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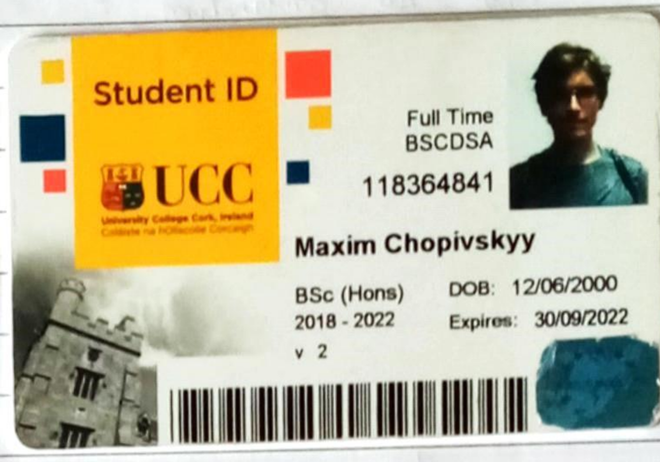

Total Number of pages: 16

Q 1

(i) ~~Min~~ Min $(P(A \cap B))$ when $P(A \cup B) = 1$

Min is $\frac{3}{4} + \frac{1}{3} - x = 1$

$x = \frac{1}{12}$

Max when $B \subseteq A$

$\downarrow$

$x = \frac{1}{3}$

Therefore $\frac{1}{12} \leq P(A \cap B) \leq \frac{1}{3}$

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(ii)

P[no cup on same pattern]

$= \dfrac{\text{number of combinations with no pattern}}{\text{all combinations}}$

$\boxed{R \quad R}$ $\boxed{W \quad W}$ $\boxed{S \quad S}$

Patterns with no combination =

W W	R R	R R
S S	R R	W W
S W	R R	R W

2 +

~~10~~

Total combinations = $\dfrac{6!}{2^3} = \dfrac{720}{8}$

$= 90$

$\boxed{\text{Answer: } \dfrac{10}{90} = \dfrac{1}{9}}$

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118364841

(iii)

According to De Morgan's Law

$$P\left[\bigcap_{r=1}^{\infty} A_r\right] = \lim_{n\to\infty} P\left[\bigcap_{r=1}^{n} A_r\right]$$

$$= \lim_{n\to\infty} \left[1 - P\left[\left(\bigcap_{r=1}^{n} A_r\right)^c\right]\right]$$

$$= 1 - \lim_{n\to\infty} P\left(\bigcup_{r=1}^{n} A_r^c\right)$$

$$\geq 1 - \lim_{n\to\infty} \sum_{r=1}^{n} P[A_r^c] =]$$

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Q2

(i) $\frac{2}{3} \times \frac{3}{4} + \frac{1}{3} \times 0$

$= \boxed{\frac{1}{2}}$

(ii)

	T	LC
T	$\frac{1}{2}$	0
F	$\frac{1}{6}$	$\frac{1}{3}$
	$\frac{2}{3}$	$\frac{1}{3}$

$\cancel{P[T \cap T \mid A]}$ $\quad \frac{2}{3} \quad \left(\frac{\frac{3}{4} \times \frac{3}{4}}{\frac{3}{4} \times \frac{3}{4} + \frac{1}{4} \times \frac{1}{4}} \right)$

$\cancel{P[T]P[T]}$ $\quad \frac{3}{5}$

$P[T \cap T] = \frac{1}{4}$

$P[F \cap F] = \frac{1}{4}$

$$\boxed{\frac{P[T \cap T]}{P[F \cap F] + P[T \cap T]} = \frac{1}{2}}$$

(iii)

$$\boxed{\frac{P[T \cap T \cap T]}{P[F \cap F \cap F] + P[T \cap T \cap T]} = \frac{1}{2}}$$

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3. (i)

$$c\int_{-\infty}^{\infty} e^{-x-e^{-x}}\,dx = 1$$

$$\downarrow$$

$$\int_{-\infty}^{\infty} e^{-x} e^{-e^{-x}}\,dx = \frac{1}{c}$$

$$c = \frac{1}{\int_{-\infty}^{\infty} e^{-x} e^{-e^{-x}}}$$

(ii)

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}, \quad \phi'(x) = \frac{-x}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

$$\phi'(x) + x\phi(x) = \frac{-x}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} + \frac{x}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} = 0$$

(iii) $\phi(x) = -\frac{1}{x}\phi'(x)$

$$\rightarrow \quad 1 - \Phi(x) = 1 - \int_{-\infty}^{x} \phi(x)\,dx$$

$$= \int_{x}^{\infty} \phi(x)\,dx = \int_{x}^{\infty} -\frac{1}{x}\phi'(x)\,dx$$

$$= -\frac{1}{x}\phi(x)\Big|_{x}^{\infty} - \int_{x}^{\infty} \phi(x)\cdot\frac{1}{x^2}\,dx$$

