CUMULANTS AND CUMULANT GENERATING FUNCTIONS

We can think of a Cumulant Generating Function (CGF) as an alternative to a Moment Generating Function, which can sometimes be more convenient to use in finding the mean and variance of a random variable, and for theoretical development.

<u>DEFINITION</u>: The CGF (denoted $C_X(s)$) of a random variable X is

$$C_X(s) = \ln M_X(s) \qquad (\text{i.e. } \log_e M_X(s))$$

where $M_X(s)$ is the moment generating function.

Thus $$M_X(s) = e^{C_X(s)}$$

As for the MGF, we can generate cumulants by differentiating.

$$C_X'(s) = \frac{M_X'(s)}{M_X(s)}$$

Note that $C_X'(0) = EX$

and $$C_X''(s) = \frac{M_X(s)\,M_X''(s) - [M_X'(s)]^2}{[M_X(s)]^2}$$

Setting $s = 0 \Longrightarrow$

$$C_X''(0) = \frac{M_X(0)\,M_X''(0) - [M_X'(0)]^2}{[M_X(0)]^2}$$

$$= \frac{EX^2 - (EX)^2}{1}$$

$$= V(X).$$

It also turns out that $C_X'''(0)$ gives the skewness $E(X-\mu)^3$

So is $C_X(s)$ simply giving us <u>Central moments</u>? <u>No</u>

The value of $C_X^{(r)}(0)$ is called the <u>r^{TH} CUMULANT</u>, and it

happens that the first three cumulants are
$E(X)$, $V(X)$ and $E(X-\mu)^3$
but the later cumulants do not correspond to $E(X-\mu)^r$

<u>DEFINITION</u>: If $C_X(s)$ is expanded in a MacLaurin series, the coefficient of $s^r/r!$ is known as the <u>r^{th} CUMULANT</u>, and is denoted by κ_r

<u>So what are these cumulants?</u> They are a collection of descriptive contants associated with a probability distribution and they are similar to moments, but for certain theoretical developments the cumulants are more useful than moments.

<u>EXAMPLES</u>: For $X \sim N(\mu, \sigma^2)$, $M_X(s) = e^{s\mu + \frac{1}{2}s^2\sigma^2}$

so that
$$C_X(s) = s\mu + \tfrac{1}{2}s^2\sigma^2$$
and so
$$E(X) = C_X'(0) = \mu$$
$$V(X) = C_X''(0) = \sigma^2$$
$$E(X-\mu)^3 = C_X^{(3)}(0) = 0$$

For $X \sim \text{Gamma}(\alpha, \beta)$ $\quad M_X(s) = \dfrac{1}{(1-\beta s)^\alpha}$

so that
$$C_X(s) = \ln\left[\frac{1}{(1-\beta s)^\alpha}\right] = -\alpha \ln(1-\beta s)$$
$(s<\cdot$

and so
$$E(X) = C_X'(0) = \left.\frac{\alpha\beta}{(1-\beta s)}\right|_{s=0} = \alpha\beta$$
$$V(X) = C_X''(0) = \left.\frac{\alpha\beta^2}{(1-\beta s)^2}\right|_{s=0} = \alpha\beta^2$$
$$E(X-\mu)^3 = C_X^{(3)}(0) = \left.\frac{2\alpha\beta^3}{(1-\beta s)^3}\right|_{s=0} = 2\alpha\beta^3$$

GENERATING MOMENTS ABOUT THE MEAN

We have seen that the MGF generates moments about the origin

We may prefer to generate moments about other points e.g the mean.

