Dynamic estimation of multinomial logit choice models

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Abstract

This paper considers the estimation of the nested multinomial logit choice model in a tightly constrained, dynamic computational setting called the data stream. I provide an approximation scheme for the model parameters that updates the current estimate according to an intuitive step rule. Stochastic convergence is guaranteed by the Robbins–Monro condition. Numerical simulations demonstrate the technique's feasibility.

Keywords: choice models, multinomial logit, dynamic parameter estimation

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1. Introduction

In a recent paper, Ho-Nguyen and Kılınç-Karzan (2021) articulate the need for *dynamic* choice model fitting techniques that can efficiently update the current estimate in light of new observations of consumer behavior, without resorting to solving a full optimization problem over the perturbed data. That paper concerned nonparametric choice models, which are hard to estimate even in a static setting. On the other hand, *parametric* choice models offer a substantial advantage in the computational tractability of both model fitting and assortment optimization tasks (Bunch, 1987; Davis and Gallego and Topaloglu, 2014). To establish parity in the larger debate over the relative merits of parametric and nonparametric choice models, it is worth examining whether this tractability advantage also holds in the dynamic setting.

The present paper answers in the affirmative by providing a dynamic approximation scheme for the multinomial logit choice model and its nested variant. When a new observation of a consumer choice given a certain consideration set comes in, the technique proposed here updates the current

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parameter estimate according to a simple rule with a sequence of decreasing step sizes. Stochastic convergence is guaranteed by the Robbins–Monro condition, and can be improved by tuning step parameters or utilizing a rolling average. I conclude by demonstrating the feasibility of the technique through numerical simulations.

2. The multinomial logit choice model and its nested variant

Consider a set of *products* $\mathcal{N} = \{1 \dots n\}$. A consumer is offered an *assortment* $\mathcal{A} \subseteq \mathcal{N}$ of products to choose from. A *choice model* defines the probability that the consumer will choose product i.

In the multinomial logit (MNL) choice model, each consumer j's preference order over the products in $\mathcal{A}$ is determined by the sum of a systematic product preferability parameter δ_i and a noise term ϵ_{ij}. The noise terms have an iid extreme value distribution, and the consumer chooses the product in $\mathcal{A}$ for which $\delta_i + \epsilon_{ij}$ is greatest. It can be shown that the probability of a given consumer choosing product i from the assortment $\mathcal{A}$ is

$$\Pr[\,\text{choose } i \mid \text{assortment } \mathcal{A}\,] = \frac{\exp \delta_i}{\sum_{h \in \mathcal{A}} \exp \delta_h} \tag{1}$$

The value of this equation does not change when each δ_i is perturbed by the same constant; hence, we may assume without loss of generality that $\sum_i \delta_i = 0$.

A known issue with the MNL choice model is the assumption that introducing a new item into the assortment does not alter the preferability of the items already in the assortment relative to one another. This assumption, known as the independence of irrelevant alternatives, is violated when some products are near substitutes. The *nested* multinomial logit model (NMNL) addresses this issue by grouping similar products into *nests*.

Under NMNL, the customer first chooses a nest $k \in \{1 \dots s\}$, then a product i within the nest. Both choices are made using MNL. Each nest's preferability parameter is written σ_k, while the products within the nest have preferability parameters δ_{ki}. For convenience, assume each nest contains n products. Hence $\mathcal{N} = \{1 \dots s\} \times \{1 \dots n\}$. We can represent a given assortment $\mathcal{A}$ as a list of (nest, product) tuples (k, i). The assortment may span the s nests and sn products in any arbitrary way. Let $\mathcal{A}^{\text{nests}}$ denote the set of nests represented in the assortment $\mathcal{A}$, and let $\mathcal{A}_k^{\text{products}}$ denote the set of products that are in both nest k and the assortment $\mathcal{A}$. Then the probability that a given customer chooses the product (k, i) is

$$\Pr[\text{choose } (k,i) \mid \text{assortment } \mathcal{A}] = \frac{\exp \sigma_k}{\sum_{l \in \mathcal{A}^{\text{nests}}} \exp \sigma_l} \cdot \frac{\exp \delta_i}{\sum_{h \in \mathcal{A}_k^{\text{products}}} \exp \delta_h} \tag{2}$$

where we may assume without loss of generality that $\sum_k \sigma_k = 0$ and $\sum_i \delta_{ki} = 0, \forall k$.

