A comparison principle for variational problems with application to OT

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What is submodularity?

History of submodularity

A function $E : \mathbb{R}^d \to \mathbb{R}$ is *submodular* if:

$$E(x \wedge y) + E(x \vee y) \leq E(x) + E(y)$$

where $x \wedge y = (\min(x_i, y_i))_i$ and $x \vee y = (\max(x_i, y_i))_i$.

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Economics [Lorentz, 1953, Topkis, 1978] to derive monotone comparative statics in $\mathbb{R}^d$,

Optimization On discrete domains [Bach, 2013], then in $\mathbb{R}^d$ [Bach, 2018],

Calculus of variations [Alter et al., 2005] to derive comparison principles for optimizers of the perimeter.

Examples of lattices

- $\mathbb{R}^n$ where $x \leq y \iff x_i \leq y_i$ for all i.
- **Sobolev space:** If $U \subset \mathbb{R}^n$ is open then $H^1(U)$, with $u \leq v$ if $u(x) \leq v(x)$ for almost every $x \in U$, is a lattice.

$$u \vee v(x) = u(x) \vee v(x), \nabla(u \vee v)(x) = \nabla u(x)\chi_{u(x) \geq v(x)} + \nabla v(x)\chi_{u(x) < v(x)}$$

- **Continuous functions:** $C(\Omega)$ is a lattice with $\phi \leq \psi$ if $\phi(x) \leq \psi(x)$ for all $x \in \Omega$.
- **Radon measures:** $\mathcal{M}(\Omega)$ is a lattice with

$$\mu \leq \nu \iff \forall \phi \in C(\Omega), \phi \geq 0, \int \phi d\mu \leq \int \phi d\nu$$

Examples of submodular functionals

- **Dirichlet's energy** on $H^1(U)$

$$u \mapsto \int_U \|\nabla u(x)\|^2 dx.$$

- **Kantorovich** functional in optimal transport

$$(\phi \in C(\Omega)) \mapsto \int \phi^c d\nu$$

where $\nu \in M_+(\Omega)$ and $(\phi^c(y) = \sup_x \phi(x) - c(x,y)$.

(Entropic) Optimal Transport

Let Ω be a compact metric space, $c \in C(\Omega \times \Omega)$ a cost function, $\epsilon \geq 0$, and $\alpha \in \mathcal{M}_+(\Omega)$ a nonzero reference measure.

For $\mu, \nu \in \mathcal{M}_+(\Omega)$ with $\mu(\Omega) = \nu(\Omega) > 0$, the entropic optimal transport problem is:

$$OT_{c,\epsilon}(\mu, \nu) = \inf_{\pi \in \Pi(\mu,\nu)} \int c \, d\pi + \epsilon \, \mathrm{KL}(\pi \mid \alpha \otimes \nu) = \sup_{\phi \in C(\Omega)} \left[\int \phi \, d\mu - K_{\epsilon,\nu}(\phi) \right]$$

where the Kullback–Leibler divergence is

$$\mathrm{KL}(\pi \mid \alpha \otimes \nu) = \int \log \left(\frac{d\pi}{d(\alpha \otimes \nu)} \right) d\pi.$$

Kantorovich functional

The semidual/Kantorovich functional is

$$K_{\epsilon,\nu}(\phi) = \int \epsilon \log \left(\int e^{[\phi(x) - c(x,y)]/\epsilon} d\alpha(x) \right) d\nu(y) \xrightarrow[\epsilon \to 0]{} K_{0,\nu}(\phi) = \int \phi^c d\nu$$

The functional $K_{\epsilon,\nu}$ is convex and submodular.

JKO Scheme (Jordan–Kinderlehrer–Otto)

Let $\mu_0 \in \mathcal{M}_+(\Omega)$ be an initial measure.

The JKO scheme defines recursively a sequence $(\mu_n^\tau)_{n\in\mathbb{N}}$ by solving:

$$\mu_{n+1}^\tau = \underset{\nu\in\mathcal{M}_+(\Omega)}{\arg\min} \left\{ \frac{1}{\tau} OT_{c,\epsilon}(\mu_n^\tau, \nu) + H(\nu) \right\}$$

where H is a functional over $\mathcal{M}_+(\Omega)$ typically, an internal energy functional.

