ALGEBRE

TD Moodle Analyse

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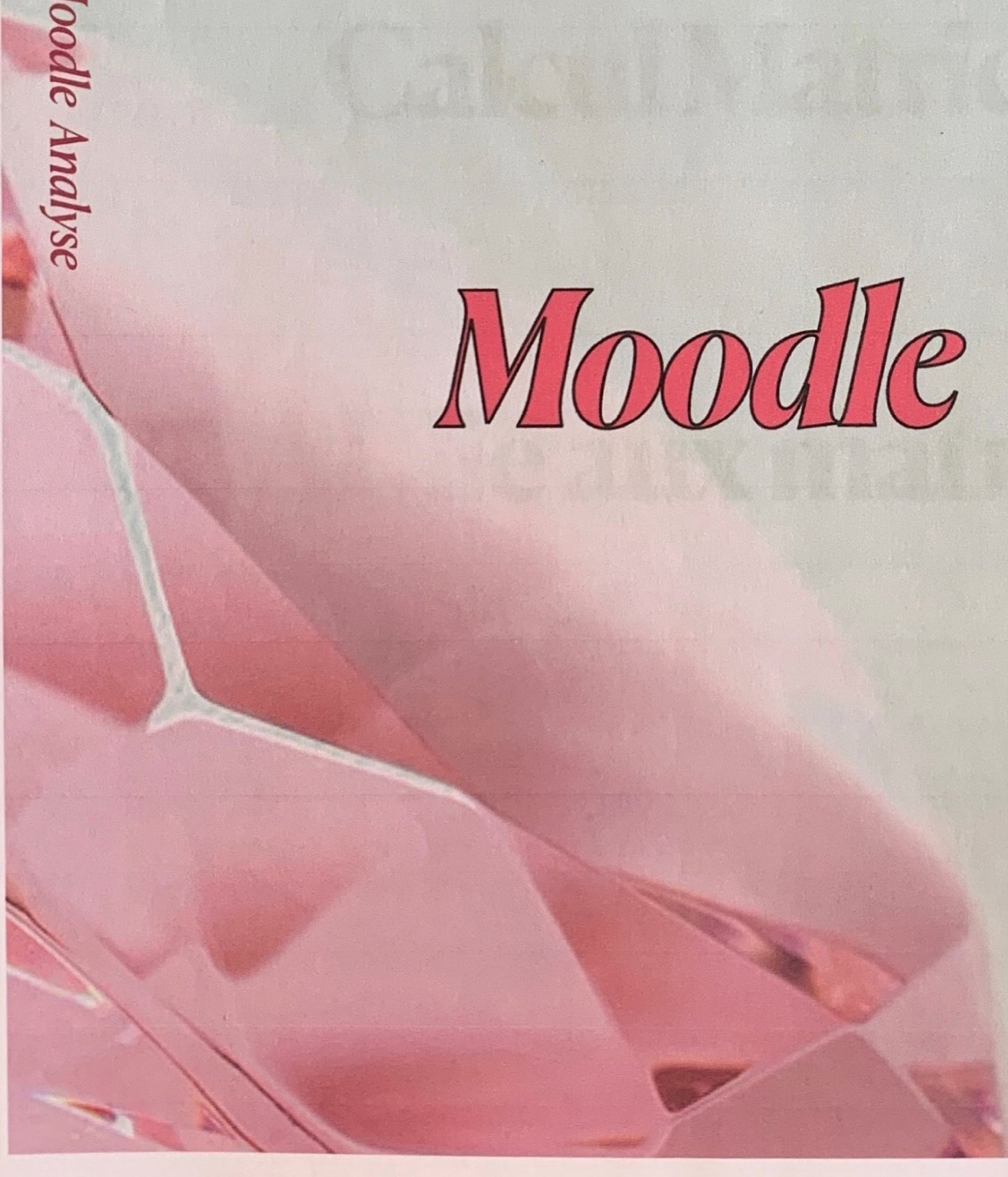
Moodle



Espaces Vectoriels

Applications linéaires

Calcul Matriciel

AL lié aux matrices

- QCM - 1 - Maxence.

Test Blanc M21A

Algèbre 1

- MAXENCE -

$_2[X] \rightarrow \leq 2.$ $\mathcal{B}\{1, X, X^2\}$

$P_1 = 1 - X,\ P_2 = 1 + X,\ P_3 = X^2,\ P_4 = 1 + X^2$

$$\begin{array}{cccc} \begin{pmatrix} 1 & 1 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \\ P_1 \quad P_2 \quad P_3 \quad P_4 \end{array} \rightarrow \begin{matrix} 1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{matrix}$$

$$\begin{matrix} 1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{matrix} \rightarrow \begin{matrix} & 1 & 1 & 1 & 0 \\ & 0 & -1 & 1 & 0 \\ L_3 - L_1 & 0 & -1 & -1 & 1 \end{matrix} \rightarrow \begin{matrix} 1 & +1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$$

Rg P_1, P_2, P_3, P_4 : 3.

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$ P_3, P_4 : Rg 2.

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 0 \\ & 1 & 0 \\ -1 & 1 & \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \rightarrow$$ Rg 3.