3. Static parameter estimation for (N)MNL

Consider the static parameter estimation task in the MNL choice model. m consumers, indexed by j, were shown assortments $\mathcal{A}_j \subseteq \mathcal{N}$ and asked to pick their favorite product y_j. For convenience, abbreviate the preferability parameter associated with consumer j's choice from δ_{y_j} to δ_j.

The likelihood of the m observations with respect to the parameters δ is

$$L(\delta) = \prod_{j=1}^{m} \frac{\exp \delta_j}{\sum_{h \in \mathcal{A}_j} \exp \delta_h} \tag{3}$$

which is concave in δ after taking the logarithm.

Now consider the NMNL choice model. Each consumer j chooses a product from an assortment $\mathcal{A}_j$, and we may write the consumer's choice as (x_j, y_j), where y_j is the product chosen and x_j is the nest to which y_j belongs. (Note that this requires that the nests be allocated in advance.) Again, $\mathcal{A}_j$ as a list of (nest, product) tuples (k, i). Let $\mathcal{A}_j^{\text{nests}}$ denote the set of nests represented in the assortment $\mathcal{A}_j$, let $\mathcal{A}_j^{\text{products}}$ denote the set of products that are in both x_j and the assortment $\mathcal{A}_j$, and abbreviate the parameters $(\sigma_{x_j}, \delta_{y_j})$ to (σ_j, δ_j). Then the likelihood function is

$$L(\sigma, \delta) = \prod_{j=1}^{m} \frac{\exp \sigma_j}{\sum_{l \in \mathcal{A}_j^{\text{nests}}} \exp \sigma_l} \cdot \frac{\exp \delta_j}{\sum_{h \in \mathcal{A}_j^{\text{products}}} \exp \delta_h} \tag{4}$$

Computational tools for static estimation of variants of the MNL choice model are widely available (e.g. Croissant, n.d.).

4. Parameter estimation in the data stream framework

The dynamic parameter estimation scheme offered below is designed for an online setting called the data stream. In the data stream, we are only allowed to store our *current* parameter estimate δ and a state variable t representing the number of observations used to form the current estimate. When a new observation $(\mathcal{A}, y)$ comes in, we can update the stored variables, but we cannot store the observation itself.

4.1. The MNL parameter update

First, consider the estimation of MNL choice parameters. At time t, the parameter estimate is $\delta^{(t)}$. A new observation arrives in which a consumer with consideration set $\mathcal{A}_t$ chose product y_t. Let $Z^{(t)}$ denote the n-vector with

$$Z_i^{(t)} = \begin{cases} -\frac{\exp \delta_i^{(t)}}{\sum_{h \in \mathcal{A}_t} \exp \delta_h^{(t)}}, & i \in \mathcal{A}_t, i \neq y_t \\ 1 - \frac{\exp \delta_i^{(t)}}{\sum_{h \in \mathcal{A}_t} \exp \delta_h^{(t)}}, & i \in \mathcal{A}_t, i = y_t \\ 0, & i \notin \mathcal{A}_t \end{cases} \tag{5}$$

I propose the following *MNL parameter update:*

$$\delta^{(t+1)} = \delta^{(t)} + \frac{\alpha}{t^r} Z^{(t)} \tag{6}$$

$\alpha > 0$ and $r \in (0, 1]$ are step parameters. The intuition behind this parameter update is as follows. Seeing that y_t was chosen from $\mathcal{A}_t$, we

- Increase the estimated preferability of the item that was picked,
- Decrease the estimated preferability of the items that were rejected, and
- Leave unchanged the preferability of items not in the presented assortment.

Notice that $\sum_{i=1}^n \delta_i$ remains constant under the parameter update, because $\sum_i Z_i = 0$.

