OLLSCOIL NA hEIREANN, CORCAIGH
The National University of Ireland, Cork

COLAISTE NA hOLLSCOILE, CORCAIGH
University College, Cork

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Semester 2 - Summer 2017

Second University Examination in Science
Financial Mathematics and Actuarial Science; Mathematical Sciences;
Higher Diploma in Statistics

ST2054 - Probability and Mathematical Statistics

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Time allowed: Three hours.

Statistical tables are available. A calculator may be used provided that it does not contain any information stored by any person prior to this examination.

Fifteen minutes of reading time are permitted prior to this examination.

PLEASE ANSWER ANY NINE QUESTIONS
All questions carry equal marks

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PLEASE
ENSURE THAT YOU HAVE THE CORRECT EXAM PAPER

Question 1

(i) State the Axioms of Probability. [3 marks]

(ii) Using the Axioms of Probability, prove that for any two events A and B, the following expression is true:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

[3 marks]

Consider two events A and B such that $P(A) = \frac{1}{3}$ and $P(B) = \frac{1}{2}$. Determine the value of $P(B \cap A^C)$, that is, the probability of the intersection of B and A complement, for each of the following conditions:

(iii) A and B are disjoint [1 mark]

(iv) $A \subset B$ [2 marks]

(v)$P(A \cap B) = \frac{1}{8}$. [1 mark]

Question 2

The probability that any child in a certain family will have blue eyes is $\frac{1}{4}$, and this feature is inherited independently by each child in the family. If there are 5 children in a particular family, calculate the following:

(i) The probability that there will be fewer than 3 children with blue eyes in the family. [3 marks]

(ii) The expected number of children in the family who'll have blue eyes. Comment on any unusual characteristics of the expected value and explain how it can still be considered to make sense. [4 marks]

(iii) It is known that at least one of the children has blue eyes. Find the probability that at least two children have blue eyes. [3 marks]

Question 3

The random variable Y has a negative exponential distribution with *pdf* as follows:

$$f(y) = \lambda e^{-\lambda y}, \;\; y > 0$$

$$= 0, \;\; otherwise$$

(i) Find $P(Y > s | Y > t)$, for $s \geq t$. [3 marks]

(ii) Find an expression for the (conditional) *pdf* of Y, conditional on $Y \leq 200$. [5 marks]

(iii) Find an expression for the (conditional) *pdf* of Y, conditional on $100 < Y \leq 200$. [2 marks]

Question 4

There are four men in a room, 1 pair of brothers, and 2 unrelated men. The probability that any man has blood-group X is $\frac{1}{4}$. The probability that if one brother has blood-group X, the other brother also has X is $\frac{3}{4}$, i.e., $P[B_{2X}|B_{1X}] = P[B_{1X}|B_{2X}] = \frac{3}{4}$. Otherwise the blood-groups are independent.

(i) Find the probabity that one brother has blood group X, conditional on the event that the other brother does not, i.e., $P[B_{2X}|B_{1X^C}]$ [1 mark]

(ii) Find the probability that exactly 2 men in the room have blood-group X. [5 marks]

Consider a sequence of Bernoulli trials, where the probability of success (S) on each trial is p and the probability of failure (F) is $1 - p$. Let X denote the completed run-length that begins on the first trial. For example $X = 3$ when you get either $SSSF$ or $FFFS$.

(iii) Find the probability mass function (pmf) of X. [2 marks]

(iv) Hence, or otherwise, find $E[X]$. [2 marks]

Question 5

(i) Explain what is meant by *convergence in probability* for a sequence of random variables. [3 marks]

Suppose X is a continuous random variable with cumulative distribution function F.

(ii) Find the distribution of the random variable $V = F(X)$. [3 marks]

(iii) If U is Uniform$(0,1)$ find the distribution of $X = F^{-1}(U)$. Comment on a potential use for this result. [4 marks]

Question 6

The continuous random variables X and Y have a joint pdf $f(x,y)$ where:

$$f(x,y) = k(x^2 + 2xy) \text{ ,for } 0 < x < 1, \ \ 0 < y < 1$$

(i) Find k. [1 mark]

(ii) Find $P[Y < 0.5 | X < 0.5]$. [3 marks]

(iii) Show that the marginal distribution of X is

$$f_X(x) = \frac{6}{5}(x^2 + x)$$

[2 marks]

(iv) Show that the conditional distribution of Y is

$$f_{Y|X}(y|x) = \frac{x + 2y}{x + 1}$$

[1 mark]

(v) Find $P[Y < 0.5 | X = 0.5]$. [1 mark]

(vi) Find $P[Y < X]$. [2 marks]

Question 7

(i) Derive expressions, in terms of a, b and c, for the principal components of a 2-dimensional random vector (X) with mean μ and covariance matrix

$$S = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$$

What are the variances of the first and second principal components? [5 marks]

(ii) A p-dimensional random variable X has a $N_p(\mu, \Sigma)$ distribution if its characteristic function is given by

$$\phi_X(t) = e^{it'\mu - \frac{1}{2}t'\Sigma t}$$

Use this result to derive the distribution of the vector $Y = AX$ when A is a $q \times p$ matrix.

