Simulating Sun-Earth-Moon System Using an N-body Orbital System

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The objective for this project is to create an N-body modelling system to simulate, and investigate the factors of, the Earth-Moon-Sun system. Within this, it will investigate the Runge-Kutta and Verlet manual integration methods, as well as methods of calculating orbital periods and eccentricities. The results will find that this is succesful, and that the system is able to be adapted for other closed-body systems.

1 Introduction

1.1 The Physics of Gravitational Orbital Systems

When considering how a closed system of bodies interacts with each other on a scale as large as astronomical bodies, the only forces that need to be considered are those from a body's gravitational attraction to other bodies. Between two bodies of masses m_1 and m_2, this can be described by Newton's law of universal gravitation:

$$\boldsymbol{F}_{12} = G\frac{m_1 m_2}{|\boldsymbol{r}_{12}|^2}\hat{\boldsymbol{r}_{12}} \tag{1}$$

Where $\boldsymbol{r}_{12}$ is the vector describing the distance between the two bodies.

When considering n-body systems, the force experienced by body k can be extrapolated to be:

$$\boldsymbol{F}_k = \sum^n \boldsymbol{F}_{kl} = \sum^n G\frac{m_k m_l}{|\boldsymbol{r}_{kl}|^2}\hat{\boldsymbol{r}_{kl}} = m_k \boldsymbol{a}_k \tag{2}$$

Also of useful note in the case of an orbit, and in this paper, is the period T of the orbit, which is the time taken for a body to complete one full orbit around another body. This can be calculated manually, or via Kepler's third law:

$$T^2 = \frac{4\pi^2 a^3}{G(M+m)} \tag{3}$$

Where a is the semi-major axis of the orbit which is the average of r_a and r_p, the orbital radius at apoapsis (furthest point in orbit) and periapis (closes point in orbit) respectively.

Additionally, the eccentricity e of an orbit describes how circular it is. This is mathematically defined by the following equation:

$$e = \frac{r_a - r_p}{r_a + r_p} \tag{4}$$

1.2 Using a manual iterative apporach

1.21 Verlet method

Since the acceleration changes as the position changes, it is necessary to use an iterative method to evolve the system. Two methods were used in this investigation, the first of which is the Verlet method.

At time t, a body will have defined position and velocity, and the acceleration can be calculated using its position. The position will then be evolved by one timestep δt for each body in the system.

$$\boldsymbol{r}_k(t+\delta t) = \boldsymbol{r}_k(t) + \boldsymbol{v}_k(t)\delta t + \frac{1}{2}\boldsymbol{a}_k(t)\delta t^2 \tag{5}$$

The time is then evolved to $t \leftarrow t + \delta t$. The new acceleration is now calculated, then the velocity for each body can be evolved.

$$\boldsymbol{v}_k(t) = \boldsymbol{v}_k(t-\delta t) + \frac{\delta t}{2}[\boldsymbol{a}_k(t) + \boldsymbol{a}_k(t-\delta t)] \tag{6}$$

1.22 Runge-Kutta method

Another method used is the Runge-Kutta method. This works by calculating four different slopes.

- $k1$ - the slope at $t = 0$
- $k2$ - the slope at $t = \frac{\delta t}{2}$ given $k1$ at $t = 0$
- $k3$ - the slope at $t = \frac{\delta t}{2}$ given $k2$ at $t = 0$
- $k4$ - the slope at $t = \delta t$ given $k3$ at $t = 0$

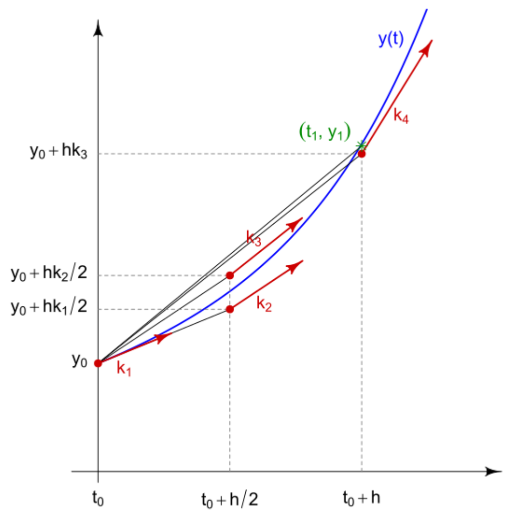

FIG. 1: The slopes used by the method. [1]

The total change then is then a weighted average of these 4 slopes:

$$\frac{\delta y}{\delta t} = \frac{\delta t}{6}[k_1 + 2k_2 + 2k_3 + k_4] \tag{7}$$

This can be used to evolve both the position and velocity of a body, by considering the slope as the velocity and acceleration respectively. It is important to note that each k value must be calculated for every body before the next can be.

