From centralized school choice problems to competitive admissions markets: On the equivalence of stable matchings and market equilibrium

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1 Introduction

The classical *school choice problem* is a one-to-many stable matching problem. Given a set of students and a set of schools, each student supplies her ranked preferences over the schools (typically without ties), and each school supplies its ranked preferences over the students (sometimes allowing for ties). Using a deferred acceptance algorithm in conjunction with a tiebreaking rule, the school board computes a *stable assignment* of students to schools, or one in which every school that rejects students fills its capacity with students equal to or better than other students who prefer that school to the school to which they are matched.

Recently, attention has shifted toward a nonatomic formulation of this problem in which individual students are replaced with a distribution of students over the space of possible preference lists and scores. The nonatomic formulation, due to Azevedo and Leshno (**supplydemandfw**), enables the characterization of assignment policies, including stable assignments, via school admissions cutoffs, which indicate a score threshold above which all students have the option of attending the school in question. Current research in school choice problems takes advantage of the nonatomic formulation because it can be interpreted as the limit of the discrete assignment problem as the number of students and seats increases to infinity; thus, score cutoffs in the nonatomic formulation are free of the "noise" associated with discretization.

However, formulating school choice problems in terms of cutoffs rather than stable assignments is also an opportunity to greatly generalize the purview of school choice research. In actuality, most admissions markets are not run by a central agency. Instead, they are dynamic: colleges can admit or reject students as they please, although it is safe to assume that to the extent that a given college can derive a partial preference order over the set of its applicants, it will admit the subset that exceeds a certain score cutoff rather than an arbitrary subset from all over the map.

The cutoff formulation also creates space for the possibility that schools may have goals other than filling their capacity. They may not even have a meaningful limit on capacity, as is the case for many online schools.

In this dynamic situation, that paper offers a theoretical description of the relationship between supply and demand and posits a few comparative statics. However, it doesn't concern itself with computation, neither of the location of the equilibrium nor of the statics themselves.

The one computable instance it does consider is with completely iid preferences on the part of the schools. then it provides expressions for comparative statics and a theoretical discussion about when negative incentives can arise.

It is this computational question to which I turn. I propose a continuous model that is both computationally tractable, moderately realistic, and versatile enough to accommodate not just centralized markets that used a DA procedure but also dynamic markets in which schools freely admit and reject students.

This model enables us to actually compute useful comparative statics.

Moreover, it is realistic xyz. I establish the equivalency between the notion of equilibrium defined in reference to the demand function and school capacities, and several notions of equilibrium rooted in quantitative economics, game theory, and mechanism design. In fact, most previous literature on school choice process considers a centralized ...

2 Interpretations of equilibrium in admissions markets

Before specifying the market used in this paper, I will define a nonatomic admissions market and summarize some important theoretical results.

2.1 Admissions markets

Definition 1. An *admissions market* consists of a set of schools $C = \{1 \dots |C|\}$ and a mass-1 continuum of students over the set S of student types. The market is characterized by four parameters:

1. The measure $\eta : 2^S \mapsto [0,1]$ over the continuum of students.
2. The score cutoff vector $p \in [0,1]^C$.
3. The demand vector $D \in [0,1]^C$.
4. The capacity vector $q \in \mathbb{R}^C_{++}$.

The model is *nonatomic* in that it represents students as a probability measure over the set of student types instead of considering individual students as discrete actors. Each point s in the set of student types S is associated with a preference list over the schools $>_s$ and a percentile score at each school $\theta_{sc} \in [0,1]$. Hence, $S = C! \times [0,1]^C$.

Schools marginally prefer students with higher scores. Their admissions decisions are represented by the score cutoff vector p. Any student for whom $\theta_{sc} \geq p_c$ is said to be *admitted* to school c.

The demand vector represents the number of students who enroll at each school. Assume that each student attends her favorite school among the set of schools to which she is admitted, which is called her *consideration set* $C^\# \in 2^C$. Then the demand for school c is a function of p and η; specifically, it is the measure of students who are admitted to school c but not admitted to any school that they prefer to c:

$$D_c \equiv \eta \left(s : c = \arg\max_{>_s} \{\hat{c} : \theta_{s\hat{c}} \geq p_{\hat{c}}\} \right) \tag{1}$$

Observe that D_c is weakly decreasing in p_c and weakly increasing in $p_{c'}$ for $c' \neq c$, and that $p_c = 1 \implies D_c = 0$ regardless of the other schools' cutoffs.

If the preference lists are independent of the score vectors, then the demand can also be expressed as the sum of the demand from each combination of preference list and consideration set:

$$D_c = \sum_{>_s \in C!} \sum_{\substack{C^\# \in 2^C: \\ c \in C^\#}} \eta\Big(s : \underbrace{\theta_{sc'} \geq p_{c'}, \forall c' \in C^\#}_{\text{got into schools in } C^\#} \\ \text{and } \underbrace{\theta_{sc''} < p_{c''}, \forall c'' \in C \setminus C^\#}_{\text{rejected elsewhere}} \\ \text{and } \underbrace{c >_s \hat{c}, \forall \hat{c} \in C^\# \setminus \{c\}}_{\text{prefers } c \text{ among } C^\#} \Big) \tag{2}$$

This expression yields immediate insight into the complexity of nonatomic admissions markets. When schools are allowed to set their own cutoffs, and students are given free choice among the schools in their consideration sets, the number of terms in the sum above is $|C|! \times 2^{|C|}$. However, such a general characterization of students is not always needed. The model considered in this paper (§3), characterizes S using only a $|C|$-vector of school preferability parameters. Alternatively, generic markets can be discretized by representing individual students using explicit preference lists and score vectors.

Each school's capacity $q_c > 0$ represents the fraction of the total mass of students that the school can accept. The capacity is used to define the notion of equilibrium below.

Assume that students prefer to be assigned to any school, even their last choice, than to remain unassigned. This is without loss of generality; if some students prefer to be unassigned than attend a particular school, then this choice can be incorporated into the model by adding a dummy school, representing nonassignment, with arbitrary large capacity.

Assume that almost no ties occur among scores at a given school. That is, for any fixed c and constant $\bar{\theta}$, $\eta(s : \theta_{sc} = \bar{\theta}) = 0$. It follows that the demand is continuous in p.

2.2 Notion of equilibrium

Let's consider a notion of equilibrium that turns out to have several realistic interpretations.

Definition 2. An admissions market is in *equilibrium* if the following conditions hold:

$$D_c \leq q_c, \quad \forall c \tag{3}$$

$$D_c = q_c, \quad \forall c : p_c > 0 \tag{4}$$

The first condition, called the *capacity condition,* says that no school's demand exceeds its capacity. The second, called the *stability condition,* says that if a school is rejecting students, it must be at full capacity.

Using the sign constraint on p and the capacity condition, an equivalent to the stability condition is $D_c < q_c \implies p_c = 0$ or $p^T (D - q) = 0$.

As shown below, a sufficient condition for the existence of the equilibrium is that the demand is continuous in p. A sufficient condition for the uniqueness of the equilibrium is that the demand is strictly decreasing in p. Both of these follow from, for example, an assumption of full support in η.

2.3 Interpretation of the equilibrium conditions

The applicability and interpretation of the equilibrium conditions depends on the design of the admissions markets. I offer four interpretations, moving from theoretical to practical. With additional assumptions on η, such as full support, all four interpretations become sufficient conditions for equilibria as well. Additionally, the first interpretation establishes a sufficient condition for the existence of the equilibrium.

In all cases, assume the distribution of student types η and the school capacities q are fixed; thus the demand $D(p)$ is determined entirely by the cutoffs p.

2.3.1 As a fixed point of a tâtonnement process

Suppose that each school has set an admissions target of q_c and observes its demand from year to year. If more than q_c students enroll, then the school attempts to reduce remand by increasing its cutoff. If fewer than q_c students enroll, the school attempts to increase demand by lowering its cutoff (but not past zero).

Let $Z(p) \equiv D(p) - q$ denote the excess demand vector. Then the process described above implies the following recursive relation between the cutoff vector in year k and in year $k+1$:

$$p_c^{(k+1)} = \max\left\{0,\ p_c^{(k)} + \Gamma_c\left[Z_c\big(p^{(k)}\big)\right]\right\}$$

where Γ is sign-preserving. A dynamic process like this one, in which prices adjust in the direction of excess demand, is called a *tâtonnement process*.

Theorem 1. *If $\bar{p}$ is a fixed point of the tâtonnement process, then it satisfies the equilibrium conditions. The converse also holds.*

Proof. Pick $\bar{p}$ such that $\bar{p} = \max\big\{0, \bar{p}_c + \Gamma_c\left[Z_c(\bar{p})\right]\big\}$. Subtract $\bar{p}_c$ from both sides to obtain $0 = \max\big\{-\bar{p}_c, \Gamma_c\left[Z_c(\bar{p})\right]\big\}$, which implies $0 \geq \Gamma_c\left[Z_c(\bar{p})\right]\big\}$. This means that the excess demand is negative, which establishes the capacity condition. Now, suppose $\bar{p}_c > 0$; then $\bar{p}_c = \bar{p}_c + \Gamma_c\left[Z_c(\bar{p})\right]$ establishes the stability condition $Z_c(\bar{p} = 0)$. Hence, any fixed point of the tâtonnement process is an equilibrium.

As for the converse, if $p^{(k)}$ satisfies the equilibrium conditions, it is easy to verify that $p^{(k+1)} = p^{(k)}$. ☐

Moreover, if the demand function is continuous in p, then Brouwer's fixed-point theorem guarantees that a fixed point exists, because the cutoff update maps the convex set $[0,1]^C$ to itself. This means that continuous demand is sufficient for the existence of an equilibrium in admissions markets.

Quantitative economists have studied tâtonnement processes extensively. For an introduction, see Codenotti and Varadarajan (**compmkteq**) or Intriligator (**mathematicaloptandecontheory**). A classical proof of various convergence conditions is Uzawa (**walrastatonnement**).

2.3.2 As a competitive (Nash) equilibrium

Suppose that each school's capacity q_c is a physical constraint on the number of students it can admit. Each school would like to recruit as many students as possible. However, if more students choose to attend the school than the school has capacity for, it must rent additional classroom space at considerable expense. Pick a school c and fix the cutoffs at the other schools $p_{c'}$. Let $u_c(p_c; p_{c'})$ denote school c's utility function, and $\hat{p}_c \equiv p_c : D_c(p_c; p_{c'}) = q_c$ denote the cutoff value that causes c to fill its capacity, if such a value exists. In the situation described, u_c is increasing in p_c when $0 \leq p_c \leq \hat{p}_c$, decreasing in p_c when $p_c > \hat{p}_c$, and the fixed costs associated with excess demand are nonnegative:

$$\lim_{p_c \to \hat{p}_c^+} u_c(p_c; p_{c'}) \leq u_c(\hat{p}_c; p_c c')$$

Another scenario in which utility functions with this shape may arise is as follows: Schools' utility functions are determined mostly by the number of students they enroll, and to a lesser extent by their average score. If the demand for a school is less than its capacity, then a marginal student is always desirable. On the other hand, if the demand exceeds the capacity, then the school has no ability to procure space for the excess demand. Instead, it allows students to register on a first come, first served basis. Because the set of registered students is a random subset of the students who attempt to enroll at the school, it provides the school with less overall utility than if it handpicked the q_c students with the highest scores. That is, u_c is increasing when $0 \leq p_c \leq \hat{p}_c$, and decreasing when $p_c > \hat{p}_c$.