How is this used to derive the distribution of the least squares estimates $\hat{\beta} = [X'X]^{-1}X'y$ in the Normal Linear Model? State the result and your assumptions.

What is the distribution of the vector of predictions and residuals, $(\hat{y}, y - \hat{y})'$?
[5 marks]

Question 8

Let $X_1, \ldots, X_n$ be independent, identically distributed random variables, with

$$X_i \sim exp(\lambda) \ \forall i.$$

Let $S = \sum_{i=1}^{n} X_i$.

(i) Find the Moment Generating Function for X_1. [3 marks]

(ii) Using the properties of Moment Generating Functions, show that S follows a gamma distribution with $\alpha = n$ and $\beta = \lambda$, i.e.:

$$S \sim \Gamma(n, \lambda)$$

. [2 marks]

(iii) Hence, or otherwise, find expressions for the mean and variance of S.[5 marks]

Question 9

The random variable Z has the Standard Normal distribution, and the random variable U has a Chi-square distribution with n degrees of freedom, and Z and U are independent. The random variable T is constructed as follows:

$$T = \frac{Z}{\sqrt{\frac{U}{n}}}$$

(i) Write down the joint probability density function (pdf) of Z and U. [3 marks]

(ii) Find the joint pdf of T and U. [3 marks]

(iii) Use this to find the pdf of T. Name the distribution of T. [4 marks]

Question 10

The Iris flower data set or Fisher's Iris data set is a multivariate data set introduced by Ronald Fisher in 1936. The following is a subset of the Sepal Lengths of each of three Iris species: Setosa, Versicolor and Virginica.

Species	Sepal Length (y_{ij})										$\sum_j y_{ij}$	$\sum_j y_{ij}^2$
Setosa	5.4	4.8	5.1	5.7	5.1	5.4	5.7	5.8	4.3	4.8	**52.1**	**273.53**
Versicolor	5.0	5.9	6.0	6.1	5.6	6.7	5.6	5.8	6.2	5.6	**58.5**	**344.07**
Virginica	7.4	7.9	6.4	6.3	6.1	7.7	6.3	6.4	6.0	6.9	**67.4**	**458.58**

$$\sum_i \sum_j y_{ij} = 178.0 \quad \sum_i \sum_j y_{ij}^2 = 1076.18$$

Consider the one-way analysis of variance model with equal numbers of observations $(j = 1, 2, \ldots, n)$ per treatment group $(i = 1, 2, \ldots, t)$:

$$Y_{ij} = \mu + \tau_i + e_{ij},, \quad i = 1, 2, \ldots, t \quad j = 1, 2, \ldots, n$$

(i) Let μ be the mean of the sepal lengths in the combined three populations. Carry out an appropriate two sided test of the hypothesis that $\mu = \mu_0 = 6$, with the assumption that the variance for the combined populations is $\sigma = 0.6913$. Clearly state your null and alternative hypotheses, along with your conclusions. [3 marks]

(ii) State the key assumptions underlining an ANOVA procedure. [2 marks]

(iii) Carry out a test of the hypothesis that the mean sepal length of each species is the same. Clearly state your null and alternative hypotheses, along with your conclusions. [5 marks]

Formulae for ST2054

Law of Total Probability: $P(A) = \sum_{i=1}^{k} P(A|E_i)P(E_i)$ when $\cup_{i=1}^{k} E_i = \Omega$ and all E_i are mutually exclusive.

$$\begin{array}{lll} X \sim Binomial(n,p) & E(X) = np & Var(X) = np(1-p) \\ U \sim Chi\text{-}Square(n) & E(X) = n & Var(X) = 2n \\ X \sim N(\mu, \sigma^2) & E(X) = \mu & Var(X) = \sigma^2 \\ X \sim Poisson(m) & E(X) = m & Var(X) = m \end{array}$$

Probability Distributions and MGFs:

Exponential pdf $f(x) = \lambda e^{-\lambda x}$ **Uniform pdf** $f(x) = \frac{1}{b-a}$

Binomial pmf $P[X = k] = \binom{n}{k} p^k (1-p)^{n-k}$

Multinomial pmf $P[X_1 = x_1, \ldots X_k = x_k] = \frac{n!}{x_1!, \ldots, x_k!} p_1^{x_1} \ldots p_k^{x_k}$, $\sum_{i=1}^{k} x_i = n$

