Calcul de $\cos(a+b)$

$$\begin{aligned}\cos(a+b) &= \Re\left(e^{i(a+b)}\right)\\ &= \Re\left(e^{ia}e^{ib}\right)\\ &= \Re((\cos a + i\sin a)(\cos b + i\sin b))\\ &= \Re(\cos a\cos b + i\cos a\sin b + i\sin a\cos b - \sin a\sin b)\\ &= \cos a\cos b - \sin a\sin b\end{aligned}$$

On en déduit accessoirement que $\cos(a-b) = \cos a\cos(-b) - \sin a\sin(-b)$

$= \cos a\cos b + \sin a\sin b$

Calcul de $\sin(a+b)$

$$\begin{aligned}\sin(a+b) &= \Im\left(e^{i(a+b)}\right)\\ &= \Im\left(e^{ia}e^{ib}\right)\\ &= \Im((\cos a + i\sin a)(\cos b + i\sin b))\\ &= \Im(\cos a\cos b + i\cos a\sin b + i\sin a\cos b - \sin a\sin b)\\ &= \cos a\sin b + \sin a\cos b\end{aligned}$$

On en déduit accessoirement que $\sin(a-b) = \cos a\sin(-b) + \sin a\cos(-b)$

$= \sin a\cos b - \cos a\sin b$

Calcul de $\tan(a+b)$

$$\tan(a+b) = \frac{\sin(a+b)}{\cos(a+b)} = \frac{\cos a\sin b + \sin a\cos b}{\cos a\cos b - \sin a\sin b}$$

Calcul de $\tan(a-b)$

$$\tan(a-b) = \frac{\sin(a-b)}{\cos(a-b)} = \frac{\sin a\cos b - \cos a\sin b}{\cos a\cos b + \sin a\sin b}$$

Calcul de $\sin a + \sin b$

On sait que $\sin(a+b) = \cos a\sin b + \sin a\cos b$ et

$\sin(a-b) = -\cos a\sin b + \sin a\cos b$

En sommant les deux égalités, on obtient

$$\begin{aligned}\sin(a+b) + \sin(a-b) &= \cos a\sin b + \sin a\cos b - \cos a\sin b + \sin a\cos b\\ &= 2\sin a\cos b\end{aligned}$$

En posant $a+b = A$ et $B = a-b = A-2b$ on arrive à

$$\begin{cases} a+b = A\\ a-b = B\end{cases} \Longrightarrow \begin{cases} a+b = A\\ 2a = A+B\end{cases} \Longrightarrow a = \frac{A+B}{2}, b = A - \frac{A+B}{2} = \frac{A-B}{2}$$

$$\begin{aligned}\sin A + \sin B &= 2\sin a\cos b\\ &= 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}\end{aligned}$$

On en déduit accessoirement que $\sin a - \sin b = \sin a + \sin(-b) = 2\sin\frac{a-b}{2}\cos\frac{a+b}{2}$

Calcul de $\cos a + \cos b$

On sait que $\cos(a+b) = \cos a \cos b - \sin a \sin b$ et $\cos(a-b) = \cos a \cos b + \sin a \sin b$

En sommant les deux égalités, on obtient

$$\begin{aligned}\cos(a+b) + \cos(a-b) &= \cos a \cos b - \sin a \sin b + \cos a \cos b + \sin a \sin b \\ &= 2 \cos a \cos b\end{aligned}$$

En posant $a+b = A$ et $B = a-b = A-2b$ on arrive à

$$\begin{cases} a+b = A \\ a-b = B \end{cases} \Longrightarrow \begin{cases} a+b = A \\ 2a = A+B \end{cases} \Longrightarrow a = \frac{A+B}{2}, b = A - \frac{A+B}{2} = \frac{A-B}{2}$$

$$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$$

Calcul de $\cos a - \cos b$

On sait que $\cos(a+b) = \cos a \cos b - \sin a \sin b$ et $\cos(a-b) = \cos a \cos b + \sin a \sin b$

En soustrayant $\cos(a-b)$ à $\cos(a+b)$ on obtient

$$\begin{aligned}\cos(a+b) - \cos(a-b) &= \cos a \cos b - \sin a \sin b - \cos a \cos b - \sin a \sin b \\ &= -2 \sin a \sin b\end{aligned}$$

En posant $a+b = A$ et $B = a-b = A-2b$ on arrive à

$$\begin{cases} a+b = A \\ a-b = B \end{cases} \Longrightarrow \begin{cases} a+b = A \\ 2a = A+B \end{cases} \Longrightarrow a = \frac{A+B}{2}, b = A - \frac{A+B}{2} = \frac{A-B}{2}$$

$$\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}$$

Calcul de $\tan a + \tan b$

$$\begin{aligned}\tan a + \tan b &= \frac{\sin a}{\cos a} + \frac{\sin b}{\cos b} \\ &= \frac{\sin a \cos b + \cos a \sin b}{\cos a \cos b} \\ &= \frac{\sin(a+b)}{\cos a \cos b}\end{aligned}$$

Calcul de $\tan a - \tan b$

$$\begin{aligned}\tan a - \tan b &= \frac{\sin a}{\cos a} - \frac{\sin b}{\cos b} \\ &= \frac{\sin a \cos b - \cos a \sin b}{\cos a \cos b} \\ &= \frac{\sin(a-b)}{\cos a \cos b}\end{aligned}$$

