Feuille 3. Exercice 6 (Utilisation de Fubini).

Calculez $\iint_D f(x,y)\,dx\,dy$ dans les cas suivants.

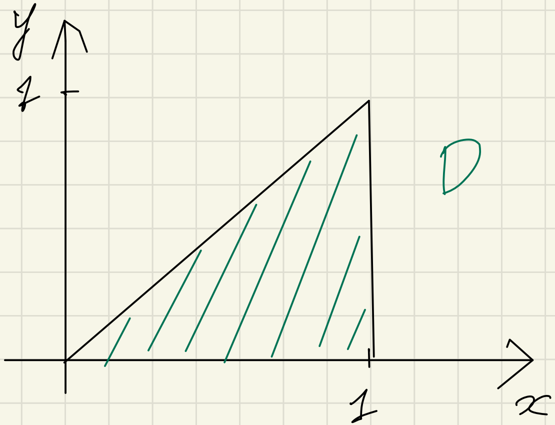
(1) $f(x,y) = xy^2$ $\quad D = [0,1] \times [1,2]$

$$\int_0^1 \int_1^2 xy^2 \, dy \, dx = \int_0^1 x \left(\int_1^2 y^2 dy \right) dx$$

$$= \left(\int_0^1 x \, dx \right) \left(\int_1^2 y^2 dy \right)$$

$$= \frac{1}{2} \times \frac{1}{3} \left[2^3 - 1^3 \right] = \frac{7}{6}$$

(2) $f(x,y) = x^2 y^3$, $D = \{(x,y) \in [0,1]^2 : y \le x\}$



$$D = \left\{(x,y) : \begin{array}{l} 0 \le x \le 1 \\ 0 \le y \le x \end{array}\right\}$$

d'où

$$\iint_D f(x,y)\,dy\,dx = \int_0^1 \left(\int_0^x f(x,y)\,dy\right) dx = \int_0^1 x^2 \int_0^x y^3\,dy\,dx$$

$$= \int_0^1 x^2 \,\frac{x^4}{4}\,dx = \frac{1}{28}$$

(3) $f(x,y) = x^2/y$ $\qquad D = [-1,1] \times [1,2]$

$$\iint_D f(x,y)\,dx\,dy = \left(\int_{-1}^{1} x^2\,dx\right)\left(\int_1^2 \frac{dy}{y}\right) = \frac{2}{3}\ln 2$$

(4) $f(x,y) = \sin(x+y)$ $\qquad D = [0,\frac{\pi}{2}]^2$.

On écrit $\sin(x+y) = \sin x \cos y + \sin y \cos x$

$$\iint_D f(x,y)\,dx\,dy = 2\left(\int_0^{\frac{\pi}{2}} \sin x\,dx\right)\left(\int_0^{\frac{\pi}{2}} \cos y\,dy\right) = 2$$

(5) $f(x,y) = \dfrac{x}{\sqrt{1+xy+x^2}}$ $\qquad D = [3,7] \times [-2,2]$

Fix once $x \in [3, 7]$

$$\int_{-2}^{2} f(x,y)\,dy = \int_{-2}^{2} \frac{x\,dy}{\sqrt{1 + x^2 + xy}}$$

$$= \left[2\sqrt{1 + x^2 + xy} \right]_{y=-2}^{y=2}$$

$$= 2\left(\sqrt{1 + x^2 + 2x} - \sqrt{1 + x^2 - 2x} \right)$$

$$= 2\left(\sqrt{(1+x)^2} - \sqrt{(1-x)^2} \right)$$

Comme $x \geq 1$ $\begin{cases} \sqrt{(1+x)^2} = 1+x \\ \sqrt{(1-x)^2} = x-1 \end{cases}$

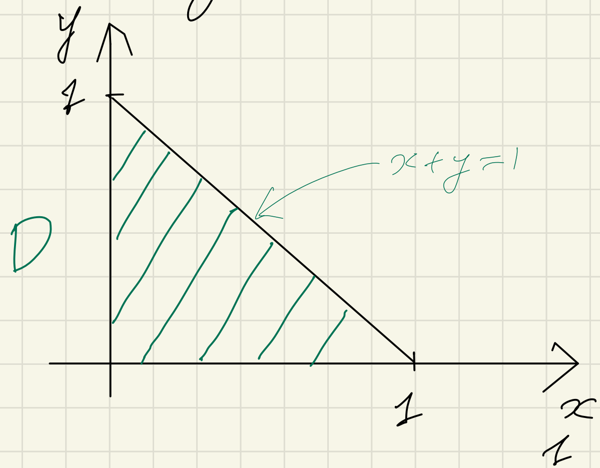
d'où pour $x \in [3, 7]$

$$\int_{-2}^{2} f(x,y)\,dy = 4$$

Donc $\displaystyle\iint_D f(x,y)\,dy = \int_3^7 4\,dx = 16.$

Exercice 8 (Utilisation de Fubini)

