Filtrations

- R ring, M graded R-module
- Filtration

$$F_0 M \subset F_1 M \subset F_2 M \dots$$

$\Rightarrow$ associated graded

$$gr(M) = \bigoplus_{i=0}^{\infty} F_{i+1}M / F_i M$$

- If R is also a filtered ring, then $gr(M)$ is a $gr(R)$-module.
- Filtration of coherent R-module M is "good" if $gr(M)$ is a finitely generated $gr(R)$-module

Our setting

- Recall: $D_{[\![\hbar]\!]}$ is the subsheaf of $D_{Z_{K[\![\hbar]\!]}}$ generated by a local section of $\partial_{Z[\![\hbar]\!]}$'s $\partial^{\hbar} := \hbar\partial$ w/ ∂ local vector field
- Not $D_{Z[\![\hbar]\!]}$, which is locally $D_{Z_K}(\mathcal{U})[\![\hbar]\!]$
- Instead, use the rescaled derivation $\partial^{\hbar}$ ← not exponent

where

$$[\partial_i^{\hbar}, g] = \hbar \partial_{z_k} g, \quad [6_{\nu_i}, 6_{\nu_j}] = 0$$

$$g \in \partial_{z_k} \qquad [6_{\nu_i}, \nu_j] = \delta_{ij} \hbar 6_{\nu_i}$$

Filtrations, cont.

- $\mathcal{D}_{[[\hbar]]}$ filtered by

$$F_k \mathcal{D}_{[[\hbar]]} := \mathcal{O}_{Z[[\hbar]]}\langle \partial_{i_1}^{\hbar}, \partial_{i_1}^{\hbar}\partial_{i_2}^{\hbar}, \ldots, \partial_{i_1}^{\hbar}\cdots\partial_{i_k}^{\hbar} \rangle$$

- $\mathcal{D}_{((\hbar))} := K((\hbar)) \otimes_{K[[\hbar]]} \mathcal{D}_{[[\hbar]]}$ filtered by

$$F_k \mathcal{D}_{((\hbar))} := K((\hbar)) \otimes_{K[[\hbar]]} F_k \mathcal{D}_{[[\hbar]]}$$

- $\mathcal{O}_{T^*Z} \simeq \mathcal{D}_{[[\hbar]]}/\hbar \mathcal{D}_{[[\hbar]]}$ filtered by

$$F_k \mathcal{O}_{T^*Z} := \underline{K[[\hbar]]/(\hbar) \otimes_{K[[\hbar]]}} F_k \mathcal{D}^{[[\hbar]]}$$

"taking limit as $\hbar \longrightarrow 0$"

$\boxed{\text{Def:}}$ The characteristic cycle of M a $D_{[[h]]}$, $D_{((h))}$ or $\mathcal{O}_{T^*Z}$-mod is

$$CC(M) := \sum_{p \in \mathrm{Supp}_0(\mathrm{gr}(M))} m_p V(p)$$

- Does not depend on choice of good filtration.
- Lives in $T^*Z_{[[h]]}$, $T^*Z_{((h))}$, & T^*Z, respectively

$\boxed{\text{Compare to:}}$ X algebraic variety, M a coherent D_X-module, then

$$\mathrm{Char}(M) = \mathrm{Supp}(\mathrm{gr}\, M) \subseteq T^*X$$

dim X for M holonomic

$$CC(M) = \sum_{\substack{\Lambda \text{ irred.} \\ \text{component of } \mathrm{char}(M)}} m_\Lambda [\Lambda]$$

- $\mathrm{Supp}_0(\mathrm{gr}(M))$ = minimal associated primes

- The multiplicity m_p is the length of $\mathrm{gr}(M)_p$ as a $\mathcal{O}_{T^*X,p}$-module

$\boxed{\text{Recall}}$

- $$\begin{array}{c} X = V(1-yf(x,z)) \subseteq Z \times T^n \\ \downarrow \pi \\ Z \end{array}$$

- M^{hyp}
  - D_{Z_K}-module
  - Iso. to $H^n((X/Z)_K, \omega)$ as a DD-mod

    where K involves ν's

- M_0^{hyp}
  - $\mathcal{O}_{T^*Z_K \times T_K}$-module
  - Defined as $(K[[\hbar]]/(\hbar)) \otimes_{K[[\hbar]]} M^{hyp}_{[[\hbar]]}$

    where $M^{hyp}_{[[\hbar]]} \simeq H^n((X/Z)_{K[[\hbar]]}, \omega)$ as $DD_{[[\hbar]]}$-mod

$\boxed{\text{Theorem 4.1}}$ "Commutative object computes hypergeom. disc."

$$CC(M^{hyp}) = CC(M_0^{hyp})$$

as algebraic cycles on T^*Z_K.

