From centralized school choice problems to competitive admissions markets: On the equivalence of stable matchings and market equilibrium

Max Kapur

June 8, 2021

Contents

1 Introduction

This paper concerns admissions markets, which are dynamic systems consisting of students, schools, and some sort of mechanism by which students are matched to schools each admissions cycle. An example of a mechanism is the application process: students apply to schools, schools admit their favorite students, and students choose their favorite school. Another example of a mechanism is a centralized admissions process in which the school board decides which students will attend which school. What are the costs and benefits of the various mechanisms available, and how do these mechanisms respond to a change in demographics, student preferences, or school resources?

Previous literature on admissions markets can be divided into one of two paradigms. First, there is a descriptive paradigm, which builds a statistical model of the distribution of students at each school using observations of past market instances and demographic data. These models are agnostic to the assignment mechanism, and therefore also to level of schooling in question. On the demand side, the descriptive paradigm is best exemplified by regression studies from the social science literature that compare admissions outcomes across socioeconomic indicators and student demographics (Sackett et al. 2012; Lucey and Saguil 2020; Ginsburg 2017) admissions. Such studies are useful tools in evaluating public policy goals such as to increase the number of disadvantaged students who attend a four-year university, but when these demand-side models appear in social science literature, they typically refrain from modeling admissions outcomes at particular schools. On the other hand, there is an international industry of admissions counselors who advise college applicants on which universities to apply to and how to write their application markets. The admissions counselor plays an especially important role in college admissions markets such as South Korea, where students are not allowed to apply to more than six colleges; to prevent a given applicant from wasting one of her applications, ruling out schools for which she is unqualified is imperative, and requires an accurate picture of each school's admissions criteria. The most important item in the admissions counselor's toolbelt is a prediction engine trained on past student data; such models can be constructed using standard data analysis techniques (**asimulationapproachtopredictingcollegeadmissions**).

Remaining within the descriptive paradigm, another question we may ask about admissions markets concerns the supply side: How do schools compare to one another? In this area, the research is wide ranging in methodology. There are regression studies that evaluate the effectiveness of particular schools by comparing of standardized test scores of marginal admits with marginal rejects after a few years of attendance (Dobbie and Roland G. Fryer 2014). But as survey results suggests, the *desirability* of a school is not a simple function of its ability to produce talented graduates. At the secondary level, parents often cite word-of-mouth testimony or target demographics as a more important factor of school desirability (Bancroft 2015) in school choice districts, whereas at the college level, students are often encouraged to choose schools based on an abstract notion of "fitness" that may not be reducible to facts about the student and school. In the case of college admissions especially, there is considerable interest in developing a model of a given school's demand, so that the school can respond strategically to demographic changes, an unexpected spike in applications, or a shift in recruitment goals. Indeed, a survey of sixteen colleges' admissions offices found that several had spent five- and six-figure sums to hire outside consulting firms to predict applicants' probability of enrollment and transfer (Primary Research Group 2014). While the particulars of these models are apparently a closely guarded secret, it is reasonable to assume that they are classifiers trained on observations from past admissions cycles, and that they model the school's *marginal* demand without directly modeling changes in the preferability of competitor schools. The 2020 coronavirus pandemic, however, caused a major perturbation in both students' preferences (Steimle et al. preprint, submitted) and colleges' admissions practices (as many colleges dropped their standardized-test requirements).

The second paradigm that has been widely applied to the study of admissions markets, more familiar to those in computational fields, is a prescriptive paradigm. It formulates the mapping of students to seats in schools as an optimization problem. In the ideal case, each student submits a preference order, or ranking, of the schools in the market, and each school likewise submits its preference order over the students. Then, the school board computes an assignment that respects schools' capacities while maximizing, in some sense, the quality of the overall allocation. Literature on this so-called *school choice problem* has yielded a number of appealing assignment mechanisms, many of which are based on the classical deferred acceptance (DA) algorithm. DA's signature property is that it produces a stable matching, meaning no student feels cheated because a less qualified student has taken her seat at a school she likes. Variations of DA include mechanisms that introduce randomness to break ties in schools' preference orders (Ashlagi and Nikzad 2020), and mechanisms that exploit the same ties to improve student utility (Abdulkadiroğlu, Che, and Yasuda 2015) or optimize for distributional goals like gender parity (Bodoh-Creed 2020). DA is itself the one-to-many form of the classical Gale–Shapley proposal algorithm for computing stable marriages, and its properties, including stability, incentive compatibility for the side of the market that takes the proposing role (usually students), and proposer optimality are well known (Gale and Shapley 1962; Roth 1982).

Mechanism designs such as DA presuppose a centralized assignment process in which students and schools are bound by the school board's decision. Because DA is incentive compatible and produces fairly high (if suboptimal) welfare outcomes, this may not seem like a hard sell. But it is also a matter of fact that many admissions markets, including most college admissions markets, are not centralized. Instead, each school sets its own admissions standards by consulting private goals regarding the number and kind of students it wishes to enroll. For example, in the United States, students are free to apply to as many colleges as time and their application-fee budget permits, and the fall admissions process concludes in April, when students observe which schools they were admitted to and register for the school they like best. In this context, the notion of a school's "capacity" is fuzzier, and represents the point at which the school believes that relaxing its admissions standards is not worth an additional student's tuition dollars. DA asserts that a centralized admissions process is "efficient," but it does so by treating capacity as a hard constraint rather than as the optimum of each school's utility function.

In theory, statistical models can predict distributional outcomes, and therefore model school incentives, in either the centralized or decentralized case. However, the typical regression model assumes a mathematical form of each school's demand curve and fits its parameters to the available data, without incorporating explicit information about the way the final assignment is achieved. Such parametric modeling is quite accurate when interpolating between known data points and when the behavior of other schools can be fixed. But in a school district that uses a centralized assignment paradigm, a change in the capacity of one school induces a shift not only along that school's demand curve, but a transition to an entirely different stable assignment; hence, the parametric model's predictions are strained in a centralized context.

Another difference between the two paradigms is that regression models are by their nature continuous, whereas classical results in stable matching assumed a discrete market. However, in mechanism design research, attention has recently shifted toward a nonatomic formulation of the school-choice problem in which individual students are replaced with a distribution of students over the space of possible preference lists and scores. The nonatomic formulation, due to Azevedo and Leshno (2016), enables the characterization of assignment policies, including stable assignments, via a compact vector of school admissions cutoffs, which indicate a score threshold above which all students have the option of attending the school in question. A favorable property of the nonatomic formulation is that it can be interpreted as the limit of the discrete assignment problem as the number of students and seats increases to infinity; thus, score cutoffs in the nonatomic formulation are free of the "noise" associated with discretization.

It appears that the principle advantage of statistical regression is that it is legible and computationally cheap, while a disadvantage is the lack of an explicit description of how the school assignment mechanism works, which must be compensated for by extensive data collection. On the other hand, the mechanistic paradigm offers a lucid picture of the interaction between students and schools, and computing stable assignments in the discrete case is rather simple. The initial promise of the nonatomic framework was that differentiating the stability conditions would allow us to compute comparative statics and colleges' incentive gradients, in principle letting us predict the effect of a perturbation in the market fundamentals on an unrealized assignment. In practice, the nonatomic framework proves to be intractably complex; even finding a compact representation of the input data is tough. It is also not obvious that classical results in stable assignment have any traction in the decentralized context, although as I will show, there is more crossover than may be apparent at first glance.

Thus, this article attempts to walk a middle path that achieves the computational approachability of regression models while retaining an actual model of students' and colleges' decisionmaking. I show that certain insights from the mechanism design literature can be carried over into the study of nonatomic markets, and use these results as the basis for quantitative techniques that can predict demand curves and comparitive statics in both centralized markets and decentralized markets in and out of equilibrium.

The body of this article is divided into two studies. The first (§2) is a theoretical treatment of the notion of equilibrium in nonatomic admissions markets. While this study's foundation is a result of Azevedo and Leshno (2016) establishing the equivalence of stable matchings and Walrasian equilibrium in matching markets, I rearrange their argument by first defining the market equilibrium in reference to the capacity and demand parameters. Then, I demonstrate four possible interpretations of the equilibrium conditions, of which stable assignment is only one. As I argue, even when we abandon the centralized school assignment paradigm implied by stable assignment algorithms in favor of a dynamic admissions market paradigm in which schools adjust their admissions standards in pursuit of arbitrary recruitment goals, the same notion of equilibrium retains interpretive meaning. Moreover, I show that deferred acceptance algorithms are a special case of a price-adjusting rule called tâtonnement that is known to converge to equibrium prices under certain conditions.

