Maxim Chopivskyy

118364841

Liang Chen

ST2054

Assignment 7

Q1 (a)

We have $m_k = E[X^k]$

$$E[e^{\theta X}] = \sum_{k=0}^{\infty} \frac{1}{k!} m_k(X)\, \theta^k$$

$$= 1 + \underbrace{\sum_{k=1}^{\infty} \frac{1}{k!} m_k(X)\, \theta^k}_{= T(\theta)}$$

$\log(1 + T(\theta))$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot T(\theta)^n = k_X(\theta)$$

We have:

(1) $k_X(\theta) = \sum_{n=1}^{\infty} \frac{1}{n!} k_n\, \theta^n$

and

(2) $k_X(\theta) = \sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r} \cdot T(\theta)^r$

$$= \sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r} \left(\sum_{k=1}^{\infty} \frac{1}{k!} m_k\, \theta^k \right)^r$$

We use polynomial coefficient matching to match coefficients of θ in above expressions

Maxim Chopivskyy
17836487

Liang Chen
ST2054

Q1 (a)

(i) We match θ^1 coefficients:

① for $n = 1$ we have:

$$\frac{1}{1} k_1 \theta^1$$

$$\boxed{= k_1 \theta}$$

② for ~~m~~ $r=1$:

$$\frac{1}{1} \left(\sum_{k=1}^{\infty} \frac{1}{k!} m_k \theta^k \right)^1$$

for $k = 1$:

$$\frac{1}{1} m_1 \theta^1$$

$$\boxed{= m_1 \theta}$$

matched:

$$k_1 \theta = m_1 \theta$$

$$k_1 = m_1$$

(ii) θ^2 coefficient

① For $n=2$:

$$\boxed{\frac{1}{2} k_2 \theta^2}$$

② for $r=1$, $k=2$:

$$\boxed{\frac{1}{2} m_2 \theta^2}$$

for $r=2$:

$$-\frac{1}{2} \left(\sum_{k=1}^{\infty} \frac{1}{k!} m_k \theta^k \right)^2$$

Maxim Chepinsky
11836484l

Liang Chen
ST2054

Q1(a)

$$-\frac{1}{2}\left(\frac{1}{1}m_1\vartheta' + \dots\right)\left(\frac{1}{1}m_1\vartheta' + \dots\right)$$

$$= \boxed{-\frac{1}{2}m_1^2\Theta^2}$$

matched

$$\frac{1}{2}k_2\Theta^2 = \frac{1}{2}m_2\vartheta^2 - \frac{1}{2}m_1^2\vartheta^2$$

$$\Downarrow$$

$$k_2 = m_2 - m_1^2$$

(iii) ϑ^3 coefficient

for $n=2$

$$\boxed{\frac{1}{6}k_3\Theta^3}$$

for $n=1$, $k=3$

$$\frac{1}{6}m_3\vartheta^3$$

for $n=2$

$$-\frac{1}{2}\left(\frac{1}{2}m_1m_2\vartheta^3 + \frac{1}{2}m_1m_2\Theta^3\right)$$

$$= \boxed{-\frac{1}{2}m_1m_2\vartheta^3}$$

For $n=3$

$$\frac{1}{3}(m_1\Theta + \dots)(m_1\vartheta + \dots)(m_1\Theta + \dots)$$

$$= \frac{1}{3}m_1^3\Theta^3$$

matched

Maxim Chopivskyy
11836484l

Liang Chen
ST2054

Q1(a)

~~mfdk~~ matched

$$\frac{1}{6}k_3\theta^3 = \frac{1}{6}m_3\theta^3 - \frac{1}{2}m_1m_2\theta^3 + \frac{1}{3}m_1^3\theta^3$$

$$k_3 = m_3 - 3m_1m_2 + 2m_1^3$$

Conclusion

~~k/k m,~~

$$k_1(X) = m_1(X)$$

$$k_2(X) = m_2(X) - m_1(X)^2$$

$$k_3(X) = m_3(X) - 3m_1(X)m_2(X) + 2m_1(X)^3$$

~~Q1~~ Q1 (b)

$$K_{X+Y}(\theta) = \sum_{n=1}^{\infty} \frac{1}{n!} k_n(X+Y)\,\theta^n$$

|| IF indep

$$\log\left[E(e^{\theta X})E(e^{\theta Y})\right]$$

||

$$k_X(\theta) \cancel{\quad} + k_Y(\theta) = \sum_{n=1}^{\infty}\frac{1}{n!}k_n(X)\theta^n + \sum_{m}^{\infty}\frac{1}{m!}k_m(Y)\theta^m$$

||

$$\sum_{n=1}^{\infty}\frac{1}{n!}k_n(X+Y)\theta^n$$

hence claim is true

Maxim Chopivskyy
118364841

Liang Chen
ST2054

Q2

$$\phi_Y(t) = E[e^{itY}]$$
$$= E[e^{it(X_1^2+X_2^2 \cdots + X_n^2)}]$$
$$= E[e^{itX_1^2}] \cdot E[e^{itX_2^2}] \cdots E[e^{itX_n^2}]$$

$$\phi_{X_i^2}(t) = E[e^{itX_i^2}]$$
$$= \int_{-\infty}^{\infty} e^{itx^2} \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}x^2}$$

$$= \frac{1}{\sqrt{1-2it}} e^{\frac{u^2 it}{1-2it}}$$

$$\phi_Y(t) = \prod_{i=1}^{n} \phi_{X_i^2}(t)$$
$$= \prod_{i=1}^{n} \left[\frac{1}{\sqrt{1-2it}} e^{\left(\frac{u_i^2 it}{1-2it}\right)} \right]$$

