Question 1

Let $\{A_i : i \in I\}$ be a collection of sets. Prove "De Morgan's Law":

(i)

$$\left(\underset{i}{\cup} A_i\right)^c = \underset{i}{\cap} A_i^c$$

(ii)

$$\left(\underset{i}{\cap} A_i\right)^c = \underset{i}{\cup} A_i^c$$

Solution: Hint: One way to prove two sets are identical, one needs to prove they are "mutually inclusive". i.e. If $A \subseteq B$ and $B \subseteq A$, then $A = B$.

(i) Need to show

$$\begin{aligned} \left(\underset{i}{\cup} A_i\right)^c &\subseteq \underset{i}{\cap} A_i^c && (0.0.1) \\ \underset{i}{\cap} A_i^c &\subseteq \left(\underset{i}{\cup} A_i\right)^c && (0.0.2) \end{aligned}$$

Recall that if $a \in A$, then $a \in B$, then $A \subseteq B$. To show (0.0.1), if $a \in \left(\underset{i}{\cup} A_i\right)^c \Rightarrow a \in \underset{i}{\cap} A_i^c$.

Given

$$\begin{aligned} & a \in \left(\underset{i}{\cup} A_i\right)^c && (0.0.3) \\ \Rightarrow\ & a \notin \underset{i}{\cup} A_i && (0.0.4) \\ \Rightarrow\ & a \notin A_i \text{ for any } i && (0.0.5) \\ \Rightarrow\ & a \in A_i^c \text{ for each } i && (0.0.6) \\ \Rightarrow\ & a \in \underset{i}{\cap} A_i^c && (0.0.7) \\ \Rightarrow\ & \left(\underset{i}{\cup} A_i\right)^c \subseteq \underset{i}{\cap} A_i^c && (0.0.8) \end{aligned}$$

To show (0.0.2), if $a \in \underset{i}{\cap} A_i^c \Rightarrow a \in \left(\underset{i}{\cup} A_i\right)^c$.

Given

$$\begin{aligned} & a \in \underset{i}{\cap} A_i^c && (0.0.9) \\ \Rightarrow\ & a \in A_i^c \text{ for each } i; && (0.0.10) \\ \Rightarrow\ & a \notin A_i \text{ for each } i; && (0.0.11) \\ \Rightarrow\ & a \notin \cup A_i \text{ for each } i; && (0.0.12) \\ \Rightarrow\ & a \in \left(\underset{i}{\cup} A_i\right)^c && (0.0.13) \\ \Rightarrow\ & \underset{i}{\cap} A_i^c \subseteq \left(\underset{i}{\cup} A_i\right)^c && (0.0.14) \end{aligned}$$

Now we have shown (0.0.1) and (0.0.2) hold, therefore $\left(\underset{i}{\cup} A_i\right)^c = \underset{i}{\cap} A_i^c$.

(ii) One can follow the idea in (i), but here is a trick. Since (i) applies to any set, let's replace the A_i in (i) by A_i^c. Then (i) can be re-written as $\left(\underset{i}{\cup} A_i^c\right)^c = \underset{i}{\cap}(A_i^c)^c = \underset{i}{\cap}A_i$. Now apply the complement operator to both sides leaves us $\underset{i}{\cup}A_i^c = \left(\underset{i}{\cap} A_i\right)^c$, prove done.

Question 2

A conventional know-out tournament (such as that Wimbledon) begins with 2^n competitors and has n rounds. There are no play-offs for the positions $2, 3, \ldots, 2^n - 1$, and the initial table of draws is specified. Give a concise description of the sample space of all possible outcomes.

Solution: This question serves as a recall of some sets operators. Recall $A \times B = \{(x, y) : x \in A, y \in B\}$ Let us number the players $1, 2, \ldots, 2^n$ in the order in which they appear in the initial table of draws. The set of victors in the first round is a point in the space $V_n = \{1, 2\} \times \{3, 4\} \times \cdots \{2^{n-1}, 2^n\}$. Then re-number these victors in the same way as done for the initial draw, the set of second round victors can be thought of as a point in the space V_{n-1}, and so on. The sample space of all possible outcomes of the tournament may therefore be taken to be $V_n \times V_{n-1} \times \cdots \times V_1$, a set containing $2^{2^{n-1}} 2^{2^{n-2}} \cdots 2^1 = 2^{2^n - 1}$ points.

Question 3

A fair coin is tossed repeatedly. Show that, with probability one, a head turns up sooner or later. Show similarly that any given finite sequence of heads and tails occurs eventually with probability one.

Solution: Hint: this question can be easily solved once we know it follows a geometric distribution. But we can still do it without this prior knowledge. Consider tossing the coin repeatedly by n times and n is extremely large. The probability of a head turns up sooner or later is equivalent to 1 less the probability that a head never turns up. Then our strategy is to show the probability that a head never turns up is zero when n is extremely large. Denote as NH as the event that no head turns up ever, and NH_n as the event that no head turns up after n tosses.

$$P[NH] = \lim_{n\to\infty} P[NH_n] \tag{0.0.15}$$

$$= \lim_{n\to\infty} \frac{1}{2^n} = 0 \tag{0.0.16}$$

Therefore the probability that a head will turn up sooner or later is $1 - P[NH] = 1$. Prove done.