Rang famille vecteurs : la dimension du SEV engendré par ces vecteurs. (nbr max de vecteurs linéair^t indépendants)

① ② 2) $f : \mathbb{R} \rightarrow \mathbb{R}$, $x \rightarrow \sin x$, $g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $(x, y) \rightarrow (y, x)$

AL ? AC ~~oui~~ NON

AC non.

- $\cancel{g(x,y)}$ $g((x,y) + (x', y'))$ $= g(x + x', y + y') = g = (y + y', x + x').$ $g \neq g(x,y) + g(x', y').$
- $g(\lambda(x,y)) = g(\lambda x, \lambda y) = (\lambda y, \lambda x).$ $= \lambda(y, x)$ $= \lambda \cdot g(x,y)$

$f(0) = ?\ \cancel{X}\ 0$

$f : \mathbb{R} \longrightarrow \mathbb{R}$
$x \longmapsto \sin x$

f n'est pas linéaire car

- $f(\pi) = f\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = 0$
- $f\left(\frac{\pi}{2}\right) + f\left(\frac{\pi}{2}\right) = 2.$

$E=\{(x,y,t)\in\mathbb{R}^3 : \begin{cases} x-y+z=0 \\ y+z=0 \end{cases}\}$ z ?

$\begin{cases} x=y-z \\ y=-z \end{cases} \Leftrightarrow \begin{cases} x=-2z \\ y=-z \end{cases}$

$\forall (x,y,z)$, $E=\text{Vect}(-2z, -z, z)$

$E=\cancel{\text{Vect}}\ z.(-2,-1,1)$

~~Trouvons ker f, soit $u=(x,y,z)$~~

~~$\forall u\in E$, $\ker f=\{u\in E \mid f(x,y,z)=0$.~~

~~Im E ? $\mathcal{B}$ de $\mathbb{R}^3$:~~

~~$e_1 : (1,0,0)$; $e_2=(0,1,0)$, $e_3=(0,0,1)$??~~

~~$f(1,0,0)=$ oUi~~

$\Rightarrow \dim E = 1$

$\Rightarrow \{(2,1,-1)\}$ est une base de E.

③

$u_3 = u_2 - u_1$

Donc $\{u_1,u_2,u_3\}$ n'est pas libre.

$=\cancel{Lu}$??

④ Dans $\mathbb{R}^3$,

$u_1=(1,1,0)$, $u_2=(0,1,-1)$, $u_3=(-1,0,$

$\begin{array}{c} \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & -1 & -1 \end{pmatrix} \\ u_1 \quad u_2 \quad u_3 \end{array} \xrightarrow{L_2-L_1} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 \\ 0 & -1 \\ 0 & -1 \end{pmatrix}$

$\xrightarrow{L_2+L_3} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ de famille libre ? $\cancel{\begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 0 & \end{pmatrix}}$

Une famille est génératrice est une famille construite de combinaisons linéaires de autres vecteurs

famille génératrice + famille libre = base

Par consq^t, $\{u_1,u_2,u_3\}$ n'est pas famille génératrice

⑦

$f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$(x,y,z) \longmapsto (x+y+z, x+y-z)$

$(x,y,z)\in \text{Ker} f \Leftrightarrow f(x,y,z)=0$

$\Rightarrow (x+y+z, x+y-z)=(0,0)$

$\begin{cases} x+y+z=0 \\ x+y-z=0 \end{cases} \Leftrightarrow \begin{cases} x=-y-z \\ x=-y+z \end{cases}$

$\Rightarrow \begin{cases} z=-x-y \\ y=-x+z \end{cases} \Rightarrow \begin{cases} x=-y \\ z=0 \end{cases}$

$\Rightarrow -y-z=-y+z$

$\Rightarrow z=0$

$\text{Ker} f=\{(-y,y,0)\in\mathbb{R}^1\}$

$\text{Ker} f=\{y(-1,1,0)\in\mathbb{R}^1\}$.

Dc Ker f n'est pas injectif car $\text{Ker} f \neq 0_E$.

oui oui

⑧

$\Rightarrow \dim \ker f = 1$

Trouvons la base de Im f.

$\mathcal{B}$ de $\mathbb{R}^3$: $\{\underset{e_1}{(1,0,0)}, \underset{e_2}{(0,1,0)}, \underset{e_3}{(0,0,1)}\}$

Alors $\text{Im} f = \text{Vect}\{f(e_1), f(e_2), f(e_3)\}$

$f(e_1)=f(1,0,0)=(1,1)$

$f(e_2)=f(0,1,0)=(1,1)$

$f(e_3)=f(0,0,1)=(1,-1)$.

Dc $\text{Im} f=\text{Vect}\{(1,1),(1,1),(1,-1)\}$.

Essayons famille libre de ces vecteurs.

$\begin{array}{c} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \\ f(e_1) \ f(e_2) \ f(e_3) \end{array} \xrightarrow{L_2-L_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

On obtient de une famille libre.

Dc $\{(1,0),(1,0),(1,2)\}$ est une base de Im f.

⑤

$f_1 : \mathbb{R}_2[X] \longrightarrow \mathbb{R}$

$P(X) \longmapsto P(0)$.