Poisson pmf $P[X = k] = \frac{m^k e^{-m}}{k!}$ $MGF = e^{m(e^s - 1)}$

Gamma pdf $f(x) = \frac{1}{\Gamma(\alpha)} (\frac{x}{\beta})^{\alpha-1} e^{\frac{-x}{\beta}} \frac{1}{\beta}$ $MGF = \frac{1}{(1-\frac{s}{\beta})^\alpha}$

Normal pdf $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}$ $MGF = e^{s\mu + \frac{1}{2}s^2\sigma^2}$

Chebyshev's Inequality:

$p[|X - \mu| > k\sigma] \leq \frac{1}{k^2}$, where μ and σ^2 are the mean and variance of X.

Linear Regression

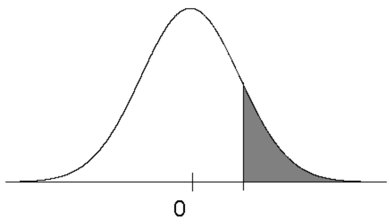
$\hat{\beta}_1 = \frac{SXY}{SXX} = \frac{\sum x_i y_i - n\bar{x}\bar{y}}{\sum x_i^2 - n(\bar{x})^2}$ $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$

Analysis of Variance:

$SST = (\sum_i \sum_j (y_{ij} - \bar{y})^2) = \sum_i \sum_j (y_{ij})^2 - \frac{(y_{..})^2}{n}$ $SSB = [\sum_i \frac{(y_{i.})^2}{n_i}] - \frac{(y_{..})^2}{n}$

Where $y_{i.} = \sum_{j=1}^{n_i} y_{ij}$ and $y_{..} = \sum_i \sum_j y_{ij}$

Table 1. Areas in the Tail of the Standard Normal Distribution

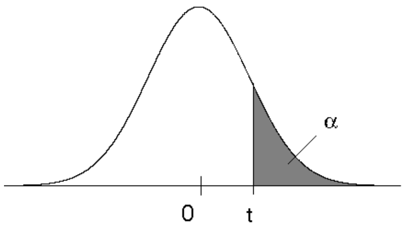


This table gives the probability that a standardised normal variable will be at least $\frac{x-\mu}{\sigma}$,

where μ is the mean and σ is the standard deviation of the normal variable.

	.00	**.01**	**.02**	**.03**	**.04**	**.05**	**.06**	**.07**	**.08**	**.09**
0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641
0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
2.0	.02275	.02222	.02169	.02118	.02068	.02018	.01970	.01923	.01876	.01831
2.1	.01786	.01743	.01700	.01659	.01618	.01578	.01539	.01500	.01463	.01426
2.2	.01390	.01355	.01321	.01287	.01255	.01222	.01191	.01160	.01130	.01101
2.3	.01072	.01044	.01017	.00990	.00964	.00939	.00914	.00889	.00866	.00842
2.4	.00820	.00798	.00776	.00755	.00734	.00714	.00695	.00676	.00657	.00639
2.5	.00621	.00604	.00587	.00570	.00554	.00539	.00523	.00508	.00494	.00480
2.6	.00466	.00453	.00440	.00427	.00415	.00402	.00391	.00379	.00368	.00357
2.7	.00347	.00336	.00326	.00317	.00307	.00298	.00289	.00280	.00272	.00264
2.8	.00256	.00248	.00240	.00233	.00226	.00219	.00212	.00205	.00199	.00193
2.9	.00187	.00181	.00175	.00169	.00164	.00159	.00154	.00149	.00144	.00139
3.0	.00135									
3.1	.00097									
3.2	.00069									
3.3	.00048									
3.4	.00034									
3.5	.00023									
3.6	.00016									
3.7	.00011									
3.8	.00007									
3.9	.00005									
4.0	.00003									

Table 2. Percentage Points of the t Distribution

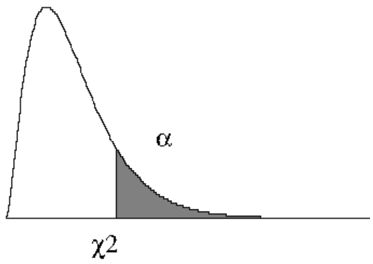


This table gives the value of the 100α percentage point of the t distribution with v degrees of freedom.