In the second study (§3), I apply the results above to a particular form of admissions market, which I call the single-score market with multinomial logic student preferences. This model is chosen for its computational tractability. Unlike the nonatomic markets theorized by previous work, in which computing the exact equilibrium requires evaluating a demand function that is exponentially complex in the number of schools, the model considered here admits a piecewise linear demand function, and each school's cutoff at equilibrium can be expressed as the solution of a triangular linear system. In this context, comparative statics at the equilibrium can be computed analytically, whereas these quantities are merely posited as theoritical constructs in previous work. I also provide an inverse optimization procedure that computes a preferability parameter for each school given the cutoff and demand vectors, and I apply the procedure to a dataset of 677 American colleges. Despite the noisy input data and the model's simplicity, this procedure yields a remarkably familiar ranking of top universities, and does so without using the costly opinion surveys or data on alumni outcomes that newspapers traditionally use to rank schools. Furthermore, it provides an estimate of each school's demand curve, which could be a useful supplement to the regression models that program planners currently use to predict their admissions yield.

2 Interpretations of equilibrium in admissions markets

Before specifying the market used in this paper, I will define a nonatomic admissions market and summarize some important theoretical results.

2.1 Admissions markets

Definition 1. An *admissions market* consists of a set of schools $C = \{1 \ldots |C|\}$ and a mass-1 continuum of students over the set S of student types. The market is characterized by four parameters:

1. The measure $\eta : 2^S \mapsto [0, 1]$ over the continuum of students.

2. The score cutoff vector $p \in [0, 1]^C$.

3. The demand vector $D \in [0,1]^C$.

4. The capacity vector $q \in \mathbb{R}^C_{++}$.

The model is *nonatomic* in that it represents students as a probability measure over the set of student types instead of considering individual students as discrete actors. Each point s in the set of student types S is associated with a preference list over the schools $>_s$ and a percentile score at each school $\theta_{sc} \in [0,1]$. Hence, $S = C! \times [0,1]^C$.

Schools marginally prefer students with higher scores. Their admissions decisions are represented by the score cutoff vector p. Any student for whom $\theta_{sc} \geq p_c$ is said to be *admitted* to school c.

The demand vector represents the number of students who enroll at each school. Assume that each student attends her favorite school among the set of schools to which she is admitted, which is called her *consideration set* $C^\# \in 2^C$. Then the demand for school c is a function of p and η; specifically, it is the measure of students who are admitted to school c but not admitted to any school that they prefer to c:

$$D_c \equiv \eta\left(s : c = \arg\max_{>_s} \{\hat{c} : \theta_{s\hat{c}} \geq p_{\hat{c}}\}\right) \tag{1}$$

Observe that D_c is weakly decreasing in p_c and weakly increasing in $p_{c'}$ for $c' \neq c$, and that $p_c = 1 \implies D_c = 0$ regardless of the other schools' cutoffs.

If the preference lists are independent of the score vectors, then the demand can also be expressed as the sum of the demand from each combination of preference list and consideration set:

$$\begin{aligned} D_c = \sum_{>_s \in C!} \sum_{\substack{C^\# \in 2^C: \\ c \in C^\#}} \eta\Big(\; s :\; & \underbrace{\theta_{sc'} \geq p_{c'}, \forall c' \in C^\#}_{\text{got into schools in } C^\#} \\ \text{and}\; & \underbrace{\theta_{sc''} < p_{c''}, \forall c'' \in C \setminus C^\#}_{\text{rejected elsewhere}} \\ \text{and}\; & \underbrace{c >_s \hat{c}, \forall \hat{c} \in C^\# \setminus \{c\}}_{\text{prefers } c \text{ among } C^\#} \;\Big) \end{aligned} \tag{2}$$

This expression yields immediate insight into the complexity of nonatomic admissions markets. When schools are allowed to set their own cutoffs, and students are given free choice among the schools in their consideration sets, the number of terms in the sum above is $|C|! \times 2^{|C|}$. However, such a general characterization of students is not always needed. The model considered in this paper (§3), characterizes S using only a $|C|$-vector of school preferability parameters. Alternatively, generic markets can be discretized by representing individual students using explicit preference lists and score vectors.

Each school's capacity $q_c > 0$ represents the fraction of the total mass of students that the school can accept. The capacity is used to define the notion of equilibrium below.

For the remainder of this article, assume that students prefer to be assigned to any school, even their last choice, than to remain unassigned. This is without loss of generality; if some students prefer to be unassigned than attend a particular school, then this choice can be incorporated into the model by adding a dummy school, representing nonassignment, with arbitrary large capacity.

Also, assume that almost no ties occur among scores at a given school. That is, for any fixed c and constant $\bar{\theta}$, $\eta(s : \theta_{sc} = \bar{\theta}) = 0$. Then, by taking transforming the marginal distribution of scores at each school to its cumulative distribution function, we may assume without loss of generality that for a random student s, $\theta_{sc} \sim \text{Uniform}[0, 1]$. It follows that the demand is continuous in p.

2.2 Notion of equilibrium

Let's consider a notion of equilibrium that turns out to have several realistic interpretations.

Definition 2. An admissions market is in *equilibrium* if the following conditions hold:

$$D_c \leq q_c, \quad \forall c \tag{3}$$

$$D_c = q_c, \quad \forall c : p_c > 0 \tag{4}$$

The first condition, called the *capacity condition,* says that no school's demand exceeds its capacity. The second, called the *stability condition,* says that if a school is rejecting students, it must be at full capacity.

Using the sign constraint on p and the capacity condition, an equivalent to the stability condition is $D_c < q_c \implies p_c = 0$ or $p^T (D - q) = 0$.

As shown below, a sufficient condition for the existence of the equilibrium is that the demand is continuous in p. A sufficient condition for the uniqueness of the equilibrium is that the demand is strictly decreasing in p. Both of these follow from, for example, an assumption of full support in η.

Before providing three interpretations of these conditions, it is worth stating a handy fact.

Theorem 1. *If $\hat{p}$ satisfies the equilibrium conditions, then it is a market-clearing cutoff vector. That is, the total measure of assigned students is* $\min\{1, \sum_c q_c\}$

Proof. Let $\hat{\eta}$ denote the measure of assigned students: $\hat{\eta} \equiv \sum_c D_c(\hat{p})$. By the capacity criterion, we have $\hat{\eta} \leq \sum_c q_c$. Since every student prefers assignment to nonassignment, $\hat{\eta} = 1 - \eta(s : \theta_{sc} < p_c, \forall c \in C) \leq 1$. If at least one school has $p_c = 0$, then $\hat{\eta} = 1 \leq q_c$ and the statement holds. Otherwise, the stability condition applies to every school, meaning $\hat{\eta} = \sum_c q_c \leq 1$. □

In reflection of this result, Azevedo and Leshno (2016) call the equilibrium conditions the market-clearing conditions. I will avoid this terminology, because it is possible for a centralized mechanism to clear the market without using score cutoffs at all (for example, by assigning students to schools completely randomly).

2.3 Interpretation of the equilibrium conditions

The applicability and interpretation of the equilibrium conditions depends on the design of the admissions markets. I offer three interpretations. With additional assumptions on η, such as full support, all three interpretations become sufficient conditions for equilibria as well. Additionally, the first interpretation establishes a sufficient condition for the existence of an equilibrium.

In all cases, assume the distribution of student types η and the school capacities q are fixed; thus the demand $D(p)$ is determined entirely by the cutoffs p.

2.3.1 As a fixed point of a tâtonnement process

Suppose that each school has set an admissions target of q_c and observes its demand from year to year. If more than q_c students enroll, then the school attempts to reduce remand by increasing its cutoff. If fewer than q_c students enroll, the school attempts to increase demand by lowering its cutoff (but not past zero).

Let $Z(p) \equiv D(p) - q$ denote the excess demand vector. Then the process described above implies the following recursive relation between the cutoff vector in year k and in year $k+1$:

$$p_c^{(k+1)} = \max\left\{0,\ p_c^{(k)} + \Gamma_c\left[Z_c\left(p^{(k)}\right)\right]\right\}$$

where Γ is sign-preserving. A dynamic process like this one, in which prices adjust in the direction of excess demand, is called a *tâtonnement process*.

Theorem 2. *If $\bar{p}$ is a fixed point of the tâtonnement process, then it satisfies the equilibrium conditions. The converse also holds.*

Proof. Pick $\bar{p}$ such that $\bar{p} = \max\{0, \bar{p}_c + \Gamma_c\left[Z_c(\bar{p})\right]\}$. Subtract $\bar{p}_c$ from both sides to obtain $0 = \max\{-\bar{p}_c, \Gamma_c\left[Z_c(\bar{p})\right]\}$, which implies $0 \geq \Gamma_c\left[Z_c(\bar{p})\right]\}$. This means that the excess demand is negative, which establishes the capacity condition. Now, suppose $\bar{p}_c > 0$; then $\bar{p}_c = \bar{p}_c + \Gamma_c\left[Z_c(\bar{p})\right]$ establishes the stability condition $Z_c(\bar{p} = 0)$. Hence, any fixed point of the tâtonnement process is an equilibrium.

As for the converse, if $p^{(k)}$ satisfies the equilibrium conditions, it is easy to verify that $p^{(k+1)} = p^{(k)}$. $\square$

Moreover, if the demand function is continuous in p, then Brouwer's fixed-point theorem guarantees that a fixed point exists, because the cutoff update maps the convex set $[0,1]^C$ to itself. This means that continuous demand is sufficient for the existence of an equilibrium in admissions markets.