AL

- $f((x)+(x')) = f(x, x')$

$f = (~~\ldots~~\ 0, 0) = f(x) + f(x')$

- $f(\lambda(x)) = \lambda . 0 = \lambda . P(x)$

$f_2 : \mathbb{R}_2[X] \longrightarrow \mathbb{R}$

$P(X) \longmapsto P(0) + 1$.

n'est pas AL

- $f(x+x') = f(x, x')$

$f = (P(0)+1, P(0)+1) = f(x) + f(x')$

- $f(\lambda(x)) = \lambda . (P(0)+1) = \lambda . f(x)$.

$f_3 : \mathbb{R}_2[X] \longrightarrow \mathbb{R}$

$P(X) \longmapsto \int_0^1 P(t)\, dt = p(1) - p(0)$.

oui AL

⑥

$f_4 : \mathbb{R}_2[X] \longrightarrow \mathbb{R}_2[X]$

$P(X) \longmapsto X . P'(X) - P(X)$.

AL

- $f(x+x') = f(x, x')$

$f = (X . P'(X) - P(X),\ X' P'(X') - P(X')$

$f = f(x) + f(x')$

- $f(\lambda . x) = \lambda (X . P'(X) - P(X))$

$- \lambda . f(X)$.

On a $\mathcal{M}_n(\mathbb{R})$ tq $AB = 0$.

Si A est inversible alors $\exists$ une matrice C,

$CA = I$. Donc $(CA)B = B$.

Or $(CA)B = C(AB)$ et $AB = 0$

Donc $B = 0$ $\boxed{c\ ?!\ c}$

$\Rightarrow$ A n'est pas inversible.

⑨

$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \overset{A}{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}} = \begin{pmatrix} 4 & 6 \\ -2 & -6 \end{pmatrix} \neq E$.

$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \overset{B}{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}} = \begin{pmatrix} 3 & -1 \\ 7 & -1 \end{pmatrix} = D$.

$- B = \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix} - \begin{pmatrix} 1 & 3 \\ 5 & 9 \end{pmatrix}$

$- B = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \neq C = B$.

$\mathcal{B} = \{e_1, e_2, e_3\}$; $\mathcal{B}' = \{u_1, u_2, u_3\}$

$u_1 = (1, 1, -1)$

$u_2 = (0, 2, 1)$

$u_3 = (0, 1, 1)$

- P : la matrice de passage de $\mathcal{B}$ à $\mathcal{B}'$.
- Q : la matrice de passage de $\mathcal{B}'$ à $\mathcal{B}$.

$\Rightarrow$ $\boxed{B = Q^{-1} . A . P}$ Th chgt de base.

⑩. Soit E : ev muni 2 bases $\mathcal{B}$ et $\mathcal{B}'$.

La matrice de passage de $\mathcal{B}$ à $\mathcal{B}'$ est

$P = \mathrm{Mat}(\mathrm{Id}_E, \mathcal{B}', \mathcal{B})$.

$P = ([e_1]_{\mathcal{B}}, [e_2]_{\mathcal{B}}, [e_3]_{\mathcal{B}})$

- $P = \mathrm{Mat}(\mathrm{Id}_E, \mathcal{B}', \mathcal{B}) = \begin{array}{c} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 1 & 1 \end{pmatrix} \\ u_1 \quad u_2 \quad u_3 \end{array}$

On cherche $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ tq $PX = Y$, $Y = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$PX = Y \iff \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ x+2y+z \\ -x+y+z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$\begin{cases} x = a \\ x + 2y + z = b \\ -x + y + z = c \end{cases} \iff \begin{cases} x = a \\ z = b - a - 2y \\ z = c + a - y \end{cases}$.

$\iff b - a - 2y = c + a - y$ $\Rightarrow z = a + c + 2a - b + c$

$\iff -2a + b - c = y$ $\quad z = 3a - b + 2c$

$\begin{pmatrix} a \\ -2a + b - c \\ 3a - b + 2c \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & -1 \\ 3 & -1 & 2 \end{pmatrix} = P^{-1}$.

$$P^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & -1 \\ 3 & -1 & 2 \end{pmatrix} \Leftrightarrow \begin{cases} e_1 = u_1 \\ e_2 = -2u_1 + u_2 - u_3 \\ e_3 = 3u_1 - u_2 + 2u_3 . \end{cases}$$

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↳ Cela revient à dire Q : ??

$B = Q^{-1} . A . P$

n) $f : \mathbb{R} \longrightarrow \mathbb{R}^2$

$x \longmapsto (x, -x)$

AL

$\mathcal{B}$ de $\mathbb{R}^2$: $\{(1,0), (0,1)\}$?

$f(1,0) = (1, -1)$?

~~$f(0,1) = ($~~

↳ Matrice $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. ? ⚠

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R9

$x . ch\, x = \frac{1}{2} x e^{x} + \frac{1}{2} x e^{-x}$

Principe de Superposiθ :

(E) $y' - 2x\, y = -2x\, ch\, x + sh\, x$.

(E_H) $y_h = k . e^{x^2}$, $k \in \mathbb{R}$.

