Ex1

Freitag, 1. Oktober 2021 10:41

1.1) $\mu_1 = -2$, $\mu_2 = 1$, $\sigma_1^2 = 6$, $\sigma_2^2 = 6$, $\rho_{12} = -\frac{1}{2}$

$$\Sigma = \begin{bmatrix} Var(X_1) & Cov(X_1,X_2) \\ Cov(X_1,X_2) & Var(X_2) \end{bmatrix}$$

① $Cor(X_1, X_2) = \rho_{12} = \frac{Cov(X_1, X_2)}{\sigma_1 \sigma_2} = \frac{\Sigma_{12}}{\sqrt{\Sigma_{11}\Sigma_{22}}}$

$\Rightarrow Cov(X_1,X_2) = \rho_{12} \cdot \sigma_1 \cdot \sigma_2 = -\frac{1}{2} \cdot \sqrt{6} \cdot \sqrt{6} = -\frac{1}{2} \cdot 6 = -3$

$\Rightarrow \underline{\underline{\Sigma = \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix}}} \quad \Rightarrow |\Sigma| = 27$

$\Rightarrow \underline{\underline{\Sigma^{-1}}} = \frac{1}{27}\begin{bmatrix} 6 & 3 \\ 3 & 6 \end{bmatrix} = \underline{\underline{\frac{1}{9}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}}}$

$$p(\vec{x}) = \frac{1}{\sqrt{(2\pi)^d \, |\Sigma|}} \exp\Big(\underbrace{-\tfrac{1}{2}(\vec{x}-\vec{\mu})^T \Sigma^{-1} (\vec{x}-\vec{\mu})}_{a}\Big)$$

$a: -\frac{1}{2}\begin{pmatrix} x_1+2 \\ x_2-1 \end{pmatrix}^T \Sigma^{-1} \begin{pmatrix} x_1+2 \\ x_2-1 \end{pmatrix}$

$\hookrightarrow -\frac{1}{2}\Big(\big[\tfrac{2}{9}(x_1+2)+\tfrac{1}{9}(x_2-1) \quad \tfrac{1}{9}(x_1+2)+\tfrac{2}{9}(x_2-1)\big] \cdot \begin{pmatrix} x_1+2 \\ x_2-1 \end{pmatrix}\Big)$

$= -\frac{1}{2}\Big(\frac{2}{9}(x_1+2)^2 + \frac{1}{9}(x_1+2)(x_2-1) + \frac{1}{9}(x_1+2)(x_2-1) + \frac{2}{9}(x_2-1)^2\Big)$

$= -\frac{1}{2}\Big(\frac{2}{9}(x_1+2)^2 + \frac{2}{9}(x_1+2)(x_2-1) + \frac{2}{9}(x_2-1)^2\Big) = -\frac{1}{2}\Big(\frac{2}{9}(x_1^2+4x_1+4) + \frac{2}{9}(x_1x_2 - x_1 + 2x_2 - 2) + \frac{2}{9}(x_2^2 - 2x_2 + 1)\Big)$

$$\Rightarrow \underline{\underline{p(\vec{x})}} = \frac{1}{\sqrt{(2\pi)^2 \cdot 27}} \exp\Big(-\frac{1}{9}(x_1^2 + x_2^2 + x_1x_2 + 3x_1 + 3)\Big)$$

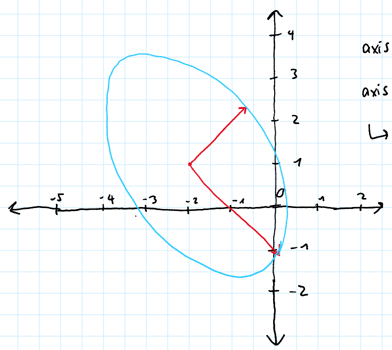
② The axes of the ellipsoid are defined by the eigenvectors of Σ.

