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Q1

joint density $= g(u,v)$

$$\frac{\partial \partial P(XY \le u,\ Z^2 \le v)}{\partial u \, \partial v} = g(u,v)$$

$$P(XY \le u) = P(XY \le u,\ Y \le u) + P(XY \le u,\ Y > u)$$

$$= P(Y \le u) + P\left(X \le \frac{u}{Y},\ Y > u\right)$$

$$= u + \int_u^1 \frac{u}{y}\, dy = u - u\log u$$

$$= u(1 - \log u)$$

$\downarrow$

$P(XY \le u,\ Z^2 \le v)$

because independent

$\downarrow$

$= P(XY \le u)\, P(Z \le \sqrt{v})$

$G(u,v) = u\sqrt{v}\,(1 - \log u)$ $0<u<1$, $0<v<1$

$$g(u,v) = \frac{\partial\partial G(u,v)}{\partial u\, \partial v} = \boxed{\frac{\log(1/u)}{2\sqrt{v}}} \ *$$

(ii)

$P(XY \le Z^2)$

$$= \int_0^1 \int_0^v \frac{\log(1/u)}{2\sqrt{v}}\, du\, dv$$

$$= \int_0^1 \frac{v - v\ln(v)}{2\sqrt{v}}\, dv$$

$$= \boxed{\frac{5}{9}} \ *$$

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Q2

Joint dist of S, R = $g(s,r)$

$$g(s,r) = f(x,y)\,|J|$$

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial s} \end{vmatrix}$$

$$f(x,y) = f(x)\,f(y)$$

$S = X + Y$
$R = X/(X+Y)$

$X = R \cdot S$
$Y = S - R \cdot S \quad = (1-r)s$

$$|J| = \begin{vmatrix} s & r \\ -s & 1-r \end{vmatrix}$$

$= s - rs + rs$
$= s$

(i)
$$\boxed{\begin{aligned} g(s,r) &= f(x)f(y)s \\ &= \lambda e^{-\lambda rs} \cdot \mu e^{-\mu(1-r)s}\, s \end{aligned}}$$

$$f_R(r) = \int_0^\infty f_{R,S}(r,s)\,ds = \int_0^\infty f_{X,Y}(rs,(1-r)s)\,s\,ds$$

$$= \int_0^\infty \lambda e^{-\lambda rs}\mu e^{-\mu(1-r)s}\,s\,ds \boxed{= \frac{\lambda\mu}{[\lambda r + \mu(1-r)]^2}}$$ (ii) $0 \le r \le 1$

Q3 (i)

$$E[S_1^2], E[S_2^2] = \sigma^2$$

Each sample variance is an unbiased estimator of σ^2.

$E[\alpha S_1^2 + B S_2^2]$ should equal σ^2

$\downarrow$

$$\alpha E[S_1^2] + B E[S_2^2] = \alpha\sigma^2 + B\sigma^2$$

$$= (\alpha + B)\sigma^2$$

$\boxed{\text{Therefore } \alpha + B \text{ should equal } 1}$

(ii)

$$Var(\alpha S_1^2 + B S_2^2)$$

$$= \alpha^2 Var(S_1^2) + B^2 Var(S_2^2) + 2\alpha B\, Cov(S_1^2, S_2^2)$$

Minimise this equation in terms of α and B

Q 4

(i) mean is 40 lb
sd is 1.15 lb

$$\bar{X} = \frac{39.6 + 40.2 + 40.9 + 40.9 + 41.4 + 39.8 + 39.4 + 43.6 + 41.8}{9}$$

$$= 40.84$$

95% CI is $\bar{X} \pm z_{0.025} \frac{\sigma}{\sqrt{n}}$

$$\left(40.84 - 1.96\left(\frac{1.15}{3}\right),\ 40.84 + 1.96\left(\frac{1.15}{3}\right)\right)$$

$$= (40.089,\ 41.59)$$

(ii)

$H_0 : \mu = \mu_0 = 40$

$H_1 : \mu > 40$

$\bar{X} = 40.84$ $\quad \mu = 40$

$n = 9$ $\quad \sigma = 1.15$

$$\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0, 1)$$

$$\frac{40.84 - 40}{1.15/\sqrt{9}} = 2.19$$

For one side: $z_{0.05} = 1.645$

↳ $2.19 > 1.645$ (outside the 95% interval)
We reject H_0 and accept H_1

Q4
(iii)

Sample variance $s^2 = \frac{(40.84 - 39.6)^2 + (40.84 - 40.2)^2 \ldots}{n-1}$

$$s^2 = \frac{13.9624}{(n-1)}$$

$$= \underline{1.7453}$$

$$\hat{\sigma}^2 = \frac{13.9624}{n}$$

$$= \underline{1.5514}$$

95% CI

$\hookrightarrow$ $P\left(\frac{n\hat{\sigma}^2}{\chi^2_8(0.025)} \leq \sigma^2 \leq \frac{n\hat{\sigma}^2}{\chi^2_8(0.975)} \right) = 0.95$

