Shape Analysis Through Diffeomorphisms

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Shape Analysis and Computational Anatomy

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  - **Comparing shapes** in the space, e.g., by finding deformations as above that requires the least "energy", so that bigger variations of shapes require higher energy.
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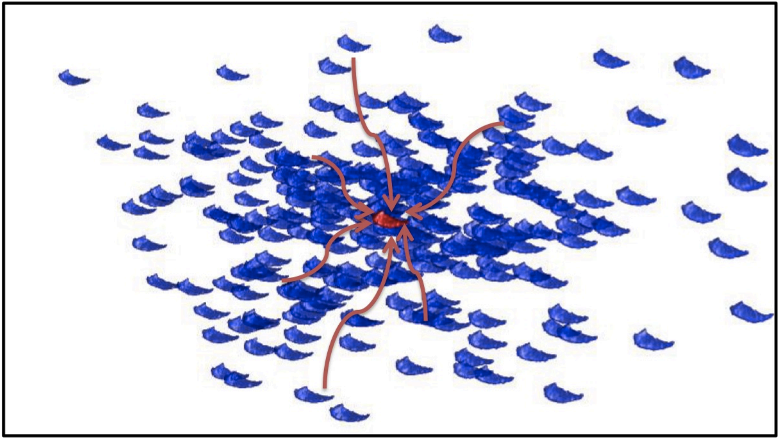
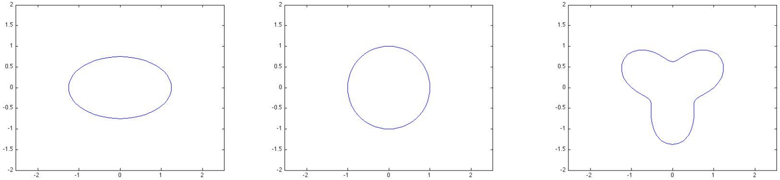
- Each of these steps requires some form of **shape registration**: finding a certain deformation from one shape onto another.

Application

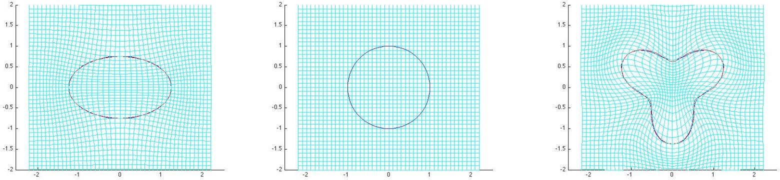


Shape registration

Goal: compare shapes while taking into account their geometric properties.



Idea: Use **diffeomorphisms**: deformations of the ambient space that preserve local and global geometric properties. The more different two shapes are, the more deformation is needed to map one close to the other.



How to build diffeomorphisms

For $t \in [0, 1]$, velocity field $v(t) : \mathbb{R}^d \to \mathbb{R}^d$. The position $x(t) \in \mathbb{R}^d$ at time t of a particle that moves along this velocity field is described by

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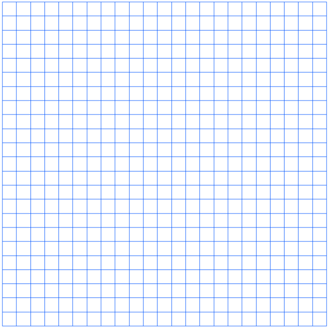
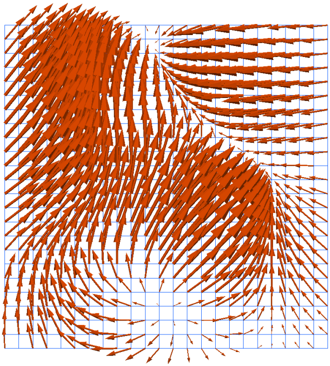
As long as we take $v(t)$ "very regular" with respect to the space variables, the transformation will be a **diffeomorphism**: it will map smooth curves onto smooth curves, corners onto corners, and preserve presence or lack of self-intersection points.