In both of these situations, the admissions market equilibrium can be interpreted as a Nash equilibrium.

Theorem 2. *Consider the game in which each school picks a cutoff $p_c \in [0, 1]$ and tries to maximize a utility function u_c that is increasing in p_c when $0 \leq p_c \leq \hat{p}_c$, decreasing in p_c when $p_c > \hat{p}_c$, and satisfies $\lim_{p_c \to \hat{p}_c^+} u_c(p_c; p_{c'}) \leq u_c(\hat{p}_c; p_c c')$.*

If p^* satisfies the equilibrium conditions, then it is a Nash equilibrium of the game above. If the utility functions are strictly increasing or decreasing, the converse is also true.

Proof. Suppose p^* satisfies the admissions market equilibrium conditions. For the schools for which $D_c = q_c$, their utility is globally maximal. For a school for which $D_c < q_c$, the only way for the school to increase its utility is to decrease its cutoff, but by assumption, $p_c^* = 0$. Hence, is incentivized to change its cutoff, and p^* is a Nash equilibrium of the game defined by the schools' utility functions and the action space $p_c \in [0, 1]$.

The converse can be shown similarly. □

As an aside, observe both of the above are natural situations in which the tâtonnement dynamics may arise.

2.3.3 As a market-clearing cutoff vector

Consider a semicentralized admissions procedure similar to the one used for college admissions in South Korea. In order to ensure an even geographic distribution of students, the government places a firm limit q_c on the number of students who can attend each university. At the beginning of the admissions cycle, colleges are given the profiles of all the students interested in attending. Each college makes admissions offers over the course of several rounds, beginning with the highest-qualified students, and at each round a subset of the admitted students tentatively commit to attending one of the colleges that admitted them. Continuing in this manner, colleges continue lowering their cutoffs and offering admission to new students until either they fill their capacity, or lowering the cutoff fails to elicit an increase in demand. That is, the process terminates when the *market clears:* There may be empty seats at some schools or some students who were admitted nowhere, but never both. The total measure of assigned students is $\min\{1, \sum_c q_c\}$.

Theorem 3. *If $\hat{p}$ is a market-clearing cutoff vector obtained from the semicentralized admissions procedure, then it satisfies the equilibrium conditions.*

Conversely, if $\hat{p}$ satisfies the equilibrium conditions, then it is a market-clearing cutoff vector. That is, the total measure of assigned students is $\min\{1, \sum_c q_c\}$.

Proof. Call the final vector of admissions cutoffs $\hat{p}$. If we assume that student preference lists are independent of their scores and that the marginal distribution of scores at each school has full support, then it follows that market-clearing cutoffs $\hat{p}$ satisfy the equilibrium conditions. The capacity condition holds at $\hat{p}$, because colleges are prohibited from lowering their cutoffs beyond the point where the demand equals the capacity. The stability condition also holds: Suppose (for a contradiction) that at the end of the final round, there is a school which has both remaining capacity (that is, $D_c < q_c$) and a positive cutoff $\hat{p}_c > 0$. By the full support assumption on the marginal score distribution, the set of marginal students at c is nonempty. Hence, if c lowers its cutoff, it can attract additional students. This contradicts the stopping criterion.

As for the converse, let $\hat{\eta}$ denote the measure of assigned students: $\hat{\eta} \equiv \sum_c D_c(\hat{p})$. By the capacity criterion, we have $\hat{\eta} \leq \sum_c q_c$. Since every student prefers assignment to nonassignment, $\hat{\eta} = 1 - \eta(s : \theta_{sc} < p_c, \forall c \in C) \leq 1$. If at least one school has $p_c = 0$, then $\hat{\eta} = 1 \leq q_c$ and the statement holds. Otherwise, the stability condition applies to every school, meaning $\hat{\eta} = \sum_c q_c \leq 1$. □

In reference to this interpretation of the equilibrium conditions, Azevedo and Leshno (**supplydemandfw**) refer to the capacity and stability constraints as the market-clearing equations.

The semicentralized admissions procedure can be interpreted as a tâtonnement process in which the initial cutoffs are a vector of ones, or as a distributed school-proposing deferred acceptance process, as discussed below (§2.4).

2.3.4 As a stable matching

The final interpretation of the equilibrium conditions comes from Lemma 1 of Azevedo and Leshno (**supplydemandfw**), which says that there is a one-to-one relationship between stable matchings and equilibrium cutoff vectors. In this section, I offer a proof of this lemma.

The notion of a stable assignment has its roots in the field of mechanism design, and thus emerges most naturally from a centralized school choice process as follows: Before the school year begins, students fill out an online form indicating their preference order over the set of schools in the district. Likewise, schools submit their preference order over the students (or equivalently, the scores they have given to each student) to the school board. Then, the school board determines the assignment of students to schools.

First, some notation. A assignment is a mapping of students to schools.

Definition 3. A school choice *assignment* is a mapping $\mu : S \to C \cup \{c_0\}$. $\mu(s) = c_0$ represents nonassignment.

The school board is interested in matchings, or assignments that respect capacity constraints.

Definition 4. A *matching* is an assignment μ that respects schools' capacity constraints; namely, $\eta(s : \mu(s) = c) \leq q_c, \forall c \in C$.

To protect itself from lawsuits and encourage honest participation in the assignment process, the school board decides to rule out matchings that create justified envy; that is, matchings in which a student s who prefers school c to c' is assigned to c' despite scoring higher than a student assigned to c, or c having remaining capacity. In such a situation, (s, c) is called a blocking pair, and a stable matching is a matching that does not admit any blocking pairs.

Definition 5. A *stable matching* (or *stable assignment*) is a matching μ that admits no blocking pairs. That is, there exist no tuples (s, c) s.t. $c >_s \mu(s)$ and one of the following holds:

- School c has remaining capacity:

$$\eta\big(s : \mu(s) = c\big) < q_c$$

- (And/or,) school c has admitted an inferior student:

$$\exists s' \neq s : \mu(s') = c \text{ and } \theta_{s'c} < \theta_{sc}$$

These are called type I and type II blocking pairs, respectively.

In a nonatomic admissions market, where the number of students is infinite, the definition above offers no guidance as to how to encode a stable matching μ. However, it turns out that stable matching are in one-to-one correspondence with equilibrium cutoff vectors. Therefore, any stable matching μ can be fully encoded by a $|C|$-vector of cutoffs p. This fact is invaluable in a computional context.

To establish this result, we must define operators that take cutoff vectors to assignments, and vice-versa. First, the assignment of students induced by instating the cutoff vector p and allowing students to choose freely among their consideration set is

$$\mu_p(s) \equiv \max_{>_s} c \in C : \theta_{sc} \geq p_c, \qquad \forall s \in S$$

Note that this assignment is not necessarily a matching because it may violate the capacity constraints.

Second, the following expression gives the admissions cutoffs implied by a given matching μ, namely, the minimum score of the students admitted to each school:

$$p_c(\mu) \equiv \min \{\theta_{sc} : \mu(s) = c\}$$

In general, these operators are not necessarily inverses of one another. However, as implied by the following theorem, they are inverses when we restrict their domains to the sets of equilibrium cutoffs and stable matchings, respectively.

Theorem 4. *If p^* satisfies the equilibrium conditions, then μ_{p^*} is a stable matching.*

Conversely, if $\bar{\mu}$ is a stable matching and η has full support, then $p_c(\bar{\mu})$ satisfies the equilibrium conditions.

Proof. Pick an equilibrium cutoff p^*. Then by the definitions of the demand function (1) and equilibrium conditions (2),

$$\begin{aligned} D_c(p^*) &= \eta\big(s : \mu_{p^*}(s) = c\big) \leq q_c, \quad \forall c \in C \\ D_c(p^*) &= \eta\big(s : \mu_{p^*}(s) = c\big) = q_c, \quad \forall c : p^*_c > 0 \end{aligned}$$

By the capacity condition, no school exceeds its capacity, so μ is an assignment. By the stability condition, there are no type I blocking pairs. And there are no type II blocking pairs, because if a student fails to meet the cutoff for a school she prefers to $\mu_{p^*}(s)$, it is because the school has replaced her with students who got higher scores. Hence, μ_{p^*} is a stable matching.

The converse is proven as follows. Fix a stable matching $\bar{\mu}$, and let $\bar{p} \equiv p(\bar{\mu})$. To get a contradiction, suppose $\bar{p}$ is not an equilibrium. This can happen in two ways:

- For some school c, $D_c(\bar{p}) > q_c$. This means $D_c(\bar{p}) > \eta\big(s : \bar{\mu}(s) = c\big)$, which implies the existence of a student s who is admitted to c at $\bar{p}$ (that is, $\theta_{sc} \geq \theta_{s'c}$ for some $s' : \bar{\mu}(s') = c$) and prefers c among her consideration set, but for whom $\bar{\mu}(s) \neq c$. Then (s, c) is a type II blocking pair; hence, $\bar{\mu}$ is not a stable matching.

- For some school c, $\bar{p}_c > 0$ and $D_c(\bar{p}) < q_c$. By the assumption of strong or full support, there is a student s for whom $c >_s \bar{\mu}(s)$ and $\theta_{sc} < \bar{p}_c$. The latter implies that $\bar{\mu}(s) \neq c$. Hence, (s, c) is a type I blocking pair, and μ is not a stable matching.

Therefore, $\bar{p}$ must satisfy the equilibrium conditions. □

2.4 Deferred acceptance algorithms as tâtonnement processes

The classical solution to the stable matching problem is known as a deferred acceptance (DA) mechanism, which comes in many flavors. When neither students' nor schools' preference lists contains ties, the student-proposing DA procedure is a deterministic algorithm for a stable matching. In this section, I first define the student- and school-proposing DA algorithms. Then, I show that DA algorithms are tâtonnement processes.

Algorithm 1. The *student-proposing deferred acceptance algorithm* is as follows. Given each student's preference order $>_s$ over the set of schools, without ties; each school's score distribution $\theta_{.c}$ over the set of students, with zero probability of ties; and the capacity q_c of each school, the following steps are repeated until no rejections take place:

1. Each student applies to the school highest on her list.
2. Each school examines the applications it received. If it received more applicants than it can seat, it rejects its least-favorite applicants such that the remaining applicants fill its capacity exactly.
3. Each rejected student removes the school that rejected her from her list.

