-defined, or the symmetric point in the middle of the state space (it would correspond to $X_t = (1/2, 1/2)$ in my context). A notable exception is Rabitsch et al. (2015), who use a global method. However, their framework is cast in discrete time so that the authors do not discuss the underlying drivers of portfolios, in particular hedging demands, the conditional (time-varying) moments of asset returns, as well as the conditional (time-varying) state variable dynamics. Under a set of assumptions, local methods could be used to study an economy further in the state space, cf. for instance Mertens and Judd (2018). However, such methods remain difficult to use in an international portfolio context due to the portfolio indeterminacy that arises in the corresponding deterministic economy.

³⁷The method currently developed in Hansen et al. (2018) could potentially help from that perspec-

naturally happen with the planned extensions discussed in Section 5.3, pose conceptually no difficulty. To be sure, computationally, traditional projection methods also are very much subject to the curse of dimensionality and scaling the number of state variables will prove limited using standard Chebyshev polynomials so that methods able to handle higher-dimensional cases will be required.³⁸ One such method consists in naturally extending the concept of projection approaches, but to replace the Chebyshev polynomials in the approximation by neural networks, which are designed specifically to handle high-dimensional contexts. I am developing these “projection methods via neural networks” for continuous-time models in [Sauzet \(2021c\)](#), and I discuss them in slightly more details in Section 5.3 and Appendix E.3.

3. Characterization of the Equilibrium

Assumption 1 (Symmetric baseline calibration). *Unless otherwise specified, the results in this section are obtained under the following calibration, $i \in \{A, B\}, j \in \{1, 2\}$:*

- *Risk aversion:* $\gamma^i = \gamma = 15$,
- *Elasticity of intertemporal substitution:* $\psi^i = \psi = 2$,
- *Bias in consumption:* $\alpha^A = \alpha = 0.75$, $\alpha^B = 1 - \alpha$, *numéraire basket:* $a = 1/2$,
- *Elasticity of substitution between goods:* $\theta^i = \theta = 2$,
- *Discount rate:* $\rho^i = \rho = 1\%$,
- *No labor income:* $\delta_j = \delta = 0$,
- *Fully integrated financial markets:* $\tau^i = \tau = 0$,
- *Output:* $\mu_{Y_j} = \mu_Y = 2\%$, $\sigma_{Y_1} = (4.1\%, 0)^T$, $\sigma_{Y_2} = (0, 4.1\%)^T$ (*no fundamental correlation*).

3.1. Evolution of the state variables

Due to the Markovian nature of the equilibrium, the laws of motion of the state variables underlie the dynamics of the economy. They are summarized in Proposition

tive.

³⁸Finer ways to construct the Chebyshev polynomials and corresponding grids, such as complete polynomials or Smolyak’s algorithm, can help. Ultimately however, they are also limited.

1. The relative supply y_t being exogenous, I focus the discussion on the endogenous state variable x_t .

Proposition 1. *Laws of motion for the wealth share x_t , and relative supply y_t :*

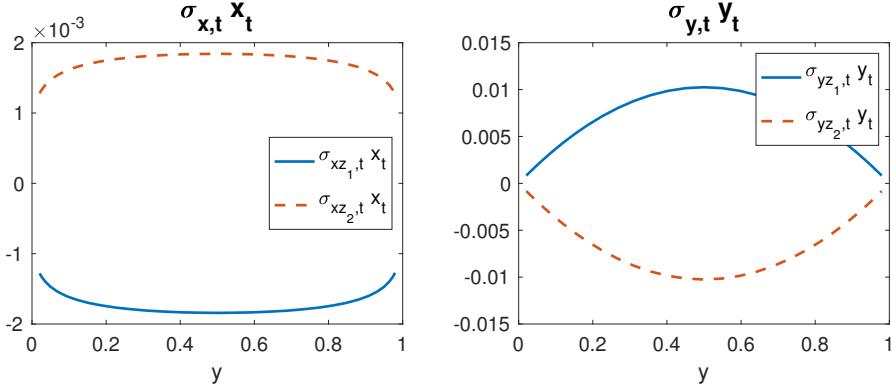
$$\begin{aligned}\frac{dx_t}{x_t} &\equiv \mu_{x,t} dt + \sigma_{x,t}^T d\vec{Z}_t \\ \frac{dy_t}{y_t} &\equiv \mu_{y,t} dt + \sigma_{y,t}^T d\vec{Z}_t\end{aligned}\tag{8}$$

where:

$$\begin{aligned}\mu_{x,t} &= (w_{1,t}^A - z_t)(\mu_{R_1,t} - r_t) + (w_{2,t}^B - (1 - z_t))(\mu_{R_2,t} - r_t) \\ &\quad + (F_{1,t}z_t + (1 - z_t)F_{2,t}) - P_t^A c_t^A + \left(\frac{\delta_1}{1 - \delta_1}\right) F_{1,t} \left(\frac{z_t}{x_t}\right) + \tau^B F_{1,t} \left(\frac{z_t}{x_t} - w_{1,t}^A\right) - \tau^A F_{2,t} w_{2,t}^A \\ &\quad - ((w_{1,t}^A - z_t)\sigma_{R_1,t} + (w_{2,t}^B - (1 - z_t))\sigma_{R_2,t})^T (z_t\sigma_{R_1,t} + (1 - z_t)\sigma_{R_2,t}) \\ \sigma_{x,t} &= ((w_{1,t}^A - z_t)\sigma_{R_1,t} + (w_{2,t}^B - (1 - z_t))\sigma_{R_2,t}) \\ \mu_{y,t} &= (1 - y_t)(\mu_{Y_1} - \mu_{Y_2}) - (1 - y_t)(\sigma_{Y_1} - \sigma_{Y_2})^T (y_t\sigma_{Y_1} + (1 - y_t)\sigma_{Y_2}) \\ \sigma_{y,t} &= (1 - y_t)(\sigma_{Y_1} - \sigma_{Y_2})\end{aligned}$$

The drift of the wealth share, $\mu_{x,t}x_t$, is shown in Figure F.24 in Appendix. It reflects the different forces impacting the budget constraints of both investors – returns on portfolios, consumptions, labor income, and the tax on foreign dividends that capture imperfect financial integration–, and drives the dispersion of the wealth share in equilibrium.

Figure 1: Diffusion terms for the state variables



Notes: Based on the symmetric calibration of Assumption 1. The figure shows a cut in which the allocation of wealth is symmetric ($x_t = 1/2$). y_t is the relative supply of the first good, which captures fundamentals. Corresponding representations as a function of both variables: Figure F.25.

Of more importance because it impacts portfolios, the diffusion of the wealth share, $\sigma_{x,t}x_t$, is shown in Figure 1, together with that of the relative supply, $\sigma_{y,t}y_t$. To fix ideas, the figure shows both terms of each diffusion as a function of the relative supply, when the allocation of wealth is symmetric ($x_t = 1/2$).³⁹ This type of representations as a function of one of the state variables are used throughout the paper when they ease the interpretation.⁴⁰ The main observation is that $\sigma_{xz,t}x_t$ is negative throughout the state space, so that a positive shock to the output of tree 1, $dZ_{1,t} > 0$, leads the wealth share of local investor A to decrease. Except when markets are imperfectly integrated as discussed in 4, this is true for any calibration and reflects the interaction of a number of underlying mechanisms that I discuss in the following sections. Namely, investor A invests more in the asset that has high payoffs when her marginal value of wealth is high, which occurs when the relative supply of her preferred good 1 is low, i.e. y_t low. When goods are good substitutes, broadly $\theta > 1$, the returns on asset 2 are higher in this situation, $\sigma_{R_1z_1,t} > \sigma_{R_2z_1,t}, \sigma_{R_1z_2,t} < \sigma_{R_2z_2,t}$, so that compared to the market portfolio, the equity portfolio of investor A exhibits a bias

³⁹Figure F.25 in Appendix shows the diffusion terms of x_t as a function of both state variables, and highlights that they vary a lot throughout the state space also in the x_t dimension. In particular, they are largest around $x_t = 1/2$, the point at which a switch occurs in which of the investors dominates the economy.

⁴⁰Note that this is only for the purpose of visualization. The equilibrium is still solved as a function of both state variables in all cases.

towards asset 2, $w_{1,t}^A - z_t < 0$, $w_{2,t}^A - (1 - z_t) > 0$. This corresponds to the standard calibration of θ in the baseline. When goods are poor substitutes instead, $\theta < 1$, asset 1 has higher returns in this situation, $\sigma_{R_1 z_1, t} < \sigma_{R_2 z_1, t}$, $\sigma_{R_1 z_2, t} > \sigma_{R_2 z_2, t}$, so that the equity portfolio of investor A exhibits a bias towards asset 1, i.e. a local bias, $w_{1,t}^A - z_t > 0$, $w_{2,t}^A - (1 - z_t) < 0$. The combination of those sets of facts yields the negative loading of the wealth share on shocks to the output of tree 1, $\sigma_{xz_1, t} x_t < 0$, and the positive loading on shocks to the output of tree 2, $\sigma_{xz_2, t} x_t > 0$, in all cases. Those patterns in turn determine the sign of the hedging of wealth risk on portfolios that I discuss in Section 3.4.

The resulting equilibrium distribution of the state variables is shown in Figure F.3 in Appendix. The sign of the diffusion terms discussed above is reflected in the strong negative relationship between the wealth share of investor A , which tends to decrease for positive output shocks to tree 1, and the relative supply of good 1, which increases in that case. The dispersion of the wealth share around this broad negative relationship is driven by the drifts and increases with the elasticity of intertemporal substitution. In this framework, the wealth share is time-varying, as soon as we move away from the log case ($\gamma = \psi = 1$), and is not purely determined by current fundamentals, y_t , as soon as we move away from the CRRA case ($\psi \neq 1/\gamma$) and introduce recursive preferences, or as we introduce imperfect financial integration. Even though the dispersion remains modest in the baseline, it increases significantly as markets become imperfectly integrated, and as investors become more heterogeneous, a point I discuss in Section 4.

3.2. Marginal values of wealth, goods prices, and risk sharing

I now turn to the marginal value of wealth of the investors, a quantity that underlies many decisions in the economy. To characterize them, note that due to the homotheticity of preferences, the value functions of the investors can be expressed as

$$V^i(W_t^i, x_t, y_t) = \left(\frac{W_t^{i1-\gamma^i}}{1 - \gamma^i} \right) J^i(x_t, y_t)^{\frac{1-\gamma^i}{1-\psi^i}} \quad (9)$$

Because W_t^i mostly have an impact in levels, the relative marginal values of wealth

of the two investors, which are obtained as the derivative of the value functions with respect to wealth, are primarily driven by the powers of J_t^i . In the remainder of the text, I therefore sometimes refer loosely to J_t^i as (monotonic transformations of) the marginal values of wealth.⁴¹ J_t^i are important economic objects in that they drive a large part of the dynamics of the stochastic discount factors of the investors, which in turn determine portfolios, asset prices, and other economic decisions. Indeed, in this context, stochastic discount factors can be expressed as⁴²

$$\xi_t^i \equiv \xi_0^i \exp \left\{ \int_0^t \left(\Theta_1^i P_u^{i1-\psi^i} J_u^i + \Theta_2^i \right) du \right\} W_t^{i-\gamma^i} J_t^{i \frac{1-\gamma^i}{1-\psi^i}} \quad (10)$$

The evolution of J_t^i are governed by two Hamilton-Jacobi-Bellman equations, summarized in Proposition A.2 in Appendix.⁴³ Figure 2 shows the result for investor A in the baseline calibration as a function of both fundamentals (y_t), shown on the horizontal axis, and the wealth share of the domestic investor (x_t), shown as different curves.⁴⁴ Results are symmetric for the investor B .

The intuition is as follows, and will be at the core of the differential tilt in the portfolio of each investor. As good 1 becomes relatively scarce, i.e. as y_t decreases, the marginal value of consumption for investor A increases given that she wishes to consume more of this good that she prefers, but cannot due to its limited supply. Following a standard envelope argument, the marginal value of wealth follows the same pattern and J_t^A therefore increases as y_t decreases, a phenomenon that occurs for any value of the wealth share and is the main driver of J_t^A . On the other hand, the marginal value of wealth increases with x_t , reflecting the fact that as she becomes dominant in the economy, investor A gets closer to holding the market portfolio, is thus unable to diversify risks as much with the other investor that becomes increasingly small, and is therefore relatively worse-off. From a macroeconomic standpoint, those patterns are consistent with the marginal value of wealth of the investor in-

⁴¹For instance, in the baseline calibration, $(1 - \gamma)/(1 - \psi) > 0$, and this is an *increasing* monotonic transformation. In terms of notation, recall that J_t^i simply denote $J^i(X_t)$, with $X_t = (x_t, y_t)'$.

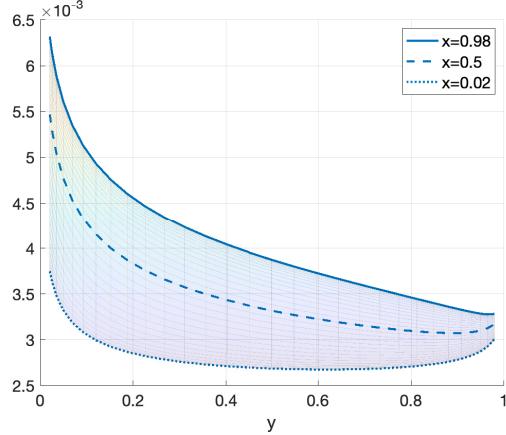
⁴²Constants Θ_1^i and Θ_2^i are provided in Appendix B.2.

⁴³These are two coupled second-order partial differential equations. The boundary conditions are the natural ones that result as the geometric drifts and diffusion terms of x_t and y_t converges to 0 when x_t and y_t approach 0 and 1, respectively.

⁴⁴A number of corresponding three-dimensional representations are also available in Appendix F.9 for the reader to whom they make the visualization more straightforward.

creasing as the price of her preferred good rises, which happens as its relative supply y_t is low, or as the investor owns a large share x_t of wealth.

Figure 2: Marginal value of wealth for investor A (J_t^A)



Notes: Based on the symmetric calibration of Assumption 1. x_t is the wealth share, which captures the share of wealth held by investor A . y_t is the relative supply of the first good, which captures fundamentals. Corresponding three-dimensional representation: Figure F.33.

From the Hamilton-Jacobi-Bellman equations in A.2, a first set of first-order conditions yield expressions for consumptions, summarized in Proposition A.3, which emphasize once again the underlying role of J_t^i : $c_t^i \equiv C_t^i/W_t^i = P_t^{i-\psi^i} J_t^i$. In the interest of space, details are shown in Appendix A.5 together with the corresponding figures, which are as expected. Combining with market-clearing conditions, one obtains Equation (11) for the terms of trade q_t , shown in Proposition 2.

Proposition 2. *The terms of trade, $q_t = q(X_t)$, solves the following non-linear equation:*

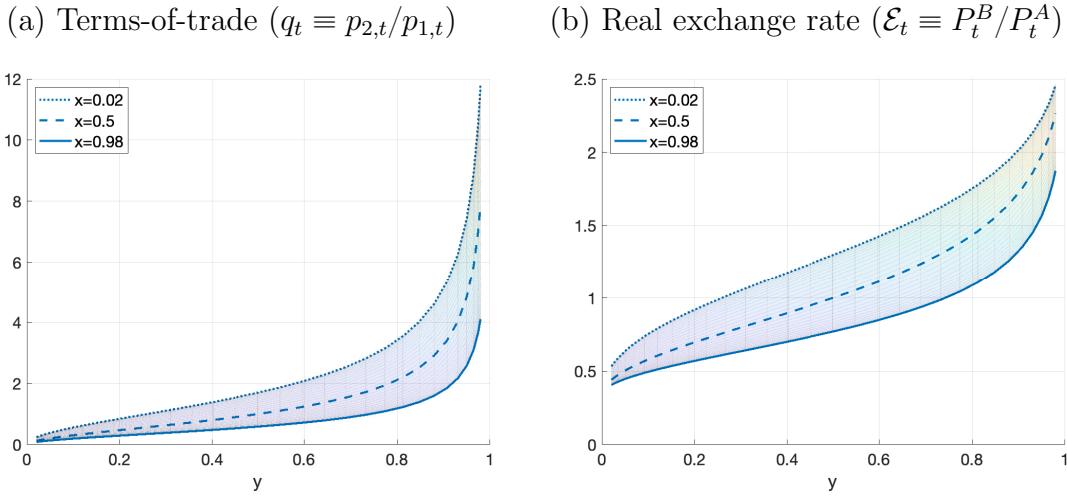
$$q_t = S_t^{1/\theta} \left(\frac{y_t}{1 - y_t} \right)^{1/\theta} \quad (11)$$

where:

$$S_t = \frac{(1 - \alpha^A) P_t^{A\theta - \psi^A} J_t^A x_t + (1 - \alpha^B) P_t^{B\theta - \psi^B} J_t^B (1 - x_t)}{\alpha^A J_t^A x_t P_t^{A\theta - \psi^A} + (1 - \alpha^B) P_t^{B\theta - \psi^B} J_t^B (1 - x_t)}$$

Prices $p_{1,t}, p_{2,t}, P_t^A, P_t^B, \mathcal{E}_t$ follow from the definition of the numéraire and Proposition A.3, and are shown in Proposition A.4.

Figure 3: Relative prices



Notes: Based on the symmetric calibration of Assumption 1. x_t is the wealth share, which captures the share of wealth held by investor A . y_t is the relative supply of the first good, which captures fundamentals.

This expression is the equivalent of Coeurdacier (2009)'s in this generalized framework, and emphasizes two main determinants of relative prices: the relative supply of the goods, captured by $y_t/(1 - y_t) = Y_{1,t}/Y_{2,t}$, and the relative demand for them, captured by S_t . The latter is akin to a transfer effect in the spirit of Keynes and Ohlin and depends on the allocation of wealth in the world economy as well as on the marginal values of wealth of both investors. The corresponding q_t is shown in Figure 3 together with the real exchange rate E_t .

Consistent with findings in Coeurdacier (2009) for an international application, who focuses on local approximations around the symmetric point $X_t = (1/2, 1/2)$, both relative prices are strongly related to the relative supply of goods, with the terms of trade worsening ($q_t \uparrow$) and the real exchange rate depreciating ($E_t \uparrow$) as the first good becomes relatively more abundant. However, in this more general framework and even around the symmetric point, the allocation of wealth in the economy also plays a role, albeit more muted than y_t . Specifically, as investor A gathers a large share of wealth, her preference for good 1 puts upward pressure on its price, which results in an improving terms of trade ($q_t \downarrow$) and appreciated real exchange rate ($E_t \downarrow$). The introduction of the wealth share in this context therefore makes the link between

relative supply and relative prices less direct, so that even though the hedging of relative supply will broadly capture the hedging of real exchange rate risk, which has been the focus of the international portfolio choice literature so far, the hedging of wealth share risk will also play a role.

Beyond relative prices themselves, which drive relative consumption decisions, the relative dividends between the two equity assets is of particular interest. They are shown in Figure F.13 and are obtained as

$$\frac{p_{2,t}Y_{2,t}}{p_{1,t}Y_{2,t}} = q_t \left(\frac{1 - y_t}{y_t} \right) = S_t^{\frac{1}{\theta}} \left(\frac{y_t}{1 - y_t} \right)^{\frac{1-\theta}{\theta}} \quad (12)$$

Contrary to relative prices, which are impacted only quantitatively by the calibration of parameters, relative dividends can flip direction. In particular, when goods are poor substitutes for one another (broadly when $\theta < 1$ ⁴⁵), the transfer effect due to relative demands is so large that relative dividends on the second asset *increase* when the relative supply of good 2 decreases, as shown in Panel (a) of Figure F.13. On the other hand, in the case in which goods are better substitutes (e.g. $\theta = 2 > 1$), relative dividends move in the same direction as the relative supply of the good underlying the payoffs of each asset, as is more standard and consistent with recent estimations of the elasticity of substitution in an international context such as that in [Imbs and Méjean \(2015\)](#). This switch in direction is consequential because it determines which of the asset has a higher payoff as a function of relative supply and will therefore be a prime determinant of how the bias in consumption translate into a bias in portfolios. Note that in both cases, the relative dividends of the second asset also decrease as the wealth share increases, consistent with the preference of investor *A* for good 1 that puts an upward pressure on the relative price of that good as she becomes dominant in the economy. This effect is more muted in the baseline calibration however.

Finally, under the assumption that there are not tax on foreign dividends so that

⁴⁵[Coeurdacier \(2009\)](#) shows that the exact value at which the switch occurs is in fact a non-linear function of all parameters. The author shows it in the CRRA case and at the symmetric point, but his findings are likely to persist in the framework of this paper with recursive preferences and globally. In practice, the switch is still close to $\theta = 1$ however, the case on which part of the seminal contribution of [Cole and Obstfeld \(1991\)](#) focuses and at which the CES aggregator of goods becomes Cobb-Douglas. In this case, relative dividends are constant and the two equity assets are perfectly correlated so that the portfolio choice is indeterminate.

risk sharing is perfect, the stochastic discount factors of both investors are perfectly correlated and we can derive in this environment a generalized version of the Backus-Smith condition of [Backus and Smith \(1993\)](#) and [Kollmann \(1995\)](#). This condition, shown in Proposition 3, emphasizes that the real exchange rate is not only determined by relative consumption, as in the usual CRRA case, but also depends on relative wealth and the marginal values of wealth of international investors. Section 4 discusses the case of imperfect financial integration in which we deviate from this condition.

Proposition 3 (Generalized Backus-Smith condition). *Under symmetric recursive preferences and perfect risk sharing*

$$\mathcal{E}_t = \phi^{\frac{1}{\gamma\psi}} \exp \left\{ \int_0^t \frac{1}{\gamma\psi} (\Theta_1 (P_u^{A1-\psi} J_u^B - P_u^{A1-\psi} J_u^A) du) \right\} \left(\frac{C_t^B}{C_t^A} \right)^{-1/\psi} \left(\frac{J_t^B}{J_t^A} \right)^{-\frac{1-1/(\gamma\psi)}{1-\psi}} \quad (13)$$

Constant Θ_1 is provided in [Appendix B.2](#), and ϕ is the relative Pareto weight of the two investors.

3.3. Asset Prices

Risk premia Starting with first moments, Proposition 4 presents the formulae for the expected risk premia on both equity assets, which are composed of three terms.

Proposition 4. *The expected risk premia on the equity assets are given by*

$$\begin{aligned} \mu_{R_1,t} - r_t &= \gamma_t \sigma_{R_1,t}^T \{ z_t \sigma_{R_1,t} + (1 - z_t) \sigma_{R_2,t} \} \\ &\quad - \gamma_t \sigma_{R_1,t}^T \left\{ x_t \left(\frac{1}{\gamma^A} \right) \left(\frac{1 - \gamma^A}{1 - \psi^A} \right) \sigma_{J^A,t} + (1 - x_t) \left(\frac{1}{\gamma^B} \right) \left(\frac{1 - \gamma^B}{1 - \psi^B} \right) \sigma_{J^B,t} \right\} \\ &\quad + \gamma_t \left(\frac{1 - x_t}{\gamma^B} \right) \tau^B F_{1,t} \end{aligned} \quad (14)$$

$$\begin{aligned} \mu_{R_2,t} - r_t &= \gamma_t \sigma_{R_2,t}^T \{ z_t \sigma_{R_1,t} + (1 - z_t) \sigma_{R_2,t} \} \\ &\quad - \gamma_t \sigma_{R_2,t}^T \left\{ x_t \left(\frac{1}{\gamma^A} \right) \left(\frac{1 - \gamma^A}{1 - \psi^A} \right) \sigma_{J^A,t} + (1 - x_t) \left(\frac{1}{\gamma^B} \right) \left(\frac{1 - \gamma^B}{1 - \psi^B} \right) \sigma_{J^B,t} \right\} \\ &\quad + \gamma_t \left(\frac{x_t}{\gamma^A} \right) \tau^A F_{2,t} \end{aligned} \quad (15)$$

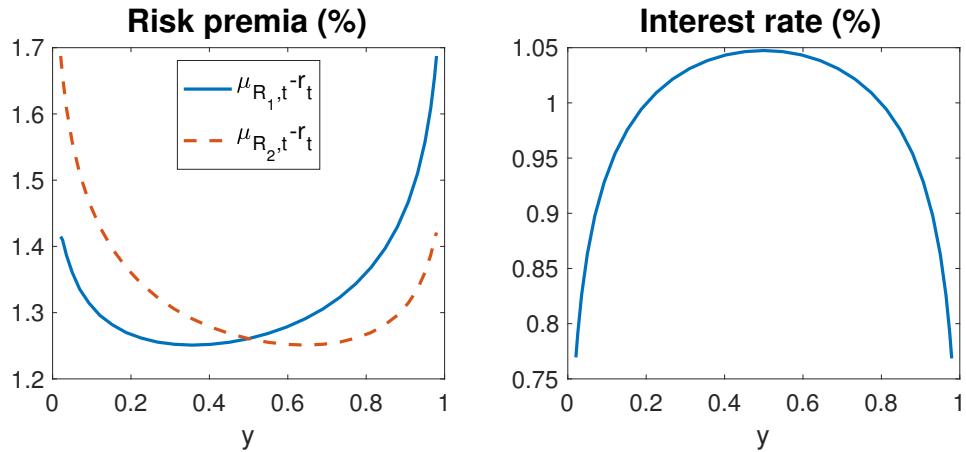
where $\gamma_t \equiv \left(\frac{x_t}{\gamma^A} + \frac{1-x_t}{\gamma^B} \right)^{-1}$ is the wealth-weighted global risk aversion.

The first term is a global component, and is driven by the covariance between each risky asset and total wealth.⁴⁶ Intuitively, an asset that comoves a lot with total wealth provide little diversification benefits, is therefore risky, and commands a high risk premium. The second term relates to how assets comove with the aggregate wealth-weighted marginal value of wealth, with the weight accounting for both differences in preferences and the allocation of wealth, and captures the fact that an asset whose payoffs are high when the aggregate marginal value of wealth is high provides a good hedge to investors, and therefore requires a lower risk premium (notice the negative sign). The third term is a general equilibrium effect arising when markets are imperfectly integrated due to the tax on dividends, and is discussed in Section 4. The price of risk on all three exposures is driven by the wealth-weighted global risk aversion, γ_t , which is constant and equal to $\gamma^i = \gamma$ in a symmetric calibration, but varies with the wealth share more generally.⁴⁷

⁴⁶Indeed, $z_t \sigma_{R_1,t} + (1 - z_t) \sigma_{R_2,t}$ is the weighted-average of the diffusions of both risky assets. The weights on the first and second assets are $z_t \equiv Q_{1,t}/(Q_{1,t} + Q_{2,t})$ and $1 - z_t$, respectively, which are the weight of each asset in the market portfolio. This is therefore nothing but the diffusion of total (economy-wide) wealth.

⁴⁷The expressions in Proposition 4 are also consistent with the expression for the price of risk κ_t , obtained from the fact that $d\xi_t^i/\xi_t^i = -r_t dt - \kappa_t^T d\vec{Z}_t$ in the baseline. For details, and an expression of r_t (pending) and κ_t , cf. Appendix B.2.

Figure 4: Returns



Notes: Based on the symmetric calibration under perfect risk sharing of Assumption 1. The figure shows a cut in which the allocation of wealth is symmetric ($x_t = 1/2$). y_t is the relative supply of the first good, which captures fundamentals. Corresponding representation as a function of both variables: F.15.

Figure 4 shows the corresponding returns as a function of fundamentals in the calibration of Assumption 1, which are representative of that for other parameter values. For both assets, the relative supply is the main determinant of expected risk premia, and those are driven almost exclusively by the global component. For instance, as good 1 becomes abundant ($y_t \uparrow$), the corresponding asset becomes dominant in total wealth ($z_t \uparrow$) so that the covariance of the asset with total wealth increases sharply. In other words, asset 1 provides increasingly poorer diversification benefits to investors, is therefore riskier, and commands a higher risk premia. The pattern for the interest rate is consistent with the evolution of the wealth-weighted marginal value of wealth.⁴⁸

Note that the risk premia on asset 2 also ultimately increases as y_t gets close to 1. This reflects the fact that even though both investors have a preference towards a specific good, they still desire both in their consumption basket given that the goods

⁴⁸In terms of levels, the average risk premia at around 1.4% remains small. This is not surprising given the relatively muted risk aversion of $\gamma^i = \gamma = 15$, and introducing portfolio constraints and other amplification mechanisms will be interesting extensions to consider to remedy this fact. On the other hand, the levels for the interest rate and the Sharpe ratio, at around 1% and 0.44 respectively, are broadly in line with the data.

are not perfectly substitutable. As one of the good becomes increasingly rare, the demand from both investors combined with a low supply put a significant upward pressure on the price of that good, so that the returns on that asset are driven up at the same time as those on the asset for which the relative supply becomes large. This phenomenon increases in magnitude as goods become more difficult to substitute (lower θ), and is also reflected in the conditional covariance of returns discussed below.

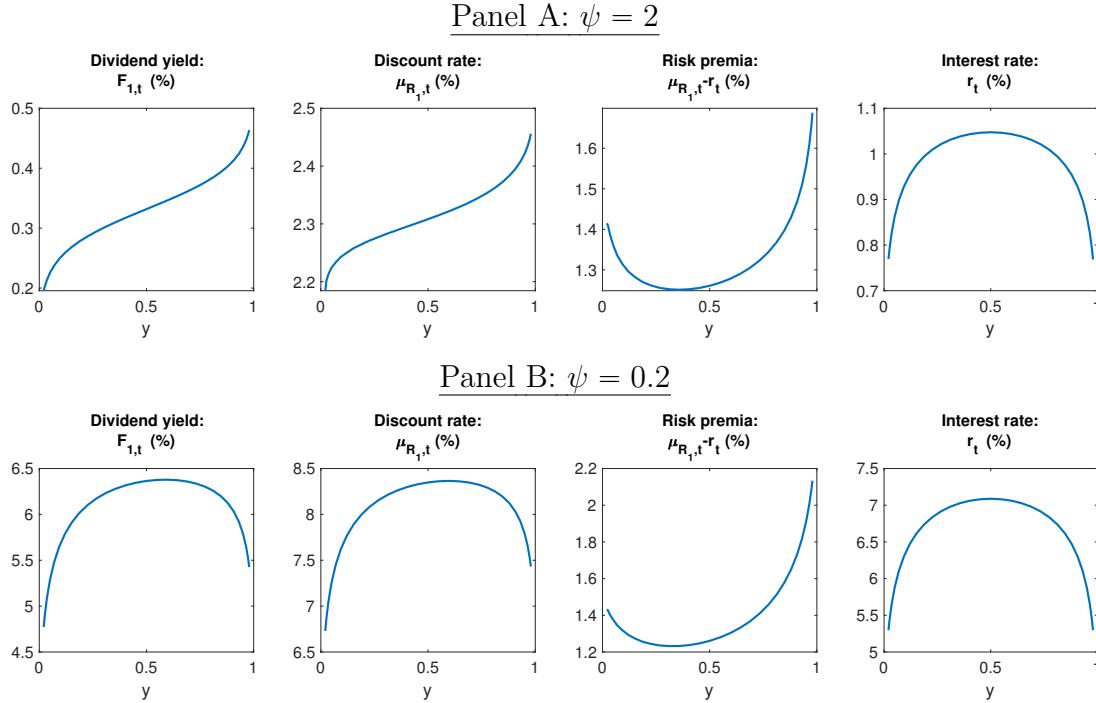
In the baseline, the impact of the wealth share on returns remains muted, as seen in Figure F.15, even though the impact on Sharpe ratios is more noticeable.⁴⁹ This impact grows significantly with imperfect risk sharing and investor heterogeneity as I discuss in Sections 4 and 5. Qualitatively, an increase in the wealth share of investor A yields an increase in the risk premium on asset 1, and a decrease in the risk premium on asset 2. In the baseline in which goods are good substitutes, this occurs because the first (second) asset is a poor (good) hedge for investor A . Indeed, the payoffs of the first asset for instance are large when her marginal value of wealth is low, which occurs primarily when her preferred good (1) is rare. Those patterns of risk premia as a function of the wealth share are reversed however when the goods are poor substitutes ($\theta < 1$) because relative dividends become inversely related to the relative supply in that case, as observed previously and shown in Figure F.13.

Dividend yields Like in a one-investor, one-good economy such as those of [Cochrane et al. \(2008\)](#) and [Martin \(2013\)](#), dividend yields $F_{1,t}, F_{2,t}$ are for the most part driven by discount rates $\mu_{R_{1,t}}, \mu_{R_{2,t}}$.⁵⁰ This is shown for asset 1 in Figure 5 and the result is symmetric for asset 2. However, the underlying drivers of the latter depends inherently on the elasticity of intertemporal substitution.

⁴⁹This is confirmed by computing the elasticities of risk premia and Sharpe ratios with respect to both state variables.

⁵⁰This is so even though the capital gain part of discount rates is quantitatively significant in this context, around 2% on average, because capital gains are less state dependent.

Figure 5: Determinants of dividend yields



Notes: Based on the symmetric calibration under perfect risk sharing of Assumption 1, except for the elasticity of intertemporal substitution ψ . The figure shows a cut in which the allocation of wealth is symmetric ($x_t = 1/2$). y_t is the relative supply of the first good, which captures fundamentals. Corresponding representation of $F_{1,t}, F_{2,t}$ as a function of both variables shown in Appendix.

When this elasticity is high (Panel A), i.e. when moving away from CRRA preferences, the riskless interest rate is low on average so that discount rates are driven mostly by risk premia, and are broadly monotonically increasing with the relative supply of the underlying tree. In that case, the relationship between risk premia and dividend yields is reminiscent of the predictability of expected returns observed empirically. When the elasticity is low however (Panel B), the riskless rate is significantly larger, so that it becomes the main driver of the evolution of discount rates. In that case, the relationship between the dividend yield of an asset and its risk premium is much weaker, and dividend yields actually predict the riskfree interest rate more.

This predictability-type pattern is confirmed by computing the regression coeffi-

cient of expected risk premia $\mu_{R_i,t} - r_t$ and interest rate r_t on $F_{i,t}$

$$\beta_{\mu_{R_i,t}-r_t,F_{i,t}} \equiv \frac{\text{cov}(\mu_{R_i,t} - r_t, F_{i,t})}{\text{var}(F_{i,t})} ; \quad \beta_{r_t,F_{i,t}} \equiv \frac{\text{cov}(r_t, F_{i,t})}{\text{var}(F_{i,t})}$$

In the baseline calibration with $\psi = 2$, $\hat{\beta}_{\mu_{R_i,t}-r_t,F_{i,t}} = 0.97$, $\hat{\beta}_{r_t,F_{i,t}} = 0.01$ on average, while when $\psi = 0.2$, $\hat{\beta}_{\mu_{R_i,t}-r_t,F_{i,t}} = -0.06$, $\hat{\beta}_{r_t,F_{i,t}} = 1.06$ on average.⁵¹

Dividend yields are also influenced by the extent of risk sharing, and the calibration of other parameters. Further, the dividend yield on the market is also related to the marginal values of wealth of both investors by the following expression:

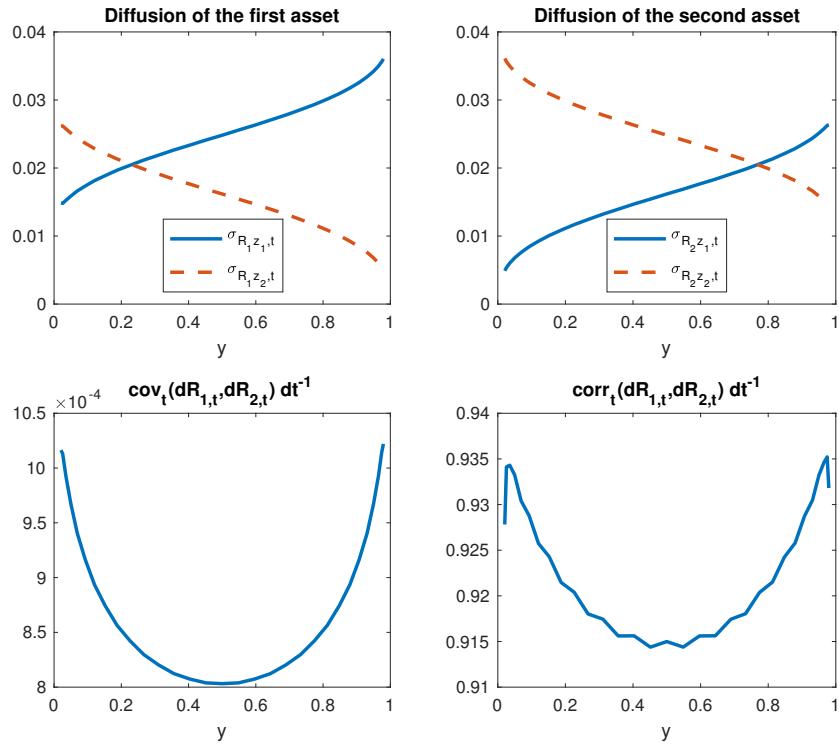
$$\left(\frac{1}{1-\delta_1} \right) F_{1,t} z_t + \left(\frac{1}{1-\delta_2} \right) F_{2,t} (1-z_t) = P_t^{A1-\psi^A} J_t^A x_t + P_t^{B1-\psi^B} J_t^B (1-x_t) \quad (16)$$

Second moments Figure 6 shows the diffusion terms for the returns on both assets in the baseline calibration, as well as the (instantaneous) conditional covariance and correlation of returns.⁵²

⁵¹In both case, the equivalent coefficient of regressing discount rate $\mu_{R_i,t}$ on $F_{i,t}$ is around 1, as expected from Figure 5.

⁵²As a side note, one of the strengths of the continuous-time framework is that it allows to express all *conditional* moments, both first and second, directly as a function of state variables.

Figure 6: Second moments of returns



Notes: Based on the symmetric calibration under perfect risk sharing of Assumption 1. The figure shows a cut in which the allocation of wealth is symmetric ($x_t = 1/2$). y_t is the relative supply of the first good, which captures fundamentals. Three-dimensional representation: Figure F.32.

