→ part of survey of OPERATIONAL METHODS widely used in theory of Diff & Int Eqns

GENERATING FUNCTIONS AND INTEGRAL TRANSFORMS

The concepts of G.F's and Integral Transform are extremely useful in deriving the prob. distrib. of sums of random variables

e.g. we may know the prob. distrib's of $\left.\begin{matrix} X \\ \text{and of } Y \end{matrix}\right\}$ indept r. vars

— and we may wish to know the prob. distrib of $X+Y$

— or indeed of $\sum_{i=1}^{n} X_i$, n indept r. vars

Other uses of these generating functions are in the derivations of the moments of random vars.

We shall see other uses as we proceed.

We shall begin with a simple problem

$\left.\begin{matrix} X_1 : \text{outcome of die throw} \\ X_2 : \text{'' of 2nd '' ''} \end{matrix}\right\}$ fair

Interested in distrib of $Z = X_1 + X_2$

$$P[Z = z] = \text{Prob}[(X_1 + X_2) = z]$$
$$= \sum_{k=1}^{z-1} \text{Prob}[X_1 = k, X_2 = (z-k)]$$
$$= \sum_{k=1}^{z-1} \text{Prob}(X_1 = k)\, \text{Prob}(X_2 = z-k)$$

Could have been written $\sum_{x_1+x_2=z} f(x_1)\, f(x_2)$

Enumeration

X_1 \ X_2	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4						
5						
6						

$\text{Prob}[Z = 1] = 0$

$P[Z = 2] = \frac{1}{36}$

etc

Now suppose we shifted our interests to

$$Z = \sum_{i=1}^{5} X_i$$

X_i = outcome of i^{th} die throw,

and we asked (e.g.)

What is $\text{Prob}[Z = 15]$?

Problem is conceptually quite simple — but an efficient procedure (apart from enumeration) is not obvious.

We can deal with problems such as this (which is essentially a COMBINATORIAL problem) using

PROB. GENERATING FN

Defn: If X is a r.var taking nonnegve integer values

Then the P.G.F. $G_X(t) = E(t^X)$

$$= \sum_{k=0}^{\infty} t^k \, P[X = k]$$

provided this converges in some interval $|t| < t_0$

In the above example (converges for $|t| \leq 1$)

$$G_X(t) = \frac{t + t^2 + t^3 + \dots + t^6}{6}$$

$$= \frac{t(1 - t^6)}{6(1 - t)}$$

NOTE: $G_X(1) = 1$

$G_X(0) = P[X = 0]$

Consider $Z = \sum^{n} X_i$, indepdt

$$G_Z(t) = E(t^{X_1 + X_2 + \dots + X_n})$$

$$= E[t^{X_1} t^{X_2} \dots t^{X_n}]$$

$$= (Et^{X_1})(Et^{X_2}) \dots (Et^{X_n})$$ by INDEPCE

Thus
$$G_Z(t) = \prod_{i=1}^{n} G_{X_i}(t)$$

Exa Return to Dice Throwing exa.

We found $G_{X_i}(t) = \dfrac{t(1-t^6)}{6(1-t)}$

Thus for $Z = \sum^{5} X_i$
$$G_Z(t) = \left[\frac{t(1-t^6)}{6(1-t)}\right]^5$$

To find $\text{Prob}[Z = 15]$, say, we only need to find coeff. of t^{15} in $G_Z(t)$

$$G_Z(t) = \frac{t^5}{6^5}\left[\sum_{k=0}^{5}\binom{5}{k}(-t^6)^k\right]\left[\sum_{j=0}^{\infty}\binom{-5}{j}(-t)^j\right]$$

$$(1-t)^{-5} = \sum_{j=0}^{\infty}\binom{-5}{j}(-t)^j$$

NOTE $a > 0$
$$\binom{-a}{k} = (-1)^k\binom{a+k-1}{k}$$

$$\binom{-4}{3} = \frac{(-4)(-5)(-6)}{1.2.3} = (-1)^3\frac{6.5.4}{1.2.3}$$

Coeff of t^{15}:

from $k = 1$ and $j = 4$

$$\Rightarrow \binom{5}{1}(-1)\binom{-5}{4} = -5\cdot\binom{8}{4}$$

$k = 0$ and $j = 10$

$$\Rightarrow \binom{-5}{10} = (-1)^{10}\binom{14}{10}$$

Soln: $$\text{Prob}[Z = 15] = \frac{\binom{14}{10} - 5\binom{8}{4}}{6^5} = .0837$$