1.3 Creating the program

While using C++ would allow the option for parallelisation (as the x and y directions can be calculated independently of each other), and therefore would be quicker, using Python allowed for implementation of the module in Jupyter notebooks. This was preferred in order to be able to run multiple simulations within the same document.

For the sake of comparison, both the Verlet and Runge-Kutta method have been tested for calculating orbital positions. In the case of periods and eccentricities, where not specified, the Verlet positions have been used.

For calculating the periods, two methods have been tested and compared. The period calculated by Kepler's third law, and the period manually calculated by finding the average amount of time steps between y-maximums in the orbit.

The reason for this is that whilst the manual method is more accurate, it necessarily needs two y-maximums, meaning that the body needs to potentially complete almost 2 full orbits, depending on where in the orbit the simulation starts. Equally, the Kepler method will return a result, although incorrect, even if an orbit has not been completed.

2 Analysis and Discussions

2.1 Earth-Moon-Sun System

2.11 Testing Starting Conditions

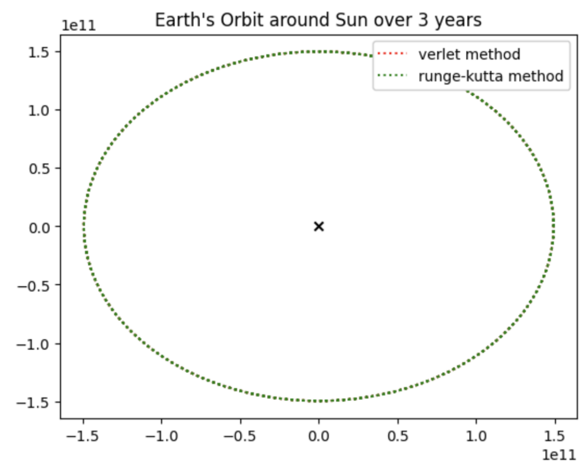

FIG. 2: Earth's orbit with recommended starting conditions

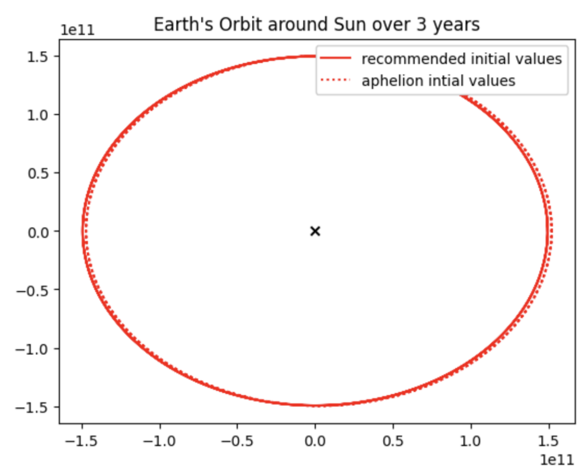

FIG. 3: Earth's orbit with aphelion starting conditions

The recommended starting conditions produce a stable orbit for both the Earth and Moon, as shown in figure 2. Both the Verlet and Runge-Kutta methods predict an orbit of 364.3 days for the Earth. Using the Verlet positions, the manual period for the moon is 27.5 days, and 27.3 for the Kepler method. These are all reasonably accurate, but the eccentricity of Earth's orbit predicted is completely off, 0.0011 for both methods. This is because the average orbital distance and velocity was used as a starting condition. If the Aphelion values [2] are used, then there is a very similar orbit seen, but with a much more accurate eccentricity of 0.0168.

