Q1.

(i) $X \sim Bin(13, 0.03)$,

Assumptions: - Immolate causes damages indep,

- The rate of triggering the burning mind effect remains unchaged.

(ii) $X \overset{app}{\sim} Poi(\lambda)$, where $\lambda = np = 13 \times 0.03 = 0.39$

$P[X \geq 2] = 1 - P[X=1] - P[X=0] = 1 - e^{-0.39} - 0.39 \times e^{-0.39} = 0.0589$

(iii) $P[X \geq 2] = 1 - P[X=1] - P[X=0]$

$= 1 - 0.97^{13} - 13 \times 0.97^{12} \times 0.03 = 0.056$

The Poisson model provides a good approximation to the Binomial dist^{n}.

Question 2.

Let X = the total # of Heads

$$P[X=x] = \sum_{n=x}^{\infty} P[X=x \mid N=n] \cdot P[N=n]$$

$$= \sum_{n=x}^{\infty} {}^{n}C_{x}\, p^{x} (1-p)^{n-x} \cdot \frac{\lambda^{n}}{n!} e^{-\lambda}$$

$$= \sum_{n=x}^{\infty} \frac{\cancel{n!}}{x!(n-x)!} p^{x}(1-p)^{n-x} \cdot \frac{\lambda^{n}}{\cancel{n!}} e^{-\lambda}$$

$$= \frac{p^{x}(1-p)^{-x}}{x!} \sum_{n=x}^{\infty} \frac{1}{(n-x)!} \cdot (1-p)^{n} \cdot \lambda^{n} e^{-\lambda}$$

$$= \frac{p^{x}(1-p)^{-x}}{x!} \sum_{n-x=0}^{\infty} \frac{[\lambda(1-p)]^{n-x} e \; e^{-\lambda(1-p)}}{(n-x)!\,(\lambda(1-p))^{-x}} \cdot e^{-p\lambda}$$

$$= \frac{p^{x}(1-p)^{-x}}{x!} \cdot e^{-\lambda p} \cdot (\lambda(1-p))^{x} \sum_{n-x=0}^{\infty} \frac{[\lambda(1-p)]^{n-x} e^{-\lambda(1-p)}}{(n-x)!}$$

$$= \frac{e^{-\lambda p} (\lambda p)^{x}}{x!}$$

$$\Rightarrow X \sim Poi(\lambda p)$$

Q3.

(a)

$f(x) = C\, 2^{-x}$

$$\sum_{x=1}^{\infty} f(x) = \sum_{x=1}^{\infty} C \cdot 2^{-x} = C \cdot \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1. \Rightarrow C = 1$$

(b) $\sum_{k=1}^{\infty} \frac{2^{-k}}{k} = \frac{2^{-1}}{1} + \frac{2^{-2}}{2} + \frac{2^{-3}}{3} + \cdots$

$= \frac{(\frac{1}{2})^1}{1} + \frac{(\frac{1}{2})^2}{2} + \frac{(\frac{1}{2})^3}{3} + \cdots$

By observation, for i^{th} element,

it is $\frac{x^i}{i} = a_i = f(x) \Rightarrow f'(x) = x^{i-1}$.

let $g(x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$

$g'(x) = 1 + x + x^2 + \cdots = \frac{1}{1-x}$

$\Rightarrow \int_0^x \frac{1}{1-t} dt = \int_0^x 1 + t + t^2 + \cdots dt = g(x)$

$-\ln(1-x) = g(x) = \sum_{k=1}^{\infty} \frac{x^{-k}}{k}$

In our case, $x = \frac{1}{2}$. $\Rightarrow \sum_{k=1}^{\infty} \frac{x^{-k}}{k} = \ln 2$

$\Rightarrow c^{-1} = \ln 2$

(c)

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\Rightarrow \frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots \qquad ①$$

Also

$$\frac{\sin x}{x} = \left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 + \frac{x}{2\pi}\right)\cdots$$

$$= \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\cdots \qquad ②$$

we are interested in the coeffirient of x^2 in ②

and it is $-\left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \cdots\right)x^2$, which should be the same with the corresponding coeff in ①.