When the algorithm terminates, return the assignment μ, where $\mu(s)$ is highest school remaining on s's preference list, or c_0 if no schools remain.

The properties of the resultant assignment are well known: In the discrete case with $|S|$ students and $|C|$ schools, the algorithm terminates in at most $|S||C|$ iterations. The resultant matching μ is stable. μ is also strongly student optimal, meaning that if another assignment μ' is chosen from the set of stable matchings, then $\mu(s) \geq_s \mu'(s)$ for all students s, and there is at least one student for whom $\mu(s) >_s \mu'(s)$. Similarly, the resultant assignment is strongly school pessimal, meaning any other stable assignment yields the same or better students at each school. Finally, the algorithm is weakly incentive compatible for individual students. That is, no student can obtain a better match than $\mu(s)$ by falsifying her preference list. Succinct proofs of these results are given in Roth (**economicsofmatching**).

It is worthwhile to compare school-proposing reverse DA.

Algorithm 2. The *school-proposing deferred acceptance algorithm* is as follows. Given each student's preference list $>_s$ over the set of schools and each school's scores θ_c over the set of students, both without ties, and the capacity q_c of each school, the following steps are repeated until no rejections take place:

1. Each school proposes to the q_c applicants in its consideration pool. If fewer than q_c students are left, the school proposes to all remaining students.

2. Each student examines the proposals she received, rejecting all but her favorite.

3. Each school removes students who rejected it from its consideration pool.

When the algorithm terminates, return the assignment μ, where $\mu(s)$ is the school s prefers among those that proposed to her, or c_0 if she received no proposals.

This algorithm has symmetrical properties to those of forward DA, including student pessimality and incentive compatibility for the schools (but not the students). For these reasons, student-proposing deferred acceptance is seldom used in school choice, and its counterpart in the National Residency Matching Program was abandoned in favor of a resident-proposing algorithm. However, in practice the differences among the resulting assignments tend to be minor.

Under the constraint of stable assignment, student and school utility (as quantified by the preference orders) trade off directly. The student-optimal stable matching from student-proposing DA and the school-optimal stable matching from school-proposing DA define two extreme points of the set of all possible stable matchings, and they are linked by exchanging cycles of students such that student utility increases and school utility decreases, or vice-versa. This means that in a discrete context, to find *all* stable matchings, it suffices to run only the student-proposing DA algorithm, then search recursively for cycles that move the assignment toward student pessimality. Such a procedure allows us to formulate the optimal stable matching problem as the maximal closure of the cycle dependency graph (**efficientalgorithmoptimal**).

Theorem 5. *When η has strong or full support, the student-proposing DA algorithm is a tâtonnement process in which the initial cutoff vector is $p = \vec{0}$, and the school-proposing DA algorithm is a tâtonnement process in which the initial cutoff vector is $p = \vec{1}$.*

Proof. Consider the case of student-proposing DA, and let $\mu^{(k)}$ denote the tentative assignment formed at each iteration of the algorithm. That is, $\mu^{(k)}(s)$ is the school at the top of s's preference list at the beginning of the kth iteration.

It suffices to prove the following three statements:

1. Each $\mu^{(k)}$ is characterized by a cutoff vector $p^{(k)}$.

   Fix k, let $p^{(k)} \equiv p\left(\mu^{(k)}\right)$, and let $m = \mu_{p^{(k)}}$. I will show that $m = \mu^{(k)}$. Pick a student s and let $c = m(s)$. Since s is among the set of students who determined the cutoff vector $p^{(k)}$, $\mu^{(k)}(s)$ is still in s's consideration set under these cutoffs; hence, $c \geq_s \mu^{(k)}(s)$. Now suppose $c >_s \mu^{(k)}(s)$. Since s prefers c, she must have been applied to s in a previous round and been rejected. This implies $p_c^{(k)} > \theta_{sc}$; hence s is not admitted to c in m, a contradiction. It follows that $m(s) = \mu^{(k)}(s)$.

2. The initial cutoff vector has $p^{(0)} = \vec{0}$.

At the beginning of the first iteration, each student is tentatively assigned to her favorite school. By the support assumption, for all schools c, the set of students who have c at the top of their list and whose score is almost zero is nonempty. Hence, the minimum score over the tentative assignment at each school is zero.

3. $p^{(k+1)}$ is related to $p^{(k)}$ by a tâtonnement update.

   At the beginning of each iteration of student-proposing DA, students who were not rejected in the previous iteration apply to the same school as before; on the other hand, students who were rejected apply to new schools. This means that students who apply to school c at the kth iteration and are *not* rejected are a subset set of students who apply at the $k+1$th iteration; hence, at every school c, $p_c^{(k+1)} \geq p^{(k)}$.

   Suppose $p_c^{(k+1)} > p_c^{(k)}$. This implies that c rejected students during iteration k, which can occur only if the number of students tentatively matched to c at k exceeds c's capacity. Hence, $D_c(p^{(k)}) > q_c$, and the excess demand $Z_c(p^{(k)})$ is positive at k. This agrees with the sign of the change in cutoff.

   Suppose $p_c^{(k+1)} = p^{(k)}$. This means that c made no rejections during the kth iteration, or equivalently, that the number of students tentatively assigned to c is less than or equal to q_c. In our notation, $\eta(s : \mu^{(k)}(s) = c) = D_c(p^{(k)}) \leq q_c$. Hence the excess demand $Z_c(p^{(k)})$ is nonpositive. If $Z_c(p^{(k)}) = 0$, then the statement holds. If $Z_c(p^{(k)}) < 0$, then the statement holds only if $p_c^{(k)} = 0$. To get a contradiction, suppose $p_c^{(k)} > 0$. By the support assumption, there are students whose score is less than $p_c^{(k)}$ who have ranked c first. These students applied to c in an earlier round—call it j—and were rejected. This implies c filled its capacity at j. Since the students not rejected at j continue to apply to c unless rejected again, c fills its capacity at all subsequent rounds, including round k. Hence $Z_c(p^{(k)}) \geq 0$, a contradiction.

The case of school-proposing DA is analogous. □

Using this result, we can rewrite the DA algorithms above in a "computational" form that uses the cutoff vector p as the state variable. In fact, allowing $p^{(0)}$ to take an arbitrary value, we can define a whole subclass of tâtonnement processes that use deferred acceptance to update the cutoff vector. I conjecture that this process converges regardless of the value of the initial cutoff vector. However, even if that conjecture is true, we still have some distance to tread before arriving at a general algorithm for admissions market equilibrium, because the process defined below does not specify how to compute the demand vector or its roots.

Algorithm 3. A *deferred acceptance tâtonnement process* is as follows. Given an initial cutoff vector $p^{(0)}$, each student's preference order $>_s$ over the set of schools, without ties; each school's score distribution $\theta_{\cdot c}$ over the set of students, with zero probability of ties; and the capacity q_c of each school, the following steps are repeated until $p^{(k+1)} = p^{(k)}$: For $k = 0, 1, \ldots,$

1. Compute the demand vector $D(p^{(k)})$.

2. For each school c for which $D_c > q_c$, increase the cutoff so that

$$p_c^{(k+1)} \equiv p_c : D_c(p_c; p_{c'}) = q_c$$

3. For each school c for which $D_c < q_c$, decrease the cutoff (but not past zero) so that

$$p_c^{(k+1)} \equiv \begin{cases} p_c : D_c(p_c; p_{c'}) = q_c, & \text{if such a } p_c \text{ exists} \\ 0, & \text{otherwise} \end{cases}$$

4. Otherwise, let $p_c^{(k+1)} \equiv p_c^{(k)}$.

When the algorithm terminates, return $p_c^{(k)}$.

Taking $p^{(0)} = \vec{1}$, it is easy to see that the the school-proposing deferred acceptance tâtonnement process is a stylized form of the market-clearing procedure described above (§2.3.3).

This algorithm bears a strong resemblance to the so-called successive tâtonnement process in which each company adjusts its price to the value that clears its supply under the assumption that other companies' prices are fixed (**walrastatonnement**).

2.5 Computing the equilibrium

With the results above in hand, let's consider a general admissions market in which η and q are fixed. We want to compute the equilibrium of this market. It is impractical to apply a DA algorithm to nonatomic admissions markets, because DA requires using exact line search to determine the new cutoff value for each school, and in general the demand is difficult to compute.

A moderate improvement over the deferred acceptance tâtonnement process is to use a *simultaneous* tâtonnement process, similar to the one proposed above (§2.3.1), that evaluates the demand vector once per iteration and updates the cutoffs in the direction of the excess demand according to a predetermined sequence of decreasing step sizes. Under a light assumption on D, such as continuity, this process can be used to compute the equilibrium to arbitrary precision.

Algorithm 4. The *admissions equilibrium tâtonnement algorithm* is as follows. Given an initial cutoff vector $p^{(0)}$, market parameters γ and q, step parameters $\alpha > 0$ and $0 \leq \beta < 1$, and a tolerance parameter ϵ:

1. Compute the excess demand $Z = D(p^{(k)}) - q$.

2. Update the cutoffs:
$$p_c^{(k+1)} \equiv p_c^{(k)} + \frac{\alpha}{(k+1)^\beta} Z_c$$

3. Terminate if $|p_c^{(k+1)} - p_c^{(k)}| < \epsilon, \forall c$; otherwise, set $k \equiv k + 1$ and repeat.

When the algorithm terminates, return $p_c^{(k)}$.

If the demand is continuous, convergence to an ϵ-approximate equilibrium is guaranteed by the fact that the sequence of step sizes satisfies the Robbins–Monro conditions (**robbinsmonro**). However, algorithm is not necessarily computationally efficient. Although a good choice of parameters can enable the algorithm to terminate in a small number of iterations, in general, evaluating the demand vector at each iteration incurs a high computational cost. For example, even under the assumption of independence between students' preference orders and score vectors, the number of terms in each school's demand can be $|C|! \times 2^{|C|}$, as shown in equation (2).

Alternatively, if we have the ability to sample student preference lists and score vectors from η, then we can exploit the relationship between equilibrium cutoffs and stable matchings to estimate the equilibrium cutoffs with high confidence in polynomial time. The technique is as follows: Draw a discrete sample from η and run a DA algorithm. Then compute the minimum score at each school in the resultant stable assignment. If student-proposing DA is used, the expected value of each the cutoffs from the student-optimal stable match approaches the equilibrium cutoff value from below as the size of the sample goes to infinity (**supplydemandfw**). Similarly, if school-proposing DA is used, the expected value of the obtained cutoffs approaches the equilibrium from above.

