Infinite Compositions

This is a document exploring infinite composition of functions.

Terminology I will use:

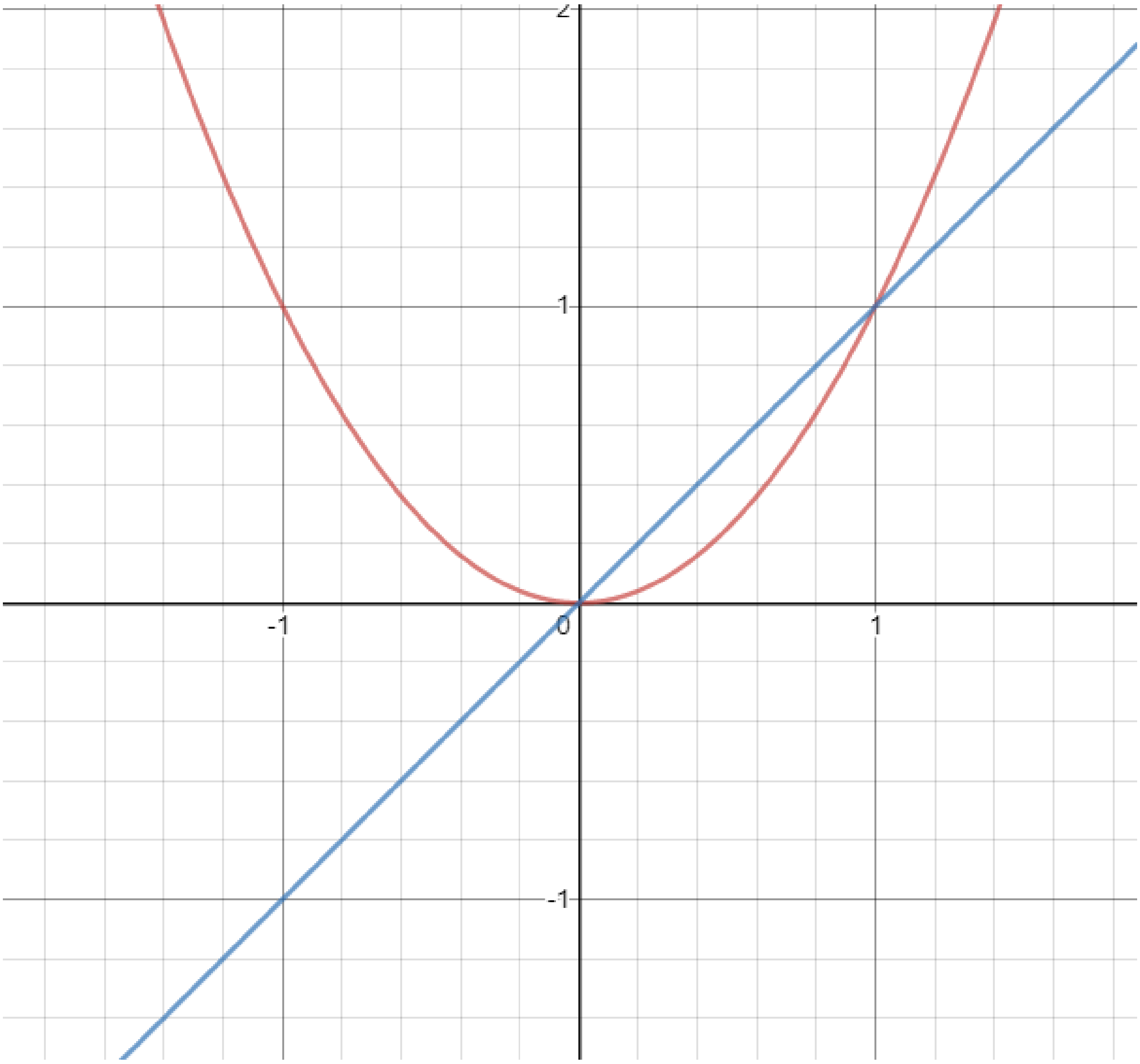
- $f(x)$ or f
    - is the initial function.
- $g(x)$ or g
    - is the final function that is produced after infinite compositions of f.
- A stable point is where $g(x) = x$
    - This also implies that $f(x) = x$
    - Also, stable points can be defined as $r(x) = 0$
    - Stable points are attractors if $r'(x) < 0$
- An attractor point is where $g(d) = c$ where d is a certain interval like: $a < d < b$.
    - All attractors are stable points, **but not vise versa**.
    - Note that the interval does not need to be symmetric!
    - An attractor has at least one slide around it, **but not vise versa**.
        - Given that f is continuous, an attractor will have two slides around it
    - An attractor is defined by: $r(x) = 0$ and $r'(x) < 0$.
- $r(x)$ or r
    - $r(x)$ is the residual function, which tells us how the function will change the next recursion.
        - $dx = r(x)$
    - It is defined as: $r(x) = f(x) - x$
- Slides
    - A slide is an range where $r(b) > 0$ or $r(b) < 0$ defined by the range $a < b < c$.
    - A slide goes down, or to the left if $r(b) < 0$
    - A slide goes up, or to the right if $r(b) > 0$
- $sign(x) = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}$

