Riemann integral (Definite integral)

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Definition (Tagged partition of an interval)

Let $[a, b]$ be a closed interval ($-\infty < a < b < +\infty$). A set of points

$$\tau = \{x_k\}_{k=0}^{n} \text{ such that } a = x_0 < x_1 < \ldots < x_n = b$$

is called a **partition** *of an interval $[a, b]$. Intervals $[x_k, x_{k+1}]$ are called* **intervals of the partition**. *We use notation $\Delta x_k = |x_{k+1} - x_k|$ for the length of an interval $[x_k, x_{k+1}]$. Then the value*

$$\lambda = \lambda_\tau = \max_{0 \le k \le n-1} \Delta x_k$$

is called a **mesh** *of the partition τ.*
We say that a pair (τ, ξ) is a **tagged partition** *of an interval $[a, b]$ if τ is a partition of an interval $[a, b]$ and $\xi = \{\xi_k\}_{k=0}^{n-1}$ is a set of tags such that $\xi_k \in [x_k, x_{k+1}]$.*

Definition (A Riemann sum, a Riemann integral, a Riemann-integrable function)

Let $f : [a, b] \to \mathbb{R}$. A **Riemann sum** *of a function f with respect to a tagged partition (τ, ξ) is defined as*

$$\sigma = \sigma(f, \tau, \xi) = \sigma(\tau, \xi) = \sum_{k=0}^{n-1} f(\xi_k)\Delta x_k.$$

The function f is called a **Riemann-integrable** *on $[a, b]$ if there exist a number $I \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every tagged partition (τ, ξ) with a mesh $\lambda_\tau < \delta$ we have $|\sigma(f, \tau, \xi) - I| < \varepsilon$, that is*

$$\forall\varepsilon > 0\ \exists\delta > 0\ \forall\,(\tau, \xi)\,(\lambda_\tau < \delta\ \to |\sigma(f, \tau, \xi) - I| < \varepsilon)\,. \tag{1}$$

The number I is called a **Riemann integral** *and is denoted as $I := \int_a^b f(x)\,\mathrm{d}x$. We denote a set of all Riemann-integrable functions on a segment $[a, b]$ as $\mathcal{R}[a, b]$.*

Example. $f(x) = 1 + x$, $x \in [0, 3]$.

$\tau = \{x_k\}_{k=0}^{n}$, $0 = x_0 < x_1 < \cdots < x_n = 3$, $\xi_k = \dfrac{x_k + x_{k+1}}{2}$.

$$\sigma = \sigma(f, \tau, \xi) = \sum_{k=0}^{n-1} f(\xi_k)\Delta x_k$$

$$= \sum_{k=0}^{n-1} \left(1 + \frac{x_k + x_{k+1}}{2}\right)(x_{k+1} - x_k) = x_n - x_0 + \frac{x_n^2 - x_0^2}{2} = \frac{15}{2}.$$

A base $\mathcal{B}$ in the set of tagged partitions (τ, ξ). The element B_d, $d > 0$, of the base $\mathcal{B}$ consists of all tagged partitions (τ, ξ) for which $\lambda_\tau < d$.

- $B_d \neq \varnothing$.
- If d_1, $d_2 > 0$, and $d = \min\{d_1, d_2\}$, then $B_{d_1} \cap B_{d_2} = B_d \in \mathcal{B}$.

We denote the base $\mathcal{B}$ by $\lim\limits_{\lambda_\tau \to 0}$. So,

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{\lambda_\tau \to 0} \sum_{k=0}^{n-1} f(\xi_k)\Delta x_k.$$

$$\left(\text{Reminder: } \lim_{\mathcal{B}} \sigma(\tau, \xi) = I \underbrace{\Leftrightarrow}_{\text{def}} \forall\ V(I)\ \exists\ (\tau, \xi) \in \mathcal{B} \quad \sigma(\tau, \xi) \subset V(I) \right)$$

Theorem (A necessary condition for intergability)

If $f \in \mathcal{R}[a, b]$, then f is bounded on $[a, b]$.

Proof. Assume the converse. Consider a partition $\tau = \{x_k\}_{k=0}^{n}$. The function f is not bounded on some $[x_r, x_{r+1}]$. Fix tags $\xi_k \in [x_k, x_{k+1}]$ for $k \neq r$. We will choose ξ_r later.

$$\sigma(f, \tau, \xi) = f(\xi_r)\Delta x_r + \underbrace{\sum_{k \neq r} f(\xi_k)\Delta x_k}_{=:\alpha} \quad \Rightarrow \quad |\sigma(f, \tau, \xi)| \geq |f(\xi_r)|\, \Delta x_r - |\alpha|.$$

Suppose $A > 0$, we choose ξ_r such that

$|\sigma(f, \tau, \xi)| \geq |f(\xi_r)|\, \Delta x_r - |\alpha| > A$, that is $|f(\xi_r)| > \dfrac{A + |\alpha|}{\Delta x_r}$.

Than $\sigma(f, \tau, \xi)$ is not bounded. $\square$

Remark. Boundedness is **not** sufficient for Riemann-integrability.

Example. The Dirichlet function

$$f_D(x) = \mathbb{1}_{\mathbb{Q}}(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \notin \mathbb{Q}. \end{cases}$$

If $\xi_k \in \mathbb{Q}$, than

$$\sigma(f_D, \tau, \xi) = \sum_{k=0}^{n-1} f(\xi_k)\Delta_k = \sum_{k=0}^{n-1} \Delta_k = b - a.$$

If $\xi_k \notin \mathbb{Q}$, than

$$\sigma(f_D, \tau, \xi) = \sum_{k=0}^{n-1} f(\xi_k)\Delta_k = 0.$$

$\sigma(f_D, \tau, \xi)$ depends on $\xi \quad \Rightarrow \quad f_D \notin \mathcal{R}[a,\ b]$.

Definition (Upper and lower Darboux sums)

Let $f : [a, b] \to \mathbb{R}$, $\tau = \{x_k\}_{k=0}^{n}$ be a partition of $[a, b]$,

$$M_k := \sup_{x \in [x_k, x_{k+1}]} f(x), \qquad m_k := \inf_{x \in [x_k, x_{k+1}]} f(x),$$

$k = 0, \ldots, n-1$. The **upper and lower Darboux sums** of a function f with respect to a partition τ are defined as

$$S = S_\tau(f) := \sum_{k=0}^{n-1} M_k \Delta x_k, \quad s = s_\tau(f) := \sum_{k=0}^{n-1} m_k \Delta x_k,$$

respectively.

Remark. The upper and lower Darboux sums might not be Riemann sums. **Why?**

Remark. f is bounded from above (from below) $\Leftrightarrow$ S (s) is finite.

The properties of Darboux sums

D1. $S_\tau(f) = \sup\limits_{\xi} \sigma(f, \tau, \xi), \quad s_\tau(f) = \inf\limits_{\xi} \sigma(f, \tau, \xi).$

Proof. We prove $s_\tau(f) = \inf\limits_{\xi} \sigma(f, \tau, \xi)$. For S_τ we can proceed the same way.

$$f(\xi_k) \geq m_k, \text{ for } k = 0, \ldots, n-1, \; \xi_k \in [x_k, x_{k+1}]$$

$$\Rightarrow \sigma(f, \tau, \xi) = \sum_{k=0}^{n-1} f(\xi_k)\Delta_k \geq \sum_{k=0}^{n-1} m_k \Delta_k = s_\tau(f).$$

Let us prove $\forall\, \varepsilon \;\exists\, \xi^0 \quad \sigma(f, \tau, \xi^0) < s_\tau(f) + \varepsilon$.

- <u>f is bounded from below.</u>
  We choose $\xi_k^0 \in [x_k, x_{k+1}]$ such that $f(\xi_k^0) < m_k + \dfrac{\varepsilon}{b-a}$, than

$$\sigma(f, \tau, \xi^0) = \sum_{k=0}^{n-1} f(\xi_k^0)\Delta x_k < \sum_{k=0}^{n-1} \left(m_k + \frac{\varepsilon}{b-a} \right) \Delta x_k$$

$$= s_\tau(f) + \sum_{k=0}^{n-1} \frac{\varepsilon}{b-a} \Delta x_k = s_\tau(f) + \varepsilon.$$

- <u>f is not bounded from below.</u> Then $s_\tau(f) = -\infty$ and $\sigma(f, \tau, \xi)$ is not bounded from below (see the proof of a necessary condition for integrability, check by yourself). $\square$

D2. The upper sum does not increase and the lower sum does not decrease when new points are added to partition.

Proof. We consider $S_\tau(f)$. Suppose $\tau = \{x_k\}_{k=0}^n$ is a partition of $[a, b]$, add a new point $c \in (x_r, x_{r+1})$, denote the new partition by T.

$$S_\tau(f) = \sum_{k=0}^{r-1} M_k \Delta x_k + M_r(x_{r+1} - x_r) + \sum_{k=r+1}^{n-1} M_k \Delta x_k,$$

$$S_T(f) = \sum_{k=0}^{r-1} M_k \Delta x_k + M'(c - x_r) + M''(x_{r+1} - c) + \sum_{k=r+1}^{n-1} M_k \Delta x_k,$$

where $M' := \sup\limits_{x \in [x_r,\, c]} f(x), \quad M'' := \sup\limits_{x \in [c,\, x_{k+1}]} f(x).$

$$M' \le M_r,\ M'' \le M_r \quad \Rightarrow \quad M'(c - x_r) + M''(x_{r+1} - c) \le M_r(x_{r+1} - x_r)$$

$$S_T(f) \le S_\tau(f). \qquad \square$$

D3. Any lower Darboux sum is not greater than any upper Darboux sum, that is

$$\forall \tau_1, \tau_2 \quad s_{\tau_1} \le S_{\tau_2}.$$

Proof. Let τ_1, τ_2 be two partitions. Denote $\tau := \tau_1 \cup \tau_2$. Then

$$s_{\tau_1}(f) \underbrace{\le}_{D2} s_\tau(f) \le S_\tau(f) \underbrace{\le}_{D2} S_{\tau_2}(f). \qquad \square$$

Definition (Darboux integrals)

The quantities

$$I^* := \inf_{\tau} S_\tau(f) \quad and \quad I_* := \sup_{\tau} s_\tau(f)$$

are called the **upper and the lower Darboux integrals**, *respectively.*

Theorem (Criterion for integrability)

Suppose f is bounded on $[a, b]$. Then the following conditions are equivalent

1. $f \in \mathcal{R}[a, b]$.

2. $\forall \varepsilon \, \exists \tau \; (S_\tau(f) - s_\tau(f) < \varepsilon)$.