$-2x . ch\, x = -e^{x} - e^{-x}$

$sh\, x = \frac{1}{2} e^{x} - \frac{1}{2} e^{-x}$

QCM-1 - RÉMY - ALGÈBRE

1) $\mathbb{R}^3$, $u_1 = (1,0,1)$, $u_2 = (-1,1,1)$, $u_3 = (1,-1,0)$

SEV, $E = Vect\{u_1, u_2\}$

$F = Vect\{u_3\}$

a) E est plan vectoriel
b) (EC) de F : $z = 0$ ← ?
c) (EC) de E : $x + 2y + z = 0$
d) F est une droite vectorielle. ←

$F = Vect\{u_3\} = Vect\{(1,-1,0)\}$

2) c) $F = \{ f : \mathbb{R} \to \mathbb{R},\ f$ dérivable sur $\mathbb{R}$, $f'(1) = 0 \}$.

a) E n'est pas EV.
b) f nulle ∈ à E
c) E est stable par additiθ ←
d. E est stable par multiplicaθ par un scalaire. ←

3) $E = \{(x,y,z,t) \in \mathbb{R}^4,\ \begin{cases} x+y+z+t = 0 \\ y + 2z = 0 \\ z + t = 0 \end{cases}$

$$\begin{cases} x = -y - z - t \\ y = -2z \\ z = -t \end{cases} \Leftrightarrow \begin{cases} x = -2t + t - t \\ y = 2t \\ z = -t \end{cases} \Leftrightarrow \begin{cases} x = -2t \\ y = 2t \\ z = -t \end{cases}$$

$E = \{(-2t, 2t, -t, t) \in \mathbb{R}^4\}$

$E = \{Vect(t(-2,2,-1,1)) \in \mathbb{R}^4\}$.

d) si $u_1, u_2 \in E$ alors $u_1 - u_2 \in E$?

4) $\mathbb{R}^4$, $E = \text{Vect}\{u_1, u_2, u_3\}$ ③

$u_1 = (1,-1,0,1)$

$u_2 = (1,0,1,0)$

$u_3 = (3,-1,1,2)$

$F = \left\{ (x,y,z,t) \in \mathbb{R}^4,\ \begin{cases} x+y-z=0 \\ y+z=0 \\ t \in \mathbb{R} \end{cases} \right\}$

$$\begin{matrix} 1 & -1 & 0 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 0 & 1 & 0 \end{matrix} \rightarrow \begin{matrix} 1 & -1 & 0 & 1 \\ 0 & -2 & -1 & 1 \\ 0 & -1 & -1 & 1 \end{matrix} \ (3L_1 - L_2) \quad \begin{matrix} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 2 & 1 & -1 \end{matrix}$$

$$\rightarrow \begin{matrix} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{matrix} \ (2L_2 - L_3)$$

Donc la famille $\{u_1, u_2, u_3\}$ est libre.

F, $\begin{cases} x + \cancel{y} = z - y \\ y = -z \\ t \in \mathbb{R} \end{cases} \iff \begin{cases} x = 2z \\ y = -z \\ t \in \mathbb{R} \end{cases}$

$F = \{ (2z, -z, z, t) \in \mathbb{R}^4$

$F = \{ \text{Vect}\,(z(2,-1,1,0) + t(0,0,0,1)) \in \mathbb{R}^4$

$E + F \neq \mathbb{R}^4$ | $E \oplus F \neq \mathbb{R}^4$

$\dim E = 3$

$\dim F = 2$.

④ $u : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$(x,y,z) \longrightarrow (-2x-4z,\ 3y,\ 2x+4z)$

$\forall\, u\,(x,y,z) \in \mathbb{R}^3$, $\text{Ker}\, u = \{0_{\mathbb{R}^3}\}$.

$\begin{cases} -2x-4z = 0 \\ 3y = 0 \\ 2x+4z = 0 \end{cases} \iff \begin{cases} x = 0 \\ y = 0 \\ z = 0 \end{cases}$

$\Rightarrow \text{Ker}\, u = \{(0,0,0) \in \mathbb{R}^3\}$

Donc Ker u est injectif.

Trouvons la base de ~~L~~ $\mathcal{B}$ de $\mathbb{R}^3$:

$\{ \underset{e_1}{(1,0,0)}, \underset{e_2}{(0,1,0)}, \underset{e_3}{(0,0,1)} \}$

Alors $\text{Im}\, f = \text{Vect}(f(e_1), f(e_2), f(e_3))$

$f(e_1) = f(1,0,0) = (-2,0,2)$

$f(e_2) = f(0,1,0) = (0,3,0)$

$f(e_3) = f(0,0,1) = (-4,0,4)$ ..

Dc $\text{Im}\, f = \text{Vect}\{(-2,0,2), (0,3,0), (-4,0,4)\}$

Extrayons une famille libre de ces vecteurs:

$$\begin{matrix} -2 & 0 & 2 \\ 0 & 3 & 0 \\ \cancel{-4} & \cancel{0} & \cancel{4} \end{matrix} \rightarrow \begin{matrix} -2 & 0 & 2 \\ 0 & 3 & 0 \end{matrix}$$

On obtient de une famil[le]

Dc $\{(-2,0,2), (0,3,0)\}$ est [une] base d[e]

$\text{Ker}\, f \cap \text{Im}\, f = \{ v \in \mathbb{R}^3 \mid v = a(0,0,0)$, $v = b(-2,0,2) + c(0,3,0)$

$\begin{cases} 0 = 0 \\ -2a + 2c = 0 \\ 3b = 0 \end{cases} \iff \begin{cases} a = c \\ b = 0 \end{cases}$

$a(0,0,0) = b(-2,0,2) + c(0,$

$y = \{ a(1,0,1) \}$

$\text{Ker}\, f \cap \text{Im}\, f = \{ a(1,0,1) \}$

! E, F de $\mathbb{R}$ EV dim finis ⑤

f : L de E dans F.