Baby Steps

Interpreting Graphs of f and finding g.

Let's get a feel for how these functions work!

Let's consider the function: $f(x) = x^2$. When the function intersects the line $y = x$, it will have a stable point. This is because: $f(x) = x$ at that point:



So the function $f(x) = x^2$, has two stable points: $x = 0, 1$.

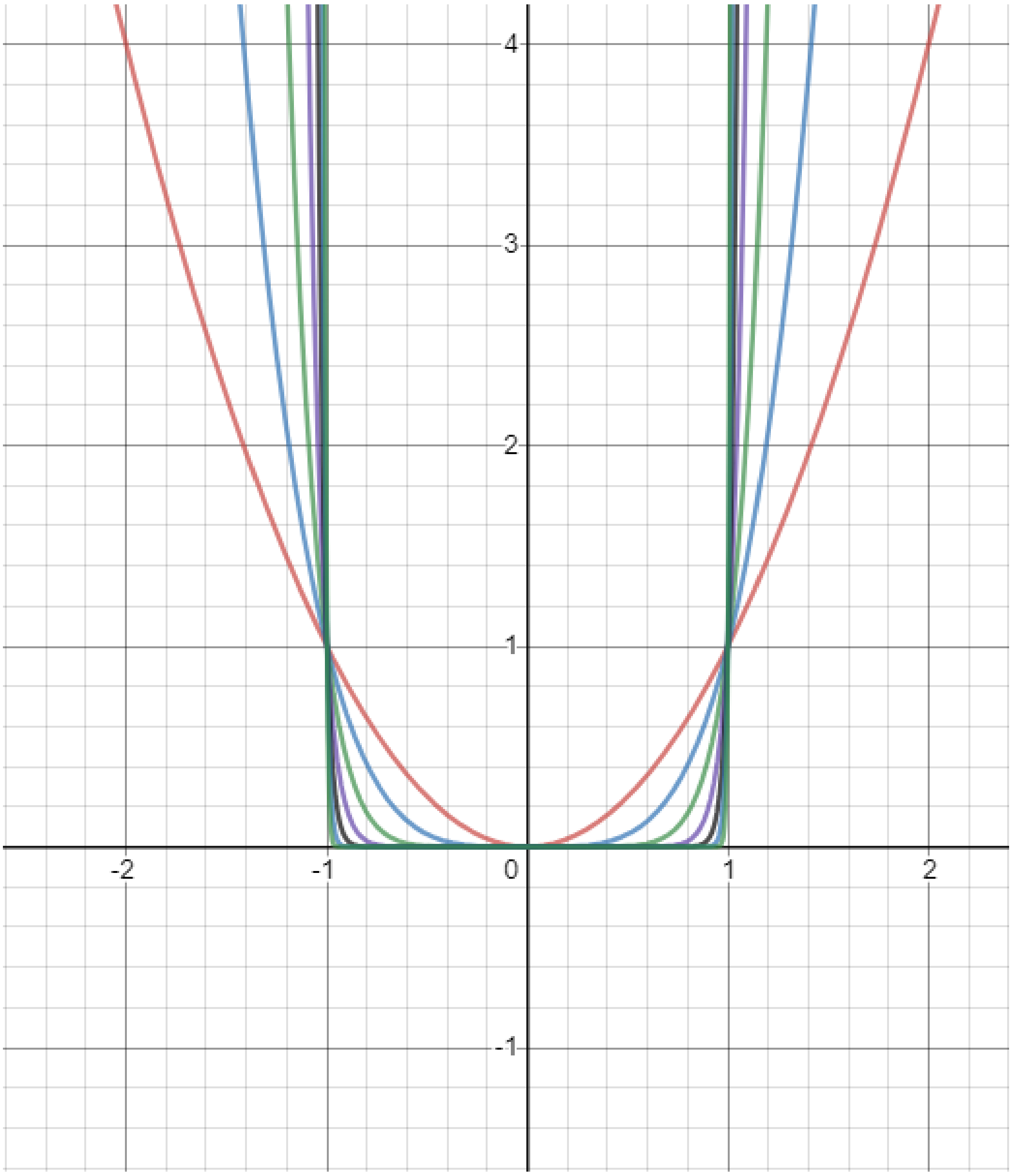
Now, let's figure out the g function.

