Question Bank 2

Question 1

There are n urns of which the r^{th} urn contains $r-1$ red balls and $n-r$ magenta balls. You pick an urn at random and remove two balls at random without replacement. Find the probability that:

(i) the second ball is magenta;

(ii) the second ball is magenta, given that the first is magenta. (Hint: $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ and $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$)

Solution:

(i) Method 1: Denote as C_i the color of the i^{th} pick.

$$
\begin{aligned}
P(C_2 = M) &= P(C_1 = M, C_2 = M) + P(C_1 = R, C_2 = M) && (0.0.1)\\
P(C_1 = R, C_2 = M) &= \frac{\sum_{r=1}^{n}(r-1)(n-r)}{n(n-1)(n-2)} && (0.0.2)\\
&= \frac{\sum_{r=1}^{n}[(r-1)n - r(r-1)]}{n(n-1)(n-2)} && (0.0.3)\\
&= \frac{n\sum_{r=1}^{n}(r-1) - \sum_{r=1}^{n}(r^2-r)}{n(n-1)(n-2)} && (0.0.4)\\
&= \frac{n\left(\frac{n(n+1)}{2} - n\right) - \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}}{n(n-1)(n-2)} && (0.0.5)\\
&= \frac{3n(n+1)^2 - 6n^2 - n(n+1)(2n+1)}{6n(n-1)(n-2)} && (0.0.6)\\
&= \frac{n^2-3n+2}{6(n-1)(n-2)} = \frac{1}{6} && (0.0.7)\\
P(C_1 = M, C_2 = M) &= \frac{\sum_{r=1}^{n}(n-r)(n-r-1)}{n(n-1)(n-2)} && (0.0.8)\\
&= \frac{\sum_{r=1}^{n}[(n-r)^2 - (n-r)]}{n(n-1)(n-2)}, \text{ variable chaneg, let k = n-r} && (0.0.9)\\
&= \sum_{k=0}^{n-1}(k^2-k) && (0.0.10)\\
&= \sum_{k=0}^{n-1}k^2 - \sum_{k=0}^{n-1}k && (0.0.11)\\
&= \frac{(n-1)n(2n-1)}{6} - \frac{n(n-1)}{2} && (0.0.12)\\
&= \frac{2n-1-3}{6(n-2)} && (0.0.13)\\
&= \frac{2(n-2)}{6(n-2)} = \frac{1}{3} && (0.0.14)\\
&\Rightarrow\ P(C_2 = M) = \frac{1}{3} + \frac{1}{6} = \frac{1}{2} && (0.0.15)
\end{aligned}
$$

Method 2: It is important to notice that the total number of balls in any urn is $r-1+n-r = n-1$. Therefore there are in total $n(n-1)$ balls in n urns. Denote as S_r and S_m the total number of red and magenta balls in these n urns. We can see

$$
\begin{aligned}
S_r &= 0 + 1 + 2 + \cdots + n - 1 && (0.0.16)\\
S_m &= n - 1 + n - 2 + \cdots + 0 && (0.0.17)
\end{aligned}
$$

and S_r is actually identical with S_m. Therefore there are $\frac{1}{2}n(n-1)$ of red and magenta balls respectively out of $n(n-1)$ balls and the probability of picking one red or magenta ball is $\frac{1}{2}$. Given we randomly pick an urn and each urn contains the same number of balls, it is equivalent to pick two balls out of $n(n-1)$ balls and each ball can be equally likely picked. Let C_i be the colour of the i^{th} pick. The first pick can be either red or magenta.

$$P(C_2 = M) = \frac{\frac{1}{2}n(n-1)}{n(n-1)} \times \frac{\frac{1}{2}n(n-1)}{n(n-1)-1} + \frac{\frac{1}{2}n(n-1)}{n(n-1)} \times \frac{\frac{1}{2}n(n-1)-1}{n(n-1)-1} \quad (0.0.18)$$

$$= \frac{1}{2} \quad (0.0.19)$$

In the above equation, the first part is the probability that Red (1st) and Magenta (2nd) balls are picked, the second part represents the probability that magenta (1st) and magenta (2nd) are picked.

(ii) Let's start with the Bays rule, the probability that the second ball is magenta, given the first ball is also magenta

$$P(C_2 = M|C_1 = M) = \frac{P(C_1, C_2 = M)}{P(C_1 = M)} = 2P(C_1, C_2 = M) = \frac{2}{3} \quad (0.0.20)$$

$$(0.0.21)$$

Note that $P(C_1 = M) = \frac{1}{2}$ is obtained by following the idea of either Method 1 or Method 2 in (i). For instance, according to Method 1,

$$P(C_1 = M) = \frac{\sum_{r=1}^{n}(n-r)}{n(n-1)} \quad (0.0.22)$$

$$= \frac{n^2 - \frac{n(n+1)}{2}}{n(n-1)} \quad (0.0.23)$$

$$= \frac{2n^2 - n^2 - n}{2n(n-1)} = \frac{1}{2} \quad (0.0.24)$$

Question 2

A Beacon Tower was built at the top of the mountain to give alarm by smoke e.g. Gondor alarmed the invasion of the Orcs to the Rohan in The Lord of the Rings. If invading enemy forces were spotted during the daytime, the patrol in one tower would burn wolf manure immediately. The smoke (also known as wolf smoke) could then be visible to the next tower, where another wolf smoke would be set. Information of an invasion was therefore delivered from one tower to the next.