Comparison principle

Theorem

Let $c \in C(\Omega \times \Omega)$ and $h\colon [0, +\infty) \to \mathbb{R}$ denote a strictly convex, l.s.c., and superlinear function. Define $H_m\colon (\nu \in \mathcal{M}_+(\Omega)) \mapsto \int h(\frac{d\nu}{dm})\, dm$ where $m \in \mathcal{M}_+(\Omega)$ is a fixed reference measure. If $\mu_1 \leq \mu_2$, then

$$\nu_1 \leq \nu_2, \quad \text{where } \nu_i = \arg\min_{\nu} OT_{c,\epsilon}(\mu_i, \nu) + H_m(\nu).$$

A similar result for $\epsilon = 0$ was obtained in [Jacobs et al., 2020] by leveraging the existence of a transport map when c is C^1_{loc} and twisted.

Submodularity and duality

Banach lattices

Definition

Let $(X, \|\cdot\|)$ be a Banach space equipped with a partial ordering $\leq$. X is a *Banach lattice* if

(i) $(X, \leq)$ is a lattice: $\phi_1 \wedge \phi_2$ and $\phi_1 \vee \phi_2$ exist in X,

(ii) $\phi_1 \leq \phi_2$ implies $\phi_1 + \phi_3 \leq \phi_2 + \phi_3$ for $\phi_3 \in X$, and $\phi_1 \leq \phi_2$ implies $\lambda\phi_1 \leq \lambda\phi_2$ for $\lambda \geq 0$,

(iii) $|\phi_1| \leq |\phi_2|$ implies $\|\phi_1\| \leq \|\phi_2\|$ where $|\phi| = \phi^+ + \phi^-$ with $\phi^+ = \phi \vee 0$ and $\phi^- = -(\phi \wedge 0)$.

$L^p(\mathbb{R}^n)$, $C(\Omega)$ and $\mathcal{M}(\Omega)$ are Banach lattices. $H^1(U)$ is a sublattice of the Banach lattice $L^2(U)$.

Submodularity and P-dominance

Definition

Let $E_1, E_2 \colon X \to \mathbb{R}$. We say that E_1 is P-dominated by E_2 denoted $E_1 \ll_{\mathrm{P}} E_2$ if for any $\phi_1, \phi_2 \in X$ we have

$$E_1(\phi_1 \wedge \phi_2) + E_2(\phi_1 \vee \phi_2) \leq E_1(\phi_1) + E_2(\phi_2).$$

If $E \ll_{\mathrm{P}} E$, then E is submodular.

Observe that if $E_1 \ll_{\mathrm{P}} E_2, E_3 \ll_{\mathrm{P}} E_4$, then $E_1 + E_3 \ll_{\mathrm{P}} E_2 + E_4$

Elementary submodular functions

Here are some elementary examples of submodular functions.

- $E : (p \in \mathbb{R}^n) \mapsto \langle p, Mp \rangle$ where M is a matrix with non-positive off diagonal elements.

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- $E : (u \in \mathbb{R}^I) \mapsto \max_i u_i$.
  In particular $(\phi \in C(\Omega)) \mapsto \max_{x \in \Omega} \phi(x) - c(x, y)$ is submodular.

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- $E : ((a, b) \in \mathbb{R}^2) \mapsto h(a - b)$ where h is a convex function.

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- $E : (u \in \mathbb{R}^l) \mapsto \max_i u_i$.
  In particular $(\phi \in C(\Omega)) \mapsto \max_{x \in \Omega} \phi(x) - c(x, y)$ is submodular.
- $E : ((a, b) \in \mathbb{R}^2) \mapsto h(a - b)$ where h is a convex function.
- If $E_i : (\phi \in C(\Omega)) \mapsto -\langle \phi, \mu_i \rangle$ for $\mu_i \in \mathcal{M}(\Omega)$, then for $\mu_1 \leq \mu_2$ we have $E_1 \ll_{\mathrm{P}} E_2$.