A controller specifies a direction at every point



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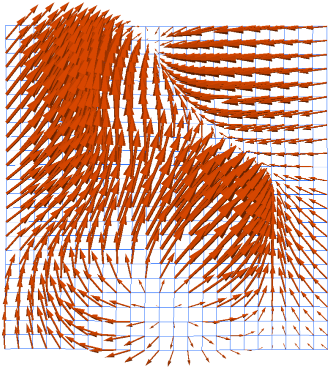

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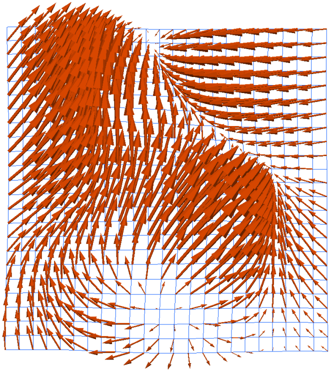

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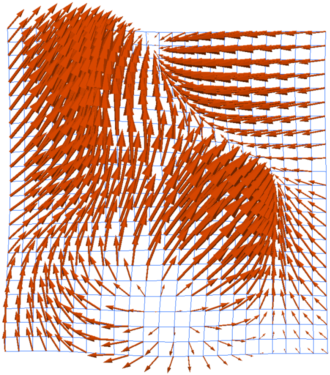

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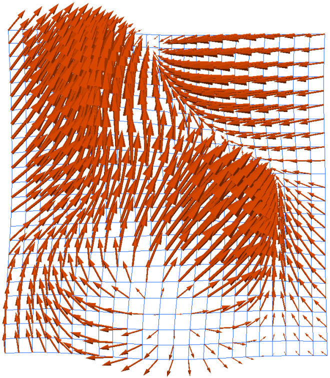
Controlling Diffeomorphisms with Vector Fields



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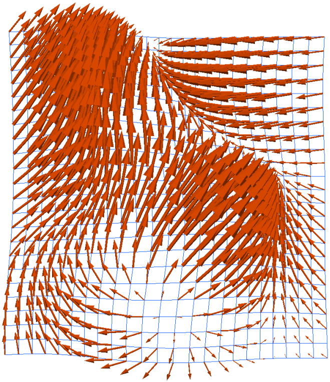

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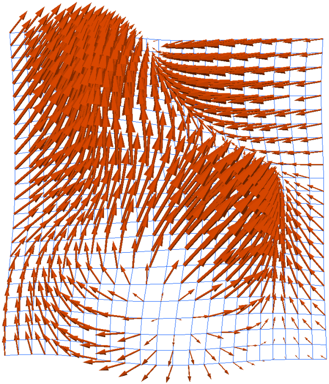

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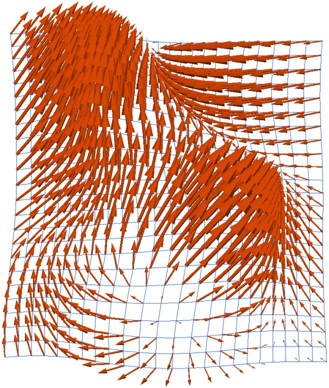

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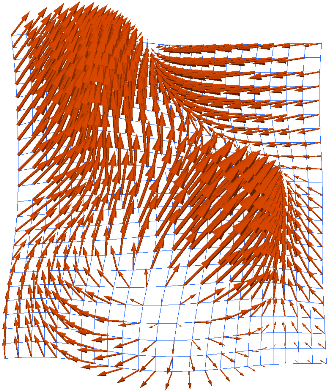

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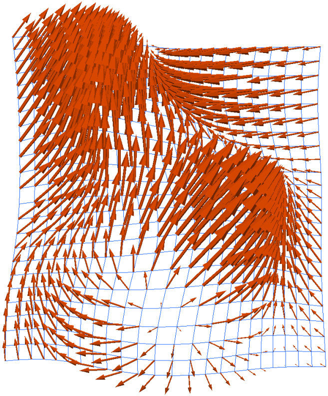

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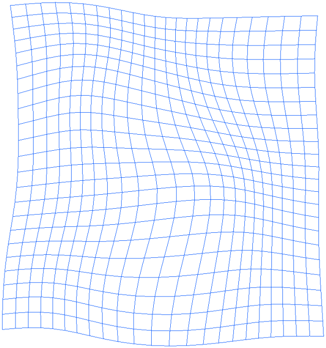


Final Grid

Energy of Deformation for Shape Registration

- Fix a shape q_0, the **template**, from which we want to register another shape q_1 (the **target**).