Above, I described an expensive, guaranteed-precision technique and a cheap, stochastic technique for computing the equilibrium. One motivation for the model considered below (§3) is the fact that the equilibrium can computed in closed form by solving a linear system in $|C|$ equations for p, providing a baseline against which to evaluate the performance of these techniques.

2.6 Equivalent formulations of the equilibrium conditions

The conditions for a market-clearing cutoff vector given in Definition 2 can be expressed in a few additional ways. Throughout this section assume η and q are fixed and use

$$F(p) \equiv -Z(p) = q - D(p)$$

to denote the excess supply vector at p.

2.6.1 Nonlinear complementarity problem

By inspection, the market-clearing cutoff problem is equivalent to the following nonlinear complementarity problem:

$$\begin{aligned} \text{find } p : \quad F(p)^T p &= 0 \\ F(p) &\geq 0 \\ p &\geq 0 \end{aligned} \tag{5}$$

2.6.2 Variational inequality problem

By a canonical result, the following variational inequality problem is also equivalent:

$$\text{find } p \geq 0: \quad F(p)^T(\pi - p) \geq 0, \quad \forall \pi \geq 0$$

If D is strictly decreasing in p, then F is strictly increasing, and p^* is unique (**theoryofvariationalinequalities**). This, combined with the argument above establishing the existence of equilibrium (§2.3.1), yields the following theorem:

Theorem 6. *If η has full support, then the admissions market equilibrium exists and is unique.*

This is Theorem 1 of Azevedo and Leshno (**supplydemandfw**).

2.6.3 Convex optimization problem

Suppose that the excess supply function F defines a conservative vector field. This means that there exists a potential function (Lyupanov function) $\Phi(p)$ whose gradient is F:

$$\exists \Phi : \nabla_p \Phi = F$$

Such a potential function does not necessarily exist for every excess supply function. In fact, in the market considered below (3), the Jacobian of F is asymmetric, which implies that F is *not* a conservative vector field.

However, supposing Φ exists, the equilibrium can be found by solving the following concave maximization problem:

$$\text{minimize} \quad \Phi(p) \qquad \text{subject to} \quad p \geq 0$$

As the feasible set has nonempty interior, Slater's condition holds, and the optimal solution (p^*, λ^*) satisfies the following KKT conditions:

$$\begin{aligned}
&F_c - \lambda^* = 0 && \text{(stationarity)} \\
&p^* \geq 0, \quad \lambda^* \geq 0 && \text{(primal, dual feasibility)} \\
&\lambda^{*T} p^* = 0 && \text{(complementarity)}
\end{aligned}$$

Eliminating the dual variables λ^* yields the nonlinear complementarity problem above (5); hence, p^* is an equilibrium. Moreover, observe that the tâtonnement procedure of Algorithm 4 is a projected gradient ascent algorithm for this convex program, and vice-versa.

2.7 Optimization tasks

With the equivalence results established above in hand, we can expand our understanding of admissions markets to encompass a range of optimization tasks that span a variety of realistic scenarios.

First, there is the canonical school-choice problem. Given η and the capacity vector q, we must compute a stable matching μ. It suffices to find the equivalent cutoff vector p, as discussed above (§4).

Second, there is the *inverse optimization* problem. Given p and demand D, we try to infer information about η such as the overall preferability of each school or the joint distribution of students' scores. This task requires many simplifying assumptions, because the number of student distributions that could induce a given stable assignment is typically infinite.

In the second half of this study, I turn to an example of a nonatomic admissions markets in which all of these problems can be solved efficiently, even when the number of schools is relatively large.

3 Single-score model with multinomial logit preferences

This section of the study considers a special kind of admissions market that has not received much attention in the school-choice literature but approximates the admissions procedure used in many systems around the world. From the standpoint of computing stable matchings, it is relatively simple: All schools have the same preference order, and students' preference orders are determined by the multinomial logit choice model. It is easy to generate a discrete sample of students.

However, this model also admits a piecewise expression for the demand that is locally invertible, which enables us not only to compute the equilibrium cutoffs exactly, but also to compute the gradient of the market parameters with respect to one another both in and out of equilibrium. This enables an interesting econometric analysis of the incentives available to schools.

3.1 Model description

3.1.1 Characterization of η

To characterize η, we must describe both how schools rank students, and how students rank schools.

In a *single-score model,* all schools share the same ranking over the students. A single-score system may arise in one of several real-world scenarios. The most obvious case is a centralized admissions market in which the government requires schools to admit students solely on the basis of a single standardized test. Alternatively, if students are scored using various dimensions of student characteristics such as test scores, GPA, and the quality of their letters of recommendation, it is common for these various dimensions to correlate tightly. If so, then principle component analysis can be used to determine a composite score whose order approximates the ordering of students at each university. Finally, in many public school systems, schools have no preference order over the students; instead students take turns picking their favorite school in an order determined by random lottery, or (equivalently) the single tiebreaking mechanism is used to generate schools' preference lists and the assignment of students to schools is computed using student-proposing DA (**whatmatters**).

Regardless of the device used to generate the single scores, taking percentile scores over the final distribution allows us to assume that the scores are uniformly distributed on the interval $[0, 1]$.

As for students' choice of school, this paper assumes students use *multinomial logit* (MNL) choice to derive their preference lists. This means that each school has a given preferability parameter δ_c. Letting $C^\# \subseteq C$ denote set of schools to which a given student is admitted, she chooses to attend school $c \in C^\#$ with

$$\frac{\exp \delta_c}{\sum_{d \in C^\#} \exp \delta_d}$$

For convenience, let $\gamma_c \equiv \exp \delta_c > 0$ and $\Gamma = \sum_c \gamma_c$. Since the equation is homogeneous in γ, we may assume without loss of generality that $\Gamma = 1$; however, I will resist this assumption since many parameter-estimation techniques for MNL choice do not use it.

Observe that in the single-score model with MNL choice, η does *not* have full support, because the probability of having different scores at any two schools is zero. However, the algebraic analysis below reveals that the equilibrium is unique.

3.1.2 Demand function

Let's derive the demand function $D(\gamma, p)$ for the single-score model with MNL student preferences.

First, sort the schools by cutoff, i.e. so that

$$p_1 \leq p_2 \leq \cdots \leq p_{|C|}$$

Ties may be broken arbitrarily, as discussed below. Since getting into school c implies getting into any school whose cutoff is less than or equal to p_c, there are only $|C| + 1$ possible consideration sets for each student:

Symbol	**Consideration set**	**Probability**
$C_{[0]}$	$\varnothing$	p_1
$C_{[1]}$	$\{c_1\}$	$p_2 - p_1$
$C_{[2]}$	$\{c_1, c_2\}$	$p_3 - p_2$
$\vdots$	$\vdots$	$\vdots$
$C_{[\lvert C\rvert-1]}$	$\{c_1, \ldots, c_{\lvert C\rvert-1}\}$	$p_{\lvert C\rvert} - p_{\lvert C\rvert-1}$
$C_{[\lvert C\rvert]}$	$\{c_1, \ldots, c_{\lvert C\rvert}\}$	$1 - p_{\lvert C\rvert}$

Hence, the demand for school c is the sum of the number of students with each of these consideration sets who choose to attend c. Letting $p_{|C|+1} \equiv 1$, that is

$$D_c = \sum_{d=c}^{|C|} \underbrace{\frac{\exp \delta_c}{\sum_{i=1}^{d} \exp \delta_i}}_{\substack{\text{prob. of choosing} \\ c \text{ from cons. set}}} \overbrace{(p_{d+1} - p_d)}^{\substack{\text{prob. of having} \\ \text{cons. } C_{[d]}}} \tag{6}$$

If at least one school has $p_c = 0$, then every student can get in somewhere, and $\sum_c D_c = 1$. Generally, there are p_1 students who get in nowhere, and $\sum_c D_c = 1 - p_1$.

3.1.3 Properties of the demand function

D is *continuous* in p. To see this, expand the equation above:

$$D_c = \gamma_c \left[\left(\frac{-1}{\sum_{i=1}^{c} \gamma_i} \right) p_c + \left(\frac{1}{\sum_{i=1}^{c} \gamma_i} - \frac{1}{\sum_{i=1}^{c+1} \gamma_i} \right) p_{c+1} + \cdots + \left(\frac{1}{\sum_{i=1}^{|C|-1} \gamma_i} - \frac{1}{\sum_{i=1}^{|C|} \gamma_i} \right) p_{|C|} + \frac{1}{\sum_{i=1}^{|C|} \gamma_i} \right]$$

Since D is linear in any neighborhood where the order of cutoffs is unambiguous, the only opportunity for discontinuity occurs when two or more cutoffs are equal. Thus, it suffices to show that the value of D_c is independent of how ties among the p_c are broken. Suppose that $p_j = \cdots = p_{j+n} = \tilde{p}$ for some $j > c$. Then (dividing by γ_c for legibility),

$$\frac{D_c}{\gamma_c} = \cdots + \left(\frac{1}{\sum_{i=1}^{j-1} \gamma_i} - \frac{1}{\sum_{i=1}^{j} \gamma_i} \right) p_j + \left(\frac{1}{\sum_{i=1}^{j} \gamma_i} - \frac{1}{\sum_{i=1}^{j+1} \gamma_i} \right) p_{j+1} + \cdots + \left(\frac{1}{\sum_{i=1}^{j+n} \gamma_i} - \frac{1}{\sum_{i=1}^{j+n+1} \gamma_i} \right) p_{j+n} + \cdots \tag{7}$$

$$= \cdots + \left(\frac{1}{\sum_{i=1}^{j-1} \gamma_i} - \cancel{\frac{1}{\sum_{i=1}^{j} \gamma_i}} \right) \tilde{p} + \left(\cancel{\frac{1}{\sum_{i=1}^{j} \gamma_i}} - \cancel{\frac{1}{\sum_{i=1}^{j+1} \gamma_i}} \right) \tilde{p} + \cdots + \left(\cancel{\frac{1}{\sum_{i=1}^{j+n} \gamma_i}} - \frac{1}{\sum_{i=1}^{j+n+1} \gamma_i} \right) \tilde{p} + \cdots \tag{8}$$

$$= \cdots + \left(\frac{1}{\sum_{i=1}^{j-1} \gamma_i} - \frac{1}{\sum_{i=1}^{j+n+1} \gamma_i} \right) \tilde{p} + \cdots \tag{9}$$

The internal sums that depend on the order of the indices $j \ldots j+n$ cancel out; hence, they may be arbitrarily reordered without changing the value of D_c. Similar canceling shows that the demand does not vary under tiebreaking when c itself is involved in a tie. Hence, D is continuous in p.