(!) Si f est bijective alors $\dim E = \dim F$.

Si $\dim E = \dim F$ alors f est bijective.

Matrices

- $(A+B)^2 = (A+B)(A+B)$
  $= A^2 + AB + BA + B^2$ ⚠
- $A(B+C) = AB + AC$ ☺
- $AB = 0 \Rightarrow A = 0$ ou $B = 0$.

Soit $M_n(\mathbb{R})$, l'ensemble des matrices carrés d'ordre n, soit $A \in M_n(\mathbb{R})$, $\exists\, m \geq 1$,

$A^m + A^{m-1} + \dots + A + I = 0$.

$AA^{-1} = I$

$A^{-1}A = I$

Le rang de A est n.

A est inversible.

$A^{-1} = -(A^{m-1} + \dots + A + I)$

⑥ $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$(x,y,z) \longmapsto (y+z,\ x+z,\ x+y)$

$\mathcal{B}' = \{u_1, u_2, u_3\}$ où $u_1 = (1,0,0)$, $u_2 = (1,1,0)$, $u_3 = (1,1,1)$.

- P la matrice de passage de $\mathcal{B}$ à $\mathcal{B}'$.

$P = \text{Mat}(\text{Id}_E, \mathcal{B}', \mathcal{B}) = ([u_1]_{\mathcal{B}}, [u_2]_{\mathcal{B}}, [u_3]_{\mathcal{B}})$

$$P = \begin{array}{c} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \\ u_1 \ u_2 \ u_3 \end{array}$$

On cherche $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ tq $PX = Y$. $Y = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$P.X = \begin{cases} x+y+z = a \\ y+z = b \\ z = c \end{cases} \iff \begin{cases} x = a-b \\ y = b-c \\ z = c \end{cases}$

D'où $AY = \begin{pmatrix} a & -b & 0 \\ 0 & -b & -c \\ 0 & 0 & c \end{pmatrix}$ Dc $P^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

$P^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \iff \begin{cases} e_1 = u_1 - u_2 \\ e_2 = -u_2 - u_3 \\ e_3 = u_3 \end{cases}$

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→ la matrice de f dans la base $\mathcal{B}$:

$f(e_1) = f(1,0,0) = (0,1,1)$

$f(e_2) = f(0,1,0) = (1,0,1)$

$f(e_3) = f(0,0,1) = (1,1,0)$

$$\mathrm{Mat}(f, \mathcal{B}, \mathcal{B}') = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

→ la matrice de f de la base $\mathcal{B}'$ dans la base $\mathcal{B}$:

$f(u_1) = f(1,0,0) = (0,1,1)$

$f(u_2) = f(1,1,0) = (1,1,2)$

$f(u_3) = f(1,1,1) = (2,2,2)$

$$\mathrm{Mat}(f, \mathcal{B}', \mathcal{B}) = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 2 & 2 \end{pmatrix}$$

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$\boxed{\text{QCM - 1 - Analyse Rémy}}$

1)

CDV : $u = \sin x$

- $\displaystyle\int \frac{\cos x}{1+\sin^2 x}\,dx = ?$ $\quad u = \sin x$, $du = \cos x \, dx$

$\displaystyle = \int \frac{du}{1+u^2}$

- $\displaystyle\int \frac{dx}{1+\sin^2 x} \neq ?$

- $\displaystyle\int \frac{\cos x}{1+\sin^2 x}\,dx = \arctan(u) + k, \; k \in \mathbb{R}.$

F une primitive de $f(x) = \arcsin x$ sur $]-1, 1[$.

$\displaystyle\int \arcsin x \, dx$ $\quad u = \arcsin x \rightarrow du = \frac{dx}{\sqrt{1-x^2}}$; $dv = 1\,dx \rightarrow v = x$

$\displaystyle\int \arcsin x \, dx = x . \arcsin x - \int \frac{x\,dx}{\sqrt{1-x^2}}$

$t = \sqrt{1-x^2}$

$dt = -x(1-x^2)^{-\frac{1}{2}}\,dx$

$\displaystyle\int \arcsin x \, dx = x . \arcsin x - \int -1 . dt$

$\displaystyle\int \arcsin x \, dx = x . \arcsin x + \int 1 \, dt$

$\displaystyle\int \arcsin x \, dx = x . \arcsin x + t$

$\displaystyle\int \arcsin x \, dx = x . \arcsin x + \sqrt{1-x^2} + k \in \mathbb{R}$

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$\frac{1}{2}t$ $\quad t = 2x$ $\quad$ D'où $dt = 2\,dx$.

- $\displaystyle\int_0^{\pi/2} \sin(2x)\,dx = \frac{1}{2}\int_0^{\pi} \sin(t)\,dt = \int_0^{\pi/2} \sin(t)\,dt.$

$t = \pi - x$ $\quad$ D'où $dt = -1\,dx$.