P-dominance for sets

Let A_1, A_2 be two subsets of X. We say that A_1 is P-dominated by A_2 denoted $A_1 \ll_{\mathrm{P}} A_2$ if for any $\phi_1 \in A_1, \phi_2 \in A_2$ we have

$$\phi_1 \wedge \phi_2 \in A_1 \text{ and } \phi_1 \vee \phi_2 \in A_2.$$

This is equvialent to the P-dominance of their indicator functions: $\iota_{A_1} \ll_{\mathrm{P}} \iota_{A_2}$ where ι_A is the indicator function of A. We recover Veinott's order or the strong set order.

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$$\phi_1 \wedge \phi_2 \in A_1 \text{ and } \phi_1 \vee \phi_2 \in A_2.$$

This is equivalent to the P-dominance of their indicator functions: $\iota_{A_1} \ll_{\mathrm{P}} \iota_{A_2}$ where ι_A is the indicator function of A. We recover Veinott's order or the strong set order.

Lemma

Let $\phi_1, \phi_2 \in X$ then $\{\phi_1\} \ll_{\mathrm{P}} \{\phi_2\}$ if and only if $\phi_1 \leq \phi_2$.

P-dominance implies comparison principle

Theorem

Let $E_1, E_2 \colon X \to \mathbb{R}$. If $E_1 \ll_{\mathrm{P}} E_2$ then

$$\arg\min E_1 \ll_{\mathrm{P}} \arg\min E_2.$$

Take $\phi_1 \in \arg\min E_1$ and $\phi_2 \in \arg\min E_2$ then by P-dominance we have

$$E_1(\phi_1 \wedge \phi_2) + E_2(\phi_1 \vee \phi_2) \leq E_1(\phi_1) + E_2(\phi_2).$$

Which implies that $\phi_1 \wedge \phi_2 \in \arg\min E_1$ and $\phi_1 \vee \phi_2 \in \arg\min E_2$.

Duality

The following two results are extensions of [Galichon et al., 2024] to Banach lattices.

Theorem

Let $E_1, E_2 \colon X \to \mathbb{R}$ be two convex l.s.c. proper functions.
Then, $E_1 \ll_{\mathrm{P}} E_2$ if and only if $E_2^* \ll_{\mathrm{Q}} E_1^*$.
In particular E is submodular if and only if $E^* \ll_{\mathrm{Q}} E^*$ (we say that E^* is exchangeable).

Recall that $E^*(\mu) = \sup_\phi \langle \phi, \mu \rangle - E(\phi)$ is the convex conjugate of E.

Exchangeability and Q-dominance

Definition

Let $F_1, F_2 \colon X^* \to \mathbb{R}$. We say that F_1 is Q-dominated by F_2 ($F_1 \ll_{\mathrm{Q}} F_2$) if :
for any $\mu_1, \mu_2 \in X^*$ and $t_{12} \in [0, (\mu_1 - \mu_2)^+]$ there is $t_{21} \in [0, (\mu_1 - \mu_2)^-]$ such that

$$F_1(\mu_1 - (t_{12} - t_{21})) + F_2(\mu_2 + (t_{12} - t_{21})) \leq F_1(\mu_1) + F_2(\mu_2).$$

If $F \ll_{\mathrm{Q}} F$ then F is said to be exchangeable. Exchangeability is stronger than convexity.

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If $F \ll_{\mathrm{Q}} F$ then F is said to be exchangeable. Exchangeability is stronger than convexity.
Variations of this definition were introduced : $M^\natural$-convexity in discrete convex analysis [Murota, 2003], S-convexity in $\mathbb{R}^n$ [Chen and Li, 2020].

Q-dominance for sets

Let B_1, B_2 be two subsets of X^*. We say that B_1 is Q-dominated by B_2 denoted $B_1 \ll_{\mathrm{Q}} B_2$ if for any $\mu_1 \in B_1, \mu_2 \in B_2$ and any $t_{12} \in [0, (\mu_1 - \mu_2)^+]$ there is $t_{21} \in [0, (\mu_1 - \mu_2)^-]$ such that

$$\mu_1 - (t_{12} - t_{21}) \in B_1 \text{ and } \mu_2 + (t_{12} - t_{21}) \in B_2.$$

This is equivalent to the Q-dominance of their indicator functions: $\iota_{B_1} \ll_{\mathrm{Q}} \iota_{B_2}$.

Probabilities are exchangeable

Lemma

The set $\mathcal{P}(\Omega)$ is exchangeable.