For a change, I focus on the second asset. In the baseline calibration, the diffusion term corresponding to the second shock ($\sigma_{R_2 z_2,t}$) is larger for most of the state space. While intuitive, given that the physical output underlying this asset $Y_{2,t}$ loads mostly on this shock, the result once again hinges on the degree of substitutability across goods due to the fact that asset payoffs also depend on goods prices. When goods are good enough substitutes like in the baseline ($\theta = 2 > 1$), Section 3.2 showed that the relative dividends on asset 2, $p_{2,t} Y_{2,t} / (p_{1,t} Y_{1,t})$, increases when the relative supply y_t decreases, so that the returns on the second asset loads more on the second shock. When goods are poor substitutes however ($\theta < 1$), the relative dividends on the second asset increases when y_t increases, due to a strong effect on goods prices coming from consumer demand, so that the second asset ultimately loads more on the

first shock.⁵³ The former case is consistent with standard modern estimations of the elasticity of intertemporal substitution in an international context, e.g. in [Imbs and Méjean \(2015\)](#), so I keep it as my baseline, but the sign of the loadings will ultimately determine the direction of the portfolio bias that obtains in equilibrium.

Beyond these differences, it is noteworthy that regardless of θ , both returns load on both shocks. This is so despite the fact that the output of each good only loads on its Brownian shock, i.e. $\sigma_{Y_1 z_2} = \sigma_{Y_2 z_1} = 0$. For instance, in the baseline, the diffusion term corresponding to the loading of the second asset returns on the first shock ($\sigma_{R_2 z_1, t}$) is positive and large throughout. In fact, as the first good becomes dominant, the latter becomes larger than the loading on its own shock! This pattern is driven by changes in goods prices, as shown in the decomposition in Appendix F, and emphasizes that the conditional moments of asset returns vary significantly throughout the space. Economically, this highlights the strong contagion taking place through asset markets: a shock on the output of a given tree has a large impact on the returns of the other tree, and can therefore impact both investors beyond its impact on goods markets.

The above is also reflected in the evolution of the covariance and correlation of returns. First, and most striking, both are large, again despite no fundamental correlation in output. Those findings are consistent with those in [Pavlova and Rigobon \(2007\)](#), who focus on a log-Cobb-Douglas case, and [Bhamra et al. \(2014\)](#), who focus on a log-CES case with no bias in consumption. They are, however, reinforced in this environment with general preferences. For instance, the correlation is above 0.9 throughout the state space and reaches as high as 0.94 depending on the state of the economy, well above that in [Bhamra et al. \(2014\)](#), who find a correlation around 0.5 in a one-good specification (with a fundamental correlation of 0.5), and slightly above 0.6 in a two-good specification with no bias in consumption and $\theta = 2$. Similarly, the magnitude is significantly larger than in the one-good specification of [Chabakauri \(2013\)](#), for whom the correlation does not increase beyond 0.5, and sharply decreases towards the boundaries. This emphasizes the impact of a two-good environment with bias in consumption for asset pricing, in which comovements in good prices have a large effect on comovements in returns. It also highlights the quantitative difference made by using different calibration of preferences. For instance, the average level of

⁵³In the limit case in which $\theta = 1$, as discussed in [Cole and Obstfeld \(1991\)](#), the payoffs on both assets are perfectly correlated so that the portfolio choice between them is indeterminate.

correlation increases with the bias in consumption α , risk aversion γ , and the elasticity of intertemporal substitution ψ , and has an inverted U-shape pattern as a function of the elasticity of substitution across goods, with a maximum around $\theta = 1$ in which both assets are perfectly correlated.

Second, covariance and correlation are also time-varying and change a lot throughout the state space, an aspect that the global solution allows to characterize. Specifically, as one of the good becomes abundant, returns increasingly comove, consistent with the evolution of diffusion terms in Figure 6. This phenomenon is the manifestation for second moments of the pattern that was also observed for expected returns. As seen in Figure F.32, the correlation of returns itself also vary strongly with the wealth share, even in the baseline, and has a saddle shape: it is significantly larger around $x_t = 1/2$, the point of the state space at which the switch in which investor dominates the economy occurs. For instance, for $y_t = 1/2$, $\text{corr}_t(dR_{1,t}, dR_{2,t})dt^{-1} = 0.879$ for $x_t \rightarrow 0$ or 1, but 0.915 for $x_t = 1/2$. The correlation also displays a slight asymmetry and reaches its minimum around $x_t = 0.35$ for low y_t and $x_t = 0.65$ for high y_t .

Taken together, those result emphasizes that the benefits of diversification provided by each asset, as measured by the comovement of their returns, depend a lot on the calibration, and are also inherently state-dependent in this context.

3.4. Portfolios

From the Hamilton-Jacobi-Bellman equations in Proposition A.2, a second set of first-order conditions yield the optimal portfolios in Proposition 5. Those are typical Merton (1973)-type portfolios and are composed of two pieces.

The first one is common to both investors when markets are perfectly integrated ($\tau^i = \tau = 0$) up to differences in risk aversion. It corresponds to the myopic portfolio that would be chosen by a one-period mean-variance investor, and is driven by the risk premia on both assets, normalized by volatilities.

The second piece is a hedging term, absent with log or myopic preferences, which captures the way investors tilt their portfolios to insure against changes in the state of the economy, captured by $X_t = (x_t, y_t)'$. They do so by overweighting assets whose payoffs are large when they find it most valuable, i.e. when their marginal values of wealth are high, so that hedging terms are governed by the covariance between risky

return, and marginal values of wealth, J_t^A, J_t^B . First, investors hedge the relative supply risk, y_t . Because the relative supply is a strong driver of the relative prices of goods, this aspect is intimately related to the hedging of real exchange rate risk that has been the focus of a large part of the portfolio choice literature in an international context: investors form their portfolios by hedging against changes in the relative prices of the goods that they desire to consume. Yet, as was visible in Figure 3, the mapping between relative supply and relative prices, although strong, is not one-for-one and is also impacted by the repartition of wealth across investors. The framework in this paper allows to disentangle those different channels: in general equilibrium, investors hedge not only against relative supply changes, i.e. changes in the physical quantity of the goods, but also against changes in their share of wealth. The latter, which had so far not been emphasized in the portfolio choice literature, matters both because it has an impact on relative prices, but also as it captures the extent to which investors are able to share and diversify risks with one another.

Overall, the common term drives the broad pattern of the portfolios of both investors throughout the state space, while the hedging term captures how investors differentially deviate from this broad pattern. Hedging terms are therefore a prime variable of interest in an economy with heterogeneous investors.

Proposition 5. *The optimal portfolios of investors A and B are given by*

$$\begin{pmatrix} w_{1,t}^A \\ w_{2,t}^A \end{pmatrix} = \frac{1}{\gamma^A} (\Sigma_t^T \Sigma_t)^{-1} \left\{ \begin{pmatrix} \mu_{R_1,t} - r_t \\ \mu_{R_2,t} - r_t - \tau^A F_{2,t} \end{pmatrix} + \left(\frac{1 - \gamma^A}{1 - \psi^A} \right) \Sigma_t^T \left(\frac{J_{x,t}^A}{J_t^A} x_t \sigma_{x,t} + \frac{J_{y,t}^A}{J_t^A} y_t \sigma_{y,t} \right) \right\}$$

$$b_t^A = 1 - w_{h,t}^A - w_{f,t}^A \quad (17)$$

$$\begin{pmatrix} w_{1,t}^B \\ w_{2,t}^B \end{pmatrix} = \frac{1}{\gamma^B} (\Sigma_t^T \Sigma_t)^{-1} \left\{ \begin{pmatrix} \mu_{R_1,t} - r_t - \tau^B F_{1,t} \\ \mu_{R_2,t} - r_t \end{pmatrix} + \left(\frac{1 - \gamma^B}{1 - \psi^B} \right) \Sigma_t^T \left(\frac{J_{x,t}^B}{J_t^B} x_t \sigma_{x,t} + \frac{J_{y,t}^B}{J_t^B} y_t \sigma_{y,t} \right) \right\}$$

$$b_t^B = 1 - w_{1,t}^B - w_{2,t}^B \quad (18)$$

where $\Sigma_t \equiv [\sigma_{R_1,t} \ \ \sigma_{R_2,t}]$.

What do portfolios look like in practice? I start by discussing average portfolios.

The portfolio choice literature in the presence of two goods, mostly in an international context, has for the most part considered so-called zero-order (i.e. steady-

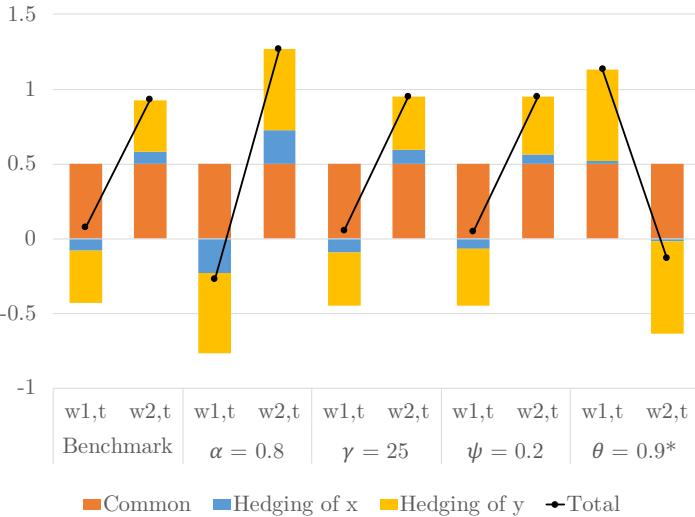
state) portfolios. These are constant and replicate locally complete markets in a small neighborhood of the symmetric point of the state space by using a second-order approximation of portfolio equations, and a first-order approximation to other equations.⁵⁴ Even though one of the main advantages of the global method I introduce in this paper is to break away from those low-order local approximations as I discuss below, I first investigate patterns at the symmetric point $X_t = (1/2, 1/2)$ to facilitate comparison with existing work. I start by focusing on the hedging of relative supply risk, to make the parallel with the hedging of real exchange risk that has been most discussed.

Average portfolios are strongly impacted by the specification of preferences. To see this, Figure 7 shows the weights allocated to the first and second equity assets in the portfolio of investor A , $w_{1,t}^A$, $w_{2,t}^A$, at $X_t = (1/2, 1/2)$ for various calibrations. Figures F.10, F.11 and F.12 in Appendix also provide additional details.⁵⁵

⁵⁴For examples of this approach, cf. Coeurdacier (2009), Tille and van Wincoop (2010), Devereux and Sutherland (2011), Coeurdacier and Rey (2013), and Coeurdacier and Gourinchas (2016), among others.

⁵⁵Note that in this symmetric calibration with perfect risk sharing, the riskless bond is not traded. I come back to this aspect in further sections.

Figure 7: Equity portfolio for investor A at $X_t = (1/2, 1/2)$



Notes: Based on the symmetric calibration under perfect risk sharing of Assumption 1, except for the specified parameters. * For $\theta = 0.9$: $\gamma = 15, \psi = 1/\gamma, \alpha \approx 0.58$ (further calibrations ongoing). The figure shows portfolios when both the allocation of wealth (x_t) and the relative supply (y_t) are symmetric, $X_t = (1/2, 1/2)$.

The most important dimension is once again the elasticity of substitution across goods, which can flip the bias in portfolio holdings. Recall that due to the bias in consumption, the marginal value of wealth of the investor A increases when the first good becomes rare (y_t decreases): she would like to consume more of her preferred good but cannot. Symmetrically, the marginal value of wealth of investor B decreases, given his preference for the second good that becomes abundant. As a result, investor A values an asset that pays in those conditions, while investor B does not. What is the asset that pays most when the first good becomes rare? When goods are good substitutes like in the baseline ($\theta = 2 > 1$), Sections 3.2 and 3.3 showed that relative dividends and returns are positively related to relative supply, so that this is the second asset. Investor A therefore overweights the second asset in her portfolio, while investor B symmetrically overweights the first asset. In other words, portfolios exhibit a bias in equity holdings towards the asset that delivers their least-preferred good (e.g. a “foreign bias” in an international context). When goods are poor substitutes instead ($\theta < 1$), relative dividends and returns are negatively related to relative supply so that

the payoffs of the *first* asset gets larger when y_t decreases, and it gets overweighted in the portfolio of investor A and underweighted in that of investor B . Portfolios therefore exhibits a bias in equity holdings towards the tree that delivers the preferred good of each investor (“home bias”). Those patterns are typically consistent with the hedging of real exchange risk that has been one of the focus of the literature in an international context, as discussed e.g. recently in [Coeurdacier \(2009\)](#). Indeed, an asset that pays well when the relative supply of an investor’s preferred good is low is also an asset that pays well when the relative price of that good is high. It is therefore valued by that investor, and overweighted in their portfolio.

In an international context, the discussion suggests that a first explanation for why portfolios might be biased towards domestic assets empirically could be that the goods produced by different countries are poor substitutes, and is consistent with findings in [Heathcote and Perri \(2002\)](#), [Kollmann \(2006\)](#), and [Corsetti et al. \(2008\)](#).⁵⁶ This calibration however can be called into questions for three reasons. First, even though it is the subject of some debate in the literature, standard modern estimations of θ typically put it above one, with [Imbs and Méjean \(2015\)](#)’s popular estimate in the range of [4, 6]. Values above one are also consistent with a large body of empirical work in international trade. Second, the case of $\theta < 1$ also has a number of counterfactual predictions: (i) growth is immesirizing, i.e. the output of a country at market value decreases for a positive supply shock so that a positive domestic shock mostly benefits the foreign country, and (ii) the introduction of other realistic aspects of international trade and macroeconomics, such as trade costs, leads to an even worse foreign bias in equity holdings in this situation, as discussed in [Coeurdacier \(2009\)](#). Third, even though a low θ could yield the right direction in terms of portfolios, the home bias obtained as a result is in fact too extreme for reasonable calibrations of the parameters. This aspect, hinted at in [Coeurdacier \(2009\)](#), is confirmed in my general setup: e.g. even for α as low as 0.58, significantly below usual calibrations such as the baseline of $\alpha = 0.75$, each investor *shorts* the foreign asset at the symmetric point ($w_{2,t}^A = w_{1,t}^B = -13\%$) in order to allocate more than 100% of their wealth to the local asset when $\theta = 0.9$. For all those reasons, I stick to the standard case of $\theta > 1$ as my baseline. To turn the foreign bias in equity holdings that obtains into a home bias like in the data, we can instead rely on the other (plausible) channel that I introduce in this environment: imperfect financial integration. I discuss this aspect in Section

⁵⁶The impact of this assumption is also discussed in [Tille \(2001\)](#) and [Coeurdacier \(2009\)](#).

4. In a more general context, one could think of having θ itself be time-varying. This could prove relevant e.g. in the environmental context touched upon in Section 5.2, as discussed in Gollier (2019).

What about the effect of wealth share hedging? When risk sharing is perfect, the hedging of wealth share risk turns out to reinforce the bias that emerges from the hedging of relative supply. To see this, let us focus on the baseline calibration and consider a negative shock to the output of the first tree ($dZ_{1,t} < 0$). In that case, the wealth share of investor A increases, as was discussed in 3.1.⁵⁷ This in turn leads her marginal value of wealth to increase, given that it is more difficult for her to diversify risk, so that she values an asset that pays in those conditions. Concurrently, such a shock tends to decrease relative output y_t as well as the relative dividends on the first equity asset, provided that goods are good enough substitutes, so that its payoffs decreases relative to those on the second asset. This is so because the mild upward pressure on the price of the good 1 coming from the increase in the size of investor A in the economy is not enough to compensate the downward pressure due to the lower supply. As a consequence, the first asset therefore does not pay off in a situation where it would be valuable, which leads investor A to tilt her portfolio further away from the first asset. The phenomenon is reversed in cases of low substitutability of goods, so that wealth share hedging reinforces the “home bias” that obtains in that case. Quantitatively, the impact of the hedging of x_t remains muted in the baseline, but grows significantly as investors become more heterogeneous and as markets become imperfectly integrated so that sharing risk is more difficult across investors. I discuss those aspects below and in Section 4.

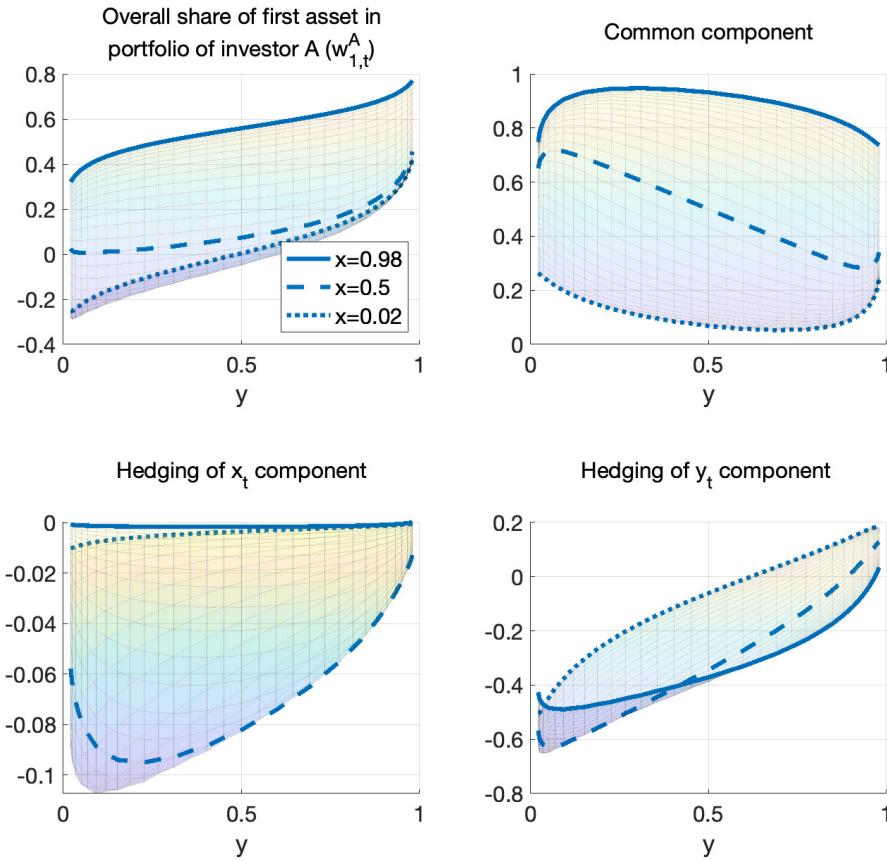
The impact of both hedging terms taken together is significant and the “foreign bias” in equity holdings that obtains in the baseline is large: at the symmetric point, investor A allocates $w_{2,t}^A = 93\%$ of her wealth to the second asset, and only $w_{1,t}^A = 7\%$ to the first asset that delivers her preferred good, compared to the 50-50% split consistent with the common term.⁵⁸ The bias is reinforced as the risk aversion increases

⁵⁷Recall that this was due to the combination of the signs of $w_{h,t} - z_t < 0$, $w_{f,t} - (1 - z_t) > 0$, which obtain mostly from the hedging of relative supply in the baseline, and of the patterns of the diffusion of risky returns, $\sigma_{R,t}, \sigma_{R^*,t}$.

⁵⁸Again, this would also be the case with a low θ so that the resulting “home bias” in equity holdings would still be counterfactually large, with the second asset not being invested in or even being shorted.

(Figure F.11), making investors more sensitive to risks in the economy, while (very) mildly reduced as the elasticity of intertemporal substitution increases (Figure F.12). For both, the hedging of wealth risk grows in importance, even though it remains mostly muted compared to that of relative supply risk. The bias in portfolios is also strongly reinforced as the bias in consumption increases (Figure F.10). This leads the investors to have a stronger preference towards their preferred good, which strengthens the hedging of y_t , but also make them more heterogeneous, which strengthens the effect of the wealth share even more. As a result, the impact of wealth share hedging is strongly reinforced and becomes as large as that of y_t . This heightened impact of the wealth share risk is a theme that will come back when I study further heterogeneity as well as imperfect financial integration in Section 4, and for the applications of Section 5. For large values of α , the portfolio bias can become extreme. For instance, with $\alpha = 0.85$, a value that is still lower than the types of values used more recently in the literature, $w_{1,t}^A = -175\%$ and $w_{2,t}^A = 275\%$, in words, investor A is willing to severely short the first asset to lever up the share of her wealth that she allocates to the second asset. Similarly, the “*home bias*” is strongly reinforced when $\theta < 1$. Overall, those results are broadly consistent with findings in Coeurdacier (2009) in an international context, even though the author focuses on a CRRA case in which the risk aversion and elasticity of substitution are inversely related to one another and in which, most importantly, the wealth share does not play a role of its own. In addition, the non-linearities that obtain as heterogeneity increases render the use of a global method particularly important as compared to first and second-order local approximations. Beyond magnitudes, hedging terms are also important in that they drive the differential tilt in the portfolios of the two heterogeneous investors.

Figure 8: Components of the portfolio of investor A



Notes: Based on the symmetric calibration under perfect risk sharing of Assumption 1. x_t is the wealth share, which captures the share of total wealth held by investor A . y_t is the relative supply of the first good, which captures fundamentals. Corresponding three-dimensional representations: Figure F.34.

What happens beyond the symmetric point? Being able to study portfolios and other variables not just for $X_t = (1/2, 1/2)$ but for any point of the state space is one of the breakthroughs allowed by the global method in this paper. Conceptually, this is a natural way forward given that even for the symmetric point, the hedging of the state variables is fundamentally about what is happening outside of this point, i.e. about dynamics throughout the state space, which cannot be visualized and studied with local low-order approximations and constant portfolios but that the method here

suddenly make completely visible.

Figure 8 shows the components of the weight of the first asset in the portfolio of investor A in the baseline calibration. What happens at $X_t = (1/2, 1/2)$ was discussed before, but the picture reveals that portfolios and their components vary substantially with the state of the economy. For instance, investor A strongly shorts the first asset, $w_{1,t}^A = -30\%$, when her preferred good (1) is rare and when her share of total wealth is small, while she allocates a large portion of her portfolio to it, $w_{1,t}^A = 80\%$, when good 1 is abundant and her share of wealth large. This is strikingly different to the $w_{1,t}^A = 7\%$ that is picked by investor A at the symmetric point.

This dependence on the state of the economy comes both from the common component, which drives the overall shape of both portfolios, and from hedging terms. The relative supply of course has a strong impact on each component, for instance with the hedging of the relative supply becoming much stronger as good 1 becomes rare. But most importantly, the picture suggests a second important role for the wealth share in addition to its impact as a pricing factor that is hedged: its role as a state variable, which captures the average investor in the economy in a given instant. Because this average investor looks very different according to whether she most resembles investor A or B , the wealth share has a strong direct impact on the common component, highlighting its effect on risk premia and the conditional variance-covariance matrix of returns, as well as on the hedging of the wealth share itself, which is largest around $x_t = 1/2$, the point around which the dominant investor in the economy switches and at which the volatility of the wealth share is largest.

What about the bias in portfolios in this global solution context? To study it, it is no longer sufficient to compare $w_{1,t}^i$ and $w_{2,t}^i$, the weight of each asset in the portfolio of a given investor. Indeed, as the state of the economy evolves, the share of each asset in the market portfolio also changes compared to the 50-50% split that obtains at the symmetric point. To study this question, I therefore compute a portfolio bias measure towards the local or “home” equity, HB_t^i , and a portfolio bias measure towards the “foreign” equity, FB_t^i .⁵⁹ Those measures have the added benefit that they are closer to those that have been used empirically, for instance in Coeurdacier

⁵⁹As a reminder: local or “home” refer to the preferred good of an investor, or the tree that produces that good. I adopt this terminology from the international context purely for clarity.

and Rey (2013).⁶⁰ They are defined as the share of the “local” tree in the equity portfolio of an investor (e.g. $w_{1,t}^A/(w_{1,t}^A + w_{2,t}^A)$ for investor A) divided by its share in the market portfolio (z_t), and the share of “foreign” tree in their equity portfolio (e.g. $w_{2,t}^A/(w_{1,t}^A + w_{2,t}^A)$ for investor A) divided by its share in the market portfolio ($1 - z_t$), i.e.

$$HB_t^A \equiv \frac{w_{1,t}^A/(w_{1,t}^A + w_{2,t}^A)}{z_t} \quad \text{and} \quad FB_t^A \equiv \frac{w_{2,t}^A/(w_{1,t}^A + w_{2,t}^A)}{1 - z_t} \quad (19)$$

Those measures are shown in the bottom two panels of Figure F.14 for investor A in the baseline calibration. They are defined analogously for investor B , and are symmetric in that calibration.⁶¹ They paint an even starker picture than the components discussed above: portfolios vary substantially with the state of the economy not only in terms of the weights themselves, but also in terms of how biased they are. For instance, as she becomes dominant in the economy, and even though equity prices adjust accordingly, investor A has to get closer to holding the market portfolio, i.e. both HB_t^A and FB_t^A converge to one. When her share of total wealth diminishes however, and if in addition the relative supply of her preferred good 1 becomes rare, she shorts the first equity asset in a magnitude that is particularly extreme when compared to the market portfolio (-1.2), while she levers up and invest about 1.6 times as much than the market portfolio on the second asset. Those observations are mirrored in the case in which a “home bias” obtains when θ is low.

Taken together, those results confirm the strong bias in equity holdings that obtained in the baseline at the symmetric point ($HB_t^A = 0.2$ at $X_t = (1/2, 1/2)$), while emphasizing that the extent of this bias is also inherently state-dependent. Note once again that both are true for high θ like in the baseline for which a “foreign bias” obtains, and for low θ in which a “home bias” obtains. The introduction of imperfect

⁶⁰The fact that $w_{1,t}^i$ and $w_{2,t}^i$ vary throughout the state space could provide a rationale for $w_{1,t}^A$ is above $w_{2,t}^A$ in practice in an international context, provided that the world economy is in some particular part of the state space. However, in the baseline calibration, the “foreign bias” is quantitatively so stark that the regions of the state space in which $w_{1,t}^A > w_{2,t}^A$ are small. Cf. Figure XX. In addition, the proper way to study the “home bias” is to compare portfolio weights to the market portfolio, as done in the rest of the paper.

⁶¹Recall that $z_t \equiv Q_{1,t}/(Q_{1,t} + Q_{2,t})$, i.e. it is the ratio of the price of tree 1 to total wealth. (Equity price is the same as equity value given that the supply of each equity asset is normalized to unity.) An equivalent approach would be to compute a measure of “home bias” as $1 - w_{2,t}^i/(w_{1,t}^i + w_{2,t}^i)/(1 - z_t)$ as in Coeurdacier and Rey (2013). I stick to my measure because it allows to look at both assets. Note also that when the bond is not traded like in the baseline calibration, $w_{1,t}^i/(w_{1,t}^i + w_{2,t}^i) = w_{1,t}^i$ because $w_{1,t}^i + w_{2,t}^i = 1$.

financial integration that I discuss in the next section will therefore be important to ultimately generate plausible portfolios, e.g. in an international context. In addition, those evolutions reveal that portfolios are fundamentally time-varying and strongly responding to shocks to both wealth and relative supply. Those aspects could so far not be discussed in the literature, given the main focus on zero-order constant portfolios, and local neighborhoods of the symmetric point.⁶² Even though empirical facts about the time evolution of portfolio bias measures remain for the moment elusive, given the limited length of this times series and their relative smoothness due to their low frequency (annual or less), the results in this section suggest that they are an important target for future research, as echoed in [Coeurdacier and Rey \(2013\)](#). To that end, the large and detailed data gathering effort undertaken e.g. for the Global Capital Allocation project of [Maggiori et al. \(2020\)](#) and [Coppola et al. \(2020\)](#) will assuredly prove invaluable.

Finally, we can revisit the impact of each component quantitatively. To do so, Table 1 decomposes the (unconditional) variance of $w_{1,t}^A$ into its three components: in the baseline calibration with $\alpha = 0.75$, hedging components drive 30% of the changes in portfolios, and this proportion increases to 69% as $\alpha = 0.85$.⁶³ This confirms the picture that emerged from the analysis of average portfolios at the symmetric point in which the hedging components were also particularly important. Although the hedging of wealth share risk itself remains muted in the baseline, it increases significantly as investors heterogeneity increases and becomes on par with the hedging of relative supply. Perhaps most importantly and as an additional reminder, even when they are quantitatively smaller and while the common component drives the broad shape of the portfolio, hedging components are conceptually responsible for the differential tilt in portfolios between investors A and B , which is often the question of interest in a portfolio choice context with heterogeneous investors. Ignoring them, or focusing on special cases such as log or myopic preferences in which hedging components are absent as been common in part of the literature, can therefore yield significantly different portfolios.

⁶²Again, as investors become more heterogeneous, portfolios also become strongly non-linear (cf. Figures F.16 and F.17), so that using a global method is also crucial from this perspective as low-order local approximations could become imprecise.

⁶³Specifically, I compute each of the component of the following decomposition: $1 = \frac{\text{var}(w_{1,t}^A)}{\text{var}(w_{1,t}^A)} = \frac{\text{cov}(w_{1,t}^{A\text{common}}, w_{1,t}^A)}{\text{var}(w_{1,t}^A)} + \frac{\text{cov}(w_{1,t}^{A\text{hedg.x}}, w_{1,t}^A)}{\text{var}(w_{1,t}^A)} + \frac{\text{cov}(w_{1,t}^{A\text{hedg.y}}, w_{1,t}^A)}{\text{var}(w_{1,t}^A)}.$

Table 1: Variance decomposition of portfolio weights

	Common component	Hedging of x_t	Hedging of y_t	Total
$\alpha = 0.5$	-42%	0%	142%	100%
$\alpha = 0.75$	70%	7%	23%	100%
$\alpha = 0.85$	31%	35%	34%	100%

Notes: The table shows the decomposition of $w_{1,t}^A$, the share of the first asset in the portfolio of investor A , into its three components. $\alpha = 0.75$ is the baseline calibration. Results are identical for $w_{2,t}^A$, which is equal to $1 - w_{1,t}^A$ in the baseline, and for the portfolio of investor B .

4. Imperfect financial integration and investor heterogeneity

Imperfect financial integration and investor heterogeneity are two dimensions that have the potential to strongly impact the equilibrium. I study both in this context, and show that their influence goes hand-in-hand with a reinforced effect of the allocation of wealth.

Imperfect financial integration The introduction of imperfectly integrated markets, modeled as a tax τ on “foreign” dividends in the spirit of [Bhamra et al. \(2014\)](#), impacts the economy because it prevents investors from perfectly sharing risk with one another.⁶⁴ Because the assets that they can trade are different, due to the direct tax that each investor has to pay on them as well as a general equilibrium effect on their risk premia, the opportunity sets faced by both investors differ. As a result, even though they individually face dynamically complete markets, their stochastic discount factors are no longer perfectly correlated. This has a number of consequences in terms of the evolution of their marginal values of wealth, interest rates, consumptions, and other variables, with the effect most visible on portfolios. The general specification

⁶⁴In this section, I assume that $\tau^i = \tau$, i.e. that the tax on “foreign” dividend is symmetric. However, the framework also allows for asymmetric taxes, which can be interesting to study in realistic applications. For details on the exact formulation of those taxes in the model, cf. Section [2.4](#).

of the model also allows us to study the impact of several dimensions of preferences, which turn out to have a strong impact on the magnitude of the effect of imperfect financial integration. I focus on the elasticity of intertemporal substitution ψ , which takes center stage.

When the elasticity of intertemporal substitution is low, $\psi = 0.2$ ⁶⁵, the introduction of a modest degree of imperfect financial integration is sufficient to make their respective “foreign” asset much less attractive to each investor. This comes both from the fact that the overall level of the “foreign” risk premium as perceived by an investor is directly decreased by the tax that has to be paid on it, $-\tau^A F_{2,t}$, $-\tau^B F_{1,t}$, and from the fact that the slope of the risk premia on both asset as a function of the wealth share flips sign driven by the tax as well as a modest general equilibrium effect.⁶⁶

As a result, both investors rapidly turn the bias in equity holdings towards the tree that produces their least preferred good in the baseline (“foreign bias” in equity holdings), to a bias towards the tree that produces their preferred good (“home bias”). For instance, at the symmetric point $X_t = (1/2, 1/2)$, the left panel of Figure 9 shows that a tax on the order of $\tau = 7$ to 10% is enough to bring the HB_t^A measure above 1, from 0.75 to 1.39, and the FB_t^A measure below 1, from 1.32 to 0.50, both reflecting a strong “home bias” in equity holdings compared to the market portfolio. Those are consistent for instance with empirical measures in an international context such as those in [Coeurdacier and Rey \(2013\)](#). To get a sense of magnitude, the raw share of the first asset in the equity portfolio of investor A increases from 42% to 78% at that point, broadly in line with the data. The fact that reasonable frictions on market integration can yield home bias in equity confirms the finding of [Bhamra et al. \(2014\)](#) in this general and global framework, provided that ψ is low.

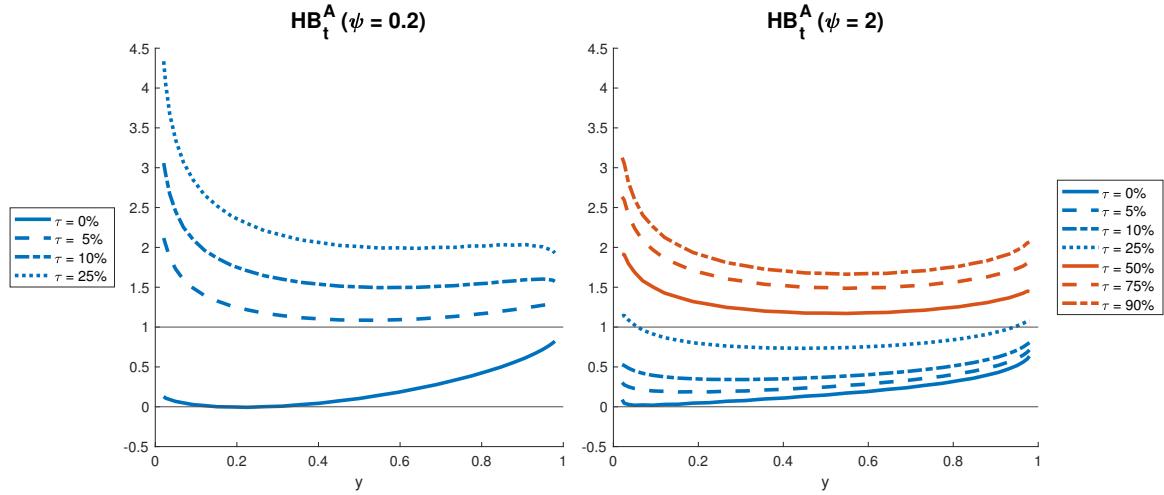
In addition, contrary to the baseline studied so far, the riskless bond is now traded in equilibrium (Panel (a) of Figure 11), reflecting the fact that less risk sharing can happen via the equity assets so that investors make use of the third asset. Bond

⁶⁵This is slightly higher than the CRRA case, $\psi = 1/\gamma \approx 0.067$, to ensure that investors still have preference for early resolution of uncertainty.

⁶⁶This is even more visible on the Sharpe ratio on the top right panel of Figure 10, which combines the effect on the risk premia and second moments. While the risk premium and Sharpe ratio on the first asset increased with x_t in the baseline, even with $\psi = 0.2$, they now both increase with it under imperfectly integrated markets, reflecting the now positive relationship between x_t and y_t discussed below.

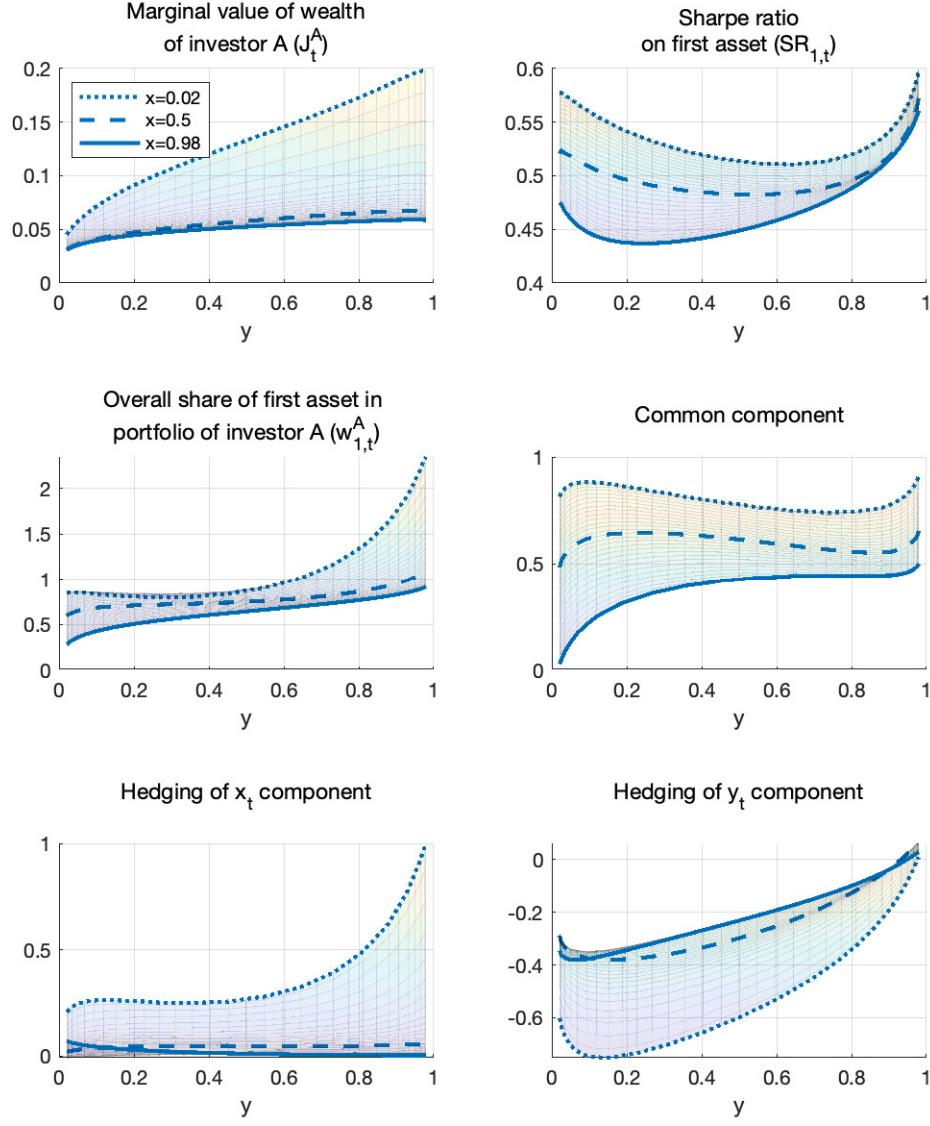
trading is also strongly asymmetric. The share of wealth allocated to the bond, b_t^A, b_t^B , strongly decreases as the wealth share of an investor increases: an investor cannot borrow from herself when she becomes dominant in the economy, a fact that participates in reinforcing the influence of the wealth share on portfolios. On the other hand, whether investors save or borrow using the bond is governed by the relative supply of their preferred good. For instance, investor A saves using the bond ($b_t^A > 0$) when good 1 is abundant. This reflects the fact that she can consume a lot of her preferred good, so that her marginal value of wealth is low. As a result, she saves some of her wealth for situations in which this is not the case, and in which her marginal value of wealth is higher. Conversely, investor A borrows as y_t decreases.