The expansion above also allows us to see that the demand vector is defined by the *matrix equation*

$$D = Ap + \frac{1}{\Gamma}\gamma \tag{10}$$

where $A \in \mathbb{R}^{|C|\times|C|}$ is the triangular matrix with

$$A_{ij} \equiv \begin{cases} 0, & i > j \\ -\gamma_i \left(\frac{1}{\sum_{k=1}^{i} \gamma_k} \right), & i = j \\ \gamma_i \left(\frac{1}{\sum_{k=1}^{j-1} \gamma_k} - \frac{1}{\sum_{k=1}^{j} \gamma_k} \right), & i < j \end{cases} \tag{11}$$

$$\implies A = \begin{bmatrix} \gamma_1 \left(\frac{-1}{\gamma_1} \right) & \gamma_1 \left(\frac{1}{\gamma_1} - \frac{1}{\gamma_1+\gamma_2} \right) & \gamma_1 \left(\frac{1}{\gamma_1+\gamma_2} - \frac{1}{\gamma_1+\gamma_2+\gamma_3} \right) & \cdots & \gamma_1 \left(\frac{1}{\sum_{i=1}^{|C|-1} \gamma_i} - \frac{1}{\Gamma} \right) \\ & \gamma_2 \left(\frac{-1}{\gamma_1+\gamma_2} \right) & \gamma_2 \left(\frac{1}{\gamma_1+\gamma_2} - \frac{1}{\gamma_1+\gamma_2+\gamma_3} \right) & \cdots & \gamma_2 \left(\frac{1}{\sum_{i=1}^{|C|-1} \gamma_i} - \frac{1}{\Gamma} \right) \\ & & \gamma_3 \left(\frac{-1}{\gamma_1+\gamma_2+\gamma_3} \right) & \cdots & \gamma_3 \left(\frac{1}{\sum_{i=1}^{|C|-1} \gamma_i} - \frac{1}{\Gamma} \right) \\ & & & \ddots & \vdots \\ & & & & \gamma_{|C|} \left(\frac{1}{\sum_{i=1}^{|C|-1} \gamma_i} - \frac{1}{\Gamma} \right) \end{bmatrix} \tag{12}$$

Since $\gamma > 0$, A is invertible.

Because the matrix A depends on the order of the p_c values, the demand function is *piecewise linear* in p. In the context of an iterative schema like those discussed below, instead of sorting p itself, it is often simpler to permute the rows and columns of A according to the inverse of the permutation that sorts p.

3.1.4 Appeal function

Another interesting indicator from Azevedo and Leshno (**supplydemandfw**) is the *appeal* of a school's entering class, or the integral of scores over the set of admitted students. The appeal of the entering class is not necessarily the school's objective function, because schools may value an abstract notion of selectivity or students' tuition dollars higher than this value.

The average score of a student with consideration set $C_{[d]}$ is $\frac{1}{2}(p_{d+1} + p_d)$, so the appeal at c is

$$L_c = \sum_{d=c}^{|C|} \underbrace{\frac{\gamma_c}{\sum_{i=1}^{d} \gamma_i}}_{\substack{\text{prob. of choosing} \\ c \text{ from cons. set}}} \overbrace{(p_{d+1} - p_d)}^{\substack{\text{prob. of having} \\ \text{cons. set } C_{[d]}}} \underbrace{\frac{1}{2}(p_{d+1} + p_d)}_{\substack{\text{avg. score of students} \\ \text{with cons. set } C_{[d]}}} = \frac{1}{2} \sum_{d=c}^{|C|} \frac{\gamma_c}{\sum_{i=1}^{d} \gamma_i} \left(p_{d+1}^2 - p_d^2 \right) \tag{13}$$

By comparison with the expression for D, the quality vector is given by

$$L = \frac{1}{2} A p.^2 + \frac{1}{2\Gamma} \gamma$$

where the notation $p.^2 = (p_1^2, \ldots, p_{|C|}^2)$ represents the entrywise square of p.

3.2 Computing the equilibrium

In the market under consideration, the equilibrium conditions are as follows:

$$
\begin{aligned}
D = Ap + \frac{1}{\Gamma}\gamma &\le q \\
D_c = A_{c.}p + \frac{1}{\Gamma}\gamma_c &= q_c, \quad \forall c : p_c > 0
\end{aligned}
\tag{14}
$$

As I will now show, it turns out that at equilibrium, the order of the school cutoffs is determined by the order of the *competitiveness ratios* γ_c/q_c. This fact enables us to compute the equilibrium directly by solving a linear system. (Below, the positive part operator x^+ works elementwise on its argument x. That is, $(x^+)_i \equiv \max\{0, x_i\}$.)

Theorem 7. *Without loss of generality, suppose that* $\frac{\gamma_1}{q_1} \le \cdots \le \frac{\gamma_{|C|}}{q_{|C|}}$. *Then* $\hat{p}_1 \le \cdots \le \hat{p}_{|C|}$, *and*

$$
\hat{p} \equiv \left[A^{-1}(q - \frac{1}{\Gamma}\gamma)\right]^+
$$

is the market equilibrium in the single-score, MNL choice model. Moreover, the equilibrium is unique.

Proof. I show the following statements:

1. $\hat{p}$ satisfies $\hat{p}_1 \le \cdots \le \hat{p}_{|C|}$. This means that the demand at $\hat{p}$ is given by the expression $A\hat{p} + \frac{1}{\Gamma}\gamma$ (which only holds if $\hat{p}$ is sorted).

2. $\hat{p}$ is the same regardless of how ties among the competitiveness ratios are resolved. This means that the equilibrium is unique, because the only opportunity for multiple equilibria occurs when there are ties among the competitiveness ratios γ_c/q_c.

3. $\hat{p}$ satisfies the equilibrium conditions given in equation (14).

For convenience, let $\bar{p} \equiv A^{-1}(q - \frac{1}{\Gamma}\gamma)$, so that $\hat{p} = \bar{p}^+$.

$\hat{p}$ is sorted. Pick any school $c < |C|$. It suffices to show that $\bar{p}_{c+1} - \bar{p}_c \ge 0$. The inverse of A is

$$
A^{-1} = \begin{bmatrix}
\frac{-1}{\gamma_1}(\gamma_1) & -1 & -1 & \cdots & -1 \\
 & \frac{-1}{\gamma_2}(\gamma_1+\gamma_2) & -1 & \cdots & -1 \\
 & & \frac{-1}{\gamma_2}(\gamma_1+\gamma_2+\gamma_3) & \cdots & -1 \\
 & & & \ddots & \vdots \\
 & & & & \frac{-1}{\gamma_{|C|}}\Gamma
\end{bmatrix}
\tag{15}
$$

It is not difficult to verify that

$$
\bar{p}_{c+1} - \bar{p}_c = \left[A^{-1}(q - \frac{1}{\Gamma}\gamma)\right]_{c+1} - \left[A^{-1}(q - \frac{1}{\Gamma}\gamma)\right]_c = \left(\sum_{j=1}^{c} \gamma_j\right)\left(\frac{q_c}{\gamma_c} - \frac{q_{c+1}}{\gamma_{c+1}}\right) \ge 0 \tag{16}
$$

which follows from the assumption that $\gamma_c/q_c \le \gamma_{c+1}/q_{c+1}$.

$\hat{p}$ is invariant under tiebreaking. Inspecting the expression for $\bar{p}_{c+1} - \bar{p}_c$ (**??**) reveals that the gap between adjacent cutoffs is zero when their competitiveness ratios are the same; therefore, the values of $\bar{p}_{c+1}$ and $\bar{p}_c$ do not change when tied indices c and $c+1$ are exchanged.

$\hat{p}$ satisfies the equilibrium conditions. The demand at $\hat{p}$ is $D = A\hat{p} + \frac{1}{\Gamma}\gamma$. Hence

$$\begin{aligned} \hat{p} = A^{-1}(D - \frac{1}{\Gamma}\gamma) = \bar{p}^{+} \geq \bar{p} = A^{-1}(q - \frac{1}{\Gamma}\gamma) \\ \implies \quad A^{-1}D \geq A^{-1}q \\ \implies \quad D \leq q \end{aligned}$$

The final statement follows from the fact that A^{-1} is triangular and its nonzero entries are strictly negative. This establishes the capacity condition.

Now, we need to show that the demand equals the capacity when $\hat{p}_c > 0$. Let b denote the first school with a nonzero cutoff. That is, $\hat{p}_1 = \cdots = \hat{p}_{b-1} = 0$, and $0 < \hat{p}_b \leq p_{b+1} \leq \cdots \leq \hat{p}_{|C|}$. Then the demand at $\hat{p}$ may be written

$$\begin{aligned} D &= A\hat{p} + \frac{1}{\Gamma}\gamma \\ &= \sum_{i=1}^{|C|} A_{.i}\hat{p}_i + \frac{1}{\Gamma}\gamma \\ &= \sum_{i=1}^{|C|} A_{.i}\left[A^{-1}\left(q - \frac{1}{\Gamma}\gamma\right)\right]_i^{+} + \frac{1}{\Gamma}\gamma \\ &= \sum_{j=b}^{|C|} A_{.j}\left[A^{-1}\left(q - \frac{1}{\Gamma}\gamma\right)\right]_j + \frac{1}{\Gamma}\gamma \\ &= \left[\sum_{j=b}^{|C|} A_{.j}A^{-1}_{j.}\right]\left(q - \frac{1}{\Gamma}\gamma\right) + \frac{1}{\Gamma}\gamma \\ &= \begin{bmatrix} 0_{b\times b} & T_{b\times(|C|-b)} \\ 0_{(|C|-b)\times b} & I_{|C|-b} \end{bmatrix}\left(q - \frac{1}{\Gamma}\gamma\right) + \frac{1}{\Gamma}\gamma \end{aligned} \tag{17}$$

where

$$T = \begin{bmatrix} \frac{-\gamma_1}{\sum_{i=1}^{b-1}\gamma_i} & \cdots & \frac{-\gamma_1}{\sum_{i=1}^{b-1}\gamma_i} \\ \vdots & \cdots & \vdots \\ \frac{-\gamma_{b-1}}{\sum_{i=1}^{b-1}\gamma_i} & \cdots & \frac{-\gamma_{b-1}}{\sum_{i=1}^{b-1}\gamma_i} \end{bmatrix} \tag{18}$$

For the schools with $\hat{p}_c > 0$, the demand is

$$D_c = \begin{bmatrix} 0 & I \end{bmatrix}_{c.}\left(q - \frac{1}{\Gamma}\gamma\right) + \frac{1}{\Gamma}\gamma = q_c$$

Hence, the stability criterion holds, and $\hat{p}$ is an equilibrium. $\square$

For reference, for the schools with $p_c = 0$, the demand at equilibrium is

$$D_c = \begin{bmatrix} 0 & T \end{bmatrix}_{c.} \left(q - \frac{1}{\Gamma}\gamma \right) + \frac{1}{\Gamma}\gamma = \frac{-\gamma_c}{\sum_{i=1}^{b-1}\gamma_i} \sum_{j=b}^{|C|} \left(q_j - \frac{1}{\Gamma}\gamma_j \right) + \frac{1}{\Gamma}\gamma_c \leq q_c$$

3.3 Validation

In this section, I offer a numerical demonstration of the cutoff sort order given by Theorem 7. Additionally, I validate the interpretation of the equilibrium cutoffs as a stationary point of a tâtonnement process, as a market-clearing price vector, and as the limit point of stable assignments. (The demonstration of the latter two interpretations follows Azevedo and Leshno (**supplydemandfw**).)

