Q1.

(a).

$E[s^X] = E[E(s^X | U)] = \int_0^1 (1+u(s-1))^n \, du = \frac{1}{n+1} \cdot \frac{1-s^{n+1}}{1-s}$,

which is the prob. generating f^n of the uniform dist^n with $a=0$, $b=n$.

(b) for $|s| < \mu+1$,

$$E[s^X] = E[E[s^X | \Lambda]] = E[e^{\Lambda(s-1)}] = \frac{\mu}{1+\mu-s}$$
$$= \frac{\mu}{1-\frac{s}{1+\mu}} \cdot \frac{1}{1+\mu} = \frac{\mu/1+\mu}{1-\frac{s}{1+\mu}},$$

which is the prob. generating f^n of a Type II Neg. Bin with $k=1$, $\Rightarrow$ A geometric dist^n.

Q2.

(a) Given $X = \sigma\rho U + \sigma\sqrt{1-\rho^2}\, V$,

$Y = \tau U$.

$$E[X \mid Y=y] = E[\sigma\rho U \mid U = y/\tau] = \frac{\sigma\rho y}{\tau}$$

(b) $$E[X^2 \mid Y=y] = E[(\sigma\rho U)^2 + \sigma^2(1-\rho^2)V^2 \mid U = y/\tau]$$
$$= \left(\frac{\sigma\rho y}{\tau}\right)^2 + \sigma^2(1-\rho^2)$$

$$\Rightarrow Var(X \mid Y=y) = \sigma^2(1-\rho^2)$$

(c) Use the conclusion that If X & Y are indep. Normal, then $E[X \mid X+Y=z] = az+b$, where $\begin{cases} a = \frac{cov(X,Z)}{\sigma_Z^2} \\ b = \mu_X - a\mu_Z \end{cases}$

$$cov(X, X+Y) = Var(X) + cov(X,Y) = \sigma^2 + \rho\sigma\tau$$
$$Var(Z) = Var(X+Y) = Var(X) + Var(Y) + 2cov(X,Y)$$
$$= \sigma^2 + \tau^2 + 2\rho\sigma\tau$$

$$E[X \mid Z] = \frac{(\sigma^2 + \rho\tau\sigma)z}{\sigma^2+\tau^2+2\rho\sigma\tau}$$

(d) $$1 - \rho(X, X+Y)^2 = \frac{\tau^2(1-\rho^2)}{\sigma^2+\tau^2+2\rho\sigma\tau}$$

$$\Rightarrow Var(X \mid Z) = \frac{\sigma^2\tau^2(1-\rho^2)}{\tau^2+\sigma^2+2\rho\sigma\tau}$$

Q3.

(a) Since V is symmetric, there exists a non-singular matrix M such that $M' = M^{-1}$ and $V = M \Lambda M^{-1}$, where Λ is the diagonal matrix with diagonal entries the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of V. Let $\Lambda^{\frac{1}{2}}$ be the diagonal matrix with diag. entries $\lambda_1^{\frac{1}{2}}, \lambda_2^{\frac{1}{2}}, \ldots \lambda_n^{\frac{1}{2}}$; $\Lambda^{\frac{1}{2}}$ is well defined since V is non-negative definite. Writing $W = M \Lambda^{\frac{1}{2}} M'$, we have that $W = W'$ and also

$$W^2 = (M \Lambda^{\frac{1}{2}} M^{-1})(M \Lambda^{\frac{1}{2}} M^{-1}) = M \Lambda M^{-1} = V.$$

Students obtain full marks for the above for Q3.

Clearly W is non-singular iff $\Lambda^{\frac{1}{2}}$ is non-singular.

This happens iff $\lambda_i > 0$ for all i, which is to say that

V is positive definite.

Q4.

a) $E[x^{X+Y}] = E\{E[x^{X+Y} \mid Y]\} = E[x^Y e^{Y(x-1)}]$

$= E[(xe^{x-1})^Y] = \exp\{\mu(xe^{x-1}-1)\}$

(b) To obtain the MGF for X firstly.

$$M_X(t) = \sum_{x=1}^{\infty} e^{xt} \frac{(1-p)^x}{x \ln(1/p)}$$

$$= \sum_{x=1}^{\infty} \frac{[e^t(1-p)]^x}{x \ln(1/p)} = \sum_{x=1}^{\infty} \frac{[1-[1-e^t(1-p)]]^x \ln \frac{1}{1-e^t(1-p)}}{x \ln(1/p) \cdot \ln \frac{1}{1-e^t(1-p)}}$$

$$= \frac{\ln \frac{1}{1-e^t(1-p)}}{\ln(1/p)}$$

$$M_S(t) = \exp[\mu(M_X(t) - 1)]$$

$$= \exp\left[\mu\left\{\frac{\ln \frac{1}{1-e^t(1-p)} - \ln(1/p)}{\ln(1/p)}\right\}\right]$$

$$= \exp\left[\mu \cdot \frac{\ln \frac{p}{1-e^t(1-p)}}{\ln \frac{1}{p}}\right] = \left\{\left\{e^{\ln \frac{p}{1-e^t(1-p)}}\right\}^{\mu}\right\}^{\frac{1}{\ln \frac{1}{p}}}$$

$$= \left[\left(\frac{p}{1 - e^t(1-p)} \right)^{r} \right]^{\frac{1}{\ln \frac{1}{p}}} = \left[\frac{1-(1-p)}{1-e^t(1-p)} \right]^{\frac{r}{\ln \frac{1}{p}}}$$

which is the MGF of a NB.

(c). $E\left[\frac{1}{1+X}\right] = E\left[\int_0^1 t^X dt\right] = \int_0^1 E[t^X] dt$

$$= \int_0^1 (q + pt)^n dt = \frac{1-q^{n+1}}{p(n+1)}.$$