We make use of the following result:

Suppose $\overline{Y} = a + b\underline{X}$

Then
$$M_Y(s) = E\left[e^{s\overline{Y}}\right]$$
$$= E\left[e^{s(a+b\overline{X})}\right]$$
$$= e^{sa}\, E\left[e^{sb\overline{X}}\right]$$
$$= e^{sa}\, M_{\overline{X}}(bs)$$

If we want to generate moments about the mean,

then choose $Y = \overline{X} - \mu$

and the MGF for $\overline{Y}$ will generate the moments of $\underline{X}$ about μ

$$E\, Y^r = E(\overline{X}-\mu)^r$$

$$M_{\overline{Y}}(s) = e^{-\mu s}\, M_{\overline{X}}(s)$$

e.g. $\overline{X} \sim N(\mu, \sigma^2)$: $\overline{Y} = \overline{X} - \mu$

$$M_{\overline{X}}(s) = e^{s\mu}\, e^{\frac{1}{2}\sigma^2 s^2}$$

$$M_{\overline{Y}}(s) = e^{\frac{1}{2}\sigma^2 s^2}$$: generates central moments of $\overline{X}$ $\underline{\underline{\mu_r}}$

<u>NOTE</u>: Consider $Z = \dfrac{\overline{X}-\mu}{\sigma} = \dfrac{\overline{X}}{\sigma} - \dfrac{\mu}{\sigma}$

Sometime ago, you were asked to accept that $Z \sim N(0,1)$

We can now prove this:

$$M_Z(s) = e^{-\frac{\mu}{\sigma}s}\, e^{\frac{1}{\sigma}\left[\mu s\right] + \frac{1}{2}\sigma^2 \frac{s^2}{\sigma^2}}$$

$$= e^{\frac{1}{2}s^2}$$ which we recognize as MGF for $Z \sim N(0,1)$

Thus $Z \sim N(0,1)$

MOMENTS FOR $N(0,\sigma^2)$

If we differentiate $M_Y(s) = e^{\frac{1}{2}\sigma^2 s^2}$ k times, we get

$$e^{\frac{1}{2}\sigma^2 s^2}\,\sigma^2 s \qquad k=1$$

$$\sigma^2 e^{\frac{1}{2}\sigma^2 s^2} + \sigma^4 s^2 e^{\frac{1}{2}\sigma^2 s^2} \qquad k=2$$

$$\sigma^2(\sigma^2 s)e^{\frac{1}{2}\sigma^2 s^2} + 2\sigma^4 s\, e^{\frac{1}{2}\sigma^2 s^2} + \sigma^6 s^2 e^{\frac{1}{2}\sigma^2 s^2} \qquad k=3$$

$$\sigma^4 e^{\frac{1}{2}\sigma^2 s^2} + \sigma^6 s^2 e^{\frac{1}{2}\sigma^2 s^2} + 2\sigma^4 e^{\frac{1}{2}\sigma^2 s^2} + \cdots \qquad k=4$$

Thus $E(X-\mu)^4 = 3\sigma^4$

<u>Moments for $N(0,\sigma^2)$</u> $\quad M_X(s) = e^{\frac{1}{2}s^2\sigma^2}$

Series expansion about $s=0$

$$1 + \tfrac{1}{2}\sigma^2 s^2 + \frac{(\frac{1}{2}\sigma^2 s^2)^2}{2!} + \frac{(\frac{1}{2}\sigma^2 s^2)^3}{3!} + \cdots$$

$$= 1 + \frac{\sigma^2 s^2}{2} + \frac{\sigma^4 s^4}{4} + \frac{\sigma^6 s^6}{12} + \cdots$$

Now for any MGF which exists,

$M(0)$

$$M(s) = 1 + \frac{M'(0)\,s}{1!} + \frac{M''(0)\,s^2}{2!} + \frac{M'''(0)\,s^3}{3!} + \cdots$$

Thus, $E X^k = E(X-\mu)^k = 0$ for k odd

(<u>for $N(0,\sigma^2)$</u>)

$$= \frac{k!\,(\frac{1}{2})^{\frac{k}{2}}\,\sigma^k}{(\frac{k}{2})!} \qquad \text{for } \underline{k \text{ even}}$$

$$EX^{2r} = \frac{(2r)!\,(\frac{1}{2})^r}{r!}\,\sigma^{2r} \qquad \text{for } r = 1,2,3\cdots$$

MGF MAY NOT ALWAYS EXIST

The existence of MGF implies that <u>moments of all orders</u> exist for the prob distrib.