$$\Rightarrow -\left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \cdots\right) = -\frac{1}{3!}$$

$$\Rightarrow -\frac{1}{6} = -\frac{1}{\pi^2}\sum \frac{1}{n^2} \quad \Rightarrow \quad \sum \frac{1}{n^2} = \frac{\pi^2}{6}$$

(d). $f(x) = C \cdot \frac{2^x}{x!}$, by observation, this is almost a Poisson dist^{n}, but $x = 1, \cdots \infty$.

$$\sum_{x=1}^{\infty} f(x) = \sum_{x=1}^{\infty} C \cdot \frac{2^x}{x!} = C \sum_{x=1}^{\infty} \frac{2^x e^{-2}}{x! e^{-2}}$$

$$= C \cdot e^2 \sum_{x=1}^{\infty} \frac{2^x e^{-2}}{x!} = Ce^2 \left(\sum_{x=0}^{\infty} \frac{2^x e^{-2}}{x!} - e^{-2} \right) = 1$$

$$Ce^2(1 - e^{-2}) = 1. \quad C = \frac{1}{e^2 - 1}$$

$$f(x) = \frac{1}{e^2 - 1} \cdot \frac{2^x}{x!}$$

Q4.

(i) (a) $P[X>1] = 1 - P(X=1) = 1 - 2^{-1} = \frac{1}{2}$.

(b) $P[X>1] = 1 - P[X=1) = 1 - (2\ln 2)^{-1}$

(c) $P[X>1] = 1 - P[X=1] = 1 - \frac{6}{\pi^2}$

(d) $P[X>1] = 1 - P[X=1] = 1 - \dfrac{2}{e^2-1}$.

(ii) (a). $f(x) = \frac{1}{2^x}$, $f'(x) = -2^{-x}\ln 2 < 0$

$\Rightarrow$ $f(x)$ is a decreasing f^{n}.

$\therefore$ $X=1$ is the most prob. value.

(b) $f(x) = \frac{1}{\ln 2} \cdot \frac{2^{-x}}{x}$,

$$\left(\frac{2^{-x}}{x}\right)' = \frac{-2^{-x}\ln 2}{x} - \frac{2^{-x}}{x^2} < 0$$

$\Rightarrow$ $f(x)$ is a decreasing f^{n}.

$\therefore$ $X=1$ is the most prob. value.

(c) $f(x) = \frac{6}{\pi^2} \cdot \frac{1}{x^2}$, $f'(x) = -\frac{1}{x^3} < 0$, $f(x) \downarrow$.

$\Rightarrow$ $X=1$ is the most prob. value.

(d)

$$f(x) = \frac{e^{-2}}{1-e^{-2}} \cdot \frac{2^x}{x!} = \frac{1}{1-e^{-2}} \cdot \underbrace{\frac{e^{-2} 2^x}{x!}}_{\text{Poisson}}$$

$\lambda = 2.$

we are looking at a constant times a R.V $X \sim Poi(2)$

$\Rightarrow$ $Y = c \cdot X$ is again a Poisson R.V.

The mode of a Poisson R.V. is $\lceil \lambda \rceil - 1$, $\lceil \lambda \rceil$

$\Rightarrow$ $X = 2$, and 1.

(iii)

(a). $P[X \text{ is even}] = \frac{1}{2^2} + \frac{1}{2^4} + \cdots + \frac{1}{2^\infty} = \frac{\frac{1}{4}}{1-\frac{1}{4}} = \frac{1}{3}$

(b) Let $g(x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$

$$g'(x) = 1 + x + x^2 + \cdots = \frac{1}{1-x}$$

$$\Rightarrow g(x) = \int_0^x \frac{1}{1-t}\,dt = -\ln(1-x) = \sum_{k=1} \frac{x^k}{k}$$

$$P[X \text{ even}] = \frac{1}{\ln 2} \cdot \sum_{k=1}^{\infty} P[X = 2k]$$

$$= \frac{1}{\ln 2} \sum_{k=1}^{\infty} \frac{2^{-2k}}{2k} = \frac{1}{2\ln 2} \cdot \sum_{k=1}^{\infty} \frac{(\frac{1}{4})^k}{k}$$

given $\sum_{k=1}^{\infty} (\frac{1}{4})^k / k = -\ln(1-\frac{1}{4}) = \ln\frac{4}{3}$

$$\Rightarrow P[X \text{ even}] = \frac{1}{\ln 4} \cdot (\ln 4 - \ln 3) = 1 - \frac{\ln 3}{\ln 4}$$