Quantitative economists have studied tâtonnement processes extensively. For an introduction, see Codenotti and Varadarajan (2007) or Intriligator (1971, chap. 9). A classical proof of various convergence conditions is Uzawa (1960).

2.3.2 As a competitive (Nash) equilibrium

Suppose that each school's capacity q_c is a physical constraint on the number of students it can admit. Each school would like to recruit as many students as possible. However, if more students choose to attend the school than the school has capacity for, it must rent additional classroom space at considerable expense. Pick a school c and fix the cutoffs at the other schools $p_{c'}$. Let $u_c(p_c; p_{c'})$ denote school c's utility function, and $\hat{p}_c \equiv p_c : D_c(p_c; p_{c'}) = q_c$ denote the cutoff value that causes c to fill its capacity, if such a value exists. In the situation described, u_c is increasing in p_c when $0 \leq p_c \leq \hat{p}_c$, decreasing in p_c when $p_c > \hat{p}_c$, and the fixed costs associated with excess demand are nonnegative:

$$\lim_{p_c \to \hat{p}_c^+} u_c(p_c; p_{c'}) \leq u_c(\hat{p}_c; p_c c')$$

Another scenario in which utility functions with this shape may arise is as follows: Schools' utility functions are determined mostly by the number of students they enroll, and to a lesser extent by their average score. If the demand for a school is less than its capacity, then a marginal student is always desirable. On the other hand, if the demand exceeds the capacity, then the school has no ability to procure space for the excess demand. Instead, it allows students to register on a first come, first served basis. Because the set of registered students is a random subset of the students who attempt to enroll at the school, it provides the school with less overall utility than if it handpicked the q_c students with the highest scores. That is, u_c is increasing when $0 \leq p_c \leq \hat{p}_c$, and decreasing when $p_c > \hat{p}_c$.

In both of these situations, the admissions market equilibrium can be interpreted as a Nash equilibrium.

Theorem 3. *Consider the game in which each school picks a cutoff $p_c \in [0, 1]$ and tries to maximize a utility function u_c that is increasing in p_c when $0 \leq p_c \leq \hat{p}_c$, decreasing in p_c when $p_c > \hat{p}_c$, and satisfies $\lim_{p_c \to \hat{p}_c^+} u_c(p_c; p_{c'}) \leq u_c(\hat{p}_c; p_c c')$.*

If p^* satisfies the equilibrium conditions, then it is a Nash equilibrium of the game above. If the utility functions are strictly increasing or decreasing, the converse is also true.

Proof. Suppose p^* satisfies the admissions market equilibrium conditions. For the schools for which $D_c = q_c$, their utility is globally maximal. For a school for which $D_c < q_c$, the only way for the school to increase its utility is to decrease its cutoff, but by assumption, $p_c^* = 0$. Hence, is incentivized to change its cutoff, and p^* is a Nash equilibrium of the game defined by the schools' utility functions and the action space $p_c \in [0, 1]$.

The converse can be shown similarly. □

As an aside, observe both of the above are natural situations in which the tâtonnement dynamics may arise.

2.3.3 As a stable matching

The final interpretation of the equilibrium conditions comes from Lemma 1 of Azevedo and Leshno (2016), which says that there is a one-to-one relationship between stable matchings and equilibrium cutoff vectors. In this section, I offer a proof of this lemma.

The notion of a stable assignment has its roots in the field of mechanism design, and thus emerges most naturally from a centralized school choice process as follows: Before the school year begins, students fill out an online form indicating their preference order over the set of schools in the district. Likewise, schools submit their preference order over the students (or equivalently, the scores they have given to each student) to the school board. Then, the school board determines the assignment of students to schools.

First, some notation. A assignment is a mapping of students to schools.

Definition 3. A school choice *assignment* is a mapping $\mu : S \to C \cup \{c_0\}$. $\mu(s) = c_0$ represents nonassignment.

The school board is interested in matchings, or assignments that respect capacity constraints.

Definition 4. A *matching* is an assignment μ that respects schools' capacity constraints; namely, $\eta(s : \mu(s) = c) \leq q_c, \forall c \in C$.

To protect itself from lawsuits and encourage honest participation in the assignment process, the school board decides to rule out matchings that create justified envy; that is, matchings in which a student s who prefers school c to c' is assigned to c' despite scoring higher than a student assigned to c, or c having remaining capacity. In such a situation, (s, c) is called a blocking pair, and a stable matching is a matching that does not admit any blocking pairs.

Definition 5. A *stable matching* (or *stable assignment*) is a matching μ that admits no blocking pairs. That is, there exist no tuples (s, c) s.t. $c >_s \mu(s)$ and one of the following holds:

- School c has remaining capacity:

$$\eta\big(s : \mu(s) = c\big) < q_c$$

- (And/or,) school c has admitted an inferior student:

$$\exists s' \neq s : \mu(s') = c \text{ and } \theta_{s'c} < \theta_{sc}$$

These are called type I and type II blocking pairs, respectively.

In a nonatomic admissions market, where the number of students is infinite, the definition above offers no guidance as to how to encode a stable matching μ. However, it turns out that stable matching are in one-to-one correspondence with equilibrium cutoff vectors. Therefore, any stable matching μ can be fully encoded by a $|C|$-vector of cutoffs p. This fact is invaluable in a computional context.

To establish this result, we must define operators that take cutoff vectors to assignments, and vice-versa. First, the assignment of students induced by instating the cutoff vector p and allowing students to choose freely among their consideration set is

$$\mu_p(s) \equiv \max_{>_s} c \in C : \theta_{sc} \geq p_c, \qquad \forall s \in S$$

Note that this assignment is not necessarily a matching because it may violate the capacity constraints.

Second, the following expression gives the admissions cutoffs implied by a given matching μ, namely, the minimum score of the students admitted to each school:

$$p_c(\mu) \equiv \min \{\theta_{sc} : \mu(s) = c\}$$

In general, these operators are not necessarily inverses of each other. However, as implied by the following theorem, they are inverses when we restrict their domains to the sets of equilibrium cutoffs and stable matchings, respectively.

Theorem 4. *If p^* satisfies the equilibrium conditions, then μ_{p^*} is a stable matching.*

Conversely, if $\bar{\mu}$ is a stable matching and η has full support, then $p_c(\bar{\mu})$ satisfies the equilibrium conditions.

Proof. Pick an equilibrium cutoff p^*. Then by the definitions of the demand function (1) and equilibrium conditions (2),

$$D_c(p^*) = \eta\big(s : \mu_{p^*}(s) = c\big) \leq q_c, \quad \forall c \in C$$
$$D_c(p^*) = \eta\big(s : \mu_{p^*}(s) = c\big) = q_c, \quad \forall c : p_c^* > 0$$

By the capacity condition, no school exceeds its capacity, so μ is an assignment. By the stability condition, there are no type I blocking pairs. And there are no type II blocking pairs, because if a student fails to meet the cutoff for a school she prefers to $\mu_{p^*}(s)$, it is because the school has replaced her with students who got higher scores. Hence, μ_{p^*} is a stable matching.

The converse is proven as follows. Fix a stable matching $\bar{\mu}$, and let $\bar{p} \equiv p(\bar{\mu})$. To get a contradiction, suppose $\bar{p}$ is not an equilibrium. This can happen in two ways:

- For some school c, $D_c(\bar{p}) > q_c$. This means $D_c(\bar{p}) > \eta\big(s : \bar{\mu}(s) = c\big)$, which implies the existence of a student s who is admitted to c at $\bar{p}$ (that is, $\theta_{sc} \geq \theta_{s'c}$ for some $s' : \bar{\mu}(s') = c$) and prefers c among her consideration set, but for whom $\bar{\mu}(s) \neq c$. Then (s, c) is a type II blocking pair; hence, $\bar{\mu}$ is not a stable matching.

- For some school c, $\bar{p}_c > 0$ and $D_c(\bar{p}) < q_c$. By the assumption of strong or full support, there is a student s for whom $c >_s \bar{\mu}(s)$ and $\theta_{sc} < \bar{p}_c$. The latter implies that $\bar{\mu}(s) \neq c$. Hence, (s, c) is a type I blocking pair, and μ is not a stable matching.

Therefore, $\bar{p}$ must satisfy the equilibrium conditions. □

2.4 Deferred acceptance algorithms as tâtonnement processes

The classical solution to the stable matching problem is known as a deferred acceptance (DA) mechanism, which comes in many flavors. When neither students' nor schools' preference lists contains ties, the student-proposing DA procedure is a deterministic algorithm for a stable matching. In this section, I first define the student- and school-proposing DA algorithms. Then, I show that DA algorithms are tâtonnement processes.