Alpha1, Beta2, Charlie3 and Delta4 are four beacon towers built strategically such that none of any three towers lie in a straight line. The visibility between any two towers can be blocked by thick fog with a probability p. Fog between each pair take place independently. If any of the tower starts the wolf smoke, it can only be seen by another with no fog in between. If Alpha1 spots an invasion, what is the probability that

(i) Delta4 is aware of the invasion given fog between Delta4 and Alpha1.

(ii) Delta4 is aware of the invasion given fog between Beta2 and Charlie3.

(iii) Delta4 is aware of the invasion given fog between Beta2 and Alpha1.

(iii) Delta4 is aware of the invasion.

Solution: This is a very very very (repeat by three times to emphasis) challenging question. A good understanding of the complement of an event is critical for solving this question. Denote as, say $A \leftrightarrow D$ for information is transferred from A to D, EF for the event that there is no fog between E and F. Note that AD^c does not necessarily imply $A \not\leftrightarrow D$ as information can be delivered by other route, e.g. ABD, ACD.

(i) Need to evaluate the probability for the event $A \leftrightarrow D|AD^c$. The possible routines are ABC, ACD or $ABCD$. Whether the information is transferred by two stages or three stages depends on if there is fog between B and C, i.e. BC with probability $(1-p)$ or BC^c with probability p.

Consider there is no fog between B and C with probability $(1-p)$. Now as long as the fire on A can be successfully seen by either B or C or both, i.e. in Stage 1: at least one of AC and AB, plus D can successfully see the fire from either B or C, i.e. in Stage 2: at least one of the BD and CD, information is then delivered from A to D, hence (at least one of the AC, AB) $\cap$ (at least one of the BD, CD). Given fog occurs independently,

$$
\begin{aligned}
P(\text{at least one of the } AC, AB) &= 1 - P(\text{none of the } AC, BC) && (0.0.25)\\
&= 1 - P(AC^c \cap BC^c) && (0.0.26)\\
&= 1 - P(AC^c) \times P(BC^c) && (0.0.27)\\
&= 1 - p^2 && (0.0.28)
\end{aligned}
$$

Similarly $P(\text{at least one of the } BD, CD) = 1 - p^2$. Again fogs occurs independently in stage 1 and 2,

$$
\begin{aligned}
& P(\text{at least one of the } AC, AB \cap \text{at least one of the } BD, CD) && (0.0.29)\\
= {} & P(\text{at least one of the } AC, AB) \times P(\text{at least one of the } BD, CD) && (0.0.30)\\
= {} & (1-p^2)^2 && (0.0.31)
\end{aligned}
$$

Now multiply by the probability that no fog between B and C, we have $(1-p^2)^2(1-p)$.

Now consider there is fog between B and C with probability p. As long as at least one of the routines ABD and ACD is open, we can have $A \leftrightarrow D$. Again we will consider its complement, i.e. none of the ABD or ACD is open.

$$
\begin{aligned}
P(ABD) &= (1-p)^2 = P(ACD) && (0.0.32)\\
P(ABD^c) &= 1 - (1-p)^2 = P(ACD^c) && (0.0.33)
\end{aligned}
$$

Now

$$
\begin{aligned}
P(\text{at least one of the } ABD \text{ and } ACD) &= 1 - P(\text{none of the } ABD \text{ or } ACD) && (0.0.34)\\
&= 1 - P(ABD^c \cap ACD^c) && (0.0.35)\\
&= 1 - P(ABD^c) \times P(ACD^c) && (0.0.36)\\
&= 1 - (1-(1-p)^2)^2 && (0.0.37)
\end{aligned}
$$

Multiply by the probability that there is fog between B and C, we have $(1-(1-(1-p)^2)^2)p$

Add these two probabilities together,

$$P(A \leftrightarrow D|AD^c) = (1-(1-(1-p)^2)^2)p + (1-p^2)^2(1-p) \tag{0.0.38}$$

(ii) Need to evaluate $P(A \leftrightarrow D|BC^c)$. The possible routines are AD, ACD and ABD. We have evaluated $P(\text{at least one of the } ABD \text{ and } ACD) = 1-(1-(1-p)^2)^2$. The probability of AD is $1-p$, hence $P(A \leftrightarrow D|BC^c) = (1-(1-(1-p)^2)^2)p + (1-p)$.

(iii) Need to evaluate $P(A \leftrightarrow D|AB^c)$. The possible routines are AD, ACD and $ACBD$.

