SAMPLING AND STATISTICAL INFERENCE

SECTION 3

TYPICAL SINGLE SAMPLE PROBLEMS

A random sample of 10 students is found to have the following SYSTOLIC BLOOD PRESSURES :

105 112 130 128 100 98 122 118 106 110

We are willing to assume that B.P. $\sim N(\mu, \sigma^2)$
— wish to estimate μ, σ^2

Point Estimators for μ and σ^2 : $\bar{X} = 112.9$

$S^2 = \frac{1}{9}\left[128{,}601 - \frac{(1129)^2}{10}\right] = 126.32 = \frac{1136.9}{9}$

Conf. Interval estimate for μ :

We know how to construct this estimate PROVIDED σ known

$\Rightarrow \quad \bar{X} \pm 1.96 \frac{\sigma}{\sqrt{n}}$ 95% Conf. Int

and this followed from the knowledge that

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

Usually, as here, σ^2 unknown. Seems sensible to use S^2 (which we know to be an unbiased estimate) instead of σ^2

But does $\bar{X} \pm 1.96 \frac{S}{\sqrt{n}}$ provide a 95% conf int?

I.E. Is $P\left[\bar{X} - 1.96 \frac{S}{\sqrt{n}} < \mu < \bar{X} + 1.96 \frac{S}{\sqrt{n}}\right] = 0.95$?

It's clear that we must investigate the prob distrib of

$$\frac{\bar{X} - \mu}{S/\sqrt{n}}$$

Now we know that

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

I.E. $\dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$

and we now know that

$$\frac{(n-1)\,s^2}{\sigma^2} \sim \chi^2_{n-1}$$

independently of $\bar{X}$ and thus of $\dfrac{\bar{X}-\mu}{\sigma/\sqrt{n}}$

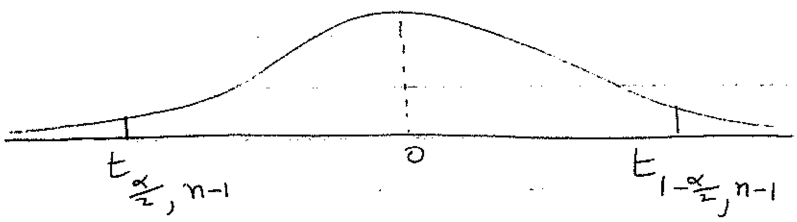
<u>Remember</u> the result that if $X \sim N(0,1)$
and $U \sim \chi^2_n$

Then $\dfrac{X}{\sqrt{U/n}} \sim t_n$ distrib

Applying this here

$$\frac{\dfrac{\bar{X}-\mu}{\sigma/\sqrt{n}}}{\sqrt{\dfrac{n-1\;\;s^2}{\sigma^2(n-1)}}} \sim t_{n-1}$$

I.E. $$\frac{\bar{X}-\mu}{s/\sqrt{n}} \sim t_{n-1}$$



Thus $$P\left[-t_{\frac{\alpha}{2},n-1} < \frac{\bar{X}-\mu}{s/\sqrt{n}} < t_{\frac{\alpha}{2},n-1}\right] = 100(1-\alpha)$$

$\Longrightarrow$

$$P\left[\bar{X} - t_{\frac{\alpha}{2},n-1}\frac{s}{\sqrt{n}} < \mu < \bar{X} + t_{\frac{\alpha}{2},n-1}\frac{s}{\sqrt{n}}\right] = 100(1-\alpha)$$

I.E. $\boxed{\bar{X} \pm t_{\frac{\alpha}{2},n-1}\dfrac{s}{\sqrt{n}}}$ is the $100(1-\alpha)\,\%$ conf. int. for μ.