$det\left(\begin{bmatrix} \lambda-6 & 3 \\ 3 & \lambda-6 \end{bmatrix}\right) = (\lambda-6)^2 - 9 = \lambda^2 - 12\lambda + 27$

$\lambda_{1/2} = \frac{12 \pm \sqrt{144 - 4 \cdot 27}}{2} = \frac{12 \pm 6}{2} = 6 \pm 3 \quad \Rightarrow \lambda_1 = 3, \ \lambda_2 = 9$

$\Rightarrow \vec{v}_1: \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \quad \Rightarrow 3x = 3y \quad \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\Rightarrow \vec{v}_2: \begin{bmatrix} -3 & -3 \\ -3 & -3 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \quad \Rightarrow \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$



axis 1: $\sqrt{3}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

axis 2: $3\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\hookrightarrow \sqrt{\lambda}$ is one standard deviation

1.2.) $X = [X_1, X_2, X_3]^T \sim \mathcal{N}(X \mid \mu, \Sigma)$

$\mu = \begin{bmatrix} -3 \\ 1 \\ 4 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & -3 \\ 0 & -3 & 6 \end{bmatrix}$

① $Cov(X_i, X_j) = \Sigma_{ij}$

$\Rightarrow Cov(X_3, X_1) = 0$

$Cov(X_3, X_1) = E[(X_1 - E(X_1))(X_3 - E(X_3))]$

$= E[X_1X_3 - X_1\underbrace{E(X_3)}_{4} - X_3\underbrace{E(X_1)}_{-3} + \underbrace{E(X_1)}_{-3}\underbrace{E(X_3)}_{4}]$

$= E[X_1X_3] - \underbrace{E[4X_1]}_{-12} + \underbrace{E[3X_3]}_{12} - 12 = 0$

$E[X_1X_3] = -12$

$\Rightarrow E[X_1 \cdot X_2] = E(X_1) \cdot E(X_2) = -12 \quad \Rightarrow X_1$ and X_3 $\underline{\underline{\text{are}}}$ independent

② $Cov(X_2, X_3) = -3$

$Cov(X_2, X_3) = E[(X_2 - E(X_2))(X_3 - E(X_3))] = E[X_2X_3 - X_2E(X_3) - X_3E[X_2] + E(X_2)E(X_3)]$

$= E[X_2X_3 - 4X_2 - X_3 + 4] = -3$

$E[X_2X_3] - 4E[X_2] - E[X_3] = -3$

$Cov(X_2,X_3) = E[(X_2-E(X_2))(X_3-E(X_3))] = E[X_2X_3 - X_2E(X_3) - X_3E[X_2] + E(X_2)E(X_3)]$

$= E[X_2X_3 - 4X_2 - X_3 + 4] = -3$

$E[X_2X_3] - \underbrace{4E(X_2)}_{4} - \underbrace{E[X_3]}_{4} = -7$

$E[X_2X_3] = 1 \quad \Rightarrow E[X_2X_3] = 1 \neq 4 = E[X_2]\cdot E[X_3]$

$\Rightarrow$ X_2 and X_3 are <u>not</u> independent

③ $(2X_1 - X_2 - X_3)$ and $(X_3 - X_2)$ $\rightarrow$ solve with $Cov(X,Y) = E[(X-E(X))(Y-E(Y))]$

$\Rightarrow E[(2X_1-X_2-X_3 - \underbrace{E[2X_1-X_2-X_3]}_{-11})(X_3-X_2-\underbrace{E[X_3-X_2]}_{3})]$

$= E[(2X_1-X_2-X_3+11)(X_3-X_2-3)]$

$= E[\cancel{2X_1X_3} - \cancel{2X_1X_2} - \cancel{6X_1} - \cancel{X_2X_3} + \cancel{X_2^2} \cancel{+3X_2} - \cancel{X_3^2} + \cancel{X_2X_3} + \cancel{3X_3} + \cancel{11X_3} - \cancel{11X_2} - 33]$