	α =	0.10	0.05	0.025	0.010	0.005	0.001	0.0005
v =	**1**	3.078	6.314	12.706	31.821	63.656	318.289	636.578
	2	1.886	2.920	4.303	6.965	9.925	22.328	31.600
	3	1.638	2.353	3.182	4.541	5.841	10.214	12.924
	4	1.533	2.132	2.776	3.747	4.604	7.173	8.610
	5	1.476	2.015	2.571	3.365	4.032	5.894	6.869
	6	1.440	1.943	2.447	3.143	3.707	5.208	5.959
	7	1.415	1.895	2.365	2.998	3.499	4.785	5.408
	8	1.397	1.860	2.306	2.896	3.355	4.501	5.041
	9	1.383	1.833	2.262	2.821	3.250	4.297	4.781
	10	1.372	1.812	2.228	2.764	3.169	4.144	4.587
	11	1.363	1.796	2.201	2.718	3.106	4.025	4.437
	12	1.356	1.782	2.179	2.681	3.055	3.930	4.318
	13	1.350	1.771	2.160	2.650	3.012	3.852	4.221
	14	1.345	1.761	2.145	2.624	2.977	3.787	4.140
	15	1.341	1.753	2.131	2.602	2.947	3.733	4.073
	16	1.337	1.746	2.120	2.583	2.921	3.686	4.015
	17	1.333	1.740	2.110	2.567	2.898	3.646	3.965
	18	1.330	1.734	2.101	2.552	2.878	3.610	3.922
	19	1.328	1.729	2.093	2.539	2.861	3.579	3.883
	20	1.325	1.725	2.086	2.528	2.845	3.552	3.850
	21	1.323	1.721	2.080	2.518	2.831	3.527	3.819
	22	1.321	1.717	2.074	2.508	2.819	3.505	3.792
	23	1.319	1.714	2.069	2.500	2.807	3.485	3.768
	24	1.318	1.711	2.064	2.492	2.797	3.467	3.745
	25	1.316	1.708	2.060	2.485	2.787	3.450	3.725
	26	1.315	1.706	2.056	2.479	2.779	3.435	3.707
	27	1.314	1.703	2.052	2.473	2.771	3.421	3.689
	28	1.313	1.701	2.048	2.467	2.763	3.408	3.674
	29	1.311	1.699	2.045	2.462	2.756	3.396	3.660
	30	1.310	1.697	2.042	2.457	2.750	3.385	3.646
	40	1.303	1.684	2.021	2.423	2.704	3.307	3.551
	60	1.296	1.671	2.000	2.390	2.660	3.232	3.460
	120	1.289	1.658	1.980	2.358	2.617	3.160	3.373
	∞	1.282	1.645	1.960	2.326	2.576	3.090	3.291

Table 3. Percentage Points of the χ^2 Distribution

This table gives the value of the 100α percentage point of the χ^2 distribution with v degrees of freedom.