3. $\forall \varepsilon \;\, \exists \delta = \delta(\varepsilon) \;\, \forall \tau \, (\lambda_\tau < \delta \; \to \;\; S_\tau(f) - s_\tau(f) < \varepsilon)$.

Proof. 1. $\Rightarrow$ 2. $\Rightarrow$ 3. $\Rightarrow$ 1.

1. $\Rightarrow$ 2.

$$I = \int_a^b f \quad \Leftrightarrow \quad \forall \varepsilon > 0 \; \exists \delta > 0 \; \forall (\tau, \xi) \left(\lambda_\tau < \delta \; \rightarrow I - \frac{\varepsilon}{3} < \sigma < I + \frac{\varepsilon}{3} \right).$$

$$\left.\begin{array}{l} S_\tau(f) = \sup\limits_\xi \sigma(f, \tau, \xi) \Rightarrow S_\tau(f) \le I + \frac{\varepsilon}{3} \\ s_\tau(f) = \inf\limits_\xi \sigma(f, \tau, \xi) \Rightarrow s_\tau(f) \ge I - \frac{\varepsilon}{3} \end{array}\right. \quad \Rightarrow \; 0 \le S_\tau(f) - s_\tau(f) \le \frac{2\varepsilon}{3}.$$

2. $\Rightarrow$ 3. Fix $\varepsilon > 0$ and a partition $\tau = \{x_k^*\}_{k=0}^n$: $S_\tau(f) - s_\tau(f) < \varepsilon/2$. We need to find $\delta = \delta(\varepsilon)$ such that for any $T = \{x_k\}_{k=0}^N$, $\lambda(T) < \delta$ the inequality $S_T(f) - s_T(f) < \varepsilon$ holds. We choose $\delta < \dfrac{\varepsilon}{8nK}$, where $K := \sup |f|([a,\ b])$.

$$S_T(f) - s_T(f) = \sum\nolimits^a (M - m)\Delta + \sum\nolimits^b (M - m)\Delta$$

$\sum^a$: intervals of T contain at least one point x_k^*,
$\sum^b$: intervals of T do not contain points x_k^*.

An amount of terms in $\sum^a$ is $\leq 2n$. $\Rightarrow$ $\sum^a (M-m)\Delta \leq 2n2K\delta < \varepsilon/2$.
$\sum^b = \sum\limits_{j=0}^{n-1} \sum^j$, where $\sum^j$ corresponds to the intervals $[x_k,\, x_{k+1}]$ of T : $[x_k,\, x_{k+1}] \subset (x_j^*,\, x_{j+1}^*)$.
$\sum\limits_{j=0}^{n-1} \sum^j (M-m)\Delta \leq \sum\limits_{j=0}^{n-1} (M_j - m_j) \sum^j \Delta \leq \sum\limits_{j=0}^{n-1} (M_j - m_j)\Delta x_j < \frac{\varepsilon}{2}$.

3. $\Rightarrow$ 1. $I^* := \inf_\tau S_\tau(f) \leq S_\tau(f)$ and $I_* := \sup_\tau s_\tau(f) \geq s_\tau(f)$.

$$S_\tau(f) \geq s_{\tau'}(f) \underbrace{\Rightarrow}_{\inf_\tau} I^* \geq s_{\tau'}(f) \underbrace{\Rightarrow}_{\sup_{\tau'}} I^* \geq I_*.$$

$$0 \leq I^* - I_* \leq S_\tau(f) - s_\tau(f) \to 0 \text{ as } \lambda_\tau \to 0 \Rightarrow I^* = I_* =: I_0.$$

$$\begin{matrix} s_\tau(f) \leq I_0 \leq S_\tau(f) \\ s_\tau(f) \leq \sigma \leq S_\tau(f) \end{matrix} \Rightarrow |\sigma - I_0| \leq S_\tau(f) - s_\tau(f) \to 0 \text{ as } \lambda_\tau \to 0$$

$$\Rightarrow I_0 = \int_a^b f. \quad \square$$

Example. The Riemann function

$$f_R(x) = \begin{cases} 1/q, & x = p/q, \\ 0, & x \notin \mathbb{Q} \text{ or } x = 0. \end{cases}$$

Let us prove $f_R \in \mathcal{R}[0,\, 1]$.
$s_\tau(f_R) = 0$. Let us prove $S_\tau(f_R) \to 0$ as $\lambda_\tau \to 0$. Fix $\varepsilon > 0$, find $N \in \mathbb{N}$: $1/N < \varepsilon/2$. The amount C_N of rational numbers $p/q \in [0,1]$, $q \le N$ is finite. We choose $\delta = \varepsilon/(4C_N)$, $\tau\ :\ \lambda_\tau < \delta$

$$S_\tau(f_R) = \sum_{M_k \ge 1/N} \underbrace{M_k}_{<1} \Delta x_k + \sum_{M_k < 1/N} M_k \Delta x_k < 2C_N \cdot 1 \cdot \delta + \frac{1}{N} \underbrace{\sum \Delta x_k}_{\le 1} < \varepsilon$$

Definition (An Oscillation of a function)

Suppose $f : D \subset \mathbb{R} \to \mathbb{R}$. *The* **oscillation** *of the function* f *on the set* D *is defined as*

$$\omega(f,\, D) = \sup_{x,y \in D} |f(x) - f(y)|\,.$$

Remark. **or** **Exercise.** $\omega(f,\, D) = \sup\limits_{x \in D} f(x) - \inf\limits_{x \in D} f(x)$.

Corollary

$$f \in \mathcal{R}[a,\, b] \quad \Leftrightarrow \quad \forall \varepsilon\ \exists \tau = \{x_k\}_{k=0}^{n} \qquad \sum_{k=0}^{n-1} \omega_k(f) \Delta x_k < \varepsilon,$$

where $\omega_k(f) := \omega(f,\, [x_k, x_{k+1}]) = M_k - m_k$.

Theorem (Integrability of a restriction)

Suppose $f \in \mathcal{R}[a, b]$, *and* $[c, d] \subset [a, b]$, *then* $f \in \mathcal{R}[c, d]$.

Proof.

$$f \in \mathcal{R}[a, b] \Rightarrow \forall \varepsilon \ \ \exists \delta = \delta(\varepsilon) \ \ \forall \tau \ (\lambda_\tau < \delta \ \to \ \ S_\tau(f) - s_\tau(f) < \varepsilon)$$

Let τ_0, τ_1, and τ_2 be partitions of $[c, d]$, $[a, c]$, and $[d, b]$, respectively, $\lambda_{\tau_j} < \delta$, $j = 0, 1, 2$. Then $\tau := \tau_0 \cup \tau_1 \cup \tau_2$ is a partition of $[a, b]$, $\lambda_\tau < \delta$.

$$\tau = x_0 = a < x_1 < \cdots < x_\mu = c < \cdots < x_\nu = d < \cdots < x_n = b.$$

$$S_{\tau_0}(f) - s_{\tau_0}(f) = \sum_{k=\mu}^{\nu-1} \omega_k(f)\Delta x_k \le \sum_{k=0}^{n-1} \omega_k(f)\Delta x_k = S_\tau(f) - s_\tau(f) < \varepsilon. \ \square$$

Theorem (Additivity of the integral w. r. t. the interval)

Suppose $a < c < b$, $f \in \mathcal{R}[a,\,c]$, $f \in \mathcal{R}[c,\,b]$. *Then* $f \in \mathcal{R}[a,\,b]$ *and*

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Proof. $f \in \mathcal{R}[a,\,c]$, $f \in \mathcal{R}[c,\,b] \Rightarrow \forall\,\varepsilon\ \ \exists\,\tau_1$, partition of $[a,\,c]$, $\exists\,\tau_2$, partition of $[c,\,b]$, $S_{\tau_1}(f) - s_{\tau_1}(f) < \dfrac{\varepsilon}{2}$ $S_{\tau_2}(f) - s_{\tau_2}(f) < \dfrac{\varepsilon}{2}$.
Then $\tau = \tau_1 \cup \tau_2$ is a partition of $[a,\,b]$ and
$S_\tau(f) - s_\tau(f) = S_{\tau_1}(f) - s_{\tau_1}(f) + S_{\tau_2}(f) - s_{\tau_2}(f) < \varepsilon \Rightarrow f \in \mathcal{R}[a,\,b]$.
Let τ_1^k, τ_2^k, $k \in \mathbb{N}$, be sequences of partitions of $[a,\,c]$ and $[c,\,b]$, respectively, $\lambda_{\tau_1^k} \to 0$, $\lambda_{\tau_2^k} \to 0$ as $k \to \infty$. Then $\tau^k := \tau_1^k \cup \tau_2^k$ is a sequence of partitions of $[a,\,b]$ and $\lambda_{\tau^k} \to 0$ as $k \to \infty$.

$$S_{\tau^k}(f) = S_{\tau_1^k}(f) + S_{\tau_2^k}(f) \underbrace{\Rightarrow}_{k\to\infty} \int_a^b f = \int_a^c f + \int_c^b f. \ \square$$

Theorem (Integrability of continuous functions)

If $f \in C[a,\ b]$*, then* $f \in \mathcal{R}[a,\ b]$.

Proof. If $f \in C[a,\ b]$, then f is uniformly continuous on $[a,\ b]$ (the Heine-Cantor theorem).

$$\forall\, \varepsilon\ \exists\, \delta\ \forall\, c, d \in [a,\ b]\ \left(|c-d| < \delta \to |f(c)-f(d)| < \frac{\varepsilon}{b-a} \right)$$

$$\text{If } \tau = \{x_k\}_{k=0}^{n}, \lambda_\tau < \delta, \text{ then } \omega_k(f) = \sup_{c,d \in [x_k,\, x_{k+1}]} |f(c)-f(d)| < \frac{\varepsilon}{b-a}$$

$$\Rightarrow \sum_{k=0}^{n-1} \omega_k(f)\Delta_k < \frac{\varepsilon}{b-a} \sum_{k=0}^{n-1} \Delta x_k = \varepsilon. \quad \square$$

Theorem (Integrability of monotone functions)

A monotone function is integrable.

Proof. Suppose f is decreasing on $[a, b]$. If $f(a) = f(b)$, then f is a constant, say, $f = C$, $\sigma(f, \tau, \xi) = C(b-a)$, $f \in \mathcal{R}[a, b]$.

