PROB. DISTRIB. OF FUNCTIONS OF A RANDOM VARIABLE

Here we are concerned with the investigation of the prob. distrib of some function of a random var. $\underline{X}$

We shall assume that the prob distrib of $\underline{X}$ is known and that the function $y = u(x)$ has been specified.
The distrib of $\underline{Y} = u(\underline{X})$ is of interest.

<u>If $\underline{X}$ is a discrete r.v.</u> then $\underline{Y}$ is also discrete (irrespective of $u(x)$)

$$\underline{X} : \quad x_1, \; x_2 \; --- \; x_k \; ---$$

Probs. $\quad f(x_1)\, f(x_2) \; --- \; f(x_k) \; ---$

Then $\underline{Y}$ takes values $\underbrace{u(x_1)}_{u_1}, \underbrace{u(x_2)}_{u_2} ----, \underbrace{u(x_k)}_{u_k} ---$
with probs $f(x_1), f(x_2) ---, f(x_k) ---$

Now it may happen that several x_i values give rise to same value of y.
Thus in general the prob. fn of $\underline{Y}$ is given by

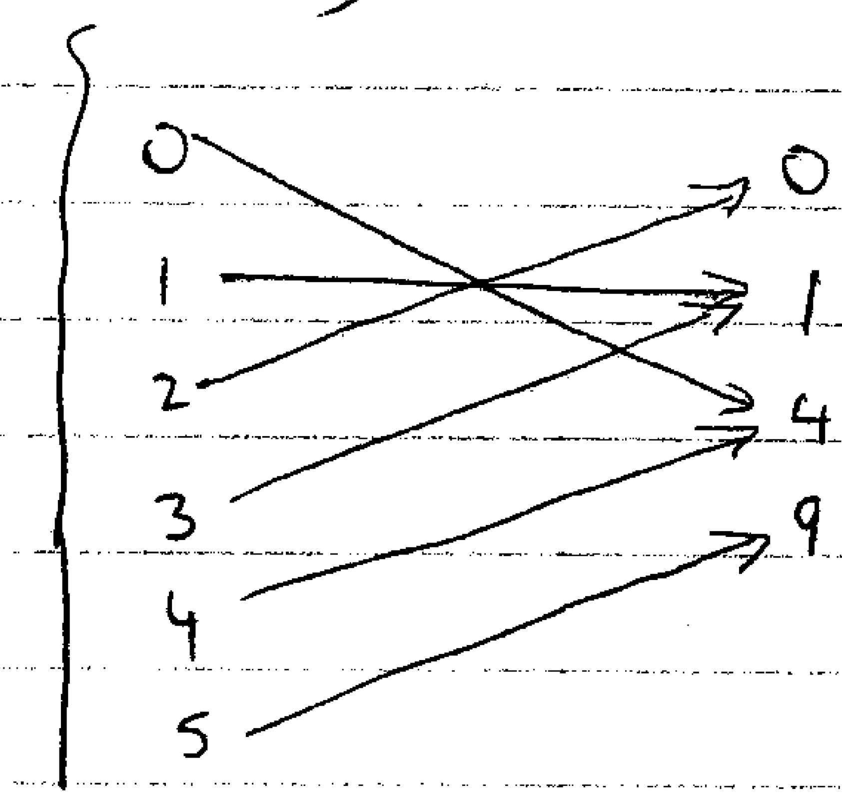
$$g(y_i) = \sum_{\substack{x_j \text{ for} \\ \text{which} \\ u(x_j) = y_i}} P[\underline{X} = x_j]$$

<u>Exa</u>:

$\underline{X}$:	0,	1,	2,	3,	4,	5
Probs	p_0	p_1	p_2	p_3	p_4	p_5

Let the fn be $u(x) = (x-2)^2$
Wish to determine prob fn of $\underline{Y} = u(\underline{X})$

