Applications of integration and functions of bounded variation

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Definition (Additive interval function)

Let (α, β) *be an ordered pair of points* $\alpha, \beta \in [a, b]$. *Suppose that to each pair* (α, β) *a number* $I(\alpha, \beta)$ *is assigned so that*

$$\forall\, \alpha, \beta, \gamma \in [a, b] \quad I(\alpha, \gamma) = I(\alpha, \beta) + I(\beta, \gamma).$$

Then the function $I(\alpha, \beta)$ *is called an* **additive (oriented) interval function** *defined on intervals contained in* $[a, b]$.

Remark. It follows from the definition that

- $\alpha = \beta = \gamma \Rightarrow I(\alpha, \alpha) = 0.$
- $\alpha = \gamma \Rightarrow I(\alpha, \beta) + I(\beta, \alpha) = 0.$

Example. If $f \in \mathcal{R}[a, b]$, then $I(\alpha, \beta) := \displaystyle\int_\alpha^\beta f.$

Theorem (The density of an additive interval function)

Let $I(\alpha,\beta)$ be an additive interval function, $\alpha,\beta\in[a,b]$. If $\exists\, f\in\mathcal{R}[a,b]\ \forall\alpha,\beta\ a\le\alpha\le\beta\le b$

$$\inf_{x\in[\alpha,\beta]} f(x)(\beta-\alpha)\le I(\alpha,\beta)\le \sup_{x\in[\alpha,\beta]} f(x)(\beta-\alpha),$$

*then $I(a,b)=\int_a^b f$. The function f is called the **density** of $I(\alpha,\beta)$.*

Proof. Let $\tau=\{t_k\}_{k=0}^n$ be a partition of $[a,b]$. As usual, $M_k:=\sup\limits_{x\in[x_k,x_{k+1}]} f(x)$, $m_k:=\inf\limits_{x\in[x_k,x_{k+1}]} f(x)$, $k=0,\dots,n-1$. Then

$$m_k\Delta x_k\le I(x_k,x_{k+1})\le M_k\Delta x_k.$$

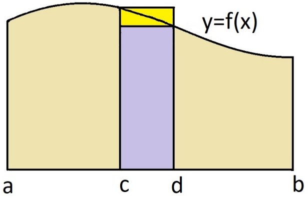
By additivity of I,

$$s_\tau(f)=\sum_{k=0}^{n-1} m_k\Delta x_k\le I(a,b)\le\sum_{k=0}^{n-1} M_k\Delta x_k=S_\tau(f).$$

It remains to pass to the limit $\lambda(\tau)\to 0$. □

The area of a curvilinear trapezoid

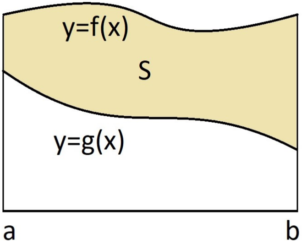
$T([a,b]) := \{(x,y) : x \in [a,b], y \in [0, f(x)]\}$ is a curvelinear trapezoid. $S(a,b)$ is the area of $T([a,b])$. $S(a,b) = S(a,c) + S(c,b)$.



$m = \inf_{x\in[c,d]} f(x),\ M = \sup_{x\in[c,d]} f(x),$

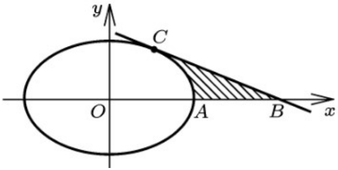
$$m(d-c) \le S(c,d) \le M(d-c),$$

$$S(a,b) = \int_a^b f.$$



$$S = \int_a^b (f-g).$$

Example. The ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ has the tangent at the point



$C\left(\dfrac{a}{2},\dfrac{b\sqrt{3}}{2}\right)$. Find the area of ABC.

$$AC:\;\; x=x_1(y)=a\sqrt{1-\frac{y^2}{b^2}},$$

$$BC:\;\; x=x_2(y)=a\left(2-\frac{y\sqrt{3}}{b}\right),$$

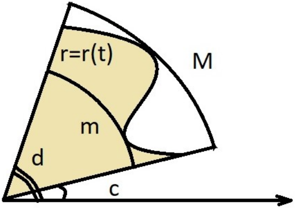
$0\le y\le\dfrac{b\sqrt{3}}{2}$. $S=\displaystyle\int_0^{b\sqrt{3}/2}(x_2(y)-x_1(y))\,\mathrm{d}y=J_2-J_1.$

$$J_2=\int_0^{b\sqrt{3}/2}x_2(y)\,\mathrm{d}y=\int_0^{b\sqrt{3}/2}a\left(2-\frac{y\sqrt{3}}{b}\right)\,\mathrm{d}y=\frac{5\sqrt{3}}{8}ab.$$

$$J_1=\left[y=b\sin t, 0\le t\le\frac{\pi}{3}\right]=\int_0^{b\frac{\sqrt{3}}{2}}x_1(y)\,\mathrm{d}y=ab\int_0^{\frac{\pi}{3}}\cos^2 t\,\mathrm{d}t$$

$$=\left(\frac{\pi}{6}+\frac{\sqrt{3}}{8}\right)ab.\; S=J_2-J_1=ab(3\sqrt{3}-\pi)/6.$$

The area of a curvilinear sector



Let (r, t) be polar coordinates.

$$T(c, d) = \{(r, t) : t \in [c, d], r \in [0, r(t)]\}$$

is a curvilinear sector. $S(c, d)$ is the area of $T([c, d])$.

