SI2024

2 DIML. RANDOM VARIABLES &

BIVARIATE PROB. DISTRIBUTIONS

L.79 - 99

The distrib. of a RANDOM VECTOR $\bar{X}, \bar{Y}$ is characterized by the distribution function

$$F(x, y) = P[X \leq x \text{ and } \bar{Y} \leq y]$$
$$= P^{\Omega}[\bar{X}(\omega) \leq x \text{ and } \bar{Y}(\omega) \leq y]$$

(also known) as JOINT DISTRIB. FN.

If we consider the interval for values of $\bar{X}$: $(x, x+h]$

" " " " $\bar{Y}$: $(y, y+k]$,

then

$$P[x < \bar{X} \leq x+h, \; y < \bar{Y} \leq y+k]$$
$$= F(x+h, y+k) - F(x+h, y) - F(x, y+k) + F(x, y)$$

which we will abbreviate as $\Delta^2 F$, known as the second difference of the function F.

Properties of bivariate distribution function $F(x, y)$:

1.) $F(x, \infty)$ is a univariate distrib fn of $\bar{X}$

2.) $F(\infty, y)$ " " " " " " $\bar{Y}$

3) $F(-\infty, y) = F(x, -\infty) = 0$

4) $\Delta^2 F \geq 0$ for every rectangle with sides parallel to the axes.

This is essential — since every rectangle must have a non-negve prob.

Problem 2.38 (L) P.87:

$$\left.\begin{aligned} F(x, y) &= 1 - e^{-x-y} && \text{for } x, y > 0 \\ &= 0 && \text{else} \end{aligned}\right]$$ is: **NOT** a distrib fn.

$$\Delta^2 F = 1 - e^{-x-y-h-k} - (1 - e^{-x-h-y}) - (1 - e^{-x-y-k}) + 1 - e^{-x-y}$$
$$= e^{-x-h-y}[1 - e^{-k}] - e^{-x-y}[1 - e^{-k}]$$
$$= (1 - e^{-k})[e^{-h} - 1]e^{-x-y}$$
$$< 0$$

2 useful types of Bivariate Distr

We shall consider 2 useful types of Bivariate distrib

DISCRETE : prob. concentrated at isolated points

Continuous : " spread over a region — and no single points OR CURVES have any positive prob.

— but there are OTHERS e.g. cont. in one variable and discrete in the other,

—— || ——

For discrete distributions it is convenient to work in terms of a PROBABILITY FUNCTION

JOINT PROB FUNCTION $\rhd$ $$f(x_i, y_j) = P(X = x_i, Y = y_j)$$

The values of $f(x_i, y_j)$ could be presented in the form of a table:

$$F(x_r, y_s) = \sum_{x_i \le x_r} \sum_{y_j \le y_s} \left[f(x_i, y_j) \right]$$

	X_1	X_2	X_3	- - -
y_1				
y_2				
y_3				
⋮				$f(x_i, y_j)$

CONDITIONS
(1) $f(x,y) \ge 0$
(2) $\sum_{x,y} f(x,y) = 1$

— or in the form of a function.

—— || ——

A bivariate distrib is said to be of continuous type if its distribution function is CONTINUOUS and has a second-order mixed partial derivative function

$$f(x, y)$$

from which F can be recovered by integration:

$$F(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u, v)\, dv\, du$$

$f(x,y)$ is the JOINT Prob DENSITY function of X, Y

The prob. that (X, Y) falls in a plane region S is computed from
$$P(S) = \iint_S f(x,y)\, dx\, dy.$$

ST2054 EXAMPLE OF A BIVARIATE DISTRIBUTION

Y X

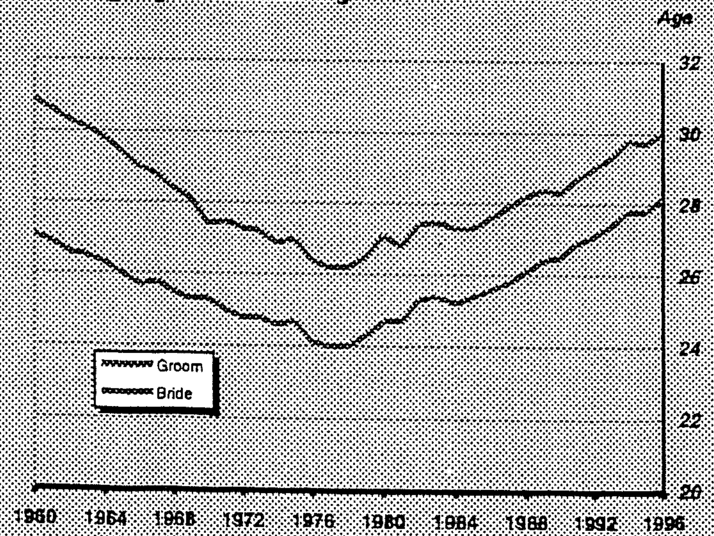
AGE OF

Table 3.11 Marriages registered in 1996 classified by age of bride and groom

Age of groom	Age of bride: Under 20	20-24	25-29	30-34	35-39	40-44	45-49	50-54	55-59	60 and over	Not stated	Total marriages
Under 20	51	24	5	–	–	1	–	–	–	–	–	**81**
20-24	81	1,123	475	54	6	–	–	–	–	–	–	**1,739**
25-29	38	1,802	4,695	834	93	8	2	1	–	–	6	**7,479**
30-34	17	450	2,340	1,586	249	26	5	–	1	–	5	**4,679**
35-39	3	74	364	553	261	57	10	2	–	–	1	**1,325**
40-44	1	13	92	128	133	55	14	3	1	–	1	**431**
45-49	1	2	15	36	47	41	28	7	–	1	–	**178**
50-54	–	1	3	11	16	14	12	11	5	1	–	**74**
55-59	–	–	3	4	4	3	9	14	4	8	–	**49**
60-64	–	–	–	–	2	3	6	3	9	10	–	**33**
65 and over	–	–	2	1	1	6	1	11	5	41	–	**68**
Age not stated	–	3	5	3	1	–	–	–	–	–	26	**38**
Total marriages	**192**	**3,492**	**7,989**	**3,210**	**813**	**214**	**87**	**52**	**25**	**61**	**39**	**16,174**