If $f(a) \neq f(b)$, then we fix ε, choose $\delta = \dfrac{\varepsilon}{f(a) - f(b)}$, and consider $\tau = \{x_k\}_{k=0}^{n}$, $\lambda_\tau < \delta$.

$$\omega_k(f) = f(x_k) - f(x_{k+1}) \;\Rightarrow\; \sum_{k=0}^{n-1} \omega_k(f)\Delta x_k = \sum_{k=0}^{n-1} \left(f(x_k) - f(x_{k+1})\right)\Delta x_k$$

$$< \frac{\varepsilon}{f(a) - f(b)} \underbrace{\sum_{k=0}^{n-1} \left(f(x_k) - f(x_{k+1})\right)}_{=f(a)-f(b)} = \varepsilon. \quad \square$$

Lemma

The integrability and the value of the integral do not change if we change values of an integrable function at a finite number of points.

Proof. Let $f \in \mathcal{R}[a,b]$, $\tilde{f} : [a,\ b] \to \mathbb{R}$, $\left\{x \,\middle|\, f(x) \neq \tilde{f}(x)\right\} = \{x_1, \ldots, x_m\}$.
f is bounded, $|f(x)| \le K$, then $|\tilde{f}(x)| \le \underbrace{\max\left\{|\tilde{f}(x_1)|, \ldots, |\tilde{f}(x_m)|, K\right\}}_{=:\tilde{K}.}$

$$\left|\sigma(f,\tau,\xi) - \sigma(\tilde{f},\tau,\xi)\right| = \left|\sum_{k: f(\xi_k) \neq \tilde{f}(\xi_k)} \Big(f(\xi_k) - \tilde{f}(\xi_k)\Big)\, \Delta x_k\right|$$

$$\le 2m(K + \tilde{K})\lambda_\tau \to 0 \text{ as } \lambda_\tau \to 0. \quad \square$$

Remark. The function f might not be defined at a finite number of points of $[a,b]$, nevertheless, f might be Riemann-integrable on $[a,b]$.

Definition (A piecewise continuous function)

A function $f : [a, b] \to \mathbb{R}$ is called **piecewise continuous** if the set of points of discontinuity is either empty or finite and all discontinuities are of a first kind (jump discontinuities).

Theorem (Integrability of piecewise continuous functions)

A piecewise continuous function is integrable.

Proof. Let $\{c_1, \dots, c_m\}$ be the points of discontinuity of f on (a, b). Denote $c_0 := a$, $c_{m+1} := b$. The function f is continuous on (c_k, c_{k+1}), $k = 0, \dots, m$, and $f(c_k \pm 0)$ are finite. So, there are at most two points, where f differs from a continuous function on $[c_k, c_{k+1}]$. By Lemma, $f \in \mathcal{R}[c_k, c_{k+1}]$. By additivity of the integral w.r.t. the interval of integration, $f \in \mathcal{R}[a, b]$. $\square$

Example. $\displaystyle\int_0^{\pi/2} \sin x \,\mathrm{d}x,$

$$\sin \in C[0,\pi/2] \Rightarrow \sin \in \mathcal{R}[0,\pi/2] \Rightarrow \int_0^{\pi/2} \sin x \,\mathrm{d}x = \lim_{n\to\infty} \sigma(f,\tau^n,\xi^n),$$

where $\lambda_{\tau^n} \to 0$ as $n \to \infty$.

$$\tau^n = \{x_k^n\}_{k=0}^n = \left\{\frac{\pi k}{2n}\right\}_{k=0}^n, \quad \xi_k^n = x_k^n, \quad \Delta x_k = \frac{\pi}{2n}, \quad \lambda_{\tau^n} \to 0 \text{ as } n \to \infty.$$

$$\sigma(\sin,\tau^n,\xi^n) = \frac{\pi}{2n}\sum_{k=0}^{n-1} \sin\frac{\pi k}{2n} = \frac{\sqrt{2}\pi}{4n}\frac{\sin\left(\frac{\pi}{4}-\frac{\pi}{4n}\right)}{\sin\frac{\pi}{4n}} \to 1 \text{ as } n \to \infty.$$

$$\left(\text{Reminder: } \sum_{k=1}^{N} \sin k\alpha = \frac{\sin\frac{Nx}{2}\sin\frac{(N+1)x}{2}}{\sin\frac{x}{2}}\right)$$

Example. $\displaystyle\int_a^b \frac{\mathrm{d}x}{x^2}$, $0 < a < b$.

$$\frac{1}{x^2} \in C[a,b] \Rightarrow \frac{1}{x^2} \in \mathcal{R}[a,b] \Rightarrow \int_a^b \frac{\mathrm{d}x}{x^2} = \lim_{n\to 0} \sigma(1/x^2, \tau^n, \xi^n),$$

where $\lambda_{\tau^n} \to 0$ as $n \to \infty$.

$$\tau^n = \{x_k\}_{k=0}^n, \quad \xi_k = \sqrt{x_k x_{k+1}}.$$

$$\sigma(1/x^2, \tau, \xi) = \sum_{k=0}^{n-1} \frac{\Delta x_k}{x_k x_{k+1}} = \sum_{k=0}^{n-1} \left(\frac{1}{x_k} - \frac{1}{x_{k+1}} \right) = \frac{1}{a} - \frac{1}{b}.$$

Example. Let f be monotonic on $[0,1]$. Let us prove that

$$\int_0^1 f(x)dx - \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) = O\left(\frac{1}{n}\right).$$

$$s_\tau(f) \le \int_0^1 f \le S_\tau(f), \quad s_\tau(f) \le \sigma \le S_\tau(f)$$

$$\Rightarrow \left|\int_0^1 f - \sigma\right| \le S_\tau(f) - s_\tau(f).$$

Let τ be a uniform partition $\tau = \left\{\frac{k}{n}\right\}_{k=0}^{n}$, f is monotonic

$$\Rightarrow S_\tau(f) - s_\tau(f) = \frac{1}{n}|f(1) - f(0)| = O\left(\frac{1}{n}\right).$$

$$\left|\frac{\int\limits_0^1 f - \sigma}{S_\tau(f) - s_\tau(f)}\right| \le 1 \Rightarrow \int\limits_0^1 f - \sigma = O\left(\frac{1}{n}\right).$$

Reminder: A set X is **countable** if it is equipollent with the set $\mathbb{N}$, that is, $\mathrm{card} X = \mathrm{card} \mathbb{N}$.

Definition (Set of measure zero)

It is said that a set $E \subset \mathbb{R}$ has **measure zero** *if for any ε there exists a covering of the set E by at most countable system $\{(a_n, b_n)\}_n$ of intervals such that $\sum_n |b_n - a_n| < \varepsilon$.*

Example. Any at most countable set has measure zero.
Indeed, let $\{x_k\}_k$ be at most countable set, than the desired covering is $(x_k - \varepsilon 2^{-k-1}, x_k + \varepsilon 2^{-k-1})$.

Theorem (Lebesgue's criterion for Riemann integrability)

$f \in \mathcal{R}[a, b] \Leftrightarrow f$ is bounded on $[a, b]$ and the points of discontinuity of f form a set of measure zero.

Example. The Riemann function f_R is bounded, the set of points of discontinuity of f_R is $\mathbb{Q} \setminus \{0\}$. It is countable. Therefore, $f_R \in \mathcal{R}[a, b]$.

Example. We give a example of integrable function such that the set of points of discontinuity for the function is not countable.

The Cantor set.

We take the interval $F_1 = [0,1]$, then we remove the interval $(1/3, 2/3)$ and get $F_2 = [0,1/3] \cup [2/3,1]$. After that we remove the intervals $(1/9, 2/9)$, $(7/9, 8/9)$ and get $F_3 = [0,1/9] \cup [2/9,1/3] \cup [2/3,7/9] \cup [8/9,1]$. We continue the process for infinitely many steps. On each step we remove the open middle third from each closed interval designed on the previous step. As a result, we have a sequence of sets $(F_n)_{n=1}^{\infty}$.

$$\text{The set } F := \bigcap_{k=1}^{\infty} F_k \text{ is called } \textbf{the Cantor set}.$$

1. F is not countable. Indeed, let $x \in [0,1]$ has ternary representation $x = 0.x_1x_2\ldots$, $x_k \in \{0,1,2\}$. Then $F = \{x = 0.x_1x_2\ldots \ : \ x_k \in \{0,2\}\}$. Using the binary representation for a real number, we conclude that F is equipollent to $[0,1]$, which is not countable.

2. F is of measure zero.

$$F = [0,1] \setminus \left(\left(\frac{1}{3}, \frac{2}{3}\right) \cup \left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right) \cup \dots \right)$$

The sum of the lengths of all open intervals is equal to

$$\frac{1}{3} + 2 \cdot \frac{1}{9} + 4 \cdot \frac{1}{27} + \dots = \frac{1}{3}\left(1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \dots\right) = \frac{1}{3}\frac{1}{1-\frac{2}{3}} = 1.$$

Consider the function $f = \mathbb{1}_F$. It is bounded and the set of points of discontinuity of f is F. Therefore, $f \in \mathcal{R}[0,1]$.

Example. Suppose $f : [a, b] \to \mathbb{R}, f \in \mathcal{R}[a, b], A \leq f(x) \leq B$ and $\psi : [A, B] \to \mathbb{R}, \psi \in C[A, B]$, $g = \psi \circ f : [a, b] \to \mathbb{R}$. Let us prove that $g \in \mathcal{R}[a, b]$.
f satisfies Lebesgue's criterion for Riemann integrability, $\psi \circ f$ continuous at every points of continuity of $f \Rightarrow \psi \circ f$ satisfies Lebesgue's criterion for integrability.
The condition $\psi \in C[A, B]$ can not be relaxed to $\psi \in \mathcal{R}[A, B]$. Indeed,

$$\psi(y) = \begin{cases} 0, & y = 0, \\ 1, & y \neq 0, \end{cases} \qquad f_R(x) = \begin{cases} 1/q, & x = p/q, \\ 0, & x \notin \mathbb{Q} \text{ or } x = 0. \end{cases}$$

$$f_R \in \mathcal{R}[a, b], \quad \psi \in \mathcal{R}[A, B], \quad f_D = \psi \circ f_R \notin \mathcal{R}[a, b].$$

Theorem (Integrability and arithmetic operations)

If $f, g \in \mathcal{R}[a,b]$, $\alpha \in \mathbb{R}$, then αf, $f+g$, $|f|$, fg $\mathcal{R}[a,b]$. In addition, if $\inf g([a,b]) > 0$, *then* $f/g \in \mathcal{R}[a,b]$.

