ST2054 (and ST3068, ST6003) Problem Set 6 Due 5pm 21st of February 2020

Question 1

(a) Let X have the binomial distribution $Bin(n, U)$, where U is uniform on $(0, 1)$. Show that X is uniformly distributed on $\{0, 1, 2, \ldots, n\}$. Hint: In this question, $X|U \sim Bin(n, U)$ as the parameter "p" of X depends on another random variable U.

(b) Similarly, let X have a Poisson distribution with parameter Λ, where Λ is exponential with parameter μ. Show that X has a geometric distribution.

Question 2

Let X and Y have a bivariate normal density with zero means, variance σ^2, τ^2, and correlation ρ. Show that:

(a) $E(X|Y) = \frac{\rho\sigma}{\tau}Y$,

(b) $var(X|Y) = \sigma^2(1 - \rho^2)$,

(c) $E(X|X + Y = z) = \frac{(\sigma^2+\rho\sigma\tau)z}{\sigma^2+2\rho\sigma\tau+\tau^2}$,

(d) $var(X|X + Y = z) = \frac{\sigma^2\tau^2(1-\rho^2)}{\tau^2+2\rho\sigma\tau+\sigma^2}$

Hint: X and Y can be written as $X = \sigma\rho U + \sigma\sqrt{1 - \rho^2}V$ and $Y = \tau U$, where U and V are independent $N(0, 1)$.

Question 3

A symmetric matrix is called non-negative definite if its eigenvalues are non-negative. Show that a non-negative definite symmetric matrix $\boldsymbol{V}$ has a square root, in that there exists a symmetric matrix $\boldsymbol{W}$ satisfying $\boldsymbol{W}^2 = \boldsymbol{V}$. Show further that $\boldsymbol{W}$ is non-symmetric if and only if $\boldsymbol{V}$ is positive definite.

Question 4

(a) Let X have the Poisson distribution with parameter Y, where Y has the Poisson distribution with parameter μ. Show that $G_{X+Y}(x) = \exp\{\mu(xe^{x-1} - 1)\}$

(b) Let $X_1, X_2, \ldots$ be independent identically distributed random variables with the following probability mass function

$$f(x) = \frac{(1 - p)^x}{x \log(1/p)}, \ x \geq 1,$$

where $0 < p < 1$. If N is independent of the X_i and has the Poisson distribution with parameter μ, show that $Y = \sum_{i=1}^{N} X_i$ has a negative binomial distribution.

(c) Let X have the binomial distribution with parameter n and p, and show that

$$E\left(\frac{1}{1+X}\right) = \frac{1-(1-p)^{n+1}}{(n+1)p}.$$

Hint: start with a function whose integral is $\frac{1}{1+X}$.