$$D = \{ (x,y) \in \mathbb{R}^2 : x \geq 0 \quad y \geq 0 \quad x+y \leq 1 \}$$



Pour intégrer sur D, on utilise la représentation:

$$D = \{ (x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1 \quad 0 \leq y \leq 1-x \}$$

(a) Calculer $I_1 = \iint_D dx\,dy$

$$I_1 = \int_0^1 \left(\int_0^{1-x} 1\,dy \right) dx = \int_0^1 (1-x)\,dx = 1 - \frac{1}{2} = \frac{1}{2}$$

(b) Calculez $I_2 = \iint_D (x^2+y^2)\,dx\,dy$

Par symétrie $I_2 = 2\iint_D x^2\,dx\,dy = 2\int_0^1 x^2 \int_0^{1-x} dy\,dx$

$$= 2\int_0^1 x^2(1-x)\,dx = 2\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{6}$$

(c) Calculer $I_3 = \iint_D xy(x+y)\,dx\,dy$

$$I_3 = 2\int_0^1 x^2 \left(\int_0^{1-x} y\,dy \right) dx = 2\int_0^1 x^2 \, \frac{1}{2}(1-x)^2\,dx$$

$$= \int_0^1 x^2\,dx - 2\int_0^1 x^3\,dx + \int_0^1 x^4\,dx = \left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right) = \frac{1}{30}.$$

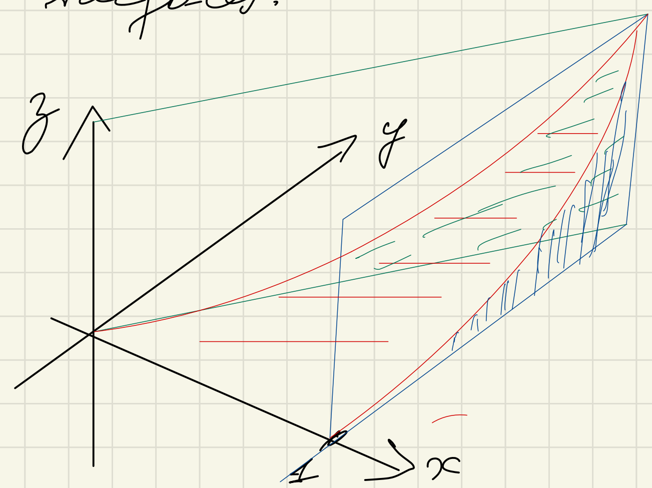
Exercice 9 (Intégrales triples)

Calculez $\iiint_V f(x,y,z)\,dx\,dy\,dz$ dans les cas suivants.

(1) $f(x,y,z) = x y^2 z^3$ où $V \subset \mathbb{R}^3$ est le domaine borné délimité par les surfaces.

$\{z = xy\}$ $\{x = y\}$

$\{x = 1\}$ $\{z = 0\}$



On écrit

$$V = \{ (x,y,z) : 0 \leq x \leq 1 , \quad 0 \leq y \leq 1-x , \quad 0 \leq z \leq xy \}$$

$$\iiint_V f(x,y,z)\, dz\, dy\, dx = \int_0^1 x \left(\int_0^{1-x} y^2 \left(\int_0^{xy} z^3\, dz \right) dy \right) dx$$

$$= \frac{1}{4} \int_0^1 x \left(\int_0^{1-x} y^2 \left(x^4 y^4 \right) dy \right) dx$$

$$= \frac{1}{4} \int_0^1 x^5 \left(\int_0^{1-x} y^6\, dy \right) dy = \frac{1}{28} \int_0^1 x^5 (1-x)^7\, dx$$

$$= \frac{1}{28} \left(\int_0^1 x^5 - 7x^6 + 21x^7 - 35x^8 + 35x^9 - 21x^{10} + 7x^{11} - x^{12} \right) dx$$