- Using the knowledge that:
  - $|x^n| < |x|$ given that $|x| < 1$.
  - Which means $\lim_{n \to \infty} x^n = 0$, given that $|x| < 1$
  - We can prove, that $x = 0$ is an attractor with its range being $(-1, 1)$.
- However, we know that if $\lim_{n \to \infty} x^{2n} = \infty$, given $|x| > 1$.
- And the final component, $x^n = 1$, given that $x = 1$

So with all of that we can define $g(x)$:

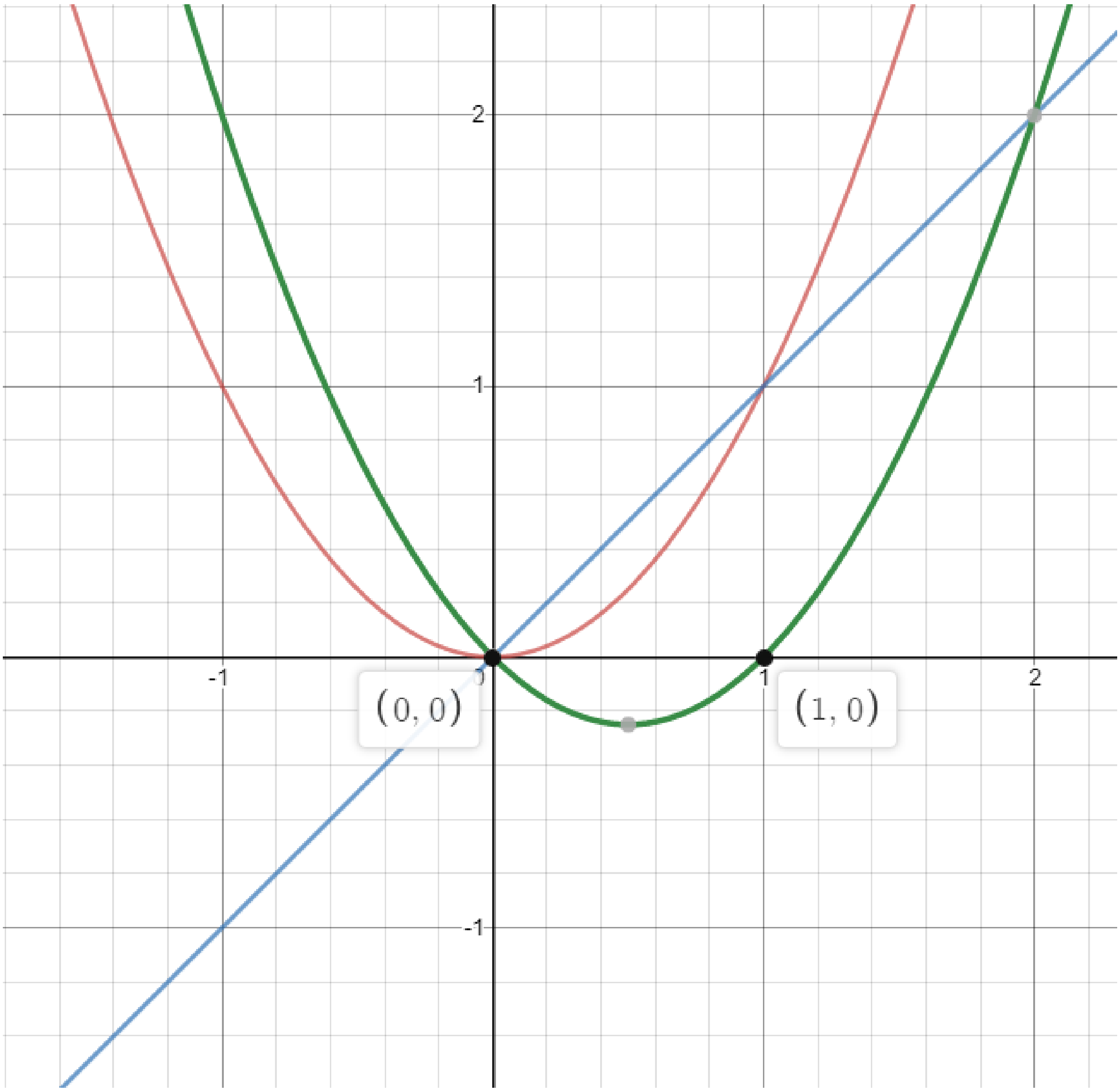
$$g(x) = \begin{cases} 0 & |x| < 1 \\ 1 & |x| = 1 \\ \infty & |x| > 1 \end{cases}$$