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$$= \frac{1}{x}\phi(x) + \int_x^\infty \frac{1}{x^3}\, d\,\phi(x)$$

$$\cancel{= \frac{1}{x}\phi(x) - \frac{1}{x^3}\phi(x) + \int_x^\infty \frac{1}{x^4}\phi(x)\,dx}$$

$$= \frac{1}{x}\phi(x) - \frac{1}{x^3}\phi(x) + \int_x^\infty \frac{1}{x^4}\phi(x)\,dx$$

$$\hookrightarrow \quad 1 - \Phi > \frac{1}{x}\phi(x) - \frac{1}{x^3}\phi(x)$$

$$\Rightarrow \quad \frac{1}{x} - \frac{1}{x^3} < \frac{1-\Phi(x)}{\phi(x)}$$

$\phi(x) > 0$

$$\frac{1}{x} - \frac{1}{x^3} < \frac{1-\Phi(x)}{\phi(x)} = \frac{1}{x}\phi(x) - \frac{1}{x^3}\phi(x) + \frac{1}{x^5}\phi(x) - \int_x^\infty \frac{1}{x^6}\phi(x)\,dx$$

$$\hookrightarrow \quad 1 - \Phi(x) < \frac{1}{x}\phi(x) - \frac{1}{x^3}\phi(x) + \frac{1}{x^5}\phi(x)$$

(iv) given $0 \leq \alpha \leq 1$, we have $\frac{H(x)}{x}$ is non decreasing

$\downarrow$

$$\frac{1}{\alpha x} H(\alpha x) \leq \frac{1}{x} H(x) \quad \text{for all } x \geq 0$$

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7

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$$\rightarrow \quad -\alpha^{-1}\ln[1-F(ax)] \leq -\ln(1-F(x))$$

$$\Updownarrow \quad \text{exp}$$

$$\hookrightarrow \quad [1-F(x)]^{a} \leq 1-F(ax)$$

(v)

If $\frac{H(x)}{x}$ is non decreasing

$$\Rightarrow \frac{H(\alpha t)}{\alpha t} \leq \frac{H(t)}{t} \Rightarrow H(at) \leq \alpha H(t)$$

For $0 \leq \alpha \leq 1$ and $t \geq 0$

Also, $H((1-a)t) \leq (1-a)H(t)$

$$\Rightarrow H(at) + H[(1-a)t] \leq H(t), \quad \text{let } x = at$$
$$y = (1-a)t$$

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Q4 (i) $P(X_1 < X_2)$

$= 1 - P(X_1 = X_2) - P(X_1 > X_2)$

$= \dfrac{1 - P(X_1 = X_2)}{2}$

$P(X_1 = X_2)$

$= P(X_1 = x, X_2 = x)$

$= \sum_{i=1}^{\infty} \left((1-p_i) p_i^{x-1} \right)^2$

=

(ii)

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(9)

Q5 (i)

$$X_n \xrightarrow{D} X$$

$$\hookrightarrow \lim_{n \to \infty} P[|X_n - X| < \varepsilon] = 1$$

$$aX_n + b \xrightarrow{D} aX + b$$

$$\hookrightarrow \lim_{n \to \infty} P[|aX_n \cancel{+b} - aX \cancel{-b}| < \varepsilon] = 1$$

$$\hookrightarrow \lim_{n \to \infty} P[|a(X_n - X)| < \varepsilon] = 1$$

is true

(ii)

From ~~Markov Inequality~~ : Chebyshev

$$P[|X - \underset{\substack{\parallel \\ 0}}{\mu}| \geq t\sigma] \leq \frac{1}{t^2}$$

$$\downarrow$$

$$P[X \geq t\sigma] \leq \frac{1}{t^2}$$

$$\downarrow$$

$$P[X \geq t] \leq \frac{\sigma^2}{t^2 + \sigma^2}$$

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Q6 (i) $c\int_0^\infty\int_0^y x(y-x)e^{-y}\,dx\,dy = 1$

$\downarrow$

$$\frac{1}{2}x^2ye^{-y} - \frac{1}{3}x^3e^{-y}\Big|_0^y$$

$$=$$

$$\frac{1}{2}y^3e^{-y} - \frac{1}{3}y^3e^{-y}$$

$$\frac{1}{6}\int_0^\infty y^3e^{-y}\,dy = \frac{1}{c}$$

$\downarrow$

$$\frac{1}{6}\times 6 = \frac{1}{c}$$

$$\boxed{c = 1}$$

(ii) $f_X(x) = \int_x^\infty x(y-x)e^{-y}\,dy$

$$= xe^{-x}$$

$$f_Y(y) = \int_0^y x(y-x)e^{-y}\,dx$$

$$= \frac{y^3e^{-y}}{6}$$

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Q6(ii)