(2) $f(x,y,z) = (1+x+y+z)^{-3}$ $\quad V$ domaine borné

par les surfaces $\quad x+y+z=1$

$x=0 \quad y=0 \quad z=0$

$$V = \{ (x,y,z) : \ 0 \leq x \leq 1, \ \ 0 \leq y \leq 1-x, \ \ 0 \leq z \leq 1-x-y \}$$

$$Q_2 := \iiint_V f(x,y,z)\, dx\, dy\, dz$$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{dz}{(1+x+y+z)^3} \, dy \, dx$$

$$= \frac{1}{2} \int_0^1 \int_0^{1-x} \left(\frac{1}{(1+x+y)^2} - \frac{1}{2^2} \right) dy \, dx$$

D'où

$$Q_2 = \frac{1}{2}\left(\int_0^1 \left(\frac{1}{1+x} - \frac{1}{2}\right) dx - \frac{1}{4}\int_0^1 (1-x)\, dx\right)$$

$$= \frac{1}{2}\left(\ln(2) - \frac{1}{2} - \frac{1}{4} + \frac{1}{8}\right)$$

$$= \frac{1}{2}\left(\ln 2 - \frac{5}{8}\right)$$

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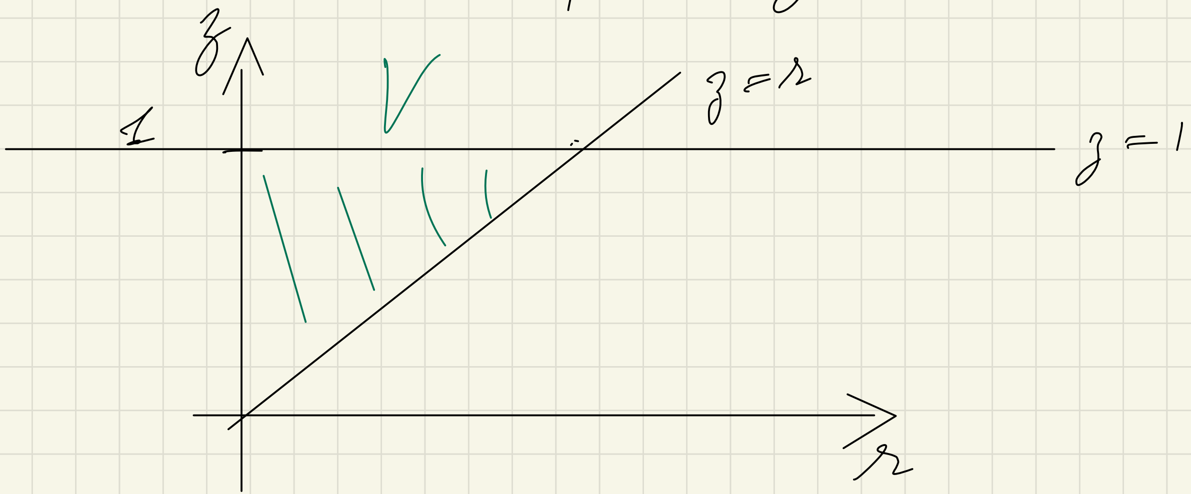
(3) $f(x,y,z) = \sqrt{x^2+y^2}$ V est le domaine borné délimité par $x^2+y^2 = z^2$ et $z = 1$.

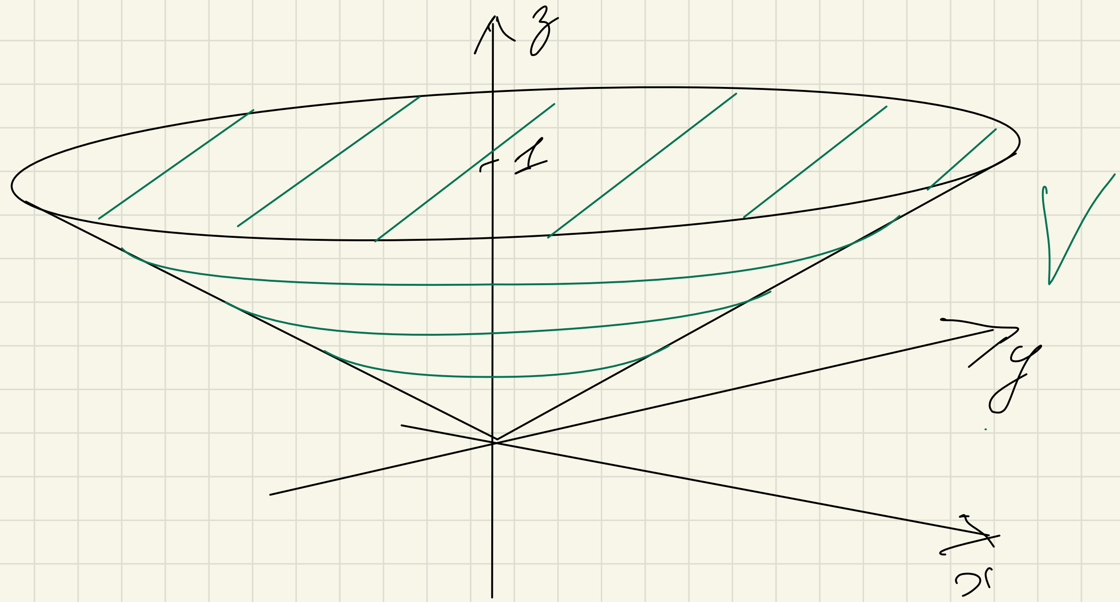
On note $r = \sqrt{x^2+y^2}$. En coordonnées cylindriques,

