Q1.

$$P[XY \leq u] = P[\underbrace{XY \leq u \cap Y \leq u}_{\text{equal to } Y\leq u \text{ as } X\in[0,1]}] + P[XY \leq u \cap Y > u]$$

$$= P[Y \leq u] + P[X \leq \frac{u}{Y} \cap Y > u]$$

$$= u + \int_X \int_Y dx\, dy \quad , \quad \text{where } x \leq \frac{u}{y}, \ u < y \leq 1$$

$$= u + \int_u^1 dy \int_0^{\frac{u}{y}} 1\, dx = u + \int_u^1 \frac{u}{y}\, dy = u(1 - \ln u)$$

Given X, Y, Z are indep. $\Rightarrow$ XY indep of Z.

$$P[XY \leq u, Z^2 \leq v] = P[XY \leq u]P[Z \leq \sqrt{v}] = u\sqrt{v}(1 - \ln u), \quad 0 < u, v < 1$$

The joint pdf is then $g(u, v) = \dfrac{\ln \frac{1}{u}}{2\sqrt{v}}$, $0 \leq u, v \leq 1$.

$$P[XY \leq Z^2] = \iint_{0 \leq u \leq v \leq 1} \frac{\ln(\frac{1}{u})}{2\sqrt{v}}\, du\, dv = \frac{5}{9}.$$

Q2.

The transformation $s = X+Y$, $r = \frac{X}{X+Y}$

$\Rightarrow \begin{cases} X = rs \\ Y = (1-r)s, \end{cases}$ the Jacobian $= J = s$.

$f_{R,S}(r,s) = f_{X,Y}(x,y) \cdot s = f_X(x) \cdot f_Y(y) \cdot s$, as $X, + Y$ are indep

$= \lambda e^{-\lambda rs} \cdot \mu e^{-\mu s(1-r)} s$.

$\Rightarrow f_R(r) = \int_0^\infty f_{R,S}(r,s)\,ds$.

$= \int_0^\infty \underbrace{\lambda \mu s e^{-(\lambda r + \mu(1-r))s}}\,ds$

this is proportional to an expected value of an exp with para. $\lambda r + \mu(1-r)$

$= \int_0^\infty \lambda \mu \frac{1}{(\lambda r + \mu(1-r))} \cdot (\lambda r + \mu(1-r)) \cdot s e^{-(\lambda r + \mu(1-r))s}\,ds$

pdf

$= \frac{\lambda \mu}{(\lambda r + \mu(1-r))^2}$, $0 \le r \le 1$.

Q3.

(i) S_1^2 & S_2^2 are both unbiased estimator (UE) of σ^2.

$\Rightarrow E[S_1^2] = E[S_2^2] = \sigma^2$.

$E[\alpha S_1^2 + \beta S_2^2] = \alpha E(S_1^2) + \beta E[S_2^2] = (\alpha+\beta)\sigma^2$

A condition s.t. $\alpha S_1^2 + \beta S_2^2$ is an UE of σ^2 implies

$\alpha + \beta = 1$.

(ii) Recall: $\frac{m-1}{\sigma^2} S_1^2 \sim \chi^2_{m-1}$ $\quad \frac{n-1}{\sigma^2} S_2^2 \sim \chi^2_{n-1}$.

$Var(\frac{m-1}{\sigma^2} S_1^2) = 2(m-1)$

$Var(\frac{n-1}{\sigma^2} S_1^2) = 2(n-1)$

$\Rightarrow Var(S_1^2) = \frac{2\sigma^4}{m-1} \qquad Var(S_2^2) = \frac{2\sigma^4}{n-1}$

$Var(\alpha S_1^2 + \beta S_2^2) \overset{indep}{=} \alpha^2 (\frac{2\sigma^4}{m-1}) + \beta^2 \frac{2\sigma^4}{n-1} = 2\sigma^4 [\frac{\alpha^2(n-1) + \beta^2(m-1)}{(m-1)(n-1)}]$ ①

as $\alpha + \beta = 1$, plug in ①

$Var(\alpha S_1^2 + \beta S_2^2) = 2\sigma^4 [\frac{\alpha^2(n-1) + (1-\alpha)^2(m-1)}{(m-1)(n-1)}]$

$\frac{d\, Var(\ )}{d\alpha} = 2\sigma^4 [\frac{2\alpha(n-1) - 2(1-\alpha)(m-1)}{(m-1)(n-1)}] = 0 \Rightarrow \alpha = \frac{m-1}{m+n-2}$.

