M

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Ch 1 : Calcul Intégral

$\boxed{\text{Th.1}}$ $\boxed{S'(x) = f(x)}$ $\boxed{\text{DM-0}}$

$$S_{[a,b]}(f(x)) = \int_a^b f(x)\,dx$$

1) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

2) $\int_a^b f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + \int_{c_2}^b f(x)\,dx$

3) $f : [a,b] \to \mathbb{R}$, $f \leq 0$; $S_{[a,b]}(f(x)) = -S_{[a,b]}(-f(x))$

FF Principale
ou
FF de Newton

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$ $\boxed{\text{DM-1}}$

$$\int f(x)\,dx = F(x) + C \quad , \quad C \in \mathbb{R}.$$

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ex $\int_0^{\pi/2} \sin x\,dx = -\cos\left(\frac{\pi}{2}\right) - (-\cos 0) = 1$

$F'(x) = \sin x$

$F(x) = -\cos x$

1) $\int x^n \, dx = \dfrac{x^{n+1}}{n+1} + C$ si $n \neq -1$

2) $\int \sqrt{x} \, dx = \int x^{1/2} \, dx = \dfrac{x^{3/2}}{3/2} + C = \dfrac{2}{3} x^{3/2} = \dfrac{2}{3}\sqrt{x^3} + C$

3) $\int \dfrac{1}{x} \, dx = \ln|x| + C$, $x > 0$.

4) $\int \sin x \, dx = -\cos x + C$

5) $\int \cos x \, dx = \sin x + C$

6) $\int e^x \, dx = e^x + C$

@ $\displaystyle\int_0^1 x^3 - 2x^2 + 3 \, dx = \left(\frac{1}{4}x^4 - \frac{2}{3}x^3 + 3x\right)\Bigg|_{x=0}^{x=1}$

@ $\displaystyle\int_0^1 x^{10} \, dx = \frac{x^{11}}{11}\Bigg|_{x=0}^{x=1} = \frac{1}{11}$

@ $\displaystyle\int_0^1 (2x+1)^{10} \, dx = \frac{1}{2} \; \frac{(2x+1)^{11}}{11}$

$$\boxed{\int f(ax+b) \, dx = \frac{1}{a} F(ax+b) + C}$$

DM-2

ISR

Intégrale aux sommes de Riemann

$\displaystyle\sum_{k=1}^{n} \frac{1}{k} > \ln(n-1) \longrightarrow +\infty$

$\forall\, c > 0,\ \exists\, N,\ \forall n \geq N,\ \displaystyle\sum_{k=1}^{n} \frac{1}{k} > c.$

DM-3

1) $\int \alpha f(x) + \int \beta g(x) \, dx = \alpha \int f(x) \, dx + \beta \int g(x) \, dx$

2) $\int (ax+b) \, dx = \dfrac{1}{a} F(ax+b) + C$, $a \neq 0$.

3) $\int f(u) . u' dx = F(u) + C$ $\left(\begin{array}{l} f : I \to \mathbb{R} \text{ : continue} \\ u : J \to I \text{ : dérivable} \end{array} \right)$

4) $\int f(u) . du = F(u) + C$ Changement de variable

5) $\int u \, dv = uv - \int v . du$ Intégra° par parties

DM-4

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1) $\int \alpha^x \, dx = \dfrac{\alpha^x}{\ln \alpha} + C$

2) $\int \dfrac{1}{x^2+1} = \arctan x + C$

3) $\int \dfrac{1}{\sqrt{1-x^2}} \, dx = \arcsin x + C$

4) $\int \dfrac{1}{x^2+a} = \dfrac{1}{a} \arctan \dfrac{x}{a} + C$

@ $\int (x^2+1)^{10} dx = \int (x^2+1)^{10} dx^2 = \dfrac{1}{2} \int (u+1)^{10} du$

$\int (x^2+1)^{10} dx = \dfrac{1}{2} \int (u+1)^{10} du = \dfrac{1}{2} \; \dfrac{(u+1)^{11}}{11} + C$

car $dx^2 = ((x)^2)' dx = 2\, dx$

éliminer le 2.

1 : Structure Linéaire

Ex 1 3)
$$\left|\begin{matrix} 2 & 1 & -3 \\ 3 & 2 & 1 \\ 7 & 4 & -5 \end{matrix}\right| \begin{matrix} m \\ m+3 \\ 2m+5 \end{matrix} \longrightarrow 2\ell_2 - 3\ell_1 \left|\begin{matrix} 2 & 1 & -3 \\ 0 & 1 & 11 \\ 7 & 4 & -5 \end{matrix}\right| \begin{matrix} m \\ -m+6 \\ 2m+5 \end{matrix}$$

$$\longrightarrow \left|\begin{matrix} 2 & 1 & -3 \\ 0 & 1 & 11 \\ 0 & 1 & 11 \end{matrix}\right| \begin{matrix} m \\ -m+6 \\ 11m-32 \end{matrix} \longrightarrow 2\ell_3 - 7\ell_1$$

$$\rightsquigarrow \left|\begin{matrix} 1 & 2 & -3 \\ 0 & -1 & 7 \\ 4 & 7 & -5 \end{matrix}\right| \begin{matrix} m \\ 3-m \\ 2-m \end{matrix} \rightsquigarrow \left|\begin{matrix} 1 & 2 & -3 \\ 0 & -1 & 7 \\ 0 & 0 & 0 \end{matrix}\right| \begin{matrix} m \\ 3-m \\ 2-m \end{matrix}$$