Figure 9: Equity portfolio of investor A vs. market portfolio



Notes: Based on the symmetric calibration of Assumption 1, except for ψ and τ . The figure shows a cut in which the allocation of wealth is symmetric ($x_t = 1/2$). y_t is the relative supply of the first good, which captures fundamentals. Effect on the “foreign bias” measure FB_t^A : Figure F.18.

Figure 10: Impact of the wealth share under imperfectly integrated markets ($\tau = 10\%$) with a low elasticity of intertemporal substitution ($\psi = 0.2$)



Notes: Based on the symmetric calibration under perfect risk sharing of Assumption 1, except that $\psi = 0.2$ and $\tau = 10\%$. x_t is the wealth share, which captures the share of total wealth held by investor A . y_t is the relative supply of the first good, which captures fundamentals.

The stark switch in the tilt of portfolios comes with a larger impact of the wealth share.⁶⁷ This stems in part from its reinforced direct effect as a state variable: the identity of the investor holding most of the wealth in the economy, captured by x_t , matters more when risk sharing is imperfect because investors have a more difficult time insuring against risks. This direct impact can be observed on portfolios as well as other variables in Figure 10 for a tax of $\tau = 10\%$.⁶⁸ Second, the impact of the hedging of wealth share risk also grows markedly. Quantitatively, the variance decomposition of $w_{1,t}^A$ yields shares of 37%, 33%, and 35%, for the common, x_t -hedging, and y_t -hedging components, respectively. The wealth share hedging therefore plays a much larger role now on par with other components, compared to the 7% it was responsible for in the baseline.⁶⁹ Further, not only does the magnitude of the hedging of x_t changes, but so does its sign. While the hedging of fundamentals still make any investor dislike their local asset (as long as goods are realistically good enough substitutes), the hedging of x_t is now positive, meaning that instead of reinforcing the “foreign bias” coming from y_t like it did in the baseline, it directly contributes to obtaining the “home bias” in equity holdings.

This happens because in this case, the loading of the wealth share on the Brownian shocks flips, with the wealth share now *increasing* for a positive shock to the output of good 1, so that the first asset provides a good hedge for changes in the allocation of wealth for investor A . This flip occurs because of the overpowering effect of imperfect financial integration on portfolios: by making the first asset more attractive, the tax yields a “home bias” in equity holdings in equilibrium so that compared to the market portfolio, $w_{1,t}^A - z_t > 0$ and $w_{2,t}^A - (1 - z_t) < 0$, the opposite of the baseline case. Following Proposition 1, this results in $\sigma_{xz_1,t}x_t > 0$, $\sigma_{xz_2,t}x_t < 0$. In words, the wealth share of investor A loads positively on shocks to the output of good 1, and negatively on shocks to the output of good 2.⁷⁰ Finally, note that instead of being

⁶⁷As before, portfolios also vary significant with the relative supply of goods. For instance, HB_t^A ranges from 1.25 to above 3, depending on whether good 1 becomes abundant ($y_t \rightarrow 1$) or scarce ($y_t \rightarrow 0$).

⁶⁸Note that with $\psi = 0.2$, the actual marginal value of wealth, $J_t^{i \frac{1-\gamma}{1-\psi}}$, is a decreasing monotonic transformation of J_t^i .

⁶⁹For the second asset, those numbers are 18%, 32%, and 27%, with the remaining 23% attributed to the tax payment itself. For the baseline those were of 70%, 7%, and 23%, for both $w_{1,t}^A$ and $w_{2,t}^A$.

⁷⁰This switch in sign can be observed by comparing Figures F.25 and F.28 in Appendix F. Figure F.4 shows the corresponding distribution, obtained as before by simulating the economy for 250

broadly symmetric around $x_t = 1/2$, the hedging of wealth share risk now tends to decrease with x_t , reflecting its larger impact on the slope of J_t closer to small values of x_t .

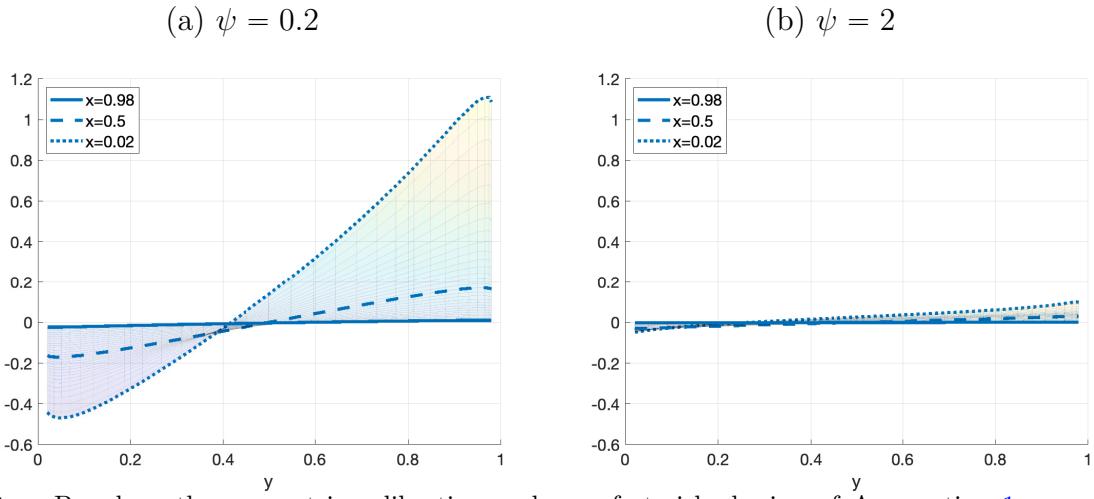
Turning now to the case where the elasticity of intertemporal substitution is high, $\psi = 2$, as in the (realistic) baseline calibration of Section 3, the picture changes drastically. As seen on Panel (b) and (d) of Figure 9, taxes on “foreign” dividends now have a much more limited impact on portfolios. As an example, reasonable taxes on the order of $\tau = 7$ or 10% do not overturn the counterfactual “foreign bias” in equity holdings in an international context, like they did for $\psi = 0.2$, let alone bringing it to the ballpark estimate observed in practice. In fact, for this to happen even only for the symmetric point of the state space, τ has to climb to values as high as 50, 75%, or more, which are clearly implausible.

Why does the elasticity of intertemporal substitution have such a central role? Quantitatively, this comes for a large part from its impact on the dividend yields, $F_{1,t}, F_{2,t}$, of the two equity assets. With a large ψ , substitution effects dominate so that investors value assets even when they pay far in the future. The resulting equity prices, which are nothing but the present value of the streams of dividend paid by the assets discounted with the appropriate stochastic discount factors, therefore tend to be larger compared to dividends given that even far-away payments are highly valued. The resulting dividend yields, $F_{1,t}, F_{2,t}$, which divide dividends at market values, $p_{1,t}Y_{1,t}, p_{2,t}Y_{2,t}$, by equity prices, $Q_{1,t}, Q_{2,t}$, are therefore significantly smaller on average (Figure F.30). Because the impact of the tax on “foreign” dividends is ultimately governed by the magnitudes of the dividend yields given that the differences in equity premia as perceived by the two investors are $-\tau^B F_{1,t}, -\tau^A F_{2,t}$, their quantitative ability to impact risk premia, and therefore portfolios, is therefore much more

years. Accordingly, the relationship between fundamentals y_t and the wealth share x_t is now positive. This switch can of course have long-term consequences in terms of the surviving agent in the very long run: e.g. if good 1 becomes dominant in the long run, investor A will tend to dominate the economy under modest degrees of imperfect financial integration, while investor B would have dominated under perfect risk sharing. Imperfect financial integration also have an effect on the dispersion of the wealth share around its broad relationship with y_t : as τ increases, x_t moves further away, underlying the fact that when financial markets are imperfectly integrated, investors have a more difficult time sharing risk with each other. This is the result of the effect of market imperfection on risk premia and portfolios, described above, but also on consumption, driven by J_t , and taxes themselves. Figures F.24 and F.26 in Appendix show $\mu_{x,t}x_t$ and its decomposition into its several components, for $\psi = 0.2$ and $\tau = 10\%$.

limited when ψ is large. Economically, this happens in part because as ψ increases, variations in the wealth share beyond the broad direction given by fundamentals (y_t) are larger (this is true even under perfect risk sharing), so that both state variables have a clear distinct role and two equity assets are required to hedge against changes in both of them. Relatedly, this result is also connected to the fact that the extent of bond trading, which becomes important when risk sharing is imperfect, becomes much more limited as ψ increases as seen in Panel (b) of Figure 11. Interestingly, the tax on “foreign” dividends has a limited effect in that case, even though the diversification benefits provided by the two equity assets appear smaller with an average correlation of returns of 0.91 against 0.77 with $\psi = 0.2$. This reflects the fact that looking at average correlations might not be an accurate enough measure of diversification benefits in contexts in which the correlation is inherently state-dependent like here.

Figure 11: Share of bond in the portfolio of investor A



Notes: Based on the symmetric calibration under perfect risk sharing of Assumption 1, except for ψ and $\tau = 10\%$. x_t is the wealth share, which captures the share of total wealth held by investor A . y_t is the relative supply of the first good, which captures fundamentals.

The impact of imperfect financial integration also depends on the bias in consumption and risk aversion that both increase the impact of the wealth share, the former by making investors more heterogeneous and therefore less able to share risk, and the latter by increasing the impact of the effect on investors’ decisions. Like the main mechanism above, this also occurs when ψ is high, but is significantly more

muted. Overall, those pieces of evidence point to the significant role played by the several dimensions of preferences in modulating the effect of the wealth share on the equilibrium, and therefore the effect of imperfect financial integration, a fact that could not have been studied so far in the literature given that it focused for the most part on special cases.

Overall, imperfectly integrated markets have a profound impact on the equilibrium, which is intimately related to the rising influence of the wealth share. This is consistent with investors no longer being able to share risk perfectly when there are frictions in market integration, so that the identity of the investor holding most assets in equilibrium, captured by the wealth share, matters more. Taken together, those results emphasize the intricate interplay between portfolio choices, asset prices, and risk sharing in this context, and imperfect financial integration has the potential to strikingly change the portfolios of investors. To be sure, matching portfolios throughout the state space e.g. in an international context is no easy fit, and as we have seen, portfolios remain strongly state-dependent, in fact even more so than in the specification with perfect risk sharing studied thus far. In addition, whether imperfect financial integration has a strong enough impact to overturn the “foreign bias” in equity holdings that obtains in the baseline (provided that goods are good enough substitutes) depends significantly on the calibration of preferences, a fact that we have been able to uncover thanks to the generality of the framework. Yet, with those caveats in mind, imperfect financial integration of the form studied in this section remains a realistic and plausible way to generate a home bias in equity holdings broadly in line with the data – both qualitatively, and quantitatively for a relevant part of the state space. It is therefore adequate for those purposes, and I focus on this specification for the international application of Section 5.1.

Investor heterogeneity The heterogeneity of investors is another factor that strongly reinforces the influence of the wealth share on the equilibrium, not only conceptually but also quantitatively.

This was already apparent in the analysis of the baseline calibration studied so far in which an increase in the bias in consumption, which constitutes the fundamental heterogeneity in the economy, increases the impact of the wealth share significantly. This is true of both the direct effect of the wealth share as a state variable capturing

the average investor in the economy, and of the hedging of wealth risk, which becomes as important a determinant of portfolios as other components (Table 1).

Labor income is another way to introduce heterogeneity in the framework, while remaining in a symmetric calibration. This happens because labor income, modeled here as a constant share (δ_1, δ_2) of the output of each tree being paid to the “local” investor who prefers that good, makes the budget constraint of each investor more dependent on local conditions. This analysis is relegated to Appendix A.7 in the interest of space but labor income has a strong impact on the equilibrium and its underpinnings. While its effect on risk premia and Sharpe ratios is somewhat modest, it significantly affects portfolios, marginal values of wealth, consumptions, and the interest rate.⁷¹ Most importantly, and in line with the emerging theme of this section, this effect goes hand-in-hand with a bolstered importance for the wealth share. As a stark example, the share of portfolio variance explained by the hedging of x_t increases from 7% in the baseline without labor income, to a whooping 70% for $\delta = 62.5\%$, a calibration roughly in line with the average labor share in the United States over the last 50 years. On the contrary, the common and y_t -hedging components now explain a mere 20% and 10%, instead of 70% and 23% in the baseline. In short: the hedging of wealth share risk becomes the main driver of the shape of portfolios. More generally, the direct effect of the wealth share as a state variable is also greatly reinforced.⁷²

⁷¹For portfolios specifically, labor income reinforces the bias in portfolio holdings. This comes from the fact that labor income is perfectly correlated with the payoff of the local asset, so that it renders each asset yet more attractive/unattractive to the local investor depending on the elasticity of substitution across goods. In the baseline for instance, if we focus on an international interpretation, labor income reinforces the foreign bias in equity holdings on average so that “The International Diversification Puzzle Is Worse Than You Think” (Baxter and Jermann, 1997). Importantly, this effect is also strongly state-dependent, and in particular relevant as the wealth share of an investor gets small.

⁷²The way those patterns change when considering a more general and realistic specification for labor income could prove an interesting exploration. One particular specification could be to construct labor income as a time-varying share of the output of each country, as explored for instance in Coeurdacier and Gourinchas (2016). As the authors suggest, the correlation of labor income with output, once computed with the proper conditioning, could in fact turn out to be negative, providing a natural way to generate a “home bias” in equity holdings. If the share is itself stochastic, it could also provide an additional hedging motive that could prove relevant in practice also as it introduces a natural degree of market incompleteness. Labor income could also take a more general form, for instance as a separate source of idiosyncratic risk in the spirit of the recent heterogeneous-agent macroeconomic literature like Kaplan et al. (2018), or by introducing a distribution of investors of each type by generalizing the overlapping generation structure of Gârleanu and Panageas (2015) to a two-good, two-country setting. I leave these promising avenues for future research.

In summary, the heterogeneity of investors therefore makes the wealth share an important variable of interest in this framework. This aspect will also be particularly apparent in the international application of Section 5.1 in which a different kind of heterogeneity, in the form asymmetries in preferences, takes center stage.

Taking stock, the characterization of asset prices and global portfolios emphasizes the importance of the allocation of wealth in this general economy. The allocation of wealth matters both as a state variable that captures the average investor, and as a pricing factor against which investors hedge. The magnitude of the impact of the allocation of wealth can grow substantially with imperfect financial integration, and when investors become more heterogeneous. In other words, “capital is back” in this context too: consistent with a broader emerging literature in economics, the allocation of capital, here across investors, has a prime role in determining economic outcomes.

In terms of portfolios, “home bias” in equity holdings can obtain in the setup either when the elasticity of substitution across goods is low, or due to imperfect financial integration provided that the elasticity of intertemporal substitution is moderate. Again, in the international application of Section 5.1, I focus on the latter case, which is both realistic – international markets are likely to be imperfectly integrated in practice –, and because it is more consistent with standard estimations of the elasticity of substitution across goods. As we have seen in that case, the home bias is amplified by the hedging of the wealth share risk. More generally, portfolios as well as other variables strongly vary throughout the state space, emphasizing the importance of the global solution. All those aspects are present and even reinforced in the applications of the model to which I now turn.

5. Applications

Because of its generality, the “222” framework in this paper represents a versatile building block towards several applications and extensions. I briefly present two applications: to the international financial system in Section 5.1, and to sustainable and environmental topics in Section 5.2. Both are developed in ongoing work ([Sauzet](#),

2021a,b). Section 5.3 mentions a number of other applications and promising extensions, some of which may require higher-dimensional resolution methods such as the “projection methods via neural networks” developed in Sauzet (2021c).

5.1. Application 1: the International Financial System

A prominent application to which I have referred throughout, and of which I have sometimes borrowed the terminology, is the international portfolio choice problem, which consists in modeling the portfolio decisions of investors in different countries, which typically have a preference towards their own local good. Taking investors A and B to be the representative investors of two countries, domestic and foreign, the advances of the framework in this paper allow me to characterize the *general* and *global* solution to this international portfolio choice problem, which had been a long-standing open issue in the literature.

This literature has a long and distinguished history, and my purpose here is not to survey it (cf. Obstfeld, 2007, Coeurdacier and Rey, 2013, for an overview). Compared to a recent and large part of that literature, such as Corsetti et al. (2008), Tille and van Wincoop (2010), Coeurdacier (2009), Devereux and Sutherland (2011), Evans and Hnatkovska (2012), Coeurdacier and Rey (2013), Coeurdacier and Gourinchas (2016), I bring (i) a solution that is global and does not rely on approximations. This allows to complete the picture and trace out the evolution of economic outcomes as we move away from the point of approximation (typically the symmetric point), which proves important in this context where variables are strongly state-dependent and potentially non-linear. I also bring (ii) general preferences, which allow to move away from special cases and study all situations under a unified framework (cf. Pavlova and Rigobon, 2007, 2008, 2010, Stathopoulos, 2017 for special cases). A limited number of contributions have relied on global methods in similar settings e.g. Kubler and Schmedders (2003) (one country), Stepanchuk and Tsyrennikov (2015) (one good), Rabitsch et al. (2015), and Coeurdacier et al. (2020) (one good). To those, I bring (iii) continuous-time methods, which make it possible to study portfolio drivers, in particular hedging demands, asset prices and their conditional first and second moments, as well as the determinants of wealth and state variable dynamics, in ways that are inaccessible in a discrete-time formulation and therefore make continuous-time the

natural tool of choice to study this type of questions. Finally (iv), to all, in addition to labor income as in [Baxter and Jermann \(1997\)](#) and asymmetries in preferences, I bring imperfect financial integration, which is an important topic in international finance but had not been studied thus far in a general international portfolio choice context.⁷³

Taken together, those innovations make it possible to revisit a number of results in the literature under a unified framework. First, the calibration of preferences can have a significant impact. For instance, the discussion of portfolios under imperfect financial integration in Section 4 confirm but qualify the findings in [Bhamra et al. \(2014\)](#) in this general setting with non-log preferences and home bias in consumption: imperfectly integrated markets *can* deliver portfolios consistent with the data *provided that* the elasticity of intertemporal substitution is moderate. Second, the global solution shows that a number of variables, chief among them portfolios, can be strongly state-dependent, even under perfectly integrated markets, so that relying on local approximations can be problematic. Lastly, my discussion throughout emphasizes the key role that can be played by the allocation of wealth for global portfolios and asset prices, an aspect on which that literature had put little emphasis thus far.

Beyond reassessing various results in the literature, the ability of the framework to handle truly general preferences, including asymmetries, allows it to reproduce a number of stylized facts about the structure and dynamics of the international financial system, and in particular the role of the United States, and of asset returns in this context. I discuss this application, as well as some of the results above, in detail in [Sauzet \(2021a\)](#), and I therefore only provide a brief summary below.

The domestic country is now taken to represent the United States, the country at the center of the international financial system, and its representative investor is assumed to display a higher tolerance for risk. This assumption, in the spirit of [Caballero et al. \(2008\)](#), [Gourinchas et al. \(2017\)](#), and [Maggiori \(2017\)](#), is meant to capture the greater development and depth of U.S. financial markets. Like in [Gourinchas et al. \(2017\)](#), and [Maggiori \(2017\)](#), by making the country as a whole better able and willing to carry financial risk in the world economy, this asymmetry naturally

⁷³More general specification of labor such as a time-varying share in the spirit of [Coeurdacier and Gourinchas \(2016\)](#) or idiosyncratic labor income risk as in [Kaplan et al. \(2018\)](#) are interesting avenues for further exploration.

replicates its average external position (Fact 1, [Gourinchas and Rey, 2007b](#)): the United States plays the role of the world banker, by borrowing in safe securities from the rest of the world, and investing in risky assets internationally. This large negative net foreign asset position is associated with higher excess returns on the external balance sheet of the country on average, given the higher share of risky assets that pay more in expectation: this is the exorbitant privilege of the world banker (Fact 2, [Gourinchas et al., 2017](#)). Importantly, the economy also still features two meaningfully different equity assets and a modest degree of imperfect financial integration delivers a home bias in equity holdings broadly consistent with empirical observations (Fact 4, [Coeurdacier and Rey, 2013](#)).

The framework does not only replicate facts about external portfolios on average however, and the asymmetry in risk tolerance yields a number of predictions about the *dynamics* of the international financial system that are strongly borne out in the data. As a crisis hits, the center country is impacted particularly severely due to its high allocation to risky assets, so that it transfers a large amount of wealth to the rest of the world. This exorbitant duty is the flip side of its exorbitant privilege in normal times: the United States must become the world insurer in times of trouble (Fact 3, [Gourinchas et al., 2017](#)). In addition, by worsening the wealth position of the risk-tolerant world banker, the shock leads to a sharp increase in global risk aversion, which in turn pushes up all risk premia and Sharpe ratios worldwide. These two markers are reminiscent of some aspects of the Global Financial Cycle (Fact 5, [Rey, 2013](#), [Miranda-Agrippino and Rey, 2020](#)), for which a general equilibrium exploration had remained elusive. Those patterns are representative of the type of global risk-off scenarios that typically occur in times of global crisis such as most recently in the Great Recession of 2008 or the Global Pandemic of 2020-2021.

In addition, the model allows to study the evolution of portfolios as a response to those shocks, and can shed light on the process of external adjustment of the center country. For the latter, while its net foreign asset position strongly deteriorates following the shock, the sharp increase in risk premia that occurs simultaneously highlights the role of valuation effects as proposed in [Gourinchas and Rey \(2007a\)](#) in this situation: the higher expected returns on its global portfolio ease some of the pressure on the domestic country to balance its external position in the short term. This negative relationship between net foreign asset position and expected risk premia therefore replicates the type of predictability relationship between the two

documented in [Gourinchas and Rey \(2007a\)](#), and extended to more recent data in [Gourinchas et al. \(2019\)](#) (Fact 6).⁷⁴

From an asset pricing perspective, the model speaks to a number of facts about asset returns dynamics in this international environment. Namely, risk premia, Sharpe ratios – and to some extent volatilities and correlations in a relevant region of the state space – are all countercyclical in the sense that they increase following the shock, consistent with a wide range of evidence notably for the United States (Fact 7, [Lettau and Ludvigson, 2010](#), among others). Those patterns are the reflection of the type of dynamics emerging in asset pricing settings with heterogeneous agents (e.g. [Weinbaum, 2009](#)), in an economy in which there are also two goods, two assets, and a home bias in consumption. Importantly, those patterns are driven for a large part not by changes in the quantity of risk but by the evolution of the compensation for risk, captured here by the time-varying global risk aversion. This is in line with a large literature that has seen changes in the price of risk emerge as a crucial explanation behind asset return predictability more generally.

Another value of studying those questions in the general framework of this paper is that it allows to perform a number of counterfactual exercises. For instance, I show that a mild decrease in the frictions in international markets can generate the secular decrease in home bias that has been documented in recent decades (Fact 4, [Cœurdacier and Rey, 2013](#)), as well as some of the increase in the financial synchronization that has been observed throughout the world over a long-time horizon but particularly in the last three decades (Fact 8, [Jordà et al., 2019](#)). A re-interpretation of the model at a lower frequency could also be used to make sense of the secular decline in interest rate that has been observed worldwide, provided that the wealth share of the domestic risk-tolerant country decreases in the long run (Fact 9, [Caballero et al., 2008](#), [Hall, 2016](#)). Finally, changes in the tax on foreign dividends, potentially asymmetric, could also be used to study the impact on global asset prices, portfolios, and risk sharing, of macroprudential policies aimed at curbing sudden international capital flows. In addition, studying the evolution of portfolios with the state of the economy could potentially provide some rationale for the so-called Great Retrenchment of capital that followed the financial crisis of 2007-2008 ([Milesi-Ferretti](#)

⁷⁴In the long run, the higher share of risky assets in the domestic portfolio also leads the domestic country to grow in world wealth and its net foreign asset position to become positive. This further alleviates the burden on the necessity of short-term adjustment in times of crisis, and allows the country to run a more negative net foreign asset position for a while.

and Tille, 2011).

In summary, a seemingly small change in the specification of the model – the introduction of asymmetries in risk tolerance – generates a vast number of facts about the structure and dynamics of the international financial system and of asset returns, which are strongly borne out in the data. Between this version of the model, and a simpler, symmetric calibration discussed previously, it is therefore particularly adapted to study portfolio choice questions in an international context.

5.2. Application 2: sustainable and environmental finance

The framework that I develop in this paper can also be used to study a completely different set of questions, in the domain of environmental finance.⁷⁵

Following Guesnerie (2004), Hoel and Sterner (2007), Sterner and Persson (2008), Gollier (2010), Traeger (2011), Barro and Misra (2016), and Gollier (2019), the two goods can indeed be taken to represent aggregate economic capital (physical capital, labor, scientific knowledge, *etc.*) on one hand, and various ecosystem services that are generated by natural capital on the other.

First and contrary to most of this literature, my framework embeds not only two goods and two trees, but also *two investors* with possibly heterogeneous preferences towards the goods. It can therefore allow to study not only how relative prices can be crucial for the pricing of the ecological services provided by natural assets, as has been discussed in the literature, but also how this pricing interacts with the allocation of wealth across investors. This can for instance make it possible to connect environmental issues to those of economic inequality in which one group of investors is holding an increasingly larger share of total wealth. One can also study the impact of having investors with different preferences towards environmental goods, which is likely relevant in practice.

Second, because I solve the model using a global solution method, it is possible to study evolutions of the economy even in corners of the state space that might be rarely visited in equilibrium. Those states might prove relevant in case of catastrophic events that could be related to climate change in the future. In addition, the

⁷⁵I am grateful to Christian Gollier for this suggestion.

introduction of imperfectly integrated markets could prove important, to allow for the possibility of imperfect risk sharing so that some states might be more difficult or costly to insure against. Because I base the resolution of the framework on the decentralized equilibrium, all of those cases are covered in the technology that I propose in this paper.

Lastly, allowing for general (and potentially asymmetric) preferences makes it possible to study the dependence of various results in this literature on their calibration. Augmenting those preferences, e.g. with a time-varying degree of substitutability across goods as in [Gollier \(2019\)](#), or along several dimensions, could prove particularly interesting.

I explore these and other related aspects in ongoing work ([Sauzet, 2021b](#)).

5.3. Other applications and extensions

The model could also be applied to a number of other topics, such as modeling different sectors of the economy, in the spirit of [Menzly et al. \(2004\)](#), [Santos and Veronesi \(2006\)](#), or assessing the resilience of the global financial system when one country, e.g. the United States, has an exorbitant privilege.⁷⁶

Beyond those applications, the framework is also a well-suited building block for many potential extensions. For instance, one can study a generalization of this economy with N investors, M trees, and L goods (“NML” model, explored in ongoing work, [Sauzet, 2021e](#)). By having more than two trees, one can explore limiting their trading to a subset of investors so as to introduce a natural source of market incompleteness that can fundamentally impact the equilibrium. Relatedly, this could also be achieved by allowing for more general specifications of the share of labor income or taxes, for instance by letting them be stochastic. Another natural extension would be to consider a production economy in which investors can also directly influence

⁷⁶This could for instance be done by studying the dynamics of volatilities and correlations in such an economy. I am grateful to Harjoat Bhamra for this suggestion. The heterogeneity in risk aversion, e.g. as studied in [Bhamra and Uppal \(2009, 2014\)](#) and [Schneider \(2021\)](#), will be crucial for this analysis.

the supply of their preferred good, or more general endowment structure such as disaster or long-run risk. Various ways of making the model stationary could also be interesting to explore.⁷⁷

Because I solve for the decentralized solution throughout, the framework is readily set to tackle more general market structures beyond imperfect risk sharing such as incomplete markets that would arise in the presence of idiosyncratic labor income risk as in [Kaplan et al. \(2018\)](#), or capital risk as in [Brunnermeier and Sannikov \(2014, 2015\)](#). Particularly interesting and relevant in this context will also be the addition of constraints on the portfolios of investors, e.g. by adapting [Gârleanu and Pedersen \(2011\)](#), [Chabakauri \(2013\)](#) to my “222” economy, which could lead to a strong reinforcement of the type of dynamics discussed in the paper. Taken together, those different channels will likely lead to a strengthening of the dispersion and role of the wealth share in equilibrium.

The framework can also be extended along more ambitious dimensions. The most promising among them relate to the introduction in this setting of the type of financial intermediaries that have been discussed in the recent intermediary asset pricing literature e.g. in [Daníelsson et al. \(2012\)](#), [He and Krishnamurthy \(2013\)](#), [Adrian and Shin \(2014\)](#), or [Adrian and Boyarchenko \(2015\)](#). Those global intermediaries, which are very relevant in practice, can be involved in the dealing of foreign currencies, in the spirit of [Hau and Rey \(2006\)](#) and [Gabaix and Maggiori \(2015\)](#), or can play the role of bankers as in [Maggiori \(2017\)](#) and [Jiang et al. \(2020\)](#). As an illustration, in ongoing work ([Sauzet, 2021d](#)), I explore a third possibility: the introduction of a global asset manager. This addition is briefly described in Appendix E.1 and could help capture additional aspects of the Global Financial Cycle of [Rey \(2013\)](#) and [Miranda-Agrippino and Rey \(2020\)](#), pertaining to the leverage and role of global financial intermediaries. The combination of global financial intermediaries with time-varying demand for safe assets, which could be generated by the introduction of multiple heterogeneous investors *within* each country, could also help make way towards a resolution for the so-called “reserve currency paradox” emphasized by

⁷⁷This could be done e.g. by adapting the share process of [Menzly et al. \(2004\)](#), [Santos and Veronesi \(2006\)](#) to y_t so that neither of the goods and assets dominates the economy in the long run, which could also ensure the survival of both investors. Another possibility could be to adapt the overlapping-generations structure of [Gârleanu and Panageas \(2015\)](#) to my multi-good multi-tree context.

[Maggiori \(2017\)](#). I briefly touch upon this question in Appendix E.2 and it is also explored in ongoing work ([Sauzet, 2021f](#)).

Finally, from a methodological standpoint, the number of state variables is likely to rapidly increase with those extensions. Because computationally traditional projection methods are very much subject to the curse of dimensionality, higher-dimensional methods will be required. For instance, even the addition of a third state variable, like in the global asset manager extension, renders the resolution significantly slower, and increasing the order of approximation much beyond $N = 10$ proves difficult.⁷⁸ One such method consists in naturally extending the concept of projection approaches, but to replace the Chebyshev polynomials in the approximation by neural networks, which are designed specifically for high-dimensional settings. I am developing these “projection methods via neural networks” for continuous-time models in [Sauzet \(2021c\)](#). I discuss them in slightly more details in Section E.3, and they should prove very useful as I pursue yet more ambitious extensions.

In summary, the framework in this paper is well-suited to handle several applications and extensions. The combination of these extensions mentioned with higher-dimensional resolution approaches such as the “projection methods via neural networks” developed in [Sauzet \(2021c\)](#) provide many promising avenues for future research.

6. Conclusion

In this paper, I characterize the global solution to the portfolio problem of two heterogeneous investors with general preferences, in a two-tree, two-good environment. One of the main economic messages that emerges from that characterization is that the allocation of wealth across investors matters in this general portfolio choice setting. This finding resonates with an emerging theme in the broader economic literature that has recently emphasized the role of the wealth distribution in determining economic outcomes in macroeconomics (e.g. [Brunnermeier and Sannikov, 2014](#), [Kaplan et al., 2018](#)), finance (e.g. [Gomez, 2017](#), [Lettau et al., 2019](#), [Greenwald et al., 2020](#)), and

⁷⁸Finer ways to construct the Chebyshev polynomials and corresponding grids, such as complete polynomials or Smolyak’s algorithm, can help. Ultimately however, they are also limited.

economics more generally (e.g. [Piketty and Zucman, 2014](#)). In other words, “capital is back” in this setting too: the allocation of wealth across investors has a prime role in driving asset prices, portfolios, and risk sharing, an aspect that had received little emphasis thus far.

To derive this result, I adapt recent advances in multi-agent continuous-time asset pricing models to a two-investor, two-tree, two-good economy in which investors have recursive preferences and a bias in consumption towards a preferred good. This allows me to move away from the special cases in terms of preferences that have been the focus of most of the (multi-good) portfolio choice literature. In addition, while most contributions especially in an international context have relied on so-called low-order local approximations in which portfolios are constant, I solve the model using a global solution method. This approach makes it possible to fully trace out the evolution of economic variables with the state of the economy, in sharp contrast to local methods that mostly capture evolutions in a small neighborhood of a specific state.

The allocation of wealth matters both as a state variable that captures the average investor in the economy and directly impacts economic outcomes, and as a pricing factor that is hedged by investors. Its effect is relevant even in a baseline with symmetric calibration and perfect risk sharing, but grows tremendously as markets become imperfectly integrated, and as investors become more heterogeneous. The results also emphasize both (i) the state-dependence of most economic variables in this environment – e.g. portfolios vary substantially with the allocation of wealth –, and (ii) the vital impact of the calibration of preferences – e.g. the potency of imperfect financial integration is strongly reduced with a high elasticity of intertemporal substitution. This makes the novel framework presented in this paper, which is based on a global solution method and allows for general recursive preferences including asymmetries, particularly adapted to study this economy.

Because of its generality, the “222” framework in this paper represents a well-suited building block towards several applications and extensions.

A first and prominent one, which I explore in [Sauzet \(2021a\)](#), is international finance. The framework indeed allows to characterize the global solution to the international portfolio problem in full generality, which had been a long-standing open issue in the literature. More generally, the ability of the framework to handle truly

general preferences, including asymmetries, also allows it to reproduce a number of stylized facts about the structure and dynamics of the international financial system, and of asset returns in this context, which are strongly borne out in the data.

The framework can also be used to study a completely different set of questions, in the domain of environmental finance, with the two goods taken to represent aggregate economic capital on one hand, and various ecosystem services generated by natural capital on the other.⁷⁹ Importantly, because my framework embeds not only two goods and two trees, but also *two investors* with possibly heterogeneous preferences towards the goods, it can allow to study not only how relative prices can be crucial for the pricing of the ecological services provided by natural assets, as has been discussed in the literature, but also how this pricing interacts with the allocation of wealth across investors. This can for instance make it possible to connect environmental issues to those of economic inequality in which one group of investors is holding an increasingly larger share of total wealth. One can also study the impact of having investors with different preferences towards environmental goods, which is likely relevant in practice. I explore this application in ongoing work ([Sauzet, 2021b](#)).

The model is also a well-suited building block for many potential extensions, such as a generalization to N investors, M trees, L goods (“NML” model, ongoing work in [Sauzet, 2021e](#)), or the addition of a production side or portfolio constraints. Promising among these extensions are those related to the introduction in this setting of financial intermediaries of the type that has been discussed in the recent intermediary asset pricing literature, and illustrations were briefly discussed in Section 5.3 e.g. with the inclusion of a global asset manager ([Sauzet, 2021d](#)). The implementation of those extensions will benefit from the fact that I am solving for the *decentralized* equilibrium of this economy so that the framework is readily set to tackle a wide range of market structures. It will also likely require higher-dimensional methods such as the “projection methods via neural networks” being developed in [Sauzet \(2021c\)](#). I leave all these promising avenues for future research.

⁷⁹I am grateful to Christian Gollier for this suggestion.

References

- Adrian, Tobias, and Nina Boyarchenko, 2015, Intermediary Balance Sheets, 2015 Meeting Papers 239, Society for Economic Dynamics.
- Adrian, Tobias, and Hyun Song Shin, 2014, Procyclical leverage and value-at-risk, *Review of Financial Studies* 27, 373–403.
- Anderson, Evan W., 2005, The dynamics of risk-sensitive allocations, *Journal of Economic Theory* 125, 93 – 150.
- Anderson, Robert M., and Roberto C. Raimondo, 2008, Equilibrium in continuous-time financial markets: Endogenously dynamically complete markets, *Econometrica* 76, 841–907.
- Backus, David K., and Gregor W. Smith, 1993, Consumption and real exchange rates in dynamic economies with non-traded goods, *Journal of International Economics* 35, 297 – 316.
- Bansal, Ravi, and Amir Yaron, 2004, Risks for the long run: A potential resolution of asset pricing puzzles, *The Journal of Finance* 59, 1481–1509.
- Barro, Robert J., and Sanjay Misra, 2016, Gold Returns, *The Economic Journal* 126, 1293–1317.
- Basak, Suleyman, and Domenico Cuoco, 1998, An Equilibrium Model with Restricted Stock Market Participation, *The Review of Financial Studies* 11, 309–341.
- Basak, Suleyman, and Michael Gallmeyer, 2003, Capital Market Equilibrium with Differential Taxation, *Review of Finance* 7, 121–159.
- Baxter, Marianne, and Urban Jermann, 1997, The international diversification puzzle is worse than you think, *American Economic Review* 87, 170–80.
- Baxter, Marianne, Urban J Jermann, and Robert G King, 1998, Nontraded goods, nontraded factors, and international non-diversification, *Journal of International Economics* 44, 211 – 229.