$\beta = 1 - \alpha = \frac{n-1}{m+n-2}$

Need to calulate 2^{nd} order derivative to justify this is truely a minimum value.

Q4.

(i) $\Sigma X_i = 367.6$. $\Sigma X^2 = 15028.38$.

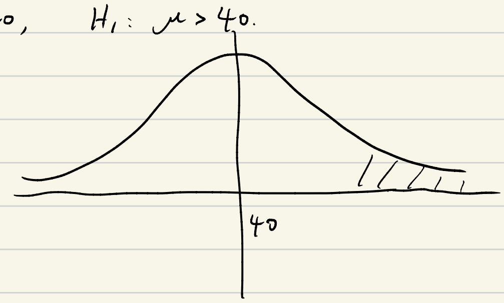
CI at $\alpha = 5\%$: $\bar{X} \pm \frac{\sigma}{\sqrt{n}} \cdot 1.96 = 367.6 \pm \frac{1.15}{3} \cdot 1.96$

$= (366.8487, 368.3513)$

(ii) $\bar{X} = \frac{\Sigma X_i}{9} = 40.84$.

$H_0 : \mu = \mu_0 = 40$, $H_1 : \mu > 40$.

Under H_0:



Test stat $= \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} = \frac{0.84}{1.15/3} = 2.19$.

95% quantile $= 1.64 < 2.19$.

Reject the H_0, that μ is greater than 40.

(iii) 95% CI for σ^2

$= \left(\frac{8 \cdot s^2}{\chi^2_{97.5\%}} , \frac{8 \cdot s^2}{\chi^2_{2.5\%}} \right)$.

Q5.

(i) $u = x+y$, $v = \frac{x}{x+y}$. $J = u$. , $x = uv$, $y = u(1-v)$

$f_{u,v}(u,v) = f_{X,Y}(uv, u(1-v))\, v$

$$\overset{\text{indep } X,Y}{=} \frac{\lambda^{m+n}}{\Gamma(m)\Gamma(n)} (uv)^{m-1} \{u(1-v)\}^{n-1} e^{-\lambda u} u.$$

$$= \underbrace{\left\{ \frac{\lambda^{m+n}}{\Gamma(m+n)} u^{m+n-1} e^{-\lambda u} \right\}}_{f^{n} \text{ of } u \text{ only}} \underbrace{\left\{ \frac{v^{m-1}(1-v)^{n-1}}{B(m,n)} \right\}}_{f^{n} \text{ of } v \text{ only}}$$

$\Rightarrow$ U, V indep.

$U \sim \Gamma(\lambda, m+n)$, $V \sim \text{Beta}(m,n)$.

(ii) According to (i), $\Rightarrow$ $Y_r = \frac{X_r}{X_1+\cdots+X_r}$ is indep

of $X_1+\cdots+X_r$, $\Rightarrow$ indep of $X_{r+1}, X_{r+2}, \ldots, X_{k+1}$,

indep of $X_1+\cdots+X_{k+1}$.

$\Rightarrow$ Y_r is indep of Y_{r+s} for $s \geq 1$.

(iii) Let $s = X_1 + \cdots + X_{k+1}$

$\Rightarrow$ $x_1 = z_1 s$, $x_2 = z_2 s$, ---, $x_k = z_k s$.