But if some moments do not exist, then MGF will not exist (i.e. divergence)

e.g. our 2 earlier examples of Prob distribs for which some moments do not exist

(a) $f(x) = \frac{1}{\pi} \frac{1}{1+x^2}$ $\quad -\infty < x < \infty$ (CAUCHY)

$$M_X(s) = \int_{-\infty}^{\infty} e^{sx} \frac{1}{\pi(1+x^2)} dx$$

$$> \int_0^{\infty} \frac{1}{\pi} e^{sx} \frac{1}{1+x^2} dx$$

$$> \frac{1}{\pi} \int_0^{\infty} \frac{sx}{1+x^2} dx$$

$$= \frac{s}{2\pi} \log(1+x^2) \Big]_0^{\infty}$$

$\boxed{e^z > z \text{ for } z > 0}$

diverges $\underline{\underline{\forall s}}$

(b) $f(x_i) = \frac{6}{\pi^2} \frac{1}{x^2}$ $\quad x = 0, 1, 2, \ldots$

$M_X(s) = \sum_0^{\infty} e^{sx} \frac{6}{\pi^2} \frac{1}{x^2}$ does not converge

THE CHARACTERISTIC FUNCTION

We have seen that the MGF for a prob distrib may not always exist.

This has diminished the usefulness of the MGF as a procedure for deriving certain theoretical results — and instead, attention is focussed on yet another Integral Transform : THE CHARACTERISTIC FUNCTION — which always exists for any prob. distrib.

Defn : For any random variable X, we define its characteristic fn as follows:

$$\phi_X(t) = E[\cos tX] + i\,E[\sin tX]$$

$$i = \sqrt{-1}$$

because of the De Moivre formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

This defn is often written as

$$\phi_X(t) = E\left[e^{itX}\right]$$

$\phi_X(t)$ exists for all real t :

because Sine and Cosine are bounded by 1

So that

$$E|\cos tX| = \int_{-\infty}^{\infty} |\cos tX|\, f(x)\,dx \leq \int_{-\infty}^{\infty} f(x)\,dx = 1$$

DERIVATION OF $\phi_X(t)$ FROM MGF

In those cases where MGF exists — and has already been computed — we can find the Char. fn merely by replacing s by it.

Thus $\phi_X(t) = M_X(it)$

(Complex Variable theory needed to prove this — but it is easy to accept)

PRINCIPAL USE

The main use of $\phi_X(t)$ is in deriving certain theoretical results on Prob. Distribs — some of which we shall touch on later.

Although it may be used for generating moments, this is not its principal role.

As in the case of other Generating functions, we have that for $Z = X + Y$ (indept R. vars)

$$\phi_Z(t) = \phi_X(t)\,\phi_Y(t)$$

with the usual extension to finite sums:

for $Z = \sum^{n} X_i$ (indept r. vars)

$$\phi_Z(t) = \prod_{1}^{n} \phi_{X_i}(t)$$

NOTE: The characteristic Fn is also referred to in other Contexts as the FOURIER TRANSFORM — and an inversion formula exists for recovering the PDF from $\phi(t)$:

i.e. $$f(x) = \lim_{t \to \infty} \frac{1}{2\pi} \int_{-t}^{t} e^{-itx}\,\phi(t)\,dt$$

We shall not derive this result — nor shall we need it in this course

Some Properties of Characteristic Fns

1 $\phi_X(0) = 1$

2 $|\phi_X(t)| \leq 1$ $\qquad [\phi(t)\,\phi(-t)]^{\frac{1}{2}}$

3 $\phi_{aX+b}(t) = e^{itb}\,\phi_X(at)$ $\qquad$ a, b real const

4 $\phi_X(t)$ is uniformly cont. in t.