Let $\mu_1, \mu_2 \in \mathcal{P}(\Omega)$. For $t_{12} \in [0, (\mu_1 - \mu_2)^+]$ set $t_{21} = \frac{t_{12}(\Omega)}{(\mu_1-\mu_2)^+(\Omega)}(\mu_1 - \mu_2)^-$. We have $t_{21} \in [0, (\mu_1 - \mu_2)^-]$ thus

$$\mu_1' = \mu_1 - (t_{12} - t_{21}) \geq \mu_1 - t_{12} \geq \mu_1 \wedge \mu_2 \geq 0$$

and $\mu_1'(\Omega) = \mu_1(\Omega) - t_{12}(\Omega) + t_{12}(\Omega) = 1$.

Q-dominance for singletons

Lemma

Let $\mu_1, \mu_2 \in X^*$ then $\{\mu_1\} \ll_{\mathrm{Q}} \{\mu_2\}$ if and only if $\mu_1 \leq \mu_2$.

Assume $\{\mu_1\} \ll_{\mathrm{Q}} \{\mu_2\}$ then for any $t_{12} \in [0, (\mu_1 - \mu_2)^+]$ there is $t_{21} \in [0, (\mu_1 - \mu_2)^-]$ such that $\mu_1 - (t_{12} - t_{21}) \in \{\mu_1\}$. Which implies that $(t_{12} - t_{21}) = 0$ and in particular $t_{12} = (t_{12} - t_{21})^+ = 0$. Thus $(\mu_1 - \mu_2)^+ = 0$ which is equivalent to $\mu_1 \leq \mu_2$.

Q-dominance implies comparison principle

Proposition

Let $F_1, F_2 \colon X^* \to \mathbb{R}$. If $F_1 \ll_{\mathrm{Q}} F_2$ then

$$\arg\min F_1 \ll_{\mathrm{Q}} \arg\min F_2.$$

Take $\mu_1 \in \arg\min F_1, \mu_2 \in \arg\min F_2$. By Q-dominance for any $t_{12} \in [0, (\mu_1 - \mu_2)^+]$ there is $t_{21} \in [0, (\mu_1 - \mu_2)^-]$

$$F_1(\mu_1 - (t_{12} - t_{21})) + F_2(\mu_2 + (t_{12} - t_{21})) \le F_1(\mu_1) + F_2(\mu_2).$$

Thus as before we have $\mu_1 - (t_{12} - t_{21}) \in \arg\min F_1, \mu_2 + (t_{12} - t_{21}) \in \arg\min F_2$.

Q-dominance and sum

Given $F_1 \ll_{\mathrm{Q}} F_2$ we will be interested in comparing

$$\arg\min_{\nu} F_1(\nu) + H(\nu) \quad \text{and} \quad \arg\min_{\nu} F_2(\nu) + H(\nu).$$

However H exchangeable is not sufficient to imply $F_1 + H \ll_{\mathrm{Q}} F_2 + H$.
We need a stronger assumption on H : total exchangeability.

Total exchangeability

Definition

H is totally exchangeable if for any μ_1, μ_2 and any $t_{12} \in [0, (\mu_1 - \mu_2)^+]$, $t_{21} \in [0, (\mu_1 - \mu_2)^-]$ we have

$$H(\mu_1 - (t_{12} - t_{21})) + H(\mu_2 + (t_{12} - t_{21})) \leq H(\mu_1) + H(\mu_2).$$

If H is totally exchangeable then H is exchangeable and if $F_1 \ll_{\mathrm{Q}} F_2$ then $F_1 + H \ll_{\mathrm{Q}} F_2 + H$.
Internal energies $H_m : \nu \mapsto \int h(\frac{d\nu}{dm})dm$ where $h : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{+\infty\}$ is a proper convex function are totally exchangeable.

Submodularity of Log-Laplace

Lemma

Let $m \in \mathcal{M}_+(\Omega)$ then $\mathcal{L}_m : \phi \mapsto \log\left(\int e^\phi dm\right)$ is submodular.

