Introduction

$$X = \{(x,z) \in (\mathbb{C}^*)^n \times Z \mid f_1(x;z) = \dots = f_\ell(x;z) = 0\} \subseteq (\mathbb{C}^*)^n \times Z$$

$\downarrow \pi$ flat family of very affine locally complete intersections

Z

$$\nabla_\chi^\pi(Z) = \overline{\{z \in Z : \chi_z := |\chi_{top}(\pi^{-1}(z))| < \chi^*\}} \subset Z$$

Euler discriminant locus

generic Euler characteristic over Z

$$\int_\Gamma f(x;z)^{-\nu_0} x_1^{\nu_1} \dots x_n^{\nu_n} \frac{dx_1 \dots dx_n}{x_1 \dots x_n} \quad \text{Euler integral} \quad (1)$$

$\ell = 1$

singular locus? (branch points or poles for some contour Γ)

new object: holonomic D_Z-module $M_\pi^{hyp}(\nu)$

Main Results (I)

- for generic ν, $\mathrm{Sing}(M_\pi^{hyp}(\nu)) = \nabla_\chi^\pi(Z)$ (Thm. 2.16)
- $\nabla_\chi^\pi(Z)$ is purely codimension 1 (or $\emptyset$) (Thm. 2.11)
- for $\ell = 1$ and generic ν, $\mathrm{Sing}(M_\pi^{hyp}(\nu))$ is also the singular locus of the Euler integral (1) (Thm. 3.2)

analogy to GKZ case: $\ell = 1$, $Z = \mathbb{C}^A$, then:

$$\mathrm{Sing}(D_{Z_A}/H_A(\nu)) = V(E_A) \quad (\S 4.3)$$

GKZ ideal

principal A-determinant

hypergeometric discriminant:

$$CC(\mathcal{M}_\pi^{hyp}(v)) = \sum_{Y \subset Z} m_Y T_Y^* Z \subseteq T^* Z \xrightarrow{p} Z$$

closed subvariety, not necessarily codim 1

this is a refinement of $\nabla_\chi^\pi(Z)$

(morally: " $p_* CC(\mathcal{M}_\pi^{hyp}(v)) = \nabla_\chi^\pi(Z)$ " (Prop. 4.7))

relation to Euler characteristic by Kashiwara index theorem:

$$\chi_z = \sum_{Y \subset Z} (-1)^{\operatorname{codim} Y} m_Y \, Eu_Y(z)$$

Euler obstruction

problem: $CC(\mathcal{M}_\pi^{hyp}(v))$ is hard to compute

(and even hard to define*)

*personal view

$$\boxed{\underline{\text{Main Result II}} \quad CC(\mathcal{M}_\pi^{hyp}(v)) = CC(\mathcal{M}_0^{hyp}) \quad (\text{Thm } 4.1)}$$

better to compute

arises from differential form

$-\omega_0 + \sum_{i=1}^n v_i \, d\log x_i$

proof ingredient: $\mathcal{M}_{[[t]]}^{hyp} \xrightarrow[t \to 0]{} \mathcal{M}_0^{hyp}$

non-commutative

commutative

§5: compute $CC(\mathcal{M}_0^{hyp})$ for several Feynman graphs

Notation & Definition of $\mathcal{M}_\pi^{hyp}(v)$

Z quasi-proj. smooth irred. complex var.

$T^n = T = \operatorname{Spec} \mathbb{C}[x_1^{\pm}, \dots, x_n^{\pm 1}]$

$\mathbb{C} \subset k$, Z_k = base change of Z along $\operatorname{Spec} k \to \operatorname{Spec} \mathbb{C}$

$\mathbb{K} = \mathbb{C}(v_1, \dots, v_n)$

X locally complete intersection, closed subvariety of $Z \times T^n$ via $\iota_X : X \hookrightarrow Z \times T^n$

all irred. comp. have same dim = dim X

$\pi = \pi_Z \circ \iota_X : X \to Z$

$X_z = \pi^{-1}(z)$, $\chi_z = |\chi_{top}(X_z)|$

π factors through $X \xrightarrow{\tilde{\pi}} U \xrightarrow{\iota_U} Z$, $U \subseteq Z$ open

assume $\pi : X \to U$ <u>flat</u>

$\Rightarrow \dim X_z = \dim X - \dim Z$ unless $X_z = \emptyset$

Y smooth alg. var., $\mathcal{D}_Y$ sheaf of rings of diff. operators

$R^{rel} = \mathbb{C}\langle v_1, \dots, v_n, \sigma_1^{\pm 1}, \dots, \sigma_n^{\pm 1} \rangle \Big/ \begin{array}{l} [\sigma_i, v_j] = \delta_{ij}\sigma_j \\ [\sigma_i^{-1}, v_j] = -\delta_{ij}\sigma_j \\ [\sigma_i, \sigma_j] = 0 \\ [v_i, v_j] = 0 \end{array}$

non-commutative ring

$R^{rel} \cong \Gamma(\mathcal{D}_T, T)$ (2)

$$v_i \mapsto -x_i \partial_{x_i}$$
$$\sigma_i^{\pm 1} \mapsto x_i^{\pm 1}$$

$$\mathcal{D}_Z \boxtimes R^{rel} := \mathcal{D}_Z \otimes_{\mathbb{C}} R^{rel}$$

from (2), we get isom. $\mathcal{M}_{Z\times T/Z} : \mathcal{D}_Z \boxtimes R^{rel} \xrightarrow{\sim} \mathcal{D}_{Z\times T} = \mathcal{D}_Z \boxtimes \mathcal{D}_T$

take $\mathcal{D}_{Z\times T}$-mod $M \rightsquigarrow \mathcal{D}_Z \boxtimes R^{rel}$-mod $N = \mathcal{M}^{rel}(M)$:

for $P \in \mathcal{D}_Z \boxtimes R^{rel}$, $m \in N$: $P \bullet m := (\mathcal{M}(P)) \cdot m$

$$R := \mathbb{K} \otimes_{\mathbb{C}[v_1,\dots,v_n]} R^{rel}$$

$\mathbb{D} := \mathcal{D}_Z \boxtimes R$ sheaf of $\mathbb{K}$-alg.