- Berrada, Tony, Julien Hugonnier, and Marcel Rindisbacher, 2007, Heterogeneous preferences and equilibrium trading volume, *Journal of Financial Economics* 83, 719 – 750.
- Bhamra, Harjoat S., Nicolas Coeurdacier, and Stéphane Guibaud, 2014, A dynamic equilibrium model of imperfectly integrated financial markets, *Journal of Economic Theory* 154, 490 – 542.
- Bhamra, Harjoat S., and Raman Uppal, 2009, The Effect of Introducing a Non-Redundant Derivative on the Volatility of Stock-Market Returns When Agents Differ in Risk Aversion, *The Review of Financial Studies* 22, 2303–2330.
- Bhamra, Harjoat S., and Raman Uppal, 2014, Asset Prices with Heterogeneity in Preferences and Beliefs, *The Review of Financial Studies* 27, 519–580.
- Brandt, Michael W., John H. Cochrane, and Pedro Santa-Clara, 2006, International risk sharing is better than you think, or exchange rates are too smooth, *Journal of Monetary Economics* 53, 671 – 698.
- Brunnermeier, Markus, and Yuliy Sannikov, 2014, A macroeconomic model with a financial sector, *American Economic Review* 104, 379–421.
- Brunnermeier, Markus K., and Lasse Heje Pedersen, 2009, Market liquidity and funding liquidity, *The Review of Financial Studies* 22, 2201–2238.
- Brunnermeier, Markus K., and Yuliy Sannikov, 2015, International credit flows and pecuniary externalities, *American Economic Journal: Macroeconomics* 7, 297–338.
- Brunnermeier, Markus K., and Yuliy Sannikov, 2019, International monetary theory: A risk perspective, *Working Paper* .
- Caballero, Ricardo J., Emmanuel Farhi, and Pierre-Olivier Gourinchas, 2008, An equilibrium model of "global imbalances" and low interest rates, *American Economic Review* 98, 358–93.
- Camanho, Nelson, Harald Hau, and Hélène Rey, 2018, Global portfolio rebalancing and exchange rates, Working paper.

- Campbell, John Y., 1999, Chapter 19 asset prices, consumption, and the business cycle, volume 1 of *Handbook of Macroeconomics*, 1231 – 1303 (Elsevier).
- Cass, David, and Anna Pavlova, 2004, On trees and logs, *Journal of Economic Theory* 116, 41 – 83.
- Chabakauri, Georgy, 2013, Dynamic Equilibrium with Two Stocks, Heterogeneous Investors, and Portfolio Constraints, *The Review of Financial Studies* 26, 3104–3141.
- Chan, Yeung Lewis, and Leonid Kogan, 2002, Catching up with the joneses: Heterogeneous preferences and the dynamics of asset prices, *Journal of Political Economy* 110, 1255–1285.
- Cochrane, John H., Francis A. Longstaff, and Pedro Santa-Clara, 2008, Two Trees, *The Review of Financial Studies* 21, 347–385.
- Coeurdacier, Nicolas, 2009, Do trade costs in goods market lead to home bias in equities?, *Journal of International Economics* 77, 86 – 100.
- Coeurdacier, Nicolas, and Pierre-Olivier Gourinchas, 2016, When bonds matter: Home bias in goods and assets, *Journal of Monetary Economics* 82, 119 – 137.
- Coeurdacier, Nicolas, and Hélène Rey, 2013, Home bias in open economy financial macroeconomics, *Journal of Economic Literature* 51, 63–115.
- Coeurdacier, Nicolas, Hélène Rey, and Pablo Winant, 2020, Financial integration and growth in a risky world, *Journal of Monetary Economics* 112, 1 – 21.
- Colacito, Ric, Max Croce, Steven Ho, and Philip Howard, 2018, Bkk the ez way: International long-run growth news and capital flows, *American Economic Review* 108, 3416–49.
- Colacito, Riccardo, and Mariano M. Croce, 2011, Risks for the long run and the real exchange rate, *Journal of Political Economy* 119, 153–181.
- Colacito, Riccardo, and Mariano M. Croce, 2013, International asset pricing with recursive preferences, *Journal of Finance* 68, 2651–2686.

- Cole, Harold L., and Maurice Obstfeld, 1991, Commodity trade and international risk sharing: How much do financial markets matter?, *Journal of Monetary Economics* 28, 3 – 24.
- Coppola, Antonio, Matteo Maggiori, Brent Neiman, and Jesse Schreger, 2020, Redrawing the map of global capital flows: The role of cross-border financing and tax havens, Working Paper 26855, National Bureau of Economic Research.
- Corsetti, Giancarlo, Luca Dedola, and Sylvain Leduc, 2008, International Risk Sharing and the Transmission of Productivity Shocks, *The Review of Economic Studies* 75, 443–473.
- Cuoco, Domenico, and Hua He, 1994, Dynamic equilibrium in infinite-dimensional economies with incomplete financial markets, Working paper, University of Pennsylvania.
- Danielsson, Jón, Hyun Song Shin, and Jean-Pierre Zigrand, 2012, Procyclical leverage and endogenous risk, Mimeo, LSE, <http://www.riskresearch.org>.
- Devereux, Michael B., and Alan Sutherland, 2011, Country portfolios in open economy macro-models, *Journal of the European Economic Association* 9, 337–369.
- Dou, Winston Wei, and Adrien Verdelhan, 2015, The volatility of international capital flows and foreign assets, *Working Paper* .
- Drechsler, Itamar, Alexi Savov, and Philipp Schnabl, 2018, A model of monetary policy and risk premia, *The Journal of Finance* 73, 317–373.
- Duarte, Victor, 2019, Machine learning for continuous-time economics, Available at SSRN: <https://ssrn.com/abstract=3012602> .
- Duffie, D., J. Geanakoplos, A. Mas-Colell, and A. McLennan, 1994, Stationary markov equilibria, *Econometrica* 62, 745–781.
- Duffie, Darrell, and Larry Epstein, 1992, Stochastic differential utility, *Econometrica* 60, 353–94.
- Dumas, Bernard, 1989, Two-Person Dynamic Equilibrium in the Capital Market, *The Review of Financial Studies* 2, 157–188.

- Dumas, Bernard, 1992, Dynamic Equilibrium and the Real Exchange Rate in a Spatially Separated World, *The Review of Financial Studies* 5, 153–180.
- Dumas, Bernard, Raman Uppal, and Tan Wang, 2000, Efficient intertemporal allocations with recursive utility, *Journal of Economic Theory* 93, 240–259.
- Ehling, Paul, and Christian Heyerdahl-Larsen, 2015, Complete and incomplete financial markets in multi-good economies, *Journal of Economic Theory* 160, 438 – 462.
- Evans, Martin D.D., and Viktoria Hnatkovska, 2012, A method for solving general equilibrium models with incomplete markets and many financial assets, *Journal of Economic Dynamics and Control* 36, 1909 – 1930.
- Fang, Xiang, 2019, Intermediary leverage and currency risk premium, *Available at SSRN: <https://ssrn.com/abstract=3290317>* .
- Fernández-Villaverde, Jesus, and Lawrence Christiano, 2011, Nber summer institute, computational tools & macroeconomic applications, Methods lectures.
- Gabaix, Xavier, and Ralph S.J. Koijen, 2020, Granular instrumental variables, *Available at SSRN: <https://ssrn.com/abstract=3368612>* .
- Gabaix, Xavier, and Matteo Maggiori, 2015, International liquidity and exchange rate dynamics*, *The Quarterly Journal of Economics* 130, 1369–1420.
- Gârleanu, Nicolae, and Stavros Panageas, 2015, Young, old, conservative, and bold: The implications of heterogeneity and finite lives for asset pricing, *Journal of Political Economy* 123, 670 – 685.
- Gârleanu, Nicolae, Stavros Panageas, and Jianfeng Yu, 2020, Impediments to Financial Trade: Theory and Applications, *The Review of Financial Studies* 33, 2697–2727.
- Gârleanu, Nicolae, and Lasse Heje Pedersen, 2011, Margin-based asset pricing and deviations from the law of one price, *The Review of Financial Studies* 24, 1980–2022.

- Geanakoplos, John, 1990, An introduction to general equilibrium with incomplete asset markets, *Journal of Mathematical Economics* 19, 1 – 38.
- Geanakoplos, John, and Andreu Mas-Colell, 1989, Real indeterminacy with financial assets, *Journal of Economic Theory* 47, 22 – 38.
- Geanakoplos, John D, and Heraklis M Polemarchakis, 1986, Existence, regularity, and constrained suboptimality of competitive allocations, *Essays in Honor of Kenneth J. Arrow: Volume 3, Uncertainty, Information, and Communication* 3, 65.
- Gehrig, Thomas, 1993, An information based explanation of the domestic bias in international equity investment, *The Scandinavian Journal of Economics* 95, 97–109.
- Gollier, Christian, 2010, Ecological discounting, *Journal of Economic Theory* 145, 812–829, Judgment Aggregation.
- Gollier, Christian, 2019, Valuation of natural capital under uncertain substitutability, *Journal of Environmental Economics and Management* 94, 54–66.
- Gomez, Matthieu, 2017, Asset prices and wealth inequality, 2017 Meeting Papers 1155, Society for Economic Dynamics.
- Gourinchas, Pierre-Olivier, and Hélène Rey, 2007a, International financial adjustment, *Journal of Political Economy* 115, 665–703.
- Gourinchas, Pierre-Olivier, and Hélène Rey, 2007b, From World Banker to World Venture Capitalist: U.S. External Adjustment and the Exorbitant Privilege, in *G7 Current Account Imbalances: Sustainability and Adjustment*, NBER Chapters, 11–66 (National Bureau of Economic Research, Inc).
- Gourinchas, Pierre-Olivier, Hélène Rey, and Nicolas Govillot, 2017, Exorbitant Privilege and Exorbitant Duty, Working paper, University of California, Berkeley, London Business School.
- Gourinchas, Pierre-Olivier, Hélène Rey, and Maxime Sauzet, 2019, The international monetary and financial system, *Annual Review of Economics* 11, 859–893.

- Greenwald, Daniel L, Martin Lettau, and Sydney C Ludvigson, 2020, How the wealth was won: Factors shares as market fundamentals, Working Paper 25769, National Bureau of Economic Research.
- Guesnerie, Roger, 2004, Calcul économique et développement durable, *Revue économique* 55, 363–382.
- Hall, Robert E., 1988, Intertemporal substitution in consumption, *Journal of Political Economy* 96, 339–357.
- Hall, Robert E, 2016, The role of the growth of risk-averse wealth in the decline of the safe real interest rate, Working Paper 22196, National Bureau of Economic Research.
- Hansen, Lars Peter, Paymon Khorrami, and Fabrice Tourre, 2018, Comparative valuation dynamics in models with financing restrictions .
- Hart, Oliver D, 1975, On the optimality of equilibrium when the market structure is incomplete, *Journal of Economic Theory* 11, 418 – 443.
- Hassan, Tarek, 2013, Country size, currency unions, and international asset returns, *The Journal of Finance* 68, 2269–2308.
- Hau, Harald, and Hélène Rey, 2006, Exchange Rates, Equity Prices, and Capital Flows, *The Review of Financial Studies* 19, 273–317.
- He, Zhiguo, and Arvind Krishnamurthy, 2013, Intermediary Asset Pricing, *American Economic Review* 103, 732–770.
- Heathcote, Jonathan, and Fabrizio Perri, 2002, Financial autarky and international business cycles, *Journal of Monetary Economics* 49, 601 – 627.
- Hoel, Michael, and Thomas Sterner, 2007, Discounting and relative prices, *Climatic Change* 84, 265–280.
- Hugonnier, J., S. Malamud, and E. Trubowitz, 2012, Endogenous completeness of diffusion driven equilibrium markets, *Econometrica* 80, 1249–1270.
- Imbs, Jean, and Isabelle Méjean, 2015, Elasticity optimism, *American Economic Journal: Macroeconomics* 7, 43–83.

- Jeanne, Olivier, and Andrew K. Rose, 2002, Noise Trading and Exchange Rate Regimes*, *The Quarterly Journal of Economics* 117, 537–569.
- Jiang, Zhengyang, Arvind Krishnamurthy, and Hanno Lustig, 2020, Dollar safety and the global financial cycle, Working Paper 27682, National Bureau of Economic Research.
- Jordà, Òscar, Moritz Schularick, Alan Taylor, and Felix Ward, 2019, Global financial cycles and risk premiums, *IMF Economic Review* 67, 109–150.
- Jordà, Òscar, Moritz Schularick, and Alan M. Taylor, 2016, *Macrofinancial History and the New Business Cycle Facts*, 213–263 (University of Chicago Press).
- Judd, Kenneth, 1992, Projection methods for solving aggregate growth models, *Journal of Economic Theory* 58, 410–452.
- Judd, Kenneth, 1998, *Numerical Methods in Economics*, volume 1, first edition (The MIT Press).
- Kaplan, Greg, Benjamin Moll, and Giovanni L. Violante, 2018, Monetary policy according to hank, *American Economic Review* 108, 697–743.
- Kargar, Mahyar, 2019, Heterogeneous intermediary asset pricing, Available at SSRN: <https://ssrn.com/abstract=3258119> .
- Koijen, Ralph S. J, and Motohiro Yogo, 2020, Exchange rates and asset prices in a global demand system, Working Paper 27342, National Bureau of Economic Research.
- Kollmann, Robert, 1995, Consumption, real exchange rates and the structure of international asset markets, *Journal of International Money and Finance* 14, 191 – 211.
- Kollmann, Robert, 2006, A dynamic equilibrium model of international portfolio holdings: Comment, *Econometrica* 74, 269–273.
- Kouri, Pentti, 1982, Balance of Payment and the Foreign Exchange Market: A Dynamic Partial Equilibrium Model, *Economic interdependence and Flexible Exchange Rates ed. Jagdeep Bhandari and Bluford Putnam* 116–56.

Kubler, Felix, and Karl Schmedders, 2003, Stationary Equilibria in Asset-Pricing Models with Incomplete Markets and Collateral, *Econometrica* 71, 1767–1793.

Lettau, Martin, and Sydney C. Ludvigson, 2010, Chapter 11 - measuring and modeling variation in the risk-return trade-off, in YACINE AÏT-SAHALIA, and LARS PETER HANSEN, eds., *Handbook of Financial Econometrics: Tools and Techniques*, volume 1 of *Handbooks in Finance*, 617 – 690 (North-Holland, San Diego).

Lettau, Martin, Sydney C. Ludvigson, and Sai Ma, 2019, Capital share risk in u.s. asset pricing, *The Journal of Finance* 74, 1753–1792.

Lucas, Robert, 1978, Asset prices in an exchange economy, *Econometrica* 46, 1429–45.

Maggiori, Matteo, 2017, Financial intermediation, international risk sharing, and reserve currencies, *American Economic Review* 107, 3038–71.

Maggiori, Matteo, Brent Neiman, and Jesse Schreger, 2020, International currencies and capital allocation, *Journal of Political Economy* 128, 2019–2066.

Martin, Ian, 2011, The forward premium puzzle in a two-country world, Working Paper 17564, National Bureau of Economic Research.

Martin, Ian, 2013, The lucas orchard, *Econometrica* 81, 55–111.

Mehra, Rajnish, and Edward C. Prescott, 1985, The equity premium: A puzzle, *Journal of Monetary Economics* 15, 145 – 161.

Menzly, Lior, Tano Santos, and Pietro Veronesi, 2004, Understanding predictability, *Journal of Political Economy* 112, 1–47.

Mertens, Thomas M., and Kenneth L. Judd, 2018, Solving an incomplete markets model with a large cross-section of agents, *Journal of Economic Dynamics and Control* 91, 349 – 368, Special Issue in Honour of Prof. Carl Chiarella.

Merton, Robert, 1973, An intertemporal capital asset pricing model, *Econometrica* 41, 867–87.

Milesi-Ferretti, Gian-Maria, and Cédric Tille, 2011, The great retrenchment: international capital flows during the global financial crisis, *Economic Policy* 26, 285–342.

- Miranda, Mario J, and Paul L Fackler, 2004, *Applied computational economics and finance*.
- Miranda-Agrivino, Silvia, and Hélène Rey, 2020, U.S. Monetary Policy and the Global Financial Cycle, *The Review of Economic Studies* rdaa019.
- Obstfeld, Maurice, 2004, External adjustment, *Review of World Economics (Weltwirtschaftliches Archiv)* 140, 541–568.
- Obstfeld, Maurice, 2007, International risk sharing and the costs of trade, *The Ohlin Lectures* Stockholm School of Economics.
- Obstfeld, Maurice, and Kenneth Rogoff, 2001, The Six Major Puzzles in International Macroeconomics: Is There a Common Cause?, in *NBER Macroeconomics Annual 2000, Volume 15*, NBER Chapters, 339–412 (National Bureau of Economic Research, Inc).
- Parra-Alvarez, Juan, 2018, A comparison of numerical methods for the solution of continuous-time dsge models, *Macroeconomic Dynamics* 22, 1555–1583.
- Pavlova, Anna, and Roberto Rigobon, 2007, Asset Prices and Exchange Rates, *The Review of Financial Studies* 20, 1139–1180.
- Pavlova, Anna, and Roberto Rigobon, 2008, The Role of Portfolio Constraints in the International Propagation of Shocks, *The Review of Economic Studies* 75, 1215–1256.
- Pavlova, Anna, and Roberto Rigobon, 2010, An asset-pricing view of external adjustment, *Journal of International Economics* 80, 144 – 156, Special Issue: JIE Special Issue on International Macro-Finance.
- Piketty, Thomas, and Gabriel Zucman, 2014, Capital is Back: Wealth-Income Ratios in Rich Countries 1700–2010*, *The Quarterly Journal of Economics* 129, 1255–1310.
- Polemarchakis, Heraklis M, 1988, Portfolio choice, exchange rates, and indeterminacy, *Journal of Economic Theory* 46, 414–421.

Rabitsch, Katrin, Serhiy Stepanchuk, and Viktor Tsyrennikov, 2015, International portfolios: A comparison of solution methods, *Journal of International Economics* 97, 404 – 422.

Rey, Hélène, 2013, Dilemma not trilemma: the global cycle and monetary policy independence, *Proceedings - Economic Policy Symposium - Jackson Hole* 1–2.

Santos, Tano, and Pietro Veronesi, 2006, Labor Income and Predictable Stock Returns, *The Review of Financial Studies* 19, 1–44.

Sauzet, Maxime, 2021a, Asset prices, global portfolios, and the international financial system, Working paper.

Sauzet, Maxime, 2021b, Asset prices, portfolios, allocation of wealth, and the environment, Work in progress.

Sauzet, Maxime, 2021c, Projection methods via neural networks for continuous-time models, Work in progress.

Sauzet, Maxime, 2021d, Global asset manager and the global financial cycle, Work in progress.

Sauzet, Maxime, 2021e, N investors, m trees, l goods, Work in progress.

Sauzet, Maxime, 2021f, Demand for bonds, global currency dealer, and the reserve currency paradox, Work in progress.

Schneider, Andrés, 2021, Risk sharing and the term structure of interest rates, *The Journal of Finance* .

Schorfheide, Frank, Dongho Song, and Amir Yaron, 2018, Identifying long-run risks: A bayesian mixed-frequency approach, *Econometrica* 86, 617–654.

Schroder, Mark, and Costis Skiadas, 1999, Optimal consumption and portfolio selection with stochastic differential utility, *Journal of Economic Theory* 89, 68 – 126.

Stathopoulos, Andreas, 2017, Asset Prices and Risk Sharing in Open Economies, *The Review of Financial Studies* 30, 363–415.

- Stepanchuk, Serhiy, and Viktor Tsyrennikov, 2015, Portfolio and welfare consequences of debt market dominance, *Journal of Monetary Economics* 74, 89 – 101.
- Sterner, Thomas, and U. Martin Persson, 2008, An even sterner review: Introducing relative prices into the discounting debate, *Review of Environmental Economics and Policy* 2, 61–76.
- Stulz, René M., 1983, The demand for foreign bonds, *Journal of International Economics* 15, 225 – 238.
- Tille, Cédric, 2001, The role of consumption substitutability in the international transmission of monetary shocks, *Journal of International Economics* 53, 421 – 444, Symposium on the Japanese economic slump of the 1990's.
- Tille, Cédric, and Eric van Wincoop, 2010, International capital flows, *Journal of International Economics* 80, 157–175.
- Traeger, Christian P., 2011, Sustainability, limited substitutability, and non-constant social discount rates, *Journal of Environmental Economics and Management* 62, 215–228.
- Uribe, Martin, and Stephanie Schmitt-Grohé, 2017, *Open economy macroeconomics* (Princeton University Press).
- Wang, Jiang, 1996, The term structure of interest rates in a pure exchange economy with heterogeneous investors, *Journal of Financial Economics* 41, 75 – 110.
- Weinbaum, David, 2009, Investor heterogeneity, asset pricing and volatility dynamics, *Journal of Economic Dynamics and Control* 33, 1379 – 1397.
- Zapatero, Fernando, 1995, Equilibrium asset prices and exchange rates, *Journal of Economic Dynamics and Control* 19, 787–811.

Appendix

A. Additional equations and results

A.1. Drift and diffusion terms for any variable

Remark A.1. By Itô's Lemma, the geometric drift and diffusion term for any function $g_t = g(X_t)$ are given by:

$$\frac{dg_t}{g_t} = \frac{dg(X_t)}{g(X_t)} \equiv \mu_{g,t} dt + \sigma_{g,t}^T d\vec{Z}_t \quad (\text{A.1})$$

where:

$$\mu_{g,t} = \frac{g_{x,t}}{g_t} x_t \mu_{x,t} + \frac{g_{y,t}}{g_t} y_t \mu_{y,t} + \frac{1}{2} \frac{g_{xx,t}}{g_t} x_t^2 \sigma_{x,t}^T \sigma_{x,t} + \frac{1}{2} \frac{g_{yy,t}}{g_t} y_t^2 \sigma_{y,t}^T \sigma_{y,t} + \frac{g_{xy,t}}{g_t} x_t y_t \sigma_{x,t}^T \sigma_{y,t} \quad (\text{A.2})$$

$$\sigma_{g,t} = \frac{g_{x,t}}{g_t} x_t \sigma_{x,t} + \frac{g_{y,t}}{g_t} y_t \sigma_{y,t} \quad (\text{A.3})$$

This result is used repeatedly throughout the paper.

As a point of notation, recall that for any function g , g_t simply denotes $g(X_t)$, not the time-derivative of g (which is zero because the model is stationary due to infinite horizon). $g_{x,t}, g_{y,t}, g_{xx,t}, g_{yy,t}, g_{xy,t}$ denote the partial derivatives of $g(X_t)$.

A.2. Returns, and risk premia

The (geometric) drifts and diffusion terms for asset returns are obtained from Itô's Lemma and are as follows, for $j \in \{1, 2\}$

$$\begin{aligned} dR_{j,t} &= \mu_{R_j,t} dt + \sigma_{R_j,t}^T d\vec{Z}_t \\ &\equiv \left(F_{j,t} + \mu_{p_j,t} + \mu_{Y_j} + \sigma_{p_j,t}^T \sigma_{Y_j} - \mu_{F_j,t} + \sigma_{F_j,t}^T \sigma_{F_j,t} - (\sigma_{p_j,t} + \sigma_{Y_j})^T \sigma_{F_j,t} \right) dt \\ &\quad + (\sigma_{p_j,t} + \sigma_{Y_j} - \sigma_{F_j,t})^T d\vec{Z}_t \end{aligned} \quad (\text{A.4})$$

where $\mu_{p_j,t}$, $\mu_{F_j,t}$, $\sigma_{p_j,t}$, $\sigma_{F_j,t}$ are obtained using Remark A.1 above.

Proposition A.1. *The expected risk premia on the equity assets are given by*

$$\begin{aligned} \mu_{R_1,t} - r_t &= \gamma_t \sigma_{R_1,t}^T \{ z_t \sigma_{R_1,t} + (1 - z_t) \sigma_{R_2,t} \} \\ &\quad - \gamma_t \sigma_{R_1,t}^T \sigma_{x,t} x_t \left\{ x_t \left(\frac{1}{\gamma^A} \right) \left(\frac{1 - \gamma^A}{1 - \psi^A} \right) \frac{J_{x,t}^A}{J_t^A} + (1 - x_t) \left(\frac{1}{\gamma^B} \right) \left(\frac{1 - \gamma^B}{1 - \psi^B} \right) \frac{J_{x,t}^B}{J_t^B} \right\} \\ &\quad - \gamma_t \sigma_{R_1,t}^T \sigma_{y,t} y_t \left\{ x_t \left(\frac{1}{\gamma^A} \right) \left(\frac{1 - \gamma^A}{1 - \psi^A} \right) \frac{J_{y,t}^A}{J_t^A} + (1 - x_t) \left(\frac{1}{\gamma^B} \right) \left(\frac{1 - \gamma^B}{1 - \psi^B} \right) \frac{J_{y,t}^B}{J_t^B} \right\} \\ &\quad + \gamma_t \left(\frac{1 - x_t}{\gamma^B} \right) \tau^B F_{1,t} \end{aligned} \quad (\text{A.5})$$

$$\begin{aligned} \mu_{R_2,t} - r_t &= \gamma_t \sigma_{R_2,t}^T \{ z_t \sigma_{R_1,t} + (1 - z_t) \sigma_{R_2,t} \} \\ &\quad - \gamma_t \sigma_{R_2,t}^T \sigma_{x,t} x_t \left\{ x_t \left(\frac{1}{\gamma^A} \right) \left(\frac{1 - \gamma^A}{1 - \psi^A} \right) \frac{J_{x,t}^A}{J_t^A} + (1 - x_t) \left(\frac{1}{\gamma^B} \right) \left(\frac{1 - \gamma^B}{1 - \psi^B} \right) \frac{J_{x,t}^B}{J_t^B} \right\} \\ &\quad - \gamma_t \sigma_{R_2,t}^T \sigma_{y,t} y_t \left\{ x_t \left(\frac{1}{\gamma^A} \right) \left(\frac{1 - \gamma^A}{1 - \psi^A} \right) \frac{J_{y,t}^A}{J_t^A} + (1 - x_t) \left(\frac{1}{\gamma^B} \right) \left(\frac{1 - \gamma^B}{1 - \psi^B} \right) \frac{J_{y,t}^B}{J_t^B} \right\} \\ &\quad + \gamma_t \left(\frac{x_t}{\gamma^A} \right) \tau^A F_{2,t} \end{aligned} \quad (\text{A.6})$$

where $\gamma_t \equiv \left(\frac{x_t}{\gamma^A} + \frac{1 - x_t}{\gamma^B} \right)^{-1}$ is the wealth-weighted global risk aversion.

All parameters can differ between the two investors. Cf. the main text for a discussion. To complete the definition of the optimization problem, the investors are subject to a standard transversality condition, and W_0^j are given. Note also that $W_t^j \geq 0$.

A.3. Equilibrium

The definition of the equilibrium is standard.

Definition 1. A competitive equilibrium is a set of aggregate stochastic processes adapted to the filtration generated by \vec{Z} : the price of the equity asset $(Q_{1,t}, Q_{2,t})$, and the interest rate (r_t) , together with a set of individual stochastic processes for each investor: consumption of each good $(C_{1,t}^A, C_{2,t}^A, C_{1,t}^B, C_{2,t}^B)$, wealth (W_t^A, W_t^B) , and portfolio shares $(w_{1,t}^A, w_{2,t}^A, w_{1,t}^B, w_{2,t}^B)$, such that, given the output of the two endowment trees $(Y_{1,t}, Y_{2,t})$:

1. Given the aggregate stochastic processes, individual choices solve the investor optimization problem given above.
2. Markets clear.
 - a) Good markets:

$$\begin{aligned} C_{1,t}^A + C_{1,t}^B &= Y_{1,t} \\ C_{2,t}^A + C_{2,t}^B &= Y_{2,t} \end{aligned} \tag{A.7}$$

- b) Equity markets:

$$\begin{aligned} w_{1,t}^A W_t^A + w_{1,t}^B W_t^B &= Q_{1,t} \\ w_{2,t}^A W_t^A + w_{2,t}^B W_t^B &= Q_{2,t} \end{aligned} \tag{A.8}$$

Most importantly, as shown in Section 2.3 of the main text, the equilibrium can be recast as a stationary recursive Markovian equilibrium in which all variables of interest are expressed as a function of a pair of state variables $X_t \equiv (x_t, y_t)'$, whose dynamics are also solely a function of X_t . x_t is the wealth share of investor A , and y_t is the relative supply of the first good (preferred by investor A).

A.4. Hamilton-Jacobi-Bellman equations

Proposition A.2. J_t^A, J_t^B satisfy the Hamilton-Jacobi-Bellman equations:

$$\begin{aligned} 0 &= \left(\frac{1}{\psi^A - 1} \right) P_t^{A1-\psi^A} J_t^A - \left(\frac{1}{1 - 1/\psi^A} \right) \rho^A + r_t + \frac{\gamma^A}{2} (w_{1,t}^A \sigma_{R1,t} + w_{2,t}^A \sigma_{R2,t}) \\ &\quad + \left(\frac{1}{1 - \psi^A} \right) \mu_{J^A,t} + \frac{1}{2} \left(\frac{1}{1 - \psi^A} \right) \left(\frac{\psi^A - \gamma^A}{1 - \psi^A} \right) \sigma_{J^A,t}^T \sigma_{J^A,t} \end{aligned} \quad (\text{A.9})$$

$$\begin{aligned} 0 &= \left(\frac{1}{\psi^B - 1} \right) P_t^{B1-\psi^B} J_t^B - \left(\frac{1}{1 - 1/\psi^B} \right) \rho^B + r_t + \frac{\gamma^B}{2} (w_{1,t}^B \sigma_{R1,t} + w_{2,t}^B \sigma_{R2,t}) \\ &\quad + \left(\frac{1}{1 - \psi^B} \right) \mu_{J^B,t} + \frac{1}{2} \left(\frac{1}{1 - \psi^B} \right) \left(\frac{\psi^B - \gamma^B}{1 - \psi^B} \right) \sigma_{J^B,t}^T \sigma_{J^B,t} \\ &\quad + \left(\frac{1}{1 - \delta^B} \right) F_{2,t} \left(\frac{1 - z_t}{1 - x_t} \right) + w_{2,t}^A \left(\frac{x_t}{1 - x_t} \right) \tau^A F_{2,t} \end{aligned} \quad (\text{A.10})$$

where $\mu_{J^i,t}, \sigma_{J^i,t}$ are the geometric drift and diffusion terms of J_t^i obtained as in Remark A.1:

$$\frac{dJ_t^i}{J_t^i} \equiv \mu_{J^i,t} dt + \sigma_{J^i,t}^T d\vec{Z}_t \quad (\text{A.11})$$

A.5. Consumptions, goods prices

Proposition A.3. The consumption of each investor is given by:

$$c_t^i \equiv \frac{C_t^i}{W_t^i} = P_t^{i-\psi^i} J_t^i \quad (\text{A.12})$$

$$c_{1,t}^i = \alpha^i \left(\frac{p_{1,t}}{P_t^i} \right)^{-\theta} c_t^i \quad (\text{A.13})$$

$$c_{2,t}^i = (1 - \alpha^i) \left(\frac{p_{2,t}}{P_t^i} \right)^{-\theta} c_t^i \quad (\text{A.14})$$

$$P_t^i = [\alpha^i p_{1,t}^{1-\theta} + (1 - \alpha^i) p_{2,t}^{1-\theta}]^{1/(1-\theta)} \quad (\text{A.15})$$

Proposition A.4. *The terms of trade, $q_t = q(X_t)$, solves the following non-linear equation:*

$$q_t = S_t^{1/\theta} \left(\frac{y_t}{1 - y_t} \right)^{1/\theta} \quad (\text{A.16})$$

where:

$$S_t = \frac{(1 - \alpha^A)P_t^{A\theta - \psi^A} J_t^A x_t + (1 - \alpha^B)P_t^{B\theta - \psi^B} J_t^B (1 - x_t)}{\alpha^A J_t^A x_t P_t^{A\theta - \psi^A} + (1 - \alpha^B)P_t^{B\theta - \psi^B} J_t^B (1 - x_t)}$$

Using the defintion of the numéraire, prices follow:

$$p_{1,t} = (a + (1 - a)q_t^{1-\theta})^{1/(\theta-1)} \quad (\text{A.17})$$

$$p_{2,t} = p_{1,t} q_t = (aq_t^{\theta-1} + (1 - a))^{1/(\theta-1)} \quad (\text{A.18})$$

$$P_t^i = [\alpha^i p_{1,t}^{1-\theta} + (1 - \alpha^i) p_{2,t}^{1-\theta}]^{1/(1-\theta)} \quad (\text{A.19})$$

$$\mathcal{E}_t = P_t^B / P_t^A \quad (\text{A.20})$$

A.6. Calibration

This section provides details on the baseline symmetric calibration of Assumption 1. It is based mostly on standard asset pricing parameters, as well as on the calibration in an international context, which has been one of the main focus of the multi-good portfolio choice literature.

At $\gamma^i = \gamma^* = 15$, risk aversion is a bit on the high side, although within the range of values that are common in asset pricing. This allows to generate slightly more realistic risk premia, given that the model only features mild frictions in the form of imperfect financial integration. (This is nothing but the equity premium puzzle of [Mehra and Prescott \(1985\)](#).) The risk aversion could be increased much further for the purpose of matching risk premia more closely to the data, given that recursive preferences decouple it from the inverse of the elasticity of intertemporal substitution. However, the focus in this paper is on the mechanisms rather than on an exact quantitative match. Moving forward, extensions of the model, some of which discussed in Section 5.3, will be the prime way to generate higher risk premia. Prominent examples include the introduction of portfolio constraints, and of non-diversifiable idiosyncratic risk.

Although the elasticity of intertemporal substitution is set to $\psi^i = \psi = 2$ in the baseline, consistent with recent estimates e.g. in [Schorfheide et al. \(2018\)](#) and with values around $\psi = 1.5$ that have been used in the asset pricing literature e.g. in [Bansal and Yaron \(2004\)](#), I discuss its effect at length in Sections 3 and 4. I contrast the cases with $\psi^i = \psi = 0.2$ and $\psi^i = \psi = 2$, and ψ turns out to have a large impact on the potency of imperfect financial integration. For the international application of Section 5 ([Sauzet, 2021a](#)), I therefore use $\psi^i = \psi = 0.5$, which allows me to generate a plausible home bias in equity holdings while matching the broad level of the interest rate in this asymmetric context. This lower value goes some way towards the much lower estimates of the elasticity of intertemporal substitution that have been used historically in the earlier literature e.g. in [Hall \(1988\)](#), [Campbell \(1999\)](#).

The bias in consumption $\alpha^A = 1 - \alpha^B = \alpha = 0.75$ is consistent with the share of import in the consumption basket of the United States and other countries in recent years. The value is therefore in line with the range of values that have been used in the international portfolio choice literature, although slightly lower given the slight increase in world trade in recent decades. In that literature, values as high as $\alpha = 0.9$ or even $\alpha = 0.975$ are sometimes necessary from a quantitative perspective, but this is not the case in the context of this paper where I study the dynamics throughout the state space instead of local neighborhoods of a steady-state. Note that, as α increases further, portfolios and other variables become very non-linear, and the impact of the wealth share is strongly reinforced even in the baseline calibration. Other values of α can be studied as they could be relevant for other applications, e.g. in the environmental application of Section 5.2 ([Sauzet, 2021b](#)).

The numéraire basket has a weight of $a = 1 - a = 1/2$ on each good. The value of a has no consequence on quantities and only tilts prices accordingly. I therefore stick to a symmetric numéraire basket to ease interpretation. In extensions of the model with more assets (e.g. multiple bonds), portfolio constraints, and additional sources of risk, the denomination of the numéraire could be of more interest, an aspect that I am planning to explore.

The elasticity of substitution between goods $\theta^i = \theta = 2$ is in line with modern standard estimates again in the international literature, e.g. in [Imbs and Méjean \(2015\)](#). Cf. among others [Tille \(2001\)](#), [Corsetti et al. \(2008\)](#), [Coeurdacier \(2009\)](#),

[Obstfeld \(2007\)](#), [Bhamra et al. \(2014\)](#) for a discussion. I take a value slightly lower than [Imbs and Méjean \(2015\)](#)'s preferred range of [4, 6], as a compromise towards the lower values that had been used in the earlier literature. From an economic standpoint, most relevant is that this elasticity is above one, a point whose impact I discuss at length throughout Section 3, and in particular in Section 3.4 on portfolios. Again, explore different values can be particularly interesting for the environmental application in which this parameter can play a key role. Cf. for instance [Gollier \(2019\)](#) for an overview. In that paper, the author introduces a stochastic elasticity of substitution, which could be interesting to study in my context.

The discount rate is standard at $\rho^i = \rho = 1\%$, and allows to match the broad level of the interest rate.

In the main text of the paper, labor income is inactive: $\delta_j = \delta = 0\%$. I briefly cover the impact of labor income, which has been discussed in the literature, in Appendix [A.7](#). In that case, I use $\delta_j = \delta = 62.5\%$, in line with the average labor share in the United States over the last 50 years.

The tax on “foreign” dividends, which captures imperfect financial integration, is set to $\tau^i = \tau = 0\%$ in the baseline. Its effect is discussed at length in Section 4, and some more in Section 5.

Output processes have a growth rate in annual terms of $\mu_{Y_j} = \mu_Y = 2\%$, and a volatility of $\sigma_{Y_1 z_1} = \sigma_{Y_2 z_2} = 4.1\%$. This is in line with typical values used in the literature, and broadly consistent with world averages e.g. in [Uribe and Schmitt-Grohé \(2017\)](#), in International Monetary Fund or World Bank data, or in longer-run series in [Jordà et al. \(2016\)](#). Asymmetries in output growth rates and volatilities could be an interesting exploration from the perspective of studying the integration of developed slower-growing countries with emerging faster-growing economies in an international context, or different growth rates of the capital good and ecosystem services in an environmental application. Importantly, the fundamental correlation between the output of each tree is assumed to be zero. This is not meant to capture empirical correlations, but allows to focus on the correlation between asset returns and goods prices that emerge purely endogenously, and which is large and state-dependent.

