§3: Euler integral and its deformation.

$$M^{hyp} = M^{hyp}_{\pi} = \mathfrak{M}\left(\mathcal{B}_{X|Z\times T}\right) \qquad X \to V\left(1 - y f(x,z)\right) \qquad T = (\mathbb{C}^*)^{n+1} \text{ coords } x, y.$$

Here $\mathcal{B}_{X|Z\times T}$ is the following $D_{Z\times T}$ - module:

$$\mathcal{B}_{X|Z\times T} = D_{Z\times T}/\mathcal{I}, \text{ where } \mathcal{I} \text{ is generated by}$$

(1) $1 - y f(x,z)$

(2) $x_j \partial_{x_j} - (y\partial_y)\, y \cdot (x_j \partial_{x_j}) f(x,z) \qquad j = 1, \dots, n.$

(3) $\partial_{z_k} - (y\partial_y) \cdot y \cdot \partial_{z_k} f(x,z). \qquad k = 1, \dots, m$

(partial) Mellin: $\mathfrak{M} : \cancel{K} \; \mathbb{C}[z, x^{\pm 1}]\langle \partial_z, \partial_x \rangle \longrightarrow K\langle \sigma_\nu, \partial_z \rangle =: DD$

$$-x_i \partial_{x_i} \longrightarrow \nu_i$$
$$x_i^{\pm 1} \longrightarrow \sigma_{\nu_i}^{\pm 1}$$

(A) $1 - \sigma_{\nu_0} f(\sigma_\nu, z) \qquad (x_0 = y)$

(B) $-\nu_j + \nu_0\, \sigma_{\nu_0}\, \nu_j\, f(\sigma_\nu, z) \qquad j = 1, \dots, n$

(C) $\partial_{z_k} - \nu_0 \cdot \sigma_{\nu_0} \cdot \partial_{z_k} f(\sigma_\nu, z) \qquad k = 1, \dots, m$

These generate a left DD - ideal J.

Observation: these operators annihilate the Euler integral

$$I(z,\nu) = \int_\Gamma \frac{x_1^{\nu_1} \cdots x_n^{\nu_n}}{f(x,z)^{\nu_0}} \frac{dx_1}{x_1} \wedge \dots \wedge \frac{dx_n}{x_n}.$$

(A) $$\sigma_{\nu_0} f(\sigma_\nu, z) \circ I(z,\nu) = \sigma_{\nu_0} \int_\Gamma f(x,z) \frac{x^\nu}{f(x,z)^{\nu_0}} \frac{dx}{x}$$

$$= I(z,\nu).$$

(B) "Four lectures on Euler integrals" Proposition 4.4.

(C) $$\nu_0 \sigma_{\nu_0} (\partial_{z_k} f(\sigma_\nu, z)) \circ I(z,\nu) = \nu_0 \sigma_{\nu_0} \int_\Gamma \partial_{z_k} f \cdot \frac{x^\nu}{f^{\nu_0}} \frac{dx}{x}$$

$$= \int_\Gamma \nu_0 \cdot \frac{x^\nu}{f^{\nu_0+1}} \partial_{z_k} f \, \frac{dx}{x} = \partial_{z_k} \circ I(z,\nu).$$

$d \log \frac{x^\nu}{f^{\nu_0}}$

Consider the de Rham complex with $\nabla_\omega = d_x + \omega \wedge$.

$$\cancel{0 \to \Omega^0(X/Z} \quad 0 \to \Omega^0_{X/Z} \xrightarrow{\nabla_\omega} \Omega^1_{X/Z} \xrightarrow{\nabla_\omega} \cdots \to \Omega^n_{X/Z} \to 0.$$

$$H^n(\Omega^\bullet_{X/Z}, \nabla_\omega) = \frac{\Omega^n_{X/Z}}{\nabla_\omega(\Omega^{n-1}_{X/Z})} = \text{"}\frac{n\text{-forms}}{n\text{-forms which integrate to 0 on any cycle}}\text{"} \qquad \eta \to \int_\Gamma \eta \cdot \frac{x^\nu}{f^{\nu_0}}$$

Think of $[\eta] \in H^n(\ldots)$ as an integral

$$\int_\Gamma \eta \frac{x^\nu}{f^{\nu_0}} \quad \text{with unspecified cycle } \Gamma.$$