Proof. The case of fg

f, g are bounded, $|f(x)| \le C_1$, $|g(x)| \le C_2$. Let $x_1, x_2 \in E \subset [a,b]$.

$$|f(x_1)g(x_1) - f(x_2)g(x_2)| \le |f(x_1) - f(x_2)||g(x_1)| + |g(x_1) - g(x_2)||f(x_2)|$$

$$\le C_2|f(x_1) - f(x_2)| + C_1|g(x_1) - g(x_2)| \underbrace{\le}_{\text{sup}} C_2\omega(f,E) + C_1\omega(g,E)$$

$$\Rightarrow \omega(fg,E) \le C_2\omega(f,E) + C_1\omega(g,E).$$

For any $\tau = \{x_k\}_{k=0}^{n}$ we get $\omega_k(fg) \le C_2\omega_k(f) + C_1\omega_k(g) \Rightarrow$

$$\sum_{k=0}^{n-1} \omega_k(fg)\Delta x_k \le C_2 \sum_{k=0}^{n-1} \omega_k(f)\Delta x_k + C_1 \sum_{k=0}^{n-1} \omega_k(g)\Delta x_k.$$

The case of $|f|$

$$\big||f(x_1)| - |f(x_2)|\big| \le |f(x_1) - f(x_2)| \le \omega(f, E) \Rightarrow \omega(|f|, E) \le \omega(f, E)$$

The case of $1/g$

$$m := \inf g\left([a, b]\right), \quad \left|\frac{1}{g(x_1)} - \frac{1}{g(x_2)}\right| = \frac{|g(x_1) - g(x_2)|}{|g(x_1)g(x_2)|} \le \frac{|g(x_1) - g(x_2)|}{m^2}$$

$$\le \frac{1}{m^2}\omega(g, E) \;\Rightarrow\; \omega(1/g, E) \le \frac{1}{m^2}\omega(g, E) \quad \square$$

Remark. $|f| \in \mathcal{R}[a, b] \not\Rightarrow f \in \mathcal{R}[a, b]$, for example

$$f_D(x) - 1/2 = \begin{cases} 1/2, & x \in \mathbb{Q}, \\ -1/2, & x \notin \mathbb{Q} \end{cases} \notin \mathcal{R}[a, b], \quad \text{while} \quad |f_D| \equiv 1/2 \in \mathcal{R}[a, b].$$

Remark. If $a > b$ and $f \in \mathcal{R}[b, a]$, then $\int_a^b f := -\int_b^a f$, $\int_a^a f := 0$.

Properties of Riemann integral

I1. Linearity of integral. If $f, g \in \mathcal{R}[a, b]$, $\alpha, \beta \in \mathbb{R}$, then

$$\int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g$$

Proof. By theorem on integrability and arithmetic operations, $\alpha f + \beta g \in \mathcal{R}[a, b]$. Let (τ^n, ξ^n) be a sequence of tagged partitions $\tau^n = \{x_k^n\}_{k=0}^{N_n}$, $\xi^n = (\xi_k^n)_{k=0}^{N_n-1}$, $\lambda_{\tau^n} \to 0$ as $n \to \infty$, then

$$\sum_{k=0}^{N_n-1} \left(\alpha f(\xi_k^n) + \beta g(\xi_k^n)\right) \Delta x_k^n = \alpha \sum_{k=0}^{N_n-1} f(\xi_k^n) \Delta x_k^n + \beta \sum_{k=0}^{N_n-1} g(\xi_k^n) \Delta x_k^n.$$

It remains to pass to the limit $n \to \infty$. $\square$

I2. Monotonicity of integral. If $a < b$, $f, g \in \mathcal{R}[a,b]$, $f(x) \le g(x)$ for $x \in [a,b]$, then $\int_a^b f \le \int_a^b g$.

Proof. Let (τ^n, ξ^n) be a sequence of tagged partitions $\tau^n = \{x_k^n\}_{k=0}^{N_n}$, $\xi^n = (\xi_k^n)_{k=0}^{N_n-1}$, $\lambda(\tau^n) \to 0$ as $n \to \infty$, then

$$\sum_{k=0}^{N_n-1} f(\xi_k^n)\Delta x_k^n \le \sum_{k=0}^{N_n-1} g(\xi_k^n)\Delta x_k^n.$$

It remains to pass to the limit $n \to \infty$. □

Corollary 1. If $m \le f(x) \le M$, $x \in [a,b]$, then

$$m(b-a) \le \int_a^b f \le M(b-a).$$

Corollary 2. If $f(x) \ge 0$, $x \in [a,b]$, then $\int_a^b f \ge 0$.

I3. If $f \in \mathcal{R}[a,b]$, $f \geq 0$, $\exists\, x_0 \in [a,b]\ \big(f(x_0) > 0, f$ is continuous at $x_0\big)$, then $\displaystyle\int_a^b f > 0$.

Proof. f is continuous at $x_0 \Rightarrow$ for $\varepsilon = \dfrac{f(x_0)}{2}$ $\exists\, \delta\, \forall\, x \in (x_0 - \delta, x_0 + \delta)$ $\big|f(x) - f(x_0)\big| < \dfrac{f(x_0)}{2} \Rightarrow f(x) > \dfrac{f(x_0)}{2}$. We denote $[c,d] := [a,b] \cap [x_0 - \delta, x_0 + \delta]$.

$$\int_a^b f = \left(\underbrace{\int_a^c}_{\geq 0} + \int_c^d + \underbrace{\int_d^b}_{\geq 0}\right) f \geq \int_c^d f \geq \int_c^d \frac{f(x_0)}{2} = \frac{f(x_0)}{2}(d-c) > 0. \square$$

Example. $f \in \mathcal{R}[a,b]$, $f > 0$ on $[a,b] \Rightarrow \int_a^b f > 0$.
By Lebesgue's criterion the points of discontinuity of f form a set of measure zero. So,

$$\exists\ x_0 \quad \text{such that} \quad f(x_0) > 0, f \text{ is continuous at } x_0$$

It remains to apply the property I3

I4. $\left|\int_a^b f\right| \le \left|\int_a^b |f|\right|$

Proof. $a < b$

$$-|f| \le f \le |f| \underbrace{\Rightarrow}_{I2} -\int_a^b |f| \le \int_a^b f \le \int_a^b |f| \Rightarrow \left|\int_a^b f\right| \le \int_a^b |f|.$$

$b < a$ $\qquad \int_a^b f = -\int_b^a f$

$$\left|\int_a^b f\right| = \left|\int_b^a f\right| \le \int_b^a |f| = \left|\int_a^b |f|\right| \quad \square$$

Theorem (The first mean value theorem)

If $f, g \in \mathcal{R}[a,b]$, $g \geq 0$ (or $g \leq 0$) on $[a,b]$, and $m \leq f \leq M$, then

$$\exists\, \mu \in [m, M] \quad \int_a^b fg = \mu \int_a^b g.$$

Proof. $g \geq 0 \Rightarrow \displaystyle\int_a^b g \geq 0$

$$m \leq f \leq M \Rightarrow mg \leq fg \leq Mg \underbrace{\Rightarrow}_{I2} m \int_a^b g \leq \int_a^b fg \leq M \int_a^b g$$

If $\displaystyle\int_a^b g = 0$, then $\displaystyle\int_a^b fg = 0$ and any μ is appropriate.

If $\displaystyle\int_a^b g > 0$, then $m \leq \dfrac{\int_a^b fg}{\int_a^b g} \leq M$. □

Corollary

If $f \in C[a,b]$, $g \in \mathcal{R}[a,b]$, $g \geq 0$ *(or* $g \leq 0$*) on* $[a,b]$, *then*

$$\exists\, c \in [a,b] \ \int_a^b fg = f(c) \int_a^b g.$$

Proof. By the Weierstrass maximum value theorem $\exists\, x_1, x_2 \in [a,b]$ $f(x_1) = \max f([a,b]) = M$, $f(x_2) = \min f([a,b]) = m$. By the Bolzano intermediate value theorem $\forall\, \mu \in [m,M]\ \exists\, c \in [a,b]\ \ f(c) = \mu$. $\square$

Corollary

If $f \in \mathcal{R}[a,b]$ *and* $m \leq f \leq M$, *then* $\exists\, \mu \in [m,M] \ \int_a^b f = \mu(b-a)$.

Corollary

If $f \in C[a,b]$, *then* $\exists\, c \in [a,b] \ \int_a^b f = f(c)(b-a)$.

Theorem (Integral with variable upper limit)

Let $f \in \mathcal{R}[a,b]$, $x \in [a,b]$, $\Phi(x) := \displaystyle\int_a^x f$, *then*

1. $\Phi \in C[a,b]$
2. *If* f *is continuous at* $x_0 \in [a,b]$, *then* Φ *is differentiable at* x_0 *and* $\Phi'(x_0) = f(x_0)$.

The function Φ *is called an* **integral with variable upper limit**.

Proof. 1. $f \in \mathcal{R}[a,b] \Rightarrow \exists\, M \ |f| \le M$. Let $x_0, x_0 + \Delta x \in [a,b]$.

$$\left|\Phi(x_0+\Delta x)-\Phi(x_0)\right| = \left|\int_a^{x_0+\Delta x} f - \int_a^{x_0} f\right| = \left|\int_{x_0}^{x_0+\Delta x} f\right| \underbrace{\le}_{I4} \left|\int_{x_0}^{x_0+\Delta x} |f|\right|$$

$$= \begin{cases} \displaystyle\int\limits_{x_0}^{x_0+\Delta x} |f|, & \Delta x \ge 0, \\ \displaystyle\int\limits_{x_0+\Delta x}^{x_0} |f|, & \Delta x \le 0, \end{cases} \le \begin{cases} \displaystyle\int\limits_{x_0}^{x_0+\Delta x} M, & \Delta x \ge 0, \\ \displaystyle\int\limits_{x_0+\Delta x}^{x_0} M, & \Delta x \le 0, \end{cases} = M|\Delta x| \to 0$$

as $\Delta x \to 0$.

2. $$\frac{\Phi(x_0+\Delta x)-\Phi(x_0)}{\Delta x}-f(x_0)=\frac{1}{\Delta x}\int\limits_{x_0}^{x_0+\Delta x} f(t)\,\mathrm{d}t-\frac{1}{\Delta x}\int\limits_{x_0}^{x_0+\Delta x} f(x_0)\,\mathrm{d}t$$

$$=\frac{1}{\Delta x}\int\limits_{x_0}^{x_0+\Delta x}\big(f(t)-f(x_0)\big)\,\mathrm{d}t$$

f is continuous at $x_0 \;\Rightarrow\; \forall\,\varepsilon\;\exists\,\delta\;\forall\,t\in(x_0-\delta,x_0+\delta)\;\;|f(t)-f(x_0)|<\varepsilon.$
We choose Δx : $x_0+\Delta x\in(x_0-\delta,x_0+\delta)$, then

$$\left|\frac{1}{\Delta x}\int\limits_{x_0}^{x_0+\Delta x}\big(f(t)-f(x_0)\big)\,\mathrm{d}t\right|\le\frac{1}{|\Delta x|}\left|\int\limits_{x_0}^{x_0+\Delta x}|f(t)-f(x_0)|\,\mathrm{d}t\right|$$

$$=\begin{cases}\frac{1}{|\Delta x|}\int\limits_{x_0}^{x_0+\Delta x}|f(t)-f(x_0)|\,\mathrm{d}t, & \Delta x\ge 0,\\ \frac{1}{|\Delta x|}\int\limits_{x_0+\Delta x}^{x_0}|f(t)-f(x_0)|\,\mathrm{d}t, & \Delta x\le 0,\end{cases}\le\frac{1}{|\Delta x|}\,\varepsilon\,|\Delta x|=\varepsilon\quad\square$$

Remark. If $f \in C[a, b]$, then $\forall\, x \in [a, b]\ \ \Phi'(x) = f(x)$.