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f(y)}$$

$$= 6\,\frac{x(y-x)\cancel{e^{-y}}}{y^3\cancel{e^{-y}}}$$

$$f_{Y|X}(y|x) = \frac{\cancel{x}(y-x)e^{-y}}{\cancel{x}e^{-x}}$$

$$= (y-x)\,e^{x-y}$$

(iii) $E[X|Y] = \int x\, f_{X|Y}(x|y)$

$$= 6\int_0^y \frac{x(y-x)}{y^3}\,dx$$

$$= 6 \times \frac{1}{6} = \boxed{1}$$

(iv) $E[Y|X] = \int_x^\infty y\,(y-x)e^{x-y}\,dy$

$$\boxed{= x+2}$$

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(12)

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Q7 (i)

$$E(Y) = E[W^{-1}(X-\mu)]$$

$$= W^{-1} E[(X-\mu)] = 0$$

$$Var(Y) = \underbrace{(W^{-1}\cdot W)}_{I} \cdot \underbrace{W\cdot(W^{-1})^T}_{m}$$

$$= I \cdot (W^T)^T (W^{-1})^T$$

$$= I \cdot (\underbrace{W^{-1} \cdot W^T}_{W})^T$$

$$= I$$

$\hookrightarrow N(0, I)$

(ii)

$\sum X_i = 18.66$

$\sum Y_i = 25.51$

$\bar{X} = 1.87$ $\qquad E(XY) = 5.82$

$\bar{Y} = 2.55$

$$cov(X,Y) = E[(X-E(X))(Y-E(Y))]$$

$$Var(Z) = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{pmatrix}$$

$$Var(Z) = \begin{pmatrix} 1.48 & 1.69 \\ 1.69 & 1.55 \end{pmatrix}$$

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7.(ii) Eigenvalues:

$$\begin{vmatrix} 1.49-\lambda & 1.69 \\ 1.69 & 1.55-\lambda \end{vmatrix} = 0$$

$$\hookrightarrow \quad \lambda^2 - 3.04\lambda + 2.3 - 2.86 = 0$$

$$\lambda^2 - 3.04\lambda - 0.56 = 0$$

$$\lambda = 3.21 \;,\; -0.17$$

Eigenvectors:

$$\begin{pmatrix} -1.72 & 1.69 \\ 1.69 & -1.66 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$-1.72 x_1 + 1.69 x_2 = 0$$
$$1.69 x_1 - 1.66 x_2 = 0$$

Eigenvectors:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Principle components are the eigenvectors. The direction which has the largest variance is the eigenvector with the largest eigenvalue, in this case, 3.21.

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Q10

(i) $49.38 \pm 1.96 \frac{11.34}{\sqrt{10}}$

$\parallel$

$(42.35, 56.41)$

(ii) $\cancel{\sigma^2 = \frac{(n-1)s^2}{\chi^9_{0.025}, 0.975}}$

$$\frac{(n-1)s^2}{\chi^9_{0.975}} \le \sigma^2 \le \frac{(n-1)s^2}{\chi^9_{0.025}}$$

$$\frac{9 \times 13.35^2}{19.023} \le \sigma^2 \le \frac{9 \times 13.35^2}{2.7}$$

$\cancel{6.316 \le \sigma^2 \le 44.5}$

$$\boxed{84.32 \le \sigma^2 \le 594.1}$$

(iii) Mean: $\frac{10 \times 49.38 + 9 \times 35.9 + 7 \times 46.1}{26}$

$=$

43.8

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210 (iii) $SD: \dfrac{10 \times 13.35 + 9 \times 10.13 + 7 \times 11.2}{27}$

$= 327.316$

(iv) $H_0: \mu_A = \mu_B = \mu_C$

H_1: at least one pair of A, B, C are different means.

$$SSB = \sum_{i=1}^{3} n_i(\mu_i - 40.7)^2 =$$

$$= 10 \times (49.38 - 40.7)^2 + \dots$$

$$= 1151.08$$

$$SSE = \sum_{i=1}^{3} \sum_{j=1}^{n} (x_{ij} - \mu_i)^2 =$$

$$= 1602.382 + 820.33 + 751.26$$

$$= 3173.972$$

$$df_B = 3 - 1 = 2$$

$$df_E = 26 - 3 = 23$$

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16

$$\frac{\frac{1161.018}{2}}{\frac{3173.972}{23}}$$

$= 4.17$

$F_{2,23} = 4.35$

$4.17 < F_{2,23}$

$\therefore$ Don't reject H_0

Anova assumptions:

1. Observations are independent
2. Residuals are distributed normally
3. Equality of variance, variance in groups are same
4. $\sum \tau_i = 0$