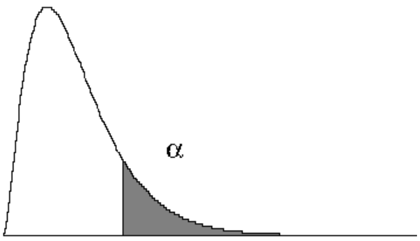


	α =	0.995	0.990	0.980	0.975	0.950	0.900	0.800	0.750	0.700	0.500
v =	1	0.0000	0.0002	0.0006	0.0010	0.0039	0.0158	0.0642	0.1015	0.1485	0.4549
	2	0.0100	0.0201	0.0404	0.0506	0.1026	0.2107	0.4463	0.5754	0.7133	1.3863
	3	0.0717	0.1148	0.1848	0.2158	0.3518	0.5844	1.0052	1.2125	1.4237	2.3660
	4	0.207	0.297	0.429	0.484	0.711	1.064	1.649	1.923	2.195	3.357
	5	0.412	0.554	0.752	0.831	1.145	1.610	2.343	2.675	3.000	4.351
	6	0.676	0.872	1.134	1.237	1.635	2.204	3.070	3.455	3.828	5.348
	7	0.989	1.239	1.564	1.690	2.167	2.833	3.822	4.255	4.671	6.346
	8	1.344	1.647	2.032	2.180	2.733	3.490	4.594	5.071	5.527	7.344
	9	1.735	2.088	2.532	2.700	3.325	4.168	5.380	5.899	6.393	8.343
	10	2.156	2.558	3.059	3.247	3.940	4.865	6.179	6.737	7.267	9.342
	11	2.603	3.053	3.609	3.816	4.575	5.578	6.989	7.584	8.148	10.341
	12	3.074	3.571	4.178	4.404	5.226	6.304	7.807	8.438	9.034	11.340
	13	3.565	4.107	4.765	5.009	5.892	7.041	8.634	9.299	9.926	12.340
	14	4.075	4.660	5.368	5.629	6.571	7.790	9.467	10.165	10.821	13.339
	15	4.601	5.229	5.985	6.262	7.261	8.547	10.307	11.037	11.721	14.339
	16	5.142	5.812	6.614	6.908	7.962	9.312	11.152	11.912	12.624	15.338
	17	5.697	6.408	7.255	7.564	8.672	10.085	12.002	12.792	13.531	16.338
	18	6.265	7.015	7.906	8.231	9.390	10.865	12.857	13.675	14.440	17.338
	19	6.844	7.633	8.567	8.907	10.117	11.651	13.716	14.562	15.352	18.338
	20	7.434	8.260	9.237	9.591	10.851	12.443	14.578	15.452	16.266	19.337
	21	8.034	8.897	9.915	10.283	11.591	13.240	15.445	16.344	17.182	20.337
	22	8.643	9.542	10.600	10.982	12.338	14.041	16.314	17.240	18.101	21.337
	23	9.260	10.196	11.293	11.689	13.091	14.848	17.187	18.137	19.021	22.337
	24	9.886	10.856	11.992	12.401	13.848	15.659	18.062	19.037	19.943	23.337
	25	10.520	11.524	12.697	13.120	14.611	16.473	18.940	19.939	20.867	24.337
	26	11.160	12.198	13.409	13.844	15.379	17.292	19.820	20.843	21.792	25.336
	27	11.808	12.878	14.125	14.573	16.151	18.114	20.703	21.749	22.719	26.336
	28	12.461	13.565	14.847	15.308	16.928	18.939	21.588	22.657	23.647	27.336
	29	13.121	14.256	15.574	16.047	17.708	19.768	22.475	23.567	24.577	28.336
	30	13.787	14.953	16.306	16.791	18.493	20.599	23.364	24.478	25.508	29.336
	40	20.707	22.164	23.838	24.433	26.509	29.051	32.345	33.660	34.872	39.335
	50	27.991	29.707	31.664	32.357	34.764	37.689	41.449	42.942	44.313	49.335
	60	35.534	37.485	39.699	40.482	43.188	46.459	50.641	52.294	53.809	59.335
	70	43.275	45.442	47.893	48.758	51.739	55.329	59.898	61.698	63.346	69.334
	80	51.172	53.540	56.213	57.153	60.391	64.278	69.207	71.145	72.915	79.334
	90	59.196	61.754	64.635	65.647	69.126	73.291	78.558	80.625	82.511	89.334
	100	67.328	70.065	73.142	74.222	77.929	82.358	87.945	90.133	92.129	99.334

	α =	0.300	0.250	0.200	0.100	0.050	0.025	0.020	0.010	0.005	0.001
v =	1	1.074	1.323	1.642	2.706	3.841	5.024	5.412	6.635	7.879	10.827
	2	2.408	2.773	3.219	4.605	5.991	7.378	7.824	9.210	10.597	13.815
	3	3.665	4.108	4.642	6.251	7.815	9.348	9.837	11.345	12.838	16.266
	4	4.878	5.385	5.989	7.779	9.488	11.143	11.668	13.277	14.860	18.466
	5	6.064	6.626	7.289	9.236	11.070	12.832	13.388	15.086	16.750	20.515
	6	7.231	7.841	8.558	10.645	12.592	14.449	15.033	16.812	18.548	22.457
	7	8.383	9.037	9.803	12.017	14.067	16.013	16.622	18.475	20.278	24.321
	8	9.524	10.219	11.030	13.362	15.507	17.535	18.168	20.090	21.955	26.124
	9	10.656	11.389	12.242	14.684	16.919	19.023	19.679	21.666	23.589	27.877
	10	11.781	12.549	13.442	15.987	18.307	20.483	21.161	23.209	25.188	29.588
	11	12.899	13.701	14.631	17.275	19.675	21.920	22.618	24.725	26.757	31.264
	12	14.011	14.845	15.812	18.549	21.026	23.337	24.054	26.217	28.300	32.909
	13	15.119	15.984	16.985	19.812	22.362	24.736	25.471	27.688	29.819	34.527
	14	16.222	17.117	18.151	21.064	23.685	26.119	26.873	29.141	31.319	36.124
	15	17.322	18.245	19.311	22.307	24.996	27.488	28.259	30.578	32.801	37.698
	16	18.418	19.369	20.465	23.542	26.296	28.845	29.633	32.000	34.267	39.252
	17	19.511	20.489	21.615	24.769	27.587	30.191	30.995	33.409	35.718	40.791
	18	20.601	21.605	22.760	25.989	28.869	31.526	32.346	34.805	37.156	42.312
	19	21.689	22.718	23.900	27.204	30.144	32.852	33.687	36.191	38.582	43.819
	20	22.775	23.828	25.038	28.412	31.410	34.170	35.020	37.566	39.997	45.314
	21	23.858	24.935	26.171	29.615	32.671	35.479	36.343	38.932	41.401	46.796
	22	24.939	26.039	27.301	30.813	33.924	36.781	37.659	40.289	42.796	48.268
	23	26.018	27.141	28.429	32.007	35.172	38.076	38.968	41.638	44.181	49.728
	24	27.096	28.241	29.553	33.196	36.415	39.364	40.270	42.980	45.558	51.179
	25	28.172	29.339	30.675	34.382	37.652	40.646	41.566	44.314	46.928	52.619
	26	29.246	30.435	31.795	35.563	38.885	41.923	42.856	45.642	48.290	54.051
	27	30.319	31.528	32.912	36.741	40.113	43.195	44.140	46.963	49.645	55.475
	28	31.391	32.620	34.027	37.916	41.337	44.461	45.419	48.278	50.994	56.892
	29	32.461	33.711	35.139	39.087	42.557	45.722	46.693	49.588	52.335	58.301
	30	33.530	34.800	36.250	40.256	43.773	46.979	47.962	50.892	53.672	59.702
	40	44.165	45.616	47.269	51.805	55.758	59.342	60.436	63.691	66.766	73.403
	50	54.723	56.334	58.164	63.167	67.505	71.420	72.613	76.154	79.490	86.660
	60	65.226	66.981	68.972	74.397	79.082	83.298	84.580	88.379	91.952	99.608
	70	75.689	77.577	79.715	85.527	90.531	95.023	96.387	100.425	104.215	112.317
	80	86.120	88.130	90.405	96.578	101.879	106.629	108.069	112.329	116.321	124.839
	90	96.524	98.650	101.054	107.565	113.145	118.136	119.648	124.116	128.299	137.208
	100	106.906	109.141	111.667	118.498	124.342	129.561	131.142	135.807	140.170	149.449