$\text{Prob}[Z = 13] = \text{Coeff of } t^{13} =: \begin{pmatrix} k=0 \\ j=8 \end{pmatrix} + \begin{pmatrix} k=1 \\ j=2 \end{pmatrix}$

$$\Rightarrow \binom{-5}{8} = (-1)^8\binom{12}{8} + (5)(-1)(-1)^2\binom{6}{2}$$

$$= \frac{(12)(11)(10)(9)}{1.2.3.4} - 5(15)$$

$$= 495 - 75$$

Prob $= \dfrac{420}{6^5} = \dfrac{420}{7776} = 0.055$

FAMILIAR EXAMPLE Random Expt with 2 outcomes S, F. Repeated n times

Suppose $X_i = 1$ if i^{th} outcome of R.E. is S
$= 0$ " " " " " " F

R.E. is performed n times so that we have
$X_1, X_2, ---, X_n$ n indept random vars

Introd. $Z = \sum_{i=1}^{n} X_i$ (just the number of successes in n trials)

Want to know the prob. distrib of Z.

We've already found this by one method.

— Now we use the Prob. Gen. function method.

let $p(X_i = 1) = p$
$p(X_i = 0) = q$

So that $G_{X_i}(t) = q + tp$ for $i = 1, 2, -- n$

$$\text{Now } G_Z(t) = \prod_{i=1}^{n} G_{X_i}(t)$$

$$= (q + tp)^n$$

$$= q^n + \binom{n}{1} q^{n-1}(tp) + \binom{n}{2} q^{n-2}(tp)^2 + --- + \binom{n}{n-1} q (tp)^{n-1} + (tp)^n$$

Coeff of t^r gives the $\text{Prob}[Z = r]$

i.e. $\text{Prob}[Z = r] = \binom{n}{r} q^{n-r} p^r$

as before.

CONVOLUTIONS

NOTE We introduce the concept of <u>convolution</u>

$a_0, a_1, a_2 \; ----, a_r, \ldots$ sequence of reals
$b_0, b_1, b_2 \; ----, b_r, \ldots$ another " " "

Then consider

$$C_r = a_0 b_r + a_1 b_{r-1} + --- + a_{r-1} b_1 + a_r b_0$$

The sequence of $\{C_r\}$ is referred to as the <u>convolution</u> of seqs. $\{a_i\}$ and $\{b_i\}$

We are here interested in Prob. distribs

X : <u>Prob Distrib</u> : $0, 1, 2 \; ---- \; m, ---$
$\qquad f_i \quad p_0 \; p_1 \; p_2 \; --- \; p_m ---$

Y : Prob Distrib $0, 1, 2, ---, n, --$
$\qquad g_j \quad q_0 \; q_1 \; q_2 \; ---- q_n --$

$Z = X + Y$: <u>wish to know prob. distrib of Z</u>

Possible values of Z :

$$0, 1, 2, 3, ---, r, ---$$

$$\text{Prob}[Z = r] = \text{Prob}[X + Y = r]$$

$$= \sum_{i=0}^{r} \text{Prob}[X = i, Y = r - i]$$

$$= \sum_{i=0}^{r} f_i \, g_{r-i} \quad (\text{using independence of } X, Y)$$

→ a <u>CONVOLUTION</u>

The prob. function of $Z \; (= X + Y)$ is the convolution of the prob. functions of X and of Y.

$$\{h_r\} = \{f_r\} * \{g_r\}$$

It is instructive to look at $G_Z(t) = G_X(t)\, G_Y(t)$ again:

$$\text{Gen. fn for } Z = \sum_{k=0}^{\infty} h_k t^k = G_Z(t)$$

$$\text{" " " } X = \sum_{i=0}^{\infty} f_i t^i = G_X(t)$$

$$\text{" " " } Y = \sum_{j=0}^{\infty} g_j t^j = G_Y(t)$$

Consider $G_X(t)\, G_Y(t)$

$$= (f_0 + f_1 t + f_2 t^2 + \cdots)(g_0 + g_1 t + g_2 t^2 + \cdots)$$

$$\begin{aligned} = {} & f_0 g_0 \\ & + t(f_1 g_0 + f_0 g_1) \\ & + t^2(f_2 g_0 + f_1 g_1 + f_0 g_0) \\ & + \cdots \\ & + t^r(f_r g_0 + \cdots + f_0 g_r) \\ & + \cdots \end{aligned}$$

$$= \sum_{k=0}^{\infty} h_k t^k$$

Thus we see that multiplying the generating fns together is an easy method of forming the convolutions of the probability functions.