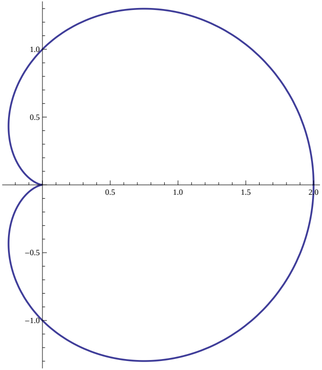
$$S(a, b) = S(a, c) + S(c, b)$$

$m := \inf_{t\in[c,d]} r(t), \quad M := \sup_{t\in[c,d]} r(t)$

$$\frac{m^2}{2}(d-c) \le S(c, d) \le \frac{M^2}{2}(d-c)$$

$$S(c, d) = \frac{1}{2}\int_c^d r^2.$$

Example. $r(t) = \cos t + 1,\ t \in [0, 2\pi]$ cardioid.



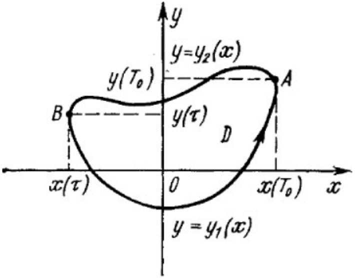
$$S = \frac{1}{2}\int_0^{2\pi} r^2 = \frac{1}{2}\int_0^{2\pi} (\cos t + 1)^2 \,\mathrm{d}t$$

$$= \int_0^{\pi} (\cos t + 1)^2 \,\mathrm{d}t = \int_0^{\pi} (cos^2 t + 2\cos t + 1) \,\mathrm{d}t$$

$$= \int_0^{\pi} \frac{1 + \cos 2t}{2} \,\mathrm{d}t + (2\sin t + t)\Big|_0^{\pi} = \frac{3\pi}{2}.$$

Let the boundary be defined by the parametric curve

$$\Gamma : x = x(t), y = y(t), t \in [T_0, T_1], x(T_0) = x(T_1), y(T_0) = y(T_1).$$



Let $[T_0, T_1]$ be divided by $\tau \in (T_0, T_1)$ into two parts $[T_0, \tau]$, $[\tau, T_1]$, and $x = x(t)$ be strictly monotone and continuously differentiable on each interval $[T_0, \tau]$, $[\tau, T_1]$. Then Γ consists of graphics of two functions $y = y_1(x)$, $y = y_2(x)$. Let $y_1(x) \le y_2(x)$ for all x. Denote by S_D the area of D. Then

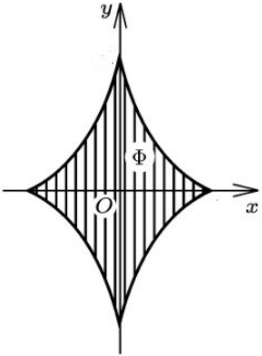
$$S_D = \int_a^b (y_2(x) - y_1(x))\,\mathrm{d}x = \int_a^b y_2(x)\,\mathrm{d}x - \int_a^b y_1(x)\,\mathrm{d}x$$

Changing the variable $x = x(t), t \in [T_0, \tau)$, $x = x(t), t \in [\tau, T_1)$, $y_2(x(t)) = y(t), t \in [T_0, \tau)$, $y_1(x(t)) = y(t), t \in [\tau, T_1)$, we get

$$S_D = -\int_{T_0}^{\tau} y(t)x'_t\,\mathrm{d}t - \int_{\tau}^{T_1} y(t)x'_t\,\mathrm{d}t = -\int_{T_0}^{T_1} y(t)x'(t)\,\mathrm{d}t.$$

Changing x and y their places, we get $S_D = \int_{T_0}^{T_1} x(t)y'(t)\,\mathrm{d}t$. Joining two formulas yields $S_D = \frac{1}{2}\int_{T_0}^{T_1} \left(x(t)y'(t) - y(t)x'(t)\right)\,\mathrm{d}t$.