Table 4. Percentage Points of the F Distribution

This table gives the value of the 100α percentage point of the F distribution with v_1 degrees of freedom in the numerator and v_2 degrees of freedom in the denominator.
F is tabulated for α = 0.050, 0.025 and 0.010, respectively.



	v_1														
v_2	1	2	3	4	5	6	7	8	9	10	12	16	20	24	∞
1	161	199	216	225	230	234	237	239	241	242	244	246	248	249	254
	648	799	864	900	922	937	948	957	963	969	977	987	993	997	1018
	4052	4999	5404	5624	5764	5859	5928	5981	6022	6056	6107	6170	6209	6234	6366
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.41	19.43	19.45	19.45	19.50
	38.51	39.00	39.17	39.25	39.30	39.33	39.36	39.37	39.39	39.40	39.41	39.44	39.45	39.46	39.50
	98.50	99.00	99.16	99.25	99.30	99.33	99.36	99.38	99.39	99.40	99.42	99.44	99.45	99.46	99.50
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.69	8.66	8.64	8.53
	17.44	16.04	15.44	15.10	14.88	14.73	14.62	14.54	14.47	14.42	14.34	14.23	14.17	14.12	13.90
	34.12	30.82	29.46	28.71	28.24	27.91	27.67	27.49	27.34	27.23	27.05	26.83	26.69	26.60	26.13
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.84	5.80	5.77	5.63
	12.22	10.65	9.98	9.60	9.36	9.20	9.07	8.98	8.90	8.84	8.75	8.63	8.56	8.51	8.26
	21.20	18.00	16.69	15.98	15.52	15.21	14.98	14.80	14.66	14.55	14.37	14.15	14.02	13.93	13.46
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.68	4.60	4.56	4.53	4.37
	10.01	8.43	7.76	7.39	7.15	6.98	6.85	6.76	6.68	6.62	6.52	6.40	6.33	6.28	6.02
	16.26	13.27	12.06	11.39	10.97	10.67	10.46	10.29	10.16	10.05	9.89	9.68	9.55	9.47	9.02
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.92	3.87	3.84	3.67
	8.81	7.26	6.60	6.23	5.99	5.82	5.70	5.60	5.52	5.46	5.37	5.24	5.17	5.12	4.85
	13.75	10.92	9.78	9.15	8.75	8.47	8.26	8.10	7.98	7.87	7.72	7.52	7.40	7.31	6.88
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.49	3.44	3.41	3.23
	8.07	6.54	5.89	5.52	5.29	5.12	4.99	4.90	4.82	4.76	4.67	4.54	4.47	4.41	4.14
	12.25	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72	6.62	6.47	6.28	6.16	6.07	5.65
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.20	3.15	3.12	2.93
	7.57	6.06	5.42	5.05	4.82	4.65	4.53	4.43	4.36	4.30	4.20	4.08	4.00	3.95	3.67
	11.26	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91	5.81	5.67	5.48	5.36	5.28	4.86
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	2.99	2.94	2.90	2.71
	7.21	5.71	5.08	4.72	4.48	4.32	4.20	4.10	4.03	3.96	3.87	3.74	3.67	3.61	3.33
	10.56	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35	5.26	5.11	4.92	4.81	4.73	4.31
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.91	2.83	2.77	2.74	2.54
	6.94	5.46	4.83	4.47	4.24	4.07	3.95	3.85	3.78	3.72	3.62	3.50	3.42	3.37	3.08
	10.04	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94	4.85	4.71	4.52	4.41	4.33	3.91
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.79	2.70	2.65	2.61	2.40
	6.72	5.26	4.63	4.28	4.04	3.88	3.76	3.66	3.59	3.53	3.43	3.30	3.23	3.17	2.88
	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63	4.54	4.40	4.21	4.10	4.02	3.60
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.60	2.54	2.51	2.30
	6.55	5.10	4.47	4.12	3.89	3.73	3.61	3.51	3.44	3.37	3.28	3.15	3.07	3.02	2.73
	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39	4.30	4.16	3.97	3.86	3.78	3.36
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.60	2.51	2.46	2.42	2.21
	6.41	4.97	4.35	4.00	3.77	3.60	3.48	3.39	3.31	3.25	3.15	3.03	2.95	2.89	2.60
	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.19	4.10	3.96	3.78	3.66	3.59	3.17
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.53	2.44	2.39	2.35	2.13
	6.30	4.86	4.24	3.89	3.66	3.50	3.38	3.29	3.21	3.15	3.05	2.92	2.84	2.79	2.49
	8.86	6.51	5.56	5.04	4.69	4.46	4.28	4.14	4.03	3.94	3.80	3.62	3.51	3.43	3.00