for a $\mathcal{D}_{Z\times T}$-mod M, the partial Mellin transform is

$$\mathcal{M}(M) := \mathbb{K} \otimes_{\mathbb{C}[v_1,\dots,v_n]} \mathcal{M}^{rel}(M) \qquad (\mathbb{D}\text{-mod})$$

$M_{\mathbb{K}} := \mathbb{K} \otimes_{\mathbb{C}} M \qquad \mathcal{D}_{Z_{\mathbb{K}}\times T_{\mathbb{K}}}$-mod

$$\pi_Z : Z_{\mathbb{K}} \times T_{\mathbb{K}} \to Z_{\mathbb{K}}$$

$$\Omega^k_{X/Z}(M) := \pi_{Z*}\Big(\bigoplus_{1\le i_1<\dots<i_k\le n} M_{\mathbb{K}}\, dx_{i_1} \wedge \dots \wedge dx_{i_k} \Big)$$

one-form: $\omega := v_1 \frac{dx_1}{x_1} + \dots + v_n \frac{dx_n}{x_n}$

relative twisted differential $\nabla_\omega : \Omega^k_{X/Z}(M) \to \Omega^{k+1}_{X/Z}(M)$

$$\phi \mapsto (d_x + \omega \wedge)\phi$$

$\rightsquigarrow$ cochain complex $(\Omega^\bullet_{X/Z}(M), \nabla_\omega)$

$H^n(\Omega^\bullet_{X/Z}(M), \nabla_\omega)$ has a DD-mod structure

Thm 2.3
- $\mathcal{M}(M) \cong H^n(\Omega^\bullet_{X/Z}(M), \nabla_\omega)$ as DD-mod
- $H^j(\Omega^\bullet_{X/Z}(M), \nabla_\omega) = 0 \quad \forall j \neq n$.

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order filtration on D_Y:

locally $F_l D_Y = \sum_{|\alpha| \leq l} \mathcal{O}_Y \partial_x^\alpha$

sheaf of graded rings $gr\, D_Y = \bigoplus_{l=0}^{\infty} F_l D_Y / F_{l-1} D_Y$ commutative

$\pi : T^*Y \to Y$, $gr\, D_Y \cong \pi_* \mathcal{O}_{T^*Y}$ (omit π_* from notation)

N coherent D_Y-mod $\rightsquigarrow$ $gr\, N$ is an $\mathcal{O}_{T^*Y}$-mod

$Supp_0(gr\, N)$: minimal associated primes

$p \in Supp_0(gr\, N)$, $(gr\, N)_p$ is Artinian

$$\rightsquigarrow \underbrace{length_{\mathcal{O}_{T^*Y}, p} (gr\, N)_p}_{\overset{!!}{m_p}} < \infty$$

$V(p) := Spec(\mathcal{O}_{T^*Y}/p)$

<u>characteristic cycle</u> of N

$$CC(N) := \sum_{p \in Supp_0(gr\, N)} m_p V(p) \quad \text{alg. cycle on } T^*Y$$

characteristic variety

$$\mathrm{Char}(N) := \mathrm{supp}(CC(N))$$

singular locus

$$\mathrm{Sing}(N) := \bar{\omega}\left(\mathrm{Char}(N) \setminus T^*_Y Y\right)$$

$\bar{\omega} : T^*Y \to Y$ 0 section

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local cohomology :

affine locally : $\tilde{X} = \mathrm{Spec}\, R$, $Y \underset{\text{closed}}{\subseteq} \tilde{X}$, $Y = \mathrm{Spec}\, R/I$

R-mod M, $\Gamma_I(M) := \bigcup_{n \geq 0} (0 :_M I^n)$

$= \{ m \in M : \exists N \text{ st } m I^N = 0 \}$

$\rightsquigarrow$ make this global + take right derived functor

$R\Gamma_X(\mathcal{O}_{Z \times T})$ local cohom. complex ($X \subseteq Z \times T$ closed) $= \tilde{X}$

" Y from above

$\mathcal{B}_{X \mid Z \times T} := H^{\mathrm{codim}\, X} R\Gamma_X(\mathcal{O}_{Z \times T})$ $D_{Z \times T}$ - mod

Def • hypergeometric D_{Z_K}-mod $M_\pi^{\mathrm{hyp}} := M(\mathcal{B}_{X \mid Z \times T})$

• hypergeometric D_Z-mod $M_\pi^{\mathrm{hyp}}(v) := H^0 \int_{\pi_Z} \mathcal{B}_{X \mid Z \times T}\, x^v$

for generic $v \in \mathbb{C}^n$.

$\int_{\pi_Z} \mathcal{B}_{X \mid Z \times T}$ direct image

$\in D^b(D_Z\text{- mod})$