Q1.

(a) $F(\frac{3}{2}) - F(\frac{1}{2}) = \frac{1}{2}$

(b) $F(2) - F(1) = \frac{1}{2}$

(c) $P[Y \le X] = P[X^2 \le X] = P[X \le 1] = \frac{1}{2}$

(d) $P[X \le 2Y] = P[X \le 2X^2] = P[X \ge \frac{1}{2}] = \frac{3}{4}$

(e) $P[X + X^2 \le \frac{3}{4}] = P[\underbrace{(X+\frac{3}{2})}_{>0}(X-\frac{1}{2}) \le 0]$

$= P[X - \frac{1}{2} \le 0] = P[X \le \frac{1}{2}] = \frac{1}{4}$

(f) $P[Z \le z] = P[\sqrt{X} \le z] = P[X \le z^2]$

$= \frac{1}{2}z^2$ if $0 \le z \le \sqrt{2}$.

Q2.

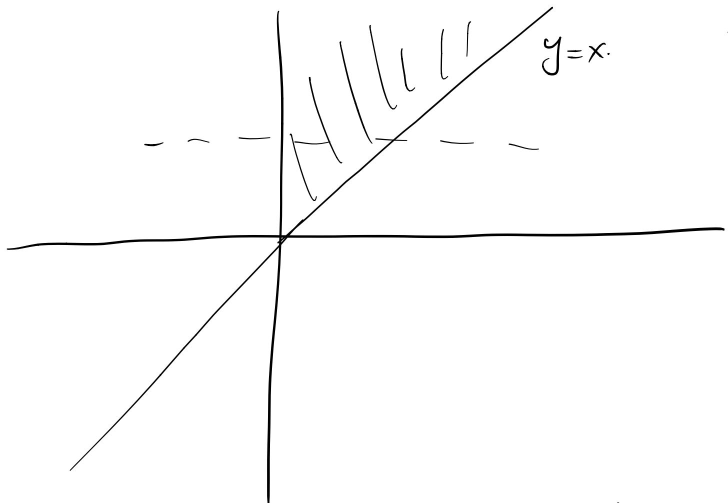
Integrate by parts

$$RHS = \int_0^\infty r x^{r-1} P[X > x] \, dx$$

$$= \int_0^\infty r x^{r-1} \left\{ \int_x^\infty f(y) \, dy \right\} dx \, ,$$

$x \leq y$

$\infty > x > 0$



change the order of integral.

$$= \int_{y=0}^\infty f(y) \left\{ \int_{x=0}^{x=y} r x^{r-1} \, dx \right\} dy = \int_0^\infty f(y) \, y^r \, dy = E(X^r)$$

Q3.

(i). The $dist^n$. f^n. F_Y of Y is

$$F_Y(y) = P[Y \le y] = P[aX \le y] = P[X \le \frac{y}{a}]$$

$$= F_X(\frac{y}{a})$$

$$f_Y(y) = \frac{1}{a} f_X(\frac{y}{a})$$

(ii)

We know that

$$F_{-X}(x) = P[-X \le x] = P[X \ge -x]$$

$$= 1 - \underbrace{P[X \le -x]}_{= F_X(-x)}, \text{ as}$$

X is a cts R.V, $P[X = -x] = 0$

$$\Rightarrow f_{-X}(x) = -f_X(-x) \times (-1) = f_X(-x)$$

Given X and $-X$ have the same $dist^n$. f^n. $\Rightarrow$

$$f_X(x) = f_{-X}(x)$$

$$\Rightarrow f_X(-x) = f_X(x).$$

Conversely, given $f_X(-x) = f_X(x)$ for all x,
Need to prove $P[-X \le y] = P[X \le y]$
then substituting $u = -x$.

$$P[-X \le y] = P[X \ge -y] = \int_{-y}^{\infty} f_X(x)\, dx$$

$$= \int_{-\infty}^{y} f_X(-u)\, du = \int_{-\infty}^{y} f_X(u)\, du$$

$$= P[X \le y]$$

$\Rightarrow$ X and $-X$ have the same dist^{n}.

Q4.

(a) By def^{n}, $r(x) = \lim_{h \to 0} \frac{1}{h} \frac{F(x+h) - F(x)}{1 - F(x)}$

$$= \frac{1}{1-F(x)} \underbrace{\lim_{h \to 0} \frac{F(x+h) - F(x)}{h}}$$

This is the def^{n}. of derivative.

$$= \frac{f(x)}{1 - F(x)}$$

$$H'(x) = - \frac{1}{1-F(x)} \cdot -f(x) = \frac{f(x)}{1-F(x)}$$

(b) $H'(x) = r(x) \Rightarrow H(x) = \int_0^x r(y)\,dy$.

$G(x) = \frac{H(x)}{x}$, $G'(x) = -\frac{H(x)}{x^2} + \frac{r(x)}{x}$

$$= \frac{r(x)}{x} - \frac{1}{x^2} \int_0^x r(y)\,dy$$

$$= \frac{x\,r(x)}{x^2} - \frac{1}{x^2} \int_0^x r(y)\,dy$$

$$= \frac{1}{x^2} \int_0^x (r(x) - r(y))\,dy,$$

given. $r(x)$ is an ↗ f^{n}. w.r.t. x.

$\Rightarrow$ for $x \geq y$, $r(x) \geq r(y) \Rightarrow G'(x) \geq 0$

(c) given $0 \leq \alpha \leq 1$, we have $\frac{H(x)}{x}$ is non decrease iff.

$$\frac{1}{\alpha x} H(\alpha x) \leq \frac{1}{x} H(x) \quad \text{for all } x \geq 0.$$

that is $-\alpha^{-1} \ln[1-F(\alpha x)] \leq -\ln[1-F(x)]$

exp to both side to get the answer.

(d) If $\frac{H(x)}{x}$ is non - decreasing

$$\Rightarrow \frac{H(\alpha t)}{\alpha \cancel{t}} \leq \frac{H(t)}{\cancel{t}} \Rightarrow H(\alpha t) \leq \alpha H(t)$$

for $0 \leq \alpha \leq 1$ and $t \geq 0$.

Similarly, $H((1-\alpha)t) \leq (1-\alpha) H(t)$

$\Rightarrow H(\alpha t) + H[(1-\alpha)t] \leq H(t)$, let $x = \alpha t$

$y = (1-\alpha) t$

(e) $r(x) = \alpha \beta x^{\beta-1}$

(f) $r(x) = \lambda$

(g) $r(x) = \dfrac{\lambda \alpha e^{-\lambda x} + \mu (1-\alpha) e^{-\mu x}}{\alpha e^{-\lambda x} + (1-\alpha) e^{-\mu x}} \to \min(\lambda, \mu)$

as $x \to \infty$