Example. Find the area of the figure bounded by $\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1$.



$$x(t) = a\cos^3 t,\;\; y(t) = b\sin^3 t,\;\; t \in [0, 2\pi].$$

$$S_\Phi = 4\cdot\frac{1}{2}\int_0^{\frac{\pi}{2}} \left(x(t)y'(t) - y(t)x'(t)\right)\,\mathrm{d}t$$

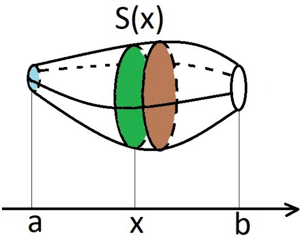
$$= 6ab\int_0^{\pi/2} \left(\cos^4 t\sin^2 t + \sin^4 t\cos^2 t\right)\,\mathrm{d}t$$

$$= 6ab\int_0^{\pi/2} \sin^2 t\cos^2 t\,\mathrm{d}t = \frac{3ab}{2}\int_0^{\pi/2}\sin^2 2t\,\mathrm{d}t$$

$$= \frac{3ab}{4}\int_0^{\pi/2}(1-\cos 4t)dt = \frac{3\pi ab}{8}.$$

The volume of a solid

Let $T \subset \mathbb{R}^3$ be a solid, $T(x) := \{(y,z) \in \mathbb{R}^2 : (x,y,z) \in T\}$ be a cross section in a coordinate x, $S(x)$ be an area of $T(x)$. Suppose that



1. S is continuous on $[a,b]$,
2. $\exists\, [a,b]\;\; T(x) = \varnothing$ outside $[a,b]$,
3. $\forall\, [c,d] \subset [a,b] \quad \exists\, x^*, x^{**} \in [a,b]$
   $\forall\, x \in [c,d] \quad T(x^*) \subset T(x) \subset T(x^{**})$.

Let $V(c,d)$ be a volume of the part of T laying between the planes $x = c$, $x = d$.

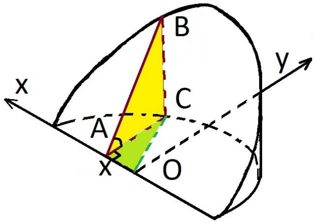
Then $V(c,d)$ is an additive interval function and

$$S(x^*)(d-c) \le V(c,d) \le S(x^{**})(d-c)$$

$$\Rightarrow V(c,d) = \int_c^d S.$$

If T is obtained by revolving the curvilinear trapezoid corresponding to the function $y = f(x)$, then $S(x) = \pi f^2(x)$, so $V(c,d) = \pi \displaystyle\int_c^d f^2$.

Example. $T = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \le a^2,\ 0 \le z \le \tan\alpha y\right\}.$



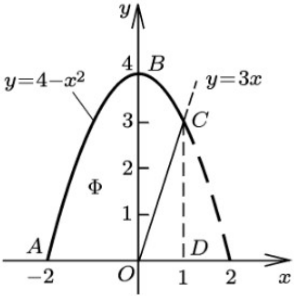
$x \in [-a, a],$

$$S(x) = S_{\Delta ABC} = \frac{1}{2}|AC|^2 \tan\alpha$$

$$= \frac{1}{2}\left(a^2 - x^2\right)\tan\alpha,$$

$$V = \int_{-a}^{a} \frac{1}{2}\left(a^2 - x^2\right)\tan\alpha \,\mathrm{d}x = \int_0^a \left(a^2 - x^2\right)\tan\alpha \,\mathrm{d}x$$

$$= \left(a^2 x - \frac{x^3}{3}\right)\tan\alpha\Big|_{x=0}^{x=a} = \frac{2}{3}a^3 \tan\alpha.$$



Example. The figure is bounded by the parabola $y = 4 - x^2$, an interval $[-2, 0]$ of Ox, and an interval of $y = 3x$. Find the volume of the solid which is a result of the revolving of the figure around Ox.

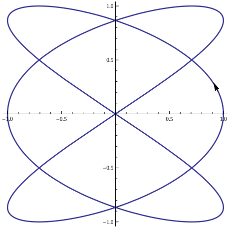
$$V_1 = \pi \int_{-2}^{1} \left(4 - x^2\right)^2 dx = \frac{153}{5}\pi, \quad V_2 = \pi \int_0^1 (3x)^2 dx = 3\pi$$

$$V = V_1 - V_2 = \frac{138}{5}\pi.$$

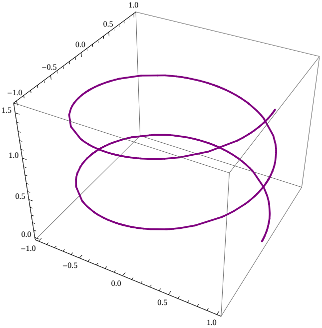
Definition (A path)

A path *in* $\mathbb{R}^d$ *is a mapping* $\gamma : [a, b] \to \mathbb{R}^d$, $\gamma : t \mapsto (\gamma_1(t), \ldots, \gamma_d(t))$, *where all coordinate functions* γ_k *are continuous on* $[a, b]$. *The points* $\gamma(a)$, $\gamma(b)$ *are called the* **initial point** *and the* **terminal point** *of the path. The path is* **closed** *if these points coincide. If* $\gamma(t_1) = \gamma(t_2)$ *implies* $t_1 = t_2$ *or* $t_1, t_2 \in \{a, b\}$, *then the path is called* **simple**. *If* $\gamma_k \in C^r[a, b]$, *then the path is called* r-**smooth** *(***smooth**, *if* $r = 1$*). If there exists a partition* $\tau = \{t_k\}_{k=0}^{n}$ *of* $[a, b]$ *and restrictions* $\gamma_{[t_k, t_{k+1}]}$, $k = 0, \ldots, n-1$ *are smooth, then the path is called* **piecewise smooth**. *The image* $\gamma([a, b])$ *is called the* **support** *of the path.*

Example. The path $\gamma(t) = (\cos 3x, \sin 2x)$, $x \in [0, 2\pi]$ is not simple. It is closed and smooth. Its support is called Lissajoux curve.