Figure 1 demonstrates the relationship between the equilibrium cutoffs p_c^* and the competitiveness ratios γ_c/q_c in randomly generated markets. Some of the markets are overdemanded, yielding $p^* > 0$, and others are underdemanded; however, the equilibrium cutoffs are always ordered according to the competitiveness ratios. As the graph indicates, the precise relationship is nonlinear and highly sensitive to variance in the market parameters. This suggests that even in the highly stylized model under consideration, it is difficult to predict the effect of small perturbation in a single γ_c or q_c value on the market as a whole simply from looking at the equilibrium assignment of students.

Figures 2 through 4 consider a fictional admissions market called Birchtown, which has the following parameters. I have sorted the schools by their optimal cutoffs:

$$\begin{aligned} \gamma &= \left(\tfrac{2}{12}, \tfrac{1}{12}, \tfrac{3}{12}, \tfrac{6}{12}\right) \\ q &= (0.3, 0.1, 0.2, 0.2) \\ p^* &= (0.2, 0.3, 0.4, 0.6) \\ D(p^*, \gamma) &= (0.3, 0.1, 0.2, 0.2) \end{aligned}$$

As the total capacity is less than one, each school fills its capacity at equilibrium.

Figure 2 shows fifty iterations of the simultaneous tâtonnement process (Algorithm 4) applied to Birchtown. At each iteration, the demand is computed directly by evaluating the expression derived above (6). Then, the cutoff vector is adjusted in the direction of excess demand according to a predetermined, decreasing sequence of step size. The cutoffs converge smoothly toward p^*, which suggests the continuity of the demand function and the stability of the equilibrium.

Figures 3 and 4 consider discrete approximations of the Birchtown admissions markets with 20, 200, or 2000 students. In Figure 3, schools admit students according to their equilibrium cutoffs, students choose their favorite school, and schools observe their demand. When there are many students, the demand at each school approximately equals its capacity. In Figure 4, a stable matching of the students is computed so that each school fills its capacity exactly. When there are many students, the implied cutoffs approximately equal p^*.

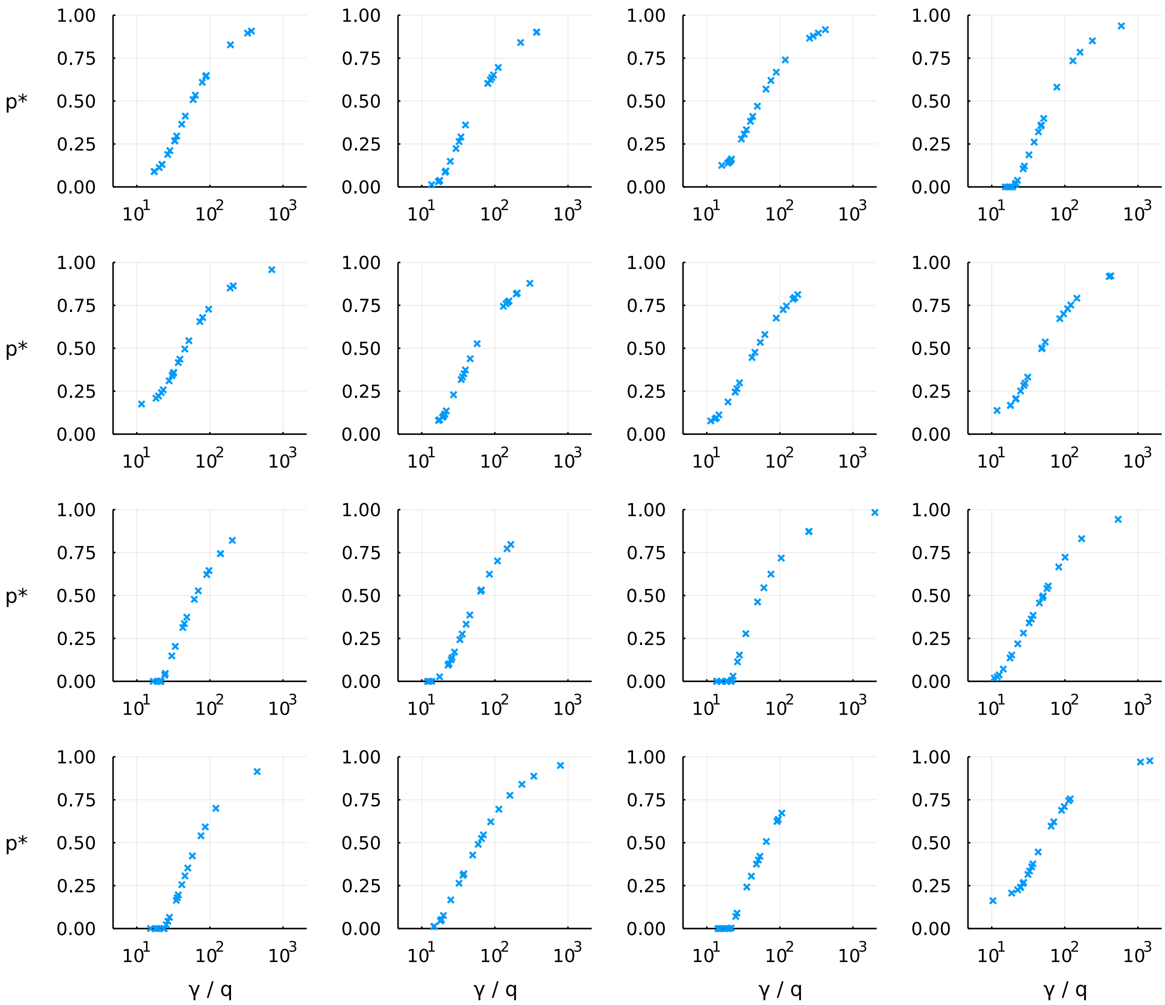

Figure 1: Competitiveness ratios γ_c/q_c and equilibrium cutoffs p_c^* in 16 randomly generated admissions markets each containing 20 schools. The preferability parameters δ_c are drawn from $\mathrm{Uniform}(0,1)$; then each $\gamma_c \equiv \exp\delta_c/\sum_{j\in C}\exp\delta_j$. The capacities are drawn from $\mathrm{Uniform}(0,1/20)$; hence the market as a whole has a 50 percent chance of being over- or underdemanded. The figure shows that the order of the equilibrium cutoffs is determined by the order of competitiveness ratios.

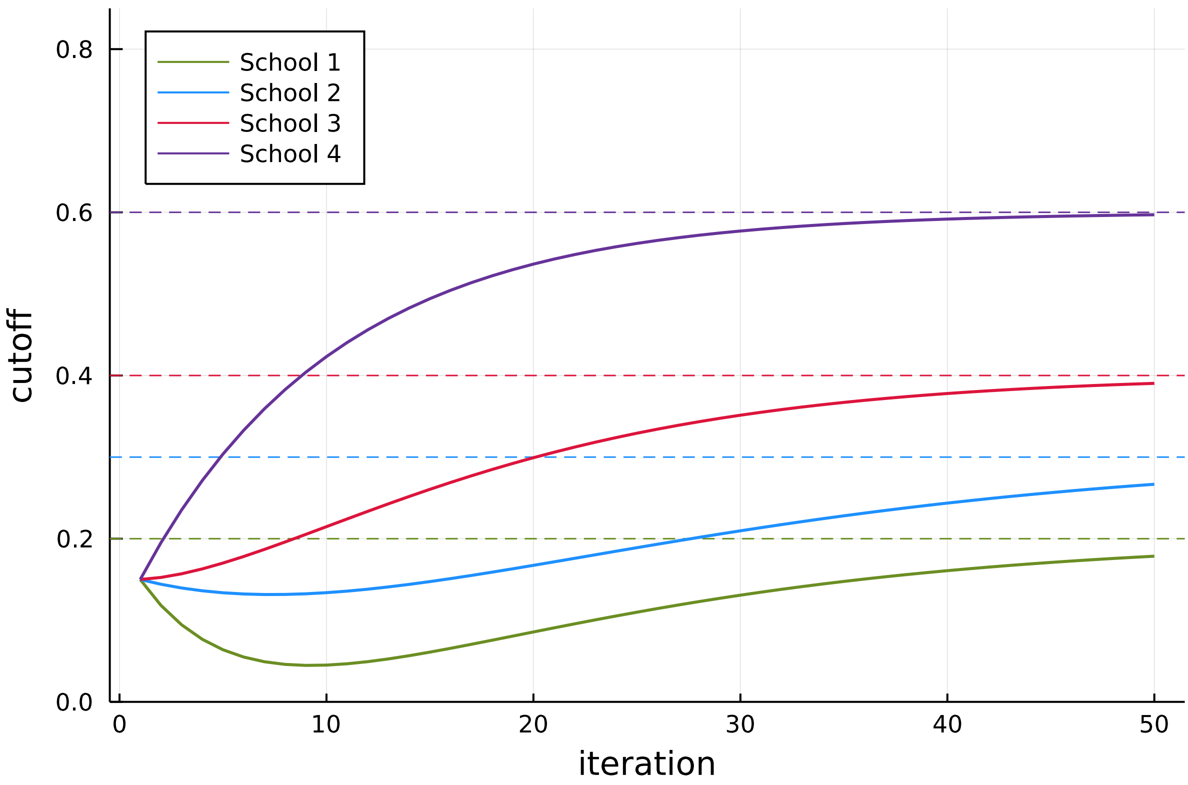

Figure 2: Convergence of fifty iterations of the simultaneous tâtonnement process (Algorithm 4) in the fictional nonatomic admissions market of Birchtown (see text). The step size parameters are $\alpha = 0.2, \beta = 0.01$, and the initial cutoff vector is $p^{(0)} = (0.15, 0.15, 0.15, 0.15)$.

Compare Figures 3 and 4 with Figure 4 of Azevedo and Leshno (**supplydemandfw**), which demonstrates the asymptotic convergence of implicit cutoffs as the number of students increases using a numerical experiment in which scores are partially correlated between schools.