V est délimité par $z = \pm r$ et $z = 1$



V est donc le cône tronqué.

$$\left\{ (x,y,z) \in \mathbb{R}^3 : \quad 0 \leq z \leq 1 \quad \sqrt{x^2+y^2} \leq z \right\}$$



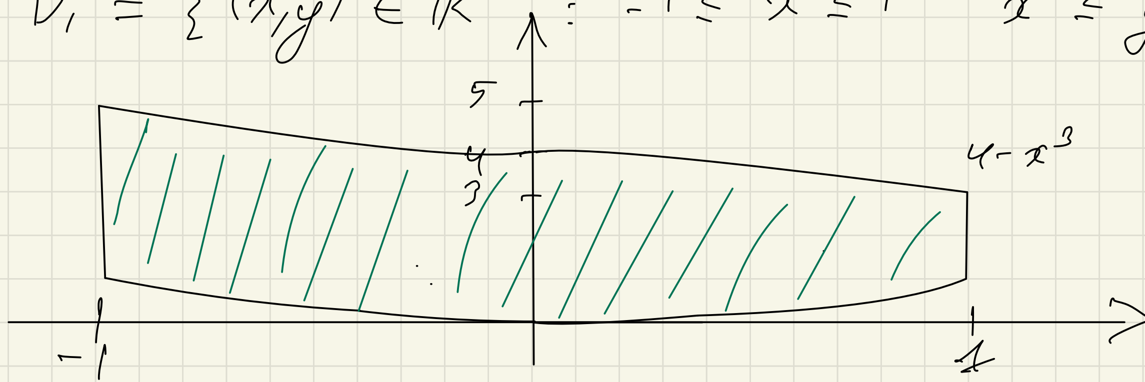
En utilisant les coordonnées polaires $dx\,dy = r\,d\theta\,dr$

$$Q_3 = \iiint f(x,y,z)\,dx\,dy\,dz = \int_0^1 \left(\int_0^z \left(\int_0^{2\pi} d\theta \right) r^2\,dr \right) dz$$

$$= \frac{2\pi}{3} \int_0^1 z^3\,dz = \frac{\pi}{6}.$$

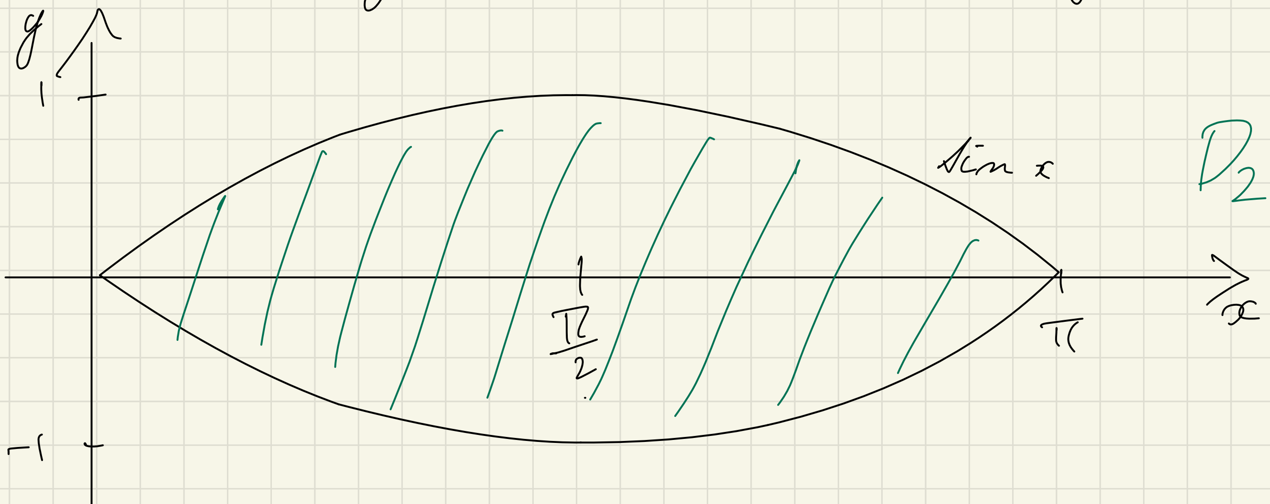
__Exercice 10__ Calculer les aires des domaines suivants.

