Question 1.

(a) Recall the Taylor expansion at zero. $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot x^n$

$e^{\theta X} = 1 + \theta X + \frac{\theta^2}{2!} X^2 + \cdots = 1 + \sum_{k=1}^{\infty} \frac{\theta^k}{k!} \cdot X^k$.

$\Rightarrow E[e^{\theta X}] = 1 + \sum_{k=1}^{\infty} \frac{\theta^k}{k!} m_k$, where $m_k = E[X^k]$

$= 1 + S(\theta)$, where $S(\theta) = \sum_{k=1}^{\infty} \frac{\theta^k}{k!} m_k$.

$\Rightarrow K_X(\theta) = \ln(1 + S(\theta))$ ①

Aim to rewrite ① into a polynomial w.r.t. θ by using the Taylor expansion again, where $S(\theta)$ is considered as a variable.

$\Rightarrow K_X(\theta) = \ln(1+S(\theta)) = \sum_{r=1}^{\infty} (-1)^{r+1} \frac{S(\theta)^r}{r}$ ②

Given $K_X(\theta) = \sum_{n=1}^{\infty} \frac{1}{n!} k_n(X) \theta^n$ provided in question,
③

② == ③, $\Rightarrow$ the coeff of θ with the same power in ② and ③ should be identical.

Now $S(\theta)^r = \left(\sum_{k=1}^{\infty} \frac{\theta^k}{k!} m_k\right)^r$.

For $k_1(X)$ in ③, $n=1$, coeff in ③ is $k_1(X)$ for θ.

in ②, the only possible way to obtain θ is to have

$k = r = 1$.

we have $(-1)^2 \cdot S(\theta) = S(\theta) = \sum_{k=1}^{\infty} \frac{\theta^k}{k!} m_k$

coeff for θ is m_1 $\Rightarrow$ $m_1 = k_1(X)$

For $k_2(x)$ in ③, $n=2$, coeff in ③ is $\frac{1}{2!}k_2(X)$ for θ^2

in ②, the only combination to have θ^2 is that

$r=1,\ k=2$

or $r=2,\ k=1$

For $r=1,\ k=2$, coeff is $(-1)^2 \cdot 1 \cdot \frac{m_2}{2!}$ for θ^2

For $r=2,\ k=1$, coeff is $(-1)^3 \cdot \frac{1}{2} \cdot m_1^2$ for θ^2

$\Rightarrow \frac{1}{2}m_2 - \frac{1}{2}m_1^2 = \frac{1}{2} \cdot k_2(x) \Rightarrow k_2(X) = m_2 - m_1^2$

For $k_3(X)$ in ③, $n=3$, coeff in ③ is $\frac{1}{3!}k_3(X)$ for θ^2

r can be 1, 2, 3.

- For $r=1,\ k=3$, coeff is $(-1)^2 \cdot \frac{m_3}{3!}$ for θ^3
- For $r=2,\ k=2$ or 1

i.e. $(m_1\theta + \frac{m_2}{2!}\theta^2 + \cdots) \times (m_1\theta + \frac{m_2}{2!}\theta^2 + \cdots)$

tims

times

coeff is $2 \cdot \frac{m_1}{1!} \cdot \frac{m_2}{2!} \cdot \frac{1}{2} \cdot (-1)^3 = -\frac{1}{2}m_1m_2$ for θ^3

- For $r=3$, coeff $= (-1)^4 \cdot \frac{1}{3} \cdot m_1^3$, for θ^3.

i.e.

$(m_1\theta + \frac{m_2}{2!}\theta^2 + \cdots) \times (\theta m_1 + \frac{m_2}{2}\theta^2 + \cdots) \times (m_1\theta + \frac{m_2}{2}\theta^2 + \cdots)$

times.

Now we have:

$\frac{m_3}{3!}(-1)^2 + (-\frac{1}{2})m_1m_2 + \frac{1}{3}m_1^3 = \frac{1}{3!}k_3(X)$

$\Rightarrow k_3 = m_3 - 3m_1m_2 + 2m_1^3$

ⓑ If X and Y are indep, $K_{X+Y}(\theta) = \ln\{E(e^{\theta X})E(e^{\theta Y})\} = K_X(\theta) + K_Y(\theta)$. #.

Question 2.

If X is $N(\mu, 1)$, then the MGF of X^2 is

$$M_{X^2}(s) = E[e^{sX^2}] = \int_{-\infty}^{\infty} e^{sx^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\mu)^2} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}(x^2 - 2\mu x + \mu^2 - 2sx^2)\right] dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left[(1-2s)x^2 - 2\mu x + \mu^2\right]\right\} dx.$$

let $\sqrt{1-2s}\,x = t$. $\quad = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left[t^2 - 2\mu \cdot \frac{1}{\sqrt{1-2s}} \cdot t + \frac{\mu^2}{1-2s} - \frac{\mu^2}{1-2s} + \mu^2\right]\right\} \frac{dt}{\sqrt{1-2s}}$

$dx = \frac{1}{\sqrt{1-2s}} dt$.

$$= \exp\left(-\frac{1}{2}\left(\mu^2 - \frac{\mu^2}{1-2s}\right)\right) \frac{1}{\sqrt{1-2s}} \cdot \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(t - \frac{\mu}{\sqrt{1-2s}}\right)^2\right\} dt}_{=1}.$$

$$= \frac{1}{\sqrt{1-2s}} \cdot \exp\left[\frac{\mu^2 s}{1-2s}\right] \quad \text{if } s < \frac{1}{2}.$$

It follows that $M_Y(s) = \prod_{j=1}^{n} \left\{ \frac{1}{\sqrt{1-2s}} \exp\left\{\frac{\mu_j s}{1-2s}\right\}\right\} = (1-2s)^{\frac{n}{2}} \cdot \exp\left(\frac{s\theta}{1-2s}\right)$,

where $\theta = \mu_1^2 + \mu_2^2 + \cdots + \mu_n^2$

Replace s by it to obtain the answer.

