Physics 3: Class 6

Max Jauregui

April 1, 2019

1 Electric potential

Let us consider a system of charges and let us suppose that we want to bring a particle with charge q from an infinite distance to a position $\mathbf{r}$ following a trajectory C. The necessary energy to perform this task, denoted by $U(q, \mathbf{r})$, is proportional to q. Then, we can define an energy per unit of charge, which is called the *electric potential* of the system, by

$$V(\mathbf{r}) = \frac{U(q, \mathbf{r})}{q}\,.$$

The unit of electric potential in the SI is the volt (V).

In terms of the electric force, the energy $U(q, \mathbf{r})$ can be written as

$$U(q, \mathbf{r}) = -\int_C \mathbf{F}_e \cdot d\mathbf{r}\,.$$

Since $\mathbf{F}_e = q\mathbf{E}$, where $\mathbf{E}$ is the electric field generated by the system, we have

$$U(q, \mathbf{r}) = -q\int_C \mathbf{E} \cdot d\mathbf{r}\,.$$

This implies that the electric potential of the system of charges is given by

$$V(\mathbf{r}) = -\int_C \mathbf{E} \cdot d\mathbf{r}\,. \tag{1}$$

The electric potential of a particle with charge Q, localized at the origin of our coordinate system, at a point $\mathbf{r}$ is

$$\begin{aligned}
V(\mathbf{r}) &= -\int_C \frac{Q}{4\pi\epsilon_0}\frac{\mathbf{r}'}{(r')^3} \cdot d\mathbf{r}' \\
&= -\int_{t_1}^{t_2} \frac{Q}{4\pi\epsilon_0[r'(t)]^3}\mathbf{r}'(\mathbf{t}) \cdot \frac{d\mathbf{r}'}{dt}\,dt \\
&= -\int_{t_1}^{t_2} \frac{Q}{4\pi\epsilon_0[r'(t)]^3}r'(t)\frac{dr'}{dt}\,dt \\
&= -\int_{\infty}^{r} \frac{Q}{4\pi\epsilon_0(r')^2}\,dr'\,.
\end{aligned}$$

Therefore,

$$V(\mathbf{r}) = \frac{Q}{4\pi\epsilon_0}\frac{1}{r}\,.$$

If the particle of charge Q is at a position $\mathbf{r}'$, the electric potential at a point $\mathbf{r}$ will be given by

$$V(\mathbf{r}) = \frac{Q}{4\pi\epsilon_0}\frac{1}{|\mathbf{r}-\mathbf{r}'|}\,.$$

If we have a system of n charges $q_1,\ldots,q_n$, the electric potential at a point $\mathbf{r}$ will be given by

$$V(\mathbf{r}) = \sum_{i=1}^{n}\frac{q_i}{4\pi\epsilon_0}\frac{1}{|\mathbf{r}-\mathbf{r}_i|}\,.$$

In the case of a continuous distribution of charges with charge density $\rho(\mathbf{r})$, the electric potential at a point $\mathbf{r}$ will be

$$V(\mathbf{r}) = \int_{\mathbb{R}^3}\frac{\rho(\mathbf{r}')}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|}\,dV'\,.$$

Let us consider a charged particle that is initially at a position $\mathbf{r}_1$ and moves slowly to a position $\mathbf{r}_2$ describing a trajectory C under the action of an external electric field $\mathbf{E}$. Then, it follows from Eq. (1) that

$$\int_C \mathbf{E}\cdot d\mathbf{r} = V(\mathbf{r}_1) - V(\mathbf{r}_2)\,.$$

In particular, if $\mathbf{r}_1 = \mathbf{r}_2$, we have

$$\oint_C \mathbf{E}\cdot d\mathbf{r} = 0\,.$$