v_2 \ v_1	1	2	3	4	5	6	7	8	9	10	12	16	20	24	∞
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.48	2.38	2.33	2.29	2.07
	6.20	4.77	4.15	3.80	3.58	3.41	3.29	3.20	3.12	3.06	2.96	2.84	2.76	2.70	2.40
	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89	3.80	3.67	3.49	3.37	3.29	2.87
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.42	2.33	2.28	2.24	2.01
	6.12	4.69	4.08	3.73	3.50	3.34	3.22	3.12	3.05	2.99	2.89	2.76	2.68	2.63	2.32
	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78	3.69	3.55	3.37	3.26	3.18	2.75
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.34	2.25	2.19	2.15	1.92
	5.98	4.56	3.95	3.61	3.38	3.22	3.10	3.01	2.93	2.87	2.77	2.64	2.56	2.50	2.19
	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.60	3.51	3.37	3.19	3.08	3.00	2.57
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.28	2.18	2.12	2.08	1.84
	5.87	4.46	3.86	3.51	3.29	3.13	3.01	2.91	2.84	2.77	2.68	2.55	2.46	2.41	2.09
	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46	3.37	3.23	3.05	2.94	2.86	2.42
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.23	2.13	2.07	2.03	1.78
	5.79	4.38	3.78	3.44	3.22	3.05	2.93	2.84	2.76	2.70	2.60	2.47	2.39	2.33	2.00
	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35	3.26	3.12	2.94	2.83	2.75	2.31
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.18	2.09	2.03	1.98	1.73
	5.72	4.32	3.72	3.38	3.15	2.99	2.87	2.78	2.70	2.64	2.54	2.41	2.33	2.27	1.94
	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26	3.17	3.03	2.85	2.74	2.66	2.21
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.15	2.05	1.99	1.95	1.69
	5.66	4.27	3.67	3.33	3.10	2.94	2.82	2.73	2.65	2.59	2.49	2.36	2.28	2.22	1.88
	7.72	5.53	4.64	4.14	3.82	3.59	3.42	3.29	3.18	3.09	2.96	2.78	2.66	2.58	2.13
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	2.12	2.02	1.96	1.91	1.65
	5.61	4.22	3.63	3.29	3.06	2.90	2.78	2.69	2.61	2.55	2.45	2.32	2.23	2.17	1.83
	7.64	5.45	4.57	4.07	3.75	3.53	3.36	3.23	3.12	3.03	2.90	2.72	2.60	2.52	2.06
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.09	1.99	1.93	1.89	1.62
	5.57	4.18	3.59	3.25	3.03	2.87	2.75	2.65	2.57	2.51	2.41	2.28	2.20	2.14	1.79
	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07	2.98	2.84	2.66	2.55	2.47	2.01
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.00	1.90	1.84	1.79	1.51
	5.42	4.05	3.46	3.13	2.90	2.74	2.62	2.53	2.45	2.39	2.29	2.15	2.07	2.01	1.64
	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89	2.80	2.66	2.48	2.37	2.29	1.80
50	4.03	3.18	2.79	2.56	2.40	2.29	2.20	2.13	2.07	2.03	1.95	1.85	1.78	1.74	1.44
	5.34	3.97	3.39	3.05	2.83	2.67	2.55	2.46	2.38	2.32	2.22	2.08	1.99	1.93	1.55
	7.17	5.06	4.20	3.72	3.41	3.19	3.02	2.89	2.78	2.70	2.56	2.38	2.27	2.18	1.68
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.92	1.82	1.75	1.70	1.39
	5.29	3.93	3.34	3.01	2.79	2.63	2.51	2.41	2.33	2.27	2.17	2.03	1.94	1.88	1.48
	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72	2.63	2.50	2.31	2.20	2.12	1.60
100	3.94	3.09	2.70	2.46	2.31	2.19	2.10	2.03	1.97	1.93	1.85	1.75	1.68	1.63	1.28
	5.18	3.83	3.25	2.92	2.70	2.54	2.42	2.32	2.24	2.18	2.08	1.94	1.85	1.78	1.35
	6.90	4.82	3.98	3.51	3.21	2.99	2.82	2.69	2.59	2.50	2.37	2.19	2.07	1.98	1.43
∞	3.84	3.00	2.61	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.75	1.64	1.57	1.52	1.02
	5.03	3.69	3.12	2.79	2.57	2.41	2.29	2.19	2.11	2.05	1.95	1.80	1.71	1.64	1.03
	6.64	4.61	3.78	3.32	3.02	2.80	2.64	2.51	2.41	2.32	2.19	2.00	1.88	1.79	1.04