5 $\left.\dfrac{d^r}{dt^r}\phi_X(t)\right|_{t=0} = i^r E(X^r)$ $\qquad$ provided $E(X^r)$ exists

6 <u>NOTE</u>: If EX^n exists then EX^r exists for $r < n$ $\quad r = 0, 1, 2, \ldots n$.

and it is possible to expand $\phi(t)$.

$$\phi(t) = 1 + (it)EX + \frac{(ib)^2}{2!}EX^2 + \ldots + \frac{(it)^n}{n!}EX^n + o(t^n)$$

where $o(u)$ is a function such that $\lim_{u\to 0}\frac{o(u)}{u} \to 0$

i.e. if EX^n exists, then $\phi(t)$ has a Taylor series exp. of order n.

NOTE <u>Inversion</u>

We should point out that the Char. fn is often termed the Fourier Transform in other contexts — and an inversion formula exists for recovering the PDF from $\phi(t)$

i.e.
$$f(x) = \lim_{t\to\infty}\frac{1}{2\pi}\int_{-T}^{T} e^{-itx}\,\phi(t)\,dt$$

We shall not derive this result — nor shall we use it in this course.

<u>Proofs for Properties</u>

$$|\phi_X(t)| \leq 1$$

$$|\phi_X(t)| = |E\, e^{itX}| = \left|\int e^{itx} f(x)\,dx\right|$$

$$\leq E|e^{itX}|$$

$$= E\mathbb{1}$$

$$= 1$$

<u>$\phi(t)$ uniformly continuous in t</u>

$$|\phi(t+h) - \phi(t)| = |E\, e^{i(t+h)X} - E\, e^{itX}|$$

$$= |E\, e^{itX}(e^{ihX} - 1)|$$

$$\leq E|e^{itX}||e^{ihX} - 1|$$

$$\leq E|e^{ihX} - 1|$$

$$\lim_{h\to 0} |\phi(t+h) - \phi(t)| \leq \lim_{h\to 0} E|e^{ihX} - 1|$$

$$= E \lim_{h\to 0} |e^{ihX} - 1|$$

$$= 0$$

last expression indept of t $\longrightarrow$ hence uniformity of convergence.

Some Examples of Char Fns

Using fact that when MGF exists (and is known), we get the char. fn by replacing s by it, we see that

Binomial: $\phi(t) = (q + pe^{it})^n$

Poisson $\phi(t) = e^{\lambda(e^{it}-1)}$

Gamma $\phi(t) = \dfrac{1}{(1-i\beta t)^{\alpha}}$

Normal $\phi(t) = e^{it\mu - \frac{1}{2}t^2\sigma^2}$

Cauchy $\phi(t) = e^{-|t|}$

$f(x) = \dfrac{1}{\pi(1+x^2)}$

↳ requires complex var. theory to show this.

Uniqueness

THM

There is a one-one correspondence between a characteristic fn. and its distrib fn.

OK ✓

Thus: sum of exponentials = Gamma.

L p. 149

To each char. fn there corresponds a unique D.F. having that char. fn

LIMITS OF DISTRIB FNS

We shall see that it can be very useful to approximate one Prob. Distrib by another, which is more convenient to handle.

— e.g. computing Binomial Probs for large n

— can we find another Prob. Distrib to approx this?

Thus we are interested in the Prob distrib of a random var X_n as n increases.

In studying this, we shall see that the approximation to the prob. distrib of X_n is best expressed in terms of the DISTRIBUTION FUNCTION (which represents a prob.) rather than the PDF

Thus let $X_1, X_2, \ldots, X_n, \ldots$ be a sequence of r. vars with corresponding Dist Fns

$$F_1(x), \; F_2(x) \; \ldots \; F_n(x) \ldots$$

Now the limit $\lim_{n\to\infty} F_n(x)$ may or may not exist — and even if it does exist — it may happen that $\lim_{n\to\infty} F_n(x)$ is not a Distrib Fn.

