Question 1.

$$E(1+X) = \int_{-1}^{1} (1+x)^{g_1+1}(1-x)^{g_2}\,dx \cdot (\text{constant})$$

$$= \frac{g_2!\,(g_1+1)!\,2^{g_1+g_2+2}}{(g_1+g_2+2)!} \times \frac{(g_1+g_2+1)!}{g_1!\,g_2!\,2^{g_1+g_2+1}}$$

$$= \frac{2(g_1+1)}{g_1+g_2+2} \quad \Rightarrow \quad E(X) = \frac{g_1-g_2}{g_1+g_2+2}$$

✓

$$Var(X+1) = E[(X+1)^2] - E(X+1)^2 = Var(X)$$

$$E[(X+1)^2] = \int_{-1}^{1} (1+x)^2(1+x)^{g_1}(1-x)^{g_2}\,dx \cdot C$$

$$= \int_{-1}^{1} (1+x)^{g_1+2}(1-x)^{g_2}\,dx \cdot (\text{constant})$$

$$= \frac{g_2!\,(g_1+2)!\,2^{g_1+g_2+3}}{(g_1+g_2+3)!} \cdot \frac{(g_1+g_2+1)!}{g_1!\,g_2!\,2^{g_1+g_2+1}}$$

$$= \frac{4(g_1+2)(g_1+1)}{(g_1+g_2+2)(g_1+g_2+3)}$$

$$Var(X) = Var(X+1) = \frac{4(g_1+2)(g_1+1)}{(g_1+g_2+2)(g_1+g_2+3)} - \frac{4(g_1+1)^2}{(g_1+g_2+2)^2}$$

$$= \frac{4(g_1+2)(g_1+1)(g_1+g_2+2) - 4(g_1+1)^2(g_1+g_2+3)}{(g_1+g_2+2)^2(g_1+g_2+3)} \quad ① \ ②$$

$$\tfrac{1}{4}① = (g_1+1+1)(g_1+1)(g_1+1+g_2+1)$$

$$= [(g_1+1)^2 + (g_1+1)][(g_1+1)+(g_2+1)]$$

$$= \cancel{(g_1+1)^3} + \cancel{(g_1+1)^2(g_2+1)} + \cancel{(g_1+1)^2} + (g_1+1)(g_2+1)$$

$$\tfrac{1}{4}② = (g_1+1)^2(g_1+1+g_2+1+1)$$

$$= \cancel{(g_1+1)^3} + \cancel{(g_1+1)^2} + \cancel{(g_1+1)^2(g_2+1)}$$

Now $\tfrac{1}{4}(① - ②)$

$$= (g_1+1)(g_2+1)$$

$$\Rightarrow Var(X) = \frac{4(g_1+1)(g_2+1)}{(g_1+g_2+2)^2(g_1+g_2+3)}$$

choose g_1 + g_2 s.t.

$$\begin{cases} E[\alpha_r] = 0.98 \\ SD(\alpha_r) = 0.01 \end{cases}$$

$$\frac{g_1-g_2}{g_1+g_2+2} = 0.98$$

$$\frac{4(g_1+1)(g_2+1)}{(g_1+g_2+2)^2(g_1+g_2+3)} = 0.01^2$$

Let $a = g_1+1$, $b = g_2+1$.

$$\begin{cases} \dfrac{a-b}{a+b} = 0.98 \\ \dfrac{4ab}{(a+b)^2(a+b+1)} = 0.01^2 \end{cases}$$

$\Rightarrow b = 3.94 \quad a = 99b = 390$

$\Rightarrow g_1 = 389, \quad g_2 = 3.$

Question 2.

$S \equiv$ total Mana required. $\sim LN(\mu, \sigma^2)$
& $E(S) = 1.5$, $Var(S) = 2$.

(Marks are still given if students have
$\mu = 1.5$ $\sigma^2 = 2$ directly)

$$\Rightarrow e^{\mu + \frac{1}{2}\sigma^2} = 1.5 \quad \& \quad e^{2\mu + \sigma^2}(e^{\sigma^2} - 1) = 2$$

$$\Rightarrow \mu = 0.087471, \quad \sigma = 0.797489.$$

Assume the mana is drawn uniformly throughout the day.

$\Rightarrow$ Yr end mana position $= (1+\theta) \times 1.5 \times 1.06^{\frac{1}{2}} + 0.8 \times 1.06$
$- S$

Require

$$P[S > (1+\theta) \times 1.5 \times 1.06^{\frac{1}{2}} + 0.8 \times 1.06] = 0.05$$

$$\Rightarrow P[\ln S > \ln[\qquad\qquad]] = 0.05$$

$$\Rightarrow \frac{\ln[(1+\theta) \times 1.5 \times 1.06^{\frac{1}{2}} + 0.8 \times 1.06] - 0.087471}{0.797489} = 1.625$$

$$\Rightarrow \theta = 1.075$$

Question 3.