Proposition 1. *If the observations are drawn from an MNL choice model and each consideration set appears with nonzero probability, then the MNL parameter update converges in expectation to the true parameters δ.*

4.2. Why the parameter update converges

Consider the multivariate root-finding problem

$$\text{find } \theta^* : F(\theta^*) = 0$$

and the iterative scheme

$$\theta^{(t+1)} = \theta^{(t)} + \frac{\alpha}{t^r} F\left(\theta^{(t)}\right)$$

where $\alpha > 0$ and $r \in (0, 1]$ are step parameters. Such an iterative scheme is known to converge to the root when the Jacobian of $F : \mathbb{R}^n \mapsto \mathbb{R}^n$ is negative definite. A typical example occurs when using gradient ascent to maximize a strictly concave function whose gradient is F. Moreover, Robbins and Monro (1951) showed (in the univariate case) that one can replace $F(\theta)$ with an *experiment* $\hat{F}(\theta)$ satisfying $\mathbb{E}\left[\hat{F}(\theta)\right] = F(\theta)$. The same iterative scheme exhibits *stochastic convergence*, meaning that with probability one,

$$\lim_{t\to\infty} \mathbb{E}\left[F\left(\theta^{(t)}\right)\right] = \theta^*$$

Many convergence rate results also generalize to stochastic interpretations.

4.3. The MNL parameter update is a Robbins–Monro algorithm

To express the parameter update as a Robbins–Monro algorithm, it suffices to show that the optimal parameters are the zero of some function F and that the parameter update accords with an experiment $\hat{F}$ which is alike in expectation.

Let δ denote the current parameter estimate and let δ^* denote the true, unknown parameters of the underlying choice model. When the probability of offering each consideration set $\mathcal{A}$ is $\Pr[\mathcal{A}] > 0$, then the expected value of Z has

$$\begin{aligned}
\mathbb{E}\left[Z(\delta)_i\right] &= \sum_{\substack{\mathcal{A}\in 2^{\mathcal{N}}:\\ i\in\mathcal{A}}} \Pr\left[\mathcal{A}\right] \cdot \mathbb{E}\left[Z(\delta)_i \mid \mathcal{A}\right] && (7)\\
&= \sum_{\substack{\mathcal{A}\in 2^{\mathcal{N}}:\\ i\in\mathcal{A}}} \Pr\left[\mathcal{A}\right] \cdot \left(\Pr\left[\,\text{choose } i \mid \mathcal{A}\,\right] - \frac{\exp\delta_i}{\sum_{h\in\mathcal{A}}\exp\delta_h}\right) && (8)\\
&= \sum_{\substack{\mathcal{A}\in 2^{\mathcal{N}}:\\ i\in\mathcal{A}}} \left(\frac{\exp\delta_i^*}{\sum_{h\in\mathcal{A}}\exp\delta_h^*} - \frac{\exp\delta_i}{\sum_{h\in\mathcal{A}}\exp\delta_h}\right) && (9)\\
&= 0 && (10)
\end{aligned}$$

if and only if $\delta^* = \delta$. Hence, the counterpart to $F(\theta)$ is the expected value of the random function $Z(\delta)$, while the counterpart to the experiment $\hat{F}(\theta)$ is the realization $Z^{(t)}(\delta)$. By observing that the Jacobian of $\mathbb{E}\left[Z(\delta)\right]$ is negative definite, we conclude that the iterative procedure converges in expectation to δ^*.

4.4. The NMNL parameter update

The NMNL parameter update is analogous to the MNL parameter update. Each new observation consists of a consideration set $\mathcal{A}_t$, a choice of nest x_t, and a choice of product y_t. Define $\mathcal{A}_t^{\text{nests}}$ and $\mathcal{A}_t^{\text{products}}$ as above, and let

$$W_k^{(t)} = \begin{cases} -\frac{\exp \sigma_k^{(t)}}{\sum_{i \in \mathcal{A}_t^{\text{nests}}} \exp \sigma_i}, & k \in \mathcal{A}_t^{\text{nests}}, k \neq x_t \\ 1 - \frac{\exp \sigma_k^{(t)}}{\sum_{i \in \mathcal{A}_t^{\text{nests}}} \exp \sigma_i}, & k \in \mathcal{A}_t^{\text{nests}}, k = x_t \\ 0, & k \notin \mathcal{A}_t^{\text{nests}} \end{cases} \tag{11}$$