<u>example</u> $X_1 \sim \text{Poisson}(\lambda_1)$: $G_{X_1}(t) = E\,t^{X_1}$
$X_2 \sim \text{Poisson}(\lambda_2)$ $\quad G_{X_2}(t) = E\,t^{X_2}$

$$G_{X_1}(t) = \sum_{r=0}^{\infty} t^r \frac{e^{-\lambda_1}(\lambda_1)^r}{r!} = e^{-\lambda_1} e^{+\lambda_1 t} = e^{\lambda_1(t-1)}$$

Sim $G_{X_2}(t) = e^{\lambda_2(t-1)}$

$Z = X_1 + X_2$: <u>What is distrib of Z?</u>

$$G_Z(t) = e^{\lambda_1(t-1)}\, e^{\lambda_2(t-1)} = e^{(\lambda_1+\lambda_2)(t-1)}$$

We recognize this as the generating fn for a Poisson r. var having mean $(\lambda_1 + \lambda_2)$

THE FACTORIAL MOMENT GENERATING FN.

This has the same formal defn as the Prob Gen Fn

i.e. $G_X(t) = E(t^X)$

— However, we are not restricting X to be discrete

Thus $G_X(t) = \int_{-\infty}^{\infty} t^x f(x)\,dx$ for X contin.

and $= \sum_i t^{x_i} f(x_i)$ " X discrete

USE

To see how $G_X(t)$ is used, consider

$$G_X'(t) = \frac{d}{dt}\left[E(t^X)\right]$$

$$= E\left[X t^{X-1}\right]$$

assuming that diff and expectation may be interchanged

(shall not examine condns under which this is possible in this course)

and $G_X''(t) = E\left[X(X-1)t^{X-2}\right]$

— general

$$G_X^{(r)}(t) = E\left[X(X-1)\dots(X-r+1)t^{X-r}\right]$$

Setting $t = 1$:

$$G_X'(1) = EX$$

$$G_X''(1) = E\left[X(X-1)\right]$$

$$\vdots$$

$$G_X^{(r)}(1) = E\left\{X(X-1)\dots(X-r+1)\right]$$

These are referred to as FACTORIAL MOMENTS

r^{TH} factorial moment

exa **Binomial**

We already found that P.G.F $= (pt+q)^n$

— of course this is also the FMGF

To get $E(X)$:

$$G_X'(t) = n(pt+q)^{n-1}p$$
$$G_X'(1) = n(p+q)^{n-1}p$$
$$= np \qquad \text{as before}$$

$E[X(X-1)]$

$$G_X''(t) = n(n-1)(pt+q)^{n-2}p^2$$
$$G_X''(1) = n(n-1)p^2 \qquad \text{as before}$$

Exa

Geometric $\quad X = \#$ trials up to 1st success

$$G_X(t) = \sum_{k=1}^{\infty}(q^{k-1}p)t^k$$
$$= \frac{p}{q}\sum_{k=1}^{\infty}(tq)^k$$
$$= \frac{p}{q}\,\frac{tq}{1-tq} \qquad \text{provided } |tq|<1$$
$$= \boxed{\frac{pt}{1-tq}}$$

$$G_X'(t) = \frac{(1-tq)p + pt(q)}{(1-tq)^2} = \frac{p}{(1-tq)^2}$$

$$E(X) = G_X'(1) = \frac{p}{(1-q)^2} = \boxed{\frac{1}{p}}$$

$$E[X(X-1)] = G_X''(1)$$
$$G_X''(t) = \frac{+2pq}{(1-tq)^3}$$
$$G_X''(1) = \frac{2pq}{(1-q)^3} = \frac{2q}{p^2} = E[X^2] - E[X]$$

Thus $\text{Var}(X) = \frac{2q}{p^2} + \frac{1}{p} - \left(\frac{1}{p}\right)^2 = \frac{2q-1+p}{p^2}$

$$\boxed{= \frac{q}{p^2} = \frac{1}{p^2} - \frac{1}{p}}$$

Negative Binomial (THE FACTORIAL M.G.F.)