(i) $D_1 = \{(x,y) \in \mathbb{R}^2 : -1 \leq x \leq 1 \quad x^2 \leq y \leq 4 - x^3\}$



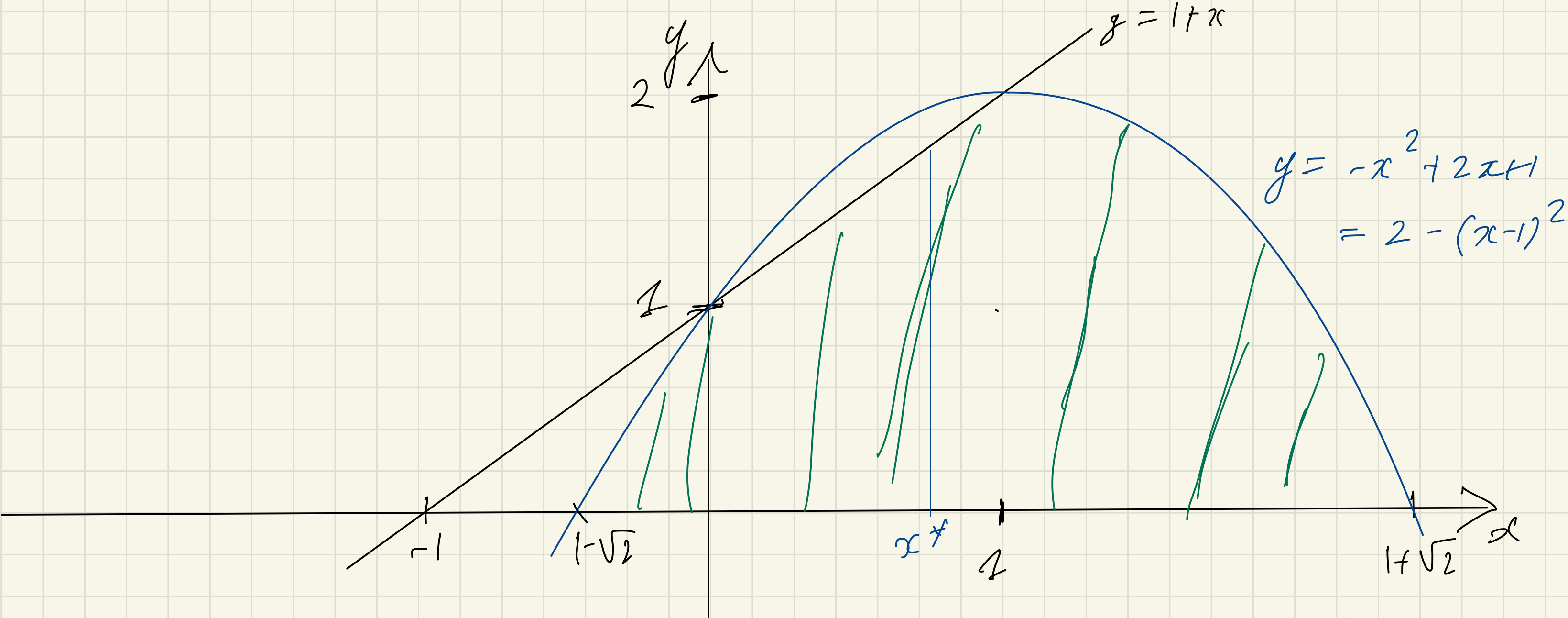
$$|D_1| = \int_{-1}^{1} (4 - x^3)\, dx - \int_{-1}^{1} x^2\, dx = 8 - \frac{2}{3} = \frac{22}{3}$$