$$Z_{ki}^{(t)} = \begin{cases} -\frac{\exp \delta_i^{(t)}}{\sum_{h \in \mathcal{A}_t^{\text{products}}} \exp \delta_h^{(t)}}, & k = x_t, i \in \mathcal{A}_t^{\text{products}}, i \neq y_t \\ 1 - \frac{\exp \delta_i^{(t)}}{\sum_{h \in \mathcal{A}_t^{\text{products}}} \exp \delta_h^{(t)}}, & k = x_t, i \in \mathcal{A}_t^{\text{products}}, i = y_t \\ 0, & \text{otherwise} \end{cases} \tag{12}$$

The parameters are updated as follows.

$$\sigma^{(t+1)} = \sigma^{(t)} + \frac{\beta}{t^r} W^{(t)} \tag{13}$$

$$\delta^{(t+1)} = \delta^{(t)} + \frac{\alpha}{t^r} Z^{(t)} \tag{14}$$

$\beta, \alpha > 0$ and $r \in (0, 1]$ are step parameters.

Proposition 2. *If the observations are drawn from an NMNL choice model and each consideration set appears with nonzero probability, then the NMNL parameter update converges to the true parameters σ and δ.*

5. Computational examples

I demonstrate the tractability of the dynamic parameter update through three simulations. The first simulation concerns the MNL case and is summarized in Figure 1; the second concerns the NMNL case and is summarized in

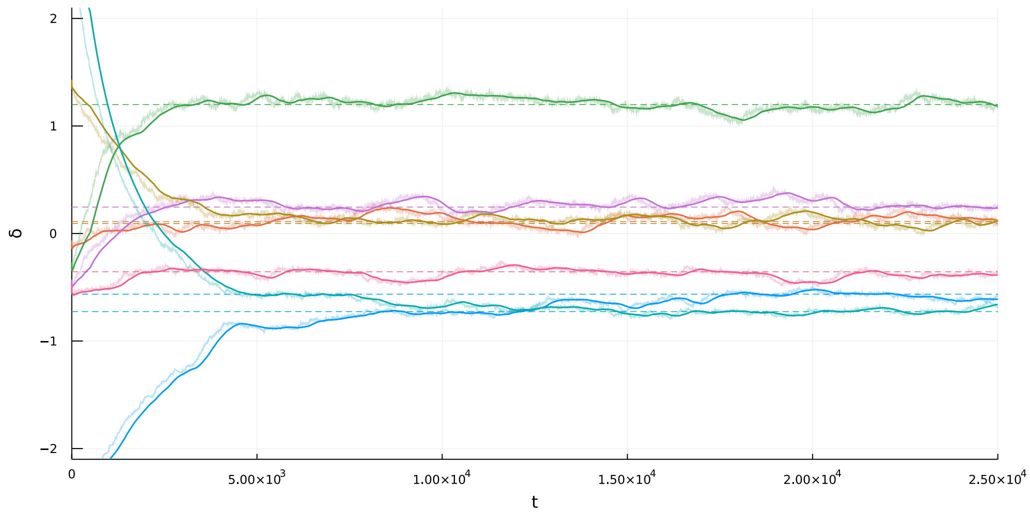

Figure 1: Dynamic estimation of MNL choice parameters for a collection of $n = 7$ products. True preferability parameters δ and the initial estimate $\delta^{(1)}$ were drawn independently from a normal distribution and centered at zero. At times $t = 1 \dots 25000$, a random assortment consisting of between two and five products was drawn, a consumer chose a product using the MNL choice model, and the parameter estimates were updated according to the MNL parameter update (Equation 6). The parameter estimate at each iteration is shown in a light stroke, while the darker lines show a moving average over the previous 500 estimates. The step parameters are $\alpha = 0.01$ and $r = 0.05$.

Figure 2. In both simulations, a small number of products was chosen to yield a legible graph; the parameter update formula is computationally tractable for much higher values of n and s. The third simulation considers a larger product line, and compares the accuracy of maximum likelihood estimation over a big dataset with the average parameter estimate over the final 1000 iterations of a dynamic estimation of the same. While the accuracy is necessarily lower in the latter case, it is acceptable given the savings in computation time. In all cases, it is possible to tune the step parameters α, β, and r to obtain smoother, slower convergence. The Julia code for all three simulations is available on Github at https://github.com/maxkapur/MultinomialLogit.