2.12 Changing Solar Mass

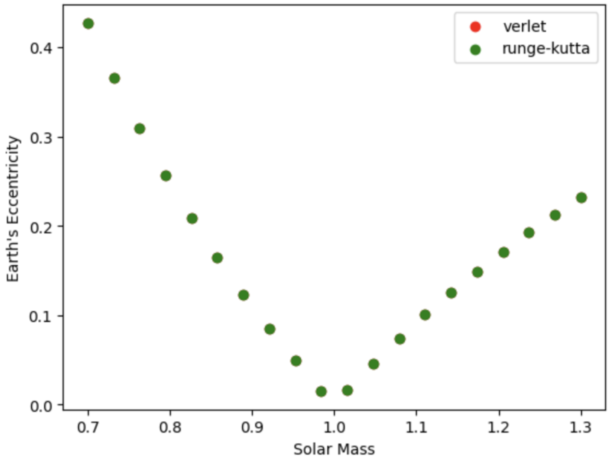

FIG. 4: Eccentricity of Earth's orbit, at different solar masses

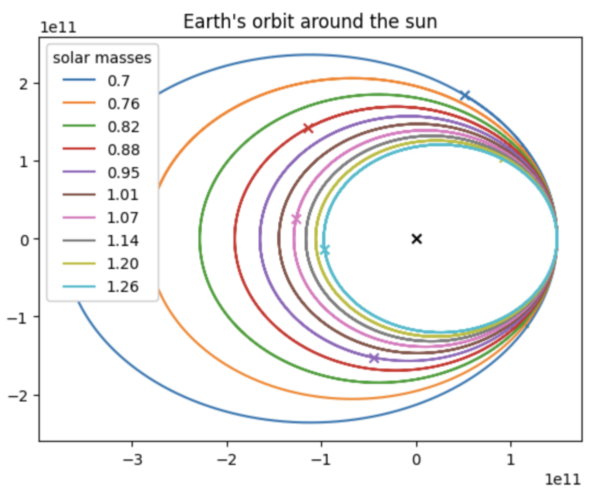

FIG. 5: Graphs depicting the orbits, the crosses depict the final points of the simulation

A change in the mass of the sun will affect its gravitational field, since $g = \frac{GM}{r^2}$. Because of this Earth's orbit ceases to be circular once the mass is changed. When the solar mass is less than 1, the starting point of the Earth is the aphelion, because of the lower gravity, whilst above 1 solar mass the starting point becomes the perihelion.

In addition to this, at higher solar masses the Earth completes multiple orbits, while at 0.7 it only just completes one, as seen in figure 5. Because of this, the simulation was run for 3 years so that an orbit could be completed for every solar mass. This means that this simulation takes around 5 minutes to run, notably longer than any other.

2.13 Changing Lunar Mass

Figure 6 clearly shows an exponential decrease in period as the lunar mass increases, with both the Verlet and Runge-Kutta methods again giving almost identical results. It can be seen in figure 8 that as the mass increases, the orbital length also decreases in tandem to the period, with the initial position becoming the perigee of the orbit.

Figure 9 gives a clear demonstration on how the gravitational force experienced by the moon towards the Earth changes as it moves around its orbit. 100 lunar masses was chosen as the large variation in orbital distance allows for a clearer picture on how the force changes. When closer to Earth, the force is much greater, and is directed towards it at all points on the orbit.

The manual and Kepler methods for calculating the orbit are compared in figure 7. While being very close to one another, the manual method is the most accurate for finding the period of the simulated orbit.