Source: CSO

Average age at first marriage



Marriage rate

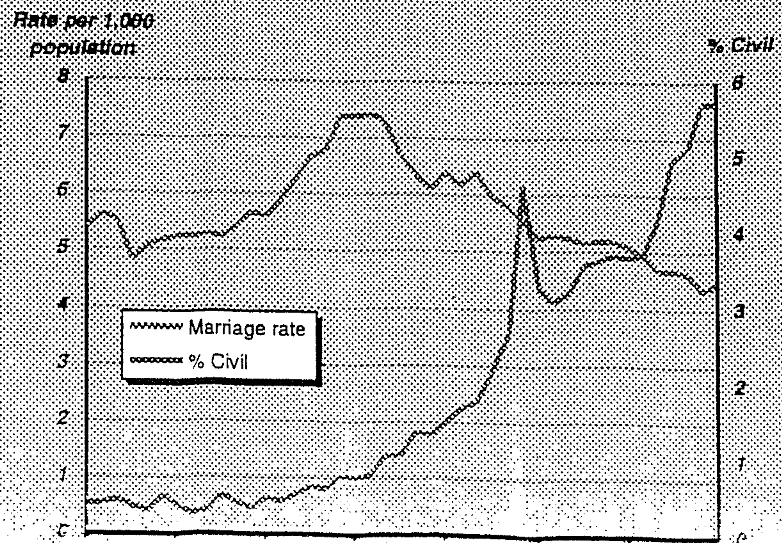


Table 25. Marriages registered in 1968, classified by age group of Groom and age group of Bride.

Age group of Groom	Age group of Bride											
	Under 20	20 - 24	25 - 29	30 - 34	35 - 39	40 - 44	45 - 49	50 - 54	55 - 59	60 and over	Not stated	All ages
Under 20 ..	307	205	8	1	1	—	—	—	—	—	4	527
20—24 ..	1,209	4,748	708	65	9	2	2	—	—	—	10	6,753
25—29 ..	274	3,258	2,273	327	47	9	5	—	—	—	16	6,249
30—34 ..	70	805	1,116	480	120	23	5	1	—	—	8	2,630
35—39 ..	19	212	388	378	182	54	24	—	1	—	7	1,261
40—44 ..	7	84	133	204	144	84	24	2	1	1	2	668
45—49 ..	4	25	87	85	94	58	42	7	6	1	8	362
50—54 ..	—	9	10	32	43	48	35	24	6	1	—	208
55—59 ..	—	4	—	6	17	22	33	20	15	5	—	122
60—64 ..	—	—	2	5	2	19	16	15	9	11	—	79
65 and over	—	1	—	1	1	1	6	6	15	27	1	58
Not stated	6	17	7	8	—	1	—	1	—	—	43	77
All ages ..	1,895	9,589	4,090	1,686	660	323	193	75	52	46	94	18,928 18933

Table 25(a). Marriages registered in 1969, classified by age group of Groom and age group of Bride.

Age group of Groom	Age group of Bride											
	Under 20	20 - 24	25 - 29	30 - 34	35 - 39	40 - 44	45 - 49	50 - 54	55 - 59	60 and over	Not stated	All ages
Under 20 ..	411	183	8	—	—	1	—	—	—	—	—	603
20—24 ..	1,234	5,305	842	79	[illegible]	[illegible]	—	[illegible]	—	—	20	7,581
25—29 ..	273	3,564	2,315	331	51	6	1	—	—	—	16	6,658
30—34 ..	61	808	1,127	511	136	22	8	1	—	—	7	2,681
35—39 ..	10	213	389	369	212	63	13	3	—	—	4	1,278
40—44 ..	11	59	121	163	158	78	26	6	—	—	1	624
45—49 ..	3	21	41	69	83	75	40	19	2	1	—	354
50—54 ..	—	10	9	15	44	47	48	16	8	3	—	200
55—59 ..	1	2	6	4	15	28	28	21	16	4	—	121
60—64 ..	—	1	3	1	4	8	14	13	4	3	—	51
65 and over	—	—	3	1	1	8	10	11	11	33	1	79
Not stated	4	10	8	6	[illegible]	—	—	—	—	—	43	78
All ages ..	2,076	10,276	4,903	1,542	713	334	186	91	41	44	98	20,304

For a bivariate density function, we have

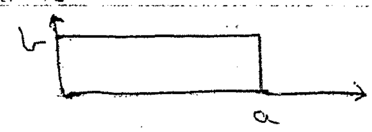
(1) $f(x, y) \geq 0$

(2) $\iint_{R_2} f(x,y) = 1$

Examples

(1) Uniform $f(x, y) = k$ for $0 \leq x \leq a$ and $0 \leq y \leq b$

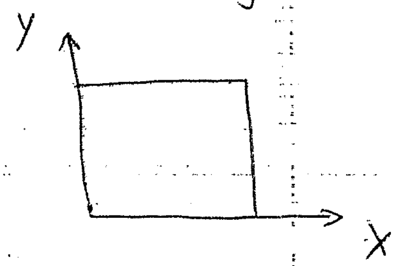
$= 0$ else

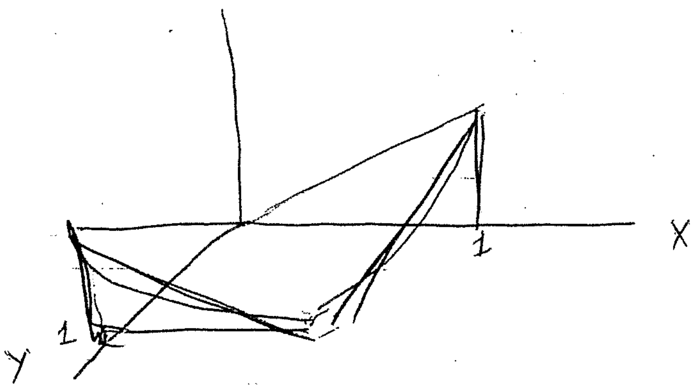


Thus require $k = \frac{1}{ab}$

(2) $f(x,y) = 2(x + y - 2xy)$ for $0 \leq x \leq 1$ and $0 \leq y \leq 1$

$= 0$ else





$4x^2 - 2x + 1$

$x > \frac{1}{2}$ $y > \frac{1}{2}$

$2x^2 < x + \frac{1}{2}$

$4 -$

~~$x+y>1$~~ x

$2x^2 -$

$2x^2 < 2x$

$x + y > 2xy$?

~~$x+y$~~ $y < 1$ $x > y$

$2xy < 2x^2$

ILLUSTRATIVE EXA. : 2 Diml Random Var.