- Si $m \neq 2 \rightarrow \mathcal{S} = \emptyset$
- Si $m = 2 \longrightarrow \begin{cases} x = 7z+1 \\ y = 4 - 11z \\ z \in \mathbb{R} \end{cases} \rightarrow \mathcal{S}(7z-1, -11z+4, z, z \in \mathbb{R})$

Ex 1 DS·1

$U_1, U_2, U_3, U_4 \in \mathbb{R}^4$.

$U_1 = (1, 1, -1, 2)$
$U_2 = (1, 2, 0, 1)$
$U_3 = (2, 1, 1, 1)$
$U_4 = (1, 3, 3, -1)$

1) Cette famille est-elle libre ?
2) Trouver une base $W = \text{Vect}(U_1, U_2, U_3, U_4)$ de SEV engendré.
3) Mq $W = (2, -1, -1, 3) \in W$.
4) Trouver vect q n'est pas élt de espace.

$$\left|\begin{matrix} \boxed{1} & 1 & -1 & 2 \\ 1 & 2 & 0 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & 3 & 3 & -1 \end{matrix}\right| \begin{matrix} \\ U_2 - U_1 \\ U_3 - 2U_1 \\ U_4 - U_1 \end{matrix} \left|\begin{matrix} 1 & 1 & -1 & 2 \\ 0 & \boxed{1} & 1 & -1 \\ 0 & -1 & 3 & -3 \\ 0 & 2 & 4 & -4 \end{matrix}\right| \begin{matrix} \\ \\ U_3 - 2U_1 + (U_3 + U_1) \\ U_4 - U_1 - 2(U_4 - U_1) \end{matrix} \left|\begin{matrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 4 & -4 \\ 0 & 0 & 2 & -2 \end{matrix}\right|$$

$\hookrightarrow \frac{1}{2}(2U_2 - U_3 - 5U_2 + 5U_1) = \vec{0}$

NON LES COEFFS NE SONT PAS NULLES = n'est pas FGL.

$$\begin{array}{r} \lambda_1 \times \\ \lambda_2 \times \\ \lambda_3 \times \\ \\ \end{array} \left| \begin{array}{cccc} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array} \right| \begin{array}{l} v_1 \\ v_2 \\ v_3 \\ \\ \end{array}$$

$W = \mathrm{Vect}(u_1, u_2, u_3, u_4) = \mathrm{Vect}\underbrace{(v_1, v_2, v_3)}_{\text{libre}}$

$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = \bar{0}$

Donc v_1, v_2, v_3 est une base : FGL.

↳ DIMENSION 3.

$$\left| \begin{array}{cccc} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{array} \right) \begin{array}{l} v_1+v_3 \\ v_2-v_3 \\ \\ \end{array} \left| \begin{array}{cccc} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array} \right| \; l_1 - l_2 \; \left| \begin{array}{cccc} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array} \right| \begin{array}{l} w_1 \\ w_2 \\ w_3 \end{array}$$

$W = \mathrm{Vect}(u_1, u_2, u_3, u_4) = \mathrm{Vect}(w_1, w_2, w_3)$

$W = \{ \lambda_1 w_1 + \lambda_2 w_2 + \lambda_3 w_3 \; , \lambda_1, \lambda_2, \lambda_3 \}$

$W = \lambda_1 (1,0,0,1) + \lambda_2 (0,1,0,0) + \lambda_3 (0,0,1,-1)$

$W = \{ (\lambda_1, \lambda_2, \lambda_3, \lambda_1 - \lambda_3) \}$

$W = \{ (x_1, x_2, x_3, x_4) \; x_4 = x_1 - x_3 \} = \{ x_1 - x_3 - x_4 = 0 \}$

Puis $w(2,-1,-1,3)$ dc $w \in W$.

(TH)

Tt ss-espace $W \subset \mathbb{R}^n$ contient une base. Le nbre de vecteurs ds chaq base de W contient m̂ nbre élt.