- $\displaystyle\int_{\pi/2}^{\pi} \sin x \, dx = -\int_{\pi/2}^{0} \sin(t)\,dt = \int_0^{\pi/2} \sin t \, dt.$

$t = 2x$ $\quad$ D'où $dt = 2\,dx$

- $\displaystyle\int_0^{\pi/2} \sin 2x \, dx = \frac{1}{2}\int_0^{\pi} \sin(t)\,dt = \int_0^{\pi/2} \sin(t)\,dt$

$$\int_1^{\sqrt{3}} \frac{dt}{t^2+1} = \arctan(\sqrt{3}) - \arctan(1) = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}.$$

$$\int_0^{1/2} \sqrt{\frac{1+x}{1-x}}\,dx$$

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QCM n° 2 - MAXI ALGÈBRE

(11)

$P_1 = 1+X$, $P_2 = -1+X$, $P_3 = 1$ | $P_1 - P_2 - 2P_3 = 0$

DONC N'EST PAS LIBRE.

$$\begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \end{vmatrix} \to \begin{vmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \end{vmatrix}$$

de $\mathbb{R}_1[X]$

FG car contient deux polynômes non-colinéaires

$\{P_1, P_2, P_3\} \to$ FG + FL = B de $\mathbb{R}_1[X]$

$$\begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix} \to \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} \quad \{P_2, P_3\} \text{ B de } \mathbb{R}_1[X]$$

$E = \{(x,y,z) \in \mathbb{R}^3 : \begin{cases} x - y + z = 0 \\ x + 2y - z = 0 \end{cases}$

$\cancel{x} - y + z = \cancel{x} + 2y - z$

$-3y + 2z = 0$.

E est une droite vectorielle ds $\mathbb{R}^3$.

$\begin{cases} x = y - z \\ 2y = z - x = z - y + z \end{cases} \iff \begin{cases} x = y - z \\ 3y = 2z \end{cases}$

$\iff \begin{cases} x = \frac{2}{3}z - \frac{3}{3}z = -\frac{1}{3}z \\ y = \frac{2}{3}z \end{cases}$

$\{-\frac{1}{2}y, y, \frac{3}{2}y\}$

Non

$E = \{(-\frac{1}{3}z, \frac{2}{3}z, z) : z \in \mathbb{R}\}$

$E = \{\mathrm{Vect}(-\frac{1}{3}, \frac{2}{3}, 1), z \in \mathbb{R}\}$.

(12) $f : \mathbb{R}^2 \to \mathbb{R}$ et $g : \mathbb{R}^2 \to \mathbb{R}^2$

$(x,y) \to |x+y|$ $\quad (x,y) \to (\max(x,y), \min(x,y))$

(A)? $f(u + \lambda v) = f(u) + \lambda f(v)$?

$u = (x,y)$, $v = (x', y')$

$f(u + \lambda v) = |x + \lambda x'$

$f((x,y) + (x',y')) = |x + x'| + |y + y'|$

$f(1,-1) = |0| = 0$

g n'est pas linéaire.

$\mathbb{R}^2 \to \mathbb{R}^3$ muni de les bases canoniques (15)

$(x,y) \to (y, x, -y)$

$\mathcal{B} = \{e_1, e_2\}$; $\mathcal{B}' = \{e_1', e_2', e_3'\}$.

$f(e_1) = f(1,0) = (0,1,0)$

$f(e_2) = f(0,1) = (1,0,-1)$

$$\mathrm{Mat} f = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$[X]$, $\mathcal{B} = \{1, X, X^2\}$; $\mathcal{B}' = \{P_1, P_2, P_3\}$.

$P_1 = X$.

$P_2 = 1 - X$.

$P_3 = (1-X)^2 = 1 - 2X + X^2$

P : la matrice de passage de $\mathcal{B}$ à $\mathcal{B}'$

$[u]_{\mathcal{B}} = [u]_{\mathcal{B}'} \times P$

$$P = \mathrm{Mat}(\mathbb{R}_{\mathcal{B}}, \mathcal{B}, \mathcal{B}') = \begin{pmatrix} 0 & 1 & 1 \\ 1 & -1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

$P^{-1} = ?$ $\quad PX = Y$.

$\begin{cases} y + z = a \\ x - y - 2z = b \\ z = c \end{cases} \iff P^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

$= Q$.

(16) $A^t C = (3\ 4)$

$A + {}^t B = (1\ 33)$

$C^t D = \begin{pmatrix} 2 & 2 & -1 \\ 0 & 1 & 1 \end{pmatrix}$

$$B^t B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

E, $\mathbb{R}$ e-v, $u_3 = 2u_1 + u_2$ (13)

$F = \mathrm{Vect}\{u_1, u_2, u_3\}$. → $\dim F = 2$

On sait $\{u_1, u_2\}$ est libre. $u_1 \in \mathrm{Vect}\{u_2, u_3\}$.

$f : E \to F$ (AL).

$n > m$: f n'est pas injective.
$n < m$: f n'est pas surjective
$m = n$: f est bijective si f est injective.