Consider there is fog between A and D with probability p. As long as one of the routines ACD and $ACBD$ are open, we have $A \leftrightarrow D$. The probability of $P(AC) = 1-p$, therefore the event reduces to as long as one of the CD and CBD are open, we have $A \leftrightarrow D$.

$$\begin{aligned}
P(\text{at least one of the } CD \text{ and } CBD) &= 1 - P(\text{none of the } CD \text{ or } CBD) && (0.0.39)\\
&= 1 - P(CD^c)P(CBD^c) && (0.0.40)\\
&= 1 - p(1-(1-p)^2) && (0.0.41)
\end{aligned}$$

Again the probability that no fog between A and D is $1-p$, eventually

$$\begin{aligned}
P(A \leftrightarrow D|AB^c) &= P(AD^c)P(AC)P(\text{at least one of the } CD \text{ and } CBD) + P(AD)\\
&= p(1-p)(1-p(1-(1-p)^2)) + (1-p) && (0.0.42)
\end{aligned}$$

(iv)

$$\begin{aligned}
P(A \leftrightarrow D) &= P(A \leftrightarrow D|AD^c)p + P(A \leftrightarrow D|AD)(1-p) && (0.0.43)\\
&= (1-(1-(1-p)^2)^2)p^2 + (1-p^2)^2(1-p)p + (1-p) && (0.0.44)
\end{aligned}$$

Question 3

On a flight from Cork to Shanghai, n students are going to spend summer in China to celebrate finishing the leaving cert. Given there are 32 counties in China and suppose each student is visiting only one county independently.

(i) What is the probability that at least two students are heading to the same county.

Zhejiang and Jiangsu are two adjacent counties in the southeast of China, also known as JiangZhe area altogether. Suzhou and Hangzhou are the capital cities of the corresponding counties respectively. Countless natural attractions and representative ancient Chinese garden-style architecture have made both cities world famous, with the proverb "Up above there is paradise, down below there are Suzhou and Hangzhou" by Chengda Fan (1126-1193 AC) in Song Dynasty and poems such as "At sunrise riverside flowers redder than fire, in spring green waves grow as blue as sapphire" by Letian Bai (772-846 AC) in Tang Dynasty.

(ii) Randomly select two students, what is the probability that both students are going to the Zhejiang province?

(iii) Randomly select two students, what is the probability that both students are travelling to the JiangZhe area?

(iv) While passing the Chinese customs control, each student's destination is recorded one by one. What is the probability that there are two students going to the JiangZhe area?

Solution:

(i) Part (i) is similar with the "birthday question" we covered in class. So

$$P(\text{at least 2 the same}) = 1 - \frac{32^{(n)}}{32^n}$$

(ii) The probability of coming from any province is $\frac{1}{32}$, hence

$$P(\text{two are from Zhejiang}) = \frac{1}{32^2}$$

(iii) As JiangZhe area contains both Zhejiang and Jiangsu, the probability of coming from Jiangzhe area is

$$P(\text{two are from JiangZhe}) = \frac{2}{32} \times \frac{2}{32} = \frac{1}{16^2}$$

(iv) There is a catch here. Given all the hometowns are recorded, order matters. The probability that a student is not from JiangZhe area is $1 - \frac{1}{16}$, the probability of one favourable pattern is

$$\left(\frac{1}{16}\right)^2 \times \left(1 - \frac{1}{16}\right)^{n-2},$$

and the total number of such patterns is $\binom{n}{2}$, hence the probability is

$$\binom{n}{2}\left(\frac{1}{16}\right)^2 \times \left(1 - \frac{1}{16}\right)^{n-2}$$

Question 4

State the definition of two independent events. Let A and B be independent events, show that A^c, B are independent events, and deduce that A^c, B^c are independent.

Solution:

(i) We say that event A is independent of B if

$$P(A|B) = P(A)$$

(ii)

$$\begin{aligned} P(A^c \cap B) &= P(B \backslash (AB)) = P(B) - P(AB) && (0.0.45) \\ &= P(B) - P(A)P(B) && (0.0.46) \\ &= P(B)P(A^c) && (0.0.47) \end{aligned}$$

Therefore, A^c and B are independent.

(iii)

$$
\begin{aligned}
P(A^c \cap B^c) &= P[(A \cup B)^c] = 1 - p[A \cup B] && (0.0.48)\\
&= 1 - P(A) - P(B) + P(AB) && (0.0.49)\\
&= 1 - P(A) - P(B) + P(A)P(B) && (0.0.50)\\
&= (1 - P(A)) - P(B)(1 - P(A)) && (0.0.51)\\
&= (1 - P(A))(1 - P(B)) && (0.0.52)\\
&= P(A^c)P(B^c) && (0.0.53)
\end{aligned}
$$

Thus A^c is independent of B^c.

Question 5

Let A and B be the events with probabilities $P(A) = \frac{3}{4}$ and $P(B) = \frac{1}{3}$. Show that $\frac{1}{12} \leq P(AB) \leq \frac{1}{3}$, and give examples to show that both extremes are possible. Find corresponding bounds for $P(A \cup B)$.