Energy of Deformation for Shape Registration

- Fix a shape q_0, the **template**, from which we want to register another shape q_1 (the **target**).
- A time-dependent velocity field $(t,x) \mapsto v(t,x)$ yields a deformation $(t,x) \mapsto \varphi(t,x)$, which acts onto q_0 as denoted by $q(t) := \varphi(t) \cdot q_0$. The goal is now to find v^* which minimizes a functional

$$J(v) = \frac{1}{2} \int_0^1 \|v(t)\|_V^2 dt + g(q(1)),$$

where $\|\cdot\|_V$ is an appropriate Hilbert norm (for instance, one can take a sufficiently smooth Sobolev norm). The data atachment $g(q(1))$ is a crude measure of the difference between the deformed shape $q(1)$ and the target q_1.

Reduction

Proposition

Assume the shape q_0 can be described by a finite family $q_0 = (x_1, \dots, x_n) \in (\mathbb{R}^d)^n$. For example, q_0 is a triangulated surface. Then there is a certain appropriate $\|\cdot\|_V$ such that for each time t, $v^*(t)$ is a sum of Gaussian vector fields centered at each x_i with fixed variance, that is,

$$v^*(t,x) = \sum_{i=1}^{n} p_i(t) e^{-\frac{|x - x_i(t)|^2}{\sigma^2}}, \quad p_i(t) \in \mathbb{R}^d, \sigma \in \mathbb{R}_+^*.$$

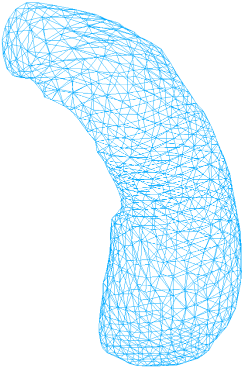
In this case,

$$\|v^*(t)\|_V^2 = \sum_{i,j=1}^{n} p_i(t)^T p_j(t) e^{-\frac{|x_i(t) - x_j(t)|^2}{\sigma^2}}.$$

We call $p(t)$ the **momentum** *of the deformation at time t.*

We can simply work on $t \mapsto (p_1(t), \dots, p_n(t))$: finite dimensional control system. This finite dimensional reduction can be performed for more general Hilbert norms $\|\cdot\|_V$, although the formula for v^* would be slightly different.

Controlling Diffeomorphisms with Vector Fields: Reduced form



The control is specified at every point of the surface, then interpolated using the kernel to the whole space.

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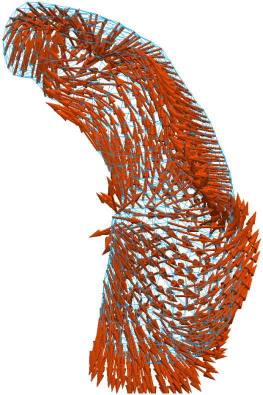

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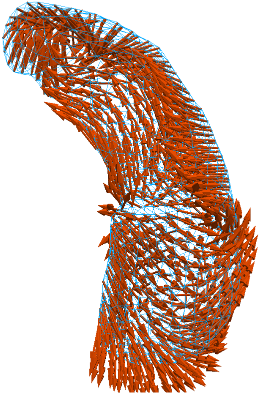

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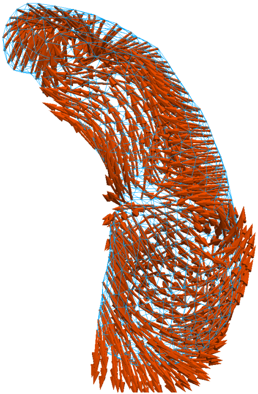

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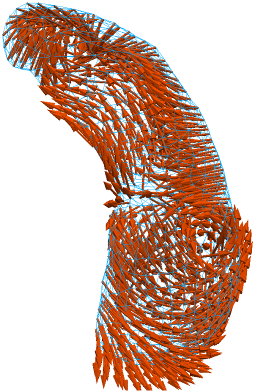

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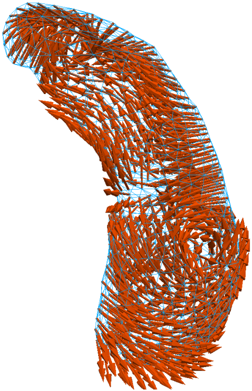