So, any continuous function has a primitive $\Phi(x) = \displaystyle\int_a^x f$.

An arbitrary primitive F of f is $F(x) = \Phi(x) + C$. For $x = a$ we get $F(a) = \displaystyle\int_a^a f + C = C$, that is $\displaystyle\int_a^x f = F(x) - F(a)$.

Theorem (The fundamental theorem of integral calculus, the Newton-Leibniz formula)

If $f \in \mathcal{R}[a,b]$, F is a primitive of f on $[a,b]$, then

$$\int_a^b f = F(b) - F(a) =: F(x)\Big|_a^b$$

Proof. Let $\tau^n = \{x_k^n\}_{k=0}^{N_n}$ be a sequence of partitions, $\lambda(\tau^n) \to 0$ as $n \to \infty$, then

$$F(b) - F(a) = \sum_{k=0}^{N_n - 1} \left(F(x_{k+1}^n) - F(x_k^n)\right) = \sum_{k=0}^{N_n - 1} F'(\xi_k^n)\Delta x_k^n$$

$$= \sum_{k=0}^{N_n - 1} f(\xi_k^n)\Delta x_k^n \to \int_a^b f, \text{ as } n \to \infty \quad \square$$

Example. $I = \int_0^1 e^x \arcsin e^{-x}\,\mathrm{d}x$.

$$F(x) = \int e^x \arcsin e^{-x}\,\mathrm{d}x = \left[t = e^{-x}\right] = -\int \frac{\arcsin t}{t^2} dt = \frac{1}{t}\arcsin t$$

$$-\int \frac{dt}{t\sqrt{1-t^2}} = \frac{1}{t}\arcsin t - \int \frac{dt}{t^2\sqrt{(1/t)^2-1}} = \frac{1}{t}\arcsin t$$

$$+\int \frac{d(1/t)}{\sqrt{(1/t)^2-1}} = \frac{1}{t}\arcsin t + \ln\left(\frac{1}{t} + \sqrt{\frac{1}{t^2}-1}\right) + C$$

$$= e^x \arcsin e^{-x} + \ln\left(e^x + \sqrt{e^{2x}-1}\right) + C.$$

$$\int_0^1 e^x \arcsin e^{-x}\,\mathrm{d}x = F(1) - F(0) = e \arcsin e^{-1} - \frac{\pi}{2} + \ln\left(e + \sqrt{e^2-1}\right).$$

Example. Prove the inequality $\frac{4}{9}(e-1) < \int_0^1 \frac{e^x \,\mathrm{d}x}{(x+1)(2-x)} < \frac{1}{2}(e-1)$.

Consider the function $f(x) = \frac{1}{(x+1)(2-x)}$, $x \in [0,1]$. Its derivative $f'(x) = \dfrac{2x-1}{(x+1)^2(2-x)^2} = 0$ for $x = 1/2$ and changes its sign from minus to plus $\Rightarrow \min_{x\in[0,1]} f(x) = f(1/2) = 4/9$, $\max_{x\in[0,1]} f(x) = f(0) = f(1) = 1/2$, So, for any $x \in [0,1]$

$$\frac{4}{9} \le \frac{1}{(x+1)(2-x)} \le \frac{1}{2},$$

and for $x \neq 0, x \neq 1/2$, and $x \neq 1$

$$\frac{4}{9}e^x < \frac{e^x}{(x+1)(2-x)} < \frac{e^x}{2}.$$

$$\frac{4}{9}\int_0^1 e^x \,\mathrm{d}x < \int_0^1 \frac{e^x}{(x+1)(2-x)}\,\mathrm{d}x < \frac{1}{2}\int_0^1 e^x \,\mathrm{d}x,$$

$$\frac{4}{9}(e-1) < \int_0^1 \frac{e^x}{(x+1)(2-x)}\,\mathrm{d}x < \frac{1}{2}(e-1).$$

Example. $\lim\limits_{n\to\infty}\left(\frac{1^3}{n^4}+\frac{2^3}{n^4}+\ldots+\frac{(n-1)^3}{n^4}\right)=\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{k=0}^{n-1}\left(\frac{k}{n}\right)^3$

$$=\lim_{n\to\infty}\sigma\left(x^3,\left\{\frac{k}{n}\right\}_{k=0}^{n-1},\left\{\frac{k}{n}\right\}_{k=0}^{n-1}\right)=\int_0^1 x^3\,\mathrm{d}x=\left.\frac{x^4}{4}\right|_0^1=\frac{1}{4}.$$

Example. $\lim\limits_{n\to\infty}S_n,\, S_n=\sum\limits_{k=1}^{n}\frac{2^{\frac{k}{n}}}{n+\frac{1}{k}}$

$$S_n=\frac{1}{n}\sum_{k=1}^{n}\frac{2^{\frac{k}{n}}}{1+\frac{1}{kn}}=S_n^{(1)}-S_n^{(2)},$$

where $S_n^{(1)}=\frac{1}{n}\sum\limits_{k=1}^{n}2^{\frac{k}{n}},\, S_n^{(2)}=\frac{1}{n}\sum\limits_{k=1}^{n}\frac{2^{\frac{k}{n}}}{1+kn}$. By $0<S_n^{(2)}<\frac{2n}{n^2}=\frac{2}{n}$, we get $\lim\limits_{n\to\infty}S_n^{(2)}=0$, so

$$\lim_{n\to\infty}S_n=\lim_{n\to\infty}S_n^{(1)}=\int_0^1 2^x dx=\left.\frac{2^x}{\log 2}\right|_0^1=\frac{1}{\log 2}.$$

Example. $I = \int_{-1}^{1} \frac{x^2+1}{x^4+1}\,\mathrm{d}x.$

The 1-st method is to find a primitive for all $x \in [-1, 1]$.

$$F(x) = \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,\mathrm{d}x = \int \frac{\mathrm{d}\left(x-\frac{1}{x}\right)}{\left(x-\frac{1}{x}\right)^2+2} = \frac{1}{\sqrt{2}}\arctan\frac{x^2-1}{x\sqrt{2}}$$

$$+\frac{\pi}{2\sqrt{2}}\mathrm{sign}x + C. \qquad I = F(1) - F(-1) = \frac{\pi}{\sqrt{2}}.$$

The 2-nd method is to apply additivity of the integral w.r.t. the interval and to exploit the Newton-Leibniz formula to each interval.

$$I = \int_{-1}^{1} \frac{x^2+1}{x^4+1}\,\mathrm{d}x = \int_{-1}^{0} \frac{x^2+1}{x^4+1}\,\mathrm{d}x + \int_{0}^{1} \frac{x^2+1}{x^4+1}\,\mathrm{d}x$$

$$= \frac{1}{\sqrt{2}}\arctan\frac{x^2-1}{x\sqrt{2}}\Bigg|_{-1}^{-0} + \frac{1}{\sqrt{2}}\arctan\frac{x^2-1}{x\sqrt{2}}\Bigg|_{+0}^{1} = \frac{\pi}{\sqrt{2}}.$$

Theorem

If $f \in \mathcal{R}[a,b]$, F is continuous on $[a,b]$, and F is a primitive of f on $[a,b]$ except a finite number of points, then $\int_a^b f = F(b) - F(a) =: F(x)\big|_a^b$.

Proof. Let $c_1, \dots, c_m$ be all points, where $F'(x) \neq f(x)$. We denote $c_0 := a$, $c_{m+1} = b$. Then

$$\int_{c_k}^{c_{k+1}} f = \lim_{\varepsilon \to 0} \int_{c_k+\varepsilon}^{c_{k+1}-\varepsilon} f = \lim_{\varepsilon \to 0} \left(F(c_{k+1}-\varepsilon) - F(c_k+\varepsilon)\right)$$

$$= F(c_{k+1}) - F(c_k). \quad \text{By additivity,}$$

$$\int_a^b f = \sum_{k=0}^m \int_{c_k}^{c_{k+1}} f = \sum_{k=0}^m F(c_{k+1}) - F(c_k) = F(b) - F(a) \quad \square$$

Example. $\int_{-1}^{1} \operatorname{sign} t \, \mathrm{d}t = |t|\Big|_{-1}^{1} = 0.$

Remark. $F \in C[a,b]$ is important. For $f(x) = 0$, $F(x) = \operatorname{sign} x$, $0 = \int_{-1}^{1} f \neq F|_{-1}^{1} = 2.$

Example. $I = \int_{-1}^{3} \frac{f'(x)\,\mathrm{d}x}{1+f^2(x)}$, where $f(x) = \frac{(x+1)^2(x-1)}{x^3(x-2)}$.

Since $\frac{f'(x)}{1+f^2(x)}$ is continuous and bounded on $D := [-1,0) \cup (0,2) \cup (2,3]$, it follows by Lemma that the integral is well-defined.

Let $F(x) := \arctan f(x)$. For every $x \in D$ we have $F'(x) = \frac{f'(x)}{1+f^2(x)}$.

F can be extended to the continuous functions from the intervals $[-1,0)$, $(0,2)$, $(2,3]$ to the intervals $[-1,0]$, $[0,2]$, $[2,3]$ respectively.

Applying consequently the additivity of the integral w.r.t. the interval and the last Theorem we get

$$I = \int_{-1}^{0} \frac{f'(x)\,\mathrm{d}x}{1+f^2(x)} + \int_{0}^{2} \frac{f'(x)\,\mathrm{d}x}{1+f^2(x)} + \int_{2}^{3} \frac{f'(x)\,\mathrm{d}x}{1+f^2(x)}$$

$$= (F(-0) - F(-1)) + (F(2-0) - F(+0)) + (F(3) - F(2+0))$$

$$= \left(-\frac{\pi}{2} - 0\right) + \left(-\frac{\pi}{2} - \frac{\pi}{2}\right) + \left(\arctan\frac{32}{27} - \frac{\pi}{2}\right) = \arctan\frac{32}{27} - 2\pi.$$

Example. Explain, why the equality is incorrect $\displaystyle\int_{-1}^{1}\frac{\mathrm{d}x}{x^2} = -\frac{1}{x}\Big|_{-1}^{1} = -2.$

1. $\dfrac{1}{x^2} \notin \mathcal{R}[-1,1]$, 2. $\left(\dfrac{1}{x^2}\right)' = -\dfrac{1}{x}$ is incorrect at $x = 0$.