Possible values:	0, 1, 4, 9
Probs	p_2, (p_1+p_3), (p_0+p_4), p_5



If $\underline{X}$ is Continuous, then $\underline{Y} = u(\underline{X})$ may be discrete or continuous depending on $u(x)$

If $\underline{X}$ is Continuous R.V. and the random var $\underline{Y} = u(\underline{X})$ is discrete

e.g. $u(x) = \text{integer part of } x$

or $\left[\begin{array}{ll} u(x) = 1 & \text{for } x > 0 \\ u(x) = 0 & \text{for } x \leq 0 \end{array}\right.$

Here it is again a simple matter to find the prob. distrib. of $\underline{Y} = u(\underline{X})$

Suppose $\underline{Y}$ may take on values $y_1, y_2 \ldots y_k \ldots$

Then $\text{Prob}[\underline{Y} = y_i] = \displaystyle\int_{\substack{x \\ \text{such that} \\ u(x) = y_i}} f(x)\,dx$

($f(x)$ PDF for $\underline{X}$)

$$\boxed{3^{rd} \text{ Situation is where both } \underline{X} \text{ and } \underline{Y} = u(\underline{X}) \text{ are continuous}}$$

HERE, There are 3 techniques that may be used in finding the PDF of $\underline{Y}$:

1) Cumulative Distrib. Fn Technique
2) M.G.F. Technique
3) Transformation Technique

We shall now present examples to illustrate the application of each approach:

Suppose $\underline{X} \sim N(0,1)$
and $u(x) = x^2$

1) Now using the CDF approach

$$G(y) = P[\underline{Y} \leq y]$$
$$= P[\underline{X}^2 \leq y]$$
$$= P[-\sqrt{y} \leq \underline{X} \leq \sqrt{y}]$$
$$= F(\sqrt{y}) - F(-\sqrt{y})$$

$$F(-\sqrt{y}) = 1 - F\sqrt{y}$$

$$= 2\,F(\sqrt{y}) - 1 \qquad \text{Symmetry}$$

Diff across w.r.t. y

$\Longrightarrow$

$$g(y) = 2\frac{d}{dy}F(\sqrt{y})$$

$$= 2\,f(\sqrt{y})\,\tfrac{1}{2}\,y^{-\frac{1}{2}}$$

$$= \frac{1}{\sqrt{2\pi}}\,e^{-\frac{1}{2}y}\,y^{-\frac{1}{2}}$$

now $\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$

for $\Gamma(p) = \int_0^\infty x^{p-1}e^{-x}\,dx$

$p > 0$

$$= \frac{1}{\sqrt{\pi}}\,\frac{y^{-\frac{1}{2}}}{2^{\frac{1}{2}}}\,e^{-y/2}$$

Remember Gamma PDF

$$f(x) = \frac{1}{\Gamma(\alpha)}\,\frac{x^{\alpha-1}}{\beta^{\alpha}}\,e^{-\frac{x}{\beta}}$$

Thus above, we have PDF of Gamma Distrib with

$$\alpha = \tfrac{1}{2} \qquad \beta = 2$$

This particular form of the Gamma Distrib is given a special Name

$\longrightarrow$ CHI SQUARE distrib with 1 degree of freedom

NOTE General form of PDF for CHI SQUARE DISTRIB is found from Gamma by taking

$$\beta = 2 \quad \text{and} \quad \alpha = \frac{n}{2}$$

Thus

$$f(x) = \frac{1}{\Gamma(\frac{n}{2})}\,\frac{x^{\frac{n}{2}-1}}{2^{n/2}}\,e^{-\frac{x}{2}} \qquad \text{for } x > 0$$

parameter of this distrib family is n

— termed the No. of degrees of freedom

shall see the reason for this later

2) <u>Using Moment Gen. Fn</u> : again $X \sim N(0,1)$
and $Y = u(X) = X^2$

$$E(e^{sY}) = E(e^{sX^2})$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{sx^2} e^{-\frac{1}{2}x^2} dx$$

$$= \int \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(1-2s)x^2} dx$$

let $t = \sqrt{1-2s}\, x$
$dt = \sqrt{1-2s}\, dx$

$$= \frac{\int \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt}{(1-2s)^{\frac{1}{2}}} = \frac{1}{(1-2s)^{\frac{1}{2}}}$$