$\boxed{\text{Strategy}}$

1. Show $M^{hyp}_{[[\hbar]]}$ is a coherent $D_{[[\hbar]]}$-module (Thm 4.2)
2. Show $\mathrm{supp}_0(\mathrm{gr}\, M^{hyp}) = \mathrm{supp}_0(\mathrm{gr}\, M_0^{hyp})$
3. Show multiplicities agree

② Equality of supports.

- First show

$$\star\ \mathrm{Supp}_0(\mathrm{gr}\, M^{hyp}_{((\hbar))}) \quad \text{— in } T^*Z_{((\hbar))}$$

$$= \{ P \in \mathrm{Supp}_0(\mathrm{gr}\, M^{hyp}_{[[\hbar]]}) : \hbar \notin P \}$$

$$\subseteq \mathrm{Supp}_0(\mathrm{gr}\, M^{hyp}_{[[\hbar]]}). \quad \text{— in } T^*Z_{[[\hbar]]}$$

- Consider the map

$$\begin{array}{ccc} (k[[\hbar]]/(\hbar)) \otimes_{k[[\hbar]]} M^{hyp}_{[[\hbar]]} & & \\ \| & & \star \\ \star\ \mathrm{Supp}_0(\mathrm{gr}\, M_0^{hyp}) & \longrightarrow & \mathrm{Supp}_0(\mathrm{gr}\, M^{hyp}_{[[\hbar]]}) \qquad (+) \\ P_0 & \longmapsto & \pi^{-1}(P_0) \\ \text{in } T^*Z_K \times T_K & & \text{in } T^*Z_{[[\hbar]]} \end{array}$$

where $\pi: D_{[[\hbar]]} \twoheadrightarrow D_{[[\hbar]]}/\hbar D_{[[\hbar]]} = \mathcal{O}_{T^*Z}$

Lemma 4.5 says $(+)$ is a bijection.

$\boxed{\text{Lem 4.5}}$ A Noetherian commutative ring, $I \subset A$ ideal contained in the Jacobson radical of A, L is a finitely-gen. A-module Hausdorff wrt the I-adic topology. Then

$$\mathrm{Supp}_{A/I}(L/I\cdot L) = \mathrm{Supp}_A(L) \cap V_{\mathrm{Spec}\, A}(I)$$

$$P \longmapsto \pi^{-1}(P) \quad \text{is a bijection,}$$

with $\pi: A \longrightarrow A/I$

② supports, cont.

- Rmk: $M^{hyp}_{((h))} \simeq K((\hbar)) \otimes_K M^{hyp}$

by comparing generators of ideals

$$\overset{?}{\Rightarrow} \underset{\text{in } T^*Z_{((h))}}{\mathrm{supp}_0(\mathrm{gr}\, M^{hyp}_{((h))})} \xrightarrow[(+)]{} \underset{\text{in } T^*Z_K}{\mathrm{supp}_0(\mathrm{gr}\, M^{hyp})}$$

(+) This equality is implied but not explicitly stated in the paper. Morally, "set $\hbar$ to 1."

③ Multiplicities.

$\boxed{\text{Prop 4.6}}$ For any p in the common supp_0,

$$\mathrm{len}_{\underline{\mathcal{O}_{T^*Z, p_0}}} \left(\mathrm{gr}(M^{hyp})_p\right)$$

$$= \mathrm{len}_{\underline{\mathcal{O}_{T^*Z[[\hbar]], p}}} \left(\mathrm{gr}(M_{[[\hbar]]}^{hyp})_p\right)$$

$$= \mathrm{len}\, \underline{\mathcal{O}_{T^*Z_{((\hbar)), p}}} \left((\mathrm{gr}\, M^{hyp}_{((\hbar))})_p\right).$$