Random Expt : 3 coins tossed

X = # heads

Y = # runs

Elem Events	→ Corresponding X, Y values
H H H	3 , 1
H H T	2 , 2
H T H	2 , 3
H T T	1 , 2
T H H	2 , 2
T H T	1 , 3
T T H	1 , 2
T T T	0 , 1

assign Probs to the elem. events $\left(\frac{1}{8}\right)$

X - Values : 0, 1, 2, 3

Y - values 1, 2, 3

Arrange Probs in 2 Way Table

X \ Y	1	2	3	$f_X(x_i)$
0	$\frac{1}{8}$	0	0	$\frac{1}{8}$
1	0	$\frac{2}{8}$	$\frac{1}{8}$	$\frac{3}{8}$
2	0	$\frac{2}{8}$	$\frac{1}{8}$	$\frac{3}{8}$
3	$\frac{1}{8}$	0	0	$\frac{1}{8}$
$f_Y(y_j)$	$\frac{2}{8}$	$\frac{4}{8}$	$\frac{2}{8}$	

Marginals

MARGINAL PROB. DISTRIBUTIONS

SKIP

$F(x, \infty)$ and $F(\infty, y)$ are in fact the distrib fns of $\underline{X}$ and of $\overline{Y}$, considered as single random variables.

i.e. $F(x,\infty) = P[\underline{X} \leq x \text{ and } \overline{Y} \leq \infty] = P[\underline{X} \leq x]$

$F(\infty, y)$ – – – – – – – $= P[\overline{Y} \leq y]$

These distribs are known as the MARGINAL distribs of $\underline{X}$ and $\overline{Y}$

refers only to their origin

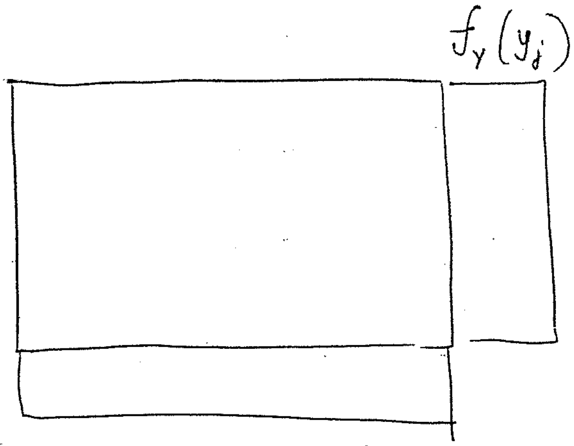
discrete case :

the marginal distrib of $\underline{X}$ is given by

$$P(\underline{X} = x_i) = \sum_{y_j} P[\underline{X} = x_i, \overline{Y} = y_j]$$

e.g.

Exa



Continuous Case

Here we obtain the density fn. of the marginal distrib. by differentiating the MARGINAL DISTRIBUTION FUNCTION :

$$\text{i.e. } f_{\underline{X}}(x) = \frac{d}{dx} F_{\underline{X}}(x) \equiv \int_{-\infty}^{\infty} f(x,y)\, dy$$

Thus <u>ONE INTEGRATES OUT</u> the unwanted variable from the joint density

Exa L. p 87

MARGINAL DISTRIBUTIONS (DISCRETE CASE)

Another Example : Introduce Multinomial Distrib : (L p. 196)

Suppose that we consider the Bernoulli trial experiment — but let us now suppose that the number of possible outcomes is k, rather than 2, (i.e. $A_1, A_2, \ldots, A_k$) — and that the prob. of outcome A_i is p_i for $i = 1, 2, \ldots, k$

(where $\sum p_i = 1$)

We are interested in the Random Variables $X_1, X_2, \ldots, X_k$ where X_i is the number of outcomes A_i observed in n repetitions of the expt.

Each individual X_i is binomially distributed (n, p_i) but we are interested in the joint distribution of the X_i's.

(Notice that $\sum X_i = n$, so that there are only $(k-1)$ variables).

The Joint Prob. Distrib of the X_i's is referred to as the MULTINOMIAL DISTRIB.

Derivation of Form of Distrib

If we consider a particular sequence of n trials which result in r_1 occurences of A_1, r_2 of A_2 - - - r_k of A_k

— then the prob. of this is just

$$p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$$

However there are many different sequences which give r_1 A_1's ; r_2 A_2's etc

— and the number of such sequences is given by the number of arrangements of n objects where r_1 are of one kind, r_2 are of another kind — and so on

— i.e.

$$\frac{n!}{r_1!\, r_2!\, r_3! \cdots r_k!}$$

So that

$$P[X_1 = r_1, X_2 = r_2, \ldots, X_k = r_k] = \frac{n!}{r_1!\, r_2! \cdots r_k!} p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$$

NOTE Term Multinomial derives from fact that these probs are given by terms in the Multinomial Expn $(p_1 + p_2 + \cdots + p_k)^n \equiv 1$

e.g. Die is tossed 10 times

A_1 : outcome is even $\quad p_1 = \frac{1}{2}$

A_2 : outcome is 3 or 5 $\quad p_2 = \frac{1}{3}$

A_3 : " " 1 $\quad p_3 = \frac{1}{6}$

Then Prob. that in 10 tosses, we will get
3 even outcomes
5 outcomes of either 3 or 5
and 2 " " 1

$$= \frac{10!}{3!\,5!\,2!}\left(\frac{1}{2}\right)^3\left(\frac{1}{3}\right)^5\left(\frac{1}{6}\right)^2$$

$$= .036008$$

Marginal Distributions for this Example

In General: Marginal Distrib of X_i is clearly Binomial (n, p_i)

$$P[X_i = r_i] = \sum_{r_1}\sum_{r_2}\sum \cdots \sum_{r_k} \frac{n!}{r_1!\,r_2!\cdots r_k!} p_1^{r_1} p_2^{r_2}\cdots p_k^{r_k}$$

Marginal Distribution of (X_1, X_2) is Trinomial

i.e. $$P[X_1 = r_1 ; X_2 = r_2] = \frac{n!}{r_1!\,r_2!\,(n-r_1-r_2)!}(p_1)^{r_1} p_2^{r_2}(1-p_1-p_2)^{n-r_1-r_2}$$

More Particularly:

Suppose we consider Trinomial Distrib:

$$P[X_1 = r_1 ; X_2 = r_2] = \frac{n!}{r_1!\,r_2!\,(n-r_1-r_2)!} p_1^{r_1} p_2^{r_2}(1-p_1-p_2)^{n-r_1-r_2}$$

Marginal Distrib of X_1:

$$P[X_1 = r_1] = \sum_{r_2} \frac{n!}{r_1!\,r_2!\,(n-r_1-r_2)!} p_1^{r_1} p_2^{r_2}(1-p_1-p_2)^{n-r_1-r_2}$$

$\boxed{\text{Exercise}}$ Show $$= \frac{n!}{r_1!\,(n-r_1)!} p_1^{r_1}(1-p_1)^{n-r_1}$$