$A = \begin{pmatrix} 1 & 2 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

$D = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$

$A^T = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$, $B^T = \begin{pmatrix} 0 & 1 & -1 \end{pmatrix}$, $C^T = \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 1 & 1 \end{pmatrix}$

$C^T D = \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix} = \begin{pmatrix} \cancel{1} & \cancel{1} \end{pmatrix}$

$3 \times 2 \times 3 \times 3$

$A^T \cdot C = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix} =$

$3 \times 1 \times 2 \times 3$.

(14) $A + B^T = \begin{pmatrix} 1 & 2 & 4 \end{pmatrix} + \begin{pmatrix} 0 & 1 & -1 \end{pmatrix}$
$= \begin{pmatrix} 1 & 3 & 3 \end{pmatrix}$

$B^T \cdot B = \begin{pmatrix} 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix}$

Inversibilité

$B = (\times)$

$C = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix}$ $CX = Y$?

$\begin{cases} x + y + z = a \\ x - z = b \\ x + y = c \end{cases} \Leftrightarrow \begin{cases} x = a - y - z \\ x = b + z \Leftrightarrow z = x - b \\ x = c - y \Leftrightarrow y = c - x \end{cases}$

$\begin{cases} x = a - c + x - x + b \\ x = a + b - c \end{cases}$ → $z = a - c$, $y = -a - b + 2c$

$C^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ -1 & -1 & 2 \\ 1 & 0 & -1 \end{pmatrix}$.

$D = \begin{pmatrix} -1 & 1 & -2 \\ 1 & 1 & 0 \\ 2 & -1 & 3 \end{pmatrix} \Leftrightarrow \begin{cases} -x + y - 2z = a \\ x + y = b \\ 2x - y + 3z = c \end{cases}$ $DX = Y$? $\Leftrightarrow \begin{cases} x = \\ y = \\ z = \end{cases}$

$D^{-1} = (\times)$

Examen M21

(A) $f_1 (x,y) \to (2x+y, x-y)$.

$f_2 (x,y,z) \mapsto (xy, x, y)$

$f_2((1,0,1) + (0,1,0)) = f(1,1,1) = (1,1,1)$

$f_2(1,0,1) + f(0,1,0) = (0,1,0) + (0,0,1)$

$f_3 (x,y,z) \longmapsto x^2 + y + 2z$?

$f_3(1,1,1) \longmapsto 1+1+2 = 4$.

$f((x,y,z) + (x',y',z')) = f(x+x', y+y', z+z')$

$f_3 = (x+x')^2 + (y+y') + 2(z+z')$
$= x^2 + x'^2 + 2xx'$

$f_4 (x,y,z) \longmapsto 2x + y - z + 1$

$f_4((x,y,z) + (x',y',z')) = f(x+x', y+y', z+z')$
$= 2(x+x') + (y+y') - (z+z') + 1$
$= 2x + y - z + 2x' + y' - z' + 1$.

(1) 2h 10'
5h 60'
1h 12'
3h 36'

(2) $E(x,y,t)$ $\begin{cases} x - y + z = 0 \\ y + z = 0 \end{cases}$

$\begin{cases} x = y - z \\ y = -z \end{cases} \Leftrightarrow \begin{cases} x = -2z \\ y = -z \end{cases}$ base de E

$E = \mathrm{Vect}((-2z, -z, z)) = z(-2, -1, 1)$
$\dim E = 1$.

$f(x) = \frac{1}{1+x} - 2e^{-x}$, $DL_3(0)$

$DL_3(0)$ $\frac{1}{1+x} = 1 - x + x^2 - x^3 + o(x^3)$

$e^u = 1 + u + \frac{u^2}{2} + \frac{u^3}{6} + o(x^3)$

$e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + o(x^3) = 2 - 2x + x^2 - \frac{x^3}{3}$.

$A - B = 1 - x + x^2 - x^3 - 2 + 2x - x^2 + \frac{x^3}{3}$
$= -1 + x - \frac{2}{3}x^3$

③

$$\begin{cases} x-y+z=1\\ 2x-3y+4z=1\\ x-2y+3z=1\end{cases} \qquad \left|\begin{array}{ccc|c}1&-1&1&1\\2&-3&4&1\\1&-2&3&1\end{array}\right.$$

$$\left|\begin{array}{ccc|c}1&-1&1&1\\0&1&-2&1\\0&-1&2&0\end{array}\right. \begin{matrix} 2L_1-L_2\\ L_3-L_1\end{matrix} \to \left|\begin{array}{ccc|c}1&-1&1&1\\0&1&-2&1\\0&0&0&1\end{array}\right. .$$

$$\begin{cases} x-y+z=1\\ y-2z=1\\ 0\neq 1\end{cases}$$

$$E\left\{(x,y,z,t) : \begin{cases} x-y+z=0\\ y-2t=0\end{cases}\right\}$$

$$\begin{cases} x=y-z\\ y=2t\end{cases} \Leftrightarrow \begin{cases} x=2t-z\\ y=2t\end{cases} \Leftrightarrow \begin{cases} z=2t-x\end{cases}$$