Algorithm 1. The *student-proposing deferred acceptance algorithm* is as follows. Given each student's preference order $>_s$ over the set of schools, without ties; each school's score distribution $\theta_{.c}$ over the set of students, with zero probability of ties; and the capacity q_c of each school, the following steps are repeated until no rejections take place:

1. Each student applies to the school highest on her list.

2. Each school examines the applications it received. If it received more applicants than it can seat, it rejects its least-favorite applicants such that the remaining applicants fill its capacity exactly.

3. Each rejected student removes the school that rejected her from her list.

When the algorithm terminates, return the assignment μ, where $\mu(s)$ is highest school remaining on s's preference list, or c_0 if no schools remain.

The properties of the resultant assignment are well known: In the discrete case with $|S|$ students and $|C|$ schools, the algorithm terminates in at most $|S||C|$ iterations. The resultant matching μ is stable. μ is also strongly student optimal, meaning that if another assignment μ' is chosen from the set of stable matchings, then $\mu(s) \geq_s \mu'(s)$ for all students s, and there is at least one student for whom $\mu(s) >_s \mu'(s)$. Similarly, the resultant assignment is strongly school pessimal, meaning any other stable assignment yields the same or better students at each school. Finally, the algorithm is weakly incentive compatible for individual students. That is, no student can obtain a better match than $\mu(s)$ by falsifying her preference list. Succinct proofs of these results are given in Roth (1982).

It is worthwhile to compare school-proposing reverse DA.

Algorithm 2. The *school-proposing deferred acceptance algorithm* is as follows. Given each student's preference list $>_s$ over the set of schools and each school's scores θ_c over the set of students, both without ties, and the capacity q_c of each school, the following steps are repeated until no rejections take place:

1. Each school proposes to the q_c applicants in its consideration pool. If fewer than q_c students are left, the school proposes to all remaining students.

2. Each student examines the proposals she received, rejecting all but her favorite.

3. Each school removes students who rejected it from its consideration pool.

When the algorithm terminates, return the assignment μ, where $\mu(s)$ is the school s prefers among those that proposed to her, or c_0 if she received no proposals.

This algorithm has symmetrical properties to those of forward DA, including student pessimality and incentive compatibility for the schools (but not the students). For these reasons, student-proposing deferred acceptance is seldom used in school choice, and its counterpart in the National Residency Matching Program was abandoned in favor of a resident-proposing algorithm. However, in practice the differences among the resulting assignments tend to be minor.

Under the constraint of stable assignment, student and school utility (as quantified by the preference orders) trade off directly. The student-optimal stable matching from student-proposing DA and the school-optimal stable matching from school-proposing DA define two extreme points of the set of all possible stable matchings, and they are linked by exchanging cycles of students such that student utility increases and school utility decreases, or vice-versa. This means that in a discrete context, to find *all* stable matchings, it suffices to run only the student-proposing DA algorithm, then search recursively for cycles that move the assignment toward student pessimality. Such a procedure allows us to formulate the optimal stable matching problem as the maximal closure of the cycle dependency graph (Irving, Leather, and Gusfield 1987).

Theorem 5. *When η has strong or full support, the student-proposing DA algorithm is a tâtonnement process in which the initial cutoff vector is $p = \vec{0}$, and the school-proposing DA algorithm is a tâtonnement process in which the initial cutoff vector is $p = \vec{1}$.*

Proof. Consider the case of student-proposing DA, and let $\mu^{(k)}$ denote the tentative assignment formed at each iteration of the algorithm. That is, $\mu^{(k)}(s)$ is the school at the top of s's preference list at the beginning of the kth iteration.

It suffices to prove the following three statements:

1. Each $\mu^{(k)}$ is characterized by a cutoff vector $p^{(k)}$.

   Fix k, let $p^{(k)} \equiv p\left(\mu^{(k)}\right)$, and let $m = \mu_{p^{(k)}}$. I will show that $m = \mu^{(k)}$. Pick a student s and let $c = m(s)$. Since s is among the set of students who determined the cutoff vector $p^{(k)}$, $\mu^{(k)}(s)$ is still in s's consideration set under these cutoffs; hence, $c \geq_s \mu^{(k)}(s)$. Now suppose $c >_s \mu^{(k)}(s)$. Since s prefers c, she must have been applied to s in a previous round and been rejected. This implies $p_c^{(k)} > \theta_{sc}$; hence s is not admitted to c in m, a contradiction. It follows that $m(s) = \mu^{(k)}(s)$.

2. The initial cutoff vector has $p^{(0)} = \vec{0}$.

   At the beginning of the first iteration, each student is tentatively assigned to her favorite school. By the support assumption, for all schools c, the set of students who have c at the top of their list and whose score is almost zero is nonempty. Hence, the minimum score over the tentative assignment at each school is zero.

3. $p^{(k+1)}$ is related to $p^{(k)}$ by a tâtonnement update.

At the beginning of each iteration of student-proposing DA, students who were not rejected in the previous iteration apply to the same school as before; on the other hand, students who were rejected apply to new schools. This means that students who apply to school c at the kth iteration and are *not* rejected are a subset set of students who apply at the $k+1$th iteration; hence, at every school c, $p_c^{(k+1)} \geq p^{(k)}$.

Suppose $p_c^{(k+1)} > p_c^{(k)}$. This implies that c rejected students during iteration k, which can occur only if the number of students tentatively matched to c at k exceeds c's capacity. Hence, $D_c(p^{(k)}) > q_c$, and the excess demand $Z_c(p^{(k)})$ is positive at k. This agrees with the sign of the change in cutoff.

Suppose $p_c^{(k+1)} = p^{(k)}$. This means that c made no rejections during the kth iteration, or equivalently, that the number of students tentatively assigned to c is less than or equal to q_c. In our notation, $\eta(s : \mu^{(k)}(s) = c) = D_c(p^{(k)}) \leq q_c$. Hence the excess demand $Z_c(p^{(k)})$ is nonpositive. If $Z_c(p^{(k)}) = 0$, then the statement holds. If $Z_c(p^{(k)}) < 0$, then the statement holds only if $p_c^{(k)} = 0$. To get a contradiction, suppose $p_c^{(k)} > 0$. By the support assumption, there are students whose score is less than $p_c^{(k)}$ who have ranked c first. These students applied to c in an earlier round—call it j—and were rejected. This implies c filled its capacity at j. Since the students not rejected at j continue to apply to c unless rejected again, c fills its capacity at all subsequent rounds, including round k. Hence $Z_c(p^{(k)}) \geq 0$, a contradiction.

The case of school-proposing DA is analogous. □

Using this result, we can rewrite the DA algorithms above in a "computational" form that uses the cutoff vector p as the state variable. In fact, allowing $p^{(0)}$ to take an arbitrary value, we can define a whole subclass of tâtonnement processes that use deferred acceptance to update the cutoff vector. I conjecture that this process converges regardless of the value of the initial cutoff vector. However, even if that conjecture is true, we still have some distance to tread before arriving at a general algorithm for admissions market equilibrium, because the process defined below does not specify how to compute the demand vector or its roots.

Algorithm 3. A *deferred acceptance tâtonnement process* is as follows. Given an initial cutoff vector $p^{(0)}$, each student's preference order $>_s$ over the set of schools, without ties; each school's score distribution $\theta_{\cdot c}$ over the set of students, with zero probability of ties; and the capacity q_c of each school, the following steps are repeated until $p^{(k+1)} = p^{(k)}$: For $k = 0, 1, \ldots,$

1. Compute the demand vector $D(p^{(k)})$.

2. For each school c for which $D_c > q_c$, increase the cutoff so that

$$p_c^{(k+1)} \equiv p_c : D_c(p_c; p_{c'}) = q_c$$

3. For each school c for which $D_c < q_c$, decrease the cutoff (but not past zero) so that

$$p_c^{(k+1)} \equiv \begin{cases} p_c : D_c(p_c; p_{c'}) = q_c, & \text{if such a } p_c \text{ exists} \\ 0, & \text{otherwise} \end{cases}$$

4. Otherwise, let $p_c^{(k+1)} \equiv p_c^{(k)}$.

When the algorithm terminates, return $p_c^{(k)}$.

Taking $p^{(0)} = \vec{1}$, it is easy to see that the the school-proposing deferred acceptance tâtonnement process is a stylized form of the market-clearing procedure described above (§**??**).

This algorithm bears a strong resemblance to the so-called successive tâtonnement process in which each company adjusts its price to the value that clears its supply under the assumption that other companies' prices are fixed (see Uzawa 1960, eqn. 6).

2.5 Computing the equilibrium

With the results above in hand, let's consider a general admissions market in which η and q are fixed. We want to compute the equilibrium of this market. It is impractical to apply a DA algorithm to nonatomic admissions markets, because DA requires using exact line search to determine the new cutoff value for each school, and in general the demand is difficult to compute.

A moderate improvement over the deferred acceptance tâtonnement process is to use a *simultaneous* tâtonnement process, similar to the one proposed above (§2.3.1), that evaluates the demand vector once per iteration and updates the cutoffs in the direction of the excess demand according to a predetermined sequence of decreasing step sizes. Under a light assumption on D, such as continuity, this process can be used to compute the equilibrium to arbitrary precision.