If the sequence of D. F's does however converge to a function $F(x)$ at each pt. of continuity of $F(x)$, where $F(x)$ is a distrib fn, then we say that the corresponding sequence of random vars $X_1, X_2 \ldots, X$ converges in DISTRIBUTION or in LAW to the random variable X whose distrib fn is $F(x)$.

— and we write

$$d \lim X_n = X$$

or

$$X_n \xrightarrow{L} X$$

NOTE: There are other types of convergence that may be defined for random variables — but these would be appropriate in a more advanced course

See LINDGREN § 2.5 for a very brief mention.

You may wonder why we demand convergence of $F_n(x)$ to $F(x$ <u>only at pts of continuity</u> of $F(x)$:

— The following example will clarify this:

Let $X_n \sim N(0, \sigma_n)$ where $\sigma_n \to 0$

$$F_n(x) = P[X_n \leq x] = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\,\sigma_n} e^{-\frac{1}{2\sigma_n^2}x^2} dx$$

$$\lim_{n\to\infty} F_n(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & x = 0 \\ 1 & x > 0 \end{cases}$$

Thus $F(x)$ is discontinuous at $x = 0$

At pts of cont. of $F(x)$, we have $F(x) = 0 \quad x < 0$

$F(x) = 1 \quad x > 0$

And we use the right continuity of $F(x)$ to say that we must have $F(0) = 1$ i.e. $F(x) = 0$ for $x < 0$

$F(x) = 1$ for $x \geq 0$

Thus $F_n(x) \to F(x)$ only at pts of continuity of $F(x)$.

and the fact that $F_n(x) \to \frac{1}{2}$ at $x = 0$ <u>does NOT MATTER</u>

<u>NOTE</u>: If $X_n \xrightarrow{L} X$ this does not imply that the pdf of X_n converges to the pdf of X

<u>Above</u>: $f_n(x) \to 0$ for $x \neq 0$

$\to \infty$ for $x = 0$

How do we find the limiting forms of Distrib. Fns.?

One way of establishing these limits for D. Fn.'s is by directly working with the D. Fn.'s themselves and attempting to find the limiting form. This can be very tedious and difficult.

A more powerful method is through the use of CHARACTERISTIC FUNCTIONS — relying on the very useful:

CONTINUITY THEOREM (P. LEVY) (LINDGREN P.149; RICE P.159)

If the sequence of distribution functions $F_n(x)$ converges to a distribution function $F(x)$, the corresponding sequence of characteristic functions $\phi_n(t)$ converges to the characteristic fn $\phi(t)$ of $F(x)$.

<u>CONVERSELY</u>: If a sequence of characteristic functions $\phi_n(t)$ converges to $\phi(t)$ which is continuous at $t=0$, then $\phi(t)$ is a characteristic function — and the corresponding sequence of distrib. fns. converges to the distribution function determined by $\phi(t)$.

<u>APPLICATION OF THE THEOREM</u>

We now illustrate the use of this theorem to show that if X_n is binomial (n, p), then the limiting distribution of

$$\frac{X_n - np}{\sqrt{npq}} \quad \text{is} \quad N(0,1)$$

<u>PROOF</u>: The characteristic function of X_n is $(pe^{it}+q)^n$

Thus, the char. fn. of $Z = \dfrac{X_n - np}{\sqrt{npq}}$ is

$$e^{-it\frac{np}{\sqrt{npq}}}\left(pe^{\frac{it}{\sqrt{npq}}}+q\right)^n$$

$$= \left(p\, e^{\frac{itq}{\sqrt{npq}}} + q\, e^{-\frac{itp}{\sqrt{npq}}} \right)^n$$

$$= \left[p\left\{ 1 + \frac{itq}{\sqrt{npq}} - \frac{t^2q^2}{2npq} + o\left(\frac{t^2}{n}\right) \right\} + q\left\{ 1 - \frac{itp}{\sqrt{npq}} - \frac{t^2p^2}{2npq} + o\left(\frac{t^2}{n}\right) \right\} \right]^n$$

$$= \left[1 - \frac{t^2}{2n} + o\left(\frac{t^2}{n}\right) \right]^n$$

As $n \to \infty$, this char. fn $\longrightarrow e^{-t^2/2}$, which we should recognize as the characteristic fn. for $N(0,1)$.