(c) $f(x) = \frac{6}{\pi^2} x^{-2}$

$$P[X \text{ is even}] = \frac{6}{\pi^2} \cdot \left(\frac{1}{2^2} + \frac{1}{4^2} + \cdots\right]$$

$$= \frac{6}{4\pi^2} \cdot \frac{\pi^2}{6} = \frac{1}{4}$$

(d) $f(x) = \frac{1}{e^2-1} \cdot \frac{2^x}{x!} = \frac{1}{1-e^{-2}} \cdot e^{-2} \cdot \frac{2^x}{x!}$

start with a poisson pmf with $\lambda = 2$.

$$\sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^{2k}}{(2k)!} = \sum_{k=1}^{\infty} \frac{e^{-\lambda} \cdot \lambda^{2k}}{(2k)!} + e^{-\lambda}$$

$$LHS = \frac{1}{2} e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} + \frac{1}{2} \cdot e^{-2\lambda} \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} e^{\lambda}$$

$$= \frac{1}{2} + \frac{1}{2} e^{-2\lambda} = RHS$$

$$\Rightarrow P[X \text{ even}] = \frac{1}{1-e^{-2}} \cdot \left[\sum_{k=1}^{\infty} \frac{e^{-2} \cdot 2^k}{(2k)!} \right]$$

$$= \frac{1}{1-e^{-2}} \left(\frac{1}{2} + \frac{1}{2} e^{-4} - e^{-2} \right)$$

$$= \frac{1}{1-e^{-2}} \cdot \frac{1+e^{-4}-2e^{-2}}{2}$$

$$= \frac{e^2-1}{2e^2}$$

Question 5.

(a) $X \sim Bin(n, p)$

$f(k) = {}^nC_k\, p^k (1-p)^{n-k}$, $\quad f(k)^2 = \left[\frac{n!}{k!(n-k)!}\right]^2 \cdot p^{2k}(1-p)^{2(n-k)}$

$f(k-1) = {}^nC_{k-1}\, p^{k-1}(1-p)^{n-k+1}$

$f(k+1) = {}^nC_{k+1}\, p^{k+1}(1-p)^{n-k-1}$

$$f(k+1)f(k-1) = \frac{n!}{(k-1)!(n-k+1)!}\cdot\frac{n!}{(k+1)!(n-k-1)!}$$

$$\times\, p^{2k}(1-p)^{2(n-k)}$$

$$= \underbrace{\frac{k(n-k)}{(k+1)(n-k+1)}}_{<1}\left[\frac{n!}{k!(n-k)!}\right]^2 p^{2k}(1-p)^{2n-2k} < f(k)^2$$

$X \sim Poi(\lambda)$

$f(k) = e^{-\lambda}\cdot\frac{\lambda^k}{k!}$, $\quad f(k)^2 = e^{-2\lambda}\frac{\lambda^{2k}}{(k!)^2}$

$f(k-1) = e^{-\lambda}\cdot\frac{\lambda^{k-1}}{(k-1)!}$, $\quad f(k+1) = e^{-\lambda}\cdot\frac{\lambda^{k+1}}{(k+1)!}$

$$f(k+1)f(k-1) = e^{-2\lambda}\frac{\lambda^{2k}}{(k!)^2}\cdot\underbrace{\frac{k}{k+1}}_{<1} < f(k)^2$$

#

2.

(b) $f(k) = \frac{90}{(\pi k)^4}$, $f(k)^2 = \frac{90^2}{\pi^8 k^8}$

$f(k-1) = \frac{90}{[\pi(k-1)]^4}$, $f(k+1) = \frac{90}{[\pi(k+1)]^4}$

$$f(k-1)f(k+1) = \frac{90^2}{\pi^8[(k-1)(k+1)]^4} = \frac{90^2}{\pi^8(k^2-1)^4}$$

$$> \frac{90^2}{\pi^8 k^8} = f(k)^2$$

(c) $X \sim$ Geometric.