Equip $H^n(\ldots)$ with DD-module structure:

$$\sigma_{\nu_j} \bullet \int_\Gamma \eta(\nu, z) \frac{x^\nu}{f^{\nu_0}} = \int_\Gamma x_j \, \eta(\nu + e_j, z) \frac{x^\nu}{f^{\nu_0}}$$

$$\rightsquigarrow \sigma_{\nu_j} \bullet [\eta(\nu, z)] = [x_j \, \eta(\nu + e_j, z)].$$

$$\partial_{z_k} \bullet [\eta(\nu, z)] = \left[\frac{\nu_0 \partial_{z_k} f(x,z)}{f(x,z)} \cdot \partial_{z_k} \eta(\nu, z) \right].$$

$$\begin{array}{ccccc} 0 \to J & \hookrightarrow & DD & \xrightarrow{\phi} & H^n(\Omega^\circ_{X/Z}, \nabla_w) \\ & & P & \longmapsto & P \bullet \left[\frac{dx}{x}\right] \end{array}$$

(maybe)
<u>Proposition 3.1</u> $M^{hyp} \simeq H^n(\Omega^\circ_{X/Z}, \nabla_w)$ as DD-modules "via ϕ" (because $[1] \longmapsto [\frac{dx}{x}]$)

That is, $H^n(\Omega^\circ_{X/Z}, \nabla_w)$ is cyclic, generated by $\left[\frac{dx}{x}\right]$ and $J = \ker \phi$.

<u>TASK</u>: eliminate σ_ν from A + B + C.

J_{elim}

The result of TASK gives a cyclic D_{Z_K}-module

$$N^{hyp} \subset M^{hyp} \text{ given by } D_{Z_K}/J_{elim} \cong D_{Z_K} \cdot \left[\frac{dx}{x}\right]$$

$\downarrow$

this inclusion might be strict, but

$$\mathrm{Sing}(N^{hyp}) = \mathrm{Sing}(M^{hyp}) \quad (p15).$$

~~For each $z \in Z$, there is a~~

Fixing $\nu \in \mathbb{C}^{n+1}$ generic: $\mathrm{Sing}\, N^{hyp}(\nu) = \mathrm{Sing}\, M^{hyp}(\nu)$.

For each z, there is a surjection

$$H_n(X_z, \mathcal{L}_\nu) \longrightarrow \mathcal{H}om_{D_Z}(N^{hyp}(\nu), \mathcal{O}_z)$$

$$[\Gamma] \longmapsto \left([\xi] \longmapsto \int_\Gamma \frac{x^\nu}{f^{\nu_0}}\, \xi\right).$$

<u>Theorem 3.2</u> For generic $\nu \in \mathbb{C}^{n+1}$, $\mathrm{Sing}(M^{hyp}(\nu))$ is where an Euler integral $\int_\Gamma \frac{x^\nu}{f^{\nu_0}} \frac{dx}{x}$ develops singularities for some Γ.

$\hbar$ deformation

likelihood equations : $d\log \frac{x^\nu}{f^{\nu_0}} = \omega = 0$

$\rightsquigarrow$ likelihood ideal $I \subset K[x^{\pm 1}, z, f(x,z)^{\pm 1}] = R$

in a <u>commutative</u> ring $= R$.

The <u>Koszul complex</u> is a free resolution of R/I :

$$\cdots \longrightarrow \Omega^{n-1}_{X/Z} \xrightarrow{\omega\wedge} \underbrace{\Omega^{n}_{X/Z}}_{\simeq R} \longrightarrow R/I \to 0$$

$d + \omega\wedge$ twisted de Rham.

VS <u>de Rham</u> :

$$\cdots \longrightarrow \Omega^{n-1}_{X/Z} \xrightarrow{d} \Omega^{n}_{X/Z} \longrightarrow H^n_{dR}(X/Z) \to 0$$

idea : view twisted de Rham as interpolating between de Rham and Koszul,(or between H^n_{dR} and R/I).

Consider $(\Omega^\bullet_{X/Z}, \hbar d + \omega\wedge)$ for small $\hbar$, work over $K[[\hbar]]$.

This amounts to replacing ν by $\frac{\nu}{\hbar}$

$$\sigma^\hbar_{\nu_i} \bullet [g(\nu,z)] = g(\nu + \hbar \cdot e_i, z)$$
$$\partial^\hbar_{z_k} \bullet [g(\nu,z)] = \hbar \partial_{z_k} g(\nu, z).$$

$$[\sigma^\hbar_{\nu_i}, \nu_j] = \delta_{ij} \cdot \hbar\, \sigma_{\nu_i}, \quad [\partial^\hbar_z, z] = \hbar.$$

For $\hbar \to 0$, the ring $K[z]\langle \sigma^{\hbar \, \pm 1}_{\nu_i}, \partial^\hbar_z \rangle$ becomes more and more commutative.

$\lim_{\hbar \to 0} H^n(\Omega_{X/Z_{\ldots}}, \nabla^\hbar_\omega) = R/I$