(P)
$$\frac{1}{(X-\alpha_1)\dots(X-\alpha_n)} = \frac{A_1}{X-\alpha_1}+\dots+\frac{A_n}{X-\alpha_n}$$

$$\boxed{A_i = \prod_{\substack{j=1\\ j\neq i}}^{m} (\alpha_i-\alpha_j)^{-1}}$$

$$\delta = q - \frac{p^2}{4}$$

$$\int \frac{dx}{x^2+px+q} = \frac{1}{\sqrt{\delta}} \cdot \arctan\left(\frac{x}{\sqrt{\delta}} + \frac{p}{2\sqrt{\delta}}\right) + C$$

$$\int \frac{cx+d}{x^2+px+q}\,dx = \frac{1}{2}\ln|x^2+px+q| - \frac{p}{2}\left[\frac{1}{\sqrt{\delta}}\cdot\arctan\left(\frac{x}{\sqrt{\delta}}+\frac{p}{2\sqrt{\delta}}\right)\right] + c$$

$$\int \frac{a_n x^n + a_{n-1}x^{n-1} + \dots + a_0}{x^2+px+q} \overset{(DE)}{=} \frac{1}{2}\ln|x^2+px+q| - \frac{p}{2}\left[\frac{1}{\sqrt{\delta}}\cdot\arctan\left(\frac{x}{\sqrt{\delta}}+\frac{p}{2\sqrt{\delta}}\right)\right]$$

(TH) $Q(X) = a_n X^n + a_{n-1}X^{n-1} + \dots + a_0 \in \mathbb{R}[X]$,
soit $\alpha \in \mathbb{C}$, racine complexe de $Q(X)$. $Q(\alpha) = 0$.
Alors $\overline{\alpha} = \overline{(x+iy)} = x - iy$ est aussi racine.
Racines de $Q(x)$ st $\alpha_1, \dots, \alpha_n \in \mathbb{R}$.
$\alpha_{r+1}, \overline{\alpha_{r+1}}, \dots, \alpha_{r+k} \in \mathbb{C}\setminus\mathbb{R}$.

$$Q(x) = a_n(x-\alpha_1)\dots(X-\alpha_r)\underbrace{[(X-\alpha_{r+1})(X-\overline{\alpha_{r+1}})]}_{x^2+px+q}$$

sans racines

$$\frac{a_n}{Q(x)} = \frac{A_1}{X-\alpha_1}+\dots+\frac{A_r}{X-\alpha_r} + \dots \; (4) + \frac{B_1X+C_1}{X^2+P_1X+Q_1} + \frac{B_KX+C_K}{X^2+P_KX+Q_K}.$$

1) $\int \frac{dx}{x^2+x-2} = ?$; $\frac{1}{x^2+x-2} = \frac{A}{x-1} + \frac{B}{x+2}$

$1 = A(x+2) + B(x-1)$
- $x = 1$, $1 = A.3 \Rightarrow A = \frac{1}{3}$
- $x = -2$, $1 = B(-3) \Rightarrow B = -\frac{1}{3}$.

$$\int \frac{dx}{x^2+x-2} = \int \frac{1}{3(x-1)} - \frac{1}{3(x+2)}\,dx = \frac{1}{3}\int\frac{1}{x-1} - \frac{1}{3}\int\frac{1}{x+2}$$
$$= \frac{1}{3}\ln|x-1| - \frac{1}{3}\ln|x+2| + C$$

2) $\int \frac{dx}{x(x^2+x-2)} = ?$; $\frac{1}{x(x^2+x-2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+2}$

$1 = A(x-1)(x+2) + B(x)(x+2) + C(x)(x-1)$
- $x = 0$; $1 = A(-1)(2) \Rightarrow A = -\frac{1}{2}$
- $x = 1$; $1 = B(1)(3) \Rightarrow B = \frac{1}{3}$
- $x = -2$; $1 = C(-2)(-3) \Rightarrow C = -\frac{1}{6}$.

$$\int \frac{dx}{x(x^2+x-2)} = -\frac{1}{2}\int\frac{dx}{x} + \frac{1}{3}\int\frac{dx}{x-1} + \frac{1}{6}\int\frac{dx}{x-2}$$
$$= -\frac{1}{2}\ln|x| + \frac{1}{3}\ln|x-1| + \frac{1}{6}\ln|x-2| + C$$

3) $\int \frac{dx}{x^2+x+2} = \int \frac{dx}{(x+\frac{1}{2})^2+\frac{7}{4}} = \frac{2}{\sqrt{7}} \arctan\left(\frac{2X}{\sqrt{7}} + \frac{1}{\sqrt{7}}\right) + C$

or $\int \frac{dx}{x^2+1} = \arctan + c$ où $S = q - \frac{p^2}{4} = \left(\sqrt{\frac{7}{4}}\right)^2$

4) $\int \frac{1}{(X-2)^2} dx = \int \frac{d(X-2)}{(X-2)^2} = \frac{(X-2)^{-2+1}}{-2+1} = \frac{-1}{X-2} + C$

5) $\int \frac{x}{x^2+x-2} dx$ or $\circledast$ $\frac{X}{x^2+x-2} = \frac{X}{3(X-1)} - \frac{X}{3(X+2)}$

$\circledast = \frac{1}{3} \frac{(X-1)+1}{X-1} - \frac{1}{3} \frac{(X-2)+2}{X+2} = \frac{1}{3}\left(1 + \frac{1}{X-1}\right) - \frac{1}{3}\left(1 - \frac{2}{X+2}\right)$