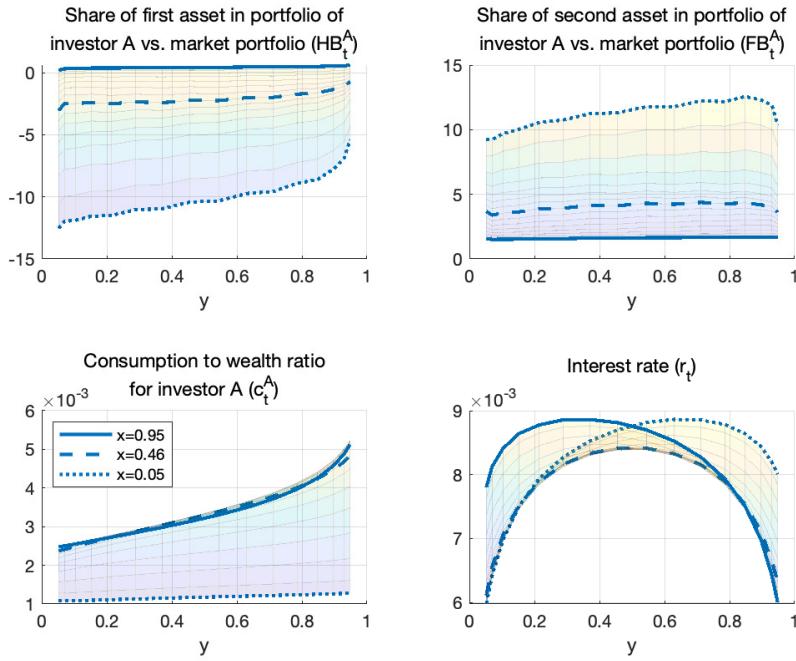
A.7. Impact of labor income

(Back to main text: Section 4.)

The heterogeneity of investors is another factor that strongly reinforces the influence of the wealth share on the equilibrium, not only conceptually but also quantitatively. This was already apparent in the analysis of the baseline calibration studied so far. As we have seen, for instance in Table 1, an increase in the bias in consumption, which constitutes the fundamental heterogeneity in the economy, increases the impact of the wealth share significantly.

Here, I briefly study the impact of heterogeneity further by staying in a symmetric calibration but introducing labor income. Heterogeneity is also partly the focus of the application of Section 5.1, albeit of a different kind, as the investors will exhibit asymmetries in tolerance for risk.

Figure A.1: Impact of the wealth share in the presence of labor income ($\delta = 62.5\%$)

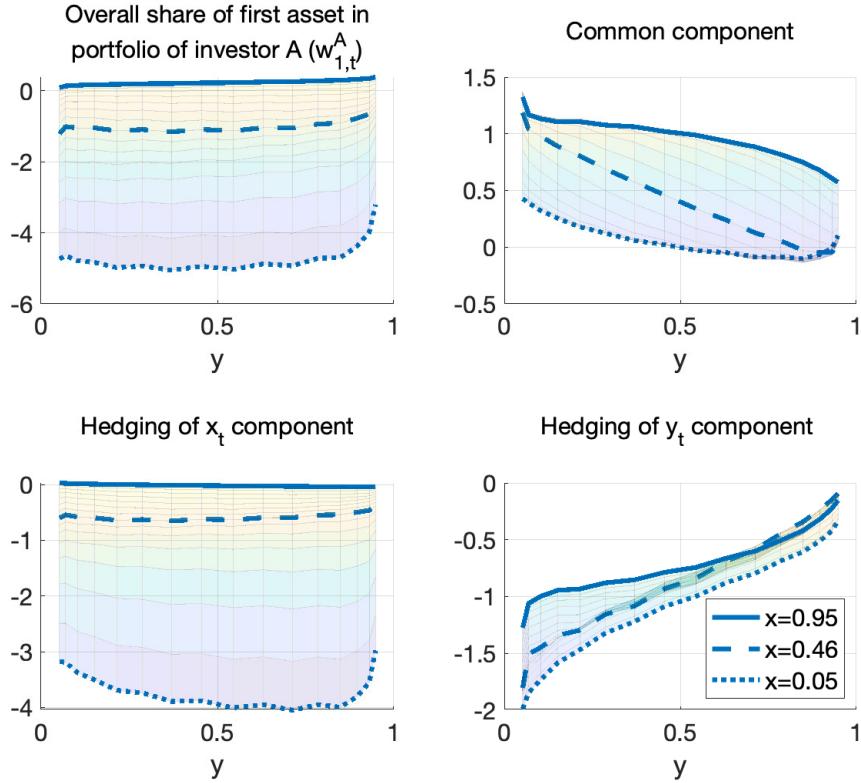


Notes: Based on the symmetric calibration under perfect risk sharing of Assumption 1, except that $\delta = 62.5\%$. x_t is the wealth share, which captures the share of total wealth held by investor A . y_t is the relative supply of the first good, which captures fundamentals. HB_t^A (FB_t^A) is the share of the first (second) asset in the portfolio of investor A divided by the share of that asset in the market portfolio. I.e. $HB_t^A \equiv w_{1,t}^A/(w_{1,t}^A + w_{2,t}^A)/z$, $FB_t^A \equiv w_{2,t}^A/(w_{1,t}^A + w_{2,t}^A)/z$.

As a reminder, labor income is modeled as a constant share ($\delta^i = \delta$) of the output of each tree being paid to the “local” investor (the investor who prefers that good). By making the budget constraint of each investor more dependent on the output of their preferred good, labor income also increases the heterogeneity between investors in the economy. While its effect on risk premia and Sharpe ratios is somewhat modest, labor income significantly affects portfolios, marginal values of wealth, consumptions, and the interest rate. Those are shown in Figure A.1 for a labor share δ of 62.5%, roughly in line with the average labor share in the United States over the last 50 years. In addition, its effect is once again going hand-in-hand with a bolstered importance for the wealth share.

The top two panels of Figure A.1 show portfolio weights for investor A as they compare to the market portfolio, HB_t^A and FB_t^A . Because her labor income is perfectly correlated with the payoff of the first asset, it renders this asset yet more unattractive to the investor, therefore reinforcing the “foreign bias” in equity holdings on average. This is in line with [Baxter and Jermann \(1997\)](#), who argue that “The International Diversification Puzzle Is Worse Than You Think” when labor takes this form. In terms of magnitude, the impact is substantial, with the measure of “home bias” now varying from -12.5 to 1 as the wealth share increases, an effect of much larger magnitude than that of fundamentals. In addition, portfolios change not only on average but also inherently in a state-dependent fashion, with the “foreign bias” reinforced in particular as an investor holds an increasingly smaller share of total wealth. Take investor A for instance: as her wealth share decreases towards zero, labor income represents an increasingly larger share of her revenues, making hedging the labor income risk increasingly important. Due to the perfect correlation between her labor income and the payoff to the first asset, this pushes investor A to tilt her portfolio away from the first asset some more.

Figure A.2: Components of the portfolio of investor A in the presence of labor income ($\delta = 62.5\%$)



Notes: Based on the symmetric calibration under perfect risk sharing of Assumption 1. x_t is the wealth share, which captures the share of total wealth held by investor A . y_t is the relative supply of the first good, which captures fundamentals.

Portfolios are not only affected in their overall shape, but also in their underlying drivers. This can be observed visually in Figure A.2, which reports the weight of the first asset in the portfolio of investor A as well as its components, and is confirmed by computing the corresponding variance decomposition of $w_{1,t}^A$ like before. From both, we observe that the share of $w_{1,t}^A$ explained by the hedging of x_t increases tremendously, going from 7.1% in the baseline without labor income, to a whooping 70.2% for $\delta = 62.5\%$. On the contrary, the common and y_t -hedging components now explain a mere 19.6% and 10.3%, instead of 69.7% and 23.1% in the baseline. In short: the hedging of wealth share risk becomes the main driver of the shape of portfolios.

Labor income also has a significant impact on marginal values of wealth, and therefore on consumptions, both becoming more dependent on the wealth share than in the baseline in which x_t affected them only modestly. For instance, the marginal value of wealth for investor A decreases more markedly as the wealth share gets smaller, due to the fact that her labor income represents an increasing amount in comparison to her wealth, ensuring that she has comparatively more resources to fund her consumption and portfolios. As a result, while the average level of consumption to wealth is broadly unchanged, the consumption of investor a significantly decreases as a fraction of wealth when $x_t \rightarrow 0$, as shown in the bottom left panel of Figure A.1. Interestingly, this pattern is reversed and her consumption increases as $x_t \rightarrow 0$, when the elasticity of intertemporal substitution ψ is small, emphasizing the impact of ψ on the relative importance of substitution and income effects.⁸⁰ When ψ is large, in particular above 1, the substitution effect is strong so that an investor ends up saving a large part of the extra labor income (as a fraction of wealth), resulting in a lower consumption as a fraction of wealth when their wealth share decreases. Conversely, as ψ is small, in particular below 1, the income effect dominates so that an investor ends up spending most of the extra labor income (as a function of wealth) on increased consumption as their wealth share decreases. This phenomenon points once again to the importance of being able to study these mechanisms in a context with general preferences, solved globally throughout the state space.

The pattern for the interest rate mirrors those for the marginal values of wealth and consumptions.⁸¹ On average, r_t slightly decreases compared to the baseline, by about 21 basis points throughout the state space, reflecting the fact that an addition risk, the labor income, needs to be hedged in this economy⁸², but more noticeable is the impact on the shape. The interest rate becomes more asymmetric as a function of relative output, going e.g. from around 0.6% to 0.8% depending on whether $y_t \rightarrow 0$ or $y_t \rightarrow 1$ when investor A holds a small share of total wealth. This represents a reinforcement of the driver of r_t in the baseline combined with a larger investor

⁸⁰I use the terms “substitution effect” and “income effect” liberally, in contrast to their more usual and restricted use that relates to the impact of the interest rate.

⁸¹This is also true for the pattern of the domestic and foreign dividend yields, $F_{1,t}$ and $F_{2,t}$, which appear in the budget constraints once we divide labor income by wealth: $\delta_1 F_{1,t} z_t / ((1 - \delta_1)x_t)$ for investor A , and $\delta_2 F_{2,t} (1 - z_t) / ((1 - \delta_2)(1 - x_t))$ for the foreign investor. Cf. Section 2.4.

⁸²This effect is limited because of the perfect correlation between labor income and the payoff of the local asset.

heterogeneity. In addition, the evolution of r_t as a function of the wealth share is also worth pointing out: as x_t gets small, the interest rate noticeably increases, which has to happen in equilibrium for the domestic investor to be willing to significantly cut down on consumption. Like before, this pattern is also reversed for small values of the elasticity of intertemporal substitution, with the interest rate *decreasing* as the wealth share gets close to zero or one in that case.

Lastly, the introduction of labor income has non-linear effects on the equilibrium distribution of state variables, as shown in Figure F.5. While the dispersion of the wealth share first decreases with δ , consistent with labor income tightening the wealth distribution by ensuring a minimum level of revenues for each investor, dispersion increases back for large values of δ . In addition, as δ increases, the steepness of the relationship between x_t and y_t increases. Those effects are the results of the interplay between the several components of the drift and diffusion of the wealth share, shown in Figure XX. Note also that the second effect, with dispersion increasing back with δ , tends to occur faster for lower level of the elasticity of intertemporal substitution ψ .

Overall, labor income has a significant impact on the equilibrium and its underpinnings due to the resulting increased heterogeneity that reinforces the impact of the wealth share. The way those patterns change when considering a more general and realistic specification for labor income could prove an interesting exploration. One particular specification could be to construct labor income as a time-varying share of the output of each country, as explored for instance in [Coeurdacier and Gourinchas \(2016\)](#). As the authors suggest, the correlation of labor income with output, once computed with the proper conditioning, could in fact turn out to be negative, providing a natural way to generate a “home bias” in equity holdings in an international context. If the share is itself stochastic, it could also provide an additional hedging motive that could prove relevant in practice also as it introduces a natural degree of market incompleteness. Labor income could also take a more general form, for instance as a separate source of idiosyncratic risk in the spirit of the recent heterogeneous-agent macroeconomic literature like [Kaplan et al. \(2018\)](#), or by introducing a distribution of investors in each country by generalizing the overlapping generation structure of [Gârleanu and Panageas \(2015\)](#) to a two-tree, two-good setting. The latter hints at

how labor income could help both (types of) investors survive in equilibrium.⁸³ I leave these promising avenues for future research.

⁸³One difficulty is that this might generate a stationary distribution between investors within a group, but it would not be sufficient *per se* to ensure a stationary distribution of wealth *between* different groups of investors, except by assuming that individual investors can switch between groups. The ability of labor income to ensure the survival of different types of agents is also used in [He and Krishnamurthy \(2013\)](#).

B. Proofs

The proof that the equilibrium can be recast as a stationary recursive Markovian equilibrium with $X = (x, y)'$ as state variables follows a guess and verify approach, e.g. as in Gărleanu and Panageas (2015).

B.1. HJBs and Propositions

Following the usual argument, (2) can be reformulated as the following Hamilton-Jacobi-Bellman equations, subject to the same budget constraints and goods aggregators

$$0 = \max_{C_t^i, w_{1,t}^i, w_{2,t}^i} f^i(C_t^i, V_t^i) dt + \mathbb{E}_t [dV_t^i] \quad (\text{B.1})$$

subject to (3) & (4)

Using the homotheticity of the value function with recursive preferences, one can show that

$$V^i(W^i, x, y) = \left(\frac{W^{i1-\gamma^i}}{1-\gamma^i} \right) J^i(x, y)^{\frac{1-\gamma^i}{1-\psi^i}} \quad (\text{B.2})$$

where $J_t^i = J^i(x_t, y_t)$ are two unknown functions to solve for. For CRRA utility, the expressions simplify to

$$V^i(W, x, y) = \left(\frac{W^{i1-\gamma^i}}{1-\gamma^i} \right) J^i(x, y)^{-\gamma} \quad (\text{B.3})$$

while for log utility, they simplify to

$$V^i(W, x, y) = \frac{1}{\rho^i} \log W^i + J^i(x, y) \quad (\text{B.4})$$

Using Itô's Lemma to compute dV_t and simplifying, we obtain the following Hamilton-Jacobi-Bellman equations for both investors

$$\begin{aligned}
0 = \max_{c_t^i, w_{1,t}^i, w_{2,t}^i} & \left(\frac{1}{1 - 1/\psi^i} \right) \left[\left(\frac{c_t^i}{J_t^{i1/(1-\psi^i)}} \right)^{1-1/\psi^i} - \rho^i \right] \\
& + (r_t + w_{1,t}^i (\mu_{R1,t} - r_t) + w_{2,t}^i (\mu_{R2,t} - r_t) - P_t^i c_t^i) \\
& + \left(\frac{1}{1 - \psi^i} \right) \mu_{J^i,t} \\
& - \frac{\gamma^i}{2} (w_{1,t}^i \sigma_{R1,t} + w_{2,t}^i \sigma_{R2,t})^T (w_{1,t}^i \sigma_{R1,t} + w_{2,t}^i \sigma_{R2,t}) \\
& + \frac{1}{2} \left(\frac{1}{1 - \psi^i} \right) \left(\frac{\psi^i - \gamma^i}{1 - \psi^i} \right) \sigma_{J^i,t}^T \sigma_{J^i,t} \\
& + \left(\frac{1 - \gamma^i}{1 - \psi^i} \right) (w_{1,t}^i \sigma_{R1,t} + w_{2,t}^i \sigma_{R2,t})^T \sigma_{J^i,t}
\end{aligned} \tag{B.5}$$

where following Remark A.1

$$\frac{dJ_t^i}{J_t^i} \equiv \mu_{J^i,t} dt + \sigma_{J^i,t}^T d\vec{z}_t \tag{B.6}$$

$$\mu_{J^i,t} = \left(\frac{J_{x,t}^i}{J_t^i} x_t \mu_{x,t} + \frac{J_{y,t}^i}{J_t^i} y_t \mu_{y,t} + \frac{1}{2} \frac{J_{xx,t}^i}{J_t^i} x_t^2 \sigma_{x,t}^T \sigma_{x,t} + \frac{1}{2} \frac{J_{yy,t}^i}{J_t^i} y_t^2 \sigma_{y,t}^T \sigma_{y,t} + \frac{J_{xy,t}^i}{J_t^i} x_t y_t \sigma_{x,t}^T \sigma_{y,t} \right) \tag{B.7}$$

$$\sigma_{J^i,t} = \frac{J_{x,t}^i}{J_t^i} x_t \sigma_{x,t} + \frac{J_{y,t}^i}{J_t^i} y_t \sigma_{y,t} \tag{B.8}$$

Taking first-order conditions with respect to $c_t^i, w_{1,t}^i, w_{2,t}^i$, respectively, yields Propositions 5 and A.3. Plugging back in the equations above delivers the Hamilton-Jacobi-Bellman equations in Proposition A.2. Prices in Propositions 2 and A.4, and risk premia in Proposition 4, are obtained by combining those expressions with the several market-clearing conditions for goods and assets.

B.2. Stochastic discount factors

The stochastic discount factors of the domestic and foreign investors are

$$\xi_t^i \equiv \xi_0^i \exp \left\{ \int_0^t \frac{\partial f^i}{\partial V^i} (C_u^i, V_u^i) du \right\} \frac{\partial V_t^i}{\partial W_t^i} = \exp \left\{ \int_0^t \frac{\partial f^i}{\partial V^i} (C_u^i, V_u^i) du \right\} W_t^{i-\gamma^i} J_t^{i \frac{1-\gamma^i}{1-\psi^i}} \quad (\text{B.9})$$

It follows that

$$\begin{aligned} \ln \xi_t &= \int_0^t \frac{\partial f^i}{\partial V} (C_u^i, V_u^i) du + \left(\frac{1-\gamma^i}{1-\psi^i} \right) \ln J_t^i - \gamma^i \ln W_t^i & (\text{B.10}) \\ \Rightarrow d \ln \xi_t^i &= \frac{\partial f^i}{\partial V} (C_t^i, V_t^i) dt + \left(\frac{1-\gamma^i}{1-\psi^i} \right) d \ln J_t^i - \gamma^i d \ln W_t^i \equiv \mu_{\ln \xi^i, t} dt + \sigma_{\ln \xi^i, t}^T d \vec{Z}_t & (\text{B.11}) \end{aligned}$$

From the definition of $f^i(C, V)$ in Equation (2), one can show that (algebra or cf. e.g. Gârleanu and Panageas (2015), Duffie and Epstein, 1992, Schroder and Skiadas (1999)):

$$\frac{\partial f^i}{\partial V} (C_t^i, V_t^i) dt = \Theta_1^i P_t^{i1-\psi^i} J_t^i + \Theta_2^i \quad (\text{B.12})$$

with constants

$$\Theta_1^i \equiv - \left(\frac{\gamma^i - \frac{1}{\psi^i}}{1 - \frac{1}{\psi^i}} \right) \quad \text{and} \quad \Theta_2^i \equiv \frac{\rho^i(\gamma^i - 1)}{1 - \frac{1}{\psi^i}}$$

In addition

$$d \ln J_t^i \equiv \mu_{\ln J^i, t} dt + \sigma_{\ln J^i, t}^T d\vec{Z}_t = \left(\mu_{J^i, t} - \frac{1}{2} \sigma_{J^i, t}^T \sigma_{J^i, t} \right) dt + \sigma_{J^i, t}^T d\vec{Z}_t \quad (\text{B.13})$$

$$d \ln W_t^i \equiv \mu_{\ln W^i, t} dt + \sigma_{\ln W^i, t}^T d\vec{Z}_t = \left(\mu_{W^i, t} - \frac{1}{2} \sigma_{W^i, t}^T \sigma_{W^i, t} \right) dt + \sigma_{W^i, t}^T d\vec{Z}_t \quad (\text{B.14})$$

$$\frac{dJ_t^i}{J_t^i} \equiv \mu_{J^i, t} dt + \sigma_{J^i, t}^T d\vec{Z}_t \quad (\text{B.15})$$

$$\begin{aligned} \mu_{J^i, t} &\equiv \frac{J_{x,t}^i}{J_t^i} x_t \mu_{x,t} + \frac{J_{y,t}^i}{J_t^i} y_t \mu_{y,t} + \frac{1}{2} \frac{J_{xx,t}^i}{J_t^i} x_t^2 \sigma_{x,t}^T \sigma_{x,t} + \frac{1}{2} \frac{J_{yy,t}^i}{J_t^i} y_t^2 \sigma_{y,t}^T \sigma_{y,t} + \frac{J_{xy,t}^i}{J_t^i} x_t y_t \sigma_{x,t}^T \sigma_{y,t} \\ \sigma_{J^i, t} &\equiv \frac{J_{x,t}^i}{J_t^i} x_t \sigma_{x,t} + \frac{J_{y,t}^i}{J_t^i} y_t \sigma_{y,t} \end{aligned}$$

and $\mu_{W^i, t}, \sigma_{W^i, t}$ are given in Equation (4) repeated here for convenience:

$$\begin{aligned} \frac{dW_t^i}{W_t^i} &= \left(r_t + w_{1,t}^i (\mu_{R_1,t} - r_t) + w_{2,t}^i (\mu_{R_2,t} - r_t) - P_t^i c_t^i \right) dt \\ &\quad + (w_{1,t}^i \sigma_{R_1,t} + w_{2,t}^i \sigma_{R_2,t})^T d\vec{Z}_t \end{aligned}$$

Therefore:

$$\mu_{\ln \xi^i, t} = \Theta_1^i P_t^{i1-\psi^i} J_t^i + \Theta_2^i + \left(\frac{1-\gamma^i}{1-\psi^i} \right) \left(\mu_{J^i, t} - \frac{1}{2} \sigma_{J^i, t}^T \sigma_{J^i, t} \right) - \gamma \left(\mu_{W^i, t} - \frac{1}{2} \sigma_{W^i, t}^T \sigma_{W^i, t} \right) \quad (\text{B.16})$$

$$\sigma_{\ln \xi^i, t} = \left(\frac{1-\gamma^i}{1-\psi^i} \right) \sigma_{J^i, t} - \gamma^i \sigma_{W^i, t} \quad (\text{B.17})$$

Finally:

$$\frac{d\xi_t^i}{\xi_t^i} \equiv \mu_{\xi^i, t} dt + \sigma_{\xi^i, t}^T d\vec{Z}_t = \left(\mu_{\ln \xi^i, t} + \frac{1}{2} \sigma_{\ln \xi^i, t}^T \sigma_{\ln \xi^i, t} \right) dt + \sigma_{\ln \xi^i, t}^T d\vec{Z}_t \quad (\text{B.18})$$

Plugging all components, we obtain:

$$\mu_{\xi^i,t} = \mu_{\ln \xi^i,t} + \frac{1}{2} \sigma_{\ln \xi^i,t}^T \sigma_{\ln \xi^i,t} \quad (\text{B.19})$$

$$\begin{aligned} &= \Theta_1^i P_t^{i1-\psi^i} J_t^i + \Theta_2^i + \left(\frac{1-\gamma^i}{1-\psi^i} \right) \left(\mu_{J^i,t} - \frac{1}{2} \sigma_{J^i,t}^T \sigma_{J^i,t} \right) - \gamma^i \left(\mu_{W^i,t} - \frac{1}{2} \sigma_{W^i,t}^T \sigma_{W^i,t} \right) \\ &\quad + \frac{1}{2} \left(\left(\frac{1-\gamma^i}{1-\psi^i} \right) \sigma_{J^i,t} - \gamma^i \sigma_{W^i,t} \right)^T \left(\left(\frac{1-\gamma^i}{1-\psi^i} \right) \sigma_{J^i,t} - \gamma^i \sigma_{W^i,t} \right) \\ &= \Theta_1^i P_t^{i1-\psi^i} J_t^i + \Theta_2^i + \left(\frac{1-\gamma^i}{1-\psi^i} \right) \left(\mu_{J^i,t} - \frac{1}{2} \sigma_{J^i,t}^T \sigma_{J^i,t} \right) \\ &\quad - \gamma \left(r_t + w_{1,t}^i (\mu_{R_1,t} - r_t) + w_{2,t}^i (\mu_{R_2,t} - r_t) - P_t^{i1-\psi^i} J_t^i \right) \\ &\quad + \frac{\gamma}{2} \left((w_{1,t}^i \sigma_{R_1,t} + w_{2,t}^i \sigma_{R_2,t})^T (w_{1,t}^i \sigma_{R_1,t} + w_{2,t}^i \sigma_{R_2,t}) \right) \\ &\quad + \frac{1}{2} \left(\left(\frac{1-\gamma^i}{1-\psi^i} \right) \sigma_{J^i,t} - \gamma^i (w_{1,t}^i \sigma_{R_1,t} + w_{2,t}^i \sigma_{R_2,t}) \right)^T \\ &\quad \left(\left(\frac{1-\gamma^i}{1-\psi^i} \right) \sigma_{J^i,t} - \gamma^i (w_{1,t}^i \sigma_{R_1,t} + w_{2,t}^i \sigma_{R_2,t}) \right) \\ \sigma_{\xi^i,t} &= \left(\frac{1-\gamma^i}{1-\psi^i} \right) \sigma_{J^i,t} - \gamma^i (w_{1,t}^i \sigma_{R_1,t} + w_{2,t}^i \sigma_{R_2,t}) \end{aligned} \quad (\text{B.20})$$

Under complete markets:

$$\frac{d\xi_t^i}{\xi_t^i} = -r_t dt - \kappa_t^T d\vec{Z}_t$$

where r_t, κ_t are the interest rate and the (two-dimensional) price of risk that are equal for both investors when markets are complete and risk sharing is perfect.

One can show that:

$$\begin{aligned} \kappa_t &= \gamma_t \{ z_t \sigma_{R_1,t} + (1-z_t) \sigma_{R_2,t} \} \\ &\quad - \gamma_t \left\{ x_t \left(\frac{1}{\gamma^A} \right) \left(\frac{1-\gamma^A}{1-\psi^A} \right) \sigma_{J^A,t} + (1-x_t) \left(\frac{1}{\gamma^B} \right) \left(\frac{1-\gamma^B}{1-\psi^B} \right) \sigma_{J^B,t} \right\} \end{aligned} \quad (\text{B.21})$$

where, $z_t = \frac{Q_{1,t}}{Q_{1,t}+Q_{2,t}} = \frac{Q_{1,t}}{W_t^A+W_t^B}$ is the ratio of the first equity price to total wealth.

Next: expression for r_t and special cases.

C. Numerical Resolution

To solve the problem numerically, I use projection methods based on Chebyshev polynomials and orthogonal collocation. The method follows [Judd \(1992, 1998\)](#). It is also presented in the NBER Summer SI Lecture by [Fernández-Villaverde and Christiano \(2011\)](#) and in [Parra-Alvarez \(2018\)](#) among others, and is applied to multi-agent asset pricing models for instance in [Drechsler et al. \(2018\)](#), [Fang \(2019\)](#), and [Kargar \(2019\)](#).

The model can be written as a system of equations

$$\mathcal{H}(\mathbf{G}) = \mathbf{0} \quad (\text{C.1})$$

where $\mathbf{G} : [0, 1] \times [0, 1] \rightarrow \mathbb{R}^M$ is a function of the state variables $X = (x, y)' : \mathbf{G}(X)$. $\mathcal{H} : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ is an operator, where $\mathcal{B}_1, \mathcal{B}_2$ are spaces of functions. $\mathbf{0}$ is the zero of \mathcal{B}_2 .

The name of the game is to solve an approximate version of (C.1)

$$\hat{\mathcal{H}}(\hat{\mathbf{G}}) \approx \mathbf{0} \quad (\hat{\mathcal{H}}(\hat{\mathbf{G}}) \approx \mathbf{0} \text{ in our case}) \quad (\text{C.2})$$

Specifically, I pick a basis $\{\Psi_{ij}(x, y)\}_{i=1, j=1}^{N, N}$ for the space of functions and use it to approximate the following variables: $\mathcal{G} \equiv \{J_t^A, J_t^B, F_{1,t}, F_{2,t}, q_t, w_{1,t}^A, w_{2,t}^A\}$. All other variables and quantities of the model can be expressed as a function of those variables.

Any $g : [0, 1] \times [0, 1] \rightarrow D^g \subset \mathbb{R}$ in \mathcal{G} is approximated at the order N as follows

$$\hat{g}(X) = \sum_{i=0}^N \sum_{j=0}^N a_{ij}^{(N)} \Psi_{ij}^{(N)}(x, y) \quad (\text{C.3})$$

where $a_{ij}^{(N)}$ are coefficients to solve for.

I use the tensor product of Chebyshev polynomials of order 0 to N as basis

$$\Psi_{ij}^{(N)}(X) = T_i(\omega(x)) T_j(\omega(y)) \quad (\text{C.4})$$

where $\omega(x) = 2(x - 1)$, $\omega(y) = 2(y - 1)$ transform x and y from $[0, 1]$ to $[-1, 1]$ over which Chebyshev polynomials are defined.

Define the residual function as

$$\mathcal{R}(X; \mathbf{a}) \equiv \hat{\mathcal{H}}(\hat{\mathbf{G}}(X)) \quad (\text{C.5})$$

Once each variable is expressed as a function of $g \in \mathcal{G}$ and state variables $X = (x, y)'$, $\hat{\mathbf{G}}$ and $\mathcal{R}(X; \mathbf{a})$ can be constructed. The last (and main) step is to find the vector of coefficients \mathbf{a} so that

$$\mathcal{R}(X; \mathbf{a}) \approx \mathbf{0} \quad (\text{C.6})$$

More precisely, for some objective function ρ , I pick

$$\hat{\mathbf{a}} = \arg \min_{\mathbf{a}} \rho(\mathcal{R}(X; \mathbf{a}), \mathbf{0}) \quad (\text{C.7})$$

There exist different methods depending on the choice of ρ (i.e. different ways to project): weighted least squares, Galerkin methods, method of moments, or collocation methods. For the latter, the weight function is the Dirac delta function, i.e. the residual is set to 0 at specific points of the state space. For the orthogonal collocation that I use here, the collocation points are picked as the zeros of the basis, i.e. the Chebyshev zeros. In practice, I use $N = 30$ in most cases, and build the basis using the *CompEcon* package of [Miranda and Fackler \(2004\)](#). The optimization is based on the *fsolve* function of Matlab, and is checked with a number of optimizers from the Global Optimization Toolbox.

Instead of using the tensor product, refined ways of constructing the basis and grid are also possible such as complete polynomials or Smolyak's algorithm. They are not necessary here but could prove useful when the number of state variables increases. The approximation can also be based on a number of other polynomials such as splines.

However, for high-dimensional settings such as the ones likely to arise for extensions of the framework in this paper, those methods rapidly become computationally too costly. This is particularly so if the order of approximation needs to be high

due to the presence of strong non-linearities (e.g. with the introduction of portfolio constraints). An alternative that seems to have promise in that context is to extend projection methods by replacing the Chebysev approximation by a neural network approximation, which is naturally able to handle high-dimensional cases. I am developing those “projection methods via neural networks” for continuous-time models in [Sauzet \(2021c\)](#). Details are provided in Appendix E.3.

D. Special Cases

D.1. Planner under symmetric CRRA preferences

The social planner problem under CRRA and symmetric preferences (same parameters, except for home bias in consumption α , which is symmetric) is as follows

$$\max_{\{C_{1,u}^A, C_{2,u}^A, C_{1,u}^B, C_{2,u}^B\}_u^\infty} \mathbb{E}_t \left[\int_t^\infty e^{-\rho(u-t)} \left(\lambda \frac{C_u^{A1-\gamma}}{1-\gamma} + (1-\lambda) \frac{C_u^{B1-\gamma}}{1-\gamma} \right) du \right] \quad (\text{D.1})$$

subject to

$$dY_{1,u} = \mu_{Y_{1,u}} du + \sigma_{Y_{1,u}}^T d\vec{z}_u \quad (\text{D.2})$$

$$dY_{2,u} = \mu_{Y_{2,u}} du + \sigma_{Y_{2,u}}^T d\vec{z}_u \quad (\text{D.3})$$

$$C_{1,u}^A + C_{1,u}^B = Y_{1,u} \quad (\text{D.4})$$

$$C_{2,u}^A + C_{2,u}^B = Y_{2,u} \quad (\text{D.5})$$

$$C_u^A = \left[\alpha^{\frac{1}{\theta}} C_{1,u}^{A\frac{\theta-1}{\theta}} + (1-\alpha)^{\frac{1}{\theta}} C_{2,u}^{B\frac{\theta-1}{\theta}} \right]^{\frac{\theta}{\theta-1}} \quad (\text{D.6})$$

$$C_u^B = \left[(1-\alpha)^{\frac{1}{\theta}} C_{1,u}^{B\frac{\theta-1}{\theta}} + \alpha^{\frac{1}{\theta}} C_{2,u}^{A\frac{\theta-1}{\theta}} \right]^{\frac{\theta}{\theta-1}} \quad (\text{D.7})$$

Plugging the market-clearing condition for the two goods, and taking first-order conditions with respect to the consumption of the first good for each gives

$$\lambda \alpha^{\frac{1}{\theta}} C_t^{A\frac{1}{\theta}-\gamma} C_{1,t}^{A-\frac{1}{\theta}} = (1-\lambda)(1-\alpha)^{\frac{1}{\theta}} C_t^{B\frac{1}{\theta}-\gamma} C_{1,t}^{B-\frac{1}{\theta}} \quad (\text{D.8})$$

$$\lambda(1-\alpha)^{\frac{1}{\theta}} C_t^{A\frac{1}{\theta}-\gamma} C_{2,t}^{A-\frac{1}{\theta}} = (1-\lambda)\alpha^{\frac{1}{\theta}} C_t^{B\frac{1}{\theta}-\gamma} C_{2,t}^{B-\frac{1}{\theta}} \quad (\text{D.9})$$

Reorganizing:

$$\frac{C_{1,t}^B}{C_{1,t}^A} = \left(\frac{1-\lambda}{\lambda} \right)^\theta \left(\frac{1-\alpha}{\alpha} \right) \left(\frac{C_t^B}{C_t^A} \right)^{1-\gamma\theta} \quad (\text{D.10})$$

$$\frac{C_{2,t}^B}{C_{2,t}^A} = \left(\frac{1-\lambda}{\lambda} \right)^\theta \left(\frac{\alpha}{1-\alpha} \right) \left(\frac{C_t^B}{C_t^A} \right)^{1-\gamma\theta} \quad (\text{D.11})$$

Also note that:

$$\frac{C_{2,t}^B}{C_{1,t}^B} = \left(\frac{\alpha}{1-\alpha} \right)^2 \frac{C_{2,t}^A}{C_{1,t}^A} \quad (\text{D.12})$$

Let us use a detour via the decentralized problem and prices to make progress easily. From the static optimization for consumption baskets:

$$C_{1,t}^A = \alpha \left(\frac{p_{1,t}}{P_t^A} \right)^{-\theta} C_t^A \quad (\text{D.13})$$

$$C_{2,t}^A = (1-\alpha) \left(\frac{p_{2,t}}{P_t^A} \right)^{-\theta} C_t^A \quad (\text{D.14})$$

$$C_{1,t}^B = (1-\alpha) \left(\frac{p_{1,t}}{P_t^B} \right)^{-\theta} C_t^B \quad (\text{D.15})$$

$$C_{2,t}^B = \alpha \left(\frac{p_{2,t}}{P_t^B} \right)^{-\theta} C_t^B \quad (\text{D.16})$$

where $p_{1,t}, p_{2,t}$ are prices of goods, and P_t^A, P_t^B are the prices of the home and foreign consumption basket:

$$P_t^A = [\alpha p_{1,t}^{1-\theta} + (1-\alpha)p_{2,t}^{1-\theta}]^{\frac{1}{1-\theta}} \quad (\text{D.17})$$

$$P_t^B = [(1-\alpha)p_{1,t}^{1-\theta} + \alpha p_{2,t}^{1-\theta}]^{\frac{1}{1-\theta}} \quad (\text{D.18})$$

Plugging (D.13) and (D.14) in (D.10) yields a relationship between C_t^B/C_t^A and the real exchange rate \mathcal{E}_t

$$\mathcal{E}_t \equiv \frac{P_t^B}{P_t^A} = \left(\frac{1-\lambda}{\lambda} \right) \left(\frac{C_t^B}{C_t^A} \right)^{-\gamma} \equiv \phi \left(\frac{C_t^B}{C_t^A} \right)^{-\gamma} \quad (\text{D.19})$$

$$\Leftrightarrow \frac{C_t^B}{C_t^A} = \phi^{\frac{1}{\gamma}} \mathcal{E}_t^{-\frac{1}{\gamma}} \quad (\text{D.20})$$

This is nothing but the Backus-Smith condition in this special case. Let us show that \mathcal{E}_t is a function of $Y_{1,t}/Y_{2,t}$ only, so that C_t^B/C_t^A is too. To do so, I first look for an equation for $q_t \equiv p_{2,t}/p_{1,t}$, the terms of trade, as a function of which \mathcal{E}_t and all other prices can be expressed.

