Q1.

Write EF for the event that there is an open road from E to F. and EF^c for the complement of this event; write $E \leftrightarrow F$ if there is an open route from E to F, and $E \not\leftrightarrow F$ if there is none. Now $\{A \leftrightarrow C\} = AB \cap BC$

$$P(AB \mid A \not\leftrightarrow C] = \frac{P(AB, A \not\leftrightarrow C]}{P(A \not\leftrightarrow C)}$$

$$= \frac{P[AB, B \not\leftrightarrow C]}{1 - P(A \leftrightarrow C)} = \frac{(1-p^2)p^2}{1-(1-p^2)^2}$$

By a similar calculation in the second case, one obtains the same answer:

$$P[AB \mid A \not\leftrightarrow C] = \frac{(1-p^2)p^3}{1-(1-p^2)^2 p - (1-p)}$$

$$= \frac{(1-p^2)p^2}{1-(1-p^2)^2}$$

Q2.

(i) Let R = event have the Real Relic

S = Saber is summoned.

$$P[R \mid S] = \frac{P(S \mid R)\,P(R)}{P(S)}$$

$$= \frac{P(S \mid R)\,P(R)}{P(S \mid R)\,P(R) + P(S \mid R^c)\,P(R^c)}$$

$$= \frac{p \cdot (1-\theta)}{(1-\theta)p + p_1 \cdot (1-p)}$$

(ii) $$P[R^c \mid NS] = \frac{P[NS \mid R^c] \cdot P(R^c)}{P(NS)}$$

$$= \frac{P(NS \mid R^c)\,P(R^c)}{P(NS \mid R^c)\,P(R^c) + P(NS \mid R)\,P(R)}$$

$$= \frac{(1-p_1)\cdot(1-p)}{(1-p_1)(1-p)+\theta\cdot p}$$

(ii) Let L = event that a Lancer is summoned.
X = # of Lancer summoned.

Assume having a real relic or not is indep between L_1, L_2, prob of summoning Not lancer, given that they have the real relic, is indep of the prob of summoning any other classes, given they have the real relic.

$$P[X=1] = 2P(L)P(\neg L)$$

$$= 2 \times \frac{p\cdot(1-\theta)}{(1-\theta)p + p_2\cdot(1-p)} \times \frac{(1-p_1)\cdot(1-p)}{(1-p_2)(1-p)+\theta\cdot p}$$

(iv) Note that $1 - P[\text{perfect match}]$

$$= P[\text{No perfect match}]$$

for all the three groups.

Let E_i represent the event of a match for the i^{th} master in a group, $i = 1, 2$.

Let Y be the number of groups with a perfect match.

$$P[Y=1] = P[S,S]^2 \cdot (1 - P(L,L)^2)(1 - P(C,C)^2)$$

$$+ (1 - P(S,S)^2)\, P(L,L)^2 (1 - P(C,C)^2)$$

$$+ (1 - P(S,S)^2)(1 - P(L,L_2)^2)\, P[C,C]^2$$

According to (i)

$$P(S,S)^2 = \left[\frac{p \cdot (1-\theta)}{(1-\theta)p + p_1 \cdot (1-p)}\right]^2$$

$$P[L,L)^2 = \left[\frac{p \cdot (1-\theta)}{(1-\theta)p + p_2 \cdot (1-p)}\right]^2$$

$$P[C_1 C]^2 = \frac{p \cdot (1-\theta)}{(1-\theta)p + p_3 \cdot (1-p)}, \quad \text{where } p_3 = 1 - p_1 - p_2.$$

plug these values to $P[Y=1]$

Q3.

(a) By indep, P[$n-1$ tails, followed by a head]

$$= 2^{-n}$$

(b) If n is odd, P[# heads = # tails] = 0.

If n is even, there are ${}^{n}C_{\frac{n}{2}}$ sequences of outcomes with $\frac{1}{2}n$ heads, $\frac{1}{2}n$ tails.

Any given sequence of heads and tails has prob. 2^{-n},

$\Rightarrow$ P[# heads = # tails] = $2^{-n}\,{}^{n}C_{\frac{n}{2}}$.

