Question Bank 4

Question 1

Let X have distribution function

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{2}x & \text{if } 0 \leq x \leq 2 \\ 1 & \text{if } x > 2 \end{cases}$$

and let $Y = X^2$. Find (a) $P(\frac{1}{2} \leq X \leq \frac{3}{2})$; (b) $P(1 \leq X < 2)$; (c) $P(Y \leq X)$; (d) $P(X \leq 2Y)$; (e) $P(X + Y \leq \frac{3}{4})$; (f) the distribution function of $Z = \sqrt{X}$.

Question 2

Let X be a non-negative random variable with density function f. Show that

$$E[X^r] = \int_0^\infty r x^{r-1} P(X > x) dx$$

for any $r \geq 1$ for which the expectation is finite.

Question 3

Find the density function of $Y = aX$, where $a > 0$, in terms of the density function of X. Show that the continuous random variable X and $-X$ have the same distribution function if and only if $f_X(x) = f_X(-x)$ for all $x \in R$.

Question 4

Let X be a positive random variable with density function f and distribution function F. Define the hazard function $H(x) = -\log[1 - F(x)]$ and the hazard rate

$$r(x) = \lim_{h \to 0} \frac{1}{h} P(X \leq x + h | X > x), x \geq 0.$$

Show that:

(a) $r(x) = H'(x) = \frac{f(x)}{1-F(x)}$,

(b) If $r(x)$ increases with x then $\frac{H(x)}{x}$ increases with x,

(c) $\frac{H(x)}{x}$ increases with x if and only if $[1 - F(x)]^\alpha \leq 1 - F(\alpha x)$ for all $a \leq \alpha \leq 1$,

(d) If $\frac{H(x)}{x}$ increases with x, then $H(x + y) \geq H(x) + H(y)$ for all $x, y \geq 0$.

Find the hazard rate when:

(e) X has the Weibull distribution, $P(X > x) \exp(-\alpha x^{\beta-1})$, $x \geq 0$,

(f) X has the exponential distribution with parameter λ,

(g) X has density function $\alpha f + (1-\alpha)g$, where $0 < \alpha < 1$ and f and g are the densities of exponential variables with respective parameters λ and μ. What happens to this last hazard rate $r(x)$ in the limit as $x \to \infty$.