Since $\mathrm{KL}(\cdot \mid m)$ is totally exchangeable and $\mathcal{P}(\Omega)$ is exchangeable we have that $\mathrm{KL}(\cdot \mid m) + \iota_{\mathcal{P}(\Omega)}$ is exchangeable. Since $\mathcal{L}_m = (\mathrm{KL}(\cdot \mid m) + \iota_{\mathcal{P}(\Omega)})^*$ we get the result by duality.

Application to Optimal transport

Reminder OT

For $\mu, \nu \in \mathcal{M}_+(\Omega)$ such that $\mu(\Omega) = \nu(\Omega)$, the entropic optimal transport problem is

$$OT_{c,\epsilon}(\mu, \nu) = \inf_{\pi \in \Pi(\mu,\nu)} \int c d\pi + \epsilon \, \mathrm{KL}(\pi \mid \alpha \otimes \nu) = \sup_{\phi \in C(\Omega)} \int \phi d\mu - K_{\epsilon,\nu}(\phi)$$

where

$$K_{\epsilon,\nu}(\phi) = \int \epsilon \log \left(\int e^{[\phi(x) - c(x,y)]/\epsilon} d\alpha(x) \right) d\nu(y), \quad K_{0,\nu}(\phi) = \int \phi^c d\nu.$$

We also have the following duality formula

$$OT_{c,\epsilon}(\mu, \nu) = \sup_{\phi,\psi} \int \phi d\mu - \int \psi d\nu - \epsilon \int e^{[\phi(x) - \psi(y) - c(x,y)]/\epsilon} d\alpha \otimes \nu(x, y)$$

Comparison principle on the potentials

For $\mu, \nu \in \mathcal{P}(\Omega)$ define the set of Kantorovich potentials

$$\Phi_{c,\epsilon}(\mu, \nu) = \arg\max_{\phi} \int \phi d\mu - K_{\epsilon,\nu}(\phi)$$

Theorem

Let U be a Borel subset of Ω, then

$$\begin{cases} \mu_1 \le \mu_2 & \text{on } U \\ \phi_1 \le \phi_2 & \text{on } \Omega \setminus U \end{cases} \implies \phi_1 \wedge \phi_2 \in \Phi_{c,\epsilon}(\mu_1, \nu), \phi_1 \vee \phi_2 \in \Phi_{c,\epsilon}(\mu_2, \nu)$$

And $\phi_1 \le \phi_2$ on the support of $\mu_2 - \mu_1$.

Comparison principle on the potentials

For $\mu, \nu \in \mathcal{P}(\Omega)$ define the set of Kantorovich potentials

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And $\phi_1 \leq \phi_2$ on the support of $\mu_2 - \mu_1$.

When ϕ_i is determined up to a constant we have $\phi_1 \leq \phi_2$.

For $\epsilon = 0$ and c quadratic we recover the comparison principle for the Monge–Ampère equation.

Submodularity in dual OT

Lemma

Let $\nu \in \mathcal{M}_+(\Omega)$, $\epsilon \geq 0$. Then $K_{\epsilon,\nu}$ is submodular.

Since $f \mapsto \log(\int e^f \, d\alpha)$ is submodular we have for any $y \in \Omega$

$$\log\left(\int e^{\phi_1(x)-c(x,y)} d\alpha(x)\right) + \log\left(\int e^{\phi_2(x)-c(x,y)} d\alpha(x)\right) \geq$$
$$\log\left(\int e^{\phi_1 \vee \phi_2(x)-c(x,y)} d\alpha(x)\right) + \log\left(\int e^{\phi_1 \wedge \phi_2(x)-c(x,y)} d\alpha(x)\right).$$

Result follows by integration.

Proof of comparison principle on potentials

The submodularity of $K_{\epsilon,\nu}$ grants

$$\begin{aligned}
&\int \phi_1 d\mu_1 - K_{\epsilon,\nu}(\phi_1) + \int \phi_2 d\mu_2 - K_{\epsilon,\nu}(\phi_2) \\
&\leq \int \phi_1 d\mu_1 - K_{\epsilon,\nu}(\phi_1 \wedge \phi_2) + \int \phi_2 d\mu_2 - K_{\epsilon,\nu}(\phi_1 \vee \phi_2) \\
&\leq \int \phi_1 \wedge \phi_2 d\mu_1 - K_{\epsilon,\nu}(\phi_1 \wedge \phi_2) + \int \phi_1 \vee \phi_2 d\mu_2 - K_{\epsilon,\nu}(\phi_1 \vee \phi_2) \\
&- \int (\phi_1 - \phi_2)^+ d(\mu_2 - \mu_1).
\end{aligned}$$