$\circledast = \frac{1}{3} \times \frac{1}{X-1} + \frac{2}{3} \cdot \frac{1}{X+2}$

$\int \frac{x}{x^2+x-2} dx = \frac{1}{3} \int \frac{dx}{X-1} + \frac{2}{3} \int \frac{dx}{x+2} = \frac{1}{3} \ln|x-1| + \frac{2}{3} \ln|x+2| + c$

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6) $\int \frac{x^3+2x^2+x+2}{x^2+x-2} dx = ?$

DE
$$\begin{array}{r|l} x^3+2x^2+x+2 & x^2+x-2 \\ \hline 2x+4 & x+1 \end{array}$$

$\int \frac{x^3+2x^2+x+2}{x^2+x-2} dx = \int \left(x+1 + \frac{2x+4}{x^2+x-2}\right) dx$

⑥

$\bullet = \frac{x^2}{2} + x + 2 \int \frac{x+2}{(x+1)(x-2)} dx$

$\bullet = \frac{1}{2} \ln|x^2+x-2| + 2\left(\int (x+2)\left(\frac{1}{3} \times \frac{1}{x-1} - \frac{1}{3} \times \frac{1}{x-2}\right) dx \right)$

$\bullet = \frac{1}{2} \ln|x^2+x-2| - \frac{2\sqrt{2}}{3} \arctan\left(\frac{-2\sqrt{2}X}{3} + \frac{\sqrt{2}}{3}\right) + C$

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7) $\int \frac{dx}{(X-1)^2(X^2+X+1)} = ?$ $\frac{1}{(X-1)^2(X^2+X+1)} = \frac{AX+B}{(X-1)^2} + \frac{CX+D}{X^2+X+1}$

$1 = (AX+B)(X^2+X+1) + (CX+D)(X-1)^2$

$1 = X^3\underbrace{(A+C)}_{0} + X^2\underbrace{(A+B-2C+D)}_{0} + X\underbrace{(A+B+C-2D)}_{0} + \underbrace{(B+D)}_{1}$

$\begin{cases} A + C = 0 \\ A + B - 2C + D = 0 \\ A + B + C - 2D = 0 \\ B + D = 1 \end{cases} \Rightarrow \begin{cases} A = -C = -D \\ -D + 1 - D - 2D + D = 0 \Rightarrow D = \frac{1}{3} \\ C = D = \frac{1}{3} \\ B = 1 - D = 2/3. \end{cases}$

$\Rightarrow \frac{1}{(X-1)^2(X^2+X+1)} = \frac{-\frac{1}{3}X + \frac{2}{3}}{(X-1)^2} + \frac{\frac{1}{3}X + \frac{1}{3}}{X^2+X+1}$

$\int \frac{dx}{(X-1)^2(X^2+X+1)} = -\frac{1}{3} \int \frac{X - 2^{\;= (X-1)-1}}{(X-1)^2} dx + \frac{1}{3} \int \frac{X+1}{X^2+X+1} dx$

$\bullet -\frac{1}{3} \int \frac{1}{X-1} - \frac{1}{(X-1)^2} dx + \frac{1}{3}\left[\frac{1}{2} \ln|x^2+x+1| - \frac{1}{2}\left[\frac{2}{\sqrt{3}} \arctan\left(\frac{2X}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right)\right]\right].$

$\bullet -\frac{1}{3} \ln|x-1| + \frac{1}{3} \frac{(X-1)^{-1}}{-1} + \frac{1}{6} \ln|x^2+x+1| - \frac{1}{6}\left[\frac{2}{\sqrt{3}} \arctan\left(\frac{2X}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right)\right] + C$

⑦

(TH) Tt ss-espace $W \subset \mathbb{R}^n$ contient une base. Le nbre de vecteurs ds chaq base de W contient m̂ nbe élt.

(P) $$\frac{1}{(x-\alpha_1)\dots(X-\alpha_m)} = \frac{A_1}{X-\alpha_1} + \dots + \frac{A_m}{X-\alpha_m}$$

$$\boxed{A_i = \prod_{\substack{j=1\\ j\neq i}}^{m} (\alpha_i - \alpha_j)^{-1}}$$

$$\delta = q - \frac{p^2}{4}$$

$$\int \frac{dx}{x^2+px+q} = \frac{1}{\sqrt{\delta}} \cdot \arctan\left(\frac{x}{\sqrt{\delta}} + \frac{p}{2\sqrt{\delta}}\right) + C$$

$$\int \frac{cx+d}{x^2+px+q}\,dx = \frac{1}{2}\ln|x^2+px+q| - \frac{p}{2}\left[\frac{1}{\sqrt{\delta}} \cdot \arctan\left(\frac{x}{\sqrt{\delta}} + \frac{p}{2\sqrt{\delta}}\right)\right] + c$$

$$\int \frac{a_m \cdot x^m + a_{m-1} x^{m-1} + \dots + a_0}{x^2+px+q} \quad \text{(DE)} \quad \frac{1}{2}\ln|x^2+px+q| - \frac{p}{2}\left[\frac{1}{\sqrt{\delta}} \cdot \arctan\left(\frac{x}{\sqrt{\delta}} + \frac{p}{2\sqrt{\delta}}\right)\right]$$