Thus, using the CONTINUITY THEOREM, the limiting prob. distrib of Z is $N(0,1)$, and the limiting prob. distrib. of X_n is $N(np, npq)$.

This means that for large n, X_n is approx. $N(np, npq)$.

NOTICE that the random variable in the above may be regarded as the sum of n independent random variables:

$$X_n = \sum_{i=1}^{n} Y_i$$

where each Y_i is a Bernoulli random variable, with mean p and variance $p(1-p)$.

Thus, we have shown that $\sum_{i=1}^{n} Y_i$ for large n is approximately $N(np, npq)$

This result is a special case of the most important theorem in STATISTICAL THEORY — the CENTRAL LIMIT THEOREM.

EXAMPLES on Normal Approx to Binomial

Let $n = 100$ $\quad p = \frac{1}{2}$

Now $E(X) = np = 50$

$V(X) = npq = 25$

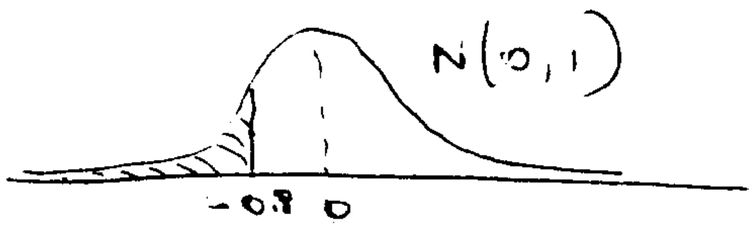
$$P[X \leq 46] = \sum_{k=0}^{46} \binom{100}{k} \left(\tfrac{1}{2}\right)^{100} \longrightarrow \text{much computation}$$

Use $X \sim N(np, npq)$

Thus $\dfrac{X - np}{\sqrt{npq}} \sim N(0,1)$

Now $P[X \leq 46] = P\left[\dfrac{X - np}{\sqrt{npq}} \leq \dfrac{46 - 50}{5}\right]$

$= P[Z \leq -0.8$

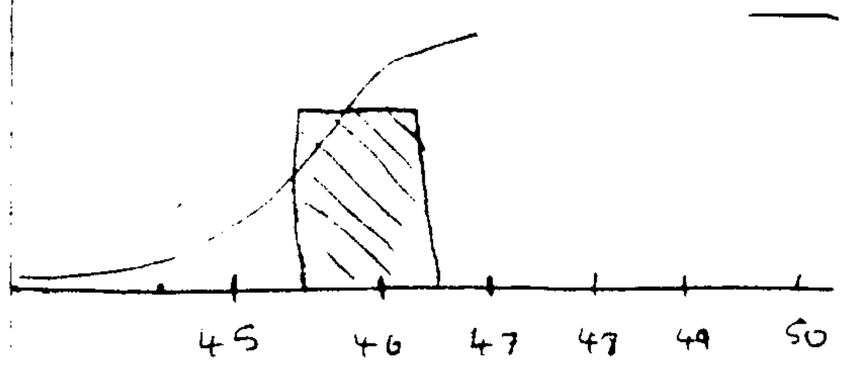


$= 0.2119$

Exact Answer from Tables of Binomial $= 0.2421$

The Approx does not seem very good
— we can improve it by using A CONTINUITY CORRECTION