Example.



Helix $\gamma(t) = (\cos t, \sin t, t/8)$, $t \in [0, 4\pi]$ is a simple smooth path in $\mathbb{R}^3$.

Example. The paths have the same support.

$$\gamma^1(t) = \left(t, \sqrt{1-t^2}\right), t \in [-1,1], \quad \gamma^2(t) = (-\cos t, \sin t), t \in [0,\pi],$$
$$\gamma^3(t) = (\cos t, \sin t), t \in [0,\pi], \quad \gamma^4(t) = (\cos t, |\sin t|), t \in [-\pi,\pi].$$

Definition (Equivalent paths)

Paths $\gamma : [a,b] \to \mathbb{R}^d$, $\gamma^* : [c,d] \to \mathbb{R}^d$ are called **equivalent** *if there exists a strictly increasing onto (or surjective) function $\theta : [a,b] \to [c,d]$ such that $\gamma = \gamma^* \circ \theta$. The function θ is called an* **admissible change of parameter**. *t is a* **parameter**.

Remark. θ is continuous.

Example. The paths γ^1 and γ^2 are equivalent: $\theta(t) = -\cos t$, $\theta : [0,\pi] \to [-1,1]$, $\gamma^1(-\cos t) = \gamma^2(t)$.

Definition (A curve)

The equivalence class of equivalent paths is called a **curve**. *An element of the class is called a* **parametrization** *of a curve. We say that a curve is* **smooth** *is there exists a smooth parametrization.*

Definition (The length of the path, rectifiable path)

Let $\tau = \{t_k\}_{k=0}^n$ be a partition of $[a, b]$, $\gamma : [a, b] \to \mathbb{R}^d$ be a path. We connect $\gamma(t_k)$, $\gamma(t_{k+1})$ by line segments to create a polygonal path. Let p_τ be the length of the polygonal path. The quantity $s_\gamma := \sup_\tau p_\tau$ is called the **length of the path** *γ. If s_γ is finite, then γ is called* **rectifiable**.

Lemma

The lengths of equivalent paths are equal.

Proof. Let $\gamma : [a, b] \to \mathbb{R}^d$, $\gamma^* : [c, d] \to \mathbb{R}^d$ be equivalent paths, $\theta : [a, b] \to [c, d]$ be an admissible change of parameter for γ, γ^*. Let $\tau = \{t_k\}_{k=0}^n$ be a partition of $[a, b]$. Then $\tau^* := \{\theta(t_k)\}_{k=0}^n$ is a partition of $[c, d]$. We use notation $|x| := \sqrt{x_1^2 + \dots x_d^2}$ for $x = (x_1, \dots, x_d) \in \mathbb{R}^d$.

$$p_\tau = \sum_{k=0}^{n-1} |\gamma(t_{k+1}) - \gamma(t_k)| = \sum_{k=0}^{n-1} |\gamma^*(\theta(t_{k+1})) - \gamma^*(\theta(t_k))| = p_{\tau^*}.$$

$$p_\tau = p_{\tau^*} \le s_{\gamma^*} \underbrace{\Rightarrow}_{\sup_\tau} s_\gamma \le s_{\gamma^*}, \qquad p_{\tau^*} = p_\tau \le s_\gamma \underbrace{\Rightarrow}_{\sup_{\tau^*}} s_{\gamma^*} \le s_\gamma.$$ $\square$

Remark. By Lemma, the length of the path does not depend on a parametrization. So, the **length of the curve** can be defined as the length of its parametrization.

Lemma (The length of the path is additive)

Suppose $\gamma : [a, b] \to \mathbb{R}^d$, $c \in (a, b)$, $\gamma^1 := \gamma\big|_{[a,c]}$, $\gamma^2 := \gamma\big|_{[c,b]}$, *then*

$$s_{\gamma_1} + s_{\gamma_2} = s_\gamma .$$

Proof. "$\leq$" Let τ_1, τ_2 be partitions of $[a, c]$, $[c, b]$. Let p_{τ_1}, p_{τ_2} be the lengths of corresponding polygonal paths. Then $\tau := \tau_1 \cup \tau_2$ is a partition of $[a, b]$ and

$$p_{\tau_1} + p_{\tau_2} = p_\tau \leq s_\gamma \underbrace{\Rightarrow}_{\sup_{\tau_1}, \sup_{\tau_1}} s_{\gamma_1} + s_{\gamma_2} \leq s_\gamma .$$

"$\geq$" Let τ be a partition of $[a, b]$.