Remark. The fundamental theorem of integral calculus is a result on a restoring a function via its derivative: If F is differentiable on $[a,b]$ and $F' \in \mathcal{R}[a,b]$, then $\displaystyle\int_a^x F' + F(a) = F(x)$ for any $x \in [a,b]$.

Example. f has a primitive on $[a,b] \not\Rightarrow f \in \mathcal{R}[a,b]$, in other words, $\exists\, F : F' \notin \mathcal{R}[a,b]$.

$$F(x) = \begin{cases} x^2\sin(1/x^3), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

$$f(x) = F'(x) = \begin{cases} 2x\sin(1/x^3) - 3/x^2\cos(1/x^3), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

f is not bounded in a neighborhood of $x = 0 \Rightarrow f \notin \mathcal{R}[a,b]$.

Theorem (Integration by parts in the Riemann integral)

If f, g are differentiable on $[a, b]$ and $f', g' \in \mathcal{R}[a, b]$, then

$$\int_a^b f'g = fg\Big|_a^b - \int_a^b fg'.$$

Proof. f, g are differentiable on $[a, b] \Rightarrow f, g$ are continuous on $[a, b]$ $\Rightarrow f, g \in \mathcal{R}[a, b] \underbrace{\Longrightarrow}_{f', g' \in \mathcal{R}[a,b]} f'g, fg' \in \mathcal{R}[a, b] \Rightarrow f'g + fg' \in \mathcal{R}[a, b]$.

By the fundamental theorem of integral calculus,

$$\int_a^b f'g + \int_a^b fg' = \int_a^b (f'g + fg') = fg\Big|_a^b \quad \square$$

Theorem (Change of Variable in the Riemann integral)

If $f \in C[a,b]$, $\varphi : [\alpha, \beta] \to [a,b]$, φ *is differentiable on* $[a,b]$, $\varphi' \in \mathcal{R}[\alpha,\beta]$, *then*

$$\int_\alpha^\beta (f \circ \varphi)\varphi' = \int_{\varphi(\alpha)}^{\varphi(\beta)} f.$$

Proof. $f \circ \varphi \in C[\alpha,\beta] \Rightarrow f \circ \varphi \in \mathcal{R}[\alpha,\beta] \underbrace{\Longrightarrow}_{\varphi' \in \mathcal{R}[\alpha,\beta]} (f \circ \varphi)\varphi' \in \mathcal{R}[\alpha,\beta]$

Let F be a primitive of f on $[a,b]$, then

$$\big(F \circ \varphi\big)'(t) = F'(\varphi(t))\varphi'(t) = f(\varphi(t))\varphi'(t).$$

So, $F \circ \varphi$ is a primitive of $(f \circ \varphi)\varphi'$ on $[\alpha,\beta]$.
By the fundamental theorem of integral calculus,

$$\int_\alpha^\beta (f \circ \varphi)\varphi' = F \circ \varphi\Big|_\alpha^\beta = F\Big|_{\varphi(\alpha)}^{\varphi(\beta)} = \int_{\varphi(\alpha)}^{\varphi(\beta)} f. \quad \square$$

Example. $\int_0^a \sqrt{a^2 - x^2}dx = [x = a\sin t] = a^2 \int_0^{\pi/2} \cos^2 t dt =$
$\frac{a^2}{2}\left(t + \frac{\sin 2t}{2}\right)\Big|_0^{\pi/2} = \frac{\pi a^2}{4}.$

Example. The polynomials $P_n(x) = \frac{1}{2^n n!}\frac{d^n\left[\left(x^2-1\right)^n\right]}{dx^n}, \quad n = 0, 1, 2, \ldots,$ are called **Legendre polynomials**. Let us prove that
$\int_{-1}^{1} Q_m(x)P_n(x)\,\mathrm{d}x = 0$ for any polynomial Q_m of order $m < n$.
Since $\frac{d^k\left[\left(x^2-1\right)^n\right]}{dx^k}$, $k = 0, \ldots, n-1$ is equal to 0 at $x = -1$ and $x = 1$, integrating by parts we get

$$\int_{-1}^{1} Q_m(x)\frac{d^n\left[\left(x^2-1\right)^n\right]}{dx^n}\,\mathrm{d}x = Q_m(x)\frac{d^{n-1}\left[\left(x^2-1\right)^n\right]}{dx^{n-1}}\Bigg|_{-1}^{1}$$

$$-\int_{-1}^{1} Q_m'(x)\frac{d^{n-1}\left[\left(x^2-1\right)^n\right]}{dx^{n-1}}\,\mathrm{d}x = \ldots$$

$$= (-1)^m \int_{-1}^{1} Q_m^{(m)}(x) \frac{d^{n-m}\left[\left(x^2-1\right)^n\right]}{dx^{n-m}} \, \mathrm{d}x$$

$$= (-1)^m Q_m^{(m)}(x) \frac{d^{n-m-1}\left[\left(x^2-1\right)^n\right]}{dx^{n-m-1}} \Bigg|_{-1}^{1} = 0,$$

$Q_m^{(m)}(x)$ is a constant.

Example. $I = \int_0^{\frac{\pi}{2}} \sin^n x \, \mathrm{d}x$. Integrating by parts, we obtain

$$I_n = \cos x \sin^{n-1} x \Big|_{\frac{\pi}{2}}^{0} + (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x dx$$

$$= (n-1) \left(\int_0^{\frac{\pi}{2}} \sin^{n-2} x dx - \int_0^{\frac{\pi}{2}} \sin^n x dx \right) = (n-1)\left(I_{n-2} - I_n\right).$$

By the recursion formula $I_n = \frac{n-1}{n} I_{n-2}$ we get

$$I_n = \begin{cases} \frac{(2k-1)!!}{(2k)!!} \cdot \frac{\pi}{2}, & n = 2k, \\ \frac{(2k)!!}{(2k+1)!!}, & n = 2k+1. \end{cases}$$

Example.

$$I = \int_{0.5}^{2} \left(1 + x - \frac{1}{x}\right) e^{x+\frac{1}{x}}\,\mathrm{d}x = \left[x + \frac{1}{x} = t, x = \frac{t \pm \sqrt{t^2 - 4}}{2}\right].$$

$$I = \int_{0.5}^{1} \left(1 + x - \frac{1}{x}\right) e^{x+\frac{1}{x}}\,\mathrm{d}x + \int_{1}^{2} \left(1 + x - \frac{1}{x}\right) e^{x+\frac{1}{x}}\,\mathrm{d}x = I_1 + I_2.$$

$$I_1 = \left[x = \frac{t - \sqrt{t^2 - 4}}{2}\right] = 0.5 \int_{2.5}^{2} e^t \left(1 - \frac{t}{\sqrt{t^2 - 4}} + t - \sqrt{t^2 - 4}\right)\,\mathrm{d}t,$$

$$I_2 = \left[x = \frac{t + \sqrt{t^2 - 4}}{2}\right] = 0.5 \int_{2}^{2.5} e^t \left(1 + \frac{t}{\sqrt{t^2 - 4}} + t + \sqrt{t^2 - 4}\right)\,\mathrm{d}t.$$

$$I = \int_{2}^{2.5} e^t \left(\frac{t}{\sqrt{t^2 - 4}} + \sqrt{t^2 - 4}\right)\,\mathrm{d}t = \int_{2}^{2.5} e^t d\sqrt{t^2 - 4}$$

$$+ \int_{2}^{2,5} e^t \sqrt{t^2 - 4} dt$$

$$= e^t \sqrt{t^2 - 4}\Big|_{2}^{2.5} - \int_{2}^{2.5} e^t \sqrt{t^2 - 4}\,\mathrm{d}t + \int_{2}^{2.5} e^t \sqrt{t^2 - 4}\,\mathrm{d}t = 1.5e^{2.5}.$$

Example. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous periodic function, T be a period of f. Prove that $\displaystyle\int_a^{a+T} f(x)dx = \int_0^T f(x)\,\mathrm{d}x$, where a is an arbitrary real number.

By additivity $\displaystyle\int_a^{a+T} f(x)\,\mathrm{d}x = \int_a^T f(x)\,\mathrm{d}x + \int_T^{a+T} f(x)\,\mathrm{d}x.$

By periodicity $\displaystyle\int_T^{a+T} f(x)\,\mathrm{d}x = \int_T^{a+T} f(x-T)\,\mathrm{d}x.$

Appying substitution $x - T = t$, we get

$$\int_T^{a+T} f(x-T)\,\mathrm{d}x = \int_0^a f(t)\,\mathrm{d}t.$$

Therefore,

$$\int_a^{a+T} f(x)\,\mathrm{d}x = \int_0^a f(x)\,\mathrm{d}x + \int_a^T f(x)\,\mathrm{d}x = \int_0^T f(x)\,\mathrm{d}x.$$

Example. Let f be a T-periodic continuous function. Prove that the function $F : x \mapsto \int_{x_0}^{x} f(t)\,\mathrm{d}t, \quad x \in \mathbb{R}$, is a sum of a linear function and a T-periodic function.

By the Theorem (Integral with variable upper limit) $\forall x \in \mathbb{R}$ $F'(x) = f(x)$. By periodicity of f, we get $F'(t+T) = f(t)$. Integrating over $[x_0, x]$, we obtain $F(x+T) - F(x_0+T) = F(x)$. Since

$$F(x_0+T) = \int_{x_0}^{x_0+T} f(t)\,\mathrm{d}t = \int_0^T f(t)\,\mathrm{d}t = C,$$

it follows that $F(x+T) - F(x) = C$. If $C = 0$, then $F(x+T) = F(x)$ and F is a T-periodic function. Let $C \neq 0$, consider the function

$$\Phi : x \mapsto F(x) - \frac{C}{T}x, \quad x \in \mathbb{R}.$$

Since Φ is T-periodic, it follows that

$$F(x) = \Phi(x) + \frac{C}{T}x, \quad x \in \mathbb{R},$$

is a sum of a periodic and a linear function.

Example. $I = \int_0^{200\pi} \sqrt{1 - \cos 2x}\, \mathrm{d}x.$

$I = \sqrt{2} \int_0^{200\pi} |\sin x|\, \mathrm{d}x$, the function $x \mapsto |\sin x|, x \in \mathbb{R}$, is π-periodic, so,

$$I = 200\sqrt{2} \int_0^{\pi} \sin x\, \mathrm{d}x = 400\sqrt{2}.$$

Example. Find the integral $I = \int_0^{\pi} \frac{\sin nx}{\sin x}\,\mathrm{d}x$, if it exists.