3.4 Unconstrained comparative statics

First, I derive comparative statics results that apply to an unconstrained market in which schools have no capacity constraints. This section makes no particular claim about what schools' objective functions are; rather, I simply compute the gradients of the demand and appeal functions with respect to p, and γ using the relations $D = Ap + \gamma$ and $L = \frac{1}{2}Ap.^2 + \frac{1}{2}\gamma$ and discuss their interpretations. These quantities prove useful in the following section, in which I quantify the incentives available to schools under a centralized admissions procedure that always produces a stable matching.

3.4.1 Cutoff effects

The change in demand in response to a change in cutoffs is the Jacobian of the demand function:

$$\mathbf{J}_p D = A$$

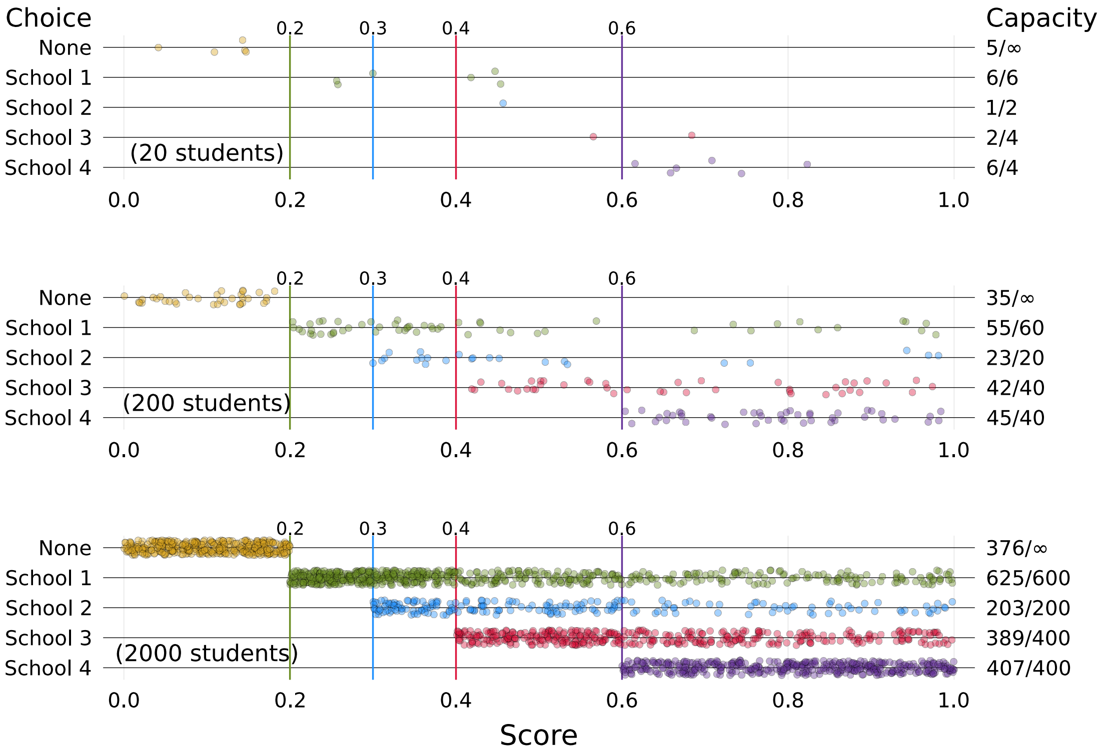

Figure 3: Simulation of a decentralized school-choice process in Birchtown. A discrete sample of student preference lists and scores is drawn from η. Each school admits students whose score exceeds its equilibrium cutoff (shown as vertical lines), then each student chooses her favorite school from her consideration set. As the sample size increases, the demand at each school approximately equals its scaled capacity.

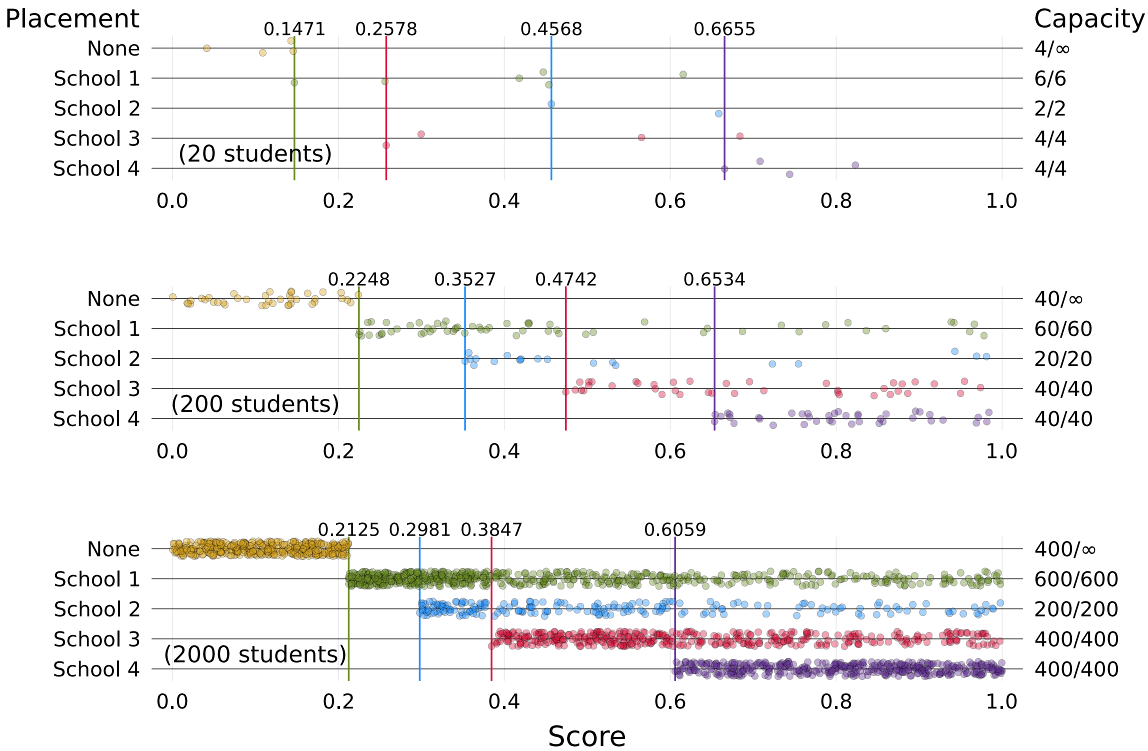

Figure 4: Simulation of a deferred acceptance process in Birchtown. A discrete sample of student preference lists and scores is drawn from η. The student-proposing deferred acceptance algorithm (Algorithm 1) is used to compute a stable matching. The score of the least-qualified admit at each school, or the school's implied score cutoff, is computed and represented as a vertical line. Regardless of the sample size, each school fills its scaled capacity, and as the sample size increases, the implied cutoffs approach the market equilibrium. Comparison with Figure 2 and Figures 3 suggests the equivalence among tâtonnement equilibrium, market clearing cutoffs, and stable assignments in nonatomic admissions markets.

The diagonal is negative, meaning that each school's demand is decreasing in its cutoff, as expected. The entries above the diagonal are positive, while those below the diagonal are zero. This means that each school c's demand is increasing in the cutoffs of the *more-selective* schools, but the cutoffs of *less-selective schools* have no local effect on the demand at c.

Intuitively, this means that if all schools are equally preferable, a highly selective school has more market power than the others: If it increases its cutoff, it will cause many students to move onto another school. On the other hand, a school c' that is less preferable than c cannot affect D_c's demand by changing its own cutoff, because any student currently admitted to c was already admitted to c', and chose c instead.

Observe also that $-1 = A_{11} < A_{22} < \cdots < A_{|C||C|} < 0$. This says that the school with the most generous cutoff has the most power to increase its demand with a marginal decrease in p_c. Intuitively, this is because a student who gets into a school with a large cutoff gets into *many* schools, so competition for this student is fiercer than for a student whose options are already limited by a low score.

Next, consider the change in the entering classes' *appeal* in response to a change in cutoffs:

$$\mathbf{J}_p L = A \operatorname{diag}(p)$$

For $p_c > 0$, the cutoff effect on appeal has the same direction as the cutoff effect on demand. Intuitively, this suggests that if a school's goal is to maximize the appeal of its entering class, it will tend to try to lower its score cutoffs as much as it can, subject to constraints on its total demand. However, the magnitude of the incentive increases when p_c is higher. This tends to counteract the market power effect described above: A school with a low cutoff has the power to attract more marginal students, but does so with little overall effect on the aggregate appeal of its entering class. In the extreme case, when $p_c = 0$, the appeal associated with a marginal student is exactly zero.

The derivatives given above are well-defined when the cutoffs are totally ordered. However, an edge case occurs when there is a tie among the cutoffs; then the subdifferential set is given by the convex hull of the Jacobians associated with the possible permutations of p. In this case, I argue that the best interpretation of the effect of an *increase* in p_c should be that associated with the permutation for which c is indexed after schools with which its cutoff is tied. That is, because p_c is "about to" become larger than the other cutoffs involved in the tie, break the tie in its favor. Likewise, to interpret a *decrease* in a tied p_c, treat p_c as the least member of the tied set.

3.4.2 Quality effects

Differentiate the demand with respect to γ to obtain the effect of a marginal change in quality:

$$(\mathbf{J}_\gamma D)_{c\hat{c}} = \frac{\partial}{\partial \gamma_{\hat{c}}} D_c = \begin{cases} \sum_{d=c}^{|C|} \frac{-\gamma_c}{\left(\sum_{i=1}^{d}\right)^2} \left(p_{d+1} - p_d\right), & \hat{c} < c \\ \sum_{d=c}^{|C|} \frac{1}{\left(\sum_{i=1}^{d}\right)} \left(1 - \frac{\gamma_c}{\left(\sum_{i=1}^{d}\right)}\right) \left(p_{d+1} - p_d\right), & \hat{c} = c \\ \sum_{d=\hat{c}}^{|C|} \frac{-\gamma_c}{\left(\sum_{i=1}^{d}\right)^2} \left(p_{d+1} - p_d\right), & \hat{c} > c \end{cases} \tag{19}$$

(Note that the $\hat{c} > c$ and $\hat{c} < c$ cases differ in the starting index.) The demand for c is predictably decreasing in the quality of the other schools and increasing in γ_c.

A similar picture emerges when we differentiate the appeal with respect to γ:

$$(\mathbf{J}_\gamma L)_{c\hat{c}} = \frac{\partial}{\partial \gamma_{\hat{c}}} L_c = \begin{cases} \frac{1}{2} \sum_{d=c}^{|C|} \frac{-\gamma_c}{\left(\sum_{i=1}^{d}\right)^2} \left(p_{d+1}^2 - p_d^2\right), & \hat{c} < c \\ \frac{1}{2} \sum_{d=c}^{|C|} \frac{1}{\left(\sum_{i=1}^{d}\right)} \left(1 - \frac{\gamma_c}{\left(\sum_{i=1}^{d}\right)}\right) \left(p_{d+1}^2 - p_d^2\right), & \hat{c} = c \\ \frac{1}{2} \sum_{d=\hat{c}}^{|C|} \frac{-\gamma_c}{\left(\sum_{i=1}^{d}\right)^2} \left(p_{d+1}^2 - p_d^2\right), & \hat{c} > c \end{cases} \tag{20}$$

By the same procedure used to show the continuity of D above, it is easy to see that the quality effects are continuous across tiebreaking permutations of p.