6. Conclusion

Recent research in choice models has undergone two shifts in emphasis: one toward nonparametric choice models, and another toward the estimation

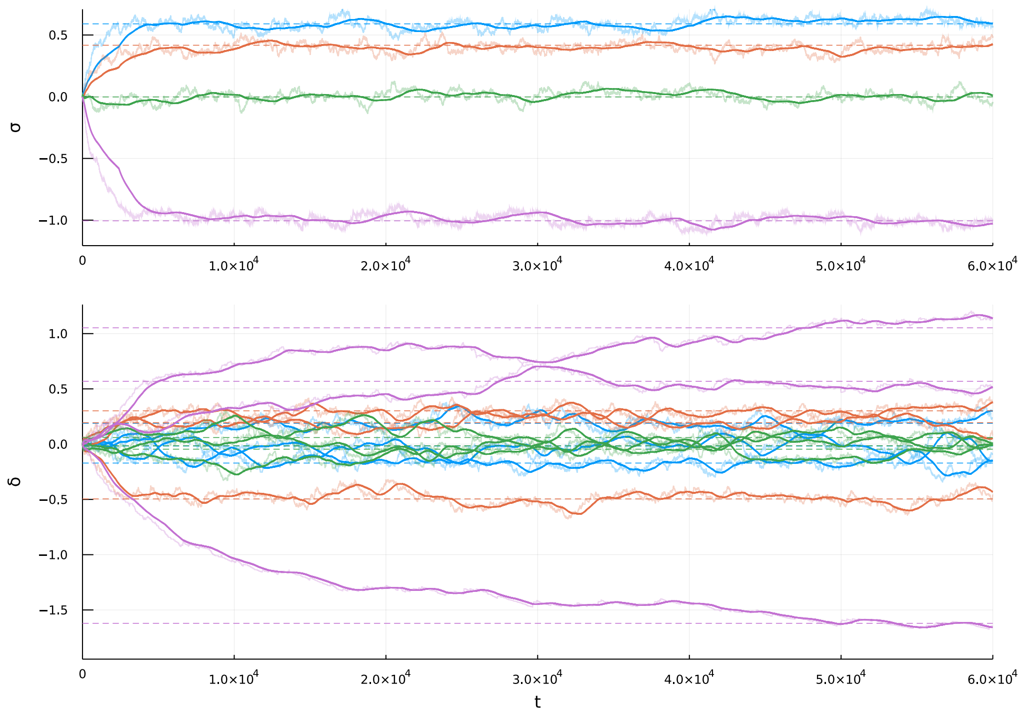

Figure 2: Dynamic estimation of MNL choice parameters for a collection of $sn = 12$ products divided across $s = 4$ nests. True preferability parameters σ and δ were drawn independently from a normal distribution and centered at zero, while the initial parameter estimates were set to zero. At times $t = 1 \ldots 60000$, a random assortment consisting of between two and five products was drawn, a consumer chose a product using the MNL choice model, and the parameter estimates were updated according to the NMNL parameter update (Equation 13). The parameter estimate at each iteration is shown in a light stroke, while the darker lines show a moving average over the previous 1200 estimates. The step parameters are $\beta = 0.01$, $\alpha = 0.03$, and $r = 0.05$. The first pane shows the time series of σ estimates, while the second shows the entries of δ, color-coded by nest membership.