But for Gamma Distrib

$$MGF = \frac{1}{(1-\beta s)^{\alpha}}$$

Thus Y has a Gamma Distrib with $\alpha = \frac{1}{2}$, $\beta = 2$
— as before.

<u>NOTE</u>: At this point it is opportune to mention that
if $Z = \sum_1^n X_i^2$ where $X_i \sim N(0,1)$
and the X_i are indept

Then $$MGF(Z) = \prod_1^n MGF(X_i^2)$$

$$= \prod_1^n \frac{1}{(1-2s)^{1/2}}$$

$$= \frac{1}{(1-2s)^{n/2}}$$

— Gamma with $\alpha = n/2$, $\beta = 2$

i.e. the CHI SQUARE distrib with n d.f.

NOTE

Mean and Var. for CHI Square Distrib

We will quote these results here
— no need to derive them, since already done (twice!) for Γ distrib

If $\underline{U} \sim \chi^2_n$

The $E(U) = n$

$Var(U) = 2n$

Remember
$E(\Gamma) = \alpha\beta$
$V(\Gamma) = \alpha\beta^2$

We saw that

If $X_i \sim N(0,1)$

Then $U = \sum_1^n X_i^2$ (for X_i indept)

has a χ^2 distrib with n d.f.

Since U is a sum of indept & id. distrib random variables, whose second $(\sigma^2 = 2)$ moments exist, we can apply the C.L.TH. to claim that

$$\frac{U}{n} \sim N\left(1, \frac{2}{n}\right)$$

for large n.

and thus $U \sim N(n, 2n)$

TRANSFORMATION TECHNIQUE

(8) In order to begin to use the Transformation technique we need the following result:

Let X be a contin. r. var with pdf $f(x)$. Let $Y = h(X)$ be a transformation such that $y = h(x)$ is strictly mononotic ~~so~~ that $x = h^{-1}(y)$ exists.
We further assume that $x = h^{-1}(y)$ has a derivative which is continuous for all y in the range of h.
Then the pdf of Y exists and is

$$g(y) = f(x)\left|\frac{dx}{dy}\right| = f[h^{-1}(y)]\left|\frac{d}{dy}h^{-1}(y)\right|$$

Proof:
Assume to begin with that $y = h(x)$ is monotonically incr i.e. $\frac{dy}{dx} > 0$ and hence $\frac{dx}{dy} > 0$

$$\begin{aligned} \text{Now } G(y) &= \text{Distrib. Fn. of } Y \\ &= P[Y \leq y] \\ &= P[h(X) \leq y] \\ &= P[X \leq h^{-1}(y)] \\ &= \int_{-\infty}^{h^{-1}(y)} f(x)\,dx \end{aligned}$$

Now $x = h^{-1}(y)$
Thus $dx = \frac{d\,h^{-1}(y)}{dy}\,dy$

$$= \int_{-\infty}^{y} f[h^{-1}(y)]\,\frac{d\,h^{-1}(y)}{dy}\,dy$$

when $x = h^{-1}(y)$ then $y = h(x) = h[h^{-1}(y)] = y$

$$\text{Thus } g(y) = f[h^{-1}(y)]\,\frac{d\,h^{-1}(y)}{dy} = f(x)\,\frac{dx}{dy} \quad \text{when } x = h^{-1}(y)$$