Q3.

$P[Y \le y] = P[X \le \ln y] = \Phi(\ln y)$ for $y > 0$. Φ is the CDF for $N(0,1)$.

Then diff. with respect to y to obtain the answer.

$f_a(x) = f(x) + a f(x) \cdot \sin(2\pi \ln x)$

For $|a| \le 1$, $\Rightarrow$ $|a \sin(2\pi \ln x)| \le 1$,

$\Rightarrow f_a(x) = f(x) + a f(x) \cdot \sin(2\pi \ln x) \ge 0$

$\int_0^\infty a \sin(2\pi \ln x) \cdot \frac{1}{x\sqrt{2\pi}} \cdot \exp[-\frac{1}{2}(\ln x)^2]\, dx$, let $\ln x = y$, $\frac{1}{x} dx = dy$

$= \int_{-\infty}^{\infty} a \underbrace{\sin(2\pi y)}_{} \cdot \frac{1}{\sqrt{2\pi}} \cdot \underbrace{\exp[-\frac{1}{2} y^2]}_{} \cdot dy$, $= 0$,

sine $\Rightarrow$ odd f^n. $\times$ even f^n. $=$ odd f^n.

$\Rightarrow \int_0^\infty f_a(x)\, dx = \int_0^\infty f(x)\, dx = 1$. $\Rightarrow$ f_a is a density f^n.

Similarly, the k^{th} moment is.

$$\int_0^\infty x^k \cdot f_a(x)\, dx = \int_0^\infty x^k f(x)\, dx + \underbrace{\int_0^\infty x^k \cdot a \sin(2\pi \ln x) \frac{1}{x\sqrt{2\pi}} \exp[\quad]\, dx}_{\text{can convert this to an odd } f^n \text{ again}}$$

$\Rightarrow$ k^{th} moment $= \int_0^\infty x^k \cdot f_a(x)\, dx = \int_0^\infty x^k f(x)\, dx$, not dep. on a.

Q4.

(a) $X_{n+1} = \begin{cases} \frac{1}{2} X_n, & \text{with prob of } \frac{1}{2} \\ \frac{1}{2} X_n + Y_n & \text{with prob. of } \frac{1}{2} \end{cases}$

(b) The char. f^n of X_{n+1} is denoted as $\phi_{n+1}(t)$

$$\phi_{n+1}(t) = E[e^{itX_{n+1}}] = \frac{1}{2} E[e^{it\frac{X_n}{2}}] + \frac{1}{2} E[e^{it\cdot(\frac{1}{2}X_n + Y_n)}]$$

$$= \frac{1}{2} \cdot \phi_n(\tfrac{t}{2}) + \frac{1}{2} \cdot \phi_n(\tfrac{t}{2}) \cdot \phi_{Y_n}(t) \qquad Y_n \overset{iid}{\sim} \exp(\lambda)$$

$$= \frac{1}{2} \phi_n(\tfrac{t}{2}) + \frac{1}{2} \phi_n(\tfrac{t}{2}) \cdot \frac{\lambda}{\lambda - it},$$

$$\phi_{n+1}(t) = \cancel{\phi_n(\tfrac{t}{2})} \left(\frac{\lambda - \frac{1}{2} it}{\lambda - it} \right) \qquad n+1 \quad (\tfrac{1}{2})^0$$

$$\cancel{\phi_n(\tfrac{t}{2})} = \phi_{n-1}(\tfrac{t}{4}) \left(\frac{\cancel{\lambda - \frac{1}{2} \cdot \frac{it}{2}}}{\lambda - \cancel{\frac{it}{2}}} \right) \qquad n \quad (\tfrac{1}{2})^1$$

$n-1 \quad (\tfrac{1}{2})^2$

$\vdots$

$2 \quad (\tfrac{1}{2})^{n-1}$

$$\cancel{\phi_2(t \cdot 2^{-(n-1)})} = \phi_1(t \cdot 2^{-n}) \cdot \left(\frac{\lambda - it \cdot 2^{-n}}{\lambda - \cancel{it \cdot 2^{-(n-1)}}} \right)$$

after cancellation

$$\phi_{n+1}(t) = \phi_1(t \cdot 2^{-n}) \cdot \frac{\lambda - it2^{-n}}{\lambda - it},$$

rule of cancellation: previous numerator is cancelled by the next denominator.

So $\left.\begin{matrix} \lambda - it2^{-n} \\ \lambda - it \end{matrix}\right\}$ can not be cancelled.

as $n \to \infty$,

$\phi_{n+1}(t) \to \frac{\lambda}{\lambda - it}$, which is an exp's char f^n.