Applying this to our set of data :

95% Conf Int : $t_{\frac{\alpha}{2},9} = 2.26$

$$\Rightarrow \quad 112.9 \pm (2.26) \frac{\sqrt{126.3}}{\sqrt{10}}$$

$$= 112.9 \pm \underbrace{(2.26)\ 3.55}$$

$$112.9 \pm 8.0$$

<u>SUMMING UP</u> : Conf. Intervals for μ

<u>σ known</u> : $\bar{X} \pm z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$

<u>σ unknown</u> $\bar{X} \pm t_{\frac{\alpha}{2}, n-1} \frac{s}{\sqrt{n}}$

<u>NOTE</u> : For large n, the shape of the t distrib approaches that of $N(0,1)$

Thus for large n, <u>even when σ unknown</u>, we can use

$$\bar{X} \pm z_{\frac{\alpha}{2}} \frac{s}{\sqrt{n}}$$

<u>Conf. Int for σ^2</u> : $\frac{(n-1)\,s^2}{\chi^2_{1-\frac{\alpha}{2}}} \longrightarrow \frac{(n-1)\,s^2}{\chi^2_{\frac{\alpha}{2}}}$

$n-1 = 9$

$\alpha = 5\%$

$\Rightarrow$ 95% Conf. Int for σ^2 : $\frac{1136.9}{17.535} \longrightarrow \frac{1136.9}{2.18}$

For σ^2 : $(64.8 \longrightarrow 521.5)$

For σ : $8.0 \longrightarrow 22.8$

HYPOTHESIS TESTING : SINGLE SAMPLE PROBLEMS

We have a random sample $X_1\ X_2\ ---\ X_n$ from a prob distn $f(x)$ and a hypothesis concerning $f(x)$ is to be tested using the sample data
— i.e. is the data consistent with the hypothesis?
— how consistent or inconsistent?

The hypothesis under test is termed the NULL HYPOTHESIS and is denoted by H_0
In this course, the hypotheses to be tested will only be concerned with parameters of the prob distn $f(x)$
e.g. that $\mu = \mu_0$ — denoted by $H_0 : \mu = \mu_0$

An example :
Jars of coffee are being filled by a Machine
The machine is set to put 225 g in each jar
A random sample of 16 jars is taken from the output from the machine
and $\overline{X}$ found to be 223.9 g

Suppose The S. Devn of the scatter in fill of the filling machine is 2 g

The packer would like to know if the machine is operating satisfactorily
i.e. he asks if $H_0 : \mu = \mu_0 = 225$ is true.
We should like to study how to test whether the data are consistent with this hypothesis.

SIMPLE & COMPOSITE HYPOTHESES
When a hypothesis fully specifies a particular prob. distrib
— it is called SIMPLE

When the hypothesis specifies a range of prob. distribs
— it is called COMPOSITE

In testing the hypothesis H_0 we are essentially trying to decide whether H_0 is CONSISTENT WITH THE DATA, or not.

It is usual to use the term ALTERNATIVE HYPOTH. in refering to the situation when H_0 is not true — and this alt. hypoth is denoted by H_1 or H_A

The terminology used is that we are testing H_0 against H_1

2 POSSIBLE ERRORS

Since we are choosing between 2 alternatives, only one of which can be true, there are 2 possible errors that we can make:

Rejecting H_0 when H_0 is true : Termed a TYPE 1 error
Accepting H_0 " " " false " " Type 2 error

We can evaluate a test procedure by determining the prob. of the 2 types of error that might be incurred using that procedure.

SERIOUSNESS OF ERRORS

In some situations one type of error may be considerably more serious than the other:
e.g. Testing a new drug.
Drug is either harmful or not
— Wish to make a decision

		States of Nature	
		Drug is Harmful	Drug is not Harmful
Decision	Reject Drug	✓	Error
	Accept Drug	Error	✓

CONVENTION is that the term Type 1 error is applied to the more serious error
— just choose the Hypothesis names accordingly

In our example, this leads to the choice

$$H_0 : \mu = 225$$
$$H_1 : \mu \neq 225$$

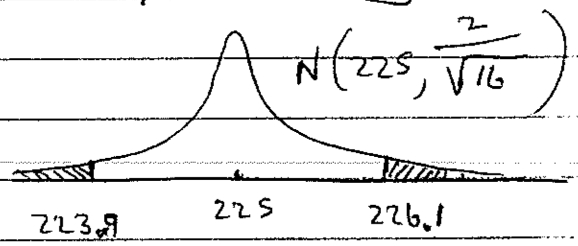
P-VALUE

A very useful way of quantifying ~~the~~ strength of the evidence against H_0 is to find the **P**-value (i.e. the PROBABILITY value) — which is the probability of observing a value for the sample statistic that is at least as extreme as the observed value, assuming H_0 true