X_r = # trials until r^{th} success

$P[X = k] =$

$\binom{-a}{k} = (-1)^k \binom{a+k-1}{k}$

$$= \binom{k-1}{r-1} p^r q^{k-r} \qquad k = r, r+1, \ldots$$

$$G_X(t) = \sum_{k=r}^{\infty} t^k \binom{k-1}{r-1} p^r q^{k-r}$$

$$= \sum_{k=r}^{\infty} (pt)^r \binom{k-1}{r-1} (qt)^{k-r}$$

It's not obvious what this summation is.

But X_r = (# trials to 1^{st} success) + (# trials from 1^{st} to 2^{nd} success) + ...
... + (# trials from $(r-1)^{th}$ to r^{th} success)

$= X + X + \cdots + X$

where each X has a geometric distib
— all the X's are indept.

Thus
$$G_{X_r}(t) = [G_X(t)]^r$$
$$= \boxed{\left(\frac{pt}{1-tq}\right)^r}$$

To find $E(X)$ and $Var(X)$ for Negve Binomial

$E(X)$:
$$G_X'(t) = r\left(\frac{pt}{1-tq}\right)^{r-1}\left[\frac{(1-tq)p + ptq}{(1-tq)^2}\right]$$
$$= r\left(\frac{pt}{1-tq}\right)^{r-1} \frac{p}{(1-tq)^2}$$
$$= rp \frac{(pt)^{r-1}}{(1-tq)^{r+1}}$$
$$G_X'(1) = rp \frac{p^{r-1}}{(1-q)^{r+1}}$$
$$= \boxed{\frac{r}{p}}$$

How could we have found this result more easily?

NOTE Agrees with $E[\Sigma X_i] = \Sigma E(X_i)$ where $X_i \sim$ Geometric

NEGATIVE BINOMIAL : $Var(X_r)$

SHOW $Var(X_r) =$

$$G''_X(t) = rp \frac{(1-tq)^{r+1}(r-1)(pt)^{r-2}p + (pt)^{r-1}(r+1)(1-tq)^r q}{(1-tq)^{2r+2}}$$

$$= rp \frac{(1-tq)^r (pt)^{r-2}\left[p(r-1)(1-tq) + (pt)(r+1)q\right]}{(1-tq)^{2r+2}}$$

$$G''_X(1) = \frac{rp \quad (1-q)^r p^{r-2}\left[p^2(r-1) + p(r+1) - p^2(r+1)\right]}{p^{2r+2}}$$

$$= \frac{r}{p^3}\left[(r+1) - 2p\right] p$$

$$E\left[X_r(X_{r-1})\right] = \frac{r}{p^2}\left[(r+1) - 2p\right] = \boxed{\frac{r(r+1)}{p^2} - \frac{2r}{p}}$$

Thus $Var(X_r) = \frac{r(r+1)}{p^2} - \frac{2r}{p} + \frac{r}{p} - \frac{r^2}{p^2}$

$$= \frac{r}{p^2} - \frac{r}{p}$$

$$= r\left[\frac{1}{p^2} - \frac{1}{p}\right]$$

$$= r\left[\frac{1-p}{p^2}\right]$$

$$= r\left(\frac{q}{p^2}\right) \quad \text{As expected}$$

NOTE : Since $X_r = \sum^r (X)$ and X indept

Then $Var\, X_r = \sum^r Var\, X$

$$= r\frac{q}{p^2}$$

The Moment Generating Function

We have seen that the factorial M.G.F. can be useful in finding factorial moments
— in particular of the 1st and 2nd order
— and hence we found the moments about the origin and the central moments

But it is possible to define a generating function which can be used to generate MOMENTS (about the origin) rather than factorial moments.

