4.2-3 Elia

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Recap singularities of $\int_\Gamma f(x;z)^{\nu_0} x_1^{\nu_1}\cdots x_n^{\nu_n} \frac{dx_1\cdots dx_n}{x_1\cdots x_n}$ Euler integral

$\hookrightarrow$ =: Landau singularity ??

Candidates: bifurcation locus, cod. one Whitney strata, Euler discriminant

GKZ-case ($\mathcal{Z} = \mathbb{C}^{A_1}$) LS = ED = principal A-determinant

X closed, equidim., locally complete intersection in $\mathcal{Z}\times T^n$
$\mathcal{Z}$ quasi-projective, smooth, irred.
π factorizes via a flat morphism $X \to U$, $U \subset \mathcal{Z}$ open

hypergeom. $D_{\mathcal{Z}}$-mod. $\mathcal{M}_\pi^{hyp}(\nu) := H^0 \int_{\pi_{\mathcal{Z}}} \mathcal{B}_{X|\mathcal{Z}\times T}\, x^\nu$

hypergeom. $D_{\mathcal{Z}_K}$-mod. $\mathcal{M}_\pi^{hyp} := \mathfrak{M}(\mathcal{B}_{X|\mathcal{Z}\times T}) \cong H^n(\Omega^\bullet_{X/\mathcal{Z}}(\mathcal{B}_{X|\mathcal{Z}\times T}), \nabla_\omega)$

partial Mellin tr. $\mathfrak{M}$: $Mod(D_{\mathcal{Z}\times T}) \to Mod(DD)$

$\mathcal{B}_{X|\mathcal{Z}\times T}$ (codim X)-th cohom. group of loc. cohom. of $R\Gamma_X(\mathcal{O}_{\mathcal{Z}\times T})$.

Cor 2.13 (Joris) $\forall \nu$ generic: $\mathrm{Sing}(\mathcal{M}_\pi^{hyp}(\nu)) = \nabla_{\mathcal{X}}^\pi(\mathcal{Z}) = \{z \in \mathcal{Z} : \mathcal{X}_z < \mathcal{X}^*\}$
pure of cod. one.

Thm 2.16 (Joris) $\forall \nu$ generic: $\nabla_{\mathcal{X}}^\pi(\mathcal{Z}) = \mathrm{Sing}(\mathcal{M}_\pi^{hyp}(\nu)) = S(\mathcal{M}_\pi^{hyp}(\nu)) = \mathrm{Sing}(\mathrm{Sol}(\mathcal{M}_\pi^{hyp}(\nu)))$

Intrinsic formula for hypergeom. discriminant $E^{hyp} := CC(\mathcal{M}_\pi^{hyp}(\nu)) = \sum_{\mathfrak{p} \subset \mathrm{Supp}_0(gr(\mathcal{M}_\pi^{hyp}(\nu)))} m_{\mathfrak{p}} V(\mathfrak{p})$
as maximum likelyhood eqs to

$$\underbrace{-\omega_0}_{\text{taut. one-form on } T^*\mathcal{Z}} + \sum_{i=1}^n \nu_i\, d\log x_i \in \Omega^1_{X\times_{T^n\times\mathcal{Z}}}(T^n_{\mathcal{X}} T^*\mathcal{Z})$$

$J_0 = \langle \omega \rangle < \mathcal{O}_{T^n\times T^*\mathcal{Z}}$, $\mathcal{M}_0^{hyp} = pr_*(\mathcal{O}_{T^n\times T^*\mathcal{Z}}/J_0) \in Mod_{coh}(\mathcal{O}_{T^*\mathcal{Z}})$

Thm 1.2 (Lizzie) $\mathcal{Z}$ affine $\Rightarrow$ $E^{hyp} = CC(\mathcal{M}_0^{hyp})$.

4.2 A geometric description of Euler discriminant

homogenized $T^*\mathcal{Z}$ is $\mathbb{P}^*\mathcal{Z} := ((\underbrace{\overset{z}{\mathbb{C}}}_{\text{trivial bundle}} \oplus T^*\mathcal{Z}) \setminus \underbrace{O_{\mathbb{C}\oplus T\mathcal{Z}}}_{\text{zero section}})/\mathbb{C}^*$ — scales fibers
with inclusion $\iota_{T^*\mathcal{Z}} : T^*\mathcal{Z} \hookrightarrow \mathbb{P}^*\mathcal{Z}$ $\{z=1\}$
& projection $\pi_{\mathbb{P}^*\mathcal{Z}} : \mathbb{P}^*\mathcal{Z} \twoheadrightarrow \mathcal{Z}$

hyperpl. at ∞ $H_\infty \cong (T^*\mathcal{Z} \setminus O_{T^*\mathcal{Z}})/\mathbb{C}^*$ $\{z=0\}$

Notation $V_0 := \overline{\pi_{T^*\mathcal{Z}}(V(J_0))}$, $\pi_{T^*\mathcal{Z}} : T^{n+1}\times T^*\mathcal{Z} \to T^*\mathcal{Z}$

$\mathbb{P}\,\mathrm{Char}(\mathcal{M}^{hyp}) := (\mathrm{Char}(\mathcal{M}^{hyp}) \setminus O_{T^*\mathcal{Z}})/\mathbb{C}^* \subset H_\infty$

identified with

Prop 4.7 $\nabla_{\mathcal{X}}^\pi(\mathcal{Z}) = \pi_{\mathbb{P}^*\mathcal{Z}}(\overline{\iota_{T^*\mathcal{Z}}(V_0)} \cap H_\infty)$.