Then optimality of ϕ_1, ϕ_2 ensures $\phi_1 \wedge \phi_2 \in \Phi_{c,\epsilon}(\mu_1, \nu), \phi_1 \vee \phi_2 \in \Phi_{c,\epsilon}(\mu_2, \nu)$ as well as $\int (\phi_1 - \phi_2)^+ d(\mu_2 - \mu_1) = 0$.

Comparison principle for JKO

Theorem

Let $c \in C(\Omega \times \Omega)$ and $h\colon [0, +\infty) \to \mathbb{R}$ denote a strictly convex l.s.c. and superlinear function. Define $H_m\colon (\nu \in \mathcal{M}_+(\Omega)) \mapsto \int h(\frac{d\nu}{dm})\, dm$ where $m \in \mathcal{M}_+(\Omega)$ is a fixed reference measure. If $\mu_1 \leq \mu_2$ then

$$\nu_1 \leq \nu_2, \quad \text{where } \nu_i = \arg\min_{\nu} OT_{c,\epsilon}(\mu_i, \nu) + H_m(\nu).$$

Exchangeability in primal OT

Lemma

The following functional defined over $\mathcal{M}_+(\Omega) \times \mathcal{M}_-(\Omega)$ is exchangeable

$$F : (\mu, \tau) \mapsto OT_{c,\epsilon}(\mu, -\tau)$$

Indeed

$$F(\mu, \tau) = OT_{c,\epsilon}(\mu, -\tau) = \sup_{\phi,\psi} \int \phi d\mu + \int \psi d\tau - \epsilon \int e^{[\phi(x) - \psi(y) - c(x,y)]/\epsilon} d\alpha \otimes \nu(x, y)$$

is the Legendre-Fenchel conjugate of a submodular function.

Application to JKO

Take $\mu_1 \leq \mu_2$ then $OT_{c,\epsilon}(\mu_1, \cdot) \ll_{\mathrm{Q}} OT_{c,\epsilon}(\mu_2, \cdot)$. Since H_m is totally exchangeable then

$$\arg\min_{\nu} OT_{c,\epsilon}(\mu_1, \nu) + H_m(\nu) \ll_{\mathrm{Q}} \arg\min_{\nu} OT_{c,\epsilon}(\mu_2, \nu) + H_m(\nu).$$

We conclude using the strict convexity of H_m which ensures the uniqueness of the minimizers.

Outlook

- The framework allows to derive comparison principles for a wide class of variational problems : in PDEs it allows for local or non local operators, in OT we also obtain results for unbalanced optimal transport.
- Proofs of the JKO comparison principle are often based on the properties of the transport map. Here we fully bypassed the use of the transport map or the Monge–Ampère equation.

Thank you for your attention !

References i

Alter, F., Caselles, V., and Chambolle, A. (2005).
A characterization of convex calibrable sets in.
Mathematische Annalen, 332(2):329–366.

Bach, F. (2013).
Learning with submodular functions: A convex optimization perspective.
Foundations and Trends® in Machine Learning, 6(2–3):145–373.

Bach, F. (2018).
Submodular functions: from discrete to continuous domains.
Mathematical Programming, 175(1–2):419–459.

Chen, X. and Li, M. (2020).
S-convexity and gross substitutability.
SSRN Electronic Journal.

Galichon, A., Hsieh, Y.-W., and Sylvestre, M. (2024).
Monotone comparative statics for submodular functions, with an application to aggregated deferred acceptance.

Jacobs, M., Kim, I., and Tong, J. (2020).
The l^1-contraction principle in optimal transport.

Lorentz, G. G. (1953).
An inequality for rearrangements.
The American Mathematical Monthly, 60(3):176–179.

References iii

Murota, K. (2003).
***Discrete Convex Analysis*.**
Society for Industrial and Applied Mathematics.

Topkis, D. M. (1978).
Minimizing a submodular function on a lattice.
Operations Research, 26(2):305–321.