Suppose now that $y = h(x)$ is monotonically decr i.e. $\frac{dx}{dy} < 0$

$$\text{Then } G(y) = P[Y \leq y] = \cancel{P[X \leq h^{-1}(y)]} = P(h(X) \leq y) = P[X \geq h^{-1}(y)]$$

$$= \int_{h^{-1}(y)}^{\infty} f(x)\,dx$$

Let $y = h(x)$
$x = h^{-1}(y)$
$dx = \frac{d\,h^{-1}(y)}{dy}\,dy$

$$= \int_{y}^{-\infty} f[h^{-1}(y)]\ \frac{d}{dy}h^{-1}(y)\,dy$$

when $x = \infty$, $y = -\infty$
$x = h^{-1}(y)$, $y = y$

$$= \int_{-\infty}^{y} f[h^{-1}(y)]\left[-\frac{d}{dy}h^{-1}(y)\right]dy$$

$$= \int_{-\infty}^{y} f[h^{-1}(y)]\left|\frac{dx}{dy}\right|dy$$

Thus for monotonic $y = h(x)$,
The PDF of $Y = h(X)$ is given by

$$g(y) = f[h^{-1}(y)]\left|\frac{dx}{dy}\right|$$

Frequently denoted by J and termed the JACOBIAN of the transformation
(Perhaps strictly the Jacobian of the inverse transformation)

EXAMPLE

Suppose $X \sim R(0,1)$ uniform distrib on 0,1
and $Y = -\log_e X$
Wish to find PDF of Y