Thus, here we would have to find

$$P\left[\left(\bar{X} \leq 223.9\right) \text{ or } \left(\bar{X} \geq 226.1\right) \middle| \mu = 225\right]$$

$$= 2P\left[\bar{X} \geq 226.1 \middle| \mu = 225\right]$$



$$= 2P\left[\frac{\bar{X} - 225}{2/\sqrt{16}} \geq \frac{226.1 - 225}{2/\sqrt{16}}\right]$$

$$= 2P\left[Z \geq 2.2\right]$$

$$= 0.028 \qquad \longleftarrow \text{ THIS IS THE P-VALUE}$$

TERMINOLOGY: The value 0.028 has also been termed the LEVEL OF STATISTICAL SIGNIFICANCE of the observation $\bar{X} = 223.9$ in relation to $H_0 : \mu = 225$.

→ TEST STATISTIC: The quantity $\frac{\bar{X} - 225}{\sigma/\sqrt{n}}$ is an example of a test statistic — which is some function of the observations that we can use to discriminate between H_0 and H_1.

Another common procedure is as follows:

The Acceptable $\left\{ \begin{matrix} \text{Size} \\ \text{level} \end{matrix} \right\}$ of Type I Error

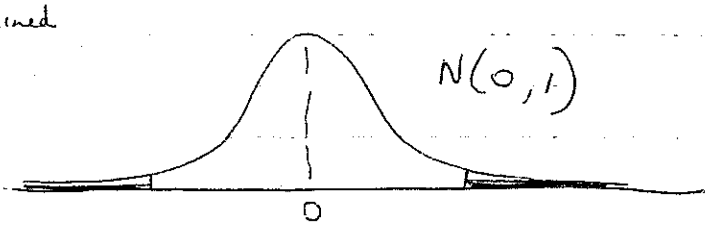
— also known as the SIGNIFICANCE LEVEL OF THE TEST

is specified: Typical figures are 5% and 1%.

The Distribution of the Test statistic under the assumption that H_0 is true is determined

— in our example:

Test Stat: $\dfrac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$



The critical region is the set of values of the Test Statistic for which H_0 is rejected.

— extent of critical region is determined by α (SIG. LEVEL)

Here we want $P\left[\dfrac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \in \text{Crit. Region} \,\middle|\, H_0 \text{ true}\right] = 0.05$

$\Rightarrow$ Critical region extends from $-\infty$ to -1.96 and $+\infty$ to $+1.96$

TEST PROCEDURE:

Evaluate the Test Statistic for given data

If in Critical Region : Reject H_0

else Accept

Here $\dfrac{\bar{X} - 225}{\frac{1}{2}} = \dfrac{223.9 - 225}{\frac{1}{2}} = -2.2$

$\Rightarrow$ Reject H_0

TEST OF HYPOTHESIS $\mu = \mu_0$ WHEN σ UNKNOWN

In carrying out the test procedure, we need to know the distribution of our test statistic under the assumption that H_0 is true.

$$\frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}} \quad (\text{for } \sigma \text{ known}) \text{ is } N(0,1)$$

under $H_0 : \mu = \mu_0$

When σ is unknown, intuitively it seems logical to use $\dfrac{\overline{X} - \mu_0}{s/\sqrt{n}}$ as test statistic

and, under H_0, this has a t_{n-1} distrib.

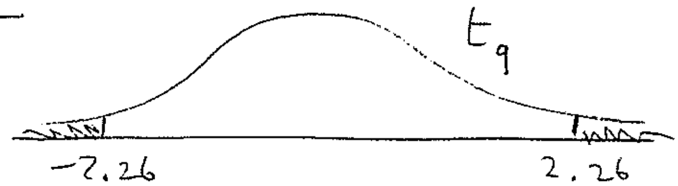
EXAMPLE: Data on student Blood pressures. (P.15)