$\mathrm{Char}(\mathcal{M}^{hyp}) = \mathrm{supp}\, CC(\mathcal{M}^{hyp}) \overset{\text{Thm 4.1}}{=} \mathrm{supp}\, CC(\mathcal{M}_0^{hyp})$

Let $\mathcal{F} = V(-\omega_0 - \nu_0 z \frac{df}{f} + \sum_{i=1}^n z\nu_i \frac{dx_i}{x_i}) \subset T^n \times \mathbb{P}^*\mathcal{Z}$,
Y any smoot toric compactification of T^n.

Notation $p_1 : Y\times\mathbb{P}^*\mathcal{Z} \twoheadrightarrow Y\times\mathcal{Z}$, $p_2 : Y\times\mathbb{P}^*\mathcal{Z}\twoheadrightarrow\mathbb{P}^*\mathcal{Z}$, $p_{\mathcal{Z}} : Y\times\mathcal{Z}\to\mathcal{Z}$.

Set $V_f := \overline{V_{T^n\times\mathcal{Z}}(f)} \subset Y\times\mathcal{Z}$. $\overline{\mathcal{F}}\subset Y\times\mathbb{P}^*\mathcal{Z}$.

Thm 4.8 $\mathbb{P}\mathrm{Char}(\mathcal{M}^{hyp}) = p_2(\overline{\mathcal{F}}\cap p_1^{-1}(V_f))$.

Cor [Conj. by Simon + Max] $\nabla_\chi^\pi(\mathcal{Z}) = p_{\mathcal{Z}}(\overline{p_1(\mathcal{F})}\cap V_f)$,

therefore independent of Y.

Proof of Cor p_1 closed $\Rightarrow \overline{p_1(\mathcal{F})} = p_1(\overline{\mathcal{F}}) \Rightarrow \overline{p_1(\mathcal{F})}\cap V_f = p_1(\overline{\mathcal{F}}\cap p_1^{-1}(V_f))$. □

Proof of 4.8 Show $\overline{\mathcal{F}}\cap p_1^{-1}(V_f) = \overline{\mathcal{F}}\cap p_2^{-1}(H_\infty)$.

"$\supset$" clear. "$\subset$" computation in coord. □

"Euler stratification $\Leftrightarrow$ E^{hyp} + Euler obstructions"

4.3 GKZ case

$f(x;z) = \sum_{a\in A} z_a x^a$ $\quad A\subset\mathbb{Z}^n$ finite, $z := (z_a)_{a\in A}\in\mathbb{C}^A = \mathcal{Z}$

A^h homog. of A : collection of $(1,a)\in\mathbb{Z}^{n+1}$, $a\in A$.

Assume A^h spans $\mathbb{Q}^{n+1}$ over $\mathbb{Q}$.

$D_{\mathcal{Z}_K[[\hbar]]} = K\langle z_a, \partial_a^\hbar : a\in A\rangle$ Weyl alg. on $\mathbb{C}^A$.

toric ideal $I_A^\hbar := \langle \partial^{\hbar\underline{u}} - \partial^{\hbar\underline{v}} : \underline{u},\underline{v}\in\mathbb{Z}^{n+1}_{\geq 0}$ s.t. $\underline{u}-\underline{v}\in\operatorname{Ker}A\rangle < D_{\mathcal{Z}_K[[\hbar]]}$

Euler oper. $\langle E_i^\hbar + v_i : i=0,\dots,n\rangle < D_{\mathcal{Z}_K[[\hbar]]}$ $\quad E_i^\hbar := A_i\cdot z\partial^\hbar$

GKZ ideal $H_A^\hbar(v) < D_{\mathcal{Z}_K[[\hbar]]}$ left ideal generated by both above.

GKZ system $\mathcal{M}_A^\hbar(v) := D_{\mathcal{Z}_K[[\hbar]]}/H_A^\hbar(v)$.

Define the $\hbar$-version of everything by $\partial\to\partial^\hbar$, $D_{\mathcal{Z}_K}\to D_{[[\hbar]]}$.

Thm 4.9 $\exists$ $D_{[[\hbar]]}$-Mod iso. $\mathcal{M}_A^\hbar(v)\cong\mathcal{M}^{hyp}_{[[\hbar]]}$, $[1]\mapsto[1]$.

Then, $[1]\in\mathcal{M}^{hyp}_{[[\hbar]]}$ cyclic. generator as $D_{[[\hbar]]}$-Mod.

Trivialize $T^*\mathcal{Z}_K\cong\mathcal{Z}_K\times(\mathcal{Z}_K)^* \Rightarrow \mathcal{O}_{T^*\mathcal{Z}_K} = K[z_a,\zeta_a ; a\in A]$

identify $\partial_a\to\zeta_a$, denote by $I_0 < \mathcal{O}_{T^*\mathbb{C}^A}$ the

GKZ ideal under this identification for $K=\mathbb{C}$.

$\mathbb{Z}^{n+1}$-grading on $\mathcal{O}_{T^*\mathbb{C}^A}$ by $\deg\zeta_a := -(1,a)$, $\deg z_a := (1,a)$.

Thm 4.1

$\Downarrow$

Cor 4.13 $\quad CC(\mathcal{M}_A(v)) = [\,\mathrm{gr}(\mathcal{O}_{T^*\mathbb{C}^A}/I_0)\,]$ alg. cycle

gets rid of v_i's