MARGINAL DISTRIBUTION IN THE CONTINUOUS CASE

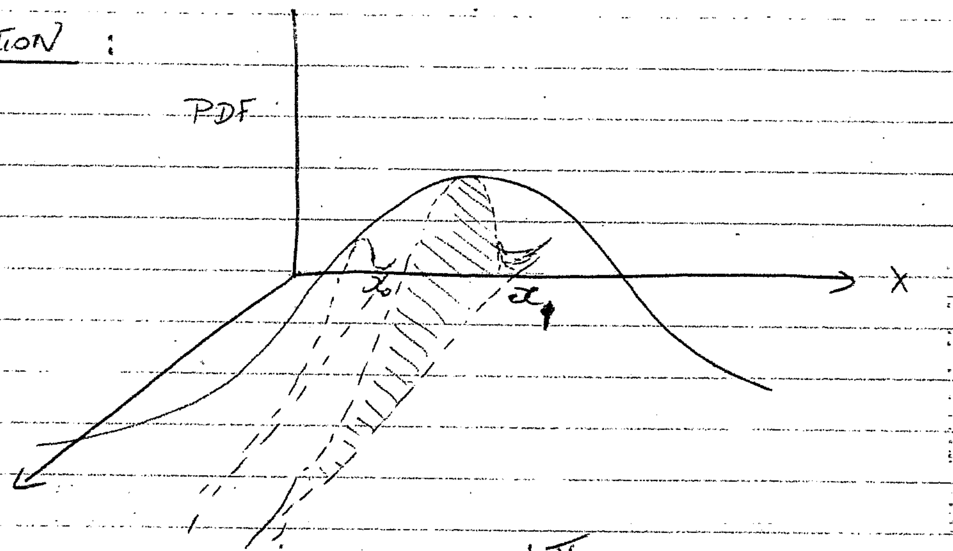
The Marginal Distrib of X is given by

$$\int_{-\infty}^{\infty} f(x,y)\,dy$$

where $f(x,y)$ is the joint P.D.F. of X, Y

and this is often written as $f_X(x)$

ILLUSTRATION :

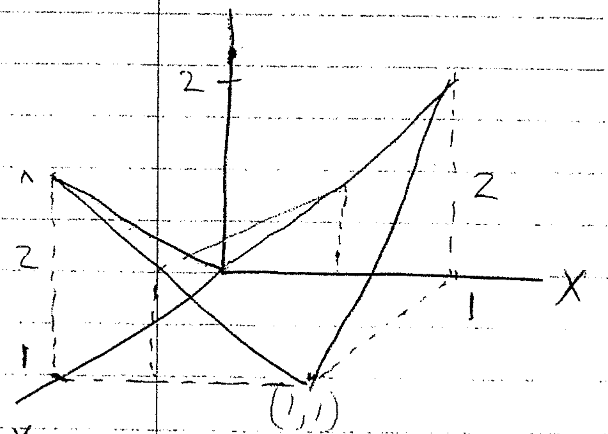


Geometrically, the marginal density of X at x_0 is the total area of the cross-section at x_0

EXAMPLE

Consider joint PDF $f(x,y) = 2(x+y-2xy)$

$0 \leq x \leq 1$
$0 \leq y \leq 1$



$$f_X(x) = \int_{-\infty}^{\infty} 2(x+y-2xy)\,dy$$

$$= \int_0^1 2(x+y-2xy)\,dy$$

$$= 2\left[xy + \frac{y^2}{2} - xy^2\right]_0^1$$

$$= 2\left[x + \tfrac{1}{2} - x\right] - 0$$

$$= 1 \quad \text{for all } x \in [0,1]$$

Similarly $f_Y(y) = 1$

ASSOCIATED

Conditional Distributions with 2 Dim'l R. Vars

Here we are concerned with prob. dist'n of X given that Y takes on a value in some set (usually a single value)

Discrete Case:

$$\text{Prob}[X = x_i \mid Y = y_j] = \frac{P[X = x_i, Y = y_j]}{P[Y = y_j]}$$

$$= \frac{f(x_i, y_j)}{f_Y(y_j)} \qquad \frac{\text{Joint Prob. Fn}}{\text{Marginal}}$$

NOTATION sometimes used $f_{X|Y}(x_i \mid y_j)$

Our example

		Y: 1	2	3
X	0	$\frac{1}{8}$	0	0
	1	0	$\frac{2}{8}$	$\frac{1}{8}$
	2	0	$\frac{2}{8}$	$\frac{1}{8}$
	3	$\frac{1}{8}$	0	0

	X: 0	1	2	3 ← VALUES
Cond'nal Probs [of $X \mid Y = 1$] :	$\frac{1}{2}$	0	0	$\frac{1}{2}$
" Probs [of $X \mid Y = 2$] :	0	$\frac{1}{2}$	$\frac{1}{2}$	0
" Prob. [of $X \mid Y = 3$]	0	$\frac{1}{2}$	$\frac{1}{2}$	0

Probs

Similarly

	Y: 1	2	3
Probs $[Y \mid X = 0]$ →	1	0	0
Prob $[Y \mid X = 1]$ →	0	$\frac{2}{3}$	$\frac{1}{3}$
Prob $[Y \mid X = 2]$ →	0	$\frac{2}{3}$	$\frac{1}{3}$
Prob $[Y \mid X = 3]$ →	1	0	0

MULTINOMIAL EXAMPLE : CONDITIONAL PROBS

We have the random variables $X_1, X_2 \ldots, X_k$ having a multinomial distrib.

— There are clearly many conditional probs that could be ~~defined~~ investigated

— but it is quite easy to see the pattern even for k variables:

Suppose we want condnal prob. distrib of $X_3, X_4 \ldots X_k$, given that $X_1 = r_1$ and $X_2 = r_2$

Well, $\text{Prob}\left[X_3 = r_3, X_4 = r_4 \ldots, X_k = r_k \mid X_1 = r_1, X_2 = r_2\right]$

$$= \frac{\text{Prob}\left[X_1 = r_1, X_2 = r_2, \ldots, X_k = r_k\right]}{P\left[X_1 = r_1, X_2 = r_2\right]}$$

$$= \frac{\dfrac{n!}{r_1!\, r_2! \ldots r_k!}\, p_1^{r_1} p_2^{r_2} \ldots p_k^{r_k}}{\dfrac{n!}{r_1!\, r_2!\, (n - r_1 - r_2)!}\, p_1^{r_1} p_2^{r_2} (1 - p_1 - p_2)^{n - r_1 - r_2}}$$

Here we are using our earlier result on Marginal Distribs.