Algorithm 4. The *admissions equilibrium tâtonnement algorithm* is as follows. Given an initial cutoff vector $p^{(0)}$, market parameters γ and q, step parameters $\alpha > 0$ and $0 \leq \beta < 1$, and a tolerance parameter ϵ:

1. Compute the excess demand $Z = D(p^{(k)}) - q$.

2. Update the cutoffs:
$$p_c^{(k+1)} \equiv p_c^{(k)} + \frac{\alpha}{(k+1)^\beta} Z_c$$

3. Terminate if $|p_c^{(k+1)} - p_c^{(k)}| < \epsilon, \forall c$; otherwise, set $k \equiv k + 1$ and repeat.

When the algorithm terminates, return $p_c^{(k)}$.

If the demand is continuous, convergence to an ϵ-approximate equilibrium is guaranteed by the fact that the sequence of step sizes satisfies the Robbins–Monro conditions (Robbins and Monro 1951). However, algorithm is not necessarily computationally efficient. Although a good choice of parameters can enable the algorithm to terminate in a small number of iterations, in general, evaluating the demand vector at each iteration incurs a high computational cost. For example, even under the assumption of independence between students' preference orders and score vectors, the number of terms in each school's demand can be $|C|! \times 2^{|C|}$, as shown in equation (2).

Alternatively, if we have the ability to sample student preference lists and score vectors from η, then we can exploit the relationship between equilibrium cutoffs and stable matchings to estimate the equilibrium cutoffs with high confidence in polynomial time. The technique is as follows: Draw a discrete sample from η and run a DA algorithm. Then compute the minimum score at each school in the resultant stable assignment. If student-proposing DA is used, the expected value of each the cutoffs from the student-optimal stable match approaches the equilibrium cutoff value from below as the size of the sample goes to infinity (Azevedo and Leshno 2016). Similarly, if school-proposing DA is used, the expected value of the obtained cutoffs approaches the equilibrium from above.

Above, I described an expensive, guaranteed-precision technique and a cheap, stochastic technique for computing the equilibrium. One motivation for the model considered below (§3) is the fact that the equilibrium can computed in closed form by solving a linear system in $|C|$ equations for p, providing a baseline against which to evaluate the performance of these techniques.

2.6 Equivalent formulations of the equilibrium conditions

The conditions for a market-clearing cutoff vector given in Definition 2 can be expressed in a few additional ways. Throughout this section assume η and q are fixed and use

$$F(p) \equiv -Z(p) = q - D(p)$$

to denote the excess supply vector at p.

2.6.1 Nonlinear complementarity problem

By inspection, the market-clearing cutoff problem is equivalent to the following nonlinear complementarity problem:

$$\begin{aligned} \text{find } p: \quad F(p)^T p &= 0 \\ F(p) &\geq 0 \\ p &\geq 0 \end{aligned} \tag{5}$$

2.6.2 Variational inequality problem

By a canonical result, the following variational inequality problem is also equivalent:

$$\text{find } p \geq 0 : \quad F(p)^T(\pi - p) \geq 0, \quad \forall \pi \geq 0$$

If D_c is strictly decreasing in p_c, then F_c is strictly increasing, and p^* is unique by a known result (Oden and Kikuchi 1980, §2).

Theorem 6. *If D_c is strictly decreasing in p_c and a market equilibrium p^* exists, then the equilibrium is unique.*

Combining this theorem with the argument above establishing the existence of equilibrium (§2.3.1) yields the following theorem:

Theorem 7. *If η has full support, then the admissions market equilibrium exists and is unique.*

This is Theorem 1 of Azevedo and Leshno (2016), where an alternative proof is given.

2.6.3 Convex optimization problem

Suppose that the excess supply function F defines a conservative vector field. This means that there exists a potential function (Lyupanov function) $\Phi(p)$ whose gradient is F:

$$\exists \Phi : \nabla_p \Phi = F$$

Such a potential function does not necessarily exist for every excess supply function. In fact, in the market considered in the second portion of this article (§3), the Jacobian of F is asymmetric, which implies that F is *not* a conservative vector field.

However, supposing Φ exists, the equilibrium can be found by solving the following concave maximization problem:

$$\text{minimize} \quad \Phi(p) \qquad \text{subject to} \quad p \geq 0$$

As the feasible set has nonempty interior, Slater's condition holds, and the optimal solution (p^*, λ^*) satisfies the following KKT conditions:

$$\begin{aligned} & F_c - \lambda^* = 0 && \text{(stationarity)} \\ & p^* \geq 0, \;\; \lambda^* \geq 0 && \text{(primal, dual feasibility)} \\ & \lambda^{*T} p^* = 0 && \text{(complementarity)} \end{aligned}$$

Eliminating the dual variables λ^* yields the nonlinear complementarity problem above (5); hence, p^* is an equilibrium. Moreover, observe that the tâtonnement procedure of Algorithm 4 is a projected gradient ascent algorithm for this convex program, and vice-versa.

2.7 Optimization tasks

With the equivalence results established above in hand, we can expand our understanding of admissions markets to encompass a range of optimization tasks that span a variety of realistic scenarios.

First, there is the canonical school-choice problem. Given η and the capacity vector q, we must compute a stable matching μ. It suffices to find the equivalent cutoff vector p, as discussed above (§4).

Second, there is the *inverse optimization* problem. Given p and demand D, we try to infer information about η such as the overall preferability of each school or the joint distribution of students' scores. This task requires many simplifying assumptions, because the number of student distributions that could induce a given stable assignment is typically infinite.

In the second half of this study, I turn to an example of a nonatomic admissions markets in which all of these problems can be solved efficiently, even when the number of schools is relatively large.

3 Single-score model with multinomial logit preferences

In this study, I consider a special kind of admissions market that has not received much attention in the school choice literature but approximates the admissions procedure used in many systems around the world. In this market, all schools have the same preference order, and students' preference orders are determined by the multinomial logit (MNL) choice model.

The primary reason for choosing this market is that it admits an expression for the demand that is invertible in the other parameters, allowing us to compute the equilibrium cutoffs analytically. We can also efficiently compute the gradient of the market parameters with respect to one another both in and out of equilibrium, which enables an interesting comparative analysis of the incentives available to schools under unconstrained school choice and when the market is confined to equilibrium by a deferred acceptance mechanism.

This model is also fairly realistic. A single-score system may arise in one of several real-world scenarios. The most obvious case arises when the government requires schools to admit students solely on the basis of a single standardized test. Alternatively, if students are scored using various dimensions of student characteristics such as test scores, GPA, and the quality of their letters of recommendation, it is common for these various dimensions to correlate tightly. If so, then principle component analysis can be used to determine a composite score whose order approximates the ordering of students at each university. Finally, in many public school systems, schools have *no* preference order over the students; instead students take turns picking their favorite school in an order determined by random lottery, or (equivalently) the single tiebreaking mechanism is used to generate schools' preference lists and the assignment of students to schools is computed using student-proposing DA (Ashlagi and Nikzad 2020). In this situation, the random numbers induce a single distribution of scores.

The MNL choice model represents a compromise between realism and computational tractability. In the worst-case scenario, there are $|C|!$ possible preference orderings, and student preferences must be encoded as a probability vector having this length. A simple way to reduce this complexity is to choose a few "representative" preference lists, but this makes it difficult to account for the exponential number of ways in which an individual student may exchange the place of two schools within the master list. In contrast, the MNL choice model gives full support to the space of all preference lists while requiring only a single parameter for each school, and its parameters can be fitted via a number of known survey methodologies.[1] The MNL choice model can also emulate to arbitrary precision the situation in which every student has the *same* preference list, by letting each school's preferability parameter differ from the next by a large order of magnitude.

[1] An interesting direction of future research would be to attempt to fit MNL parameters to student preference lists in a jurisdiction that uses deferred acceptance, like New York City.

3.1 Model description

In this section I descrive the single-score model with multinomial logit preferences, derive a closed-form expression for the demand function, and show that the demand is piecewise linear continuous and each school's demand D_c is strictly decreasing in p_c.

3.1.1 Characterization of η

To characterize η, we must describe both how schools rank students, and how students rank schools. In this model, all schools share the same ranking over the students. Assuming there are no ties, assume without loss of generality that the scores are uniformly distributed on the interval $[0, 1]$.

As for students' choice of school, this model assumes students use MNL choice to derive their preference lists. Each school has a preferability parameter $\delta_c \in \mathbb{R}$. Letting $C^\# \subseteq C$ denote set of schools to which a given student is admitted, she chooses to attend school $c \in C^\#$ with probability

$$\frac{\exp \delta_c}{\sum_{d \in C^\#} \exp \delta_d}$$

For convenience, let $\gamma_c \equiv \exp \delta_c > 0$ and $\Gamma = \sum_c \gamma_c$. Since the equation is homogeneous in γ, we may assume without loss of generality that $\Gamma = 1$; however, I will resist this assumption, since in a large market, taking a large Γ-value can yield more legible parameters.