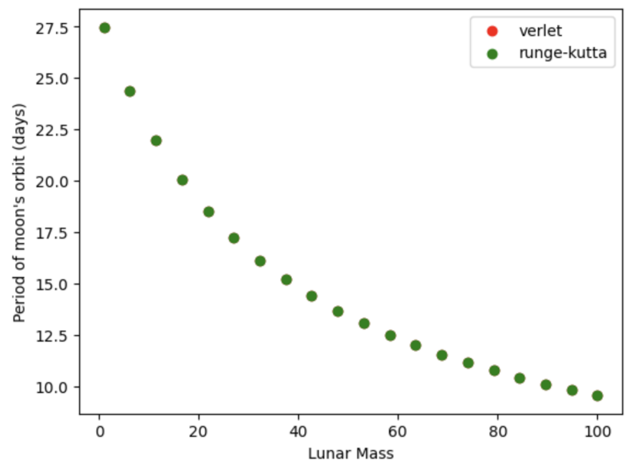

FIG. 6: Period of Moon's orbit, at different lunar masses

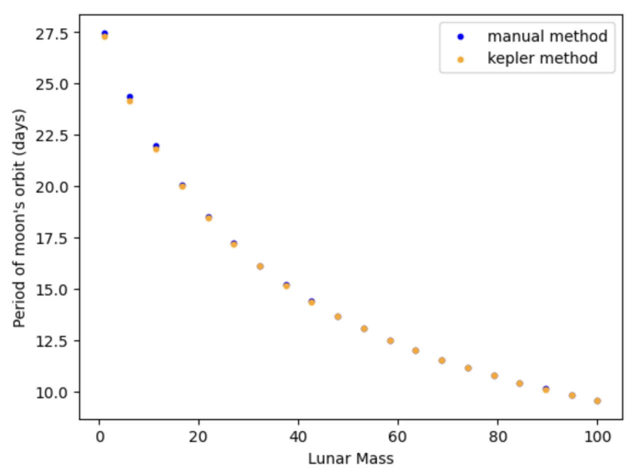

FIG. 7: The periods, calculated with Kepler vs Manual methods

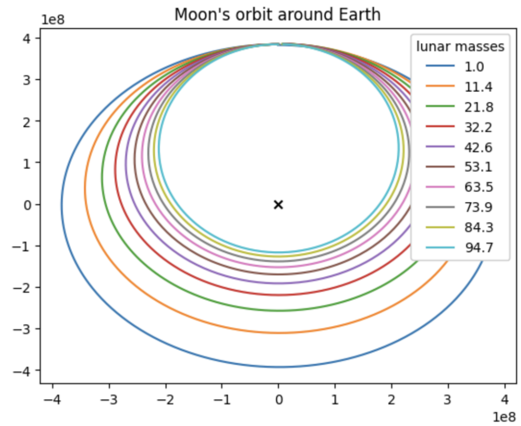

FIG. 8: Graphs depicting the orbits

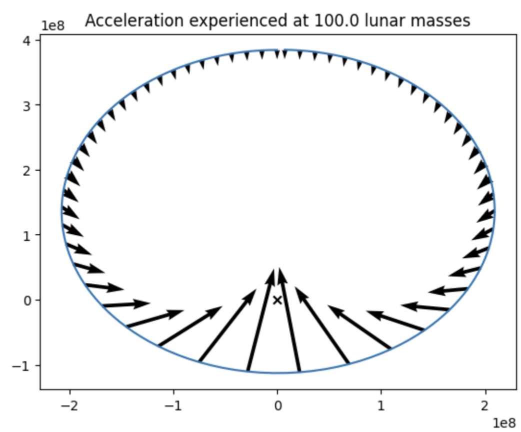

FIG. 9: Quiver graph showing the acceleration experienced during the orbit

2.2 Other Orbital Systems

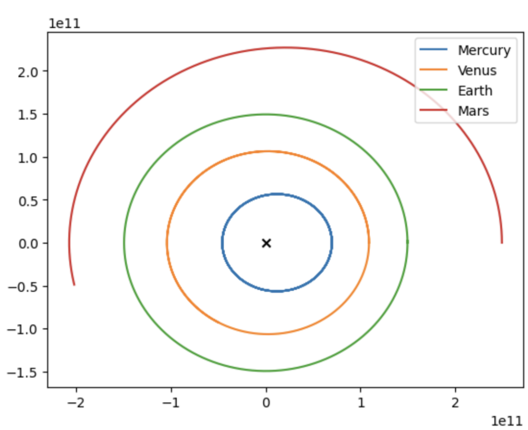

FIG. 10: The orbits of the terrestrial planets of the solar system over the course of an Earth year

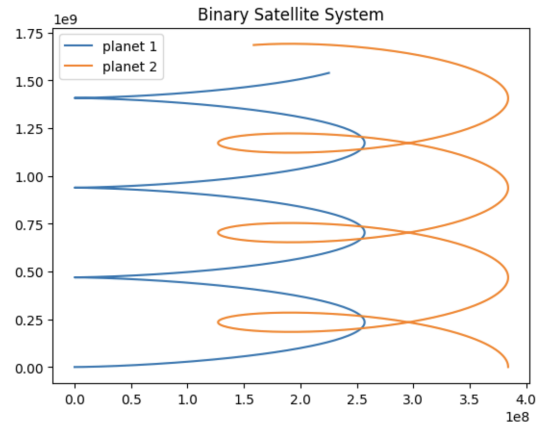

FIG. 11: The orbits of two planet-sized objects in a binary orbital

3 Conclusion

Whilst the Runge-Kutta method is a more accurate iterative method than the Verlet method, the two have given identical results throughout these simulations; the only difference being of the order of 10^{-6} in the value of the Earth's eccentricity. The Runge-Kutta method, however, is more computationally expensive and so for systems at the level of the relatively small complexity of an astronomical orbit, it seems that the Verlet method would be preferable.

Investigations into the two methods of finding the period of the orbit, via Kepler's third law and manually, showed that whilst Kepler's method may be acceptable in many cases, the manual method is superior. This is for two reasons, primarily it is more accurate, but also it will not give a false result if a full orbit hasn't been completed.

Although the primary goal of this project was to investigate the Earth-Moon-Sun system, section 2.2 demonstrates how the n-body module created can just as easily be used to simulate any other orbital system. In these two examples, the orbits of the 4 terrestrial planets in our solar system have been simulated, as well as a binary planet system. It is possible to use this not just to plot planetary orbital systems, but for simulating the movement of any closed system of interacting bodies e.g. by adapting the formula for the acceleration to account for atomic forces instead (electromagnetic, strong etc.).

4 References

[1] Wikipedia, https://en.wikipedia.org/wiki/Runge%E2%80%93Kutta_methods, (accessed January 2024)

[2] Nasa, https://nssdc.gsfc.nasa.gov/planetary/factsheet/earthfact.html, (accessed January 2024)