Since $\lim_{x\to+0} \frac{\sin nx}{\sin x} = n$, $\lim_{x\to\pi-0} \frac{\sin nx}{\sin x} = (-1)^{n+1}n$, it follows that

$$\int_0^{\pi} \frac{\sin nx}{\sin x}\,\mathrm{d}x = \int_0^{\pi} f(x)\,\mathrm{d}x, \text{ where } f(x) = \begin{cases} \frac{\sin nx}{\sin x}, & x \in (0,\pi), \\ n, & x = 0, \\ (-1)^{n+1}n, & x = \pi. \end{cases}$$

By the Euler formula $\sin kx = \frac{1}{2i}\left(e^{ikx} - e^{-ikx}\right)$, $k = 1, n$, so

$$f(x) = \frac{e^{inx} - e^{-inx}}{e^{ix} - e^{-ix}} = \sum_{k=1}^{n} e^{i((n+1)-2k)x}$$

$$= \begin{cases} 2(\cos(n-1)x + \cos(n-3)x + \ldots + \cos x), & n \text{ is even}, \\ 2(\cos(n-1)x + \cos(n-3)x + \ldots + \cos 2x) + 1, & n \text{ is odd}. \end{cases}$$

By $\int_0^{\pi} \cos(n-k)x\,\mathrm{d}x = \left.\frac{\sin(n-k)x}{n-k}\right|_0^{\pi} = 0$, $k = 1, 2, \ldots, n-1$, we finally get $I = \begin{cases} 0, & n \text{ is even}, \\ \pi, & n \text{ is odd}. \end{cases}$

Example. $\displaystyle\int_0^x e^{t^2}\,\mathrm{d}t \sim \frac{e^{x^2}}{2x}$ as $x \to +\infty$.

Applying L'Hôpital's rule we get

$$\lim_{x\to+\infty} \frac{2x\int\limits_0^x e^{t^2}dt}{e^{x^2}} = \lim_{x\to+\infty} \frac{\frac{d}{dx}\left(2x\int\limits_0^x e^{t^2}dt\right)}{\frac{d}{dx}e^{x^2}} = \lim_{x\to+\infty} \frac{2\int\limits_0^x e^{t^2}dt + 2xe^{x^2}}{2xe^{x^2}}$$

$$= \lim_{x\to+\infty} \left(\frac{\int\limits_0^x e^{t^2}dt}{xe^{x^2}} + 1 \right) = \lim_{x\to+\infty} \left(\frac{\frac{d}{dx}\int\limits_0^x e^{t^2}dt}{\frac{d}{dx}\left(xe^{x^2}\right)} + 1 \right)$$

$$= \lim_{x\to+\infty} \left(\frac{e^{x^2}}{e^{x^2} + 2x^2e^{x^2}} + 1 \right) = 1.$$

Lemma (Summation by parts or the Abel transformation)

$$\sum_{k=1}^{n} a_k b_k = A_n b_n + \sum_{k=1}^{n-1} A_k (b_k - b_{k+1}), \ \textit{where } A_k := \sum_{i=1}^{k} a_i.$$

Proof. $A_0 := 0$, $\displaystyle\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n} (A_k - A_{k-1}) b_k = \sum_{k=1}^{n} A_k b_k - \sum_{k=1}^{n} A_{k-1} b_k$

$$= \sum_{k=1}^{n} A_k b_k - \sum_{k=0}^{n-1} A_k b_{k+1} = \sum_{k=1}^{n-1} A_k (b_k - b_{k+1}) + A_n b_n - \underbrace{A_0 b_1}_{=0}. \ \square$$

Lemma (∗)

If $m \le A_k \le M$, $b_i \ge 0$, $b_i \ge b_{i+1}$, *then* $mb_1 \le \displaystyle\sum_{k=1}^{n} a_k b_k \le Mb_1$.

Proof.

$$\sum_{k=1}^{n} a_k b_k \le M b_n + \sum_{k=1}^{n-1} M(b_k - b_{k+1}) = M b_n + M(b_1 - b_n) = M b_1. \square$$

Lemma

If $f \in \mathcal{R}[a,b]$, $g \geq 0$, g *is nonincreasing on* $[a,b]$, *then*

$$\exists\, \xi \in [a,b] \ \int_a^b fg = g(a) \int_a^{\xi} f$$

Proof. Let $\tau = \{x_k\}_{k=0}^n$ be a partition of $[a,b]$, $L := \sup f([a,b])$,

$$\int_a^b fg = \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} fg = \sum_{k=0}^{n-1} g(x_k) \int_{x_k}^{x_{k+1}} f(x)\,\mathrm{d}x$$

$$+ \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} f(x)\,(g(x) - g(x_k))\,\mathrm{d}x =: S_1 + S_2.$$

$$|S_2| \leq \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} \underbrace{|f(x)|}_{\leq L} \underbrace{|g(x) - g(x_k)|}_{\leq \omega_k(g)} \,\mathrm{d}x \leq L \sum_{k=0}^{n-1} \omega_k(g) \Delta x_k \to 0$$

as $\lambda_\tau \to 0$. $\Rightarrow S_1 \to \displaystyle\int_a^b fg$, as $\lambda_\tau \to 0$.

$$S_1 = \sum_{k=0}^{n-1} g(x_k) \int_{x_k}^{x_{k+1}} f(x)\,\mathrm{d}x = \sum_{k=0}^{n-1} g(x_k)\left(F(x_{k+1}) - F(x_k)\right)$$

$= \sum_{k=1}^{n} g(x_{k-1})\left(F(x_k) - F(x_{k-1})\right)$, where $F(x) := \int_a^x f$. We denote

$a_k := F(x_k) - F(x_{k-1})$, $b_k := g(x_{k-1})$. Then $A_k = \sum_{i=1}^{k} a_i = F(x_k)$.

$F \in C[a,b] \Rightarrow \min F([a,b]) =: m \le F(x_k) \le M := \max F([a,b])$.

By Lemma $(*)$, $mg(a) \le \sum_{k=1}^{n} g(x_{k-1})\left(F(x_k) - F(x_{k-1})\right) \le Mg(a) \Rightarrow$

$mg(a) \le \int_a^b fg \le Mg(a)$.

If $g(a) = 0$, then $g = 0$, $\int_a^b fg = 0$, any ξ is appropriate.

If $g(a) \ne 0$, then $m \le \frac{1}{g(a)} \int_a^b fg \le M$. By the Bolzano intermediate value theorem for F, $\exists\, \xi \in [a,b]$ $\;F(\xi) = \frac{1}{g(a)} \int_a^b fg$. $\square$

Theorem (The second mean value theorem for the Riemann integral)

If $f \in \mathcal{R}[a,b]$, g is monotonic on $[a,b]$, then

$$\exists\, \xi \in [a,b] \quad \int_a^b fg = g(a)\int_a^\xi f + g(b)\int_\xi^b f.$$

Proof. Let g be nondecreasing on $[a,b]$. Then $g_1(x) := g(b) - g(x)$ is nonnegative and nonincreasing on $[a,b]$. By the last Lemma, $\exists\, \xi \in [a,b]$

$$\int_a^b fg_1 = g_1(a)\int_a^\xi f \Leftrightarrow g(b)\int_a^b f - \int_a^b fg = \big(g(b)-g(a)\big)\int_a^\xi f$$

$$\Leftrightarrow g(b)\left(\int_a^b f - \int_a^\xi f\right) + g(a)\int_a^\xi f = \int_a^b fg.$$

If g is nonincreasing on $[a,b]$. Then $g_1(x) := g(x) - g(b)$. $\square$

Example. Define the sign of the integral $I = \int_0^{2\pi} \frac{\sin x}{x}\,\mathrm{d}x$ via the mean value theorems.

Since $\lim_{x\to 0} \frac{\sin x}{x} = 1$, it follows that I is well-defined.

$$I = \int_0^{\pi} \frac{\sin x}{x}\,\mathrm{d}x + \int_{\pi}^{2\pi} \frac{\sin x}{x}\,\mathrm{d}x = \int_0^{\pi} \frac{\sin x}{x}\,\mathrm{d}x + \int_0^{\pi} \frac{\sin(t+\pi)}{t+\pi}\,\mathrm{d}t$$

$$= \pi \int_0^{\pi} \frac{\sin x}{x(x+\pi)}\,\mathrm{d}x = \pi \frac{\sin\xi}{\xi} \int_0^{\pi} \frac{\mathrm{d}x}{x+\pi} = \pi \frac{\sin\xi}{\xi} \log(x+\pi)\Big|_0^{\pi}$$

$$= \pi \frac{\sin\xi}{\xi} \log 2, \quad 0 < \xi < \pi \Rightarrow I > 0.$$

Example. Estimate the integral $I = \displaystyle\int_0^{2\pi} \frac{\mathrm{d}x}{1 + 0.5\cos x}$.

By the first mean value theorem

$$I = \frac{2\pi}{1 + 0.5\cos\xi}, \quad 0 < \xi < 2\pi.$$

$$-1 \le \cos\xi \le 1 \quad \Rightarrow \quad \frac{1}{2} \le 1 + 0.5\cos\xi \le \frac{3}{2} \quad \Rightarrow \quad \frac{4\pi}{3} \le I \le 4\pi.$$

Example. Estimate the integral $I = \displaystyle\int_{100}^{200} \sin \pi x^2 \,\mathrm{d}x$.

$$I = \left[\pi x^2 = t\right] = \frac{1}{2\sqrt{\pi}} \int_{100^2\pi}^{200^2\pi} \frac{\sin t}{\sqrt{t}} \,\mathrm{d}t$$

$$= \frac{1}{2\sqrt{\pi}} \left(\frac{1}{100\sqrt{\pi}} \int_{100^2\pi}^{\xi} \sin t \,\mathrm{d}t + \frac{1}{200\sqrt{\pi}} \int_{\xi}^{200^2\pi} \sin t \,\mathrm{d}t \right) = \frac{1 - \cos\xi}{400\pi},$$

$$100^2\pi < \xi < 200^2\pi, \quad 0 < I < \frac{1}{200\pi}.$$

Example. $\lim\limits_{n\to\infty} \int_n^{n+p} \frac{\sin x}{x}\,\mathrm{d}x = 0, p > 0.$

$$I_n = \int_n^{n+p} \frac{\sin x}{x}\,\mathrm{d}x = \frac{1}{n}\int_n^{\xi_n} \sin x\,\mathrm{d}x = \frac{\cos n - \cos \xi_n}{n}, \quad n < \xi_n < n + p.$$

By the estimate $|I_n| = \dfrac{|\cos n - \cos \xi_n|}{n} \leq \dfrac{2}{n}$, we obtain $\lim_{n\to\infty} I_n = 0$.