$y = h(x) : y = -\log_e x \implies x = e^{-y} = h^{-1}(y)$

$f(x) = 1$ Monotonic $\frac{dx}{dy} = -e^{-y}$

Thus $g(y) = 1\left|\frac{dx}{dy}\right| = e^{-y}$

Thus Y has exp. distrib on $(0,\infty)$

NOTE: **CHANGE OF VAR. IN AN INTEGRAL:**

$$\int_a^b f(x)\,dx$$

Suppose $y = h(x)$

$\Rightarrow x = h^{-1}(y)$

$dx = \frac{d}{dy} h^{-1}(y)\, dy$

$$= \int\left[f\left[h^{-1}(y)\right] \frac{d}{dy} h^{-1}(y)\right] dy$$

FUNCTION OF A RANDOM VARIABLE

— WHEN THE TRANSFORMATION $Y = h(X)$ IS NOT ONE-ONE

Suppose the pdf of X is $f(x)$ for $X \in S$

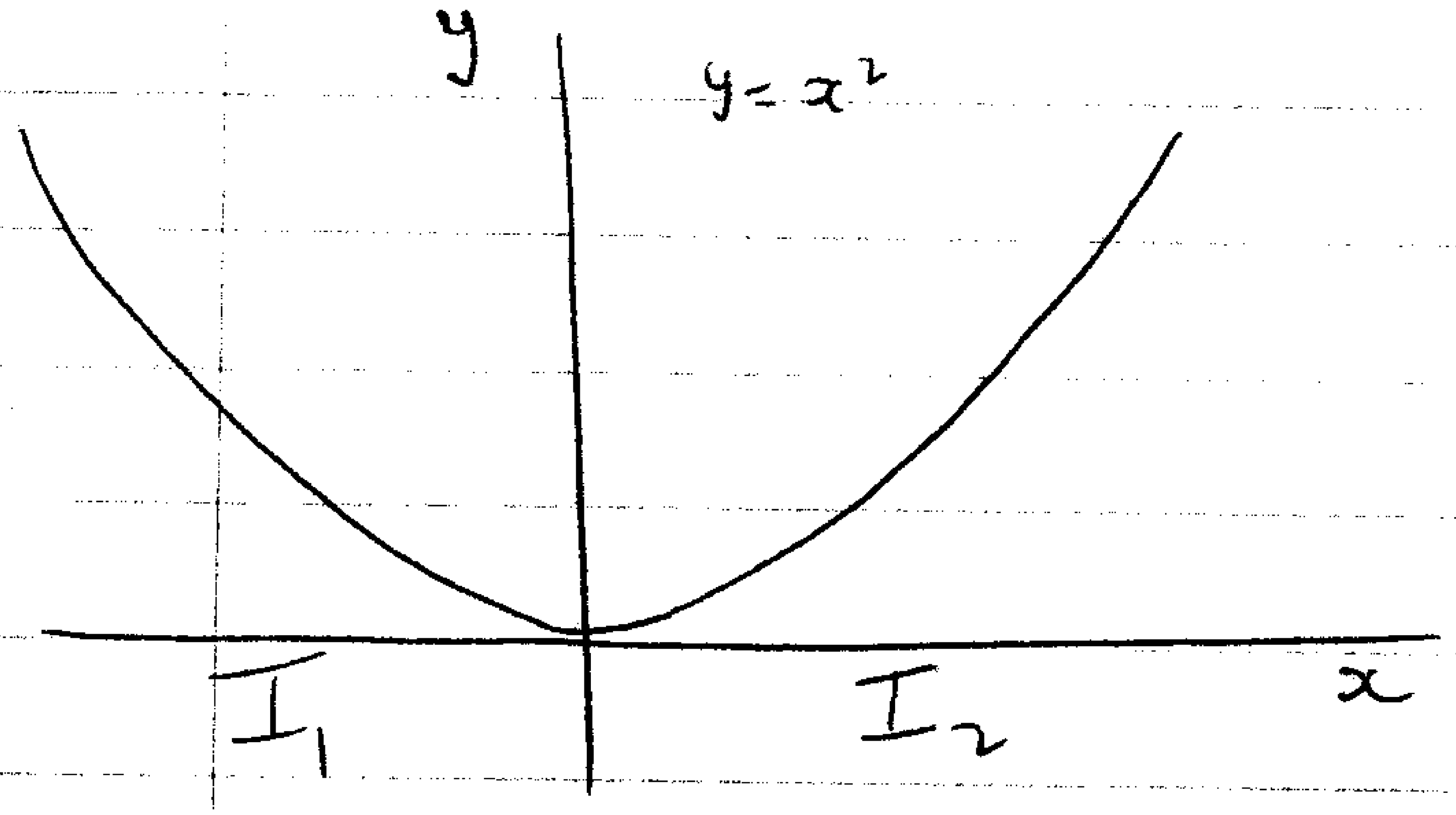
and $Y = h(X)$ — not monotone

so that the inverse $x = h^{-1}(y)$ is multiple-valued.

Procedure: Partition S into intervals so that $y = h(x)$ is strictly monotone in each interval. $\Longrightarrow$ in each interval, the inverse is unique.

Each such interval contributes to the PDF of Y.

— we add the contributions.



$Y = (X)^2$

$$g(y) = \sum_{I_i} f(h^{-1}(y)) \left|\frac{d\,h^{-1}(y)}{dy}\right|$$

$X \sim N(0,1)$

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

For I_1: $g_1(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}}\left[\frac{1}{2} y^{-\frac{1}{2}}\right]$

$\frac{d}{dy}h^{-1}(y) = \frac{1}{2} y^{-\frac{1}{2}}$

For I_2: $g_2(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}}\left[\frac{1}{2} y^{-\frac{1}{2}}\right]$

$$g(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} y^{-\frac{1}{2}}$$

MULTIVARIATE TRANSFORMATIONS

There are corresponding results for the tranformation of p random variables $X_1, X_2, \ldots, X_p$ into the random vars $Y_1, Y_2, \ldots, Y_p$.