Observe that in the single-score model with MNL choice, η does *not* have full support, because the probability of having different scores at any two schools is zero. Nonetheless, the algebraic analysis below reveals that the equilibrium is unique.

3.1.2 Demand function

Let us determine the demand function $D(\gamma, p)$ for the single-score model with MNL student preferences. First, sort the schools by cutoff, so that

$$p_1 \leq p_2 \leq \cdots \leq p_{|C|}$$

Ties may be broken arbitrarily, as discussed below. Since getting into school c implies getting into any school whose cutoff is less than or equal to p_c, there are only $|C| + 1$ possible consideration sets for each student:

Symbol	**Consideration set**	**Probability**
$C_{[0]}$	$\varnothing$	p_1
$C_{[1]}$	$\{c_1\}$	$p_2 - p_1$
$C_{[2]}$	$\{c_1, c_2\}$	$p_3 - p_2$
$\vdots$	$\vdots$	$\vdots$
$C_{[\|C\|-1]}$	$\{c_1, \ldots, c_{\|C\|-1}\}$	$p_{\|C\|} - p_{\|C\|-1}$
$C_{[\|C\|]}$	$\{c_1, \ldots, c_{\|C\|}\}$	$1 - p_{\|C\|}$

Hence, the demand for school c is the sum of the number of students with each of these consideration sets who choose to attend c. Letting $p_{|C|+1} \equiv 1$, the demand function is as follows.

$$D_c = \sum_{d=c}^{|C|} \underbrace{\frac{\exp \delta_c}{\sum_{i=1}^{d} \exp \delta_i}}_{\substack{\text{prob. of choosing} \\ c \text{ from cons. set}}} \overbrace{(p_{d+1} - p_d)}^{\substack{\text{prob. of having} \\ \text{cons. } C_{[d]}}} \tag{6}$$

If at least one school has $p_c = 0$, then every student can get in somewhere, and $\sum_c D_c = 1$. Generally, there are p_1 students who get in nowhere, and $\sum_c D_c = 1 - p_1$.

3.1.3 Continuity and piecewise linearity of the demand function

D is *continuous* in p. To see this, expand the equation above:

$$D_c = \gamma_c \left[\left(\frac{-1}{\sum_{i=1}^{c} \gamma_i} \right) p_c + \left(\frac{1}{\sum_{i=1}^{c} \gamma_i} - \frac{1}{\sum_{i=1}^{c+1} \gamma_i} \right) p_{c+1} \right. \tag{7}$$

$$\left. + \cdots + \left(\frac{1}{\sum_{i=1}^{|C|-1} \gamma_i} - \frac{1}{\sum_{i=1}^{|C|} \gamma_i} \right) p_{|C|} + \frac{1}{\sum_{i=1}^{|C|} \gamma_i} \right] \tag{8}$$

Since D is linear in any neighborhood where the order of cutoffs is unambiguous, the only opportunity for discontinuity occurs when two or more cutoffs are equal. Thus, it suffices to show that the value of D_c is independent of how ties among the p_c are broken. Suppose that $p_j = \cdots = p_{j+n} = \tilde{p}$ for some $j > c$. Then (dividing by γ_c for legibility),

$$\frac{D_c}{\gamma_c} = \cdots + \left(\frac{1}{\sum_{i=1}^{j-1} \gamma_i} - \frac{1}{\sum_{i=1}^{j} \gamma_i} \right) p_j + \left(\frac{1}{\sum_{i=1}^{j} \gamma_i} - \frac{1}{\sum_{i=1}^{j+1} \gamma_i} \right) p_{j+1} \tag{9}$$

$$+ \cdots + \left(\frac{1}{\sum_{i=1}^{j+n} \gamma_i} - \frac{1}{\sum_{i=1}^{j+n+1} \gamma_i} \right) p_{j+n} + \cdots \tag{10}$$

$$= \cdots + \left(\frac{1}{\sum_{i=1}^{j-1} \gamma_i} - \cancel{\frac{1}{\sum_{i=1}^{j} \gamma_i}} \right) \tilde{p} + \left(\cancel{\frac{1}{\sum_{i=1}^{j} \gamma_i}} - \cancel{\frac{1}{\sum_{i=1}^{j+1} \gamma_i}} \right) \tilde{p} \tag{11}$$

$$+ \cdots + \left(\cancel{\frac{1}{\sum_{i=1}^{j+n} \gamma_i}} - \frac{1}{\sum_{i=1}^{j+n+1} \gamma_i} \right) \tilde{p} + \cdots \tag{12}$$

$$= \cdots + \left(\frac{1}{\sum_{i=1}^{j-1} \gamma_i} - \frac{1}{\sum_{i=1}^{j+n+1} \gamma_i} \right) \tilde{p} + \cdots \tag{13}$$

The internal sums that depend on the order of the indices $j \ldots j+n$ cancel out; hence, they may be arbitrarily reordered without changing the value of D_c. Similar canceling shows that the demand does not vary under tiebreaking when c itself is involved in a tie. Hence, D is continuous in p.

The expansion above also allows us to see that the demand vector is defined by the matrix equation

$$D = Ap + \frac{1}{\Gamma}\gamma \tag{14}$$

where $A \in \mathbb{R}^{|C| \times |C|}$ is the triangular matrix with

$$A_{ij} \equiv \begin{cases} 0, & i > j \\ -\gamma_i \left(\frac{1}{\sum_{k=1}^{i} \gamma_k} \right), & i = j \\ \gamma_i \left(\frac{1}{\sum_{k=1}^{j-1} \gamma_k} - \frac{1}{\sum_{k=1}^{j} \gamma_k} \right), & i < j \end{cases} \tag{15}$$

$$\implies A = \begin{bmatrix} \gamma_1 \left(\frac{-1}{\gamma_1} \right) & \gamma_1 \left(\frac{1}{\gamma_1} - \frac{1}{\gamma_1+\gamma_2} \right) & \gamma_1 \left(\frac{1}{\gamma_1+\gamma_2} - \frac{1}{\gamma_1+\gamma_2+\gamma_3} \right) & \cdots & \gamma_1 \left(\frac{1}{\sum_{i=1}^{|C|-1} \gamma_i} - \frac{1}{\Gamma} \right) \\ & \gamma_2 \left(\frac{-1}{\gamma_1+\gamma_2} \right) & \gamma_2 \left(\frac{1}{\gamma_1+\gamma_2} - \frac{1}{\gamma_1+\gamma_2+\gamma_3} \right) & \cdots & \gamma_2 \left(\frac{1}{\sum_{i=1}^{|C|-1} \gamma_i} - \frac{1}{\Gamma} \right) \\ & & \gamma_3 \left(\frac{-1}{\gamma_1+\gamma_2+\gamma_3} \right) & \cdots & \gamma_3 \left(\frac{1}{\sum_{i=1}^{|C|-1} \gamma_i} - \frac{1}{\Gamma} \right) \\ & & & \ddots & \vdots \\ & & & & \gamma_{|C|} \left(\frac{1}{\sum_{i=1}^{|C|-1} \gamma_i} - \frac{1}{\Gamma} \right) \end{bmatrix} \tag{16}$$

Since $\gamma > 0$, A is invertible. This A will reappear throughout the analysis.

The matrix A depends on the order of the p_c values, so the demand function is *piecewise linear* in p.[2] Because the main diagonal of A is strictly negative, the demand at each school c is strictly decreasing in p_c. By Theorem 6, it follows that that the equilibrium is unique.

3.1.4 Appeal function

An interesting indicator from Azevedo and Leshno (2016) is the *appeal* of a school's entering class, or the integral of scores over the set of admitted students. The appeal of the entering class is not necessarily the school's objective function, because schools may value an abstract notion of selectivity or students' tuition dollars higher than this value.

The average score of a student with consideration set $C_{[d]}$ is $\frac{1}{2}(p_{d+1} + p_d)$, so the appeal at c is

$$L_c = \sum_{d=c}^{|C|} \underbrace{\frac{\gamma_c}{\sum_{i=1}^{d} \gamma_i}}_{\substack{\text{prob. of choosing} \\ c \text{ from cons. set}}} \overbrace{(p_{d+1} - p_d)}^{\substack{\text{prob. of having} \\ \text{cons. set } C_{[d]}}} \underbrace{\frac{1}{2}(p_{d+1} + p_d)}_{\substack{\text{avg. score of students} \\ \text{with cons. set } C_{[d]}}} = \frac{1}{2} \sum_{d=c}^{|C|} \frac{\gamma_c}{\sum_{i=1}^{d} \gamma_i} \left(p_{d+1}^2 - p_d^2 \right) \tag{17}$$

[2]In the context of an iterative schema such as the tâtonnement process simulated below, instead of sorting p itself, it is often simpler to permute the rows and columns of A according to the inverse of the permutation that sorts p.

By comparison with the expression for D, the appeal vector is given by

$$L = \frac{1}{2}Ap.^2 + \frac{1}{2\Gamma}\gamma$$

where the notation $p.^2 = (p_1^2, \dots, p_{|C|}^2)$ represents the entrywise square of p.