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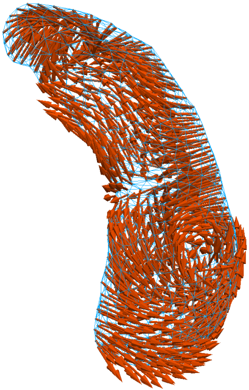

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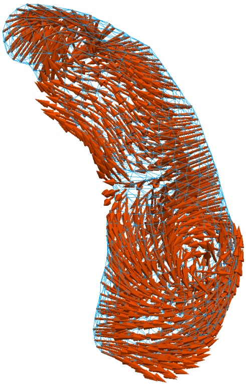

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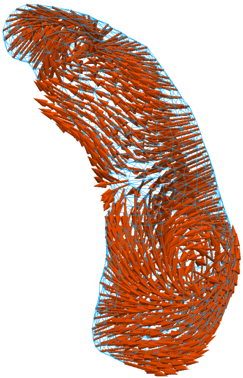

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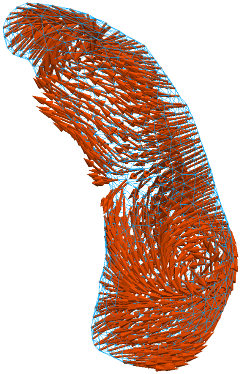

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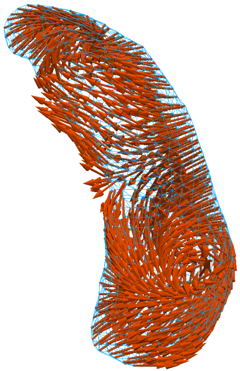

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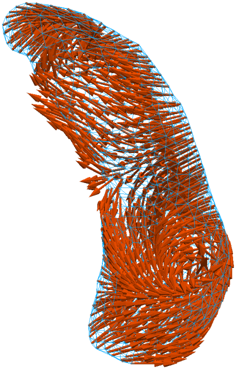

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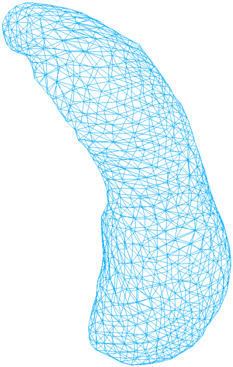

Final Deformed Surface

Pontryagin Principle

Consider the Hamiltonian

$$H(q,p) = H(x_1,\dots,x_n,p_1,\dots,p_n) = \frac{1}{2}\sum_{i,j=1}^{n} p_i(t)^T p_j(t) e^{-\frac{|x_i(t)-x_j(t)|^2}{\sigma^2}}.$$

Theorem

For reasonnable $\|\cdot\|_V$ and g, minimizers $(t,x)\mapsto v(t,x)$ of J are completely determined by the value of $p(0)=(p_1(0),\dots,p_n(0))$, through the Hamiltonian equation

$$\dot{x}_i(t) = v(t,x_i(t)) = \nabla_{p_i} H(q(t),p(t)) = \sum_{j=1}^{n} p_j(t) e^{-\frac{|x_i(t)-x_j(t)|^2}{\sigma^2}},$$

$$\dot{p}_i(t) = -\nabla_{x_i} H(q(t),p(t)).$$

Moreover, in this case,

$$J(v) = H(q_0,p(0)) + g(q(1)).$$

Registration

Registration problem reduced to minimizing

$$\tilde{J}(p_0) = H(q_0, p_0) + g(q(1)),$$

where $q(1)$ is obtained by solving the previous Hamiltonian equation with $q(0) = q_0$ and $p(0) = p_0$. The corresponding minimizing initial momentum $p_0^* = p_0(q_1)$ completely encodes the deformation from q_0 to q_1.

Karcher Mean and Hypertemplate

Assume we have $\tilde{q}_1, \ldots, \tilde{q}_k$, k distinct shapes, representing same organ/part of the brain among various patients. First step for statistical analysis: compute average shape, which will be used as the template from which all shapes are registered.