$\hookrightarrow$ $\left(\frac{9(1.5514)}{19.023} , \frac{9(1.5514)}{2.7} \right)$

$$= \boxed{(0.734, 5.17)} *$$

Q5 (i)

$X \sim \Gamma(\lambda, m)$
$Y \sim \Gamma(\lambda, n)$

Say $A = X+Y$
$B = X/(X+Y)$

$X = ab$
$Y = a(1-b)$

$$J = \begin{vmatrix} \frac{\partial x}{\partial a} & \frac{\partial x}{\partial b} \\ \frac{\partial y}{\partial a} & \frac{\partial y}{\partial b} \end{vmatrix} = \begin{vmatrix} b & a \\ 1-b & -a \end{vmatrix} = -a$$

Joint density is:

$$f_{A,B}(a,b) = f_{X,Y}(ab, a(1-b))\,|-a|$$

$$= \left\{ \frac{1}{\Gamma(m)} e^{-ab(\lambda)} (ab)^{m-1} \lambda^m \right\} \left\{ \frac{1}{\lambda(n)} e^{a(1-b)(\lambda)} (a(1-b))^{n-1} \lambda^n \right\} |a|$$

$$= \left[\frac{\lambda^{m+n}}{\Gamma(m)\Gamma(n)} (ab)^{m-1} [a(1-b)]^{n-1} e^{-\lambda a} \right] a$$

$$= \left[\frac{\lambda^{m+n}}{\Gamma(m+n)} a^{m+n-1} e^{-\lambda u} \right] \left[\frac{b^{m-1}(1-b)^{n-1}\Gamma(m+n)}{\Gamma(m)\Gamma(n)} \right]$$

$$\cancel{= \left[\lambda^{m+n} a^{m+n-1} e^{-\lambda u} \right.}$$

$$\boxed{= \left[\frac{\lambda^{m+n}}{\Gamma(m+n)} a^{m+n-1} e^{-\lambda u} \right] \left[\frac{b^{m-1}(1-b)^{n-1} \cancel{\Gamma(m+n)}}{B(m,n)} \right] \quad \text{For } 0 \le a \le 1,\ 0 \le b \le 1}$$

$\hookrightarrow$ A, B are indep

Q5 (ii)

We have shown that

$\frac{X_r}{(X_1 + \dots X_r)}$ is independent with $(X_1 + \dots X_r)$

from the first ~~[illegible]~~ part.

↳ Therefore

$\frac{X_r}{(X_1 + \dots + X_r)}$ is indep with $(X_1 + \dots X_{r+m})$, $m \geq 1$

↳

$$\boxed{\frac{X_r}{(X_1 + \dots + X_r)} \text{ is indep with } \frac{X_{r+m}}{(X_1 + \dots + X_{r+m})} \quad 2 \leq r \leq k+1, \ m \geq 1}$$

(iii)

$$Z_r = \frac{X_r}{(X_1 + \dots + X_{k+1})} \qquad 1 \leq r \leq k$$

Set $S = (X_1 + \dots + X_{k+1})$

$$Z_r = \frac{X_r}{S}$$

$X_1 = Z_1 S$, $X_2 = Z_2 S$ $\dots$, $X_k = Z_k S$

$X_{k+1} = S - Z_1 S - Z_2 S - \dots - Z_k S$

$$J = \begin{vmatrix} \frac{\partial X_1}{\partial Z_1} & \dots & \frac{\partial X_1}{\partial Z_k} & \frac{\partial X_1}{\partial S} \\ \vdots & & & \\ \frac{\partial X_{k+1}}{\partial Z_1} & \dots & \frac{\partial X_{k+1}}{\partial Z_k} & \frac{\partial X_{k+1}}{\partial S} \end{vmatrix}$$

Q 5(iii)

$$J = \begin{vmatrix} s & 0 & 0 & \cdots & 0 & z_1 \\ 0 & s & 0 & \cdots & 0 & z_2 \\ \vdots & 0 & s & \ddots & \vdots & \vdots \\ 0 & 0 & \vdots & & \ddots & z_k \\ -s & -s & -s & \cdots & & 1-z_1-\cdots-z_k \end{vmatrix}$$

$$J = \underline{s^k}$$

Joint density of $Z_1, Z_2, \ldots, Z_k, S$ is:

$$f_{Z_1, Z_2 \ldots S}(z_1, z_2 \ldots s) = f(x_1) f(x_2) \ldots F(x_{k+1}) \cdot s^k$$

$$= \left\{ \prod_{r=1}^{k} F(z_r) \right\} \cdot s^k \cdot f(s)$$

Integrate this $f(z_1, z_2 \ldots, z_k, s)$ over s to get:

$$f(z_1, z_2 \ldots, z_k)$$

$$= \frac{\Gamma(B_1 + \cdots + B_{k+1})}{\Gamma(B_1)\Gamma(B_2)\cdots\Gamma(B_{k+1})} \cdot z_1^{B_1 - 1} z_2^{B_2 - 1} \cdots z_k^{B_k - 1} (1 - z_1 - z_2 - \cdots z_k)^{B_{k+1} - 1}$$