Remember that the Binomial D. is discrete
— while the Normal is Continuous



Regard prob associated with 46 as running from 45.5 to 46.5

Thus $P[X_B \leq 46] = P[X_N \leq 46.5]$

$= P[Z \leq -0.7]$

$= \underline{0.2420}$

Compares extremely well with exact answer

2nd Exa Much smaller $n = 10$ $p = \frac{1}{2}$

$$P[X \leq 3] = \sum_{0}^{3} \binom{10}{k} \left(\tfrac{1}{2}\right)^{10}$$

$$= 0.1719 \quad \text{(Tables}$$

Normal Approx with Continuity Correction

$$P[X_B \leq 3] = P[X_N \leq 3.5]$$

$$= P\left[Z \leq \frac{3.5 - 5}{\sqrt{10\left(\frac{1}{4}\right)}}\right]$$

$$= P[Z \leq -0.9486]$$

For $P[Z \leq -0.95] \Longrightarrow 0.1711$

~ 0.1715 by interpoln.

excellent approxn.

Suppose p were 0.1

Would the approxn for $n = 10$ be as good?

CENTRAL LIMIT THEOREM

We shall (in this course) state the theorem in its simplest form

Let $X_1, X_2, \ldots, X_n$ be a sequence of indept and identically distributed random variables such that $E(X_i) = \mu$ and $Var(X_i) = \sigma^2$ both exist (i.e. are finite)
Then the limiting distribution of

$$\left(\frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \right) \text{ is } N(0,1) \quad \text{as } n \to \infty$$

where $\overline{X}_n = \dfrac{\sum^n X_i}{n}$

<u>Proof</u> : Let $\phi(t)$ be the char fn of $Z_i = \dfrac{X_i - \mu}{\sigma}$

Now $E(Z_i) = 0$

$Var(Z_i) = 1 = E\left[\dfrac{X_i - \mu}{\sigma}\right]^2 = \dfrac{1}{\sigma^2}\sigma^2 = 1$

and since the 1st and 2nd moments of Z_i exist then the first and second derivatives of $\phi(t)$ exist — and at $t = 0$, we have

$$\phi'(t)\Big|_{t=0} = 0$$

$$\phi''(t)\Big|_{t=0} = -1$$

And Taylor's theorem will allow an expansion of $\phi(t)$ around the pt $t = 0$ up to terms of order (t^2),

i.e. $\phi(t) = 1 - \dfrac{t^2}{2} + o(t^2)$

Now we are interested in

$$\frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \text{ which } = \sqrt{n} \sum_1^n \frac{X_i - \mu}{\sigma n}$$

$$= \frac{1}{\sqrt{n}} \sum_1^n \frac{X_i - \mu}{\sigma}$$

$$= \frac{1}{\sqrt{n}} \sum^n Z_i = \sum^n \frac{Z_i}{\sqrt{n}}$$

Since the X_i are indept, so also are the Z_i and the $\dfrac{Z_i}{\sqrt{n}}$

Now The char fn of $\dfrac{Z_i}{\sqrt{n}} = \phi\left(\dfrac{t}{\sqrt{n}}\right)$

Thus the " " " $\dfrac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} = \left[\phi\left(\dfrac{t}{\sqrt{n}}\right)\right]^n$

$$= \left[1 - \frac{t^2}{2n} + o\left(\frac{t^2}{n}\right) \right]^n$$

as $n \to \infty$, the above char fn

$$\longrightarrow \exp\left(-\frac{t^2}{2}\right)$$

Thus by the LEVY CONTINUITY THEOREM, we conclude that the distrib fn of $\dfrac{\overline{X}_n - \mu}{\sigma/\sqrt{n}}$ tends to the Normal Distrib fn $\displaystyle\int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt$

as $n \to \infty$

<u>NOTE</u> We should stress that in any event The random variable $\overline{X}_n = \Sigma X_i / n$ will have mean μ (whether or not indepce) and variance $\dfrac{\sigma^2}{n}$ (i.e. S.D $\dfrac{\sigma}{\sqrt{n}}$) $\longrightarrow$ due to the indepce.

