Logical Arguments

rules of inference

formal logical derivations

Motivation

We have seen examples of system specifications, flow of control through programs, etc. which we need to specify in logic to avoid ambiguity

But how do we check that an implementation meets the specification?

We could construct a truth table, and show that the result is true for the current situation

- but truth tables get very large very quickly ...

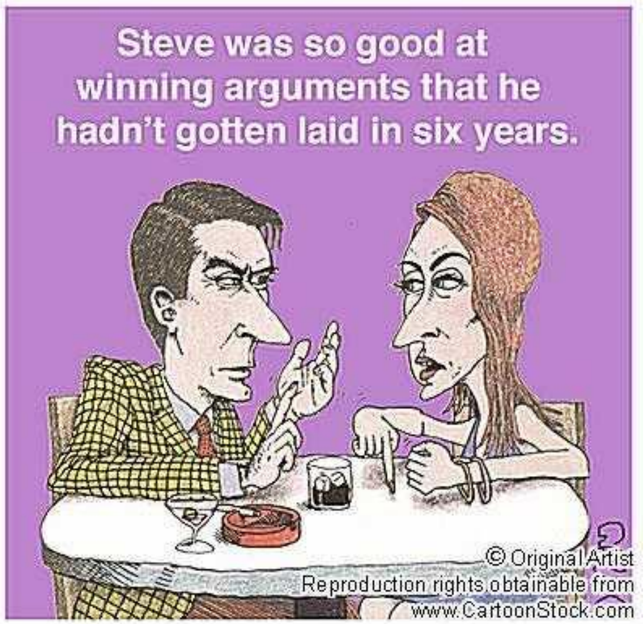
We could apply logical equivalences, rewriting the implementation to show it is logically equivalent

- but that is too limited ...

When someone wants to convince you of something, they argue with you, trying to show that their desired conclusion follows from some set of facts that you agree on

We will do the same, but we will do it formally, staying within the rules of logic

Note: *formal logical argument* ...



Logical Argument

- An argument is a sequence of statements leading from some initial facts to a conclusion
- A valid argument is one in which if the initial facts are true, then the conclusion also has to be true
- We will develop a set of rules, which you can use to generate new statements from old ones, without worrying about truth tables
- These rules will depend only on the form of the statements (and so this is called formal reasoning)

Informal Example

Suppose we start with two statements:
1) if your swipe card is accepted, you can enter E4
2) your swipe card is accepted

From those two statements, we deduce that
3) you can enter E4

This seems like a good argument: if the first two statements are correct, then the third statement should also be correct.

But we know that English is imprecise and ambiguous. Are we sure that we have understood the sentences?

Translating the informal argument into logic

Let p = "your swipe card is accepted"
Let q = "you can enter E4"

The two statements in the example then become

1) $p \rightarrow q$
2) p

and we want to deduce

3) q

Can we say that *3)* is always true whenever *1)* and *2)* are true?

Suppose we start with two statements:
1) if your swipe card is accepted, you can enter E4
2) your swipe card is accepted

From those two statements, we deduce that
3) you can enter E4

Verifying the argument

We can show that this argument is valid by building the truth table:

p	q	$p \to q$
T	T	T
~~T~~	~~F~~	~~F~~
~~F~~	~~T~~	~~T~~
~~F~~	~~F~~	~~T~~

The only scenario in which both p and $p \to q$ are true is the first line of the table. In that scenario, q is also true. Therefore, whenever both p and $p \to q$ are true, q is true.

We can express this rule as:

$$\begin{array}{rl} & p \to q \\ & \underline{p \quad\quad} \\ \therefore & q \end{array}$$

Sometimes you will see it written as $p \to q, p \vdash q$

Note: *valid* does not mean *true*

Note: even though I say something is a valid argument, it does not mean that I claim the conclusion is true. I am saying that the steps of the argument are logical, so <u>if the initial facts</u> (or premises) <u>are true</u>, then the conclusion is true.

The argument below is valid, but the conclusion is not true (because the premises are false).

1. If Enda Kenny is from Mars, then Enda Kenny has two heads
2. Enda Kenny is from Mars

Therefore

3. Enda Kenny has two heads

$$\begin{array}{l} p \rightarrow q \\ \underline{p \quad\quad} \\ \therefore\ q \end{array}$$

The form of the argument is correct.
The facts are false.

Understanding the rule

If I have a collection of statements, and I can find one of those statements that matches the pattern

$$X \rightarrow Y$$

and another statement that matches the pattern

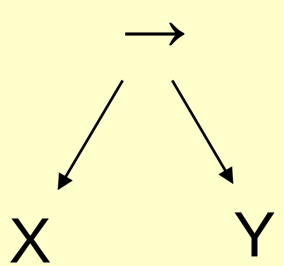
$$X$$

where the X is exactly the same in both statements, then I am entitled to add the new statement

$$Y$$

to my collection.

$$\begin{array}{l} X \rightarrow Y \\ \underline{X \quad\;} \\ \therefore\ Y \end{array}$$



Note: the syntax tree of the 1st statement must be as pictured. The X and Y can be complex trees, but we must have → at the top.

$$\begin{array}{rl} & X \rightarrow Y \\ & \underline{X \quad\quad} \\ \therefore & Y \end{array}$$

$$\boxed{x \rightarrow y,\ x \vdash y}$$

A rule of this type is called a rule of inference, since it allows us to infer new statements.

