ST2054 (and ST3068, ST6003) Problem Set 8 Due 5pm 3rd of April 2020

To submit: Please send your work in a pdf. file to my email address: liang.chen@ucc.ie by the above due time. No need to drop a written version to the school box.

Question 1

Let X, Y and Z be independent and uniformly distributed on $[0,1]$. Find the joint density function of XY and Z^2, and show that $P(XY < Z^2) = \frac{5}{9}$.

Question 2

Let X and Y be independent and exponentially distributed with parameters λ and μ. Find the joint distribution of $S = X + Y$ and $R = X/(X+Y)$. What is the density of R?

Question 3

Suppose that $\{X_i, i = 1, 2, \ldots, m\}, m > 1$, is a random sample from the distribution $N(\mu_1, \sigma^2)$, and that $\{Y_i, i = 1, 2, \ldots, n\}, n > 1$, is an independent random sample from the distribution $N(\mu_2, \sigma^2)$. Let $S_1^2 = \sum_{i=1}^{m}(X_i - \bar{X})^2/(m-1)$ and $S_2^2 = \sum_{j=1}^{n}(Y_j - \bar{Y})^2/(n-1)$.

(i) Find a condition on α and β such that $(\alpha S_1^2 + \beta S_2^2)$ is an unbiased estimator of σ^2.

(ii) Determine the values of α and β such that $(\alpha S_1^2 + \beta S_2^2)$ is an unbiased estimator of σ^2 with minimum variance.

Question 4

A production process gives components whose strengths are normally distributed with mean 40lb and standard deviation 1.15lb. A modification has been made to the process in such a way that the mean strength of the components cannot be reduced but it is not certain it will lead to any appreciable increase in mean strength. The modifications may also alter the standard deviation of the strength measurements. The strengths of 9 components selected from the modified process are: 39.6, 40.2, 40.9, 40.9, 41.4, 39.8, 39.4, 43.6, 41.8.

(i) Calculate a 95% confidence interval for μ assuming μ is unchanged at 1.15lb.

(ii) Carry out an appropriate one-sided test of the hypothesis that $\mu = \mu_0 = 40$, with the assumption that $\sigma = 1.15$.

(iii) Calculate a 95% confidence interval for σ^2.

Question 5

Let X and Y be independent variables, $\Gamma(\lambda, m)$ and $\Gamma(\lambda, n)$ respectively

(i) Find the joint density function of $X+Y$ and $X/(X+Y)$, and deduce that they are independent.

Let $\{X_r : 1 \leq r \leq k+1\}$ be independent $\Gamma(\lambda, \beta_r)$ random variables (respectively).

(ii) Show that $Y_r = X_r/(X_1 + \cdots + X_r)$, $2 \leq r \leq k+1$, are independent random variables.

(iii) Show that $Z_r = X_r/(X_1 + \cdots + X_{k+1})$, $1 \leq r \leq k$, have the joint density:

$$\frac{\Gamma(\beta_1 + \cdots + \beta_{k+1})}{\Gamma(\beta_1) \cdots \Gamma(\beta_{k+1})} z_1^{\beta_1 - 1} z_2^{\beta_2 - 1} \cdots z_k^{\beta_k - 1} (1 - z_1 - z_2 - \cdots - z_k)^{\beta_{k+1} - 1}$$