(a) Suppose $s < t$. Then

$$E[N(s)N(t)] = E[N(s)^2] + E[N(s)(N(t)-N(s))]$$
$$= E[N(s)^2] + E[N(s)]E[N(t)-N(s)]$$

since N has indep. increments. Therefore,

$$\text{cov}(N(s), N(t)) = E[N(s)N(t)] - E(N(s))E(N(t))$$
$$= (\lambda s)^2 + \lambda s + \lambda s\{\lambda(t-s)\} - (\lambda s)(\lambda t)$$
$$= \lambda s.$$

$\Rightarrow$ In general, $\text{cov}(N(s), N(t)) = \lambda \min(s, t)$

(b) $N(t+h) - N(t)$ has the same dist^n as $N(h)$, if $h>0$.
hence

$$E[\{N(t+h)-N(t)\}^2] = E[N(h)^2] = (\lambda h)^2 + \lambda h,$$

which tends to zero as $h \to 0$.

(c) By Markov Inequality.

$$P[|N(t+h) - N(t)| > \varepsilon] \leq \frac{1}{\varepsilon^2} E(\{N(t+h)-N(t)\}^2),$$

which tends to zero as $h \to 0$, if $\varepsilon > 0$.

(d) Not required.

Q4.

(i). For $x = 1, 2, \ldots$

$$P[X=x] = \sum_{y=1}^{\infty} P[X=x, Y=y]$$

$$= \sum_{y=1}^{\infty} \frac{c}{2}\left\{ \frac{1}{(x+y-1)(x+y)} - \frac{1}{(x+y)(x+y+1)} \right\}$$

$$= \frac{c}{2x(x+1)} = \frac{c}{2}\left(\frac{1}{x} - \frac{1}{x+1} \right)$$

$$\sum_{x=1}^{\infty} P[X=x] = \sum_{x=1}^{\infty} \frac{c}{2}\left(\frac{1}{x} - \frac{1}{x+1} \right)$$

$$= \frac{c}{2}\left(1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} \cdots \right) = \frac{c}{2} = 1 \Rightarrow c = 2.$$

$$P[Y=y] = \frac{1}{y} - \frac{1}{y+1} \quad \text{for } y = 1, 2, \ldots$$

(ii)

If $z \geq 2$. $P[X+Y=z] = \sum_{k=1}^{z-1} P[X=k, Y=z-k] = \frac{c}{z(z+1)}$

Also, if $z \geq 0$, $P[X-Y=z] = \sum_{k=1}^{\infty} P[X=k+z, Y=k]$

$$= c \sum_{k=1}^{\infty} \frac{1}{(2k+z-1)(2k+z)(2k+z+1)}$$

$$= \frac{1}{2} c \sum_{k=1}^{\infty} \left\{ \frac{1}{(2k+z-1)(2k+z)} - \frac{1}{(2k+z)(2k+z+1)} \right\}$$

$$= \frac{1}{2} c \sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{(r+z)(r+z+1)}$$

By symmetry, if $z \leq 0$, $P[X-Y=z] = P[X-Y=-z]$
$= P[X-Y=|z|]$

Q5.

$$f_X[j] = c \sum_{k=0}^{\infty} \left\{ \frac{j}{j!} a^j \frac{a^k}{k!} + \frac{k a^j a^k}{k!\, j!} \right\}$$

$$= c\, \frac{e^a (j+a) a^j}{j!} = P[X=j]$$

$1 = \sum_j f_X(j) = 2ace^{2a}$, whence $c = e^{-2a}/2a$

$$f_{X+Y}(r) = \sum_{j=0}^{r} \frac{c r a^r}{j!\,(r-j)!} = \frac{c r a^r 2^r}{r!}, \quad r \geq 1.$$

$$E[X+Y-1] = \sum_{r=1}^{\infty} \frac{c r(r-1)(2a)^r}{r!} = 2a,$$

since $E(X) = E(Y) \Rightarrow E(X) = a + \frac{1}{2}$.

Q6.

By integration, for $x, y > 0$.

$$f_Y(y) = \int_0^y f(x,y)\,dy = \frac{1}{6} c y^3 e^{-y}$$

$$f_X(x) = \int_x^\infty f(x,y)\,dy = c x e^{-x},$$

$\Rightarrow$ $c = 1$.

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}, \quad f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)}$$