3.5 Comparative statics at equilibrium

Now, let's analyze the incentives available to schools and government planners when the market is constrained to equilibrium, for example, by a centralized admissions process that uses a DA algorithm to produce a stable matching. Throughout this section, I assume that schools are indexed in ascending order by the competitiveness ratios γ_c/q_c, and that no ties occur among these ratios or among the equilibrium cutoffs, to avoid the undefined derivatives that can occur in the edge cases discussed above.

3.5.1 Quality effects at equilibrium

First, I will focus on the effect of a marginal change in quality on the allocation of students at equilibrium. Since, in theory, schools have the power to change their own quality by investing in their programs or marketing, the analysis below enables us to quantify the extent to which these investments are "worth it" with respect to the school's interest in maintaining high admissions standards or increasing its demand.

First, I provide yet another expression for the equilibrium cutoff vector $\hat{p}$, which can be verified by expanding the equation given in Theorem 7. $\hat{p}_c = \bar{p}_c^+$, where

$$\bar{p}_c = \frac{1}{\gamma_c}\left(\frac{\gamma_c}{\Gamma} - q_c\right)\sum_{i=1}^{c}\gamma_i + \sum_{j=c+1}^{|C|}\left(\frac{\gamma_j}{\Gamma} - q_j\right) \tag{21}$$

and, in the $c = |C|$ case, I take $\sum_{j=|C|+1}^{|C|} \left(\frac{\gamma_j}{\Gamma} - q_j\right) = 0$. This assumes the schools are indexed in ascending order by the competitiveness ratios γ_c / q_c.

Differentiating the optimal cutoffs with respect to the quality and simplifying, we have

$$(\mathbf{J}_\gamma \hat{p})_{c\hat{c}} = \frac{\partial}{\partial \gamma_{\hat{c}}} \hat{p}_c = \begin{cases} 0, & \bar{p}_c < 0 \\ \text{undefined}, & \bar{p}_c = 0 \\ -\frac{q_c}{\gamma_c}, & \bar{p}_c > 0 \text{ and } \hat{c} < c \\ \frac{q_c}{\gamma_c^2} \sum_{i=1}^{c-1} \gamma_i, & \bar{p}_c > 0 \text{ and } \hat{c} = c \\ 0, & \bar{p}_c > 0 \text{ and } \hat{c} > c \end{cases} \tag{22}$$

The Jacobian is lower triangular–any change in the quality of a school whose competitiveness ratio is already higher than that of c induces no change in the cutoff at c. As above, in the $c = 1$ case, we interpret the empty set as summing to zero: $\sum_{i=1}^{0} \gamma_i = 0$.

Applying the chain rule to the demand at equilibrium $D = A\hat{p} + \frac{1}{\Gamma}\gamma$, and letting b denote the index of the first school with a nonzero cutoff (as above), the derivative of the equilibrium demand at c with respect to the quality of $\hat{c}$ is

$$(\mathbf{J}_\gamma D(\hat{p}))_{c\hat{c}} = \frac{\partial}{\partial \gamma_{\hat{c}}} D(\hat{p}_c) = \begin{cases} -\gamma_c \frac{1 - \sum_{j=b}^{|C|} q_j}{\left(\sum_{i=1}^{b-1} \gamma_i\right)^2}, & \bar{p}_c < 0 \text{ and } \hat{c} \neq c \\ \left(-\gamma_c + \sum_{k=1}^{b-1} \gamma_k\right) \frac{1 - \sum_{j=b}^{|C|} q_j}{\left(\sum_{i=1}^{b-1} \gamma_i\right)^2}, & \bar{p}_c < 0 \text{ and } \hat{c} = c \\ \text{undefined}, & \bar{p}_c = 0 \\ 0, & \bar{p}_c > 0 \end{cases} \tag{23}$$

The appeal function can be similarly differentiated, but does not admit a legible representation for an arbitrary number of schools. However, supposing that schools treat their appeal function as an objective function to be maximized, we can determine the signs along the Jacobian of the appeal function's main diagonal by inspection.

Disregarding the knife-edge case in which $\bar{p}_c = 0$, the two derivatives above suggest that schools in the single-test model fall into one of two clear categories. For the schools for which $\bar{p}_c < 0$, a marginal improvement in quality increases the *size* of the entering class but has no effect on its *minimum score* (and, in general, the effect on the average score is small). On the other hand, for the schools for which $\bar{p}_c > 0$, their capacity is always filled at equilibrium, and any investment in quality translates into an improvement in the minimum score of the entering class. Although I have not specified the objective functions of any schools, if the objective functions are a combination of cutoff and demand, this analysis suggests that competition within these two broad groups of schools is close to zero-sum. Underdemanded schools compete for the finite pool of tuition dollars available in the market, whereas overdemanded schools compete for a choice slice of a fixed distribution of student talent.

3.5.2 Capacity and population effects

Consulting the cutoff sortation result of Theorem 7, it is easy to see that the derivative of the equilibrium cutoffs with respect to a given school's capacity is

$$(\mathbf{J}_q\hat{p})_{c\hat{c}} = \frac{\partial}{\partial q_{\hat{c}}}\hat{p}_c = \begin{cases} 0, & \bar{p}_c < 0 \\ \text{undefined}, & \bar{p}_c = 0 \\ A^{-1}_{c\hat{c}}, & \bar{p}_c > 0 \text{ and } \hat{c} < c \end{cases} \tag{24}$$

The derivative of the demand, by inspecting Equation (17), has

$$\mathbf{J}_q D(\hat{p}) = \begin{bmatrix} 0_{b\times b} & T_{b\times(|C|-b)} \\ 0_{(|C|-b)\times b} & I_{|C|-b} \end{bmatrix} \tag{25}$$

where the entries of T are negative as given in (18).

This confirms the intuitive result that only schools that are overdemanded at equilibrium can make use of excess capacity. In addition, observe that because $\mathbf{J}_q\hat{p}$ is upper triangular, adding capacity to a school whose competitiveness ratio is lower than that of c has no marginal effect on the equilibrium cutoff at c.

Effect of a change in total student population (i.e. scaling capacity) Effect of a change in capacity

3.6 Inverse optimization of student preferences

In this section, I consider the inverse optimization task, in which the demand of the market and the cutoff vector is provided and we attempt to compute the quality parameters. That is, we must solve the following system for γ:

$$D_c = \sum_{d=c}^{|C|} \frac{\gamma_c}{\sum_{i=1}^{d}\gamma_i}\left(p_{d+1} - p_d\right), \quad \forall c \in C \tag{26}$$

Assume the schools are sorted in ascending order of cutoffs, and by homogeneity, let $\sum_{c\in C}\gamma_c = 1$.

Consider the demand for $|C|$, the school with the highest index and therefore highest cutoff. Students who get into this school necessarily get into every school, so the outer sum of the system (3.6) has only one term, and the equation becomes

$$D_{|C|} = \frac{\gamma_{|C|}}{\sum_{i=1}^{|C|}\gamma_i}\left(1 - p_{|C|}\right)$$
$$\implies \gamma_{|C|} = \frac{D_{|C|}}{1 - p_{|C|}}$$

Now suppose that $\gamma_{c+1}, \gamma_{c+2}, \ldots, \gamma_{|C|}$ are known. Then γ_c can also be calculated from the observation that

$$\sum_{i=1}^{d} \gamma_i = 1 - \sum_{j=d+1}^{|C|} \gamma_j$$

where I take $\sum_{j=|C|+1}^{|C|} \gamma_j \equiv 0$.

Hence, the following recursive relation allows us to compute all the γ_c values in reverse order, starting with $\gamma_{|C|}$ and moving down:

$$\gamma_{|C|} = \frac{D_{|C|}}{1 - p_{|C|}} \tag{27}$$

$$\gamma_c = \frac{D_c}{\displaystyle\sum_{d=c}^{|C|} \frac{p_{d+1} - p_d}{1 - \sum_{j=d+1}^{|C|} \gamma_j}}, \quad \forall c \in \{|C| - 1, |C| - 2, \ldots, 1\} \tag{28}$$

Because this expression requires repeated division by small numbers, it is numerically unstable when the number of schools is large, and the error accumulates with each iteration, with the result that low-popularity schools clip to $\gamma_c = 0$. Thus, it is often more effective to solve the system using a generic root-finding procedure; this spreads the numerical error out over all the schools while preserving granularity at the low end.

3.6.1 Why it's nice to know γ

During college admissions season in the US, news media often report the following measures as indicators of school popularity. However, I argue that none of the following measure a school's underlying preferability to students as well as γ does:

- **Total enrollment** is a function of how selective the school is. For example, Stanford and CSU Chico both enroll about 17,000 students and serve a global market.
- **Class statistics** like average GPA or cutoff scores are also a function of selectivity. It is easy for a school to admit 100 students with perfect grades, and it only has to recruit one of them to obtain a high class statistic.
- **Acceptance rate** overlooks that not all applicant pools are the same. High-achieving students typically don't bother applying to more than one or two safety schools, because they know they will be able to get in someone better. At the other end, many students remove themselves from the applicant pool at selective schools because they know they don't have a chance of getting in, or know they will not be able to afford to attend.
- **Yield**, the percentage of admitted students who choose to enroll, is commonly used as one factor in rankings of colleges in the US. However, this measure unfairly punishes safety schools, and is sensitive to applicant behavior and recruiting practices. If widely

adopted, it also creates a perverse incentive for a midtier school to reject highly qualified applicants whom it expects to enroll elsewhere (so-called yield protection).

- **True yield**, the percentage of qualified students (whether they apply or not) who choose to enroll at a given school (this is my own term). This measure is better than yield, but it systematically overestimates the popularity of lenient schools, which face less competition for students. For example, consider a market with only two schools, Antarctic University and Oxvard University. Oxvard-tier students can choose between two schools, while Antarctic University–tier students have only one choice. Both schools may have similar true yield, but clearly Oxvard is globally preferable when controlling for selectivity.

γ is free of bias at the extreme ends of the distribution, so it allows us to compare the preferability of Oxford and Antarctic U even though very few students actually make this binary choice.

3.6.2 A demonstration

Maybe?

4 Extensions

Propose my general form of the dynamic admissions market.

Equivalent problems to equilibrium: - variational inequality - convex program

Its complexity; the number of potential preference lists and consideration sets.

Computable instances include mine and iid scores.

Admissions coalitions and clusters.