Calcul de $\cos(\arcsin x)$

$$\begin{aligned}\cos^2(x)+\sin^2(x)=1 &\Longrightarrow \cos^2(\arcsin x)+\sin^2(\arcsin x)=1, x\in[-1;1]\\ &\Longrightarrow \cos^2(\arcsin(x))=1-\sin^2(\arcsin x)\\ &\Longrightarrow \cos(\arcsin x)=\sqrt{1-x^2}\end{aligned}$$

Sur le même principe, on calcule $\sin(\arccos x)$:

$$\begin{aligned}\cos^2(x)+\sin^2(x)=1 &\Longrightarrow \cos^2(\arccos x)+\sin^2(\arccos x), x\in[-1;1]\\ &\Longrightarrow \sin^2(\arccos x)=1-\cos^2(\arccos x)\\ &\Longrightarrow \sin(\arccos x)=\sqrt{1-x^2}\end{aligned}$$

Calcul de $\tan(\arccos x)$ (donc d'office sur $[-1;1]$)

$$\tan(\arccos x)=\frac{\sin\arccos x}{\cos\arccos x}=\frac{\sqrt{1-x^2}}{x}$$

Même principe, calcul de $\tan(\arcsin x)$ (sur $[-1;1]$)

$$\tan(\arcsin x)=\frac{\sin\arcsin x}{\cos\arcsin x}=\frac{x}{\sqrt{1-x^2}}$$

Calcul de $\sin(\arctan x)$

On pose $\sin(\arctan x)=u$,

$$\begin{aligned}\sin(\arctan x)=u &\Longleftrightarrow \arctan x=\arcsin u\\ &\Longleftrightarrow x=\tan(\arcsin u)\\ &\Longleftrightarrow x=\frac{u}{\sqrt{1-u^2}}\\ &\Longrightarrow x^2=\frac{u^2}{1-u^2}\\ &\Longrightarrow x^2=-1+\frac{1}{1-u^2}\\ &\Longrightarrow \frac{1}{x^2+1}=1-u^2\\ &\Longrightarrow 1-\frac{1}{1+x^2}=u^2\\ &\Longrightarrow u^2=\frac{x^2}{1+x^2}\\ &\Longrightarrow u=\frac{x}{\sqrt{1+x^2}}\end{aligned}$$

Calcul de $\cos(\arctan x)$

On pose $\cos(\arctan x)=u$,

$$\begin{aligned}
\cos(\arctan x) = u &\iff \arctan x = \arccos u \\
&\iff x = \tan(\arccos u) \\
&\iff x = \frac{\sqrt{1-u^2}}{u} \\
&\implies x^2 = \frac{1-u^2}{u^2} \\
&\implies x^2 = \frac{1}{u^2} - 1 \\
&\implies u^2 = \frac{1}{1+x^2} \\
&\implies u = \frac{1}{\sqrt{1+x^2}}
\end{aligned}$$

Calcul de $\frac{d}{dx}\arccos(f(x))$

On sait que $\frac{d}{dx}f(g(x)) = g'(x)f'(g(x))$. De plus, $f(x) = \cos(\arccos f(x))$.

Donc

$$\begin{aligned}
\frac{d}{dx}f(x) = \frac{d}{dx}\cos(\arccos f(x)) &\implies f'(x) = -\arccos{}'(f(x))\sin(\arccos f(x)) \\
&\implies \arccos{}'(f(x)) = -\frac{f'(x)}{\sqrt{1-f(x)^2}}
\end{aligned}$$

Jusque là on a le tableau d'égalité suivant :

Valeur	**Égale à**	**Remarque**
$\cos(a+b)$	$\cos a\cos b - \sin a\sin b$	
$\cos(a-b)$	$\cos a\cos b + \sin a\sin b$	
$\sin(a+b)$	$\cos a\sin b + \sin a\cos b$	
$\sin(a-b)$	$\sin a\cos b - \cos a\sin b$	
$\tan(a+b)$	$\frac{\cos a\sin b + \sin a\cos b}{\cos a\cos b - \sin a\sin b}$	
$\tan(a-b)$	$\frac{\sin a\cos b - \cos a\sin b}{\cos a\cos b + \sin a\sin b}$	

$\cos a + \cos b$	$2\cos\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$	
$\cos a - \cos b$	$-2\sin\dfrac{A+B}{2}\sin\dfrac{A-B}{2}$	
$\sin a + \sin b$	$2\sin\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$	
$\sin a - \sin b$	$2\sin\dfrac{a-b}{2}\cos\dfrac{a+b}{2}$	
$\tan a + \tan b$	$\dfrac{\sin(a+b)}{\cos a\cos b}$	
$\tan a - \tan b$	$\dfrac{\sin(a-b)}{\cos a\cos b}$	
$\cos(\arcsin x)$	$\sqrt{1-x^2}$	$x \in [-1;1]$
$\cos(\arctan x)$	$\dfrac{1}{\sqrt{1+x^2}}$	
$\sin(\arccos x)$	$\sqrt{1-x^2}$	$x \in [-1;1]$
$\sin(\arctan x)$	$\dfrac{x}{\sqrt{1+x^2}}$	
$\tan(\arcsin x)$	$\dfrac{x}{\sqrt{1-x^2}}$	$x \in [-1;1]$
$\tan(\arccos x)$	$\dfrac{\sqrt{1-x^2}}{x}$	$x \in [-1;1]$
$\arccos'(f(x))$	$-\dfrac{f'(x)}{\sqrt{1-f(x)^2}}$	$f(x) \in [-1;1]$