$= E[X_2^2 - X_3^2 - 6X_1 - 2X_1X_2 + 2X_1X_3 - 8X_2 + 14X_3 - 33]$

$= E(X_2^2) - E(X_3^2) - 6E(X_1) - 2E(X_1X_2) + 2E(X_1X_3) - 8E(X_2) + 14E(X_3) - 33$

- $V[X_2] = 6 = E(X_2^2] - E(X_2)^2$

  $6 = E[X_2^2) - 1 \Rightarrow E[X_2^2] = 7$
- $V[X_3] = 6 = E(X_3^2] - E[X_3]^2$

  $6 = E[X_3^2] - 16 \Rightarrow E[X_3^2] = 22$
- $2\cdot E[X_1X_2] \longrightarrow X_1$ and X_2 are independent

  $\Rightarrow 2\cdot E(X_1]\cdot E[X_2) = 2(-3) = -6$
- $2\cdot E[X_1X_3] \rightarrow X_1$ and X_3 are independent

  $\Rightarrow 2\cdot E[X_1]\cdot E[X_3] = 2(-12) = -24$

$= 7 - 22 + 18 + 6 - 24 - 8 + 56 - 33 = 0$

$\Rightarrow Cov(2X_1-X_2-X_3, X_3-X_2) = 0$

$\Rightarrow$ $2X_1-X_2-X_3$ and X_3-X_2 <u>are</u> independent

1.3.) we want: $P(X_1 | X_2 = x_2)$, distributed $\sim \mathcal{N}(\mu, \Sigma)$

$\mu = \begin{bmatrix} -2 \\ 10 \end{bmatrix}$, $\Sigma = \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix}$

conditional of a Gaussian: $\vec{x} \sim \mathcal{N}(\vec{\mu}, \Sigma)$

$p(\vec{x}_a | \vec{X}_b = \vec{x}_b) = \mathcal{N}(\vec{x}_a | \vec{\mu}_{a|b}, \Sigma_{a|b})$

$\hookrightarrow \underline{\underline{\Sigma_{a|b}}} = \Sigma_{aa} - \Sigma_{ab}\Sigma_{bb}^{-1}\Sigma_{ba} = 6-(-3)\cdot 6^{-1}(-3) = 6 - 9\cdot\frac{1}{6} = 6 - \frac{9}{6} = \underline{\underline{\frac{27}{6}}}$

$\hookrightarrow \underline{\underline{\vec{\mu}_{a|b}}} = \vec{\mu}_a + \Sigma_{ab}\Sigma_{bb}^{-1}(\vec{x}_b - \vec{\mu}_b) = -2 + (-3)\frac{1}{6}(x_2 - 10)$

$= -2 - \frac{1}{2}(x_2 - 10) = -2 - \frac{1}{2}x_2 + 5 = \underline{\underline{-\frac{1}{2}x_2 + 3}}$

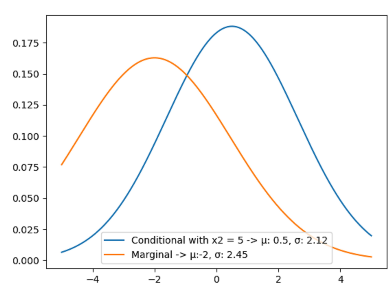
$\Rightarrow p(X_1 | X_2 = x_2) \sim \mathcal{N}(X_1 | -\frac{1}{2}x_2 + 3, \frac{27}{6})$

To compare the conditional of $X_2 = 5$ with the marginal of X_1, we first put in 5 for x_2.

$\hookrightarrow p(X_1 | X_2 = 5) \sim \mathcal{N}(X_1 | \frac{1}{2}, \frac{27}{6})$

We compare this in a plot with the marginal: $p(\vec{x}_a) \sim \mathcal{N}(\vec{x}_a | \vec{\mu}_a, \Sigma_{aa})$

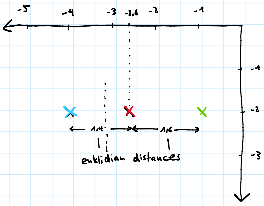
$\hookrightarrow$ in our case: $p(X_1) \sim \mathcal{N}(\vec{x}_a | -2, 6)$