The 1-st case: $c \in \tau$. Then $\tau = \tau_1 \cup \tau_2$, where τ_1, τ_2 are the partitions of $[a, c]$, $[c, b]$ $\Rightarrow$ $p_\tau = p_{\tau_1} + p_{\tau_2} \leq s_{\gamma_1} + s_{\gamma_2}$.

The 2-nd case: $c \notin \tau$. Add the point c to the partition τ: $\tau^* = \tau \cup \{c\}$. Let $\tau = \{t_k\}_{t=0}^{n}$, $c \in (t_r, t_{r+1})$. Then

$$
\begin{aligned}
p_\tau &= \sum_{k=0}^{r-1} |\gamma(t_{k+1}) - \gamma(t_k)| + |\gamma(t_{r+1}) - \gamma(t_r)| + \sum_{k=r+1}^{n-1} |\gamma(t_{k+1}) - \gamma(t_k)| \\
&\leq \sum_{k=0}^{r-1} |\gamma(t_{k+1}) - \gamma(t_k)| + |\gamma(c) - \gamma(t_r)| + |\gamma(t_{r+1}) - \gamma(c)| \\
&\quad + \sum_{k=r+1}^{n-1} |\gamma(t_{k+1}) - \gamma(t_k)| = p_{\tau^*} = p_{\tau_1} + p_{\tau_2} \leq s_{\gamma_1} + s_{\gamma_2}.
\end{aligned}
$$

In both cases,

$$
p_\tau \leq s_{\gamma_1} + s_{\gamma_2} \underbrace{\Rightarrow}_{\sup_\tau} s_\gamma \leq s_{\gamma_1} + s_{\gamma_2}. \qquad \square
$$

Theorem (The length of a smooth path)

If $\gamma : [a,b] \to \mathbb{R}^d$, $\gamma_j \in C^1[a,b]$, $j = 1, \ldots, d$, *then* γ *is rectifiable and*

$$s_\gamma = \int_a^b \left|\gamma'(t)\right| \, \mathrm{d}t = \int_a^b \left(\sum_{j=1}^d \left|\gamma_j'(t)\right|^2 \right)^{1/2} \mathrm{d}t.$$

Proof. 1. Let us prove that γ is rectifiable. Let $\tau = \{t_k\}_{t=0}^n$ be a partition of $[a,b]$. Then by Lagrange's theorem for γ_j,

$$p_\tau = \sum_{k=0}^{n-1} |\gamma(t_{k+1}) - \gamma(t_k)| = \sum_{k=0}^{n-1} \left(\sum_{j=1}^d |\gamma_j(t_{k+1}) - \gamma_j(t_k)|^2 \right)^{1/2}$$

$$= \sum_{k=0}^{n-1} \left(\sum_{j=1}^d \left|\gamma_j'(t_k^*)\Delta t_k\right|^2 \right)^{1/2} = \sum_{k=0}^{n-1} \left(\sum_{j=1}^d \left|\gamma_j'(t_k^*)\right|^2 \right)^{1/2} \Delta t_k,$$

where $t_k^* \in (t_k, t_{k+1})$.

We denote $M_{j,[a,b]} := \sup\limits_{t\in[a,b]} \left|\gamma_j'(t)\right|$, $m_{j,[a,b]} := \inf\limits_{t\in[a,b]} \left|\gamma_j'(t)\right|$. Then

$$p_\tau = \sum_{k=0}^{n-1} \left(\sum_{j=1}^{d} \left|\gamma_j'(t_k^*)\right|^2 \right)^{1/2} \Delta t_k \le \sum_{k=0}^{n-1} \left(\sum_{j=1}^{d} M_{j,[a,b]}^2 \right)^{1/2} \Delta t_k$$

$$= \left(\sum_{j=1}^{d} M_{j,[a,b]}^2 \right)^{1/2} (b-a) \;\Rightarrow\; s_\tau < \infty \Rightarrow \gamma \text{ is rectifiable.}$$

Moreover,

$$\left(\sum_{j=1}^{d} m_{j,[a,b]}^2 \right)^{1/2} (b-a) \le s_\gamma \le \left(\sum_{j=1}^{d} M_{j,[a,b]}^2 \right)^{1/2} (b-a).$$

2. Let us prove the formula. Let $s(t)$ be the length of $\gamma\big|_{[a,t]}$ Given t, $t+\Delta t \in [a,b]$, WLOG $\Delta t > 0$. By the additivity, the length of the part of the path from t to $t+\Delta t$ is $s(t+\Delta t) - s(t) = \Delta s(t)$. By **1.**

$$\left(\sum_{j=1}^{d} m^2_{j,[t,t+\Delta t]}\right)^{1/2} \Delta t \le \Delta s(t) \le \left(\sum_{j=1}^{d} M^2_{j,[t,t+\Delta t]}\right)^{1/2} \Delta t.$$