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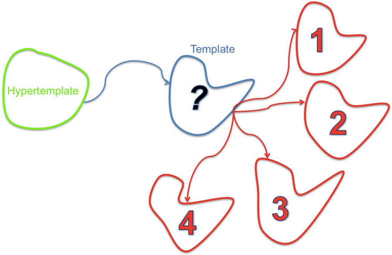
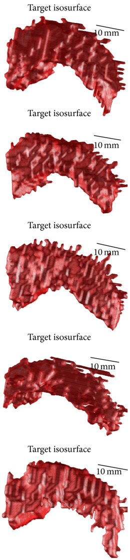


Take nice hypertemplate q_0. Minimize w.r. to $v_0, \tilde{v}_1, \ldots, \tilde{v}_k$ the functional

$$\lambda \int_0^1 \|v_0(t)\|^2 dt + \frac{1}{2} \sum_{i=1}^{k} \int_0^1 \|\tilde{v}_i(t)\|^2 dt + g_i(\tilde{\varphi}_i(1) \circ \varphi_0(1) \cdot q_0).$$

The average shape will be $\bar{q} = \varphi_0(1) \cdot q_0$, and we will simultaneously have registered every $\tilde{q}_k$ from $\bar{q}$ through some initial momentum p_k from $\bar{q}$, from which we can deduce corresponding the velocity fields that bring $\bar{q}$ to $\tilde{q}_k$.

Example of Application: Smoothing Data



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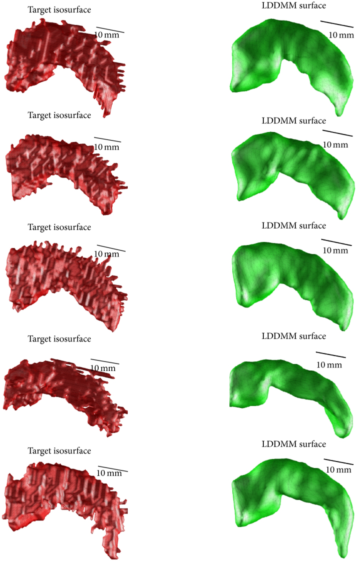

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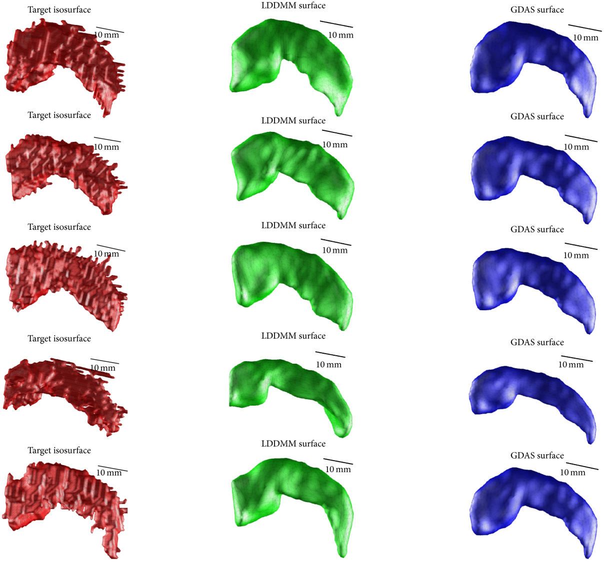

Other Geometric Markers

- We have an average shape $\bar{q}$ over a data set $\tilde{q}_1, \ldots, \tilde{q}_k$, registered as initial momenta $p_1, \ldots, p_k$ along $\bar{q}$. However, it is hard to directly give them a geometric meaning.
- Instead, each p_k yields a corresponding minimizing vector field $\tilde{v}_k(t)$ which integrates into a deformation of the space $\tilde{\varphi}_k(1)$. These have a precise geometric interpretation.

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- For example, when studying degenerative diseases, one can compute the change of volume between the average shape $\bar{q}$ and each deformed shape $\tilde{\varphi}_k \cdot \bar{q}$. This can be used to differentiate controls from sick patients in a study.

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- For example, when studying degenerative diseases, one can compute the change of volume between the average shape $\bar{q}$ and each deformed shape $\tilde{\varphi}_k \cdot \bar{q}$. This can be used to differentiate controls from sick patients in a study.
- We can be even more precise: compute the (total, surface, normal) jacobian of each deformation $\tilde{\varphi}_k$ at each point of the template, or the elastic strain along certain direction when getting from the template to one of the data points...