$e^{\left(\frac{u_i^2 it}{1-2it}\right)} \rightarrow e^{\left(\frac{(u_1^2+u_2^2 \ldots) it}{1-2it}\right)}$

$\sqrt{1-2it} \rightarrow (1-2it)^{\frac{n}{2}}$

$$= \frac{1}{(1-2it)^{\frac{n}{2}}} \cdot e^{\left(\frac{\theta it}{1-2it}\right)}$$

Maxim Chopivskyy

118364841

Liang Chen

ST2054

Q 3

We use transformation techniques:

(i)

Density Function of X:

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

$$Y = h(X) = e^X$$

$$X = h^{-1}(Y) = \ln Y$$

We see e^x is a monotonically increasing Function.

$$\frac{dx}{dy} = \frac{d h^{-1}(y)}{dy} = \frac{d \ln Y}{dy}$$

$$= \frac{1}{y}$$

Density of Y:

$$g(y) = f_X(x) \cdot \frac{dx}{dy}$$

$$= f_X(\log y) \cdot \frac{1}{y}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\log y)^2} \cdot \frac{1}{y}$$

$$= \frac{1}{y\sqrt{2\pi}} e^{-\frac{1}{2}(\log y)^2}$$

(ii)

$$\int_{-\infty}^{\infty} \{1 + a \sin(2\pi \log x)\} F(x) \, dx$$

must equal 1

Maxim Chopivskyy

118364841

Liang Chen

ST2054

Q3 (ii)

$$\underbrace{\int_{-\infty}^{\infty} F(x)\,dx}_{=1} + \underbrace{\int_{-\infty}^{\infty} a\sin(2\pi \log x)\,F(x)\,dx}_{=0}$$

because sin is an odd function
$(\sin(-x) = -\sin(x))$

$$= 1 + 0 = \boxed{1}$$

For k^{th} moment of f_a equals:

$$\int_{-\infty}^{\infty} x^k f_a(x)$$

$$= \int_{-\infty}^{\infty} x^k F(x)\,dx + \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} a\sin(2\pi y)\,e^{ky-\frac{1}{2}y^2}\,dy}_{=0}$$

because sin is odd

$$= \int_{-\infty}^{\infty} x^k F(x)\,dx$$

Same moments as log-distribution

Maxim Chopivsky

11856481

Liang Chen

ST2054

QB4

(a) For X_n , $\frac{1}{2}$ chance $=\left(\frac{1}{2}X_{n-1}\right)$, $\frac{1}{2}$ chance $=\left(\frac{1}{2}X_{n-1} + Y_{n-1}\right)$

$$X_n = \begin{cases} \frac{1}{2}X_{n-1} & \text{with prob } \frac{1}{2} \\ \frac{1}{2}X_{n-1} + Y_{n-1} & \text{with prob } \frac{1}{2} \end{cases}$$

Probability mass function of X_n :

$$P(X_n = x_{nk}) = P(X_{n-1} = x_{(n-1)k}) \ldots = P(X_1 = x_{1k})$$

(b) $$M_{X_n}(t) = E[e^{tX_n}] \cancel{= E\left[e^{t\left(\frac{1}{2}\frac{1}{2}X_{n-1} + \frac{1}{2}\left(\frac{1}{2}X_{n-1} + Y_{n-1}\right)\right)}\right]}$$

$$\cancel{= E\left[e^{\frac{1}{2}t\left(\frac{1}{2}X_{n-1}\right)} e^{\frac{1}{2}t\left(\frac{1}{2}X_{n-1} + Y_{n-1}\right)}\right]}$$

(1) $$= \frac{1}{2}E\left[e^{t\left(\frac{1}{2}X_{n-1}\right)}\right] + \frac{1}{2}E\left[e^{t\left(\frac{1}{2}X_{n-1} + Y_{n-1}\right)}\right]$$

half chance each

$$\cancel{= \frac{1}{2}M_{X_{n-1}}\left(\frac{1}{2}t\right) + \frac{1}{2}}$$

$$\frac{1}{2}E\left[e^{t\left(\frac{1}{2}X_{n-1} + Y_{n-1}\right)}\right] = \frac{1}{2}E\left[e^{\frac{1}{2}tX_{n-1}}\right]E\left[e^{tY_{n-1}}\right]$$

$$= \frac{1}{2}M_{X_{n-1}}\left(\frac{1}{2}t\right) \cdot \frac{\lambda}{\lambda - t}$$ ~ exponential MGF

(1) $$= \frac{1}{2}M_{X_{n-1}}\left(\frac{1}{2}t\right)\left\{1 + \frac{\lambda}{\lambda - t}\right\}$$

Maxim Chopivskyy
118364841

Q4 (b)

$$= M_{X_{n-1}}\left(\tfrac{1}{2}t\right)\left\{\frac{\lambda-\frac{1}{2}t}{\lambda-t}\right\}$$

$$= M_{X_{n-2}}\left(\tfrac{1}{4}t\right)\left\{\frac{\lambda-\frac{1}{4}t}{\lambda-t}\right\}$$

$$\vdots$$

approach zero

$$= M_{X_1}\left(\frac{1}{2^{n-1}}t\right)\left\{\frac{\lambda-\frac{1}{2^{n-1}}t}{\lambda-t}\right\}$$

as $n \to \infty$ then
~~M_{X_n}~~ M_{X_n} ~~approaches~~ MGF approaches

$$M_{X_1}(0)\left\{\frac{\lambda-0t}{\lambda-t}\right\}$$

$$= \boxed{\frac{\lambda}{\lambda-t}}$$

It approaches the exponential distribution