To visually verify this, we can just graph a ton of compositions of f and see if they approach our function:



As we can see, they start forming a 'bucket,' where all of the sides pass through the point $(1, 1)$ and then both ranges $(-\infty, -1)$ and $(1, \infty)$ go to ∞.

Considering r



The new green function is $r(x)$, which in our case is $r(x) = x^2 - x$. The roots of r represent all the stable points of our function. Another way to think of r is that it is the change of x given one composition. Obviously, if $y = x$ is above our function $f(x)$, then $r(x) < 0$ and we are on a downward slide. And, if $y = x$ is below our function $f(x)$, then $r(x) > 0$ and we are on a upward slide.

Simple Polynomials

Given a simple polynomial like: $f(x) = x^n$

Given the rules:

- $|x^n| < |x|$ given that $|x| < 1$.
  - Which means $\lim_{n\to\infty} x^n = 0$, given that $|x| < 1$
  - We can prove, that $x = 0$ is an attractor with its range being $(-1, 1)$.
- If n is even:
  - $g(x) = \begin{cases} 0 & |x| < 1 \\ 1 & |x| = 1 \\ \infty & |x| > 1 \end{cases}$
- If n is odd:
  - $g(x) = \begin{cases} 0 & |x| < 1 \\ sign(x) & |x| = 1 \\ sign(x) \cdot \infty & |x| > 1 \end{cases}$

Another rule can be made for negatives:

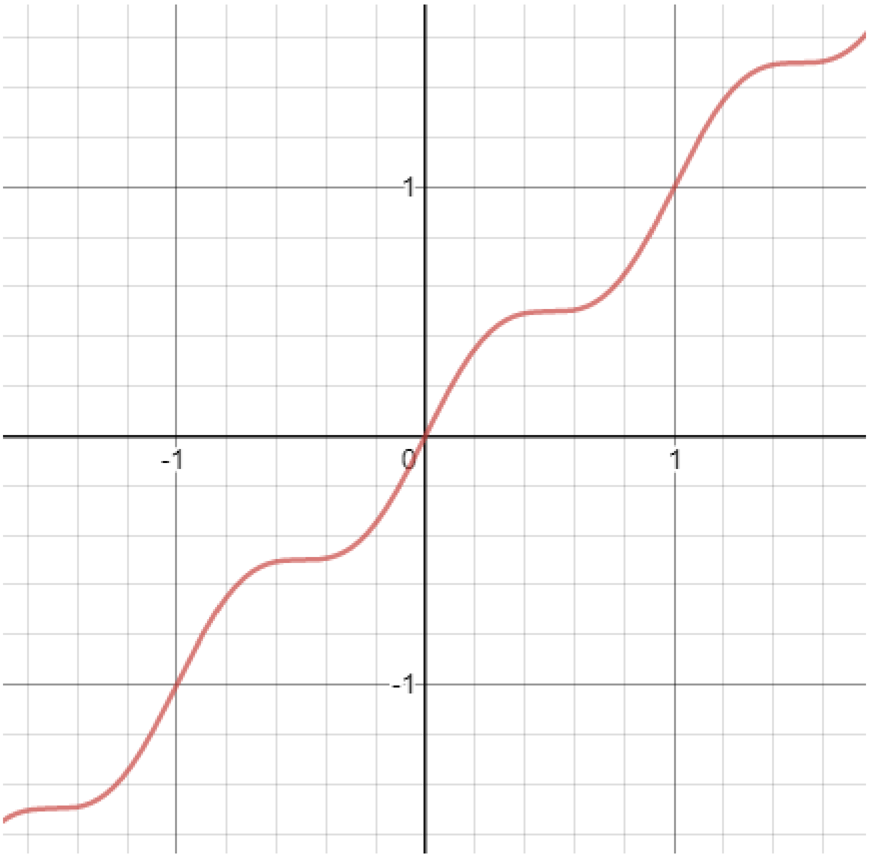
Given a simple polynomial like: $f(x) = kx^n$, where $k \in \{-1, 1\}$.