Defn The MGF of the distrib of a R. Var X is defined as follows

$$M_X(s) = E\left(e^{sX}\right)$$

$$= \int_{-\infty}^{\infty} e^{sx} f(x)\, dx \qquad \text{for contin r. var}$$

$$= \sum_{x_i} e^{sx_i} f(x_i) \qquad \text{for discrete r. var}$$

'provided we have convergence' — which may depend on S — and so we may only have convergence for a range of S values such that $|s| \leq S_0$

For $s=0$, of course, there is always convergence

USE:

To see how $M_X(s)$ can generate Moments, we differentiate w.r.t. S :

$$M_X'(s) = \frac{d}{ds}\left[E\left(e^{sX}\right)\right]$$

$$= E\left[X e^{sX}\right] \qquad \text{assuming interchange of diff and } E \text{ is permitted}$$

Set $s=0$

$$M_X'(0) = E[X]$$

Now $M''_X(s) = E[X^2 e^{sX}]$ same assumptions

and $M''_X(0) = E[X^2]$

Generally: $M^{(r)}_X(0) = E[X^r] = \mu'_r$

USE: M.G.F. FOR SUMS OF INDEPT. R. VARS

2 r.var X, Y indept

Let $Z = X + Y$

$$M_Z(s) = E[e^{s(X+Y)}]$$
$$= E[e^{sX} e^{sY}]$$
$$= [E e^{sX}][E e^{sY}] \quad \text{because of indepce}$$
$$= M_X(s) \, M_Y(s)$$

By finite induction this extends to any finite sum

$$Z = \sum^n X_i$$

$$\Rightarrow \quad M_Z(s) = M_{X_1}(s) \, M_{X_2}(s) \cdots M_{X_n}(s)$$

If the X_i are identically distributed,

then $M_Z(s) = [M_X(s)]^n$

NOTICE: That the MGF differs from FMGF only in replacement of t by e^s

Thus it follows that for Binomial Distib (e.g.)

$$M_X(s) = (pe^s + q)^n \quad \text{conv. for any finite } s$$

<u>exponential distrib</u>

$$M_X(s) = \int_0^\infty e^{sx}\,\lambda e^{-\lambda x}\,dx$$

$$= \left[\frac{\lambda\, e^{(s-\lambda)(x)}}{s-\lambda}\right]_0^\infty$$

$$= \frac{\lambda}{\lambda - s}$$

$\boxed{\text{integral converges provided } s < \lambda}$

$$M'_X(0) = \left.\frac{\lambda}{(\lambda-s)^2}\right|_{s=0} = \frac{1}{\lambda} = E(X)$$

$$M''_X(0) = \left.\frac{+2\lambda}{(\lambda-s)^3}\right|_{s=0} = \frac{2}{\lambda^2} = E(X^2)$$

Thus $Var(X) = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$ as before.

<u>Poisson Distrib</u>

$$Prob[X = k] = \frac{\lambda^k e^{-\lambda}}{k!}$$

$$M_X(s) = \sum_{k=0}^\infty e^{sk}\frac{\lambda^k e^{-\lambda}}{k!}$$

$$= e^{-\lambda}\sum_{k=0}^\infty \frac{(\lambda e^s)^k}{k!}$$

$$= e^{-\lambda} e^{\lambda e^s}$$

$$= e^{\lambda(e^s-1)}$$

Conv. for any finite s

<u>Gamma Distrib</u> $\qquad f(x) = \frac{1}{\Gamma(\alpha)\left(\beta\right)^{\alpha}}\left(x\right)^{\alpha-1} e^{-\frac{x}{\beta}}$

$$M_X(s) = \int_0^\infty e^{sx} \frac{1}{\Gamma(\alpha)} \frac{x^{\alpha-1}}{\beta^\alpha} e^{-\frac{x}{\beta}}\, dx \qquad x > 0$$

$$= \int_0^\infty \frac{1}{\Gamma(\alpha)} \frac{x^{\alpha-1}}{\beta^\alpha} e^{-x\left[\frac{1}{\beta} - s\right]}\, dx$$