$\gamma_j' \in C[a,b]$, by the Weierstrass maximum value theorem,

$$\exists\, t_j^*, t_j^{**} \in [t, t+\Delta t] \;\; m_{j,[t,t+\Delta t]} = \left|\gamma_j'(t_j^*)\right|, \;\; M_{j,[t,t+\Delta t]} = \left|\gamma_j'(t_j^{**})\right|.$$

$$t_j^* = t_j^*(\Delta t), \;\; t < t_j^*(\Delta t) < t + \Delta t \Rightarrow \lim_{\Delta t\to 0} t_j^*(\Delta t) = t.$$

By the theorem on the limit of a composite function

$$\lim_{\Delta t\to 0} \gamma_j'\left(t_j^*(\Delta t)\right) = \gamma_j'(t).$$

In the same manner,

$$\lim_{\Delta t\to 0} \gamma_j'\left(t_j^{**}(\Delta t)\right) = \gamma_j'(t).$$

So, passing to the limit $\Delta t \to 0$ in

$$\left(\sum_{j=1}^{d} \left|\gamma_j'\left(t_j^*\right)\right|^2\right)^{1/2} \leq \frac{\Delta s(t)}{\Delta t} \leq \left(\sum_{j=1}^{d} \left|\gamma_j'\left(t_j^{**}\right)\right|^2\right)^{1/2},$$

we get $s'(t) = \left(\sum_{j=1}^{d} \left|\gamma_j'(t)\right|^2\right)^{1/2} = \left|\gamma'(t)\right|.$

$\gamma_j' \in C[a, b] \Rightarrow |\gamma'| \in C[a, b]$. By the fundamental theorem of integral calculus,

$$s_\gamma = s(b) = \int_a^b s'(t)\,\mathrm{d}t = \int_a^b \left|\gamma'(t)\right|\,\mathrm{d}t. \qquad \square$$

Example.

1. $d = 2$, the path γ is given by an explicit function $y = y(x), x \in [a, b]$. Then $\gamma(t) = (t, y(t)),\ t \in [a, b]$,

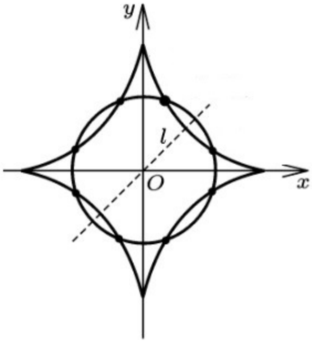
$$s_\gamma = \int_a^b \sqrt{1 + (y'(x))^2}\,\mathrm{d}t.$$

2. $d = 2$, the path γ is given by polar coordinates $r = r(t), t \in [\alpha, \beta]$. Then $\gamma(t) = (r(t)\cos t,\ r(t)\sin t)\,,\ t \in [\alpha, \beta]$.

$$\left|\gamma'(t)\right|^2 = \left((r(t)\cos t)'\right)^2 + \left((r(t)\sin t)'\right)^2 =$$
$$\left(r'(t)\cos t - r(t)\sin t\right)^2 + \left(r'(t)\sin t + r(t)\cos t\right)^2 = \left(r'(t)\right)^2 + (r(t))^2\,.$$

$$s_\gamma = \int_a^b \sqrt{(r'(t))^2 + (r(t))^2}\,\mathrm{d}t.$$

Example. Find the radius of the circle centered at the origin. The circle divides astroid $x^{2/3}+y^{2/3}=a^{2/3}$, $x\geq 0$, $y\geq 0$ into three parts of equal length.



$$x(t)=a\cos^3 t,\ y(t)=a\sin^3 t,\ t\in[0,\pi/2].$$

$$s\left(t_0\right)=\int_0^{t_0}\sqrt{x'^2+y'^2}\,\mathrm{d}t$$

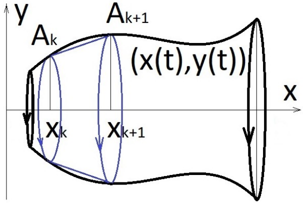
$$=3a\int_0^{t_0}\sin t\cos t\,\mathrm{d}t=\frac{3a}{2}\sin^2 t_0.$$

$$s\left(\frac{\pi}{2}\right)=\frac{3a}{2},\ 3s\left(t_0\right)=s\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \sin^2 t_0=\frac{1}{3}\Rightarrow \sin t_0=\frac{1}{\sqrt{3}},\ \cos t_0=\sqrt{\frac{2}{3}},$$

$$x_0=\frac{a}{3\sqrt{3}},\quad y_0=\frac{2}{3}\sqrt{\frac{2}{3}}a,\quad r=\sqrt{x_0^2+y_0^2}=\frac{a}{\sqrt{3}}.$$

Surface of revolution



Let $(x(t), y(t))$, $t \in [\alpha, \beta]$ be a parametric representation of a function, $y \geq 0$,
$\tau = \{t_k\}_{k=0}^{n}$ be a partition of $[\alpha, \beta]$,
$A_k = (x(t_k), y(t_k)) = (x_k, y_k)$,
$p_k = \sqrt{(x_{k+1} - x_k)^2 + (y_{k+1} - y_k)^2}$
be a length of $A_k A_{k+1}$.