(TH) $Q(x) = a_m X^m + a_{m-1} X^{m-1} + \dots + a_0 \in \mathbb{R}[x]$,
soit $\alpha \in \mathbb{C}$, racine complexe de $Q(x)$. $Q(\alpha) = 0$.
Alors $\bar{\alpha} = \overline{(x+iy)} = x - iy$ est aussi racine.
Racines de $Q(x)$ st $\alpha_1, \dots, \alpha_m \in \mathbb{R}$.
$\alpha_{r+1}, \bar{\alpha}_{r+1}, \dots, \alpha_{r+k} \in \mathbb{C} \setminus \mathbb{R}$.

sans racines

$$Q(x) = a_m (x-\alpha_1)\dots(X-\alpha_r)\left[\underbrace{(X-\alpha_{r+1})(X-\bar{\alpha}_{r+1})}_{x^2+px+q}\right]$$

$$\frac{a_m}{Q(x)} = \frac{A_1}{X-\alpha_1} + \dots + \frac{A_r}{X-\alpha_r} + \dots + \frac{B_1 X + C_1}{X^2 + P_1 X + Q_1} + \frac{B_K X + C_K}{X^2 + P_K X + Q_K}.$$

(P) Soit $(U_1, \dots, U_k)$ une base $W \subset \mathbb{R}^n$. ⓔ $W = \mathrm{Vect}(\underbrace{U_1, \dots, U_k}_{\text{libre}})$
soit $\{V_1, \dots, V_m\} \subset W$.
si $m > k$ alors $\{V_1, \dots, V_m\}$ est $\boxed{\text{LIÉE}}$

DM 6

(TH) Tt ss espace $W \subset \mathbb{R}^n$ possède une base.
Tte base de W contient m̂ nbe vecteurs = dim W.

DM 7

Corol 1 $W_1 \subsetneq W_2$: deux sous espaces de $\mathbb{R}^n$ alors $\dim W_1 < \dim W_2$

DM 8

Corol 2 $W_1 \subset W_2$; $\dim W_1 = \dim W_2 \Rightarrow W_1 = W_2$.

DM 9

Corol 3 Soit $W_1 \subset W_2 \subset \mathbb{R}^n$ 2 SEV de $\mathbb{R}^n$
Alors tte base $(u_1, \dots, u_k)$ de W_1 à une base de W_2.

DM 10

OPÉRATEURS $U, V \subset \mathbb{R}^n$ deux SE.

$U \cap V$ est sous-espace de $\mathbb{R}^n$.

DM 11

$U + V = \{u + v \mid u \in U, v \in V\}$

DM 12

(TH) $\dim(U+V) = \dim U + \dim V - \dim(U \cap V)$

1) $U \cap V = \{\vec{0}\} \Rightarrow U + V = U \oplus V$.

2) si $U \oplus V = \mathbb{R}^n$, on dit que U & V st supplémentaires.

DM 13

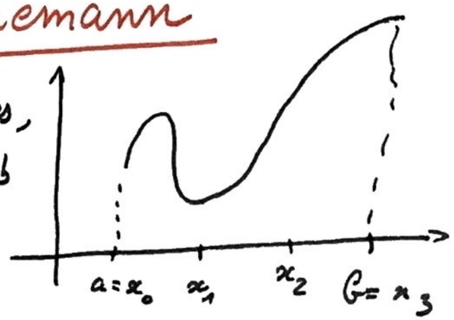
Intégrale de Riemann

→ On divise $[a, b]$ en n parties,

$x_0 = a$ $\quad$ $x_m = a+(b-a) = b$

$x_1 = a + \frac{b-a}{n}$

$x_k = a + k\frac{b-a}{n}$



On fixe $\xi_k \in [x_{k-1}, x_k]$; $\xi^{(n)} = (\xi_1, \xi_2, \dots, \xi_n)$

$$S(f, \xi^{(n)})_{[a,b]} = \sum_{k=1}^{N} \frac{b-a}{N} \cdot f(\xi_k)$$

Sommes de Darboux :

$(f \in C([a,b]))$

$m_k = \min_{x \in [x_{k-1}, x_k]} f(x)$ $\quad$ $M_k = \max_{x \in [x_{k-1}, x_k]} f(x)$ $\quad$ $\bar{s} = \sum_{k=1}^{n} m_k \frac{b-a}{n}$ $\quad$ $\underline{S} = \sum_{k=1}^{n} M_k \frac{b-a}{n}$.