We shall state (without proof) the result for $p=2$:

Let X_1, X_2 be a pair of contin r. vars with joint p.d.f. $f(x_1, x_2)$ and let

defined for $(x_1, x_2) \in S$

$$y_1 = h_1(x_1, x_2)$$
$$y_2 = h_2(x_1, x_2)$$

be a 1-1 transformation of $(x_1, x_2) \in S$ onto $(y_1, y_2) \in T$ (which is one-one) such that the partial derivatives of $x_1 = h_1^{-1}(y_1, y_2)$ and $x_2 = h_2^{-1}(y_1, y_2)$ exist and are continuous (for all $(y_1, y_2) \in T$)

Then the joint p.d.f. of (Y_1, Y_2) exists and is

$$g(y_1, y_2) = f\left[h_1^{-1}(y_1, y_2),\ h_2^{-1}(y_1, y_2)\right] |J|$$

where $J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix}$ is the Jacobian of the (inverse) transformation.

(or the Jacobian of (X_1, X_2) w.r.t. (y_1, y_2)).

provided $J \neq 0$ on T.

REF: APOSTOL ~~P.196~~ 216
P.271

Example

Let X_1, X_2 be 2 r.vars having Gamma distribs

i.e. $X_1 \sim \Gamma(\alpha_1, \beta)$

$X_2 \sim \Gamma(\alpha_2, \beta)$

and let us assume that X_1 and X_2 are indept.

Suppose wish to find the joint distrib. of the random vars

$$Y_1 = X_1 + X_2$$

$$\text{and } Y_2 = X_1 / X_2$$

$\boxed{\text{Q.: What is distrib of } Y_1 \text{?}}$

Need to find x_1, x_2 in terms of y_1, y_2 (so that can find J)

Solving: $x_1 = \dfrac{y_1 y_2}{1+y_2}$

$$x_2 = \frac{y_1}{1+y_2}$$

Thus $J = \begin{vmatrix} \dfrac{\partial x_1}{\partial y_1} & \dfrac{\partial x_1}{\partial y_2} \\ \dfrac{\partial x_2}{\partial y_1} & \dfrac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} \dfrac{y_2}{1+y_2} & \dfrac{y_1}{1+y_2} - \dfrac{y_1 y_2}{(1+y_2)^2} \\ \dfrac{1}{1+y_2} & \dfrac{-y_1}{(1+y_2)^2} \end{vmatrix}$

$$= \frac{-y_1 y_2}{(1+y_2)^3} - \frac{y_1}{(1+y_2)^2} + \frac{y_1 y_2}{(1+y_2)^3}$$

$$= -\frac{y_1}{(1+y_2)^2}$$

Notice $J \neq 0$ for any (y_1, y_2) in T

and $|J| = \dfrac{y_1}{(1+y_2)^2}$

Thus

$$g(y_1, y_2) = f(x_1, x_2)\,|J|$$

and $f(x_1, x_2) = f_1(x_1)\, f_2(x_2)$, since X_1, X_2 indept

Thus

$g(y_1, y_2) \longrightarrow$

Thus

$$g(y_1, y_2) = \left[\frac{1}{\Gamma(\alpha_1)} e^{-\frac{x_1}{\beta}} \frac{(x_1)^{\alpha_1 - 1}}{\beta^{\alpha_1}}\right]\left[\frac{1}{\Gamma(\alpha_2)} e^{-\frac{x_2}{\beta}} \frac{(x_2)^{\alpha_2 - 1}}{\beta^{\alpha_2}}\right] \frac{y_1}{(1+y_2)^2}$$

$$= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} e^{-\frac{(x_1+x_2)}{\beta}} \left(\frac{y_1 y_2}{1+y_2}\right)^{\alpha_1 - 1} \left(\frac{y_1}{1+y_2}\right)^{\alpha_2 - 1} \frac{y_1}{(1+y_2)^2} \frac{1}{\beta^{\alpha_1+\alpha_2}}$$

$$= \underbrace{\frac{1}{\Gamma(\alpha_1+\alpha_2)} e^{-\frac{y_1}{\beta}} \frac{(y_1)^{\alpha_1+\alpha_2-1}}{\beta^{\alpha_1+\alpha_2}}} \underbrace{\frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \frac{y_2^{\alpha_1 - 1}}{(1+y_2)^{\alpha_1+\alpha_2}}}$$

$$= g_1(y_1)\, g_2(y_2)$$

I.E. we can factor the joint PDF of y_1, y_2 into the form $g_1(y_1)\, g_2(y_2)$.