Test $H_0 : \mu = 120$

against $H_1 : \mu \neq 120$

Test Statistic : $\dfrac{\overline{X} - 120}{s/\sqrt{n}}$

5% Signific Level



Value of Test Statistic

$$= \frac{112.9 - 120}{\sqrt{\dfrac{126.32}{10}}} = \frac{-7.1}{3.55} = -2$$

$\Longrightarrow$ Accept H_0. $(P = 0.038)$

(SHALL HAVE MORE TO SAY ON ACCEPTING H_0 LATER

USEFUL TO COMBINE THE 2 :

95% Conf. Int for μ : $112.9 \pm \overset{8.023}{2.26(3.55)} \Longrightarrow 104.9 \rightarrow 120.9$

ONE TAIL & TWO TAIL TESTS

The 2 tests that we have studied so far are usually referred to as 2 TAIL TESTS

— clearly because the CRITICAL REGION IS MADE up of 2 areas in the tails of the sampling distrib of our test statistic.

⟶ this follows from the form of

$H_1: \mu \neq 225$ & $H_1: \mu \neq 120$

In other testing problems, the appropriate form of the alternative hypothesis may be of the following forms:

$H_1: \mu > \mu_0$ or $H_1: \mu < \mu_0$

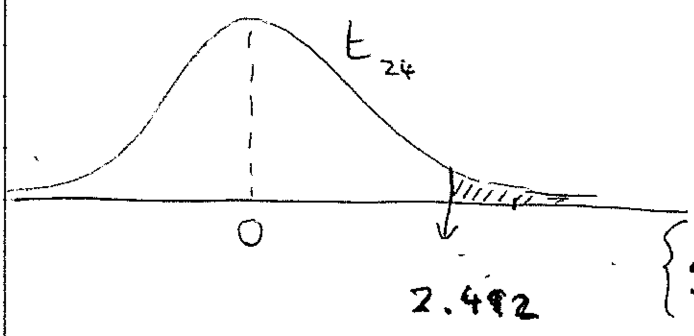
e.g. Rope manufacturer : ⟶ traditionally breaking strength = 8000 lbs
New process introduced which it is hoped will increase the breaking strength
Sample mean of 25 ropes ⟹ 8120
and sample variance, $S^2 = 40{,}000 \Rightarrow S = 200$

Using the 1% level of significance, should we conclude the rope breaking strength has increased?
(Expense attached to changing production equipment for new process)
— so concerned with error of changing when there's really no improvement

$H_0: \mu = 8000$
$H_1: \mu > 8000$

$$\frac{\bar{X}-\mu}{S/\sqrt{n}} \sim t_{n-1}$$

$$\frac{\bar{X}-\mu}{S/\sqrt{n}} = \frac{8120-8000}{200/5} = \frac{120}{40} = 3$$



P-value = 0.003

⟹ Reject H_0.

{P-VALUE
Significance Level} of the observed result

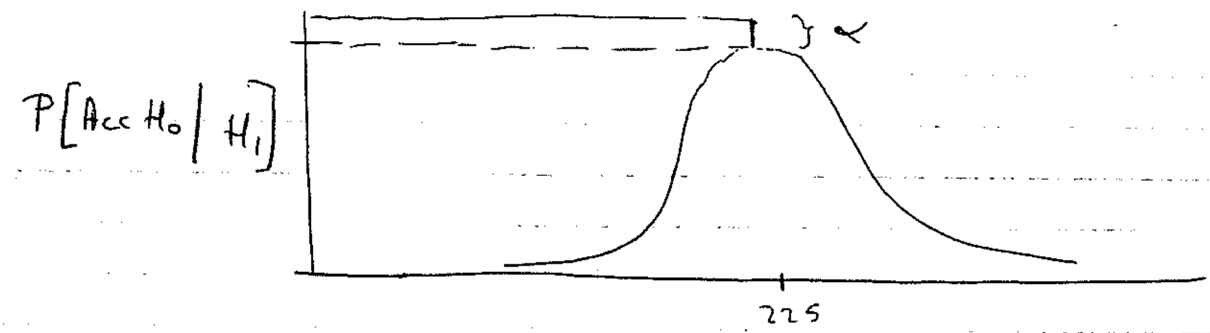
CONF INT FOR μ : $8120 \pm (2.492)40$ 8020.3 ⟶ 8219.7

OPERATING CHAR CURVE

We have said little of the (Type 2 Error) Prob.