3.2 Computing the equilibrium

In the market under consideration, the equilibrium conditions are as follows:

$$\begin{aligned} D &= Ap + \frac{1}{\Gamma}\gamma \leq q \\ D_c &= A_{c.}p + \frac{1}{\Gamma}\gamma_c = q_c, \quad \forall c : p_c > 0 \end{aligned} \tag{18}$$

As I will now show, it turns out that at equilibrium, the order of the school cutoffs is determined by the order of the *competitiveness ratios* γ_c/q_c. This fact enables us to compute the equilibrium directly by solving a linear system. Below, the positive part operator x^+ works elementwise on its argument x. That is, $(x^+)_i \equiv \max\{0, x_i\}$.

Theorem 8. *Without loss of generality, suppose that $\frac{\gamma_1}{q_1} \leq \cdots \leq \frac{\gamma_{|C|}}{q_{|C|}}$. Then $\hat{p}_1 \leq \cdots \leq \hat{p}_{|C|}$, and*

$$\hat{p} \equiv \left[A^{-1}(q - \frac{1}{\Gamma}\gamma)\right]^+$$

is the market equilibrium in the single-score, MNL choice model. Moreover, the equilibrium is unique.

Proof. I show the following statements:

1. $\hat{p}$ satisfies $\hat{p}_1 \leq \cdots \leq \hat{p}_{|C|}$. This means that the demand at $\hat{p}$ is given by the expression $A\hat{p} + \frac{1}{\Gamma}\gamma$ (which only holds if $\hat{p}$ is sorted).

2. $\hat{p}$ satisfies the equilibrium conditions given in equation (18).

For convenience, let $\bar{p} \equiv A^{-1}(q - \frac{1}{\Gamma}\gamma)$, so that $\hat{p} = \bar{p}^+$.

$\underline{\hat{p}\text{ is sorted.}}$ Pick any school $c < |C|$. It suffices to show that $\bar{p}_{c+1} - \bar{p}_c \geq 0$. The inverse of A is

$$A^{-1} = \begin{bmatrix} \frac{-1}{\gamma_1}(\gamma_1) & -1 & -1 & \cdots & -1 \\ & \frac{-1}{\gamma_2}(\gamma_1+\gamma_2) & -1 & \cdots & -1 \\ & & \frac{-1}{\gamma_2}(\gamma_1+\gamma_2+\gamma_3) & \cdots & -1 \\ & & & \ddots & \vdots \\ & & & & \frac{-1}{\gamma_{|C|}}\Gamma \end{bmatrix} \tag{19}$$

It is not difficult to verify that

$$\bar{p}_{c+1} - \bar{p}_c = \left[A^{-1}(q - \frac{1}{\Gamma}\gamma)\right]_{c+1} - \left[A^{-1}(q - \frac{1}{\Gamma}\gamma)\right]_c = \left(\sum_{j=1}^{c}\gamma_j\right)\left(\frac{q_c}{\gamma_c} - \frac{q_{c+1}}{\gamma_{c+1}}\right) \geq 0 \tag{20}$$

which follows from the assumption that $\gamma_c/q_c \le \gamma_{c+1}/q_{c+1}$.

$\hat{p}$ satisfies the equilibrium conditions. The demand at $\hat{p}$ is $D = A\hat{p} + \frac{1}{\Gamma}\gamma$. Hence

$$\begin{aligned}\hat{p} = A^{-1}(D - \frac{1}{\Gamma}\gamma) = \bar{p}^{+} \ge \bar{p} = A^{-1}(q - \frac{1}{\Gamma}\gamma)\\ \implies \quad A^{-1}D \ge A^{-1}q\\ \implies \quad D \le q\end{aligned}$$

The final statement follows from the fact that A^{-1} is triangular and its nonzero entries are strictly negative. This establishes the capacity condition.

Now, we need to show that the demand equals the capacity when $\hat{p}_c > 0$. Let b denote the first school with a nonzero cutoff. That is, $\hat{p}_1 = \cdots = \hat{p}_{b-1} = 0$, and $0 < \hat{p}_b \le p_{b+1} \le \cdots \le \hat{p}_{|C|}$. Then the demand at $\hat{p}$ may be written

$$\begin{aligned} D &= A\hat{p} + \frac{1}{\Gamma}\gamma \\ &= \sum_{i=1}^{|C|} A_{.i}\hat{p}_i + \frac{1}{\Gamma}\gamma \\ &= \sum_{i=1}^{|C|} A_{.i}\left[A^{-1}\left(q - \frac{1}{\Gamma}\gamma\right)\right]_i^{+} + \frac{1}{\Gamma}\gamma \\ &= \sum_{j=b}^{|C|} A_{.j}\left[A^{-1}\left(q - \frac{1}{\Gamma}\gamma\right)\right]_j + \frac{1}{\Gamma}\gamma \\ &= \left[\sum_{j=b}^{|C|} A_{.j}A^{-1}_{j.}\right]\left(q - \frac{1}{\Gamma}\gamma\right) + \frac{1}{\Gamma}\gamma \\ &= \begin{bmatrix} 0_{b\times b} & T_{b\times(|C|-b)} \\ 0_{(|C|-b)\times b} & I_{|C|-b}\end{bmatrix}\left(q - \frac{1}{\Gamma}\gamma\right) + \frac{1}{\Gamma}\gamma \end{aligned} \tag{21}$$

where

$$T = \begin{bmatrix} \frac{-\gamma_1}{\sum_{i=1}^{b-1}\gamma_i} & \cdots & \frac{-\gamma_1}{\sum_{i=1}^{b-1}\gamma_i} \\ \vdots & \cdots & \vdots \\ \frac{-\gamma_{b-1}}{\sum_{i=1}^{b-1}\gamma_i} & \cdots & \frac{-\gamma_{b-1}}{\sum_{i=1}^{b-1}\gamma_i}\end{bmatrix} \tag{22}$$

For the schools with $\hat{p}_c > 0$, the demand is

$$D_c = \begin{bmatrix} 0 & I \end{bmatrix}_{c.}\left(q - \frac{1}{\Gamma}\gamma\right) + \frac{1}{\Gamma}\gamma = q_c \tag{23}$$

Hence, the stability criterion holds, and $\hat{p}$ is an equilibrium. $\square$

For reference, for the schools with $\hat{p}_c = 0$, the demand at equilibrium is

$$D_c = \begin{bmatrix}0 & T\end{bmatrix}_{c.} \left(q - \frac{1}{\Gamma}\gamma\right) + \frac{1}{\Gamma}\gamma = \frac{-\gamma_c}{\sum_{i=1}^{b-1}\gamma_i} \sum_{j=b}^{|C|} \left(q_j - \frac{1}{\Gamma}\gamma_j\right) + \frac{1}{\Gamma}\gamma_c \leq q_c \tag{24}$$

With these results in hand, I turn to a comparative analysis of the incentives that two different assignment mechanisms provide to schools in this market.

3.3 Incentive gradients under decentralized assignment mechanisms

In this section, I consider the incentives available to schools in an unconstrained market in which schools have no capacity constraints. Then, in the following section, I consider these incentives in a centralized market that always produces a stable matching. In theory, schools can have any objective functions, but the analysis below assumes that each school's utility is increasing in its cutoff, its demand, and its appeal. Under decentralized assignment, both the cutoff and quality can be interpreted as variables within each school's control; under centralized assignment, schools can only affect their quality. Hence, the set of derivatives that afford a meaningful interpretation differs somewhat between the centralized and decentralized cases.

In the unconstrained market, schools do not have meaningful capacity constraints—there is ample room for any student at any school, as long as she meets its admissions standards. Thus, each school can set its own cutoff p_c in reflection of its own admissions goals. And, to the extent it can, each school can try to increase its preferability γ_c by advertising, updating its curriculum, and so on.

3.3.1 Cutoff effects

The change in demand in response to a change in cutoffs is the Jacobian of the demand function:

$$\mathbf{J}_p D = A$$

The diagonal is negative, meaning that each school's demand is decreasing in its cutoff, as expected. The entries above the diagonal are positive, while those below the diagonal are zero. This means that each school c's demand is increasing in the cutoffs of the *more-selective* schools, but the cutoffs of *less-selective schools* have no local effect on the demand at c.

Intuitively, this means that if all schools are equally preferable, a highly selective school has more market power than the others: If it increases its cutoff, it will cause many students to move onto another school. On the other hand, a school c' that is less preferable than c cannot affect D_c's demand by changing its own cutoff, because any student currently admitted to c was already admitted to c', and chose c instead.

Observe also that $-1 = A_{11} < A_{22} < \cdots < A_{|C||C|} < 0$. This says that the school with the most generous cutoff has the most power to increase its demand with a marginal decrease in p_c. Intuitively, this is because a student who gets into a school with a large cutoff gets into *many* schools, so competition for this student is fiercer than for a student whose options are already limited by a low score.