Thus we conclude (from our ealier work) that Y_1 and Y_2 are indep't of each other

and $g_1(y_1)$ is the (marginal) PDF of Y_1
and $g_2(y_2)$ is " " " " Y_2

We recognize $g_1(y_1)$ as a GAMMA PDF
(No surprize since $Y_1 = X_1 + X_2$

Remember $G_{X_1} = \frac{1}{(1-s\beta)^{\alpha_1}}$ $\quad G_{X_2} = \frac{1}{(1-s\beta)^{\alpha_2}}$

So that $G_{(X_1+X_2)} = \frac{1}{(1-s\beta)^{\alpha_1+\alpha_2}}$

SPECIAL CASE OF THIS RESULT (IMPORTANT IN APPLICATIONS)

We have already seen that for $\alpha = \frac{n}{2}$ and $\beta = 2$ n integer > 0

The Gamma Distrib $\Gamma(\frac{n}{2}, 2)$ is termed the CHI SQUARE Distrib with n degrees of freedom

Thus if $X_1 \sim \chi^2_{n_1}$ and $X_2 \sim \chi^2_{n_2}$ } indep't

Then for $Y_1 = X_1 + X_2$

and $U_2 = \dfrac{X_1/n_1}{X_2/n_2} = \dfrac{n_2}{n_1} Y_2$

We have the joint pdf of (Y_1, Y_2), (Simple extension)

Further, that $Y_1 \sim \chi^2_{n_1+n_2}$ $Y = \frac{n_2}{n_1} X$

easily shown that U_2 has P.d.f. $\dfrac{\Gamma\left(\frac{n_1+n_2}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} \dfrac{\left(\frac{n_1}{n_2}u_2\right)^{\frac{n_1}{2}-1}}{\left(1+\frac{n_1}{n_2}u_2\right)^{\frac{n_1+n_2}{2}}} \dfrac{n_1}{n_2}$

This prob distrib for U_2 has a name:

The r. var U_2 is said to have an F distribution (Fisher)

With n_1 and n_2 degrees of freedom

i.e. This F distrib. requires specification of 2 parameters n_1, n_2

Thus for the F distrib

$$g(u) = \frac{\Gamma\left(\frac{n_1+n_2}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} \left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}} \frac{u^{\frac{n_1}{2}-1}}{\left(1+\frac{n_1}{n_2}u\right)^{\frac{n_1+n_2}{2}}}$$

for $u > 0$

NOTE: $E(u)$ exists for $n_2 > 2$, $= \dfrac{n_2}{n_2-2}$

SHOW $\quad Var(u) = \dfrac{2 n_2^2 (n_1+n_2-2)}{n_1(n_2-2)^2(n_2-4)}$ for $n_2 > 4$

THE t DISTRIB (STUDENT'S t Distrib)

Suppose the r.var. $\bar{X} \sim N(0,1)$
 and the r.v. $U \sim \chi^2_n$, indept of $\bar{X}$

Suppose we form the new random var

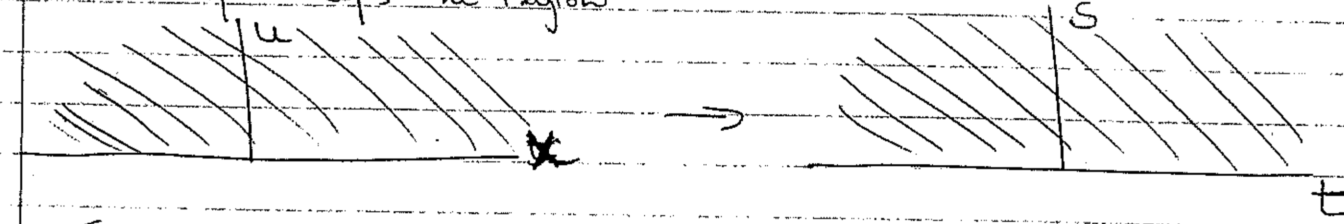
$$\bar{T} = \bar{X} \Big/ \sqrt{\frac{U}{n}}$$

We shall see ~~that~~ this r.var will occur quite naturally in applicns — for now it looks rather artificial.

