Singular Locus

"Recall"

- Z quasi-proj. complex alg. variety, smooth, irred.
- $T = T^n = \mathrm{Spec}\, \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$
- field extension : $\mathbb{C} \subset k$
- base change : $Z_k := Z \times_{\mathrm{Spec}\,\mathbb{C}} \mathrm{Spec}\, k$
- $K = \mathbb{C}(\nu)$, $\nu = (\nu_1, \dots, \nu_n)$
- X compl. alg. variety, $\pi : X \longrightarrow Z$ morphism
- $\chi_z = |$ Euler characteristic of $\pi^{-1}(z)| \geq 0$
- $\chi^* \mathrel{\hat=}$ generic value
- Y smooth alg. variety

Just a bit more...

$$CC(\mathcal{N}) = \sum_{\mathfrak{p} \in \mathrm{Supp}(\mathrm{gr}(\mathcal{N}))} \underbrace{m_{\mathfrak{p}}}_{\mathrm{length}_{\mathcal{O}_{T^*Y,\mathfrak{p}}}(\mathrm{gr}(\mathcal{N})_{\mathfrak{p}})} \underbrace{V(\mathfrak{p})}_{\mathrm{Spec}\,(\mathcal{O}_{T^*Y}/\mathfrak{p})}$$ - already records singularities

$$\mathrm{Char}(\mathcal{N}) = \mathrm{supp}\; CC(\mathcal{N})$$ "forget about multiplicities"

zero sections $s(x) = (x, 0)$

$$\mathrm{Sing}(\mathcal{N}) = \varpi(\mathrm{Char}(\mathcal{N}) \setminus T^*_Y Y), \quad \varpi : T^*Y \to Y$$

"for integrable connections $\mathrm{Char}(\mathcal{N}) = T^*_Y Y$ - Prop 2.2.5 perv. sheafs"

2.2. Euler discriminant

Goal: $\nabla_\chi(Z, \mathcal{M}^{hyp}_\pi(\nu)) = \mathrm{Sing}(\mathcal{M}^{hyp}_\pi(\nu)) \subseteq Z$

$\overset{\shortparallel}{\{z \in Z \mid \chi_z(\mathcal{M}) < \chi^*\}}$ $\overset{\text{a priori}}{\neq}$ $\nabla^\pi_\chi(Z) = \{z \in Z \mid \chi_z < \chi^*\}$

conv. series at z

Set $\chi_z(\mathcal{M}) \overset{\text{Kashiwara, index at } z \in Z}{=} \sum_{i=0}^{\infty} (-1)^i \dim_{\mathbb{C}} \mathrm{Ext}^i_{D_Z}(\mathcal{M}, \mathcal{O}^{an}_z)$,

and the generic Euler char. value $\chi^* =$ "holonomic rank $\mathcal{M}$",

and $\iota_z : \{z\} \hookrightarrow Z$.

Lem 2.3/10 $\mathcal{M}$ holonomic D_Z-module, $\mathrm{Sing}(\mathcal{M})^1$ the union of all codimension 1 components of $\mathrm{Sing}(\mathcal{M})$. Then,

$$\underset{\underset{Z}{\cap}}{\mathrm{Sing}(\mathcal{M})^1} = \nabla_\chi(Z, \mathcal{M})^1 \subset \nabla_\chi(Z, \mathcal{M})$$

Proof: "$\subset$" Let $z \in \mathrm{Sing}(\mathcal{M})^1$ be a generic i.e. smooth point in the singular locus. Then, by construction

$$\exists\, p \in \mathrm{supp}_0(gr(\mathcal{M})) : \varpi^{-1}(z) \in \mathrm{Spec}(\mathcal{O}_{T^*Z}/p) \subset T^*Z$$

with $m_p > 0$. Then, by Kashiwara's index formula

$$\chi_z(\mathcal{M}) = \sum_j (-1)^{d_j} \overset{\text{"Milnor number"}}{c_z(Z_j)}\, m_j$$

"and in the very affine case the number can only drop"

it follows that $\chi_z(\mathcal{M}) < \chi^*$.

Then, $z \in \nabla_\chi(Z, \mathcal{M})$ by definition, and the claim follows by taking Zariski closure.

"$\supset$" General D-module theory shows integrable connections e.g. holonomic D-modules on the smooth locus have generic Euler characteristic.

"Lemma: Index is Euler characteristic"

$$\underset{\mathcal{M}_n^{hyp}(\nu)}{\chi_z(\mathcal{M})} \overset{\text{Kashiwara, index at } z\in Z}{=} \sum_{i=0}^{\infty} (-1)^i \dim_{\mathbb{C}} \mathrm{Ext}^i_{D_z}(\mathcal{M}, \overset{\text{conv. series at } z}{\mathcal{O}_z^{an}})$$

$$\left(\overset{\text{p.4}}{=} \sum_{i=0}^{\infty} (-1)^i \dim_{\mathbb{C}} H^i(\overset{\text{"solution space"}}{\overline{R\mathcal{H}om_{D_z}(\mathcal{M}, \mathcal{O}_z^{an})}}) \right)$$

$$= \sum_{i=0}^{\infty} (-1)^i \dim_{\mathbb{C}} H^i(\underbrace{i_z^{-1}(R\pi_! }_{\text{base change}}\mathcal{L}_{-\nu}[\dim X_z]))$$

$$= \sum_{i=0} (-1)^i \dim_{\mathbb{C}} R^i \underline{\pi_{z!}\, i_{X_z}^{-1}} \mathcal{L}_{-\nu}[\dim X_z]$$

$$= \chi_z \quad \hat{=} \quad \text{Euler discriminant at } \pi^{-1}(z)$$

where $i_{X_z}: X_z \longrightarrow Z \times T$, $\pi_z: X_z \longrightarrow \{z\}$.