$P[X=k] = (1-p)^{k-1} p$

$f(k) = p(1-p)^{k-1}$, $f(k)^2 = p^2(1-p)^{2k-2}$

$f(k-1) = (1-p)^{k-2} \cdot p$. $f(k+1) = (1-p)^k p$

$f(k-1)f(k+1) = p^2(1-p)^{2k-2} = f(k)^2$

Q.6.

$$p(x) \propto (1+x)^{g_1}(1-x)^{g_2}, \quad |x| \le 1.$$

$$\int_{-1}^{1} p(x)\,dx = \int_{-1}^{1} (1+x)^{g_1}(1-x)^{g_2}\,dx$$

$$= \int_{-1}^{1} (1-x)^{g_2} \cdot \frac{1}{1+g_1}\, d(1+x)^{g_1+1}$$

$$= \frac{1}{g_1+1}(1+x)^{g_1+1}(1-x)^{g_2}\Big|_{-1}^{1} + \int_{-1}^{1} \frac{g_2}{g_1+1}(1+x)^{g_1+1}(1-x)^{g_2-1}\,dx$$

$$= \int_{-1}^{1} \frac{g_2}{g_1+1}(1+x)^{g_1+1}(1-x)^{g_2-1}\,dx$$

$$= \int_{-1}^{1} \frac{g_2(g_2-1)}{(g_1+1)(g_1+2)}(1+x)^{g_1+2}(1-x)^{g_2-2}\,dx$$

$$\vdots$$

$$= \int_{-1}^{1} \frac{g_2(g_2-1)\cdots(g_2-(g_2-1))}{(g_1+1)(g_1+2)\cdots(g_1+g_2)}(1+x)^{g_1+g_2}(1-x)^{0}\,dx$$

$$= \int_{-1}^{1} \frac{g_2!\,g_1!\,(1+x)^{g_1+g_2}}{(g_1+g_2)!}\,dx$$

$$= \frac{g_2!\,g_1!\,2^{g_1+g_2+1}}{(g_1+g_2+1)!}$$

$$\therefore f(x) = \frac{\Gamma(g_1+g_2+2)}{\Gamma(g_1+1)\Gamma(g_2+1)\,2^{g_1+g_2+1}}(1+x)^{g_1}(1-x)^{g_2}$$

Q7.

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}, \quad f'(x) = \frac{-x}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}x^2}$$

$$f'(x) + xf(x) = \frac{-x}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} + \frac{x}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} = 0$$

$$f'(x) + xf(x) = 0. \Rightarrow f(x) = -\frac{1}{x} f'(x)$$

$$1 - F(x) = 1 - \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_{x}^{\infty} f(x)\,dx = \int_{x}^{\infty} -\frac{1}{x} f'(x)\,dx$$

$$= -\frac{1}{x} f(x) \Big|_{x}^{\infty} - \int_{x}^{\infty} f(x) \cdot \frac{1}{x^2}\,dx$$

$$= \frac{1}{x} f(x) + \int_{x}^{\infty} \frac{1}{x^3}\,df(x)$$

$$= \frac{1}{x} \cdot f(x) - \frac{1}{x^3} f(x) + \int_{x}^{\infty} \frac{1}{x^4} f(x)\,dx.$$

Thus $1 - F(x) > \frac{1}{x} f(x) - \frac{1}{x^3} f(x) \Rightarrow \frac{1}{x} - \frac{1}{x^3} < \frac{1-F(x)}{f(x)}$

$\because (f(x) > 0)$

$$= \frac{1}{x} f(x) - \frac{1}{x^3} f(x) + \frac{1}{x^5} f(x) - \int_{x}^{\infty} \frac{1}{x^6} f(x)\,dx$$

$$\therefore 1 - F(x) < \frac{1}{x} f(x) - \frac{1}{x^3} f(x) + \frac{1}{x^5} f(x).$$

$$\frac{1-F(x)}{f(x)} < \frac{1}{x} - \frac{1}{x^3} + \frac{1}{x^5} \quad (f(x) > 0)$$

$$\therefore \frac{1}{x} - \frac{1}{x^3} < \frac{1-F(x)}{f(x)} < \frac{1}{x} - \frac{1}{x^3} + \frac{1}{x^5}$$