1.4.) to be classified: $X = [-2.6\ ,\ -2]$

each class normally distributed with $\mu_1 = [-4\ ,\ -2]$ and $\mu_2 = [-1\ ,\ -2]$

a) isotropic and identical covariance matrices



On the plot we clearly see, that the distance from X to μ_1 is smaller than the one from X to μ_2. The covariance matrices are isotropic (spheres) and identical and thus we can argue without calculation that X has to be classified into class 1.

$$d_{E1} = \|X - \mu_1\| = \left\|\begin{smallmatrix}1.4\\0\end{smallmatrix}\right\| = \sqrt{(1.4)^2} = \underline{\underline{1.4}}$$

$$d_{E2} = \|X - \mu_2\| = \left\|\begin{smallmatrix}-1.6\\0\end{smallmatrix}\right\| = \sqrt{(1.6)^2} = \underline{\underline{1.6}}$$

b.) Here we have non-diagonal covariance matrices, which would suggest to use the Mahalanobis-distance instead of the Euklidian one

↯ but the two matrices are not equal $\Rightarrow$ calculation of the multivariate Gaussian distribution!

$$\Sigma_1 = \begin{bmatrix}1.5 & 1.8\\ 1.8 & 6\end{bmatrix},\quad \Sigma_2 = \begin{bmatrix}1.5 & 0.6\\ 0.6 & 0.6\end{bmatrix},\quad d = 2 \text{ (dimensions)}$$

$$\mu_1 = [-4, -2],\quad \mu_2 = [-1, -2],\quad X = [-2.6, -2]$$

$$p(\vec{x}\,|\,\omega_i) = \frac{1}{(2\pi)^{d/2}\,|\Sigma_i|^{1/2}} \exp\left(-\tfrac{1}{2}(\vec{x}-\mu_i)^T\,\Sigma_i^{-1}(\vec{x}-\mu_i)\right)$$

① $$\Sigma_1^{-1} = \frac{1}{9 - 3.24}\begin{pmatrix}6 & -1.8\\ -1.8 & 1.5\end{pmatrix} \approx \begin{bmatrix}1.042 & -0.313\\ -0.313 & 0.26\end{bmatrix}$$

$$|\Sigma_1| = 9 - (1.8)^2 = 5.76$$

$$\Rightarrow p(X\,|\,\omega_1) = \frac{1}{(2\pi)\sqrt{5.76}} \exp\left(-\tfrac{1}{2}\begin{pmatrix}1.4 & 0\end{pmatrix}\begin{bmatrix}1.042 & -0.313\\ -0.313 & 0.26\end{bmatrix}\begin{pmatrix}1.4\\0\end{pmatrix}\right)$$ 2

$$\approx \frac{1}{2\pi\sqrt{5.76}} \exp\left(-\tfrac{1}{2}\cdot 2.042\right) \approx 0.024$$

② $$\Sigma_2^{-1} = \frac{1}{0.9 - 0.36}\begin{pmatrix}0.6 & -0.6\\ -0.6 & 1.5\end{pmatrix} = \begin{bmatrix}1.\bar{1} & -1.\bar{1}\\ -1.\bar{1} & 2.\bar{7}\end{bmatrix}$$

$$|\Sigma_2| = 0.9 - (0.6)^2 = 0.54$$

$$\Rightarrow p(X\,|\,\omega_2) = \frac{1}{(2\pi)\sqrt{0.54}} \exp\left(-\tfrac{1}{2}\begin{pmatrix}-1.6 & 0\end{pmatrix}\begin{bmatrix}1.\bar{1} & -1.\bar{1}\\ -1.\bar{1} & 2.\bar{7}\end{bmatrix}\begin{pmatrix}-1.6\\0\end{pmatrix}\right)$$

$$\approx \frac{1}{2\pi\sqrt{0.54}} \exp\left(-\tfrac{1}{2}\cdot 2.8\bar{4}\right) \approx 0.052$$

$$\Rightarrow p(X|\omega_1) \approx 0.024 > 0.052 \approx p(X|\omega_2)$$

$\Rightarrow$ X can be classified into class 2