Now $(n - r_1 - r_2) = r_3 + r_4 + \ldots + r_k$

So that we have

$$= \frac{(n - r_1 - r_2)!}{r_3!\, r_4! \ldots r_k!} \left(\frac{p_3}{1 - p_1 - p_2}\right)^{r_3} \left(\frac{p_4}{1 - p_1 - p_2}\right)^{r_4} \ldots \left(\frac{p_k}{1 - p_1 - p_2}\right)^{r_k}$$

I.E. the multinomial distrib. with $(n - r_1 - r_2)$ trials, $(k-2)$ possible outcomes on each trial and probabilities

$$\left(\frac{p_i}{1 - p_1 - p_2}\right) \quad \text{for } i = 3, 4 \ldots, k \quad \text{for each outcome } i$$

Conditional Probs in the Continuous Case

If we consider the case of X, Y continuous random vars and we investigate the prob. distrib of X given $Y = y$, say — it would appear that there are difficulties since $P(Y = y) = 0$ for continuous random vars and conditional probabilities are therefore not defined.

Despite this it is useful to define a conditional probability density for the continuous case also.

— this conditional distrib of X given $Y = y$ should have the following property: (by analogy with discrete case)

$$\int_{-\infty}^{\infty} f(x|y) f_Y(y)\, dy = f_X(x)$$

and we can see that this property holds when we define

$$f(x|y) = \frac{f(x,y)}{f_Y(y)}$$

This defn results in a conditional pdf which satisfies the properties
(1) $f(x|y) \geq 0$
(2) $\int_{-\infty}^{\infty} f(x|y) = 1$

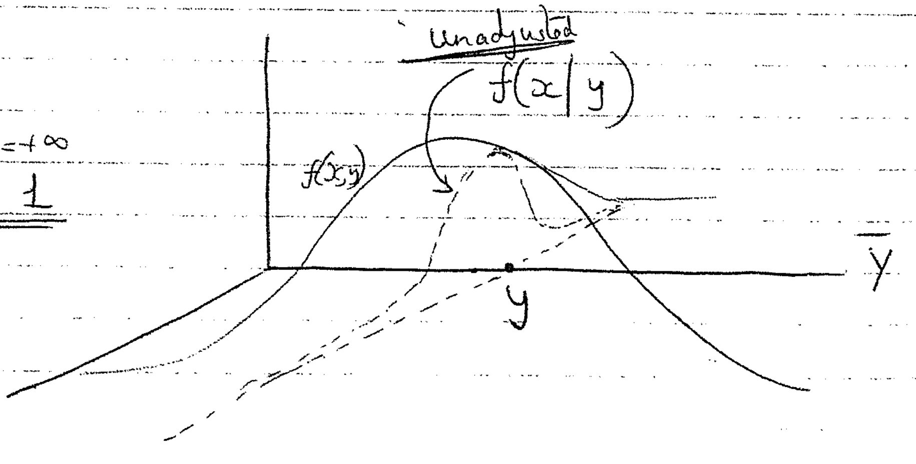
Geometric Interpretn:

Integral from $x = -\infty$ to $x = +\infty$ of cross sectional area is NOT 1

But rather $f_Y(y)$

Thus it's necessary to adjust "height" of cross section by dividing by $f_Y(y)$

so that $\int_{-\infty}^{\infty} f(x|y)\, dx = 1$



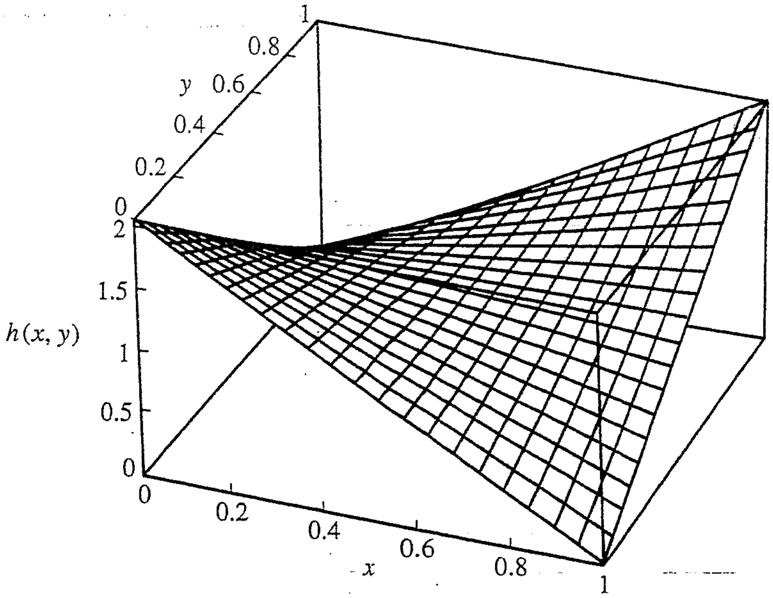


FIGURE 3.6
The joint density $h(x, y) = 2 - 2x - 2y + 4xy$, where $0 \le x \le 1$ and $0 \le y \le 1$, which has uniform marginal densities.

We have just constructed two different bivariate distributions, both of which have uniform marginals. ∎

EXAMPLE D Consider the following joint density:

$$f(x, y) = \begin{cases} \lambda^2 e^{-\lambda y}, & 0 \le x \le y, \lambda > 0 \\ 0, & \text{elsewhere} \end{cases}$$

This joint density is plotted in Figure 3.7. To find the marginal densities, it is helpful to draw a picture showing where the density is nonzero to aid in determining the limits of integration (see Figure 3.8). We then have

$$f_X(x) = \int_x^\infty \lambda^2 e^{-\lambda y}\, dy = \lambda e^{-\lambda x}, \qquad x \ge 0$$

The marginal distribution of X is exponential. The marginal density of Y is

$$f_Y(y) = \int_0^y \lambda^2 e^{-\lambda y}\, dx = \lambda^2 y e^{-\lambda y}, \qquad y \ge 0$$

The marginal distribution of Y is a gamma distribution. ∎

In some applications, it is useful to analyze distributions that are uniform over some region of space. For example, in the plane, the random point (X, Y) is uniform over a region, R, if for any $A \subset R$,

$$P((X, Y) \in A) = \frac{|A|}{|R|}$$

where $|\ |$ denotes area.

FIGURE 3.7
The joint density of Example D.

