→ Aadher

$n = 100$
$p = .0278$

k	Binomial Probability	Poisson Approximation ← $m = 2.78$
0	.0596	.0620
1	.1705	.1725
2	.2414	.2397
3	.2255	.2221
4	.1564	.1544
5	.0858	.0858
6	.0389	.0398
7	.0149	.0158
8	.0050	.0055
9	.0015	.0017
10	.0004	.0005
11	.0001	.0001

The approximation is quite good. ■

The Poisson frequency function can be used to approximate binomial probabilities for large n and small p. This suggests how Poisson distributions can arise in practice. Suppose that X is a random variable that equals the number of times some event occurs in a given interval of time. Heuristically, let us think of dividing the interval into a very large number of small subintervals of equal length, and let us assume that the subintervals are so small that the probability of more than one event in a subinterval is negligible relative to the probability of one event, which is itself very small. Let us also assume that the probability of an event is the same in each subinterval and that whether an event occurs in one subinterval is independent of what happens in the other subintervals. The random variable X is thus nearly a binomial random variable, with the subintervals consitituting the trials, and, from the limiting result above, X has nearly a Poisson distribution.

The preceding argument is not formal, of course, but merely suggestive. But, in fact, it can be made rigorous. The important assumptions underlying it are (1) what happens in one subinterval is independent of what happens in any other subinterval, (2) the probability of an event is the same in each subinterval, and (3) events do not happen simultaneously. The same kind of argument can be made if we are concerned with an area or a volume of space rather than with an interval on the real line.

The Poisson distribution is of fundamental theoretical and practical importance. It has been used in many areas, including the following:

- The Poisson distribution has been used in the analysis of telephone systems. The number of calls coming into an exchange during a unit of time might be modeled as a Poisson variable if the exchange services a large number of customers who act more or less independently.
- One of the earliest uses of the Poisson distribution was to model the number of alpha particles emitted from a radioactive source during a given period of time.
- The Poisson distribution has been used as a model by insurance companies. For example, the number of freak acidents, such as falls in the shower, for a large

population of people in a given time period might be modeled as a Poisson distribution, since the accidents would presumably be rare and independent (provided there was only one person in the shower.)

- The Poisson distribution has been used by traffic engineers as a model for light traffic. The number of vehicles that pass a marker on a roadway during a unit of time can be counted. If traffic is light, the individual vehicles act independently of each other. In heavy traffic, however, one vehicle's movement may influence another's, so the approximation might not be good.

EXAMPLE B hardly This amusing classical example is from von Bortkiewicz (1898). The number of fatalities that resulted from being kicked by a horse was recorded for 10 corps of Prussian cavalry over a period of 20 years, giving 200 corps-years worth of data. These data and the probabilities from a Poisson model with $\lambda = .61$ are displayed in the following table. The first column of the table gives the number of deaths per year, ranging from 0 to 4. The second column lists how many times that number of deaths was observed. Thus, for example, in 65 of the 200 corps-years, there was one death. In the third column of the table, the observed numbers are converted to relative frequencies by dividing them by 200. The fourth column of the table gives Poisson probabilities with the parameter $\lambda = .61$. In chapters 8 and 9 we discuss how to choose a parameter value to fit a theoretical probability model to observed frequencies and methods for testing goodness of fit. For now we will just remark that the value $\lambda = .61$ was chosen to match the average number of deaths per year.

Number of Deaths per Year	*Observed*	*Relative Frequency*	*Poisson Probability*
0	109	.545	.543
1	65	.325	.331
2	22	.110	.101
3	3	.015	.021
4	4	.005	.003

■

The Poisson distribution often arises from a model called a **Poisson process** for the distribution of random events in a set S, which is typically one-, two-, or three-dimensional, corresponding to time, a plane, or a volume of space. Basically, this model states that if $S_1, S_2, \ldots, S_n$ are disjoint subsets of S, then the numbers of events in these subsets, $N_1, N_2, \ldots, N_n$, are independent random variables that follow Poisson distributions with parameters $\lambda|S_1|, \lambda|S_2|, \ldots, \lambda|S_n|$, where $|S_i|$ denotes the measure of S_i (length, area, or volume, for example). The crucial assumptions here are that events in disjoint subsets are independent of each other and that the Poisson parameter for a subset is proportional to the subset's size. Later, we will see that this latter assumption implies that the average number of events in a subset is proportional to its size.

EXAMPLE C Suppose that an office receives telephone calls as a Poisson process with $\lambda = .5$ per min. The number of calls in a 5-min. interval follows a Poisson distribution with