- If n is even:
  - $g(x) = \begin{cases} 0 & |x| < 1 \\ k & |x| = 1 \\ k \cdot \infty & |x| > 1 \end{cases}$
- If n is odd:
  - $k = -1$:
    - $g(x) = \{ 0 \quad |x| < 1$
  - $k = 1$:
    - $g(x) = \begin{cases} 0 & |x| < 1 \\ sign(x) & |x| = 1 \\ sign(x) \cdot \infty & |x| > 1 \end{cases}$

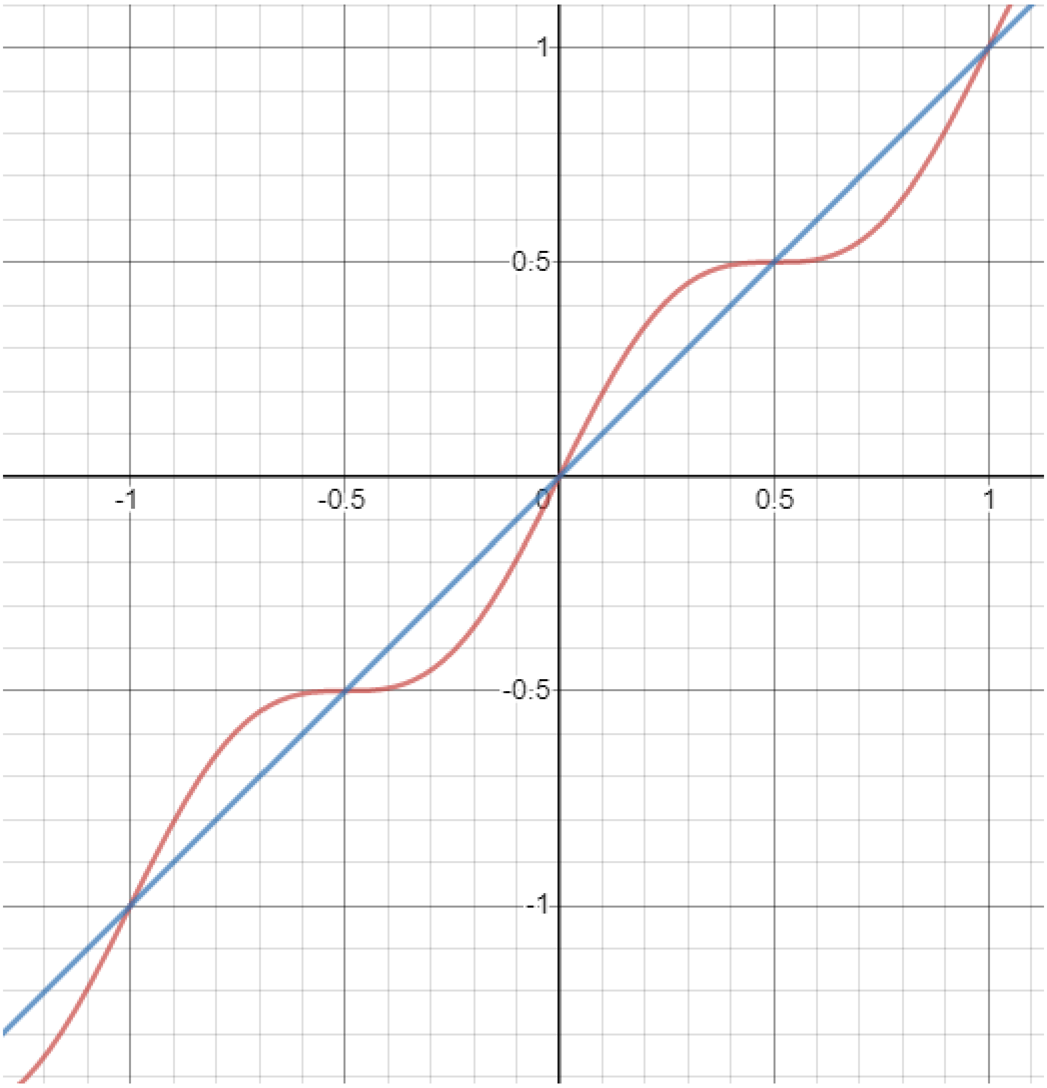
Intriguing Functions

One intriguing choice of f is:

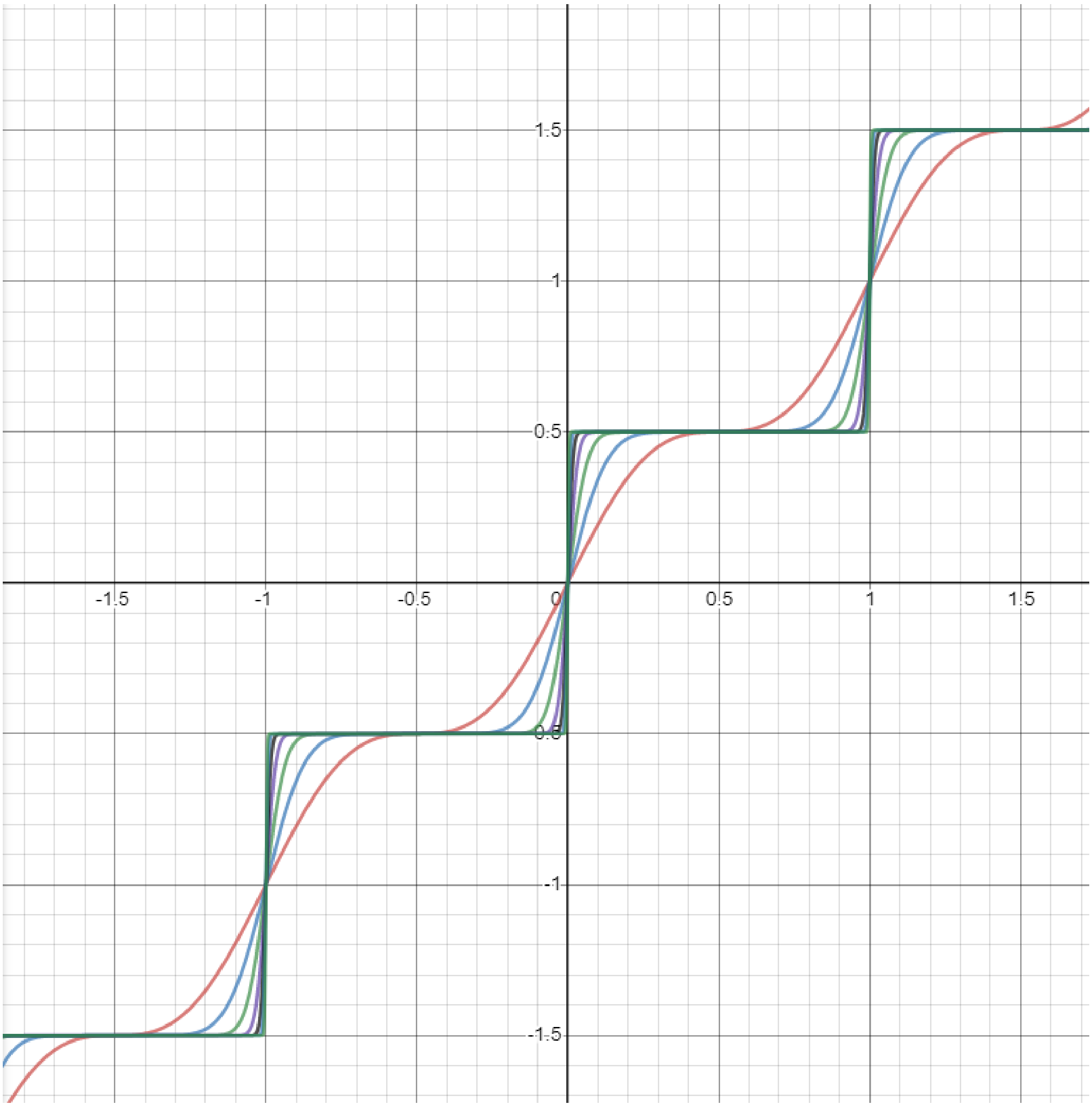
$f(x) = \frac{sin(2\pi x)}{2\pi} + x$



Now, if we plot $y = x$:



We can see that $f(x)$ has slides, and if we graph a bunch of compositions, we see that it forms plateaus.



The actual formula for g is $g(x) = floor(x) + \frac{1}{2}$.