Application: BIOCARD Study

By M. Miller, M. Albert, L. Younes et al.

- 1995-2008: Alzheimer's disease longitunal study at NIH
- 350 healthy subjects with large proprotion at risk of dementia and AD
- 1-6 MRI scans per subject
- Goal: Identify shape structures that are primarily affected (ERC, hippocampus, amygdala).

- All subjects were healthy at beginning of study
- At end of study, 66 patients diagnosed with mild cognitive impairment or dementia.
- Longitudinal model comparing differences between controls (healthy until end of study) and MCI patients.

Application: BIOCARD Study

Table 3
Annualized atrophy rates for normal group and preclinical AD group.

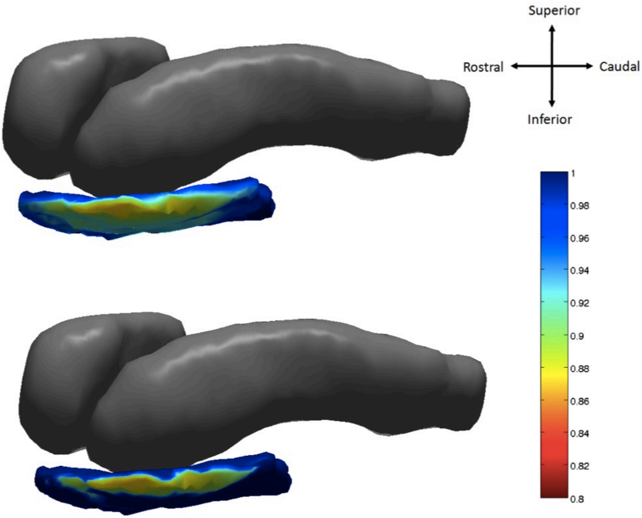
Groups	Amygdala mm^3/year	Amygdala %/year	Hippocampus mm^3/year	Hippocampus %/year	ERC mm^3/year	ERC %/year	ERC thickness mm/year	ERC thickness %/year
L controls (n = 81)	4.6 ± 39.1	0.2 ± 2.7	14.0 ± 25.2	0.5 ± 0.9	4.9 ± 19.8	0.8 ± 4.3	.008 ± 0.043	0.34 ± 1.89
L preclinical (n = 20)	16.8 ± 25.3	1.0 ± 1.6	14.4 ± 29.1	0.5 ± 1.1	8.1 ± 15.3	1.7 ± 3.2	.022 ± 0.050	0.92 ± 2.20
L ApoE4+ (n = 44)	4.9 ± 45.3	0.2 ± 3.2	12.9 ± 29.6	0.6 ± 1.0	4.8 ± 20.8	0.7 ± 4.6	.019 ± 0.047	0.76 ± 2.02
L ApoE4− (n = 73)	8.2 ± 32.4	0.4 ± 2.1	11.9 ± 24.3	0.4 ± 0.9	5.1 ± 19.5	0.5 ± 6.5	.004 ± 0.054	−0.01 ± 3.22
R controls (n = 81)	14.2 ± 29.8	0.9 ± 2.0	21.2 ± 31.8	0.9 ± 1.9	5.5 ± 19.7	0.9 ± 4.2	.007 ± 0.039	0.28 ± 1.78
R preclinical (n = 20)	22.0 ± 27.1	1.4 ± 1.8	4.6 ± 28.5	1.1 ± 1.4	13.2 ± 19.2	3.3 ± 3.8	.024 ± 0.040	1.08 ± 1.79
R ApoE4+ (n = 44)	14.5 ± 28.6	1.0 ± 2.0	20.8 ± 33.4	0.8 ± 1.8	8.4 ± 24.1	1.6 ± 4.9	.014 ± 0.047	0.60 ± 2.10
R ApoE4− (n = 73)	16.7 ± 37.1	1.0 ± 2.4	14.5 ± 30.4	0.9 ± 2.3	4.3 ± 17.3	0.6 ± 4.7	.001 ± 0.040	−0.03 ± 2.18
B controls (n = 81)	9.4 ± 27.6	0.6 ± 1.8	17.6 ± 22.4	0.7 ± 0.9	5.2 ± 14.8	1.0 ± 3.3	.008 ± 0.031	0.33 ± 1.42
B preclinical (n = 20)	19.4 ± 19.2	1.2 ± 1.2	9.5 ± 20.6	0.3 ± 0.8	10.6 ± 14.4	2.7 ± 3.1	.023 ± 0.039	1.04 ± 1.73
B ApoE4+ (n = 44)	9.7 ± 29.4	0.6 ± 2.0	16.9 ± 25.1	0.7 ± 0.9	6.6 ± 16.2	1.3 ± 3.4	.016 ± 0.034	0.71 ± 1.48
B ApoE4− (n = 73)	12.5 ± 30.0	0.7 ± 1.9	13.2 ± 20.7	0.5 ± 0.8	4.7 ± 15.9	0.6 ± 5.5	.002 ± 0.042	0.01 ± 2.47