Note:

$F_{1-\alpha\ v1,v2} = 1/F_{\alpha\ v2\ v1}$

Example:
$F_{0.975,\ 6,\ 8} = 1/F_{0.025,\ 8,\ 6} = 1/5.60 = 0.179$

Table: Q scores for Tukey's method

$\alpha = 0.05$

k / df	2	3	4	5	6	7	8	9	10
1	18.0	27.0	32.8	37.1	40.4	43.1	45.4	47.4	49.1
2	6.08	8.33	9.80	10.88	11.73	12.43	13.03	13.54	13.99
3	4.50	5.91	6.82	7.50	8.04	8.48	8.85	9.18	9.46
4	3.93	5.04	5.76	6.29	6.71	7.05	7.35	7.60	7.83
5	3.64	4.60	5.22	5.67	6.03	6.33	6.58	6.80	6.99
6	3.46	4.34	4.90	5.30	5.63	5.90	6.12	6.32	6.49
7	3.34	4.16	4.68	5.06	5.36	5.61	5.82	6.00	6.16
8	3.26	4.04	4.53	4.89	5.17	5.40	5.60	5.77	5.92
9	3.20	3.95	4.41	4.76	5.02	5.24	5.43	5.59	5.74
10	3.15	3.88	4.33	4.65	4.91	5.12	5.30	5.46	5.60
11	3.11	3.82	4.26	4.57	4.82	5.03	5.20	5.35	5.49
12	3.08	3.77	4.20	4.51	4.75	4.95	5.12	5.27	5.39
13	3.06	3.73	4.15	4.45	4.69	4.88	5.05	5.19	5.32
14	3.03	3.70	4.11	4.41	4.64	4.83	4.99	5.13	5.25
15	3.01	3.67	4.08	4.37	4.59	4.78	4.94	5.08	5.20
16	3.00	3.65	4.05	4.33	4.56	4.74	4.90	5.03	5.15
17	2.98	3.63	4.02	4.30	4.52	4.70	4.86	4.99	5.11
18	2.97	3.61	4.00	4.28	4.49	4.67	4.82	4.96	5.07
19	2.96	3.59	3.98	4.25	4.47	4.65	4.79	4.92	5.04
20	2.95	3.58	3.96	4.23	4.45	4.62	4.77	4.90	5.01
24	2.92	3.53	3.90	4.17	4.37	4.54	4.68	4.81	4.92
30	2.89	3.49	3.85	4.10	4.30	4.46	4.60	4.72	4.82
40	2.86	3.44	3.79	4.04	4.23	4.39	4.52	4.63	4.73
60	2.83	3.40	3.74	3.98	4.16	4.31	4.44	4.55	4.65
120	2.80	3.36	3.68	3.92	4.10	4.24	4.36	4.47	4.56
∞	2.77	3.31	3.63	3.86	4.03	4.17	4.29	4.39	4.47