ST2054 (and ST3068, ST6003) Problem Set 7 Due 5pm 13th of March 2020

Question 1

The cumulant generating function $K_X(\theta)$ of the random variable X is defined by $K_X(\theta) = \log E(e^{\theta X})$, the logarithm of the moment generating function of X. If the latter is finite in a neighbourhood of the origin, then K_X has a convergent Taylor expansion:

$$K_X(\theta) = \sum_{n=1}^{\infty} \frac{1}{n!} k_n(X) \theta^n$$

and $k_n(X)$ is called the nth cumulant of X.

(a) Express $k_1(X)$, $k_2(X)$ and $k_3(X)$ in terms of the moments of X.

(b) If X and Y are independent random variables, show that $k_n(X+Y) = k_n(X) + k_n(Y)$.

Hint: Use the polynomial coefficient matching techniques discussed in class.

Question 2

Let $X_1, X_2, \ldots, X_n$ be independent variables, X_i being $N(\mu_i, 1)$, and let $Y = X_1^2 + X_2^2 + \cdots + X_n^2$. Show that the characteristic function of Y is

$$\phi_Y(t) = \frac{1}{(1-2it)^{n/2}} \exp\left(\frac{it\theta}{1-2it}\right)$$

where $\theta = \mu_1^2 + \mu_2^2 + \cdots + \mu_n^2$.

Question 3

Let X be $N(0,1)$, and let $Y = e^X$; Y is said to have a log-normal distribution. Show that the density function of Y is:

$$f(x) = \frac{1}{x\sqrt{2\pi}} \exp\{-\frac{1}{2}(\log x)^2\}, x > 0.$$

For $|a| \leq 1$, define $f_a(x) = \{1 + a\sin(2\pi \log x)\} f(x)$. Show that f_a is a density function with finite moments of all (positive) order, none of which depends on the value of a.

Hint: Fully use the fact that $\sin x$ is an odd function.

Question 4

Consider a set of random variables $\{X_n : n \geq 1\}$ decays geometrically fast in that, in the absence of external input, $X_{n+1} = \frac{1}{2} X_n$. However, at any time n the corresponding random variable is also increased by Y_n with probability $\frac{1}{2}$, where $\{Y_n : n \geq 1\}$ is a sequence of independent exponential random variables with parameter λ. Find:

(a) The probability mass function of X_n. You do not need to simplify the expression.

(b) The limiting distribution of X_n as $n \to \infty$.