The local system $\mathcal{L}_\nu$ on $T = T^n$ with monodromy in the i-th axis $e^{2\pi\sqrt{-1}\nu_i}$ is given by

$$d - d\log(x^{\nu_1} \cdots x^{\nu_n}) \wedge$$

$$\left(= \left(d - \frac{\nu_1}{x_1} dx_1 + \cdots + \frac{\nu_n}{x_n} dx_n \wedge \right) x_1^{\nu_1} \cdots x_n^{\nu_n} \right)$$

encoding solutions to differential equations.

Recall the "monodromies" for $\mathcal{M}_n^{hyp}(\nu)$ were given as

$$\mathcal{M}_n^{hyp}(\nu) = H^0 \int_{\pi_z} \mathcal{B}_{X|Z\times T}\, x^{\nu}$$

2.3 Pure codimensionality

Goal: " $\nabla^{\Pi}_{x}(Z)$ is of codim 1 (unless empty)." - 2.11.

<u>Lemma 2.12:</u> Let $\mathcal{M}$ be a holonomic D_Z-module, $Z' \subset Z$ smooth closed subvariety of codim at least two. Assume the local cohomology groups $H^0_{Z'}(\mathcal{M}) = H^1_{Z'}(\mathcal{M}) = 0$ vanish and $\mathcal{M}$ is integrable on $Z \setminus Z'$. Then,

$\nabla^2 = 0$

$\mathcal{M}$ integrable on Z and $\mathrm{Sing}(\mathcal{M}) \cap Z' = \emptyset$.

Proof: Let $j : Z \setminus Z' \hookrightarrow Z$ be the natural inclusion, then $j^*\mathcal{M} = E$: $j^*\mathcal{M} \longrightarrow j^*\mathcal{M} \otimes_{\mathcal{O}_Z} \Omega^1_Z$ is an integrable connection on Z. Consider the LES

$$\underbrace{H^0_{Z'}(\mathcal{M})}_{0} \longrightarrow \mathcal{M} \longrightarrow H^0\left(\int_j j^*\mathcal{M}\right) \longrightarrow \underbrace{H^1_{Z'}(\mathcal{M})}_{0}$$

giving the iso $\mathcal{M} \simeq H^0 \int_j j^*\mathcal{M}$, and as codim $Z' > 1$ we also have $H^0_{Z'}(E) = H^1_{Z'}(E) = 0$, and thus $E \simeq H^0(\int_j j^*E)$. It follows that $\mathcal{M} \simeq E$ is an integrable connection on all of Z.

Then, integrable implies it has generic Euler characteristic and hence $\mathrm{Sing}\,\mathcal{M} \cap Z' = \emptyset$, by 2.2.5.

<u>Thm 2.11:</u> $\nabla^{\Pi}_{x}(Z)$ is pure of codim 1

Proof idea: For $Z' \subset \mathrm{Sing}\,\mathcal{M}$ with codim of two or greater we show the assumptions of 2.12 are satisfied, hence $\mathrm{Sing}\,\mathcal{M} \cap Z' = \emptyset$ ↯.

2.4 Singularity of solutions

generic

solutions to $\mathcal{M}$ fail to be holomorphic

Goal : $\nabla_Z^{\Pi}(Z) = \mathrm{Sing}(\mathcal{M}_{\Pi}^{hyp}(\nu)) = S(\mathcal{M}_{\Pi}^{hyp}(\nu))$

failure of solutions to be holomorphic is a pure codim 1 constraint

Idea : " Hartogs theorem for D-modules"

Prop 2.14: Let $\dim_{\mathbb{C}} Z = 1$. $\mathcal{M}$ is a regular holonomic D_Z-module of $\mathrm{rk} > 1$ and $0 \in \mathrm{Sing}(\mathcal{M}) \subset Z$, and suppose $L^{-1} i_0^{*} \mathcal{M} = 0$. Then, a holomorphic solution to $\mathcal{M}$ exists which can not be analytically continued to a neighborhood of 0.

Proof: By 4.16 in [41],

$$\mathbb{R}\mathcal{H}om_{D_0}(\mathbb{D}\, i_0^{*}\mathcal{M}, \mathbb{C}) \simeq \mathbb{R}\mathcal{H}om_{i_0^{-1}D_Z}(i_0^{-1}\mathcal{M}, \hat{\mathcal{O}}^{an}_{\mathbb{C}^{3}/Z})$$

The assumption $L^{-1} i_0^{*}\mathcal{M} = 0$, implies that

$$\dim_{\mathbb{C}} \mathrm{Hom}_{D_Z}(\mathcal{M}, \mathcal{O}_0^{an}) < \chi^{*} = \mathrm{hol.rk}\, \mathcal{M}$$

Prop 2.15: Extends 2.14 for $\dim_{\mathbb{C}} Z > 1$

Thm 2.16: "Euler discriminant is the locus of non-holomorphicity of solutions"

For $\nu \in \mathbb{C}^n$ generic we get

$$\nabla_{\chi}^{\Pi}(Z) = \mathrm{Sing}(\mathcal{M}_{\Pi}^{hyp}(\nu)) = S(\mathcal{M}_{\Pi}^{hyp}(\nu))$$