(ii) $D_2 = \{ (x,y) \in \mathbb{R}^2 : 0 \leq x \leq \pi, \ |y| \leq \sin x \}$



$$|D_2| = 2 \int_0^{\pi} \sin x \, dx = 4.$$

(iii) $D_3 = \{(x,y) \in \mathbb{R}^2 : \ y \geq 0, \ y \leq 1+x, \ y \leq 1+2x-x^2$



$\left\{ 1+x = 1+2x-x^2 \iff x = 1 \right.$ ou $x = 0$ $\Big|$ $2-(x-1)^2 = 0 \iff x = 1+\sqrt{2}$ ou $x = 1-\sqrt{2}$

D_3 se décompose en 3 sous domaines

$\{-\sqrt{2}+1 \leq x \leq 0 \quad 0 \leq y \leq 1+2x-x^2\}$; $\{0 \leq x \leq 1 ; 0 \leq y \leq 1+x\}$;

et $\{1 \leq x \leq 1+\sqrt{2}, \quad 0 \leq y \leq 1+2x-x^2\}$.

$$|D_3| = \int_{1-\sqrt{2}}^{0} (1+2x-x^2)\,dx + \int_0^1 (1+x)\,dx + \int_1^{1+\sqrt{2}} (1+2x-x^2)\,dx$$

$$= \sqrt{2}-1 - (\sqrt{2}-1)^2 - \frac{(\sqrt{2}-1)^3}{3} + \frac{3}{2} + \sqrt{2} + (1+\sqrt{2})^2 - 1 - \frac{(1+\sqrt{2})^3-1}{3}$$

$$= \sqrt{2} \cancel{-1} - \cancel{2} + 2\sqrt{2} - 1 - \frac{1}{3}\left(2\sqrt{2} - \cancel{6} + 3\sqrt{2} - 1\right) + \frac{3}{2} + \sqrt{2}$$
$$+ \cancel{1} + 2\sqrt{2} \cancel{+2} - 1 - \frac{1}{3}\left(2\sqrt{2} \cancel{+6} + 3\sqrt{2} + \cancel{1} \cancel{-1}\right)$$

$$= \sqrt{2}\left(6 - \frac{10}{3}\right) + \left(-2 + \frac{1}{3} + \frac{3}{2}\right) = \frac{8\sqrt{2}}{3} - \frac{1}{6}$$

Exercice 11 Calculer le volume du domaine borné de $\mathbb{R}^3$ V délimité par les surfaces suivantes

(1) $\{z = x^2+y^2\}$, $\{z = 2(x^2+y^2)\}$ $\{y = x\}$; $\{y = x^2\}$

$$V = \{(x,y,z) : 0 \leq x \leq 1,\ x^2 \leq y \leq x,\ x^2+y^2 \leq z \leq 2(x^2+y^2)\}$$

$$|V| = \int_0^1 \left[\int_{x^2}^{x} \left(\int_{x^2+y^2}^{2(x^2+y^2)} dz\right) dy\right] dx$$

$\cancel{3 \times 4} \times 5 \times 7$

$\frac{1}{420}$ $\cancel{3 \times 2 \times 2} \times 3$

$$= \int_0^1 \left[\int_{x^2}^{x} (x^2+y^2)\, dy\right] dx = \int_0^1 x^2(x - x^2) + \frac{1}{3}(x^3 - x^6)\, dx$$

$$= \frac{1}{4} - \frac{1}{5} + \frac{1}{3}\left(\frac{1}{4} - \frac{1}{7}\right) = \frac{3}{35}.$$

(2) $\{z = x+y\}$ $\{z = xy\}$ $\{x+y = 1\}$ $\{x=0\}$ $\{y=0\}$

Tout d'abord dans V, on a $x, y \geq 0$ $x+y \leq 1$

et donc $0 \leq x, y \leq 1$

Pour de tels x, y on a :

$$xy \leq \frac{1}{2}(x^2 + y^2) \leq \frac{1}{2}(x+y) \leq x+y$$

D'où

$$V = \{(x,y,z) : 0 \leq x \leq 1 \quad 0 \leq y \leq 1-x \\ xy \leq z \leq x+y\}$$

$$|V| = \int_0^1 \left(\int_0^{1-x} (x + y - xy)\, dy \right) dx$$

$$= \int_0^1 \left[x(1-x) + \frac{(1-x)^3}{2} \right] dx = \frac{1}{2} - \frac{1}{3} + \frac{1}{2} \times \frac{1}{4}$$

$$= \frac{7}{24}.$$