Neural Uncertainty

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Chapter 1

Model definition

1.1 Notations

We consider the prediction task of a set of observations $(x^1, \cdots x^T)$ given a set of input $(u^1, \cdots u^T)$.

1.2 Model

We define a L layer RNN followed by a fully connected layer. At time step t,

$$\begin{cases} x_{t+1} = \tanh(W_x y_{t+1}^L + b_x) \\ y_{t+1}^l = \tanh(W_{yu}^l y_{t+1}^{l-1} + W_{yy}^l y_t^l + b_y^l) \quad \forall 1 \leq l \leq L \end{cases}$$

with $y_t^0 \equiv u_t \; \forall t$ and $y_0^l \equiv 0 \; \forall 1 \leq l \leq L$.

Let's consider the weights of the last RNN and fully connected layers as $\Theta \equiv (W_{yu}^L, W_{yy}^L, b_y^L, W_x, b_x)$. We can define a new matrix y_t at each time step corresponding to the concatenation of all RNN layers: $y_t \equiv (y_t^1 \cdots y_t^L)$. We also introduce two sequences of random noises as i.i.d real valued random variables ϵ and η. We can now write our model in terms of two functions f and g as:

$$\begin{cases} x_{t+1} = f_\Theta(y_{t+1}) + \epsilon_{t+1} & \text{observation model} \\ y_{t+1} = g_\Theta(y_t, u_{t+1}) + \eta_{t+1} & \text{state model} \end{cases} \tag{1.1}$$

$$Q(\hat{\Theta}_p, \Theta) = \mathbb{E}_{\hat{\Theta}_p}\left[\log\ p_\Theta(X_{1:T}, y_{1:T}, u_{1:T}) | y_{1:T}\right] \tag{1.2}$$

Let's develop the interior of expectation:

$$
\begin{aligned}
p_\Theta(X_{1:T}, y_{1:T}, u_{1:T}) &= \frac{1}{T}\log\left(\prod_{k=1}^{T} p_\Theta(y_k|y_{k-1}, u_k) p_\Theta(x_k|y_k)\right) \\
&= \frac{1}{T}\sum_{k=1}^{T} \log\ p_\Theta(y_k|y_{k-1}, u_k) + \log\ p_\Theta(x_k|y_k) \\
&= \frac{1}{T}\sum_{k=1}^{T} \log\left(\det(2\pi\Sigma_y)^{-1/2}\exp(-\frac{1}{2}(y_k - g_\Theta(y_{k-1}, u_k))^T\Sigma_y^{-1}(y_k - g_\Theta(y_{k-1}, u_k))\right) \\
&+ \log\left(det(2\pi\Sigma_x)^{-1/2}\exp(-\frac{1}{2}(x_k - f_\Theta(y_k))^T\Sigma_x^{-1}(x_k - f_\Theta(y_k))\right) \\
&= -\frac{1}{2}log|\Sigma_y| - \frac{1}{2}log|\Sigma_x| \\
&- \frac{1}{2T}\sum_{k=1}^{T}(y_k - g_\Theta(y_{k-1}, u_k))^T\Sigma_y^{-1}(y_k - g_\Theta(y_{k-1}, u_k)) \\
&- \frac{1}{2T}\sum_{k=1}^{T}(x_k - f_\Theta(y_k))^T\Sigma_x^{-1}(x_k - f_\Theta(y_k))
\end{aligned}
$$