Theorem (Taylor's formula with the reminder in integral form)

If $f \in C^{n+1}(a,b)$, $n \in \mathbb{Z}_+$, $x, x_0 \in (a,b)$, *then*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{1}{n!}\int_{x_0}^{x} f^{n+1}(t)(x-t)^n \,\mathrm{d}t$$

Proof. Induction on n. $n = 0 :\ f \in C^1(a,b),\ f(x) = f(x_0) + \int_{x_0}^{x} f'$. By the fundamental theorem of integral calculus, it is true. Suppose $f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{1}{(n-1)!}\int_{x_0}^{x} f^n(t)(x-t)^{n-1}\,\mathrm{d}t$. Then

$$\frac{1}{(n-1)!}\int_{x_0}^{x} f^n(t)(x-t)^{n-1}\,\mathrm{d}t = \frac{-1}{n!}\int_{x_0}^{x} f^n(t)\,\mathrm{d}(x-t)^n$$

$$= \frac{-1}{n!}\left(f^{(n)}(t)(x-t)^n\Big|_{t=x_0}^{t=x} - \int_{x_0}^{x} f^{n+1}(t)(x-t)^n\,\mathrm{d}t \right)$$

$$= \frac{1}{n!}\left(f^{(n)}(x_0)(x-x_0)^n + \int_{x_0}^{x} f^{n+1}(t)(x-t)^n\,\mathrm{d}t \right). \quad \square$$

Remark. By the first mean value theorem there exists $c \in [x_0, x]$

$$\frac{1}{n!}\int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n \,\mathrm{d}t = \frac{f^{(n+1)}(c)}{n!}\int_{x_0}^{x}(x-t)^n \,\mathrm{d}t$$

$$= \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}.$$

So, integral form of the reminder implies the Lagrange form, however assumptions are more restrictive:

$$C^{n+1} \text{ versus } C^n \text{ and existence of } f^{n+1}.$$

Example. The Wallis formula $\pi = \lim\limits_{n\to\infty} \frac{1}{n}\left(\frac{(2n)!!}{(2n-1)!!}\right)^2$.

For $x \in \left(0, \frac{\pi}{2}\right)$, $0 < \sin x < 1$, so for $n \in \mathbb{N}$

$$\sin^{2n+1} x < \sin^{2n} x < \sin^{2n-1} x,$$

$$\int_0^{\pi/2} \sin^{2n+1} x \, \mathrm{d}x < \int_0^{\pi/2} \sin^{2n} x \, \mathrm{d}x < \int_0^{\pi/2} \sin^{2n-1} x \, \mathrm{d}x,$$

$$\frac{(2n)!!}{(2n+1)!!} < \frac{(2n-1)!!}{(2n)!!} \cdot \frac{\pi}{2} < \frac{(2n-2)!!}{(2n-1)!!}$$

$$\left(\frac{(2n)!!}{(2n-1)!!}\right)^2 \frac{1}{2n+1} < \frac{\pi}{2} < \left(\frac{(2n)!!}{(2n-1)!!}\right)^2 \frac{1}{2n}.$$

Denote by $x_n = \left(\frac{(2n)!!}{(2n-1)!!}\right)^2 \frac{1}{n}$.

$$\pi < x_n < \frac{2n+1}{2n}\pi, \quad x_n \to \pi.$$

Theorem (Hölder's inequality for integrals)

Suppose $f, g \in C[a,b]$, $1/p + 1/q = 1$ (p and q are called conjugate exponents), then

$$\left|\int_a^b fg\right| \leqslant \left(\int_a^b |f|^p\right)^{1/p} \left(\int_a^b |g|^q\right)^{1/q}$$

Proof. Let $x_k = a + \frac{k(b-a)}{n}$, $k = 0, \ldots, n$, $a_k = f(x_k)(\Delta x_k)^{1/p}$, $b_k = g(x_k)(\Delta x_k)^{1/q}$. Then $a_k b_k = f(x_k) g(x_k) \Delta x_k$ by $1/p + 1/q = 1$. Applying Hölder's inequality for sums

$$\left|\sum_{k=0}^{n-1} a_k b_k\right| \leqslant \left(\sum_{k=0}^{n-1} |a_k|^p\right)^{1/p} \left(\sum_{k=0}^{n-1} |b_k|^q\right)^{1/q}, \text{ we get}$$

$$\left|\sum_{k=0}^{n-1} f(x_k) g(x_k) \Delta x_k\right| \leqslant \left(\sum_{k=0}^{n-1} |f(x_k)|^p \Delta x_k\right)^{1/p} \left(\sum_{k=0}^{n-1} |g(x_k)|^q \Delta x_k\right)^{1/q}.$$

It remains to pass to the limit $n \to \infty$. $\square$

Corollary (Cauchy's inequality for integrals)

Let $f, g \in C[a, b]$, *then*

$$\left|\int_a^b fg\right| \leq \sqrt{\int_a^b f^2} \cdot \sqrt{\int_a^b g^2}.$$

Theorem (Minkowski's inequality for integrals)

Suppose $f, g \in C[a, b]$, $p \geq 1$, *then*

$$\left(\int_a^b |f+g|^p\right)^{1/p} \leqslant \left(\int_a^b |f|^p\right)^{1/p} + \left(\int_a^b |g|^p\right)^{1/p}.$$

Prove by yourself.

Theorem (Chebyshev's inequality for integrals)

Suppose f increases, g decreases on $[a,b]$. Then

$$\frac{1}{b-a}\int_a^b fg \le \left(\frac{1}{b-a}\int_a^b f\right)\cdot\left(\frac{1}{b-a}\int_a^b g\right).$$

In other words, the arithmetical mean of the product of two dissimilar monotonic functions does not exceed the product of the means.

Proof. Let $A = \dfrac{1}{b-a}\displaystyle\int_a^b f, \quad E = \{x \in [a,b] : f(x) \le A\}$. $E \ne \varnothing$, otherwise, $f > A$ on $[a,b]$, and integrating we obtain $A > A$.
Let $c = \sup E$. Then $A - f \ge 0, g \ge g(c)$ on $[a,c)$ and $A - f \leqslant 0$, $g \leqslant g(c)$ on $(c,b]$. Then

$$\int_a^b (A-f)g = \int_a^c (A-f)g + \int_c^b (A-f)g$$

$$\ge g(c)\int_a^c (A-f) + g(c)\int_c^b (A-f) = g(c)\int_a^b (A-f) = 0,$$

that has to be proved.

Corollary (Chebyshev's inequality for sums)

Let $n \in \mathbb{N}$, $a, b \in \mathbb{R}^n$,

$$a_1 \le \ldots \le a_n, \quad b_1 \ge \ldots \ge b_n.$$

Then

$$\frac{1}{n}\sum_{k=1}^{n} a_k b_k \le \left(\frac{1}{n}\sum_{k=1}^{n} a_k\right) \cdot \left(\frac{1}{n}\sum_{k=1}^{n} b_k\right).$$

Proof. We apply Chebyshev's inequality for integrals for piecewise constant functions $f, g : [0,1] \to \mathbb{R}$, taking the values a_k and b_k on $\left(\frac{k-1}{n}, \frac{k}{n}\right)$. (Values of f and g on the finite set of points do not affect the integrals.)

Example. Let $f \in C^1[a,b]$ and $f(a) = 0$. Prove the inequality

$$M^2 \leq (b-a)\int_a^b f'^2(x)\,\mathrm{d}x,$$

where $M = \sup_{x\in[a,b]}\{|f(x)|\}$.
Cauchy's inequality

$$\left|\int_a^x \tilde{f}(t)g(t)\,\mathrm{d}t\right| \leq \sqrt{\int_a^x \tilde{f}^2(t)\,\mathrm{d}t}\sqrt{\int_a^x g^2(t)\,\mathrm{d}t}$$

where $g(t) = f'(t), \tilde{f}(t) = 1, x \in [a,b]$, takes the form

$$\sqrt{\int_a^x (f')^2(t)\,\mathrm{d}t}\sqrt{\int_a^x \mathrm{d}t} \geq \left|\int_a^x f'(t)\,\mathrm{d}t\right| = |f(x) - f(a)| = |f(x)|\,,$$

$$\sqrt{\int_a^b f'^2(t)\,\mathrm{d}t}\sqrt{b-a} \geq |f(x)| \geq M.$$

Definition (Average of a function)

Let $f \in \mathcal{R}[a,b]$ for any $[a,b] \subset \mathbb{R}$. The function

$$F_\delta(x) = \frac{1}{2\delta} \int_{x-\delta}^{x+\delta} f(t)\mathrm{d}t$$

is called the **average** *(or the* **Steklov average***) of f.*

Properties of F_δ.

1. $F_\delta \in C(\mathbb{R})$.

Let $|f(x)| \le C$, and $|h| < \delta$. Then

$$|F_\delta(x+h) - F_\delta(x)| = \frac{1}{2\delta} \left| \int_{x+\delta}^{x+\delta+h} f(t)\mathrm{d}t + \int_{x-\delta+h}^{x-\delta} f(t)\mathrm{d}t \right|$$

$$\le \frac{1}{2\delta}(C|h| + C|h|) = \frac{C}{\delta}|h|.$$

2. If $f \in C^k(\mathbb{R})$, then $F_\delta(x) \in C^{k+1}(\mathbb{R})$.
By the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_a^{\varphi(x)} f(t)\mathrm{d}t = \frac{\mathrm{d}}{\mathrm{d}\varphi} \int_a^{\varphi} f(t)\mathrm{d}t \cdot \frac{\mathrm{d}\varphi}{\mathrm{d}x} = f(\varphi(x))\varphi'(x).$$

Since

$$F_\delta(x) = \frac{1}{2\delta} \int_a^{x+\delta} f(t)\mathrm{d}t - \frac{1}{2\delta} \int_a^{x-\delta} f(t)\mathrm{d}t,$$

it follows that

$$F_\delta'(x) = \frac{f(x+\delta) - f(x-\delta)}{2\delta}.$$

3. If $f \in C(\mathbb{R})$, than $\lim\limits_{\delta \to +0} F(\delta)(x) = f(x)$.

$$F_\delta(x) = [t = x + u] = \frac{1}{2\delta} \int_{-\delta}^{\delta} f(x+u)\mathrm{d}u.$$

The first mean-value theorem yields

$$F_\delta(x) = \frac{1}{2\delta} f(x+\tau) \cdot 2\delta = f(x+\tau),$$

where $|\tau| \leq \delta$.