$x_{k+1} = s - z_1 s - z_2 s - \cdots - z_k s$.

$$J = \begin{vmatrix} \overset{=\frac{dx_1}{dz_1}}{s} & \overset{\frac{dx_1}{dz_{i\neq 1}}=0}{0} & \cdots & & \overset{=\frac{dx_i}{ds}}{z_1} \\ 0 & s & 0 & \cdots & z_2 \\ 0 & 0 & s & \cdots\ 0 & z_3 \\ & & & & \vdots \\ 0 & \cdots & & & z_k \\ -s & -s & -s & \cdots & 1-z_1-\cdots-z_k \end{vmatrix} = s^k.$$

The joint density of $z_1, z_2, \ldots, z_k, s$ is.

$f_{z_1 \cdots z_k, s} = f_{X_1, \ldots X_{k+1}}(x_1, \ldots x_{k+1}) \cdot s^k$

$$= \left\{ \prod_{r=1}^{k+1} \frac{\lambda^{\beta_r}}{\Gamma(\beta_r)} x_r^{\beta_r - 1} e^{-\lambda x_r} \right\} \cdot s^k$$

$$= \left\{ \prod_{r=1}^{k} \frac{\lambda^{\beta_r} \cdot (z_r s)^{\beta_r - 1} e^{-\lambda z_r s}}{\Gamma(\beta_r)} \right\} \cdot \frac{\lambda^{\beta_{k+1}} \{ s(1-z_1-z_2-\cdots-z_k) \}^{\beta_{k+1}-1} e^{-\lambda s(1-z_1-\cdots-z_k)}}{\Gamma(\beta_{k+1})} \cdot s^k$$

$$= f(\lambda, \beta, s) \left\{ \prod_{r=1}^{k} z_r^{\beta_r - 1} \right\} \cdot (1-z_1-z_2-\cdots-z_k)^{\beta_{k+1}-1} \quad ①$$

Then the joint pdf of $z_1, z_2 - -, z_k$

is obtained by integrating ① w.r.t. s., i.e. integrate $f(\lambda, \beta, s)$ w.r.t. s.

$$f(\lambda, \beta, s) = \frac{\lambda^{\sum_{r=1}^{k+1} \beta_r} s^{\sum_{r=1}^{k+1} (\beta_r - 1)} e^{-\lambda s} \cdot s^k}{\Gamma(\beta_1) - - \Gamma(\beta_{k+1})} = \frac{\lambda^{\sigma} \cdot s^{\sigma} \cdot s^{-(k+1)} \cdot s^k e^{-\lambda s}}{\Gamma(\beta_1) \cdots \Gamma(\beta_{k+1})}, \text{ let } \sigma = \sum_{r=1}^{k+1} \beta_r$$

$$= \frac{\lambda \cdot \lambda^{\sigma - 1} s^{\sigma - 1} e^{-\lambda s}}{\Gamma(\beta_1) - - \Gamma(\beta_{k+1})} = \frac{(\lambda s)^{\sigma - 1} e^{-\lambda s}}{\Gamma(\ ) - - \Gamma(\ )} \cdot \lambda$$

$$\int_0^\infty f(\lambda, \beta, s)\, ds = \int_0^\infty \frac{(\lambda s)^{\sigma - 1} e^{-\lambda s} \cdot \lambda}{\Gamma(\beta_1) - - \Gamma(\beta_{k+1})}\, ds$$

let $\lambda s = t$, $\lambda ds = dt$

$$= \int_0^\infty \frac{t^{\sigma - 1} e^{-t}}{\Gamma(\beta_1) - - \Gamma(\beta_{k+1})}\, dt$$

$$= \frac{1}{\Gamma(\beta_1) - - \Gamma(\beta_{k+1})} \cdot \underbrace{\int_0^\infty t^{\sigma - 1} e^{-t}\, dt}_{\text{gamma } f^n} = \frac{\Gamma(\sigma)}{\Gamma(\beta_1) \cdots \Gamma(\beta_{k+1})}$$

$$\Rightarrow f_{z_1, \cdots z_k} = \frac{\Gamma(\beta_1 + \cdots + \beta_{k+1})}{\Gamma(\beta_1) \cdots \Gamma(\beta_{k+1})} \left\{ \prod_{r=1}^{k} z_r^{\beta_r - 1} \right\} \cdot (1-z_1-z_2-\cdots-z_k)^{\beta_{k+1}-1}$$

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