Plugging (D.13) into (D.16) in the market-clearing condition for goods yields

$$\alpha \left(\frac{p_{1,t}}{P_t^A} \right)^{-\theta} C_t^A + (1 - \alpha) \left(\frac{p_{t1,t}}{P_t^B} \right)^{-\theta} C_t^B = Y_{1,t} \quad (\text{D.21})$$

$$(1 - \alpha) \left(\frac{p_{2,t}}{P_t^A} \right)^{-\theta} C_t^A + \alpha \left(\frac{p_{2,t}}{P_t^B} \right)^{-\theta} C_t^B = Y_{2,t} \quad (\text{D.22})$$

Dividing the two:

$$q_t^{-\theta} \left(\frac{\alpha + (1 - \alpha) \mathcal{E}_t^{\theta} \frac{C_t^B}{C_t^A}}{(1 - \alpha) + \alpha \mathcal{E}_t^{\theta} \frac{C_t^B}{C_t^A}} \right) = \frac{Y_{1,t}}{Y_{2,t}} = \frac{y_t}{1 - y_t} \quad (\text{D.23})$$

$$\Rightarrow q_t = S_t^{\frac{1}{\theta}} \left(\frac{Y_{1,t}}{Y_{2,t}} \right)^{\frac{1}{\theta}} = S_t^{\frac{1}{\theta}} \left(\frac{y_t}{1 - y_t} \right)^{\frac{1}{\theta}} \quad (\text{D.24})$$

where

$$S_t = \frac{(1 - \alpha) + \alpha \mathcal{E}_t^{\theta} \frac{C_t^B}{C_t^A}}{\alpha + (1 - \alpha) \mathcal{E}_t^{\theta} \frac{C_t^B}{C_t^A}} = \frac{(1 - \alpha) + \alpha \phi^{\frac{1}{\gamma}} \mathcal{E}_t^{\theta - \frac{1}{\gamma}}}{\alpha + (1 - \alpha) \phi^{\frac{1}{\gamma}} \mathcal{E}_t^{\theta - \frac{1}{\gamma}}} \quad (\text{D.25})$$

As a side note, if the IES is equal to the elasticity of substitution between goods ($\psi = \gamma^{-1} = \theta$)

$$q_t = \bar{S}^{\frac{1}{\theta}} \left(\frac{Y_{1,t}}{Y_{2,t}} \right)^{\frac{1}{\theta}} \quad \text{with} \quad \bar{S} = \frac{(1 - \alpha) + \alpha \phi^{\frac{1}{\gamma}}}{\alpha + (1 - \alpha) \phi^{\frac{1}{\gamma}}} \quad (\text{D.26})$$

To find an equation for q_t in the general case, let us use the expression for \mathcal{E}_t as a function of q_t

$$\mathcal{E}_t = \frac{P_t^B}{P_t^A} = \left(\frac{(1 - \alpha) + \alpha q_t^{1-\theta}}{\alpha + (1 - \alpha) q_t^{1-\theta}} \right)^{\frac{1}{1-\theta}} \quad (\text{D.27})$$

Plugging this expression in the above, this yields a non-linear equation for q_t as a

function of $Y_{1,t}/Y_{2,t} = y_t/(1 - y_t)$

$$q_t^\theta = \frac{(1 - \alpha) + \alpha \phi^{\frac{1}{\gamma}} \left(\frac{(1-\alpha)+\alpha q_t^{1-\theta}}{\alpha+(1-\alpha)q_t^{1-\theta}} \right)^{\frac{\gamma\theta-1}{\gamma(1-\theta)}}}{\alpha + (1 - \alpha) \phi^{\frac{1}{\gamma}} \left(\frac{(1-\alpha)+\alpha q_t^{1-\theta}}{\alpha+(1-\alpha)q_t^{1-\theta}} \right)^{\frac{\gamma\theta-1}{\gamma(1-\theta)}}} \left(\frac{y_t}{1 - y_t} \right) \quad (\text{D.28})$$

I solve for q_t as a function of $y_t = Y_{1,t}/(Y_{1,t} + Y_{2,t})$ because this variable is in $[0, 1]$. This is more stable than to solve for a function on $[0, \infty)$. It also makes comparing this solution to the decentralized one easier. To do so, I approximate $q(y_t)$ using Chebyshev polynomials of order $N = 100$, on $N + 1$ grid points.

Once I obtain $q_t = q(y_t)$, \mathcal{E}_t follows from (D.27), C_t^B/C_t^A follows from (D.20), $C_{1,t}^B/C_{1,t}^A$ and $C_{2,t}^B/C_{2,t}^A$ from (D.10) and (D.11), and $C_{2,t}^A/C_{1,t}^A$ and $C_{2,t}^B/C_{1,t}^B$ from

$$\frac{C_{2,t}^A}{C_{1,t}^A} = \frac{(1 - \alpha) \left(\frac{p_{2,t}}{P_t^A} \right)^{-\theta} C_t^A}{\alpha \left(\frac{p_{1,t}}{P_t^A} \right)^{-\theta} C_t^A} = \left(\frac{1 - \alpha}{\alpha} \right) q_t^{-\theta} \quad (\text{D.29})$$

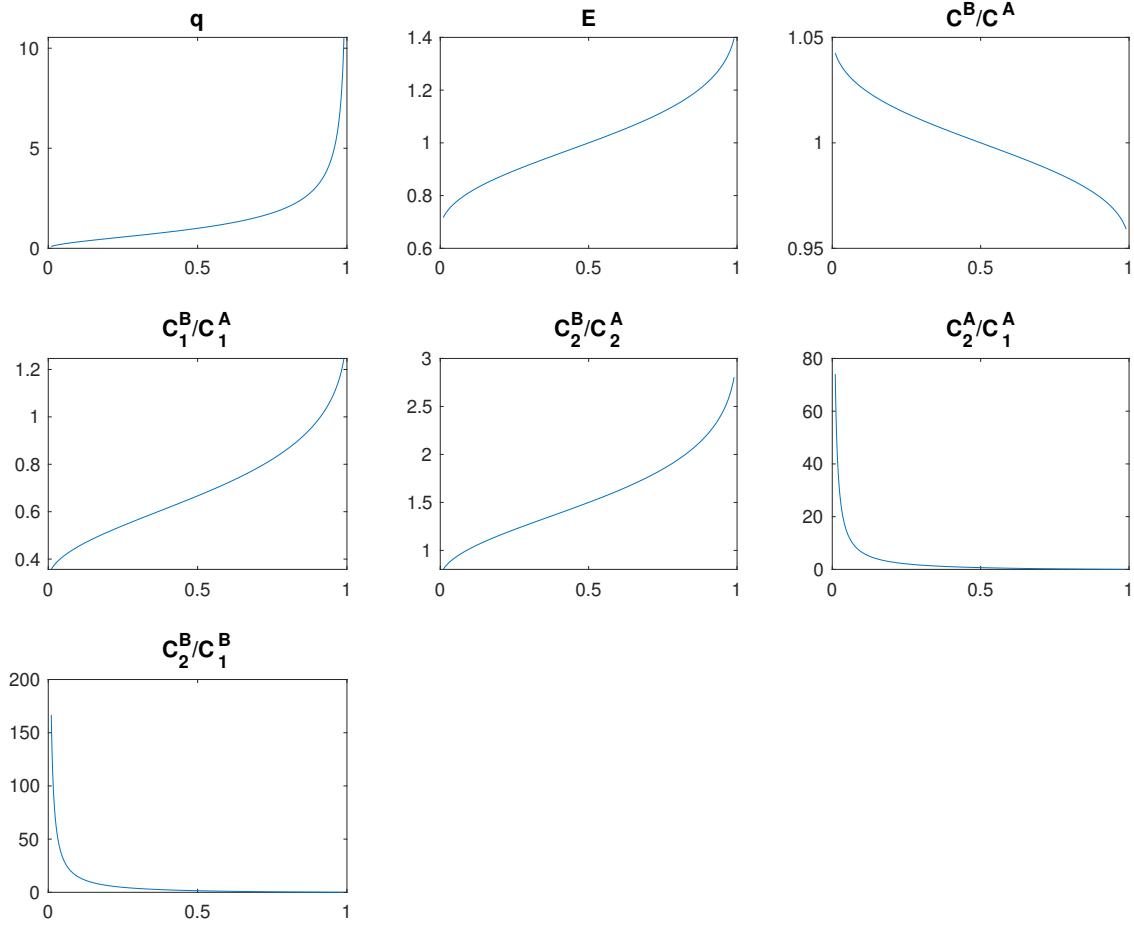
$$\frac{C_{2,t}^B}{C_{1,t}^B} = \frac{\alpha \left(p_{2,t}/P_t^B \right)^{-\theta} C_t^B}{(1 - \alpha) \left(\frac{p_t}{P_t^B} \right)^{-\theta} C_t^B} = \left(\frac{\alpha}{1 - \alpha} \right) q_t^{-\theta} \quad (\text{D.30})$$

To obtain the variables in levels, we can also used the formulas derived above. Denote

$$\frac{C_{1,t}^B}{C_{1,t}^A} \equiv g_1(y_t) \quad (\text{D.31})$$

$$\frac{C_{2,t}^B}{C_{2,t}^A} \equiv g_2(y_t) \quad (\text{D.32})$$

The resulting functions are shown in the Figure below.



Using the market-clearing condition:

$$C_{1,t}^A = \left(\frac{1}{1 + g_1(y_t)} \right) Y_t \equiv h_1(y_t) Y_{1,t} \quad (\text{D.33})$$

$$C_{1,t}^B = \left(\frac{g_1(y_t)}{1 + g_1(y_t)} \right) Y_{1,t} \equiv (1 - h_1(y_t)) Y_{1,t} \quad (\text{D.34})$$

$$C_{2,t}^A = \left(\frac{1}{1 + g_2(y_t)} \right) Y_{2,t} \equiv (1 - h_2(y_t)) Y_{2,t} \quad (\text{D.35})$$

$$C_{2,t}^B = \left(\frac{g_2(y_t)}{1 + g_2(y_t)} \right) Y_{2,t} \equiv h_2(y_t) Y_{2,t} \quad (\text{D.36})$$

Aggregate consumptions can be obtained by plugging the above in their definitions

$$C_t^A = \left[\alpha^{\frac{1}{\theta}} (h_1(y_t) Y_{1,t})^{\frac{\theta-1}{\theta}} + (1-\alpha)^{\frac{1}{\theta}} ((1-h_2(y_t)) Y_{2,t})^{\frac{\theta-1}{\theta}} \right]^{\frac{\theta}{\theta-1}} \quad (\text{D.37})$$

$$C_t^B = \left[(1-\alpha)^{\frac{1}{\theta}} ((1-h_1(y_t)) Y_{1,t})^{\frac{\theta-1}{\theta}} + \alpha^{\frac{1}{\theta}} (h_2(y_t) Y_{2,t})^{\frac{\theta-1}{\theta}} \right]^{\frac{\theta}{\theta-1}} \quad (\text{D.38})$$

Let us now focus on further variables of interest for asset pricing: equity prices, and wealth.

$$Q_{1,t} = \mathbb{E}_t \left[\int_t^\infty \frac{\xi_u^A}{\xi_t^A} p_{1,u} Y_{1,u} du \right] \quad (\text{D.39})$$

$$Q_{2,t} = \mathbb{E}_t \left[\int_t^\infty \frac{\xi_u^B}{\xi_t^B} p_{2,u} Y_{2,u} du \right] \quad (\text{D.40})$$

$$W_t^A = \mathbb{E}_t \left[\int_t^\infty \frac{\xi_u^A}{\xi_t^A} P_u^A C_u^A du \right] \quad (\text{D.41})$$

$$W_t^B = \mathbb{E}_t \left[\int_t^\infty \frac{\xi_u^B}{\xi_t^B} P_u^B C_u^B du \right] \quad (\text{D.42})$$

ξ_t^A, ξ_t^B are the stochastic discount factors for investors A and B

$$\xi_t^A \equiv e^{-\rho t} P_t^{A-1} C_t^{A-\gamma} \quad (\text{D.43})$$

$$\xi_t^B \equiv e^{-\rho t} P_t^{B-1} C_t^{B-\gamma} \quad (\text{D.44})$$

In this complete-market world, they are related by the following relation

$$\xi_t^A = \phi \xi_t^B = \left(\frac{1-\lambda}{\lambda} \right) \xi_t^B \quad (\text{D.45})$$

which is nothing but equation (D.20) above, i.e. the Backus-Smith condition.

From here, I then obtain ODEs for the following functions (in fact I obtain it for

$J_t = \frac{P_t^\psi C_t}{W_t}$ to match the decentralized solution)

$$F_{1,t}^{-1} \equiv \frac{Q_{1,t}}{p_{1,t} Y_{1,t}} = \mathbb{E}_t \left[\int_t^{\infty} \frac{\xi_u^A}{\xi_t^A} \frac{p_{1,u}}{p_{1,t}} \frac{Y_{1,u}}{Y_{1,t}} du \right] \quad (\text{D.46})$$

$$F_{2,t}^{-1} \equiv \frac{Q_{2,t}}{p_{2,t} Y_{2,t}} = \mathbb{E}_t \left[\int_t^{\infty} \frac{\xi_u^B}{\xi_t^B} \frac{p_{2,u}}{p_{2,t}} \frac{Y_{2,u}}{Y_{2,t}} du \right] \quad (\text{D.47})$$

$$J_t^{A-1} \equiv \frac{W_t^A}{P_t^A C_t^A} = \mathbb{E}_t \left[\int_t^{\infty} \frac{\xi_u^A}{\xi_t^A} \frac{P_u^A}{P_t^A} \frac{C_u^A}{C_t^A} du \right] \quad (\text{D.48})$$

$$J_t^{B-1} \equiv \frac{W_t^B}{P_t^B C_t^B} = \mathbb{E}_t \left[\int_t^{\infty} \frac{\xi_u^B}{\xi_t^B} \frac{P_u^B}{P_t^B} \frac{C_u^B}{C_t^B} du \right] \quad (\text{D.49})$$

After deriving and solving those ODEs, the equilibrium obtained is the same as the one from the decentralized solution under CRRA preferences.

Solving for the equilibrium using the planner could be extended to recursive preferences following the approach in [Dumas et al. \(2000\)](#).

E. Extensions

E.1. Extension 1: global asset manager and the Global Financial Cycle ([Sauzet, 2021d](#))

From the perspective of modeling the international financial system, an aspect that is increasingly being recognized as primordial is the role of global financial intermediaries. Those global intermediaries can be involved in the dealing of foreign currencies, in the spirit of [Hau and Rey \(2006\)](#) and [Gabaix and Maggiori \(2015\)](#), can play the role of bankers as in [Maggiori \(2017\)](#) and [Jiang et al. \(2020\)](#), or can play the role of global asset managers, like below. The main intuition is that because of their different preferences and limited risk-bearing capacity, the capitalization of those financial intermediaries is a prime determinant of asset prices, interest rates, exchange rates, and other economic outcomes worldwide. The presence of such global intermediaries is not only relevant from the perspective of realism, but could introduce a mechanism through which to capture additional aspects of the Global Financial Cycle of [Rey \(2013\)](#) and [Miranda-Agrippino and Rey \(2020\)](#), pertaining to the leverage and role of intermediaries. By way of an example, I briefly present one, the addition of a global asset manager, that I am exploring in ongoing work [Sauzet \(2021d\)](#). Figure F.2 summarizes the set-up.

The global asset manager constitutes a third type of investor, whose preferences, albeit still recursive and over the two goods, have the following specificities: (i) because she is a global citizen, the global asset manager has no particular bias towards any of the goods, and (ii) she is significantly more risk-tolerant than the consumer-investor of each country. The last point is in the spirit of the intermediary asset pricing literature, which typically models bankers as agents with lower risk aversion. Even though the current version of this work does not feature them, the limited risk-bearing capacity of the global asset manager, in the form for instance of portfolio constraints, will be an important addition.

I use the notation for the international application of Section 5.1 and [Sauzet \(2021a\)](#) in which variables for the domestic investor (investor A) are denoted without superscript, while those of the foreign investor (investor B) are denoted by *. The

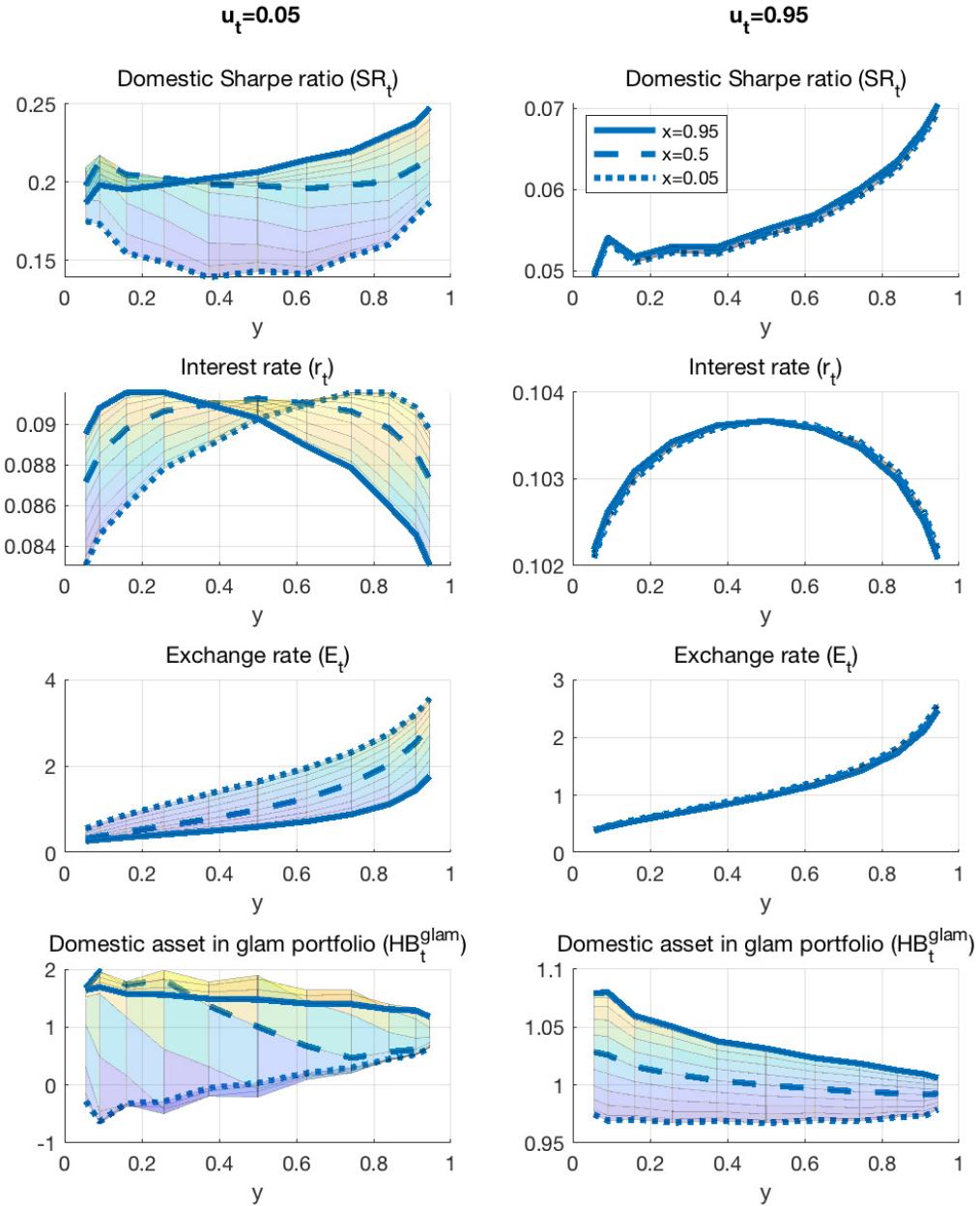
equilibrium can be represented as a function of three state variables, $X_t \equiv (x_t, y_t, u_t)'$. x_t is the wealth share of the domestic investor and is defined as before with the caveat that now, $W_t + W_t^*$ does not sum up to total world wealth, which is $W_t + W_t^* + W_t^{glam}$ and includes the wealth of the global asset manager W_t^{glam} . y_t still captures the relative supply of the goods. u_t , the new state variable, captures the share of world wealth held by the global asset manager.⁸⁴ In summary:

$$x_t \equiv \frac{W_t}{W_t + W_t^*} ; \quad y_t \equiv \frac{Y_t}{Y_t + Y_t^*} ; \quad u_t \equiv \frac{W_t^{glam}}{W_t + W_t^* + W_t^{glam}} \quad (\text{E.1})$$

Equations are presented in [Sauzet \(2021d\)](#), and Figure E.1 shows the results. The preference heterogeneity of the global asset manager, coupled with that of the investor of each country, is able to generate rich patterns in global asset prices, interest rates, goods prices, and portfolios, even without portfolio constraints. For instance, the Sharpe ratio on the domestic asset is much larger when the global asset manager is poorly capitalized (u_t small), reflecting the higher compensation for risk required by the domestic and foreign consumer-investors to hold the domestic equity asset. This is also true for foreign equity, and points to the fact that a poorly capitalized global asset manager, a proxy more generally for the global financial system, leads to increased risk premia throughout the world, in a pattern reminiscent of a Global Financial Cycle. This mechanism could complement the one stemming from the role of the domestic country as world banker discussed in the international application of Section 5.1 and [Sauzet \(2021a\)](#), by introducing financial intermediaries in the picture. When this happens, the risk premia on equity assets are also more dependent on the repartition of wealth across the remaining investors, captured by x_t , consistent with a crisis situation in which the identity of the average holder of an asset matters more and assets rapidly changing hands are accompanied by large swings in returns. The capitalization of the global asset manager also matters for interest rate, which tends to decrease as u_t gets small, reflecting a lower average risk tolerance in the economy, which corresponds with a higher demand for the safe asset (the international riskless bond). Goods prices are also affected, with the exchange rate depending significantly more on the allocation of wealth across consumer-investor.

⁸⁴The share of the domestic and foreign investor in world wealth are now obtained as $x_t(1 - u_t)$ and $x_t u_t$.

Figure E.1: Equilibrium in the presence of a global asset manager



Notes: Calibration: $\gamma^{glam} = 2 < \gamma = \gamma^* = 8, \psi = 0.2, \alpha = 0.85, \rho = 1\%$. x_t is the wealth share of the domestic investor as a fraction of $W_t + W_t^*$. y_t is the relative supply of the domestic good, which captures fundamentals. u_t is the share of world wealth held by the global asset manager.

Note also the impact on portfolios: not only is the portfolio of the global asset manager getting further from the market portfolio as u_t decreases, but it is also increasingly affected by the allocation of wealth among the remaining consumer-investors. This reflects the fact that because she is not biased towards any particular asset, the global asset manager is here to pick up the opposite side of the trades for the other two investors, and this leads to wild changes in her portfolios especially as she gets less well-capitalized.

This brief illustration shows the promise of introducing global financial intermediaries in the framework of this paper, and highlights how it can complement the mechanisms discussed previously in the main application.

E.2. Extension 2: towards a solution to the reserve currency paradox ([Sauzet, 2021f](#))

In addition to global financial intermediaries, further extensions of the framework in an international context could help make way towards resolving the so-called “reserve currency paradox” emphasized by [Maggiori \(2017\)](#) and to which the reader is referred for details. The paradox appears as follows in the framework of my paper. Consider again that the domestic country represents the United States, the risk-tolerant country at the center of the international financial system. As is discussed in Section 5.1 and [Sauzet \(2021a\)](#), and consistent with [Gourinchas et al. \(2017\)](#): in normal times, the country enjoys an exorbitant privilege by earning higher returns on average due to its riskier position, but in crisis times, it bears the exorbitant duty of insuring the rest of the world through a wealth transfer. In turn, because of the home bias in consumption, this wealth transfer towards the rest of the world tends to increase the price of foreign goods, which pushes up the price of the foreign basket and lead the domestic currency, the US dollar in this case, to depreciate. The reserve currency paradox resides in the fact that this is clearly counterfactual: empirically, the US dollar tends to *appreciate* in crisis, which is one of the main reasons why it is the world’s major reserve currency in the first place. As discussed in [Maggiori \(2017\)](#), this paradox does not depend on the specifics of the underlying model – for instance, the framework in this paper is quite different from his. Instead, it is deeply rooted in

the presence of the home bias in consumption, an aspect that goes back all the way to the classical “transfer problem” of Keynes and Ohlin discussed previously.

[Maggiori \(2017\)](#) presents a potential resolution based on trade costs depending negatively on the capitalization of financial intermediaries. Another part of the story, that I plan to implement in the current framework, relies on the importance of trade in bonds. Specifically, times of crisis are periods in which the demand for safe assets usually skyrockets (“risk-off” episodes). Because the United States is the main provider of safe asset worldwide, this sudden increase in the demand for US Treasuries goes hand-in-hand with a strong upward pressure on the currency in which they are denominated. This, in my view, is one of the main ultimate drivers of US dollar appreciation in times of crisis. To introduce such channels in the framework developed in this paper, I plan to include the following elements in future extensions ([Sauzet \(2021f\)](#), ongoing work). First, the demand for safe asset must be meaningfully time-varying, which I plan to generate from time-varying risk aversion in the form of heterogeneous investors with varying degrees of risk aversion *within countries*. A risk-off episode would therefore correspond to an event in which the risk-tolerant investor of a country is poorly capitalized. Second, the bond of the center country should be particularly attractive in difficult times⁸⁵, which could come from an ad-hoc feature or potentially by assuming that the size of the center country is larger so that its bond ensures against a larger share of world shocks, in the spirit of [Hassan \(2013\)](#).⁸⁶ Third, for this “trade in assets” channel to matter enough for exchange rates so as to reverse the reserve currency paradox driven by the trade in goods, the introduction of global financial intermediaries will be important quantitatively. They could take the form of global asset managers as presented above, intermediating trade in assets, or of global foreign currency dealers in the spirit of [Hau and Rey \(2006\)](#) and [Gabaix and Maggiori \(2015\)](#). Their role would be to ensure that, like in practice, the increased demand for bonds is met with limited capacity, which ultimately leads to an upward pressure on the price of the US currency. Finally, the introduction of portfolio constraints, for both global intermediaries and for the different investors within each country, as well as other sources of market incompleteness, will also prove

⁸⁵A related and subtle point is to disentangle the extent to which the upward pressure on US Treasuries is itself driven by the safety of the US dollar in times of crisis.

⁸⁶To do this, reformulating the output share y_t by adapting the share process of [Menzly et al. \(2004\)](#); [Santos and Veronesi \(2006\)](#) as mentioned previously could be particularly useful.

important for the mechanism to have bite quantitatively.

E.3. Extension 3: projection methods via neural networks ([Sauzet, 2021c](#))

The extensions above make clear that the number of state variables is likely to rapidly increase with additions to the framework. Projection methods are conceptually well-suited to contexts with multiple state variables, and are typically better able to handle a larger number of them than other approaches like finite-difference methods, which become rapidly computationally too costly.⁸⁷ As a result, they are well-adapted to the environment in this paper. To be sure however, computationally, traditional projection methods also are very much subject to the curse of dimensionality, and scaling the number of state variables further up will prove limited using standard Chebyshev polynomials. For instance, even the addition of a third state variable, like in the global asset manager extension above, renders the resolution significantly slower, and increasing the order of approximation much beyond $N = 10$ proves difficult. More refined ways to construct the Chebyshev polynomials and corresponding grids, such as complete polynomials or Smolyak’s algorithm, could help. Ultimately however, they are also limited and methods able to handle higher-dimensional cases will be required.

One such method consists in naturally extending the concept of projection approaches, but to replace the Chebyshev polynomials in the approximation by neural networks. In ongoing work ([Sauzet, 2021c](#)), I am developing these “projection methods via neural networks” to be applied to continuous-time problems like the one in this paper. The use not only of neural networks, but of the whole eco-system of related packages, proves of tremendous importance. First, those packages and environments, like Tensor Flow on which my implementation is based, are specifically designed for very high-dimensional contexts such as computer vision or other artificial-intelligence-type problems. As such, they are able to handle billions of observations and multiple millions of parameters. Even in the framework of this paper, this would allow me to focus on a much finer grid than do Chebyshev polynomials. Second, provided that one is judicious in the choice of the specification of the neural networks (typically in the choice of activation functions), they are naturally able to handle very non-linear

⁸⁷The method currently developed in [Hansen et al. \(2018\)](#) could potentially help from that perspective.

functions. This aspect will prove particularly important when introducing portfolio constraints, which typically lead to sharp non-linearities, and are not necessarily handled well by Chebyshev polynomials especially of low order. Third, fitting neural networks conceptually in a projection framework is also particularly useful. Contrary to other methods based on neural networks that are more akin to value function iteration, e.g. [Duarte \(2019\)](#), a method expressed in a projection approach framework is able to naturally handle even cases for which value function iteration is difficult to adapt. For instance, economies with multiple agents and incomplete markets, for which there are several value functions as well as other unknown functions, would be difficult to cast in a value function iteration framework, but pose no particular problem for projection methods via neural networks.

Overall, the method has promise. For instance, I solve a “Ten Trees” equivalent to [Cochrane et al. \(2008\)](#)’s “Two Trees” without particular difficulty, a fit that would prove impossible for Chebyshev polynomials, and even less so for finite-different methods.⁸⁸

⁸⁸On this problem, [Martin \(2013\)](#) proposes an alternative method that proves promising even with five or six trees, and possibly more. The method also allows for jumps.

F. Additional Figures

F.1. Economic set-up

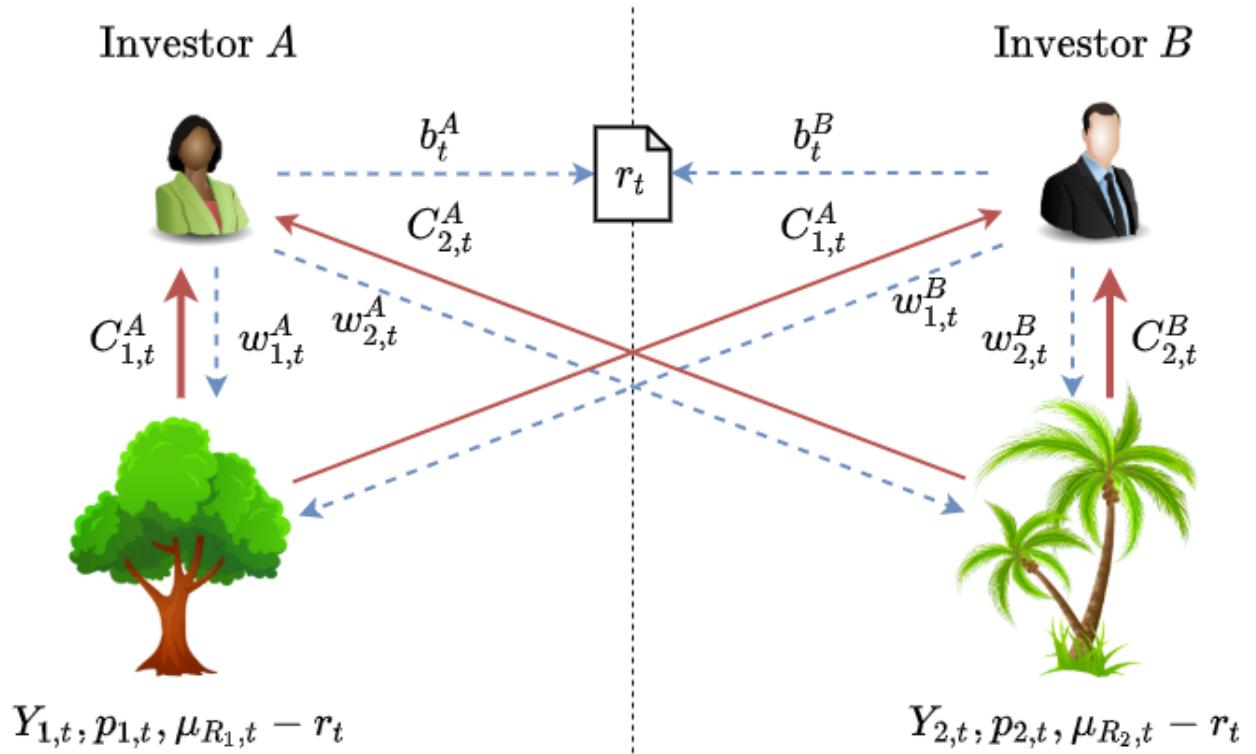


Figure F.1: Baseline economy

Notes: Good 1 (2) is the preferred (“local”) good of investor *A* (*B*). Back to main text: Section 2.

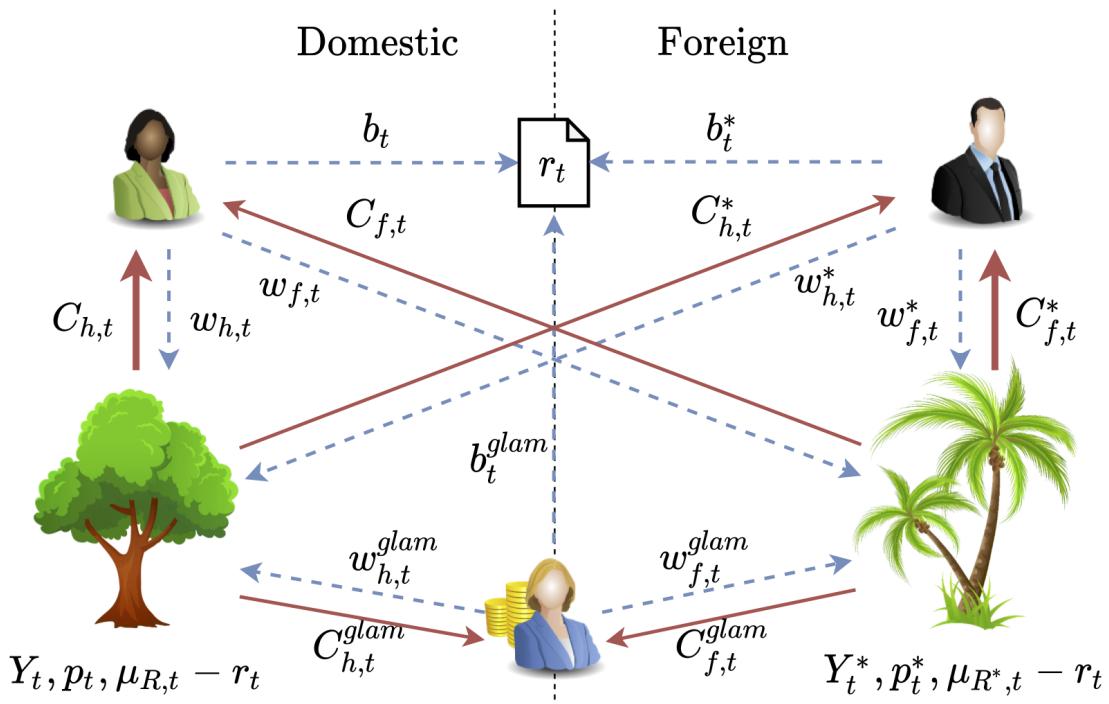


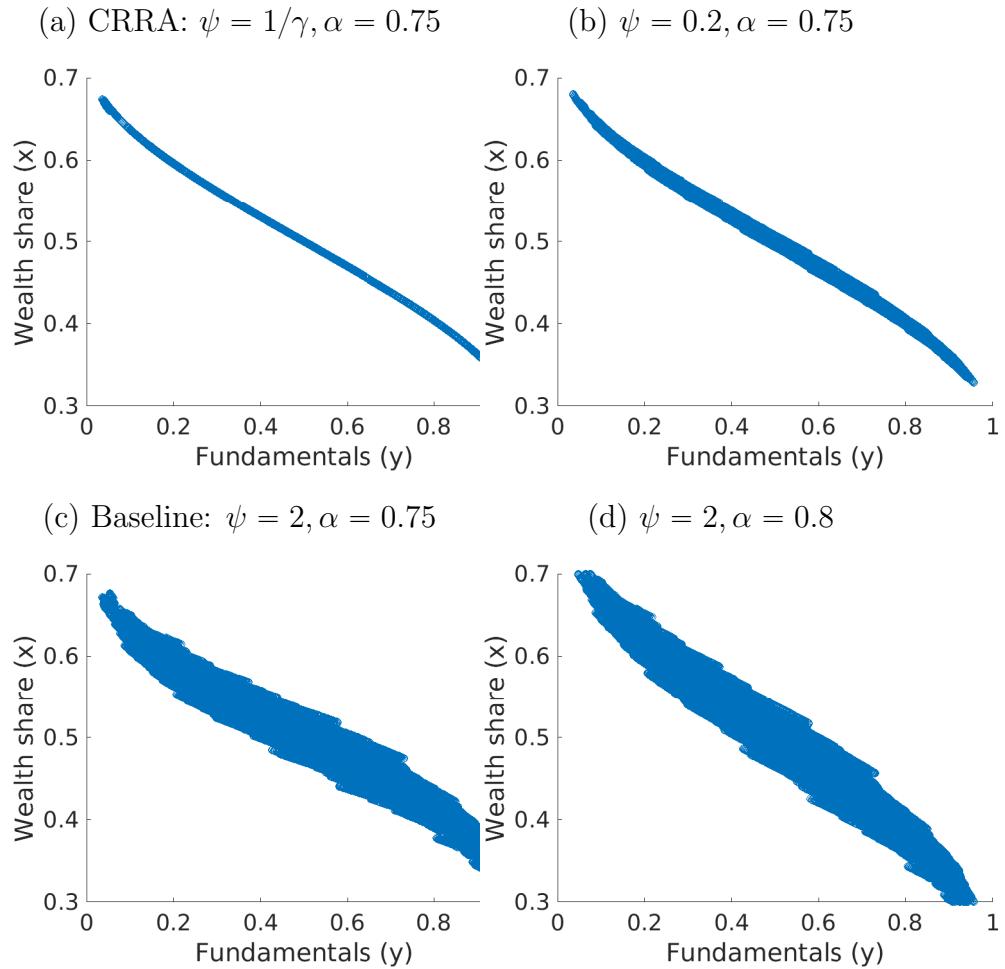
Figure F.2: International economy in the presence of a global asset manager

Notes: Back to main text: Section 5.3, back to Appendix: Section E.1.

F.2. Distributions

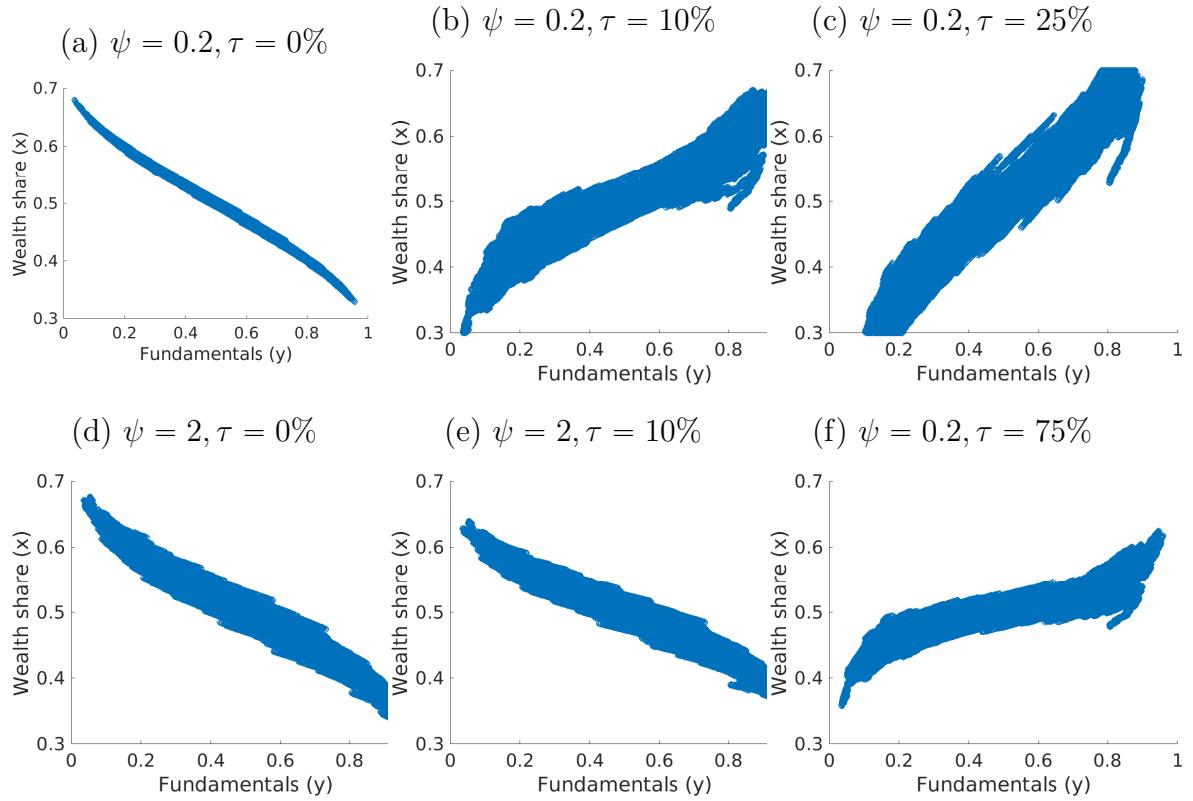
All distributions, unless otherwise specified are obtained from $nsim = 1,000$ paths of length $T = 250$ years, with $dt = 0.01$ (biweekly frequency), starting from $X_0 = (1/2, 1/2)$. The distributions are shown from the top, and for visibility each point visited during the simulation is shown with the same intensity.