→ $\bar{s}_n(f) \leqslant S(f, \xi^{(n)}) \leqslant \underline{S}_n(f)$

(D) $f(x)$ est intégrable si :

$$\lim_{n\to+\infty} \bar{s}_n(f) = \lim_{n\to+\infty} \underline{S}_n(f) = \lim_{n\to+\infty} S(f, \xi^{(n)}) = \int_a^b f(x)\,dx$$

(TH) soit $f \in C([a,b])$; alors l'intégrale de Riemann $\int_a^b f(x)$ ∃.

(C) $f'(x)$ ∃ & $f'(x)$ est continue in $[a, b]$

DM SDR

$$S_n = \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + k\,\frac{(b-a)}{n}\right)$$

$$\lim_{n\to+\infty} S_n = \int_a^b f(t)\,dt$$

Fractions Rationnelles $\cos x$ $\sin x$

M1 Changement de variable universel

$t = \tan\frac{x}{2}$ $\qquad$ $dx = \frac{2\,dt}{1+t^2}$

$$\sin x = \frac{2t}{1+t^2}$$

$$\cos x = \frac{1-t^2}{1+t^2}$$

M2 Règles de Bioche

$$w(x) = f(\cos x, \sin x)\,dx$$

si $w(-x) = w(x)$, on pose $t = \cos x$

si $w(\pi - x) = w(x)$, on pose $t = \sin x$

si $w(\pi + x) = w(x)$, on pose $t = \tan x$.

$$\int f(\cos x, \sin x)\,dx .$$

@ Trouver SEV engendré par $Vect(v_1, v_2, v_3) = ?$

$$\begin{array}{c|ccc} v_1 & 1 & 1 & 1 \\ v_2 & 1 & 2 & 1 \\ v_3 & 1 & 2 & 3 \end{array} \in \mathbb{R}^3 \qquad \begin{array}{c|ccc} v_1 & 1 & 1 & 1 \\ v_2' & 0 & 1 & 0 \\ v_3' & 0 & 1 & 2 \end{array} \qquad \begin{array}{c|ccc} v_1 & \boxed{1} & 1 & 1 \\ v_2' & 0 & \boxed{1} & 0 \\ v_3'' & 0 & 0 & \boxed{1} \end{array}$$

la famille LIBRE.

$\rightarrow \dim Vect(v_1, v_2, v_3) = 3$
$Vect(v_1, v_2, v_3) \subset \mathbb{R}^3$ $\Big\}$ $Vect(v_1, v_2, v_3) = \mathbb{R}^3$

@ $$\begin{array}{c|cccc} v_1 & 1 & 1 & -1 & 0 \\ v_2 & 0 & 1 & 2 & 3 \\ v_3 & 3 & 1 & 1 & 1 \end{array}$$

Compléter une base de SEV engendré par $Vect(v_1, v_2, v_3)$ à une base de $\mathbb{R}^4$!

$$\begin{array}{c|cccc} v_1 & 1 & 1 & -1 & 0 \\ v_2 & 0 & 1 & 2 & 3 \\ v_3' & 0 & -2 & 4 & 1 \end{array} \qquad \begin{array}{c|cccc} v_1 & 1 & 1 & -1 & 0 \\ v_2 & 0 & 1 & 2 & 3 \\ v_3'' & 0 & 0 & 8 & 7 \\ v_4 & 0 & 0 & 0 & \boxed{1} \end{array} \neq 0$$

$\overbrace{Vect(v_1, v_2, v_3)}^{\text{libre}} = \underbrace{Vect(v_1, v_2', v_3')}_{\dim 3}$

$\rightarrow v_4 = \lambda_1 v_1 + \lambda_2 v_2' + \lambda_3 v_3' = (\cdot, 0, 0, 1)$

$\lambda_1 = 0 \quad \lambda_2 = 0 \quad \lambda_3 = 0$

@ Supposons W_1,

$$W_1 = Vect \begin{vmatrix} \boxed{2} & 1 & 1 & 3 & 4 \\ 0 & 0 & \boxed{3} & 1 & 1 \\ 0 & 0 & 0 & 0 & \boxed{3} \\ 0 & 1 & \times & \times & \times \\ 0 & 0 & 0 & 1 & \times \end{vmatrix}$$

dc $\dim W_1 = 3$ $\Updownarrow$ 3 pivots non nuls $\Rightarrow$

$$\begin{vmatrix} 2 & 1 & 1 & 3 & 4 \\ 0 & 1 & \times & \times & \times \\ 0 & 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 1 & \times \\ 0 & 0 & 0 & 0 & 3 \end{vmatrix} \Big\}$$ une base de $\mathbb{R}^5$.

$W_1 \subset \mathbb{R}^5$.

SE supplémentaire ! $W_1 \oplus W_2 = \mathbb{R}^5$.

$W_1 + W_2 = \mathbb{R}^5$; $W_1 \cap W_2 = \{\vec{0}\}$ où $W_2 = Vect\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$

Pourquoi ? $W_1 \cap W_2 = \{\vec{0}\}$

$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3$

$\alpha(0,1,0,0,0) + \beta(0,0,0,1,0) = (0, \alpha, 0, \beta, 0) \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0$

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▷ LIBRE si ρ = rang = dim. NBR vecteurs.