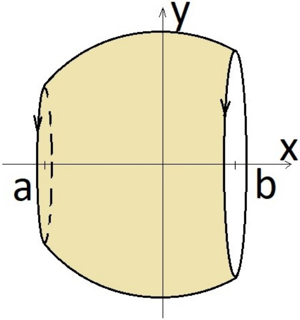
Rotating $A_k A_{k+1}$ around x-axis we get a surface of a truncated cone, the corresponding area is $s_k = \pi(y_k + y_{k+1})p_k$. If there exists $\lim\limits_{\lambda(\tau)\to 0} \sum\limits_{k=0}^{n-1} s_k$, it is called the **area of a surface of revolution**.

Theorem (The area of a surface of revolution)

If $x, y \in C^1[\alpha, \beta]$, then the area of a surface of revolution is

$$S = 2\pi \int_{\alpha}^{\beta} |y(t)| \sqrt{(x'(t))^2 + (y'(t))^2}\,\mathrm{d}t.$$

Example.



Let us find the surface area of the spherical zone, the height of the zone is h, the radius of the sphere is R.

$$y(x) = \sqrt{R^2 - x^2}, \; x \in [a, b] \subset [-R, R],$$
$$b - a = h.$$

$$y'(x) = -\frac{x}{\sqrt{R^2 - x^2}} \Rightarrow 1 + \left(y'(x)\right)^2 = \frac{R^2}{R^2 - x^2}, \quad y(x)\sqrt{1 + \left(y'(x)\right)^2} = R$$

$$S = 2\pi \int_a^b y(x)\sqrt{1 + \left(y'(x)\right)^2}\,\mathrm{d}x = 2\pi \int_a^b R = 2\pi Rh.$$

$$a = -R, \; b = R \;\Rightarrow\; S = 4\pi R.$$

The concept of the length of a path in $\mathbb{R}^m$ turns put to be meaningful for $m = 1$ as well, however it makes sense to drop the continuity requirement.

Definition

Let $f : [a, b] \to \mathbb{R}$. The quantity

$$\bigvee_a^b f = \sup_\tau \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)|,$$

where sup *is taken over all partitions $\tau = \{x_k\}_{k=0}^n$ of $[a, b]$, is called a* **variation** *of the function f on $[a, b]$. If $\bigvee_a^b f < +\infty$, then f is referred to as the* **function of bounded variation** *on $[a, b]$. The set of all functions of bounded variation on $[a, b]$ is denoted by $V[a, b]$.*

Variation is a length of a one-dimensional path. A function of bounded variation is a one-dimensional rectifiable mapping (under the assumption of continuity).

Properties

V1. Variation is additive. If $f : [a, b] \to \mathbb{R}$, $a < c < b$, then

$$\bigvee_a^b f = \bigvee_a^c f + \bigvee_c^b f$$

V2. If f is piecewise smooth on $[a, b]$, then

$$\bigvee_a^b f = \int_a^b |f'| .$$

V1. is the particular case of the Lemma on the additivity of the length of a path. **V2.** is a formula for the length of a piecewise smooth path.

V3. Variation is monotone. If $f : [a,b] \to \mathbb{R}, [\alpha,\beta] \subset [a,b]$, then

$$\bigvee_{\alpha}^{\beta} f \leq \bigvee_{a}^{b} f.$$

Proof. By additivity,

$$\bigvee_{a}^{b} f = \bigvee_{a}^{\alpha} f + \bigvee_{\alpha}^{\beta} f + \bigvee_{\beta}^{b} f \geq \bigvee_{\alpha}^{\beta} f. \qquad \square$$

Monotonicity provides the correctness of the following definition of variation for a function defined on non-closed interval. If $f : \langle a,b \rangle \to \mathbb{R}$,

$$\bigvee_{a}^{b} f := \sup_{[\alpha,\beta] \subset \langle a,b \rangle} \bigvee_{\alpha}^{\beta} f.$$

V4. Let $\gamma = (\gamma_1, \ldots, \gamma_m) : [a,b] \to \mathbb{R}^m$. Then $s_\gamma < +\infty$ iff $\gamma_i \in V[a,b]$ for all $i = 1, \ldots, m$.