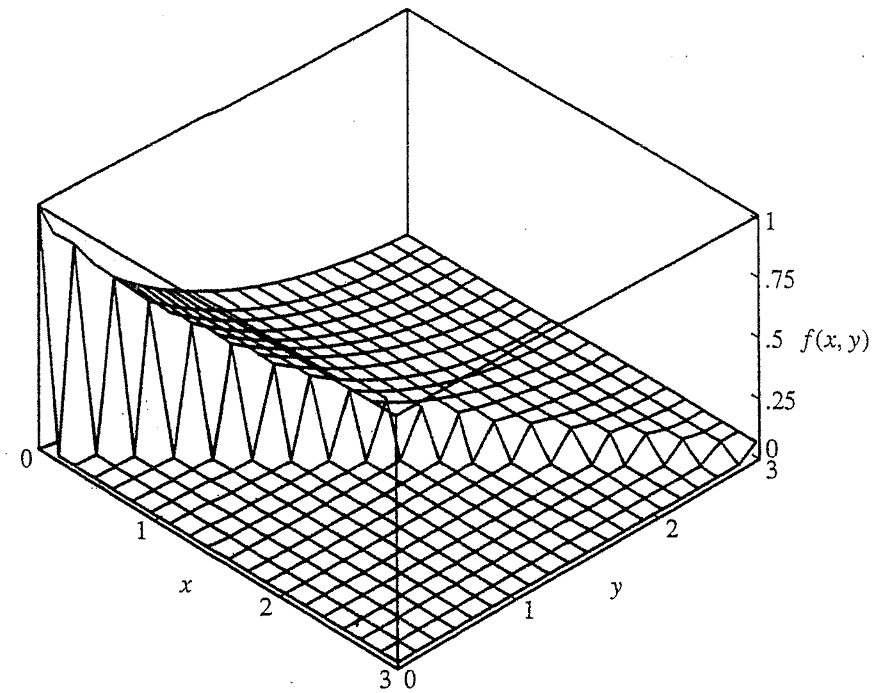
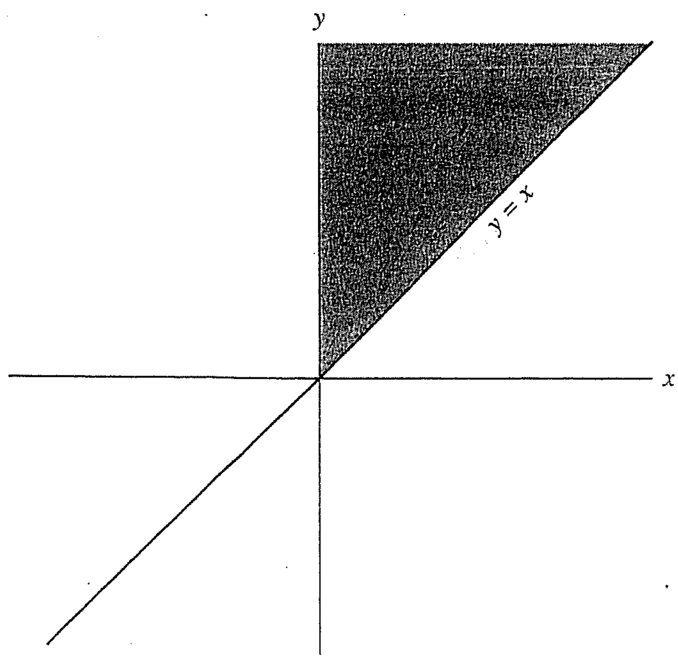


FIGURE 3.8
The joint density of Example D is nonzero over the shaded region of the plane.



Marginal Probs in terms of Condnal Probs

DISCRETE

We have
$$f_X(x_i) = \sum_j f(x_i, y_j)$$
$$= \sum_j f(x_i \mid y_j)\, f_Y(y_j)$$

Which can be quite useful
— and is yet another example of the Law of Total Prob.

EXAMPLE :

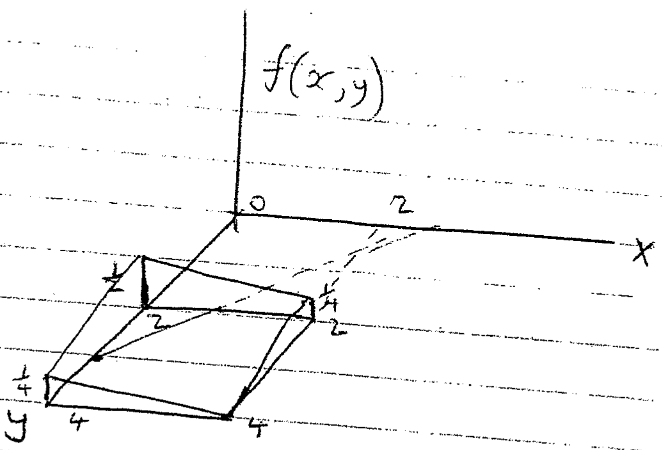
Suppose $f(x,y) = \frac{1}{8}(6-x-y)$ for $0 \leq x \leq 2$ and $2 \leq y \leq 4$

(Simple to show that 1) $f(x,y) \geq 0$
2) $\iint f(x,y) = 1$)

It is easily seen that

$f_X(x) = \frac{1}{4}(3-x)$

$f_Y(y) = \frac{1}{4}(5-y)$



Thus $f(x|y) = \dfrac{\frac{1}{8}(6-x-y)}{\frac{1}{4}(5-y)} = \dfrac{[(6-x-y)]}{2(5-y)}$

and $f(y|x) = \dfrac{(6-x-y)}{2(3-x)}$

To help in understanding these various distribs, let us evaluate

$P[0 < X < 1 ; 3 < Y < 4]$

$P[Y < 3]$

$P[X < 1 \mid Y = 3]$ and $P[X < 1 \mid Y < 3]$

$P[X + Y < 3]$

$$P[0 < X < 1 ; 3 < Y < 4] = \int_3^4 \left[\int_0^1 \frac{1}{8}(6-x-y)\,dx\right] dy$$

$$= \frac{1}{8}\int_3^4 \left[\frac{11}{2} - y\right] dy$$

$$= \frac{2}{8} = \frac{1}{4}$$

12.9

$$P[X + Y < 3] = \int_t P[X \le t \text{ and } Y = 3 - t]\, dt$$

$$= \int_0^1 \tfrac{1}{8}\left[6 - t - (3 - t)\right] dt$$

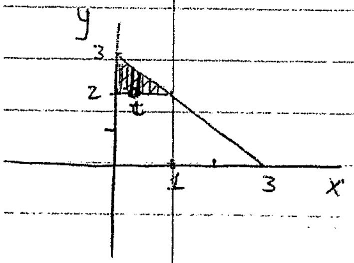
$$= \cancel{\tfrac{1}{8}\left[6t - \tfrac{t^2}{2}\right.}$$

$$= \tfrac{1}{8}\int_0^1 [3]\, dt$$

$$= \frac{3}{8}$$

(3)