▷ ρ = nbr lignes non-nulles à fin PDG

▷ $\boxed{M}$ FL : $\lambda_1 v_1 + \lambda_2 v_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \overset{?}{\Rightarrow} \lambda_1 = \lambda_2 = 0$

▷ FG : $\forall$ vecteur s'écrit ĉ CBL.

▷ $\vec{v}\begin{pmatrix} x \\ x \\ 3x \end{pmatrix} = x\vec{v_1}$; $\vec{v_1}\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \neq 0 \Rightarrow \vec{v_1}$ libre.

↳ vect libre ssi non-nul.

(D) U & V st supplémentaires si

1) $U + V = \mathbb{R}^n$
2) $U \cap V = \{\vec{0}\}$ $\Big\}$ $\mathbb{R}^n = U \oplus V$ ← la somme directe

1) $\dim \underbrace{(U+V)}_{\mathbb{R}^n} = \dim U + \dim V - \underbrace{\dim U \cap V}_{\{\vec{0}\}}$

$n = \dim U + \dim V$.

↳ si $\dim U = k$ alors $\dim V = n - k$.

(TH) $\forall$ sous-espace $U \subset \mathbb{R}^n$, $\exists$ un sous-espace supplt^n à U.

(Cod) Algorithme : $\boxed{\text{DM 14}}$

1) $V_{k+1} \notin U$
2) $V_{k+2} \notin Vect(U, V_{k+1}) = Vect(u_1, u_k, V_{k+1})$
3) $V_{k+3} \notin Vect(u_1, \dots, u_k, V_{k+1}, V_{k+2})$
4) ...

$n-k$) $V_n \notin Vect(u_1, \dots, u_k, V_{k+1}, V_{n-k})$.

Comment trouver un ss-espace supplémentaire dans la pratique ?

Ex/ $\mathcal{U} = \{(x_1,x_2,x_3,x_4) \in \mathbb{R}^4 \mid \begin{cases} x_1 + 2x_2 - 3x_3 - x_4 = 0 \\ 2x_1 + 5x_2 - 2x_3 + x_4 = 0 \end{cases}\} \subset \mathbb{R}^4$

Base de $\mathcal{U}$?

PDG $\longrightarrow \begin{cases} x_1 + 2x_2 - 3x_3 - x_4 = 0 \\ x_2 + 4x_3 + 3x_4 = 0 \end{cases}$, $x_3, x_4 \in \mathbb{R}$ $\begin{cases} x_1 = 11x_3 + 7x_4 \\ x_2 = -4x_3 - 3x_4 \end{cases}$

$(x_1,x_2,x_3,x_4) = (11x_3 + 7x_4, -4x_3 - 3x_4, x_3, x_4)$

$(x_1,x_2,x_3,x_4) = x_3(11,-4,1,0) + x_4(7,-3,0,1)$

On a mq' $\mathcal{U} = \mathrm{Vect}(\underbrace{(11,-4,1,0)}_{u_1}, \underbrace{(7,-3,0,1)}_{u_2}) \subset \mathbb{R}^4$ Base

libre

où $\dim \mathcal{U} = 2$

On cherche V : $\begin{cases} \mathcal{U} + V = \mathbb{R}^4 \\ \mathcal{U} \cap V = \{\bar{0}\} \end{cases}$

$\begin{vmatrix} 11 & -4 & 1 & 0 \\ 7 & -3 & 0 & 1 \end{vmatrix} \longrightarrow \begin{vmatrix} 7 & -3 & 0 & 1 \\ 11 & -4 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}$ $\quad V = \mathrm{Vect}((0,1,0,0),(1,0,0,0))$

$w \in V = \mu_1(0,1,0,0) + \mu_2(1,0,0,0) = (\mu_2, \mu_1, 0, 0)$

SuppoS $w = \lambda_1 u_1 + \lambda_2 u_2$,

$\exists \lambda_1, \lambda_2$ tq $w = \lambda_1 u_1 + \lambda_2 u_2 = (*, *, \lambda_2, \lambda_1)$

$(\mu_1, \mu_2, 0, 0) = (*, *, \lambda_2, \lambda_1) \implies \lambda_2 = \lambda_1 = 0$

$w = \bar{0} \implies \mathcal{U} \cap V = \{\bar{0}\}$.

Exercice $P_1 = 1 + X^2 + X^3$; $P_2 = 2 + X^2 + 3X^3 \in \mathbb{R}_3[X]$

espace de polynômes de degré ≤ 3

Sous-espace Supplémentaire à $\mathrm{Vect}(P_1, P_2) \subset \mathbb{R}_3[X]$?

$\mathbb{R}_3[X] = \{a_0 + a_1 X + a_2 X^2 + a_3 X^3 \mid a_0, a_1, a_2, a_3 \in \mathbb{R}^4\}$.