Let $u = \frac{x(1-\beta s)}{\beta}$

$du = dx\frac{(1-\beta s)}{\beta}$

$$= \int_0^\infty \frac{1}{\Gamma(\alpha)} \frac{\left(\frac{\beta}{1-\beta s}\right)^{\alpha-1}}{\beta^\alpha} u^{\alpha-1} e^{-u}\, du \frac{\beta}{(1-\beta s)}$$

$$= \frac{1}{(1-\beta s)^\alpha} \int_0^\infty \frac{1}{\Gamma(\alpha)} u^{\alpha-1} e^{-u}\, du$$

$$= \frac{1}{(1-\beta s)^\alpha}$$

provided $\frac{1-\beta s}{\beta} > 0$

i.e. $\frac{1}{\beta} - s > 0$

$s < \frac{1}{\beta}$

<u>$E(X)$ and $Var(X)$ for GAMMA DIST</u>

$$M'_X(s) = \frac{+\alpha\beta}{(1-\beta s)^{\alpha+1}}$$

$$M'_X(0) = \alpha\beta = E(X)$$

$$M''_X(s) = \frac{\alpha\beta^2(\alpha+1)}{(1-\beta s)^{\alpha+2}}$$

$$M''_X(0) = \alpha(\alpha+1)\beta^2 = E(X^2)$$

Thus $Var(X) = \underbrace{\alpha^2\beta^2 + \alpha\beta^2} - (\alpha\beta)^2$

$$= \alpha\beta^2$$

CONVOLUTIONS — CONTINUOUS CASE

If we have 2 real-valued fns $f(x)$ and $g(x)$ defined for x in $(-\infty, \infty)$, then we can define a 3rd function $h(x)$ as follows

$$h(z) = \int_{-\infty}^{\infty} f(x)\, g(z-x)\, dx$$

for z in $(-\infty, \infty)$

$h(z)$ is known as the convolution of fns f and g and is written as

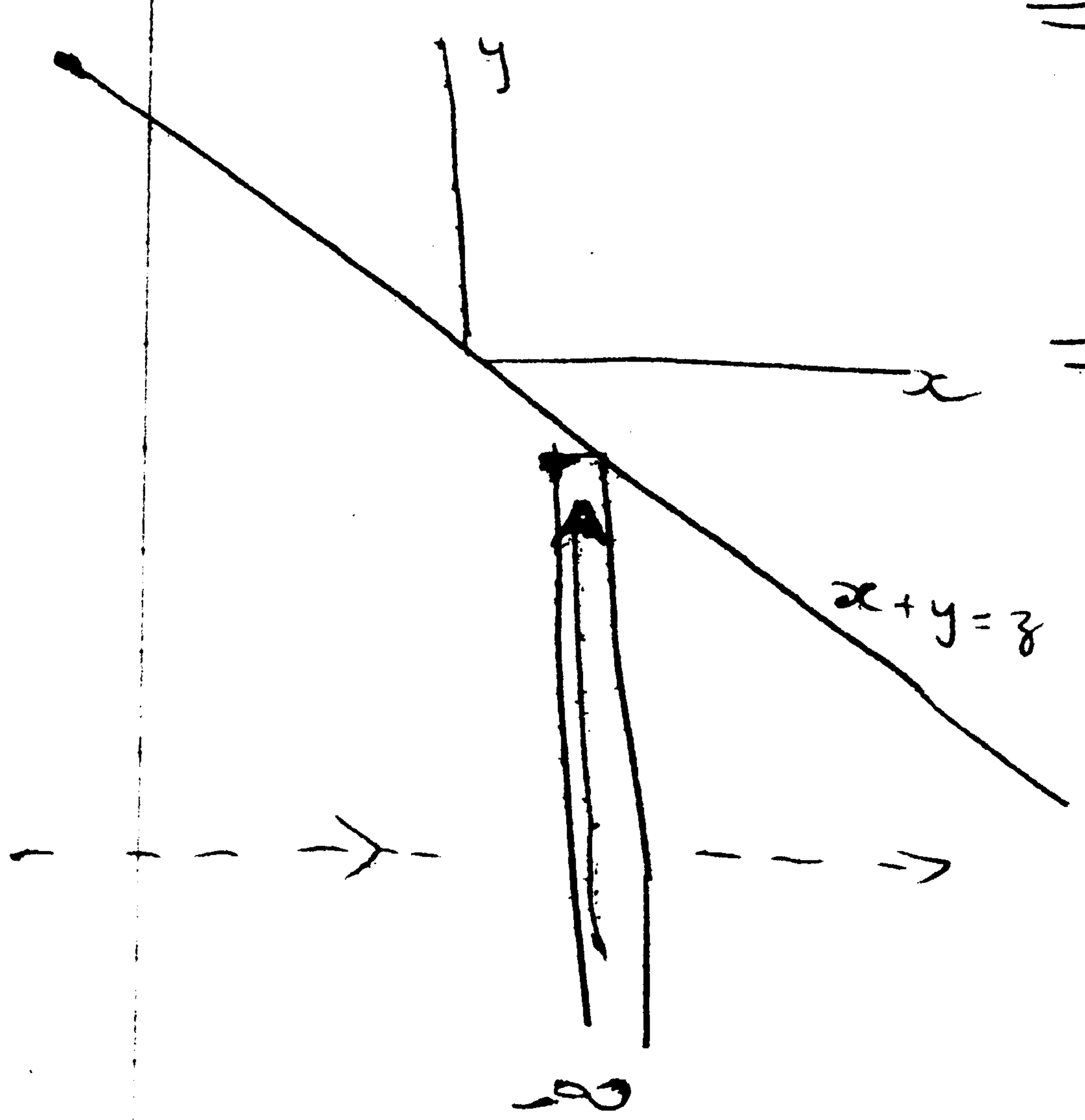
$$h = f * g$$ as in the discrete case.