Can we find other rules that always work?

Using logical equivalences in arguments

Any logical equivalence gives us a rule of inference.

If x is in my collection of statements, for some statement x, and we have a proven logical equivalence $x \equiv y$ (for some y) then I can add y to the collection.

This is useful, but it is too restrictive – the logical equivalence tells us

"if x is true then y is true" AND "if x is false then y is false".

but we don't care about the case when x is false – we have already agreed that x is true.

So can we find other rules?

Rules of inference

$$\begin{array}{l} x \to y \\ \underline{x \qquad} \\ \therefore\ y \end{array}$$

Modus Ponens

$$\begin{array}{l} \underline{x \wedge y} \\ \therefore\ x \end{array}$$

Simplification

$$\begin{array}{l} x \to y \\ \underline{y \to w} \\ \therefore\ x \to w \end{array}$$

Chaining

$$\begin{array}{l} x \to y \\ \underline{\neg y \qquad} \\ \therefore\ \neg x \end{array}$$

Modus Tollens

$$\begin{array}{l} \underline{x \qquad} \\ \therefore\ x \vee y \end{array}$$

Addition

$$\begin{array}{l} x \leftrightarrow y \\ \underline{x \qquad} \\ \therefore\ y \end{array}$$

Elimination

$$\begin{array}{l} x \vee y \\ \underline{\neg x \qquad} \\ \therefore\ y \end{array}$$

Unit Resolution

$$\begin{array}{l} x \\ \underline{y \qquad} \\ \therefore\ x \wedge y \end{array}$$

Conjunction

$$\begin{array}{l} x \vee y \\ \underline{\neg y \vee w} \\ \therefore\ x \vee w \end{array}$$

Resolution

Chaining rules together to make an argument

An argument starts with a set of assumptions or premises. Our aim is to show that our conclusion follows from the premises.

If we cannot do this by applying one of our known rules of inference, then we can apply multiple rules in sequence – each rule will introduce a new fact, that we know must follow from what we have already. We can then use that new fact to deduce new ones.

As long as we stick to verified rules of inference, or apply known logical equivalences, we know that we will produce a valid argument.

Example

From the premises

1. $p \rightarrow q$
2. $\neg q \wedge r$
3. $\neg p \rightarrow s$
4. $s \rightarrow t$

derive the conclusion

t

1. $p \rightarrow q$
2. $\neg q \wedge r$
3. $\neg p \rightarrow s$
4. $s \rightarrow t$

Give each new statement a number. Say what other statements it came from, using what rule, and with what substitutions.

5. $\neg q$	*from 2*	*by Simplification*	$[x=\neg q,\ y=r]$
6. $\neg p$	*from 1 & 5*	*by Modus Tollens*	$[x=p, y=q]$
7. s	*from 3 & 6*	*by Modus Ponens*	$[x=\neg p,\ y=s]$
8. t	*from 4 & 7*	*by Modus Ponens*	$[x=s,\ y=t]$

Modus Ponens	*Simplification*	*Modus Tollens*
$x \rightarrow y$		$x \rightarrow y$
$\underline{x \quad\quad}$	$\underline{x \wedge y}$	$\underline{\neg y \quad\quad}$
$\therefore\ y$	$\therefore\ x$	$\therefore\ \neg x$

Example

Given the facts:

1. "if the user does not provide the correct password then the user is given access to the open area"
2. "if the user can read sensitive data then the user must have access to the restricted area"
3. "the user cannot have access to the open area at the same time as the user has access to the restricted area"

4. "the user has not provided the correct password"

show that the conclusion

"the user cannot read the sensitive data"

must follow, by representing the facts and conclusions as propositional statements and constructing a valid argument

Example (continued)

p = the user provides the correct password
o = the user has access to the open area
r = the user has access to the restricted area
s = the user can read the sensitive data

Now construct the valid argument

Premises:

1. $\neg p \rightarrow o$
2. $s \rightarrow r$
3. $\neg (o \wedge r)$
4. $\neg p$

Desired conclusion:

$\neg s$

Soundness and Completeness

If we are careful with the rules of inference, and we assert a small number of (true) facts, then we can prove

- if we can apply the rules of inference to derive the conclusion *p* from a set of premises *S,* then in all situations where *S* is true, p is also true *(and we say the rules are sound)*

- if in every situation where a set of premises *S* is true a formula *p* is also true, then it will be possible to derive *p* from *S* using the rules (*and we say the rules are complete*)

... but we won't prove this in CS1112

Next lecture ...

translation

inverse, converse and contrapositive

bad arguments and logical fallacies