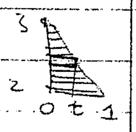
$$= \tfrac{1}{8}\int_0^1 \left(\int_0^t [6 - x - 3 + t]\, dx \right) dt$$



$$= \tfrac{1}{8}\int_0^1 \left[6t - \frac{t^2}{2} - 3t + t^2\right] dt$$

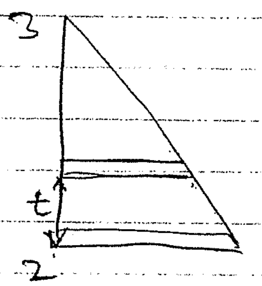
$$= \tfrac{1}{8}\left[\frac{6t^2}{2} - \frac{t^3}{6} - \frac{3t^2}{2} + \frac{t^3}{3}\right]_0^1$$

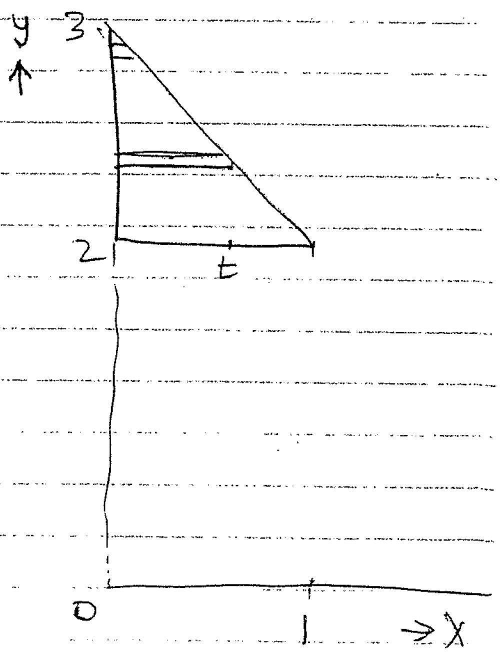
$$= \tfrac{1}{8}\left[3 - \tfrac{1}{6} - \tfrac{3}{2} + \tfrac{1}{3}\right] = \tfrac{1}{8}\left[\frac{10}{6}\right]$$



$$= \frac{10}{48} = \frac{5}{24}$$

$$\int^3\int_0^{3-y} f(x,y)\, dx\, dy$$





$y \to t$

$x \to (3-t)$

$$= \int_2^3 \left[\int_0^{3-y} \tfrac{1}{8}(6 - x - y)\, dx\right] dy$$

$$= \tfrac{1}{8}\int_2^3 \left[6x - \frac{x^2}{2} - xy\right]_0^{3-y} dy$$

$$\tfrac{1}{8}\int_2^3 6(3-y) - \frac{(3-y)^2}{2} - (3-y)y \quad dy$$

$$= \tfrac{1}{8}\left[18y - 3y^2 + \frac{(3-y)^3}{6} - \frac{3y^2}{2} + \frac{y^3}{3}\right]_2^3$$

$$= \tfrac{1}{8}\left[18 - 3(5) - \tfrac{1}{6} - \tfrac{3}{2}(5) + \tfrac{19}{3}\right] = \frac{108 - 90 - 1 - 45 + 38}{48}$$

$\cancel{108 - 180 - 1 - 15} + 38$

$$= \frac{146 - 136}{} = \frac{5}{24}$$

$P(Y < 3)$: use Marginal $f_Y(y)$

$$\int_2^3 f_Y(y)\,dy = \int_2^3 \frac{1}{4}(5-y)\,dy$$

$$= \frac{5}{8}$$

$P[X < 1 \mid Y = 3]$ use $f(x|y) = \dfrac{6-x-y}{2(5-y)}$

using $y = 3$

i.e. $f(x|y) = \dfrac{3-x}{2(2)} = \frac{1}{4}(3-x)$

Thus $P(X < 1 \mid Y = 3) = \int_0^1 \frac{1}{4}(3-x)\,dx = \frac{1}{4}\left[3x - \frac{x^2}{2}\right]_0^1$

$$= \frac{5}{8}$$

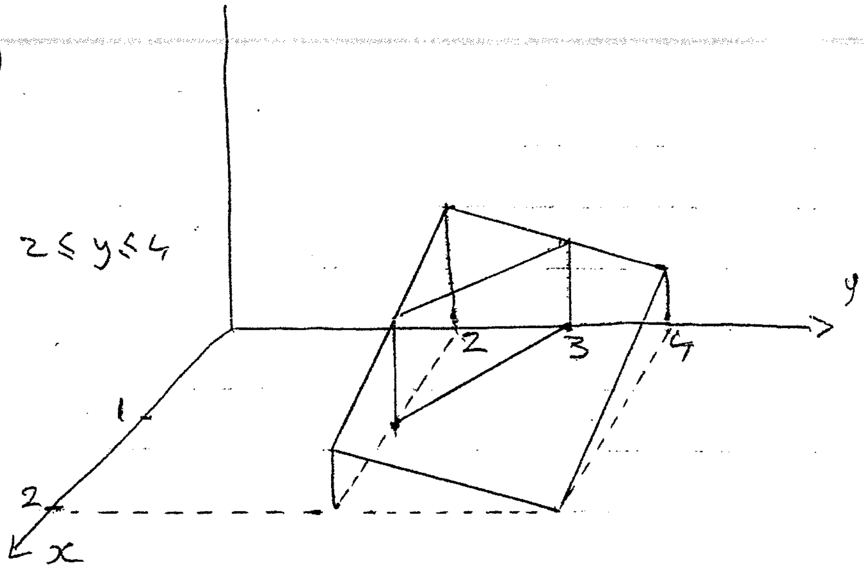
$$P[X < 1 \mid Y < 3] = \frac{P[X < 1 \text{ and } Y < 3]}{P[Y < 3]}$$

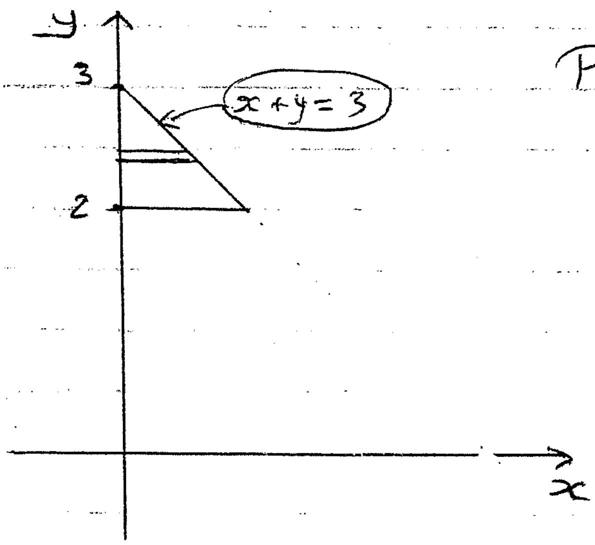
Evaluation of Probability using Double Integration