(c) There are ${}^{n}C_2$ sequences containing 2 heads and $n-2$ tails. Each sequence has prob 2^{-n}.

$\Rightarrow$ P[exactly two H] = ${}^{n}C_2 \times 2^{-n}$

(d)

$$P[\geq 2\ H] = 1 - P(\text{No } H) - P(\text{one } H)$$

$$= 1 - 2^{-n} - {}^{n}C_{1} \cdot 2^{-n}.$$

Q4.

(a) $$P[N=2 \mid S=4] = \frac{P[N=2 \cap S=4]}{P(S=4)}$$

$$= \frac{P[S=4 \mid N=2)\, P(N=2)}{\sum_i P[S=4 \mid N=i)\, P(N=i)}$$

$$= \frac{\frac{1}{12} \times \frac{1}{4}}{\frac{1}{6} \times \frac{1}{2} + \frac{1}{12} \times \frac{1}{4} + \frac{3}{216} \times \frac{1}{8} + \frac{1}{6^4} \cdot \frac{1}{16}}$$

(b) $$P[S=4 \mid N \text{ even}]$$

$$= \frac{P[S=4 \mid N=2]\,\frac{1}{4} + \frac{1}{16} \times P[S=4 \mid N=4]}{P(N \text{ even})}$$

$$= \frac{\frac{1}{12} \times \frac{1}{4} + \frac{1}{6^4} \times \frac{1}{16}}{\frac{1}{4} + \frac{1}{16} + \cdots\cdots} = \frac{4^2 3^3 + 1}{4^4 3^3}$$

(c) Let D for the # shown by the first die.

$$P[N=2 \mid S=4, D=1] = \frac{P[S=4, N=2, D=1]}{P(S=4, D=1)}$$

$$= \frac{\frac{1}{6} \times \frac{1}{6} \times \frac{1}{4}}{\frac{1}{6} \times \frac{1}{6} \times \frac{1}{4} + \frac{1}{6} \times \frac{2}{36} \times \frac{1}{8} + \frac{1}{64} \times \frac{1}{16}}$$

(d) Writing M for the max number shown.

if $1 \leq r \leq 6$.

$$P[M \leq r] = \sum_{j=1}^{\infty} P[M \leq r \mid N=j] 2^{-j}$$

$$= \sum_{j=1}^{\infty} \left(\frac{r}{6}\right)^{j} \times \frac{1}{2^{j}} = \frac{r}{12}\left(1-\frac{r}{12}\right)^{-1}$$

$$= \frac{r}{12-r}.$$

$$\Rightarrow P[M = r] = P[M \le r] - P[M \le r-1]$$

$$= \frac{r}{12-r} - \frac{r-1}{12-r+1}$$

Q5.

Think about the experiment as laying down the b+r balls from left to right in a random order. The # of possible orderings equals the number of ways of placing the blue balls, namely ${}^{b+r}C_b$.

The # of ways of placing the balls so that the first k are blue, and the next red, is the # of ways of placing the red balls so that the first is in position k+1 and the remainders are amongst the

r+b−k−1 places to the right,

namely ${}^{r+b-k-1}C_{r-1}$. The required result follows.

The prob. that the last ball is red

is $\frac{r}{b+r}$.

Q6.

$P[A \cup B \cup C]$

$= P[(A^c B^c C^c)^c] = 1 - P[A^c B^c C^c]$

$= 1 - P(A^c | B^c C^c) P(B^c | C^c) P(C^c)$

Q7.

(a) If A is indep of itself, then

$P(A) = P(AA) = P(A)^2$, $\Rightarrow P(A) = 0$ or 1.

(b) If $P(A) = 0$, $\Rightarrow 0 = P(AB) = P(A) P(B)$ for all B. If $P(A) = 1 \Rightarrow P(AB) = P(B)$, so that $P(AB) = P(A) P(B)$

Q8.

(a) $P(B|A) = P(AB)/P(A) = P(A|B)P(B) / P(A)$

$> P(B)$

(b) $P[A| B^c] = P(AB^c) / P(B^c)$

$$= \frac{P(A) - P(A \cap B)}{P(B^c)} < P(A)$$

(c) No. Consider the case

$A \cap C = \emptyset$.