Students need to be careful that there are two sub-questions here!!

Lemma: If $A \subseteq B$, then $P(B) \geq P(A)$. Prove it. We will need this lemma to prove the second part.

Denote as $s = \{H, \ldots, H, T, \ldots, T\}$ a fixed sequence of heads and tails. The length of s is k. Note that the permutation of the heads and tails in s is not concerned us in this question. Let us consider repeatedly toss many times and split these tosses into disjoint sequences of tosses $s_1, s_2, \ldots,$ where each has a length of k.

The probability that $P[s = s_i] = \frac{1}{2^k}$ for any $i = 1, 2, \ldots$.

Denote as

- A = s turns up eventually,
- B = s occurs in the first nk tosses
- C = s occurs at least as one of the first n groups

By definition, $C \subseteq B$ (why?).

$$
\begin{aligned}
P[A] &= \lim_{n\to\infty} P(B) && (0.0.17)\\
&\geq \lim_{n\to\infty} P(C) && (0.0.18)\\
&= 1-\lim_{n\to\infty} P(C^c) && (0.0.19)\\
&= 1-\lim_{n\to\infty} (1-2^{-k})^n && (0.0.20)\\
&= 1 && (0.0.21)
\end{aligned}
$$

Question 4

A biased coin is tossed repeatly. Each time there is a probability p of a head turning up. Let p_n be the probability that an even number of heads has occurred after n tosses (zero is an even number).

(i) What is the value of p_0, explain why.

(ii) Show that $p_n = p(1-p_{n-1}) + (1-p)p_{n-1}$ if $n \geq 1$. Solve this difference equation.

Solution:

(i) $n = 0$ **means no tosses and the number of heads is zero, an even number in definite. Therefore,** $p_0 = 1$**.**

(ii) It is the best to consider the first toss. If the first toss is a head with probability p, we then need an odd number of heads in the next $n-1$ tosses with probability $1-p_{n-1}$ by definition. Similarly, if the first toss is a tail with probability $1-p$, we need an even number of heads in the next $n-1$ tosses with probability p_{n-1} by definition. Therefore $p_n = p(1-p_{n-1})+(1-p)p_{n-1} = (1-2p)p_{n-1}+p$.

Find an expression with respect to p and n

$$
\begin{aligned}
p_n &= (1-2p)p_{n-1}+p && (0.0.22)\\
&= (1-2p)[(1-2p)p_{n-2}+p]+p && (0.0.23)\\
&= (1-2p)^2 p_{n-2}+(1-2p)p+p && (0.0.24)\\
&\vdots && (0.0.25)\\
&= (1-2p)^n p_0 + [(1-2p)^{n-1}+(1-2p)^{n-2}+\cdots+1]p && (0.0.26)\\
&= (1-2p)^n + p\frac{(1-(1-2p)^n)}{1-1+2p} && (0.0.27)\\
&= (1-2p)^n + \frac{1}{2}[1-(1-2p)^n] && (0.0.28)\\
&= \frac{1}{2}(1-2p)^n + \frac{1}{2} && (0.0.29)
\end{aligned}
$$

Question 5

Let A and B be the events with probabilities $P(A) = \frac{3}{4}$ and $P(B) = \frac{1}{3}$. Show that $\frac{1}{12} \leq P(AB) \leq \frac{1}{3}$, and give examples to show that both extremes are possible. Find corresponding bounds for $P(A \cup B)$.

Solution:

$$P(AB) = P(A) + P(B) - P(A \cup B) \geq P(A) + P(B) - 1 = \frac{1}{12},$$

Given $AB \subseteq A$ and $AB \subseteq B$, based on Lemma in Question 3, $P(AB) \leq min(P(A), P(B)) = \frac{1}{3}$.

To attain the bounds, consider picking numbers randomly from a set $\{1, 2, \ldots, 12\}$. Take $A = \{1, 2, \ldots, 9\}$ and $B = \{9, 10, 11, 12\}$. Now $AB = \{9\}$, and so $P(AB) = \frac{1}{12}$. To attain the upper bound, take $A = \{1, 2, \ldots, 9\}$ and $B = \{1, 2, 3, 4\}$, so $AB = \{1, 2, 3, 4\}$ and so $P(AB) = \frac{1}{3}$.

For $P(A \cup B)$, we know that

$$P(A \cup B) = P(A) + P(B) - P(AB) \leq P(A) + P(B),$$

but $P(A \cup B) \leq 1$ based on Axiom 1, hence $P(A \cup B) \leq min(P(A) + P(B), 1) = 1$ in this circumstance. The above equation also implies $P(A \cup B) \geq P(A)$ and $P(A \cup B) \geq P(B)$, and therefore $P(A \cup B) \geq max(P(A), P(B)) = \frac{3}{4}$ in this circumstance. To summarize, $\frac{3}{4} \leq P(A \cup B) \leq 1$.