NB : $\underset{\deg \leq n}{\mathbb{R}_n[X]} \overset{1:1}{\longleftrightarrow} \mathbb{R}^{n+1}$.

D'où $\mathbb{R}_3[X] \overset{1:1}{\longleftrightarrow} \mathbb{R}^4$ & $P_1 = 1 + X^2 + X^3 \longmapsto (1,0,1,1)$

$P_2 = 2 + X^2 + 3X^3 \longmapsto (2,0,1,3)$.

$\mathcal{U} = \mathrm{Vect}(P_1, P_2) \longmapsto \mathrm{Vect}((1,0,1,1),(2,0,1,3))$.

$\begin{vmatrix} 1 & 0 & 1 & 1 \\ 2 & 0 & 1 & 3 \end{vmatrix} \longrightarrow \begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{vmatrix}$ est une autre base de $\mathcal{U}$.

$\begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix}$ est une base de $\mathbb{R}^4$ $\implies V = \mathrm{Vect}((0,1,0,0),(0,0,0,1)) = \mathrm{Vect}(X, X^3)$

2: Applica° linéaire

$$(S)\begin{cases} x_1 - x_2 + 2x_3 = b_1 \\ 2x_1 + x_2 - x_3 = b_2 \\ 4x_1 - x_2 + 3x_3 = b_3 \end{cases}$$

Pour quels $(b_1, b_2, b_3) \in \mathbb{R}^3$, (S) a-t-il une solu°.

Une Applica°:

$$\underbrace{\mathrm{Vect}\left(\begin{pmatrix}1\\2\\4\end{pmatrix}, \begin{pmatrix}-1\\1\\-1\end{pmatrix}, \begin{pmatrix}2\\-1\\3\end{pmatrix}\right)}$$

$$\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} \in \mathbb{R}^3 \longmapsto \begin{pmatrix}x_1 - x_2 + 2x_3\\2x_1 + x_2 - x_3\\4x_1 - x_2 + 3x_3\end{pmatrix} = x_1\underbrace{\begin{pmatrix}1\\2\\4\end{pmatrix}}_{a_1} + x_2\underbrace{\begin{pmatrix}-1\\1\\-1\end{pmatrix}}_{a_2} + x_3\underbrace{\begin{pmatrix}2\\-1\\3\end{pmatrix}}_{a_3} \in \mathbb{R}^3$$

$$\underbrace{\begin{pmatrix}1 & -1 & 2\\2 & 1 & -1\\4 & -1 & 3\end{pmatrix}}_{A''} \cdot \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$$

Soit $A : \mathbb{R}^3 \to \mathbb{R}^3$; on a $A\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = A\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = a_1 x_1 + a_2 x_2 + a_3 x_3$

ie Pr quel $\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$, il existe $\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$ tq $A\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$.

$$\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} \overset{?}{\in} \underbrace{A(\mathbb{R}^3)}_{\text{l'image de l'applica° } A.} \quad , \quad \underbrace{\mathrm{Im}\,A}_{\text{image de l'applica° } A} = \mathrm{Vect}\left(\begin{pmatrix}1\\2\\4\end{pmatrix}, \begin{pmatrix}-1\\1\\-1\end{pmatrix}, \begin{pmatrix}2\\-1\\3\end{pmatrix}\right)$$

$$\begin{pmatrix}1 & 0 & 0\\2 & 3 & -5\\4 & 3 & -5\end{pmatrix} \to \begin{pmatrix}1 & 0 & 0\\2 & 1 & 0\\4 & 1 & 0\end{pmatrix}$$ dc $\mathrm{Im}\,A = \mathrm{Vect}\left(\begin{pmatrix}1\\2\\4\end{pmatrix}, \begin{pmatrix}-1\\1\\-1\end{pmatrix}, \begin{pmatrix}2\\-1\\3\end{pmatrix}\right) = \mathrm{Vect}\left(\begin{pmatrix}1\\2\\4\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}\right)$

$$\mathrm{Im}\,A = \lambda_1\begin{pmatrix}1\\2\\4\end{pmatrix} + \lambda_2\begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}\lambda_1\\2\lambda_1 + \lambda_2\\4\lambda_1 + \lambda_2\end{pmatrix} = \begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$$

$$\Leftrightarrow \begin{cases} \lambda_1 = b_1 \\ \lambda_2 = b_2 - 2b_1 \\ b_3 = 4b_1 + b_2 - 2b_1 = 2b_1 + b_2 \end{cases} \Rightarrow 2b_1 + b_2 - b_3 = 0$$

Il y a une solu° ssi $2b_1 + b_2 - b_3 = 0$.

D Soit $E \subset \mathbb{R}^m$, $F \subset \mathbb{R}^n$: deux SEV

Une applica° linéaire $A . E \to F$

1) $\forall \lambda \in \mathbb{R}, \forall v \in E$, $A(\lambda v) = \lambda A(v)$

2) $\forall v, u \in E$,

① $A(v+u) = A(v) + A(u)$