The proof follows from the estimate

$$\left|\gamma_i\left(t_{k+1}\right) - \gamma_i\left(t_k\right)\right| \leqslant \left|\gamma\left(t_{k+1}\right) - \gamma\left(t_k\right)\right| \leqslant \sum_{j=1}^{m} \left|\gamma_j\left(t_{k+1}\right) - \gamma_j\left(t_k\right)\right|. \quad \square$$

V5. If f is monotone on $[a,b]$, then $f \in V[a,b]$ and

$$\bigvee_a^b f = |f(b) - f(a)|.$$

Proof. For any partition

$$\sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| = \left| \sum_{k=0}^{n-1} (f(x_{k+1}) - f(x_k)) \right| = |f(b) - f(a)|. \quad \square$$

V6. If $f \in V[a,b]$, then f is bounded on $[a,b]$.
Proof. For all $x \in [a,b]$

$$2|f(x)| \le |f(x)-f(a)|+|f(b)-f(x)|+|f(a)+f(b)| \le |f(a)+f(b)|+\bigvee_a^b f. \quad \square$$

Theorem (Functions of bounded variations and arithmetic operations)

Let $f, g \in V[a,b]$, then
1. $f+g \in V[a,b]$, *2.* $fg \in V[a,b]$, *3.* $\alpha f \in V[a,b] (\alpha \in \mathbb{R})$,
4. $|f| \in V[a,b]$, *5. if* $\inf_{x\in[a,b]} |g(x)| > 0$, *then* $\frac{f}{g} \in V[a,b]$.

The proof is analogue to the proof of the Theorem [Integrability and arithmetic operations]. 2.-5. are left for a homework.
Proof. 1. $\Delta_k f := f(x_{k+1}) - f(x_k)$. Summing up over all k the inequalities

$$|\Delta_k(f+g)| \leq |\Delta_k f| + |\Delta_k g|,$$

we get

$$\sum_{k=0}^{n-1} |\Delta_k(f+g)| \leqslant \sum_{k=0}^{n-1} |\Delta_k f| + \sum_{k=0}^{n-1} |\Delta_k g| \leq \bigvee_a^b f + \bigvee_a^b g.$$

Taking sup over all partitions we obtain $\bigvee\limits_a^b (f+g) \leqslant \bigvee\limits_a^b f + \bigvee\limits_a^b g$.

Theorem (Criterion for a bounded variation)

Let $f : [a, b] \to \mathbb{R}$. Then $f \in V[a, b]$ iff f is represented as a difference of two increasing functions on $[a, b]$.

Proof. The sufficiency follows from **V5** and the last Theorem. To check the necessity we set

$$g(x) = \bigvee_a^x f, \quad x \in [a, b], \quad h = g - f.$$

Let $a \le x_1 < x_2 \le b$, by additivity

$$g(x_2) - g(x_1) = \bigvee_{x_1}^{x_2} f \ge 0,$$

$$h(x_2) - h(x_1) = \bigvee_{x_1}^{x_2} f - (f(x_2) - f(x_1)) \ge 0.$$

V7. $V[a,b] \subset R[a,b]$.
A monotonic function is integrable and a difference of integrable functions is integrable.

V8. The function of bounded variation can not have discontinuities of the second kind.
It follows from the criterion for a bounded variation.

V9. $V[a,b] \not\subset C[a,b]$ and $C[a,b] \not\subset V[a,b]$.
Proof. Since there are discontinuous monotone functions it follows that $V[a,b] \not\subset C[a,b]$. Let us give an example of continuous function of unbounded variation. Consider

$$f(x) = \begin{cases} x\cos\frac{\pi}{x}, & x \in (0,1], \\ 0, & x = 0. \end{cases}$$

$f \in C[0,1]$. We set $x_k = \frac{1}{k}$ $(k \in \mathbb{N})$, then

$$f(x_k) = \frac{(-1)^k}{k}, \quad |f(x_k) - f(x_{k+1})| = \frac{1}{k} + \frac{1}{k+1}.$$

Let $n \in \mathbb{N}$ be given, consider the partition $0 < x_n < \ldots < x_1 = 1$. (The different order of points is not essential.)

$$\sum_{k=1}^{n-1} \left| f\left(x_{k+1}\right) - f\left(x_k\right) \right| + \left| f\left(x_n\right) - f(0) \right| = -1 + 2 \sum_{k=1}^{n} \frac{1}{k}.$$

The last sum is not bounded

$$\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n-1} \geq n \frac{1}{2n-1} > \frac{1}{2}. \qquad \square$$

The function f gives an example of non-rectifiable path in $\mathbb{R}$, its graph is an example of non-rectifiable path in $\mathbb{R}^2$.

Example. Represent $f(x) = \cos^2 x$ as a difference of two increasing functions on $[0, \pi]$.

$f(x) = \bigvee\limits_a^x f(x) - \varphi(x)$, where $\varphi(x) = \bigvee\limits_a^x f(x) - f(x)$.

$$\bigvee_a^x f(x) = \int_0^x |f'| = \int_0^x |\sin 2t|\,\mathrm{d}t = \begin{cases} \sin^2 x, & 0 \le x \le \frac{\pi}{2}, \\ 1 + \cos^2 x, & \frac{\pi}{2} < x \le \pi, \end{cases}$$

$$\varphi(x) = \begin{cases} \sin^2 x - \cos^2 x, 0 \le x \le \frac{\pi}{2}, \\ 1, \quad \frac{\pi}{2} < x \le \pi, \end{cases}$$