Figure F.3: Distribution of the state variables in the baseline calibration



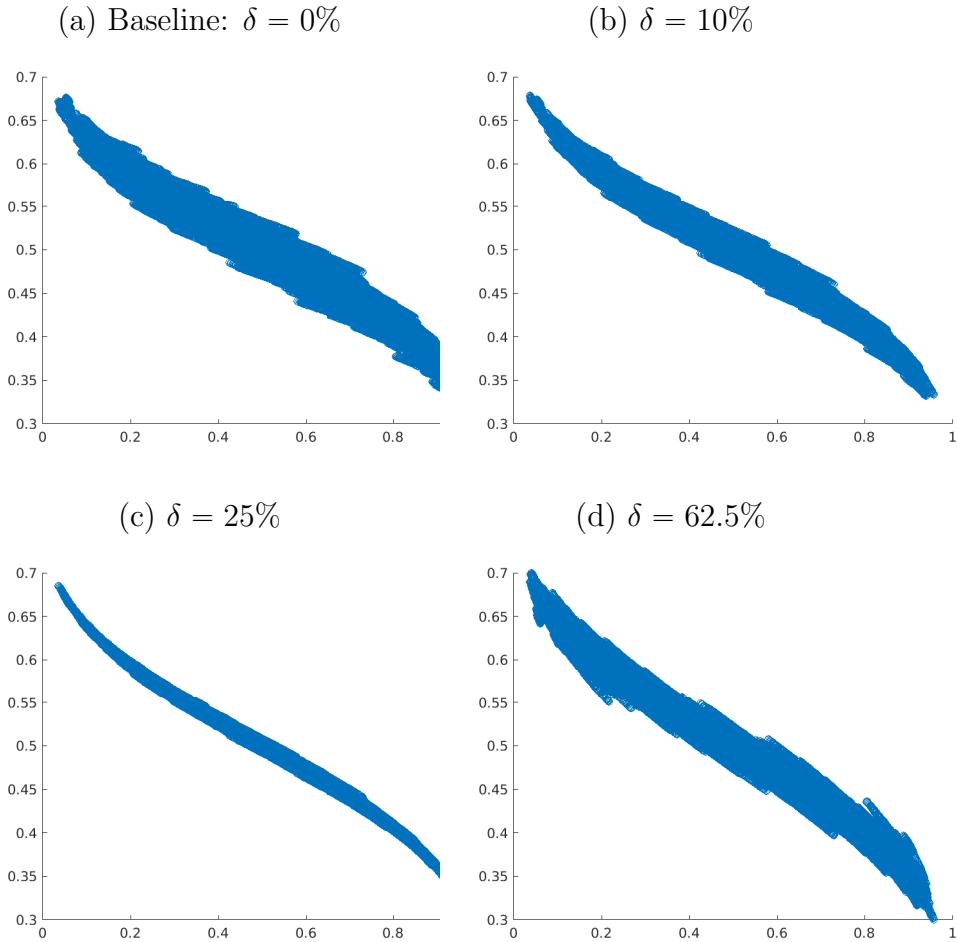
Notes: x_t , the wealth share, which captures the share of total wealth held by investor A , is shown on the vertical axis. y_t , the output share of the first good, which captures fundamentals, is shown on horizontal axis. Distribution seen from the top, and obtained from $nsim = 1,000$ paths of length $T = 250$, with $dt = 0.01$, starting from $X_0 = (1/2, 1/2)$.

Figure F.4: Distribution of the state variables under imperfect financial integration



Notes: Based on the symmetric calibration of Assumption 1 (specifically $\psi = 2$), except for imperfect financial integration (τ). x_t , the wealth share, which captures the share of total wealth held by investor A , is shown on the vertical axis. y_t , the output share of the first good, which captures fundamentals, is shown on horizontal axis. Distribution seen from the top, and obtained from $nsim = 1,000$ paths of length $T = 250$, with $dt = 0.01$, starting from $X_0 = (1/2, 1/2)$.

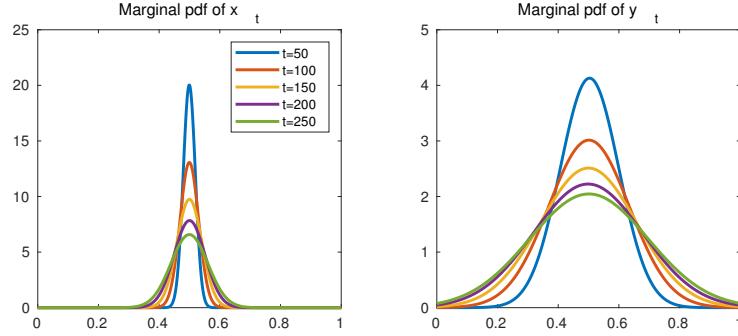
Figure F.5: Distribution of the state variables in the presence of labor income (δ)



Notes: Based on the symmetric calibration under perfect risk sharing of Assumption 1 (specifically $\psi = 2$), except for labor income (δ). x_t , the wealth share, which captures the share of total wealth held by investor A , is shown on the vertical axis. y_t , the output share of the first good, which captures fundamentals, is shown on horizontal axis. Distribution seen from the top, and obtained from $nsim = 1,000$ paths of length $T = 250$, with $dt = 0.01$, starting from $X_0 = (1/2, 1/2)$.

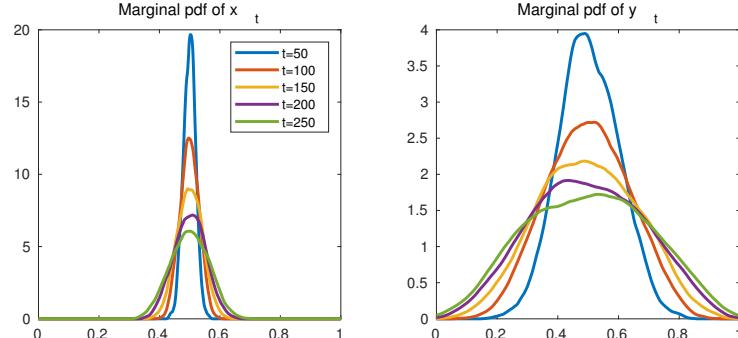
F.3. Evolution of the distribution of X_t over time

Figure F.6: Marginal distributions for x_t and y_t over time (Normal kernel, baseline calibration)



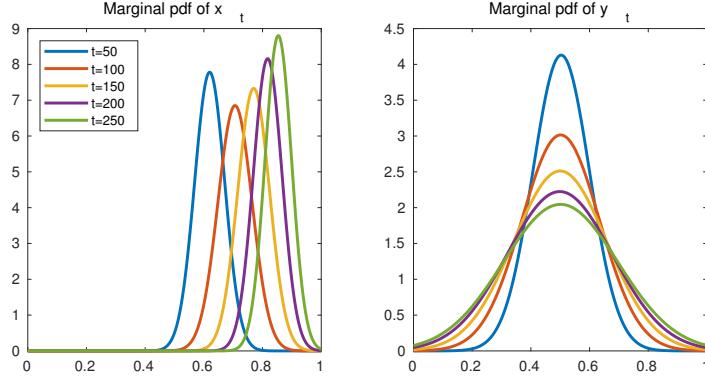
Notes: x_t is the wealth share, which captures the share of total wealth held by investor A. y_t is the relative supply of the first good, which captures fundamentals. Distribution obtained from $nsim = 1,000$ paths of length $T = 250$, with $dt = 0.01$, starting from $X_0 = (1/2, 1/2)$.

Figure F.7: Marginal distributions for x_t and y_t over time (Epanechnikov kernel, baseline calibration)



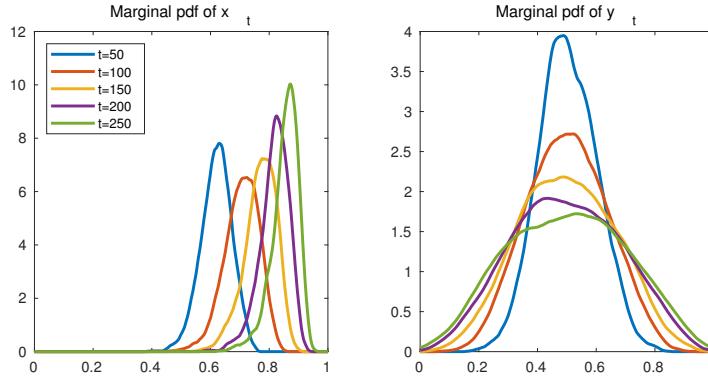
Notes: x_t is the wealth share, which captures the share of total wealth held by investor A. y_t is the relative supply of the first good, which captures fundamentals. Distribution obtained from $nsim = 1,000$ paths of length $T = 250$, with $dt = 0.01$, starting from $X_0 = (1/2, 1/2)$.

Figure F.8: Marginal distributions for x_t and y_t over time (Normal kernel, $\gamma = 7.5 < \gamma^* = 15$)



Notes: x_t is the wealth share, which captures the share of total wealth held by investor A. y_t is the relative supply of the first good, which captures fundamentals. Distribution obtained from $nsim = 1,000$ paths of length $T = 250$, with $dt = 0.01$, starting from $X_0 = (1/2, 1/2)$.

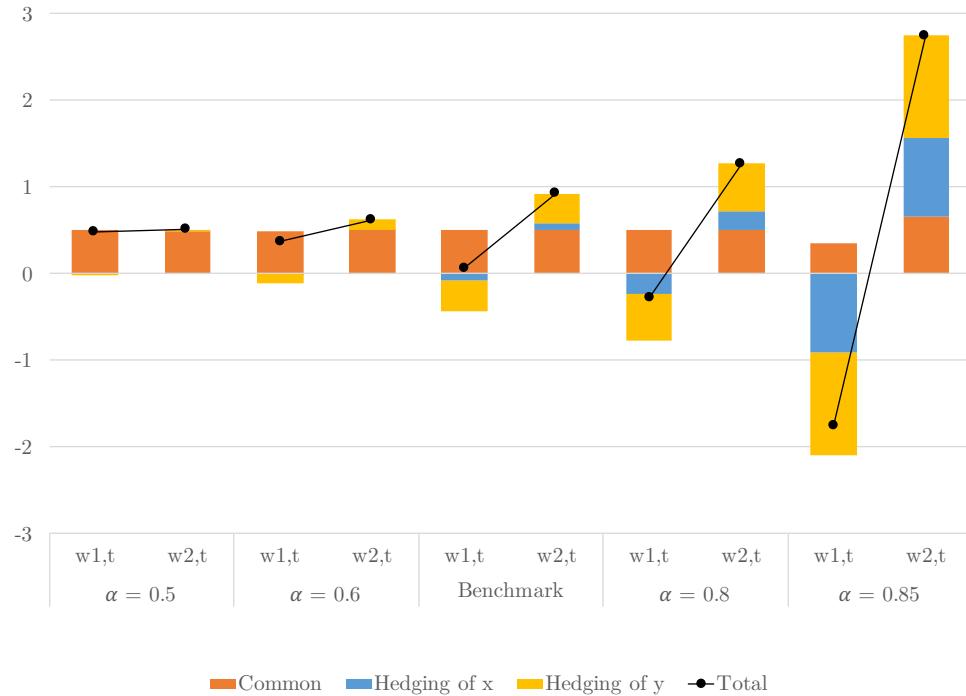
Figure F.9: Marginal distributions for x_t and y_t over time (Epanechnikov kernel, $\gamma = 7.5 < \gamma^* = 15$)



Notes: x_t is the wealth share, which captures the share of total wealth held by investor A. y_t is the relative supply of the first good, which captures fundamentals. Distribution obtained from $nsim = 1,000$ paths of length $T = 250$, with $dt = 0.01$, starting from $X_0 = (1/2, 1/2)$.

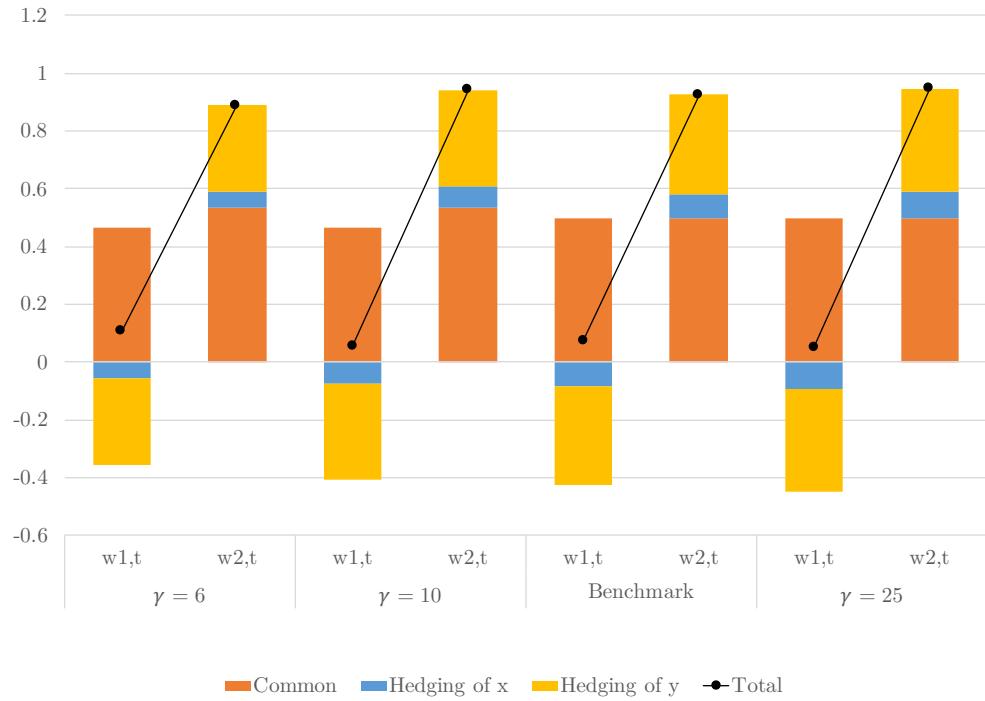
F.4. Portfolios at the symmetric point

Figure F.10: Equity portfolio for investor A at $X_t = (1/2, 1/2)$ and bias in consumption α



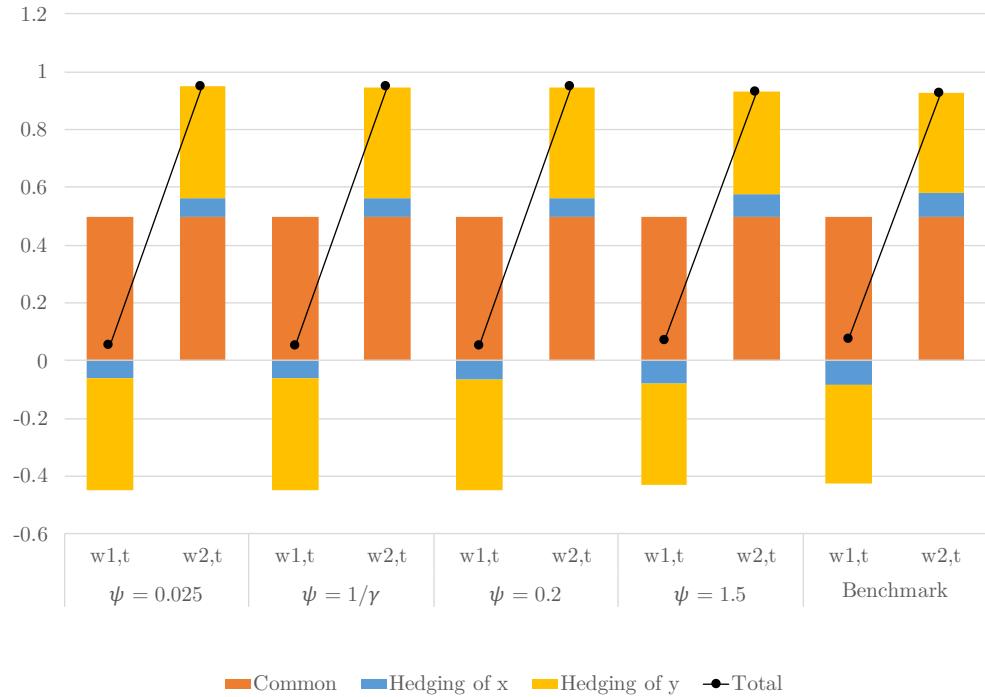
Notes: Based on the symmetric calibration under perfect risk sharing of Assumption 1, except α . The figure shows portfolios when both the allocation of wealth (x_t) and the relative supply (y_t) are symmetric, $X_t = (1/2, 1/2)$.

Figure F.11: Equity portfolio for investor A at $X_t = (1/2, 1/2)$ and risk aversion γ



Notes: Based on the symmetric calibration under perfect risk sharing of Assumption 1, except γ . The figure shows portfolios when both the allocation of wealth (x_t) and the relative supply (y_t) are symmetric, $X_t = (1/2, 1/2)$.

Figure F.12: Equity portfolio for investor A at $X_t = (1/2, 1/2)$ and elasticity of intertemp. substitution ψ

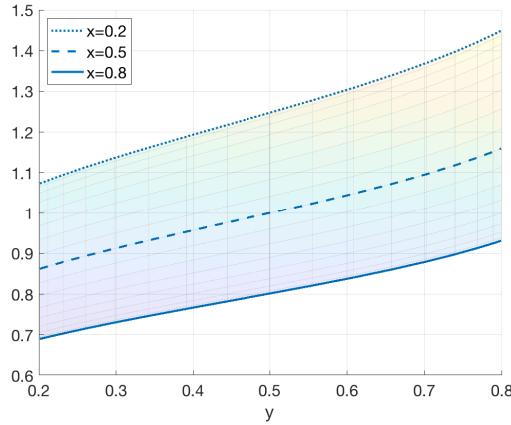


Notes: Based on the symmetric calibration under perfect risk sharing of Assumption 1, except ψ . The figure shows portfolios when both the allocation of wealth (x_t) and the relative supply (y_t) are symmetric, $X_t = (1/2, 1/2)$.

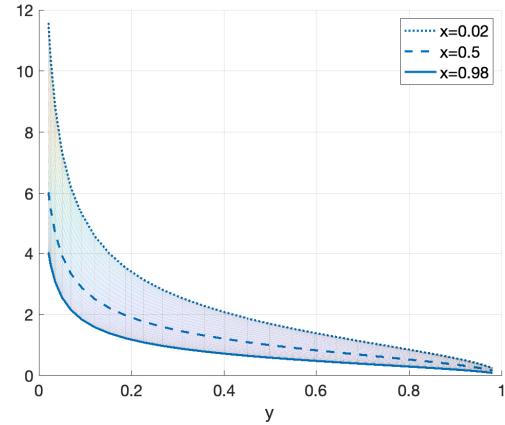
F.5. Representations as a function of both state variables

Figure F.13: Relative dividends: $p_{2,t}Y_{2,t}/(p_{1,t}Y_{1,t})$

(a) $\theta = 0.9^* < 1$

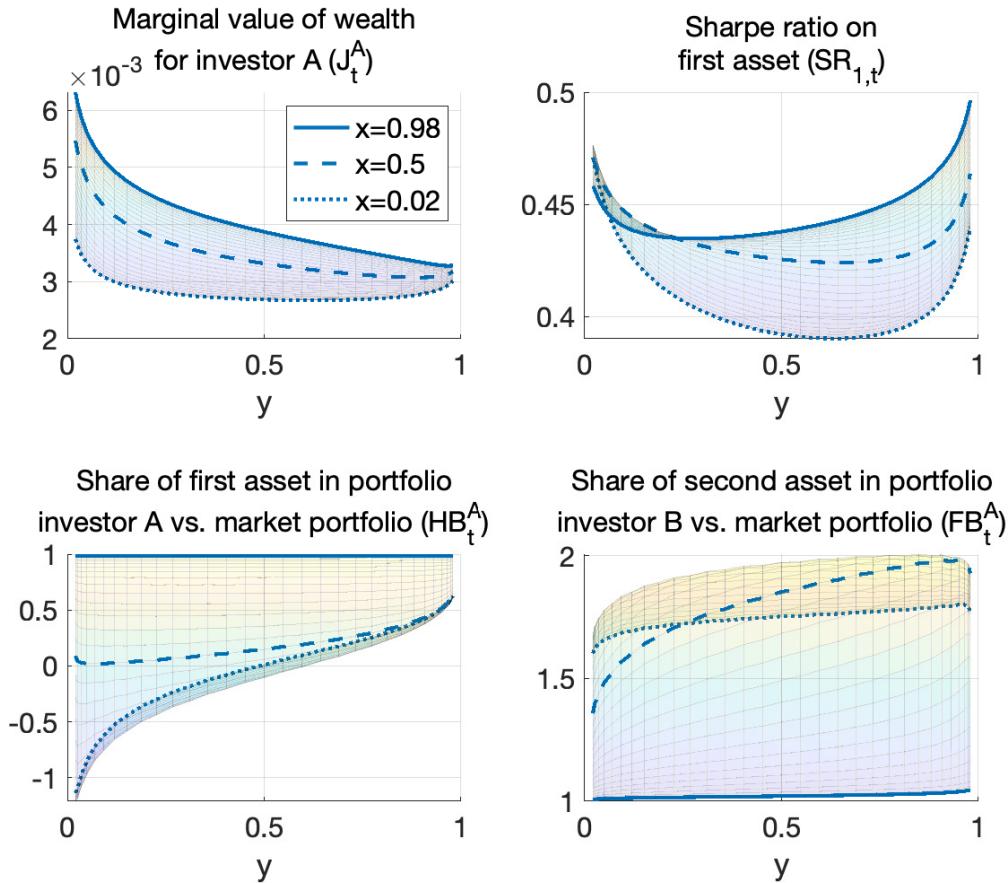


(b) $\theta = 2 > 1$



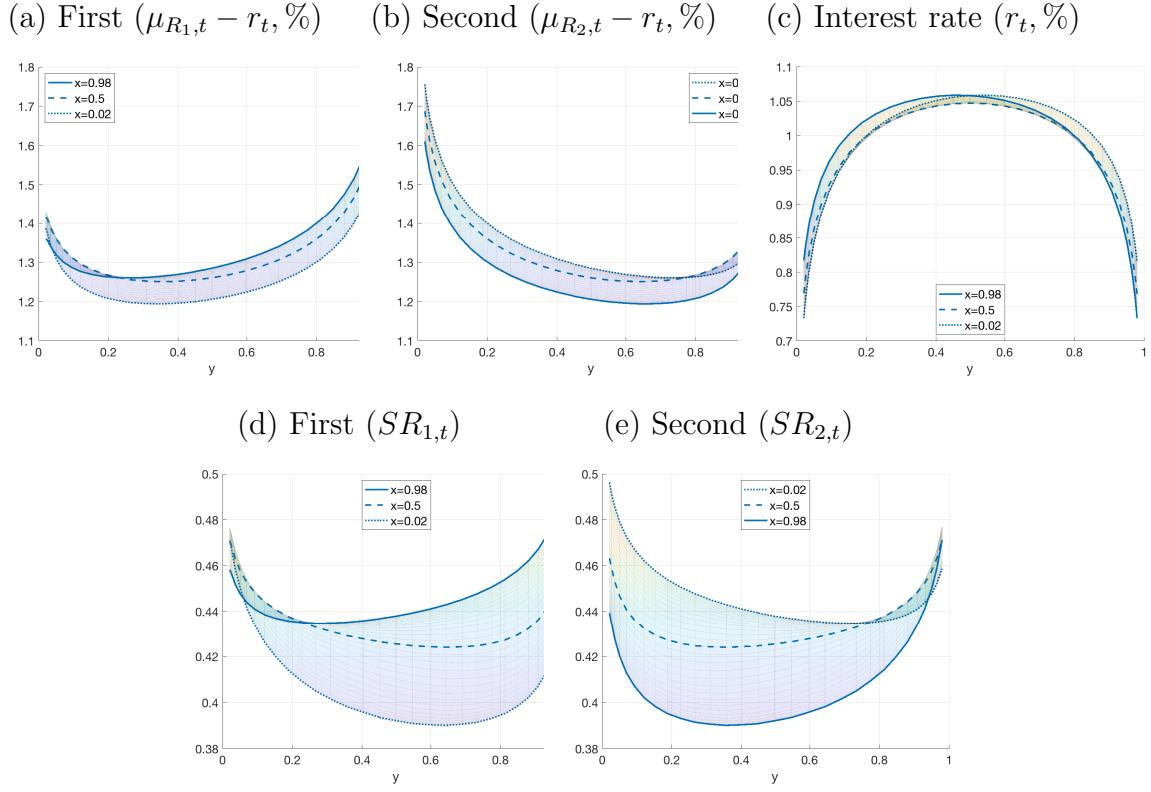
Notes: Based on the symmetric calibration of Assumption 1, except for the elasticity of substitution across goods, θ . * For Panel (a), $\gamma = 15, \psi = 1/\gamma, \alpha = 0.58$ (final calibration ongoing). x_t is the wealth share, which captures the share of total wealth held by investor A . y_t is the relative supply of the first good, which captures fundamentals.

Figure F.14: Direct impact of the wealth share



Notes: Based on the symmetric calibration under perfect risk sharing of Assumption 1. x_t is the wealth share, which captures the share of total wealth held by investor A . y_t is the relative supply of the first good, which captures fundamentals. Corresponding three-dimensional representations: Figure F.33.

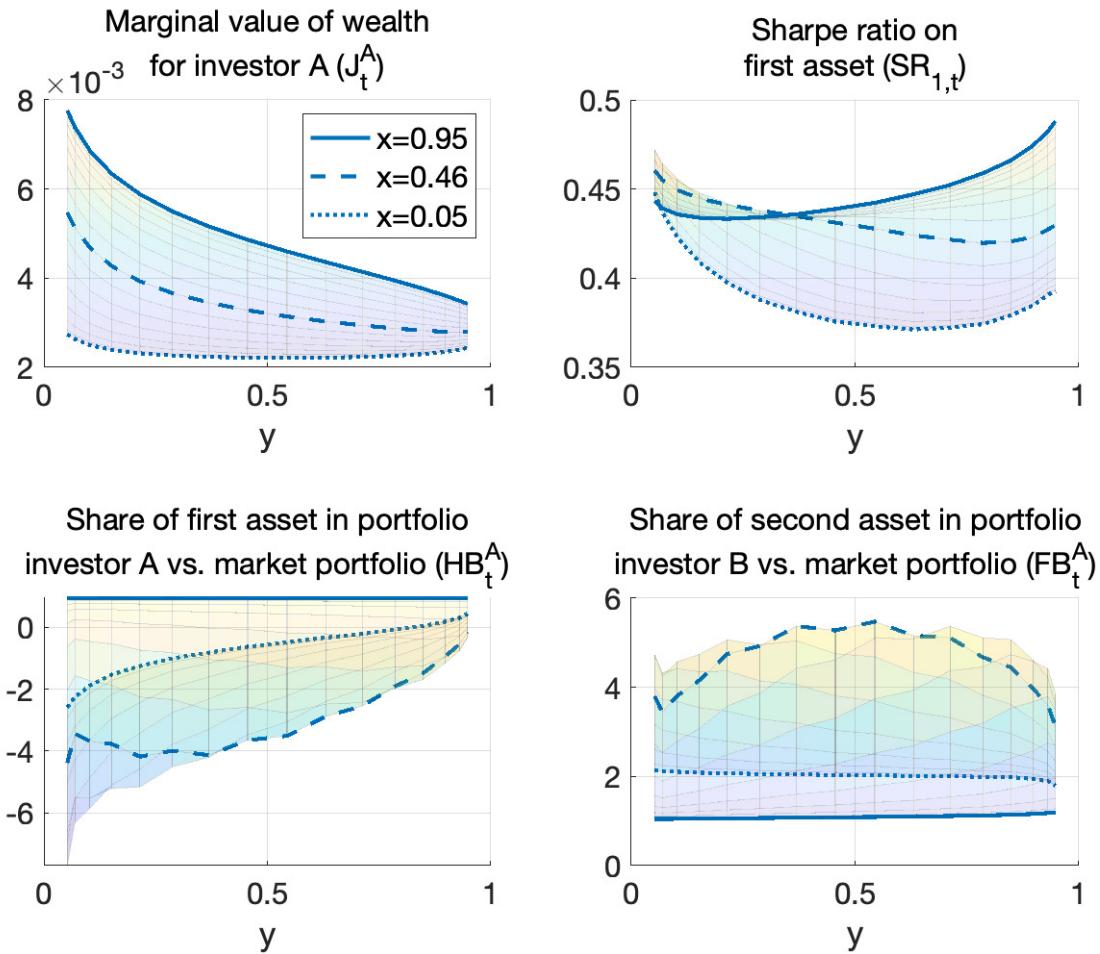
Figure F.15: Expected risk premia, Sharpe ratios for first and second assets, and interest rate



Notes: Based on the symmetric calibration of Assumption 1. x_t is the wealth share, which captures the share of total wealth held by investor A . y_t is the relative supply of the first good, which captures fundamentals. Corresponding representation when $x_t = 1/2$: Figure 4.

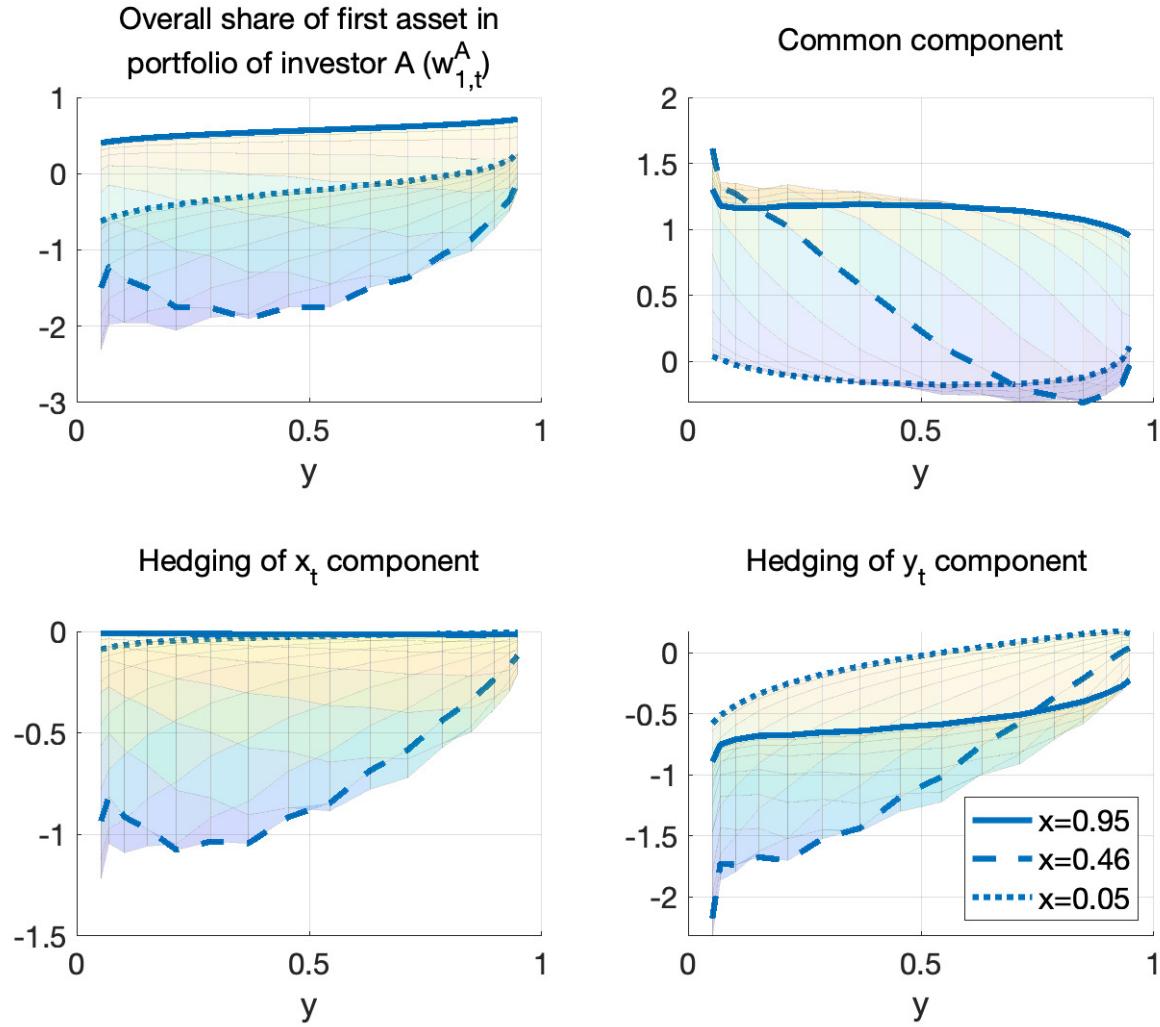
F.6. Effect of the home bias in consumption

Figure F.16: Direct impact of the wealth share for $\alpha = 0.85$



Notes: Based on the symmetric calibration under perfect risk sharing of Assumption 1, except that $\alpha = 0.85$. x_t is the wealth share, which captures the share of total wealth held by investor A . y_t is the relative supply of the first good, which captures fundamentals.

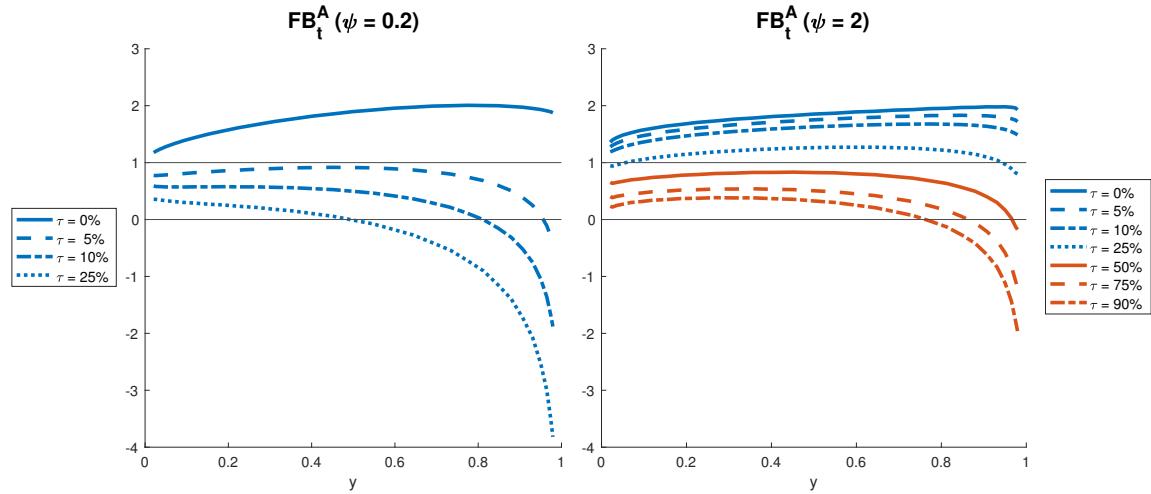
Figure F.17: Components of the domestic portfolio with $\alpha = 0.85$



Notes: Based on the symmetric calibration under perfect risk sharing of Assumption 1, except that $\alpha = 0.85$. x_t is the wealth share, which captures the share of total wealth held by investor A . y_t is the relative supply of the first good, which captures fundamentals.

F.7. Effect of imperfect financial integration

Figure F.18: Equity portfolio of investor A vs. market portfolio

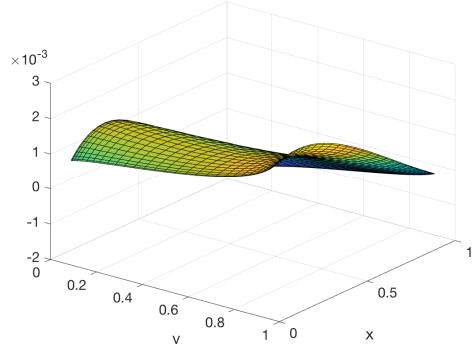


Notes: Based on the symmetric calibration of Assumption 1, except for ψ and τ . The figure shows a cut in which the allocation of wealth is symmetric ($x_t = 1/2$). y_t is the relative supply of the first good, which captures fundamentals. Effect on the home bias measure HB_t^A : Figure 9.