We shall find the form of the Prob. Distrib of T

We have a transfn from the 2 r.vars $\bar{X}, \bar{U}$
 to the new r.vars T, S
as follows: $t = \dfrac{x}{\sqrt{\frac{u}{n}}}$; $S = u$

This transfn maps the region



Inverse Transfn:

$$x = t\sqrt{\frac{S}{n}} \qquad u = S$$

Jacobian of x, u wrt t, S

$$J = \begin{vmatrix} \frac{\partial x}{\partial t} & \frac{\partial x}{\partial s} \\ \frac{\partial u}{\partial t} & \frac{\partial u}{\partial s} \end{vmatrix} = \begin{vmatrix} \sqrt{\frac{S}{n}} & \frac{1}{2}t\left(\frac{S}{n}\right)^{-\frac{1}{2}} \\ 0 & 1 \end{vmatrix}$$

$$= \sqrt{\frac{S}{n}}$$

Thus joint PDF of $\underline{T}$ and $\underline{S}$ is

$$g(t,s) = f(x,u)\,|J| \qquad \text{with } x = t\sqrt{\tfrac{s}{n}},\ u = s$$

$$= \frac{1}{\sqrt{2\pi}} e^{-t^2\frac{s}{2n}} \frac{1}{\Gamma(\frac{n}{2})\,2^{n/2}} s^{n/2-1} e^{-\frac{s}{2}} |J|$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\Gamma(\frac{n}{2})\,2^{n/2}} s^{n/2-1} e^{-\frac{s}{2}\left(1+\frac{t^2}{n}\right)} \sqrt{\frac{s}{n}}$$

Marginal PDF of $T = \int_{s=0}^{\infty} g(t,s)\,ds.$

$$= A\int_0^\infty s^{\frac{n-1}{2}} e^{-\frac{s}{2}\left(1+\frac{t^2}{n}\right)}\,ds$$

$$z = \frac{s}{2}\left(1+\frac{t^2}{n}\right)$$

$$dz = \frac{1}{2}\left(1+\frac{t^2}{n}\right)ds$$

$$= A\int_0^\infty \left(\frac{2z}{1+\frac{t^2}{n}}\right)^{\frac{n-1}{2}} e^{-z} \left(\frac{2}{1+\frac{t^2}{n}}\right) dz$$

$$= A\int_0^\infty \frac{2^{\frac{n+1}{2}}}{\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}} z^{\frac{n-1}{2}} e^{-z}\,dz$$

[Defn $\Gamma(r) = \int_0^\infty x^{r-1} e^{-x}\,dx$]

$$= \frac{1}{\sqrt{2\pi n}} \frac{1}{\Gamma(\frac{n}{2})\,2^{n/2}} \frac{2^{\frac{n+1}{2}}}{\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}} \Gamma\left(\frac{n+1}{2}\right)$$

$$= \frac{\Gamma(n+1/2)}{\sqrt{\pi n}\,\Gamma(\frac{n}{2})} \frac{1}{\left(1+\frac{t^2}{n}\right)^{n+1/2}} \qquad -\infty < t < \infty$$

— Known as the <u>t Distrib</u>

One parameter; $n \rightarrow$ # d.f.

Symmetric! $E(T) = 0$

$$V(T) = \frac{n}{n-2} \quad \text{for } n > 2$$