Example: Find $P(X+Y<3)$

for $f(x,y) = \frac{1}{8}(6-x-y)$ for $0 \le x \le 2,\ 2 \le y \le 4$

$= 0$ elswhere





$$P(X+Y<3) = \iint_{\{x+y<3\}} f(x,y)\,dx\,dy$$

$$= \int_{y=2}^{y=3} \left[\int_{x=0}^{x=3-y} \frac{1}{8}(6-x-y)\,dx \right] dy$$

$$= \frac{1}{8} \int_2^3 \left[6x - \frac{x^2}{2} - xy \right]_0^{3-y} dy$$

$$= \frac{1}{8} \int_2^3 \left[6(3-y) - \frac{(3-y)^2}{2} - (3-y)y \right] dy$$

$$= \frac{1}{8} \int_2^3 \left[18 - 6y - \frac{(3-y)^2}{2} - 3y + y^2 \right] dy$$

$$= \frac{1}{8} \left[18y - 3y^2 + \frac{(3-y)^3}{6} - \frac{3y^2}{2} + \frac{y^3}{3} \right]_2^3$$

$$= \frac{1}{8} \left[18 - 3(9-4) - \frac{1}{6} - \frac{3}{2}(9-4) + \frac{1}{3}(27-8) \right]$$

$$= \frac{1}{48} \left[108 - 90 - 1 - 45 + 38 \right]$$

$$= \frac{5}{24}$$

INDEPT RANDOM VARS

It can happen that the condnal distrib of $\bar{X}$ is indept of the value of the other random variable $\bar{Y}$

e.g. $f(x,y) = e^{-x-y}$

for which $f_{\bar{X}}(x) = e^{-x}$

$f_{\bar{Y}}(y) = e^{-y}$

and $f(x|y) = e^{-x}$ $\quad f(y|x) = e^{-y}$

DEFN We shall say that the contin random vars $\bar{X}$ and $\bar{Y}$ are indept iff

$\frac{f(x,y)}{f_Y(y)}$

$$f(x|y) = f_{\bar{X}}(x) \iff f(y|x) = f_{\bar{Y}}(y)$$

or equivalently

$$f(x,y) = f_{\bar{X}}(x)\, f_{\bar{Y}}(y)$$

(with a similar defn for the case of 2 discrete r. vars)

This implies that for any event A in the value space of $\bar{X}$
and " " " B " " " " $\bar{Y}$

$$P[\bar{X} \in A \text{ and } \bar{Y} \in B] = P[\bar{X} \in A]\, P[\bar{Y} \in B]$$

Generalizing

DEFN We shall say that n random vars $\bar{X}_1, \bar{X}_2, \dots, \bar{X}_n$ are indept if

$$f(x_1, x_2, \dots, x_n) = f_{\bar{X}_1}(x_1)\, f_{\bar{X}_2}(x)\, \dots\, f_{\bar{X}_n}(x_n)$$

where $f_{\bar{X}_i}(x_i)$ are the marginal pdfs.

Examples of Indept and Dept. R. Vars.

X_1 : score on first throw of die
X_2 : " " 2^{nd} throw of die

These R. Vars are intuitively indept.

Marginal Distb Fns are $\left(\frac{1}{6}\ \frac{1}{6}\ \frac{1}{6}\ \cdots\ \frac{1}{6}\right)$ for X_1 and X_2

so that $f(x_i, y_j)$ is

Second

1st	1	2	3	4	5	6
1						
2						
3			$\frac{1}{36}$			
4						
5						
6						

Dept R. Vars

X_1 = Score on 1^{st}
X_2 = larger of scores on 1^{st} and Second throws

X_2

X_1	1	2	3	4	5	6	
1	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{6}$
2	0	$\frac{2}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{6}$
3	0	0	$\frac{3}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{6}$
4	0	0	0	$\frac{4}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{6}$
5	0	0	0	0	$\frac{5}{36}$	$\frac{1}{36}$	$\frac{1}{6}$
6	0	0	0	0	0	$\frac{6}{36}$	$\frac{1}{6}$
	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{39}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$	

clearly

$$f(x_i, y_j) \neq f(x_i)\,f(y_j)$$

OR

Y

X	1	2	3	
0	$\frac{1}{8}$	0	0	$\frac{1}{8}$
1	0	$\frac{2}{8}$	$\frac{1}{8}$	$\frac{3}{8}$
2	0	$\frac{2}{8}$	$\frac{1}{8}$	$\frac{3}{8}$
3	$\frac{1}{8}$	0	0	$\frac{1}{8}$
	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	

Expectations for 2 Dim'l R. Vars (Meyer p.127)

Let $\mathbb{X}, \mathbb{Y}$ be a 2 dim'l r.v.
Suppose $\mathbb{Z} = G(\mathbb{X}, \mathbb{Y})$ real valued fn
Then $\mathbb{Z}$ is a one-dim'l r.var

We could determine the prob. dist'n of $\mathbb{Z}$ from a knowledge of $f(x, y)$ and of G
— this could be quite difficult
— Suppose it were $g(z)$

Then
$$E(\mathbb{Z}) = \sum_{z_i} z_i \, g(z_i) \quad \text{if } \mathbb{Z} \text{ discrete}$$
$$= \int_z z \, g(z) \quad \text{" " contin.}$$

More conveniently

The following result is true

$$E(\mathbb{Z}) = \sum_{x_i} \sum_{y_j} G(x_i, y_j) \, f(x_i, y_j) \quad \text{for } \mathbb{X}, \mathbb{Y} \text{ discrete}$$
$$= \iint G(x, y) \, f(x, y) \, dx\,dy \quad \text{for } \mathbb{X}, \mathbb{Y} \text{ contin.}$$

Some Properties of Expected Value

1) $E(a\mathbb{X} + b) = a\,E(\mathbb{X}) + b$ a, b consts

2) $G_1(x, y)$, $G_2(x, y)$ two real valued fns of x, y

Then
$$E\left[G_1(\mathbb{X}, \mathbb{Y}) + G_2(\mathbb{X}, \mathbb{Y})\right] = E\left[G_1(\mathbb{X}, \mathbb{Y})\right] + E\left[G_2(\mathbb{X}, \mathbb{Y})\right]$$

3) $E(\mathbb{X} + \mathbb{Y}) = E(\mathbb{X}) + E(\mathbb{Y})$

4) $E\left(\sum^{n} \mathbb{X}_i\right) = \sum^{n} E(\mathbb{X}_i)$

5) $E(\mathbb{X}\mathbb{Y}) = (E\mathbb{X})(E\mathbb{Y})$ for indep't $\mathbb{X}, \mathbb{Y}$

$$= \iint x\,y\, f(x, y)\, dx\,dy = \iint x\,y\, f_{\mathbb{X}}(x)\, f_{\mathbb{Y}}(y)\, dx\,dy$$