The table presents the volume atrophy rates and standard deviations in % and mm^3/year for amygdala (columns 2 and 3), hippocampus (columns 4 and 5) and entorhinal cortex (ERC) (columns 6 and 7), for time series with at least 3 scans. The top group of four rows is for L = Left; the middle group of four rows is for R = Right; the bottom group of four rows is for B = Bilateral; three preclinical subjects with hippocampal volume atrophy rates were outliers and were removed.

Morphometry measures comparing normal group vs preclinical AD group.

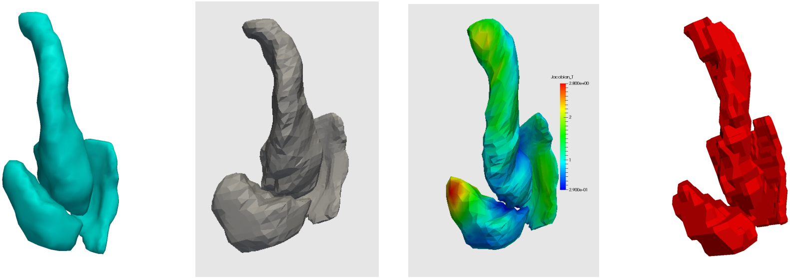
Structures examined	p-Values based on vertex measure Control vs. preclinical AD	p-Values based on Laplace measure Controls vs. preclinical AD	p-Values based on volume measure Controls vs. preclinical AD
Amygdala (L)	0.17	0.13	0.0086
Hippocampus (L)	0.022	0.33	0.073
ERC (L)	<0.0001	0.0001	0.51
Amygdala (R)	0.031	0.029	0.0043
Hippocampus (R)	0.0025	0.08	0.79
ERC (R)	0.0067	0.0003	0.17

Application: BIOCARD Study



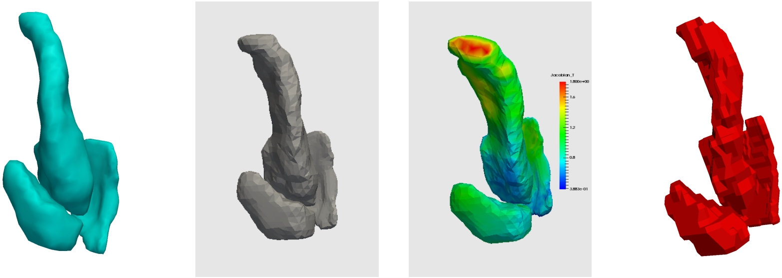
Thank you for your attention!

Single diffeomorphism



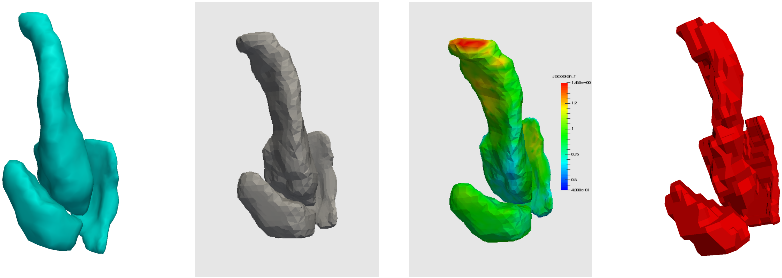
Single diffeomorphism matching: Halfway point and final point.

Identity constraints



Identity constraints matching: Halfway point and final point.

Sliding constraints



Identity constraints matching: Halfway point and final point.