Q1.

$$E\left[\frac{1}{1+X}\right] = E\left[\int_0^1 t^x \, dx\right] = \int_0^1 E(t^X)\, dx$$

$$= \int_0^1 (q+pt)^n \, dt = \frac{1-q^{n+1}}{p(n+1)}, \text{ where } q = 1-p.$$

given $np \to \lambda$

$$E\left[\frac{1}{1+x}\right] = \frac{1-\left(1-\frac{np}{n}\right)^{n+1}}{(n+1)/n \cdot p \cdot n} = \frac{1-\left(1-\frac{\lambda}{n}\right)^{n+1}}{\lambda \cdot \frac{n+1}{n}} \quad (1) \quad (2)$$

$$(1) = 1-\left(1-\frac{\lambda}{n}\right)^{n+1 \times -\frac{n}{\lambda} \cdot -\frac{\lambda}{n}}$$

$$= 1-\left(1-\frac{\lambda}{n}\right)^{-\frac{n}{\lambda} \times -\frac{n+1}{n}\lambda} \to 1 - e^{-\frac{\lambda n}{n+1}} \text{ when } n \to \infty$$

$$\to 1 - e^{-\lambda}.$$

$(2) \to \lambda$ when $n \to \infty$

$$\Rightarrow E\left(\frac{1}{1+x}\right) \to \frac{1-e^{-\lambda}}{\lambda} \text{ as } n \to \infty$$

Q2.

Conditioning on the outcome of the first toss, we obtain $h_n = q h_{n-1} + p(1-h_{n-1})$ for $n \geq 1$ where $p = 1-q$, $h_0 = 1$.

$$s^n \cdot h_n = q h_{n-1} \cdot s^n + p(1-h_{n-1}) \cdot s^n$$

$$\sum_{n=0}^{\infty} s^n \cdot h_n = H(s) = \sum_{n=0}^{\infty} q h_{n-1} s^n + s^n p (1-h_{n-1})$$

$$= s\left[\sum_{n=0}^{\infty} q \cdot s^{n-1} h_{n-1} + p(1-h_{n-1}) s^{n-1}\right]$$

$$= s\left[(q-p) H(s)\right] + \sum_{n=0}^{\infty} p \cdot s^n$$

$$H(s) = (q-p) s H(s) + p \cdot \frac{1-s+s}{1-s}$$

$$\Rightarrow H(s) - p = (q-p) s H(s) + p \frac{s}{1-s}$$

$\Rightarrow$ Solve $\uparrow$ for $H(s)$

Q3.

Fix $w=1$,

$G_{X,Y,Z}(x,y,z) = G(x,y,z,1)$

$= \frac{1}{8}(xyz + xy + yz + zx + x + y + z + 1)$

$= \frac{1}{2}(x+1)\frac{1}{2}(y+1)\frac{1}{2}(z+1) = G_X(x)G_X(y)G_Z(z)$

$\Rightarrow X, Y, Z$ are indep.

The same conclusion holds for any other set of exactly three R.Vs.

However, can check

$$G(x,y,z,w) \neq G_X(x)\,G_Y(y)\,G_Z(z)\cdot G_W(w).$$