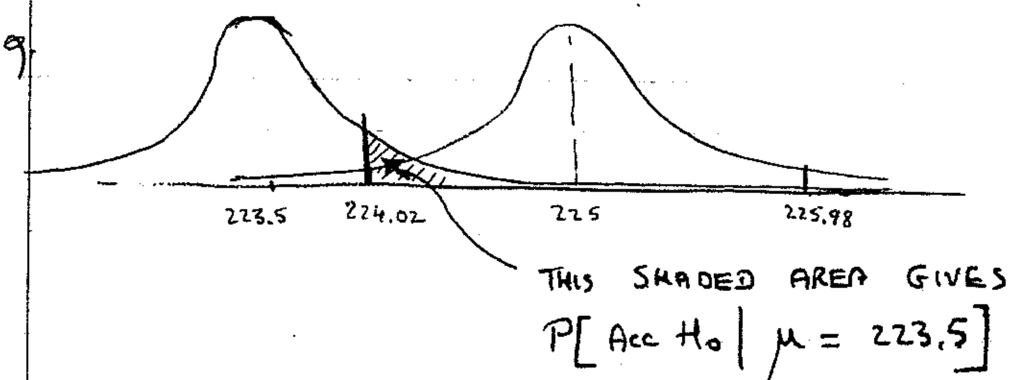
In the case of hypotheses: $H_0 : \mu = 225$
$H_1 : \mu \neq 225$

- Suppose wish to compute $P[\text{Accept } H_0 \mid H_1 \text{ true}]$



- Must be done for range of μ values within H_1
- Usually plotted against μ as shown and referred to as O.C. curve.

e.g.

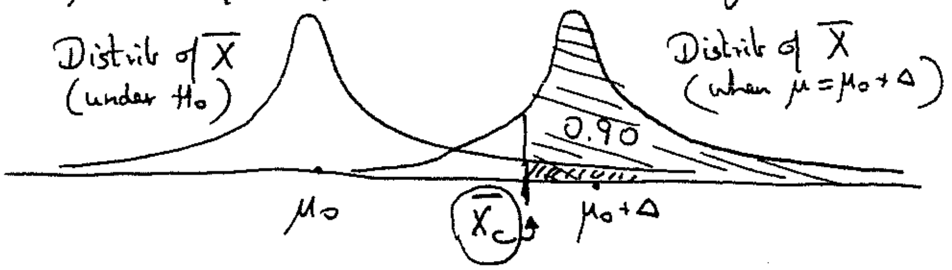


FINDING THE SAMPLE SIZE REQUIRED IN SINGLE-SAMPLE HYPOTHESIS-TESTING

Suppose we wish to test $H_0: \mu = \mu_0$

versus $H_A: \mu > \mu_0$

using a significance level of 1%, and suppose we want to have a 90% chance of detecting an increase above μ_0 of size Δ. What size n is needed?



Let's put numbers on μ_0 and Δ: Say $\mu_0 = 3000$

$\Delta = 200$

We require n so that $P\left(\bar{X} \geq \bar{X}_c \,\middle|\, \mu = 3000\right) = .01$ ①

and $P\left(\bar{X} \geq \bar{X}_c \,|\, \mu = 3200\right) = 0.90$ ②

From ① $\quad \bar{X}_c = 3000 + 2.33 \frac{\sigma}{\sqrt{n}}$

From ② $\quad \bar{X}_c = (3000 + 200) - 1.28 \frac{\sigma}{\sqrt{n}}$

$$\Rightarrow \quad 2.33 \frac{\sigma}{\sqrt{n}} + 1.28 \frac{\sigma}{\sqrt{n}} = 200$$

$$3.61 \frac{\sigma}{\sqrt{n}} = 200$$

It's clear that we need <u>some estimate for σ</u>

Let's suppose that we take σ to be about 800

$$\Rightarrow \quad \sqrt{n} = (3.61)(4) = 14.44$$

$$\Rightarrow \quad n = 208.51$$

round up to 209.

NOTICE that the value taken for σ had a strong influence, and so too did the size of Δ (i.e. 200) and the significance level (1%) and the <u>power</u>-level (90%)