$$E = \mathrm{Vect}\big((2t-z,\ 2t,\ 2t-x,\ t) \big)$$

$$t(2,2,2,1) + z(-1,0,0,0) + x(0,0,-1,0)$$

④

$$E=\mathrm{Vect}\big((2t-z),\ 2t,\ z,\ t\big) = t(2,2,0,1)+z(-1,0,1,0)$$

$$u=(1,2,-1,3)$$

$$\begin{cases} 1-2-1\neq 0\\ 2-6\neq 0.\end{cases}$$

$$E\neq \mathbb{R}^4.$$

$$I=\int \frac{dx}{x.\ln x} \qquad u=\ln x \quad du=\frac{dx}{x}$$

$$I=\int\frac{du}{u} = \ln(u)=\ln(\ln x)$$

$$I_2=\int\frac{\ln x}{x}\,dx \qquad \tan=\frac{\sin}{\cos} \quad \tan^{-1}=\frac{\cos}{\sin}$$

$$\cancel{u=\ln x \to du=\frac{dx}{x}}$$

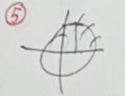
$$\cancel{v=\frac{dx}{x}\to v=-\frac{1}{x^2}.}$$

$$\cancel{I_2=-\frac{\ln x}{x^2}-\int -\frac{1}{x^2}\times\frac{dx}{x}}$$

$$\cancel{I_2=-\frac{\ln x}{x^2}+\int\frac{dx}{x^3}}$$

$$I_2=\int u\,du=\frac{1}{2}u^2=\frac{1}{2}\ln^2 x.$$

⑤



⑥

$$\int_0^{\pi/3}\sin x.\,e^{\cos x}\,dx \qquad u=\cos x \quad du=-\sin x\,dx \quad -du=\sin x\,dx$$

$$-\int_{\cos(0)}^{\cos(\frac{\pi}{3})} e^u\,du = \int_{1/2}^{1} e^u\,du = +e^u = \left[e^u\right]_{1/2}^{1}$$

$$=\left[e^{\cancel{\cos x}\,u}\right]_{1/2}^{1} = e^1 - e^{\frac{1}{2}} = e-\sqrt{e}.$$

$$\int_{\pi/6}^{\pi/4}\frac{dx}{\sin x\ \tan x}=? \qquad t=\sin x \quad dt=\cos x\,dx$$

$$I=\int \frac{dt}{\cos x}\times\frac{1}{t}\times\frac{\cos}{t} = \int_{\sin(\frac{\pi}{6})=\frac{1}{2}}^{\sin(\frac{\pi}{4})=\frac{\sqrt{2}}{2}} \frac{dt}{t^2} = \left[\frac{-1}{t}\right]_{1/2}^{\frac{\sqrt{2}}{2}}$$

$$I=-\frac{2}{\sqrt{2}}+2=2-\sqrt{2}.$$

E, un $\mathbb{R}$ ev, $u_1, u_2, u_3 \in E$ tq ⑦

$u_3 = 2u_1 + u_2$

$F = \text{Vect}\{u_1, u_2, u_3\}$

On suppose que $\{u_1, u_2\}$ est libre.

$\dim F = 2$ $\qquad -\frac{1}{2} + \frac{1}{3}$

$u_1 \in \text{Vect}\{u_2, u_3\}$ $\qquad -\frac{3}{6} + \frac{2}{6}$

Le système $\{u_1, u_2\}$ est une base de F.

$f(x) = \ln(2 + 2x + x^2) = \ln(1 + (x+1)^2)$ $\quad DL_3(0)\, f(x)$

$\ln\left(2\left(1 + x + \frac{x^2}{2}\right)\right) = \ln(2) + \ln\left(1 + x + \frac{x^2}{2}\right)$

$\ln(1+u) = u - \frac{u^2}{2} + \frac{u^3}{3} + o(u^3)$ $\quad$ "u"

$\ln\left(1 + x + \frac{x^2}{2}\right) = \left(x + \frac{x^2}{2}\right) - \frac{1}{2}\left(x + \frac{x^2}{2}\right)^2 + \frac{1}{3}\left(x + \frac{x^2}{2}\right)^3 + o(x^3)$

$= x + \cancel{\left(\frac{x^2}{2}\right)} - \cancel{\frac{1}{2}\left(x^2\right.} + \left.x^3\right) = x - \frac{1}{2}x^3 + \frac{1}{3}x^3$

$+ \ln(2) = x - \frac{x^3}{6} + o(x^3)$

⑧ $T : y = \ln(2) + x$

$f(x) - y$

$= \ln(2 + 2x + x^2) - \ln(2) - x$

→ f traverse sa tangente au point $(0, \ln(2))$ car $a_0 + a_1(x - x_0)^p$

p : impair.

⑨ ⑩

$\int e^{-x} \cos x \, dx$

$u = \cos x \rightarrow du = -\sin x \, dx$

$dv = e^{-x} dx \rightarrow v = -e^{-x}$

$I = -e^{-x} . \cos x + \int e^{-x} (-\sin x \, dx)$

$u = -\sin x \longrightarrow du = -\cos x \, dx$

$dv = e^{-x} dx \longrightarrow v = -e^{-x}$

$I = -e^{-x} . \cos x - e^{-x}(-\sin x) + \int e^{-x} (-\cos x) dx$

$= -e^{-x} . \cos x + e^{-x} . \sin x - \int e^{-x} . \cos x \, dx$

$\int e^{-x} \cos x \, dx = \frac{1}{2}\left(e^{-x}(-\cos x + \sin x)\right)$

□