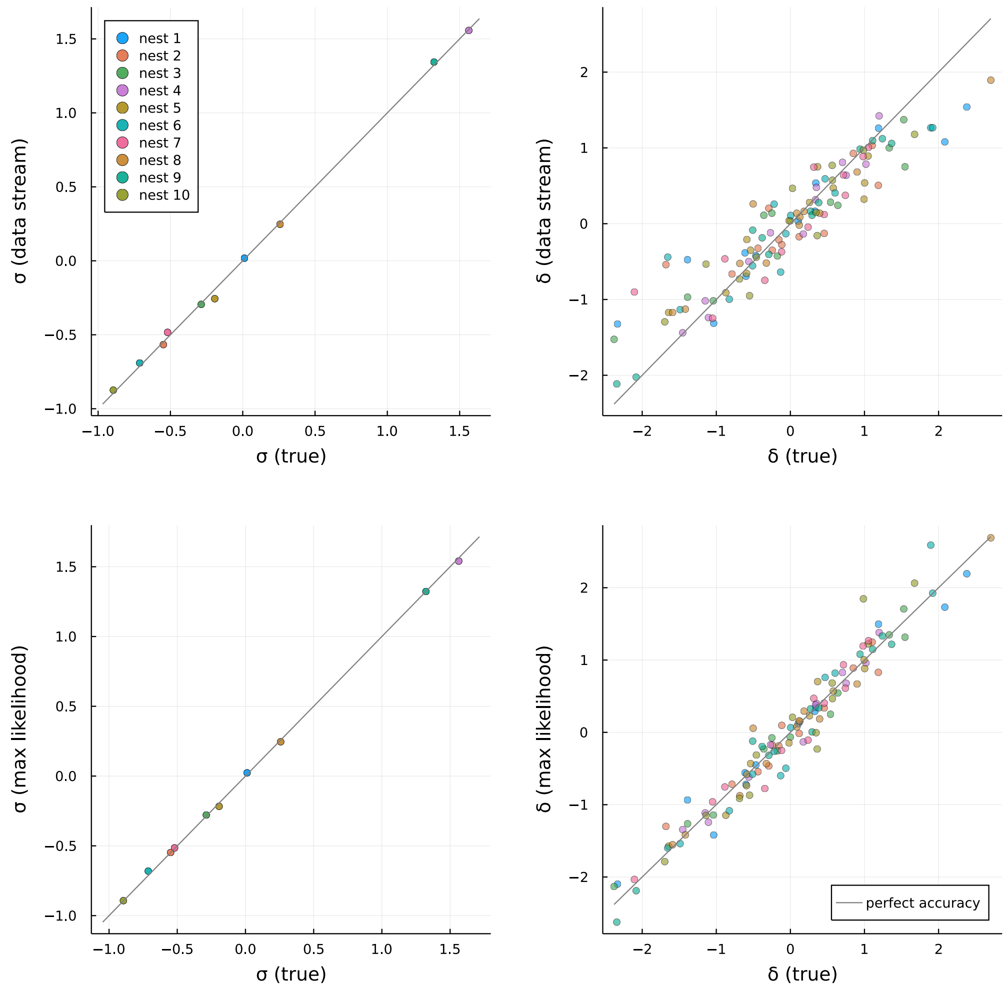

Figure 3: Performance of the dynamic parameter estimate versus maximum likelihood estimation. Preferability parameters for $sn = 120$ products divided into $s = 10$ nests were drawn from the normal distribution and centered at zero. $m = 50000$ assortments and consumer choices were generated according to the NMNL model. In the upper two panes, the observations were treated as a time series, and I report the average of the final 1000 estimates produced by the dynamic parameter update with step parameters $\beta = 0.01$, $\alpha = 0.03$, and $r = 0.05$. In the lower two panes, maximum likelihood estimation was used to estimate the parameters. Dynamic parameter estimation offers acceptable accuracy at substantially lower computational cost.

thereof in a dynamic data setting. This paper has attempted to decouple these trends by considering the dynamic estimation of *parametric* choice models in the MNL family. Preliminary results suggest that the tractability advantage of parametric choice models is pronounced in the dynamic data setting. Whereas the dynamic algorithms suggested by Ho-Nguyen and Kılınç-Karzan (2021) require solving a combinatorial subproblem at each iteration, the parameter update proposed here is linear in the number of items and does not require any subiterative optimization. Of course, all parametric choice models impose assumptions on the underlying distribution of consumer preference orders that may not be appropriate in all settings (Keane, 1997). Thus, the larger debate about the relative merits of parametric and nonparametric models is far from settled.

A natural extension of the ideas proposed here is the dynamic parameter estimation of other choice models within and beyond the MNL family.

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