This convolution arises quite naturally when we attempt to find the prob. distrib of $Z = X + Y$ by DIRECT EVALUATION ;

Suppose X a contin r.v. with PDF $f(x)$
Y " " " " $g(z)$
We wish to find the PDF of $Z = X + Y$.

Now $H(z) = \text{Prob}[Z \leq z] = \text{Prob}[X + Y \leq z]$

$$= \iint_{x+y \leq z} f(x)\, g(y)\, dy\, dx \quad \{ \text{using } \underline{\text{indepce}}$$



$$= \int_{-\infty}^{\infty} f(x) \left[\int_{y=-\infty}^{y=z-x} g(y)\, dy \right] dx$$

$$= \int_{-\infty}^{\infty} f(x)\, G(z-x)\, dx$$

Now diff w.r.t. z

$$\Rightarrow h(z) = \int_{-\infty}^{\infty} f(x)\, g(z-x)\, dx$$

In studying the distrib of $Z = X + Y$, it is often extremely difficult to evaluate the convolution. However, we can find the MGF for Z (and hence investigate the prob. distrib of Z) merely by using

$$M_Z(s) = M_X(s)\, M_Y(s)$$

Example

$$\left.\begin{array}{l} X_1 \sim \text{exponential } \lambda_1 \\ X_2 \sim \text{exponential } \lambda_2 \end{array}\right\} \text{same } \lambda \text{ (maybe)}$$

Distrib of $(X_1 + X_2)$? $\qquad Z = X_1 + X_2$

$$M_Z(s) = M_{X_1}(s)\, M_{X_2}(s)$$

$$= \left(\frac{\lambda_1}{\lambda_1 - s}\right)\left(\frac{\lambda_2}{\lambda_2 - s}\right)$$

$$= \frac{\lambda_1 \lambda_2}{(\lambda_1 - s)(\lambda_2 - s)}$$

$$M'_Z(s) = \frac{-\lambda_1 \lambda_2}{[(\lambda_1 - s)(\lambda_2 - s)]^2}\left[2s - (\lambda_1 + \lambda_2)\right]$$

$$M'_Z(s)\Big|_{s=0} = \frac{-\lambda_1 \lambda_2}{(\lambda_1 \lambda_2)^2}\left[-(\lambda_1 + \lambda_2)\right] = \frac{\lambda_1 + \lambda_2}{\lambda_1 \lambda_2}$$

$$= \frac{1}{\lambda_1} + \frac{1}{\lambda_2}$$

If **Same λ's**

$$M_Z(s) = \frac{\lambda^2}{(\lambda - s)^2} = \frac{1}{\left(1 - \frac{s}{\lambda}\right)^2}$$

Remember the MGF for Gamma Distrib (α, β): $\dfrac{1}{(1 - \beta s)^\alpha}$

Thus Z has a Gamma Distrib with $\alpha = 2$ and $\beta = \frac{1}{\lambda}$

(Assuming that there is a 1-1 correspondence between each Prob Distrib and its MGF)

Similarly $Z = \sum_{i=1}^{n} X_i$ where $X_i \sim$ Exponential (λ) has a Gamma Distrib with $\alpha = n$, $\beta = \frac{1}{\lambda}$