F.8. Application: The International Financial System ([Sauzet, 2021a](#))

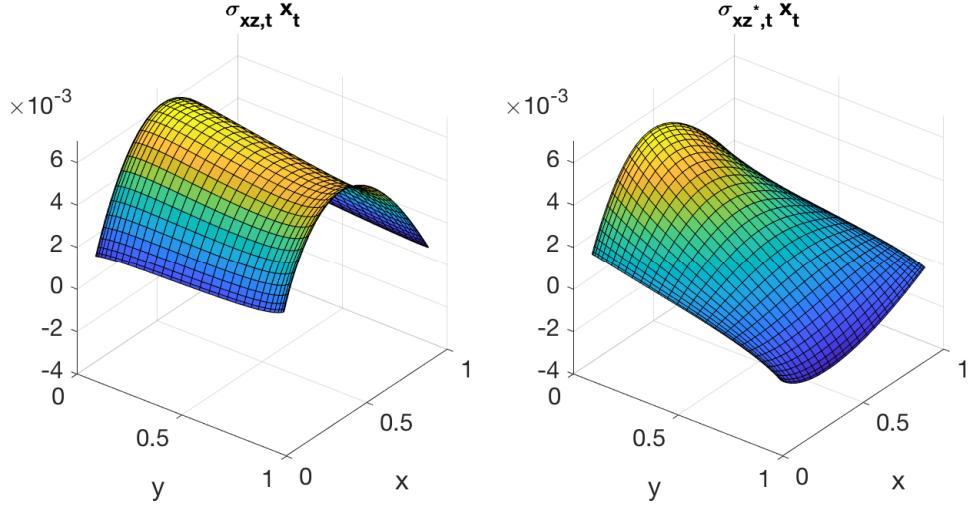
Note: notation for the international application are similar to those in the main text except that (i) variables for the domestic investor (investor A) omit the A superscript, while those of the foreign investor (investor B) are denoted by $*$, and (ii) goods and assets are denoted h (“home”) for good 1, and f (“foreign”) for good 2. For instance, $w_{h,t}$ denotes the share invested in the domestic asset (h) by the domestic investor in her portfolio (equivalent of $w_{1,t}^A$ in the notation of the main text), while $C_{f,t}^*$ denotes the consumption of the foreign good (f) by the foreign investor (*) (equivalent of $C_{2,t}^B$ in the notation of the main text).

Figure F.19: Drift of the wealth share ($\mu_{x,t}x_t$)



Notes: Based on the symmetric calibration of Assumption 1, except that $\gamma = 8 < \gamma^* = 15$, $\psi = 0.5$, and $\tau = 15\%$. x_t is the wealth share, which captures the share of worldwide wealth held by the domestic investor. y_t is the relative supply of the domestic good, which captures fundamentals.

Figure F.20: Diffusion of the wealth share ($\sigma_{x,t} x_t$)

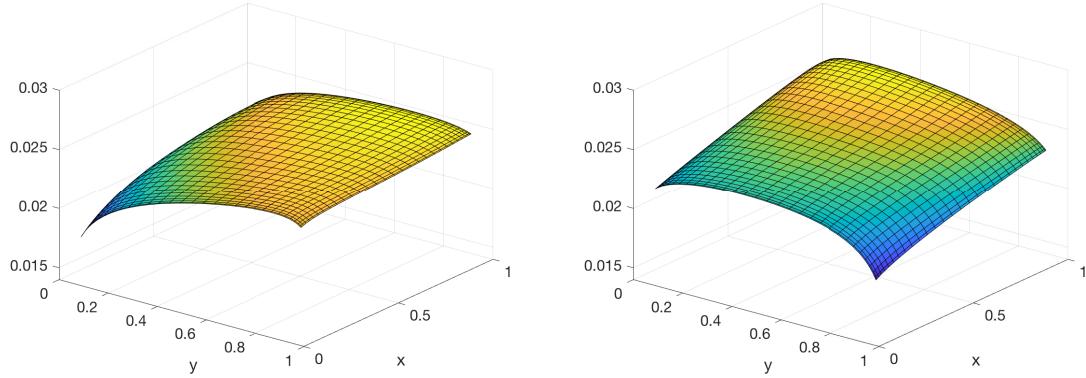


Notes: Based on the symmetric calibration of Assumption 1, except that $\gamma = 8 < \gamma^* = 15$, $\psi = 0.5$, and $\tau = 15\%$. x_t is the wealth share, which captures the share of worldwide wealth held by the domestic investor. y_t is the relative supply of the domestic good, which captures fundamentals.

Figure F.21: Dividend yields in the application of 5.1 ([Sauzet, 2021a](#))

(a) Domestic equity asset: F_t

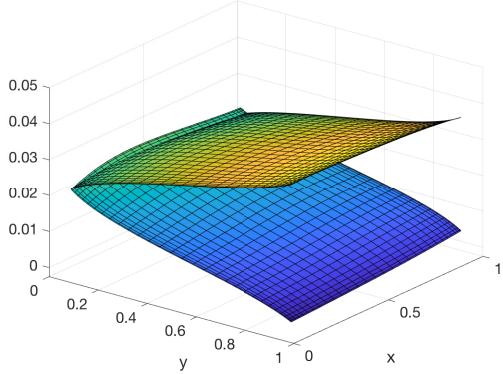
(b) Foreign equity asset: F_t^*



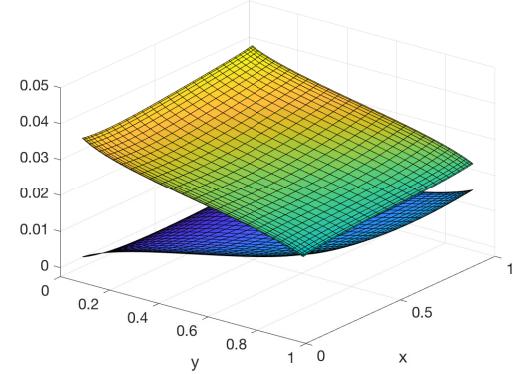
Notes: Based on the symmetric calibration of Assumption 1, except that $\gamma = 8 < \gamma^* = 15$, $\psi = 0.5$, and $\tau = 15\%$. x_t is the wealth share, which captures the share of worldwide wealth held by the domestic investor. y_t is the relative supply of the domestic good, which captures fundamentals.

Figure F.22: Second moments of returns in the application of 5.1 (Sauzet, 2021a)

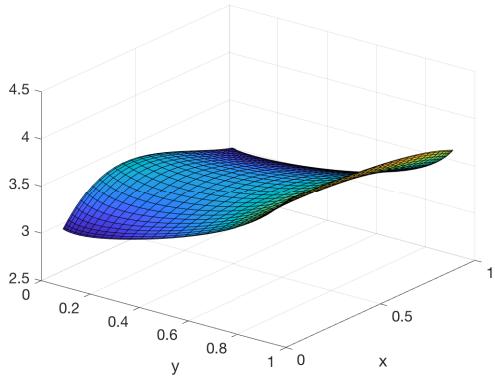
(a) Diffusion of domestic returns: $\sigma_{R,t}$



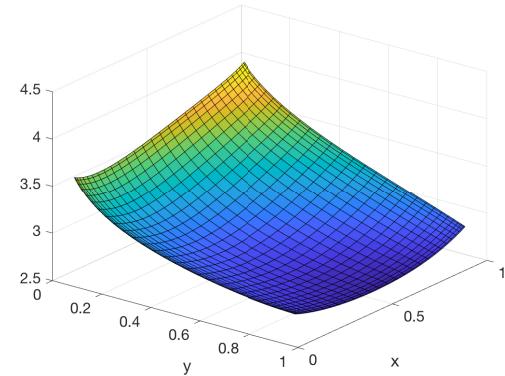
(b) Diffusion of foreign returns: $\sigma_{R^*,t}$



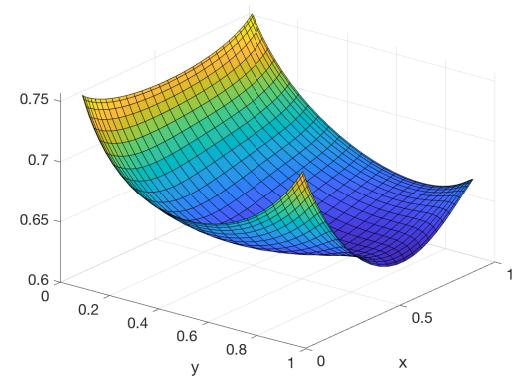
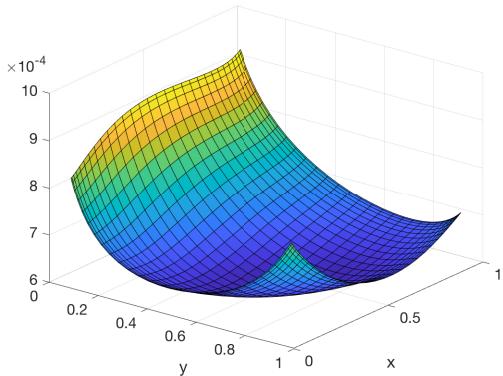
(c) Dom. volatility (%): $(\sigma_{R,t}^T \sigma_{R^*,t})^{-1/2}$



(c) Foreign volatility (%): $(\sigma_{R,t}^T \sigma_{R^*,t})^{-1/2}$



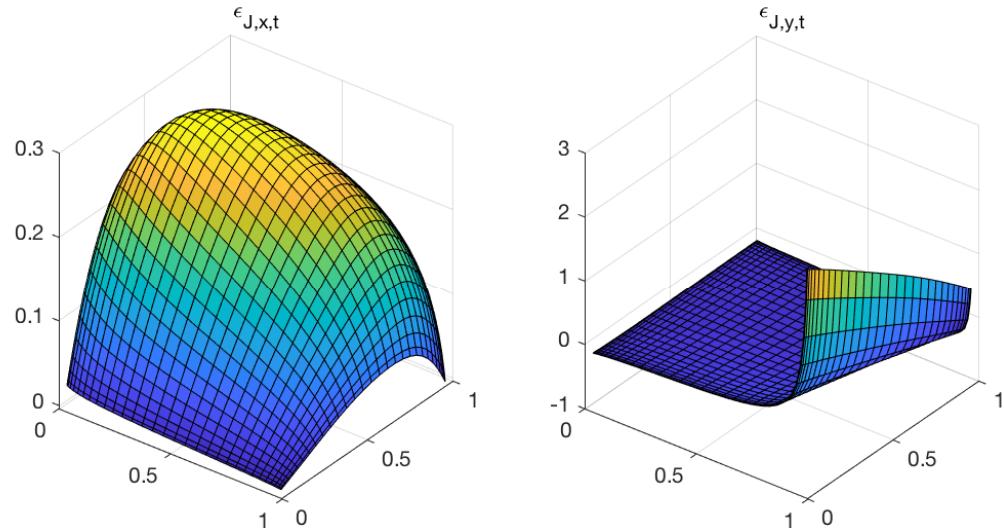
(e) Conditional cov.: $\text{cov}_t(dR_t, dR_t^*)dt^{-1}$ (f) Conditional corr.: $\text{corr}_t(dR_t, dR_t^*)dt^{-1}$



Notes: Based on the symmetric calibration of Assumption 1, except that $\gamma = 8 < \gamma^* = 15$, $\psi = 0.5$, and $\tau = 15\%$. x_t is the wealth share, which captures the share of worldwide wealth held by the domestic investor. y_t is the relative supply of the domestic good, which captures fundamentals.

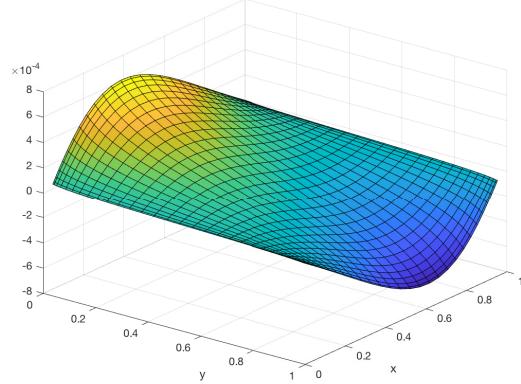
F.9. Other three-dimensional figures

Figure F.23: Conditional elasticities of the domestic marginal value of wealth (J_t)



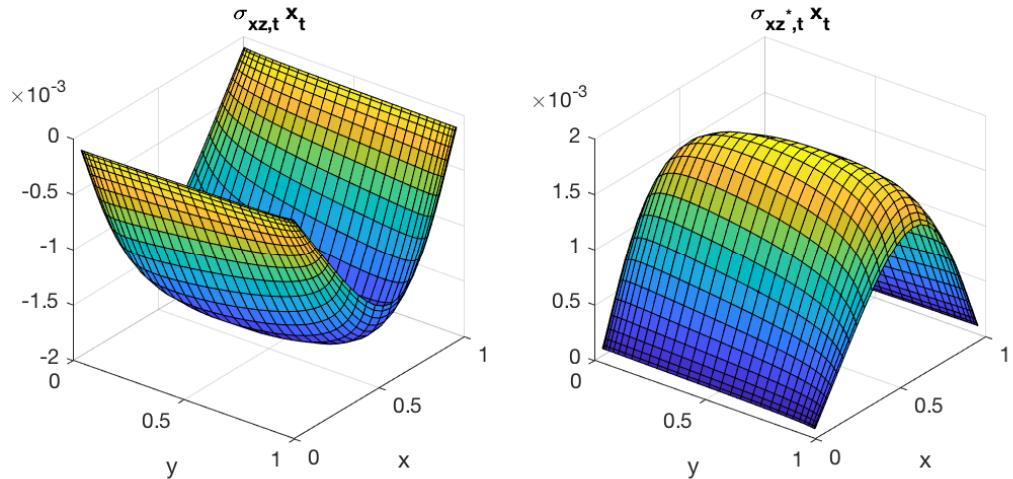
Notes: Based on the symmetric calibration under perfect risk sharing of Assumption 1. x_t is the wealth share, which captures the share of total wealth held by investor A . y_t is the relative supply of the first good, which captures fundamentals.

Figure F.24: Drift of the wealth share ($\mu_{x,t}x_t$)



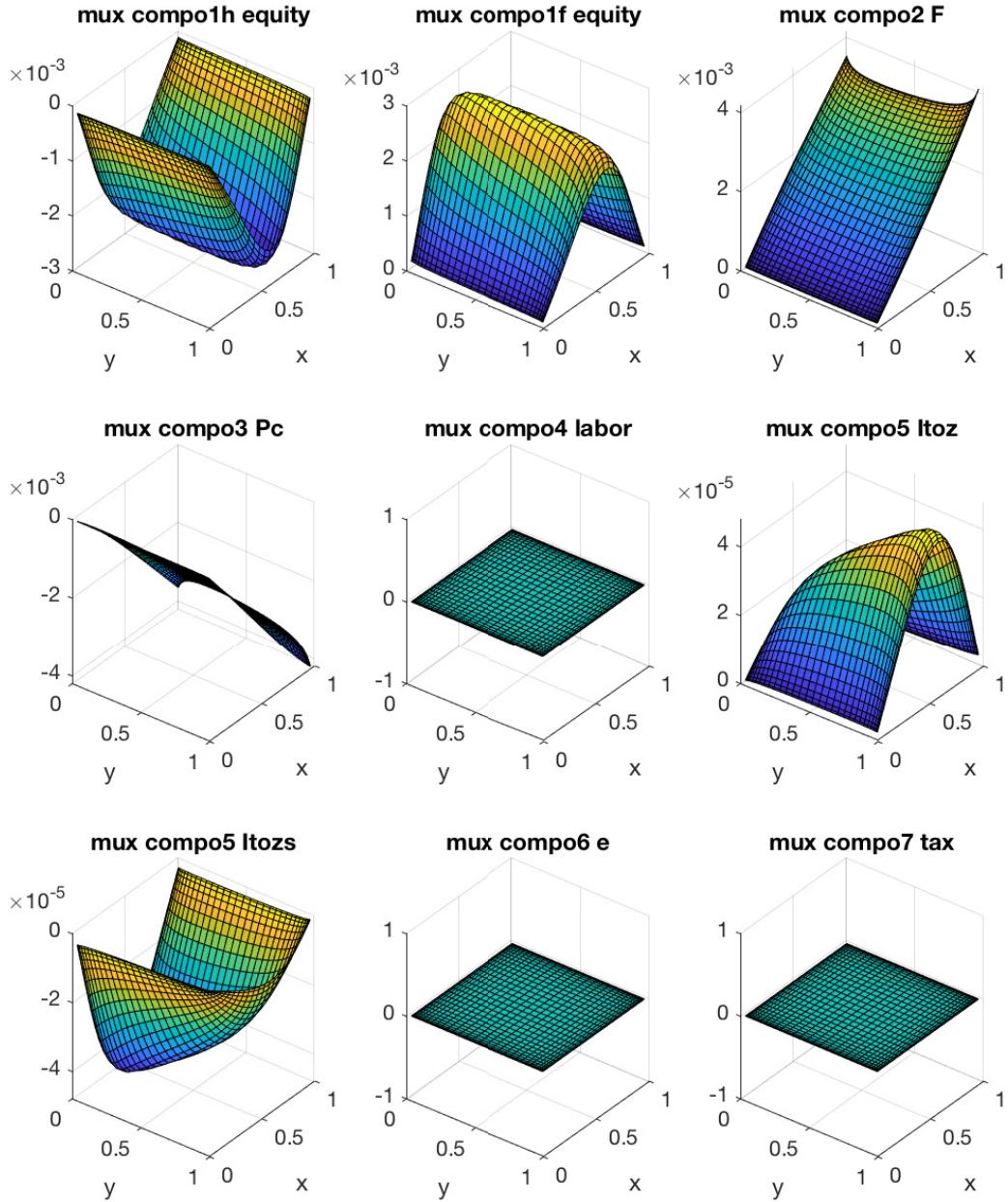
Notes: Based on the symmetric calibration under perfect risk sharing of Assumption 1. x_t is the wealth share, which captures the share of total wealth held by investor A . y_t is the relative supply of the first good, which captures fundamentals.

Figure F.25: Diffusion of the wealth share ($\sigma_{x,t}x_t$)



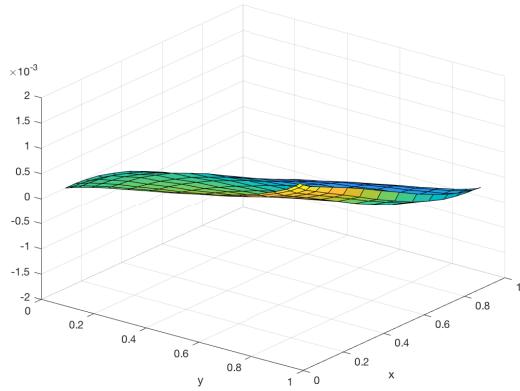
Notes: Based on the symmetric calibration under perfect risk sharing of Assumption 1. x_t is the wealth share, which captures the share of total wealth held by investor A . y_t is the relative supply of the first good, which captures fundamentals. Corresponding two-dimensional representation: Figure 1.

Figure F.26: Components of the drift of the wealth share ($\mu_{x,t}x_t$)



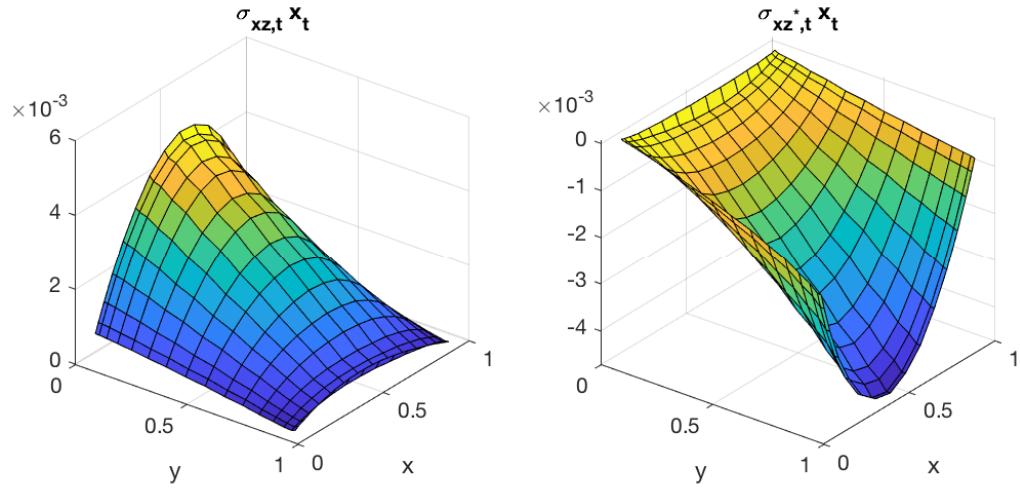
Notes: Based on the symmetric calibration under perfect risk sharing of Assumption 1. x_t is the wealth share, which captures the share of total wealth held by investor A . y_t is the relative supply of the first good, which captures fundamentals.

Figure F.27: Drift of the wealth share ($\mu_{x,t}x_t$) under imperfect financial integration ($\psi = 0.2, \tau = 10\%$)



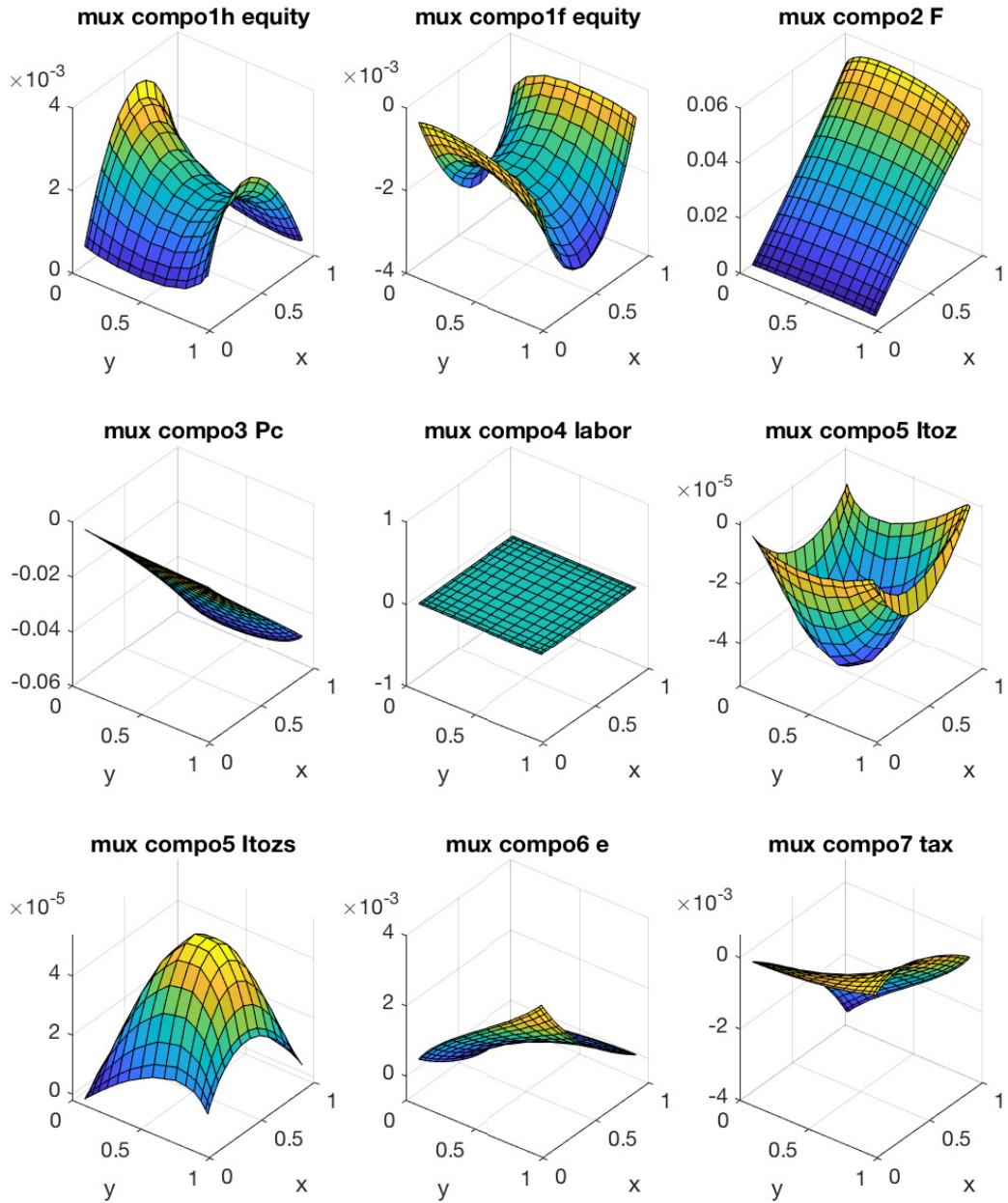
Notes: Based on the symmetric calibration of Assumption 1 except for imperfect financial integration (τ) and elasticity of intertemporal substitution (ψ). x_t is the wealth share, which captures the share of total wealth held by investor A . y_t is the relative supply of the first good, which captures fundamentals.

Figure F.28: Diffusion of the wealth share ($\sigma_{x,t} x_t$) under imperfect financial integration ($\psi = 0.2, \tau = 10\%$)



Notes: Based on the symmetric calibration of Assumption 1 except for imperfect financial integration (τ) and elasticity of intertemporal substitution (ψ). x_t is the wealth share, which captures the share of total wealth held by investor A . y_t is the relative supply of the first good, which captures fundamentals.

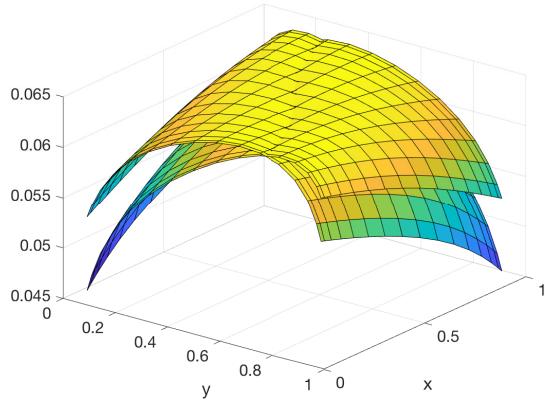
Figure F.29: Components of the drift of the wealth share ($\mu_{x,t}x_t$) under imperfect financial integration ($\psi = 0.2, \tau = 10\%$)



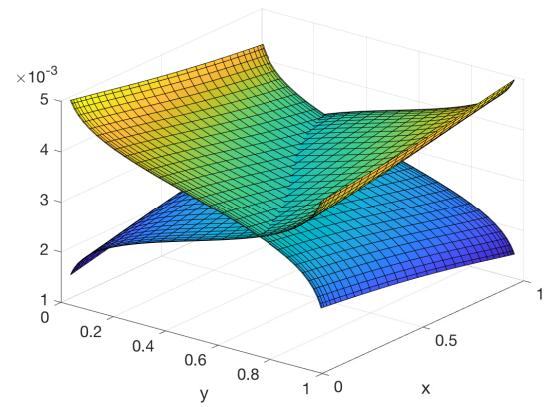
Notes: Based on the symmetric calibration of Assumption 1 except for imperfect financial integration (τ) and elasticity of intertemporal substitution (ψ). x_t is the wealth share, which captures the share of total wealth held by investor A . y_t is the relative supply of the first good, which captures fundamentals.

Figure F.30: Dividend yields

(a) $\psi = 0.2$

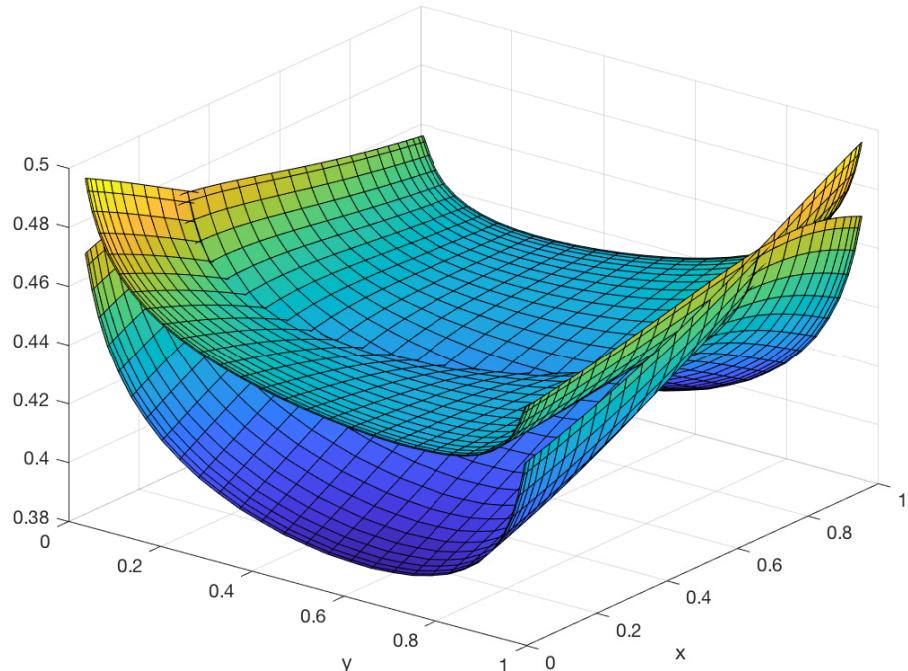


(b) $\psi = 2$



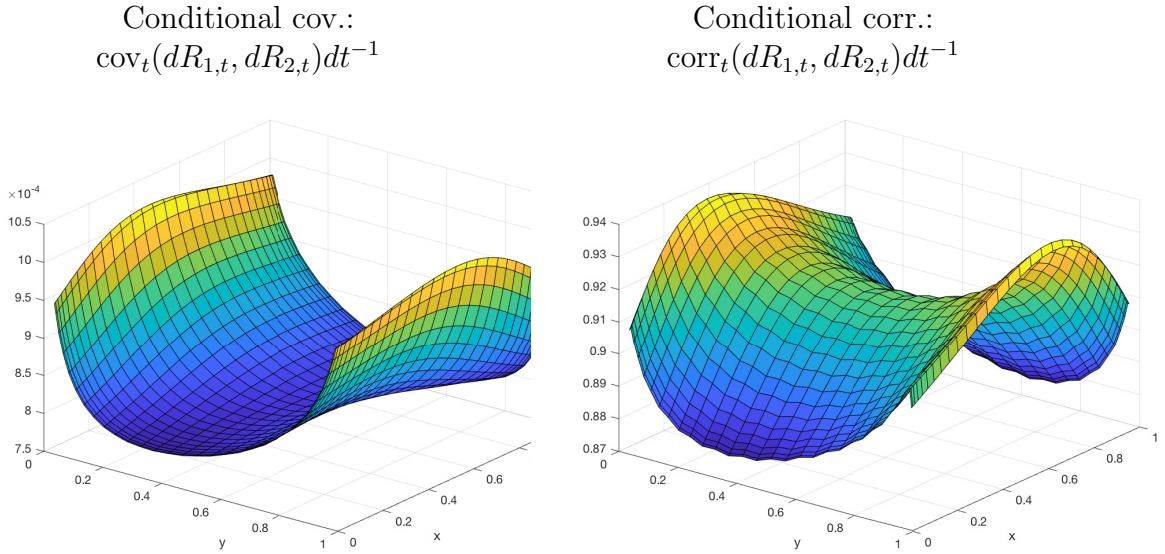
Notes: Based on the symmetric calibration under perfect risk sharing of Assumption 1. x_t is the wealth share, which captures the share of total wealth held by investor A . y_t is the relative supply of the first good, which captures fundamentals.

Figure F.31: Sharpe ratios



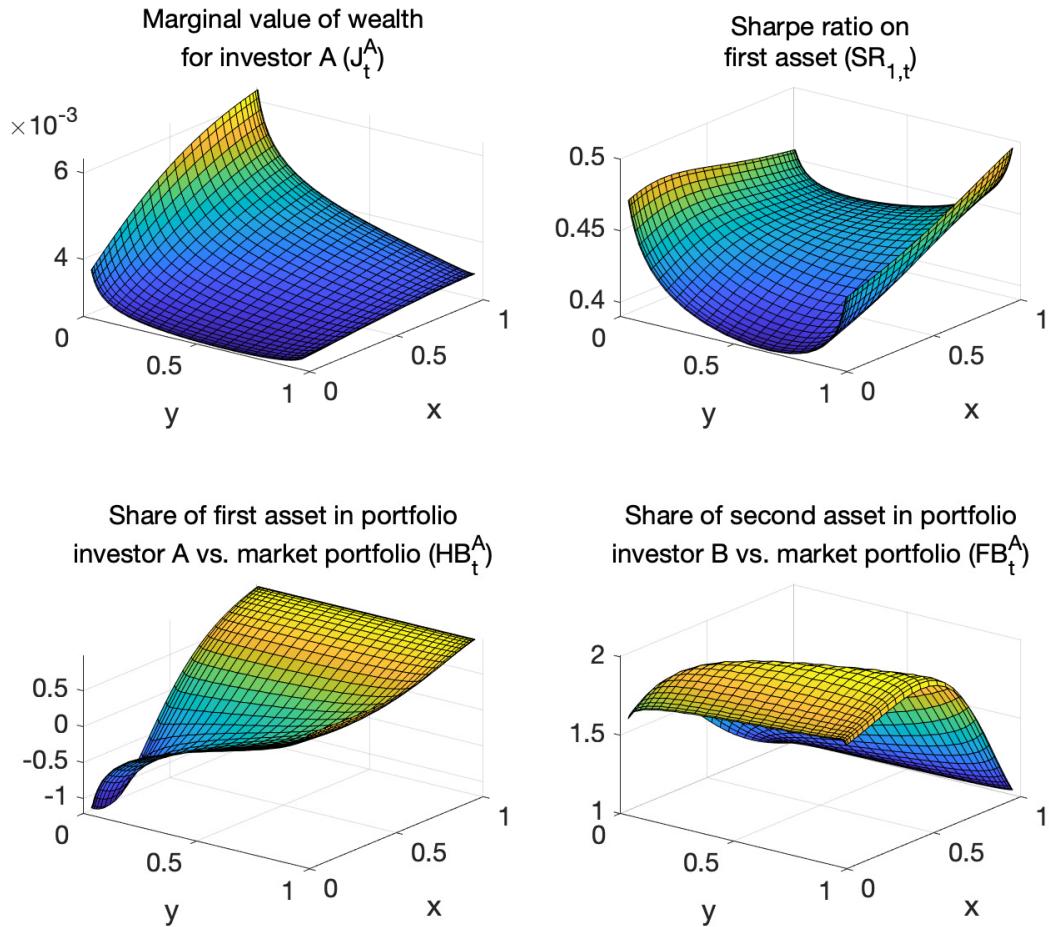
Notes: Based on the symmetric calibration under perfect risk sharing of Assumption 1. x_t is the wealth share, which captures the share of total wealth held by investor A . y_t is the relative supply of the first good, which captures fundamentals.

Figure F.32: Comovement of returns



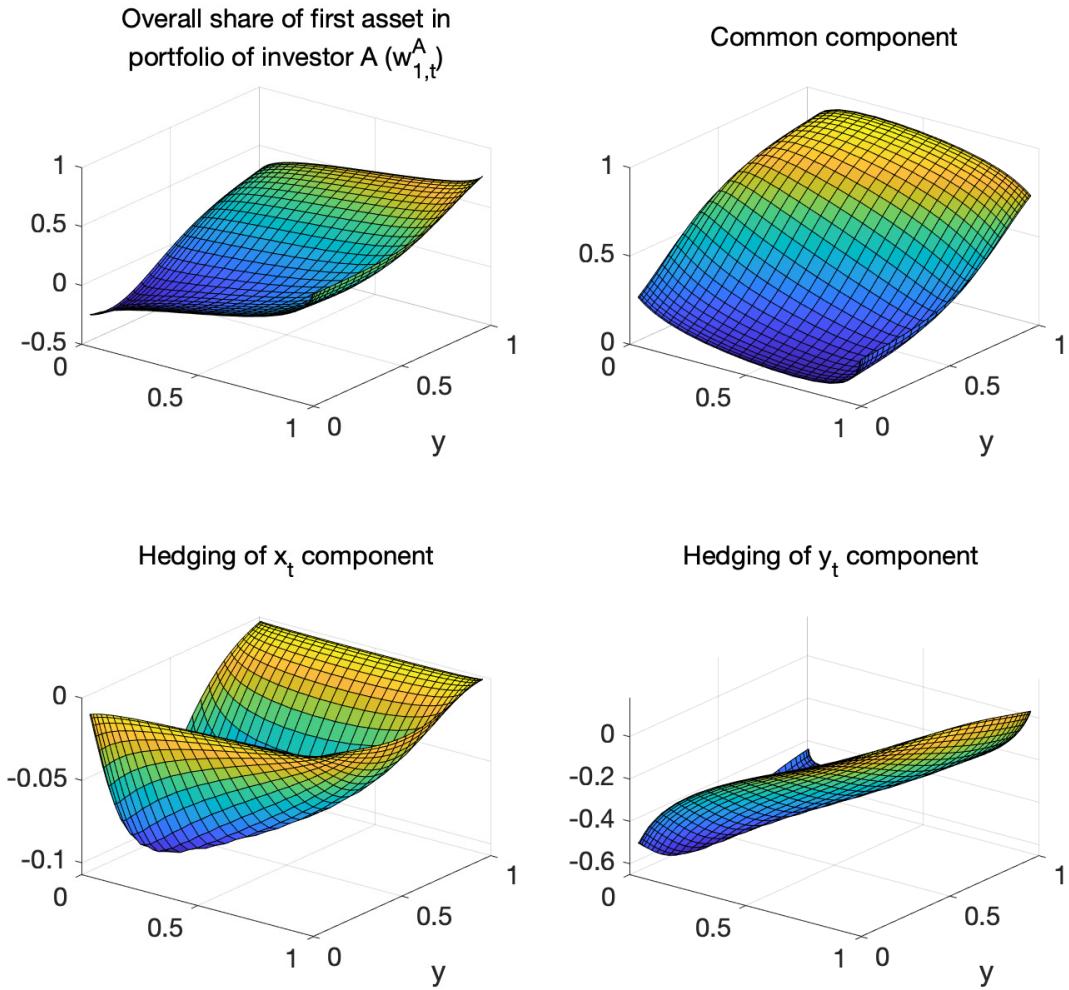
Notes: Based on the symmetric calibration under perfect risk sharing of Assumption 1. x_t is the wealth share, which captures the share of total wealth held by investor A . y_t is the relative supply of the first good, which captures fundamentals. Corresponding two-dimensional representation: Figure 6.

Figure F.33: Direct impact of the wealth share



Notes: Based on the symmetric calibration under perfect risk sharing of Assumption 1. x_t is the wealth share, which captures the share of total wealth held by investor A . y_t is the relative supply of the first good, which captures fundamentals. Corresponding two-dimensional representation: Figure F.14.

Figure F.34: Components of the portfolio of investor A (as compared to the market portfolio)



Notes: Based on the symmetric calibration under perfect risk sharing of Assumption 1. x_t is the wealth share, which captures the share of total wealth held by investor A . y_t is the relative supply of the first good, which captures fundamentals. Corresponding two-dimensional representation: Figure 8.