For large n, the Central Limit Theorem tells us something about the <u>shape</u> of the distrib of $\overline{X}_n$ — namely that it is approx. Normal in shape.

ST2054

Application of MGFs to Compound Distributions

Sums of random variables arise in both Statistical applications (of Probability) and in Actuarial Science.
In Actuarial science (for the case of short-term insurance contracts — say of one-year's duration), one is interested in

$$S = \sum_{i=1}^{N} X_i$$

where X_i denotes the claim-amount arising on policy i, and N denotes the number of claims arising in the year.
[If N turns out to be zero, then S is also zero.]

The random variable S is said to have a <u>Compound Distribution</u>.
If random variable N has a Poisson distribution, then the distribution of S is termed a <u>Compound Poisson Distribution</u>.

To find the distribution of S, we need information on the distribution of N, and of the individual X_i's.
We assume throughout that:
(i) the random variables X_i are independent and identically distributed (i.i.d.).
(ii) the random variable N is independent of the X_i's.

We have already developed methods for finding $E(S)$ and $V(S)$ (check back on this ...).

$$E(S) = E_N\left[E(S|N)\right] = E_N\left[N(EX_i)\right] = (EN)(EX_i)$$

$$\begin{aligned} V(S) &= E_N\left[V(S|N)\right] + V_N\left[E(S|N)\right] \\ &= E_N\left[N\,V(X_i)\right] + V_N\left[N(EX_i)\right] \\ &= (EN)\left[V(X_i)\right] + (EX_i)^2 V(N) \end{aligned}$$

It is also very useful to apply MGFs to the distrib of S, and this is a topic considered in further detail in MF4055 (which is related to <u>FIA subject CT6</u>).

The MGF of S is given by

$$M_S(t) = E\left[e^{tS}\right] = E_N\left[E(e^{tS}|N)\right]$$

Now $E\left(e^{tS} \mid N=n\right) = E\left[e^{t(X_1+X_2+\dots+X_n)}\right]$

$$= \prod_{i=1}^{n} E\left(e^{tX_i}\right)$$

$$= \left[M_X(t)\right]^n$$

Thus $M_S(t) = E_N\left[\{M_X(t)\}^N\right]$ Replace $M_X(t)$ by $e^{\log\{M_X(t)\}}$

$$= E_N\left[e^{N \log M_X(t)}\right]$$

This looks like the MGF for N, except that we have $\log M_X(t)$ in place of t. We write this as:

$$\boxed{M_S(t) = M_N\{\log M_X(t)\}}$$

This expression is true for any COMPOUND DISTRIBUTION.

In the case of a COMPOUND POISSON DISTRIBUTION, we make use of the MGF of the Poisson distribution:

$$M_N(t) = \exp\{m(e^t - 1)\}$$

So that we can write

$$M_S(t) = \exp\left\{m\left\{e^{\log M_X(t)} - 1\right\}\right\}$$

$$= \exp\{m[M_X(t) - 1]\}$$

<u>CUMULANT GENERATING FUNCTION FOR S</u>, I.E. $C_S(t)$

Remember that $C_S(t) = \ln\left[M_S(t)\right]$

Thus $C_S(t) = m\left[M_X(t) - 1\right]$ for the Compound Poisson Distribution.

<u>EXAMPLE</u>: Suppose that the random variable X is $N(\mu, \sigma^2)$ {I.E. all $X_i \sim N(\mu, \sigma^2)$}

And N is Poisson(m), then

$$M_S(t) = e^{m\left[e^{t\mu + \frac{1}{2}t^2\sigma^2} - 1\right]}$$

and $C_S(t) = m\left[e^{t\mu + \frac{1}{2}t^2\sigma^2} - 1\right]$

{WE COULD USE DIFFERENTIATION TO FIND $E(S)$ AND $V(S)$.}