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MGF for Normal P. Distrib

$X \sim N(\mu, \sigma^2)$

$$M_X(s) = \int_{-\infty}^{\infty} e^{sx} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} dx$$

Let $z = \frac{x-\mu}{\sigma} \Rightarrow x = \mu + \sigma z$ and $dx = \sigma dz$

$$M_X(s) = \frac{1}{\sqrt{2\pi}} \int \exp[s(\mu + \sigma z)]\, e^{-z^2/2} dz$$

$$= e^{s\mu} \frac{1}{\sqrt{2\pi}} \int \exp\left[-\tfrac{1}{2}(z^2 - 2\sigma s z)\right] dz$$

$$= e^{s\mu} \frac{1}{\sqrt{2\pi}} \int \exp\left[-\tfrac{1}{2}(z - \sigma s)^2 + \frac{\sigma^2 s^2}{2}\right] dz$$

$$= e^{s\mu + \sigma^2 s^2/2} \int \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-\sigma s)^2}}_{N(\sigma s,\, 1)} dz$$

$$= e^{s\mu + \sigma^2 s^2/2}$$

Suppose 2 r.v. $X_1 \sim N(\mu_1, \sigma_1^2)$, $X_2 \sim N(\mu_2, \sigma_2^2)$ } indept

$Z = X_1 + X_2$

$$M_Z(s) = M_{X_1}(s)\, M_{X_2}(s)$$

$$= e^{s\mu_1 + \sigma_1^2 \frac{s^2}{2}} e^{s\mu_2 + \sigma_2^2 \frac{s^2}{2}}$$

$$= e^{s(\mu_1 + \mu_2) + (\sigma_1^2 + \sigma_2^2)\frac{s^2}{2}}$$

Thus $Z \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

a remarkable result

Sum of Two Exponential Random Variables

$Z = X + Y \qquad X$ is $Exp(\lambda_1)$

$\qquad\qquad\qquad Y$ is $Exp(\lambda_2)$

Get the Distribution Function of Z

$$P(Z \le z) = P(X+Y \le z) = \iint_{\{x,y \text{ s.t. } x+y \le z\}} f(x,y)\,dx\,dy$$

$$= \int_0^z \lambda_1 e^{-\lambda_1 x} \int_0^{z-x} \lambda_2 e^{-\lambda_2 y}\,dy\,dx$$

$$= \int_0^z \lambda_1 e^{-\lambda_1 x} \left[-e^{-\lambda_2 y}\right]_0^{z-x} dx$$

$$= \int_0^z \lambda_1 e^{-\lambda_1 x}\left[1 - e^{-\lambda_2(z-x)}\right] dx$$

$$= \int_0^z \left[\lambda_1 e^{-\lambda_1 x} - \lambda_1 e^{-\lambda_2 z} e^{-(\lambda_1-\lambda_2)x}\right] dx$$

$$= \left[-e^{-\lambda_1 x}\right]_0^z - \frac{\lambda_1 e^{-\lambda_2 z}}{(\lambda_1-\lambda_2)}\left[-e^{-(\lambda_1-\lambda_2)x}\right]_0^z$$

$$= 1 - e^{-\lambda_1 z} - \frac{\lambda_1}{\lambda_1-\lambda_2}\left[e^{-\lambda_2 z} - e^{-\lambda_1 z}\right]$$

To get the pdf for Z, differentiate this w.r.t. z

$$f(z) = \lambda_1 e^{-\lambda_1 z} + \left(\frac{\lambda_1}{\lambda_1-\lambda_2}\right)\left[\lambda_2 e^{-\lambda_2 z} - \lambda_1 e^{-\lambda_1 z}\right]$$

a few steps of algebra show that this becomes

$$f(z) = \frac{\lambda_1\lambda_2}{(\lambda_1-\lambda_2)} e^{-\lambda_2 z} - \frac{\lambda_1\lambda_2}{(\lambda_1-\lambda_2)} e^{-\lambda_1 z}$$

SPECIAL CASE: $\lambda_1 = \lambda_2 = \lambda$ (say)

$$f(z) = \lambda^2 z e^{-\lambda z}$$

which is the Gamma pdf with $\alpha = 2$, $\beta = 1/\lambda$