Next, consider the change in the entering classes' *appeal* in response to a change in cutoffs:

$$\mathbf{J}_p L = A \operatorname{diag}(p)$$

For $p_c > 0$, the cutoff effect on appeal has the same direction as the cutoff effect on demand. Intuitively, this suggests that if a school's goal is to maximize the appeal of its entering class, it will tend to try to lower its score cutoffs as much as it can, subject to constraints on its total demand. However, the magnitude of the incentive increases when p_c is higher. This tends to counteract the market power effect described above: A school with a low cutoff has the power to attract more marginal students, but does so with little overall effect on the aggregate appeal of its entering class. In the extreme case, when $p_c = 0$, the appeal associated with a marginal student is exactly zero.

The derivatives given above are well-defined when the cutoffs are totally ordered. However, an edge case occurs when there is a tie among the cutoffs; then the subdifferential set is given by the convex hull of the Jacobians associated with the possible permutations of p. In this case, I argue that the best interpretation of the effect of an *increase* in p_c should be that associated with the permutation for which c is indexed after schools with which its cutoff is tied. That is, because p_c is "about to" become larger than the other cutoffs involved in the tie, break the tie in its favor. Likewise, to interpret a *decrease* in a tied p_c, treat p_c as the least member of the tied set.

3.3.2 Quality effects

Differentiate the demand with respect to γ to obtain the effect of a marginal change in quality:

$$(\mathbf{J}_\gamma D)_{c\hat{c}} = \frac{\partial}{\partial \gamma_{\hat{c}}} D_c = \begin{cases} \sum_{d=c}^{|C|} \frac{-\gamma_c}{\left(\sum_{i=1}^{d} \gamma_i\right)^2} \left(p_{d+1} - p_d\right), & \hat{c} < c \\ \sum_{d=c}^{|C|} \frac{1}{\sum_{i=1}^{d} \gamma_i} \left(1 - \frac{\gamma_c}{\sum_{i=1}^{d} \gamma_i}\right) \left(p_{d+1} - p_d\right), & \hat{c} = c \\ \sum_{d=\hat{c}}^{|C|} \frac{-\gamma_c}{\left(\sum_{i=1}^{d} \gamma_i\right)^2} \left(p_{d+1} - p_d\right), & \hat{c} > c \end{cases} \tag{25}$$

(Note that the $\hat{c} > c$ and $\hat{c} < c$ cases differ in the outer sum's starting index.) The demand for c is predictably decreasing in the quality of the other schools and increasing in γ_c.

A similar picture emerges when we differentiate the appeal with respect to γ:

$$(\mathbf{J}_\gamma L)_{c\hat{c}} = \frac{\partial}{\partial \gamma_{\hat{c}}} L_c = \begin{cases} \frac{1}{2} \sum_{d=c}^{|C|} \frac{-\gamma_c}{\left(\sum_{i=1}^{d} \gamma_i\right)^2} \left(p_{d+1}^2 - p_d^2\right), & \hat{c} < c \\ \frac{1}{2} \sum_{d=c}^{|C|} \frac{1}{\sum_{i=1}^{d} \gamma_i} \left(1 - \frac{\gamma_c}{\sum_{i=1}^{d} \gamma_i}\right) \left(p_{d+1}^2 - p_d^2\right), & \hat{c} = c \\ \frac{1}{2} \sum_{d=\hat{c}}^{|C|} \frac{-\gamma_c}{\left(\sum_{i=1}^{d} \gamma_i\right)^2} \left(p_{d+1}^2 - p_d^2\right), & \hat{c} > c \end{cases} \tag{26}$$

By the same procedure used to show the continuity of D above, it is possible to show that the quality effects are continuous across tiebreaking permutations of p.

3.4 Comparative statics at equilibrium

Now, I analyze the incentives available to schools and government planners when the market is constrained to equilibrium, for example, by a centralized admissions process that uses a DA algorithm to produce a stable matching. Throughout this section, I assume that schools are indexed in ascending order by the competitiveness ratios γ_c/q_c. The quantities derived here were proposed by Azevedo and Leshno (2016); however, I believe that expressing them in an analytic form represents a novel contribution.

3.4.1 Quality effects at equilibrium

First, I will focus on the effect of a marginal change in quality on the allocation of students at equilibrium. Since, in theory, schools have the power to change their own quality by investing in their programs or marketing, the analysis below enables us to quantify the extent to which these investments are "worth it" with respect to the school's interest in maintaining high admissions standards or increasing its demand.

First, I provide yet another expression for the equilibrium cutoff vector $\hat{p}$, which can be verified by expanding the equation given in Theorem 8. $\hat{p}_c = \bar{p}_c^+$, where

$$\bar{p}_c = \frac{1}{\gamma_c}\left(\frac{\gamma_c}{\Gamma} - q_c\right)\sum_{i=1}^{c}\gamma_i + \sum_{j=c+1}^{|C|}\left(\frac{\gamma_j}{\Gamma} - q_j\right) \tag{27}$$

and, in the $c = |C|$ case, I take $\sum_{j=|C|+1}^{|C|}\left(\frac{\gamma_j}{\Gamma} - q_j\right) = 0$. This assumes the schools are indexed in ascending order by the competitiveness ratios γ_c/q_c.

Differentiating the optimal cutoffs with respect to the quality and simplifying, we have

$$(\mathbf{J}_\gamma \hat{p})_{c\hat{c}} = \frac{\partial}{\partial \gamma_{\hat{c}}}\hat{p}_c = \begin{cases} 0, & \bar{p}_c < 0 \\ \text{undefined}, & \bar{p}_c = 0 \\ -\frac{q_c}{\gamma_c}, & \bar{p}_c > 0 \text{ and } \hat{c} < c \\ \frac{q_c}{\gamma_c^2}\sum_{i=1}^{c-1}\gamma_i, & \bar{p}_c > 0 \text{ and } \hat{c} = c \\ 0, & \bar{p}_c > 0 \text{ and } \hat{c} > c \end{cases} \tag{28}$$

As above, in the $c = 1$ case, we interpret the empty set as summing to zero: $\sum_{i=1}^{0}\gamma_i = 0$. This means that the entry in the top left is always zero. The Jacobian is lower triangular: any change in the quality of a school whose competitiveness ratio is already higher than that of c induces no change in the cutoff at c. This calculation is applied to Pallet Town and verified graphically in Figure 6.

Applying the chain rule to the demand at equilibrium $D = A\hat{p} + \frac{1}{\Gamma}\gamma$, and letting b denote the index of the first school with a nonzero cutoff (as above), the derivative of the equilibrium demand at c with respect to the quality of $\hat{c}$ is

$$\left(\mathbf{J}_{\gamma} D\left(\hat{p}\right)\right)_{c\hat{c}} = \frac{\partial}{\partial \gamma_{\hat{c}}} D(\hat{p}_c) = \begin{cases} -\gamma_c \frac{1-\sum_{j=b}^{|C|} q_j}{\left(\sum_{i=1}^{b-1} \gamma_i\right)^2}, & \bar{p}_c < 0 \text{ and } \hat{c} \neq c \\ \left(-\gamma_c + \sum_{k=1}^{b-1} \gamma_k\right) \frac{1-\sum_{j=b}^{|C|} q_j}{\left(\sum_{i=1}^{b-1} \gamma_i\right)^2}, & \bar{p}_c < 0 \text{ and } \hat{c} = c \\ \text{undefined}, & \bar{p}_c = 0 \\ 0, & \bar{p}_c > 0 \end{cases} \tag{29}$$

Disregarding the knife-edge case in which $\bar{p}_c = 0$, the two derivatives above suggest that schools in the single-test model fall into one of two clear categories. For the schools for which $\bar{p}_c < 0$, a marginal improvement in quality increases the *size* of the entering class but has no effect on its *minimum score* (and, in general, the effect on the average score is small). On the other hand, for the schools for which $\bar{p}_c > 0$, their capacity is always filled at equilibrium, and any investment in quality translates into an improvement in the minimum score of the entering class. If the objective functions are a combination of cutoff and demand, this analysis suggests that competition within these two broad groups of schools is close to zero-sum. Underdemanded schools compete for the finite pool of tuition dollars remaining in the market after the best students have chosen the top schools, whereas overdemanded schools compete for the top slice of the fixed distribution of student talent.

The appeal function also be differentiated in γ, although it does not admit a legible representation for an arbitrary number of schools. However, we can determine the signs along the Jacobian's main diagonal by inspection.

3.4.2 Capacity and population effects

Consulting the cutoff sortation result of Theorem 8, it is easy to see that the derivative of the equilibrium cutoffs with respect to a given school's capacity is

$$\left(\mathbf{J}_{q} \hat{p}\right)_{c\hat{c}} = \frac{\partial}{\partial q_{\hat{c}}} \hat{p}_c = \begin{cases} 0, & \bar{p}_c < 0 \\ \text{undefined}, & \bar{p}_c = 0 \\ A^{-1}_{c\hat{c}}, & \bar{p}_c > 0 \text{ and } \hat{c} < c \end{cases} \tag{30}$$

The derivative of the demand, by inspecting Equation (21), has

$$\mathbf{J}_q D(\hat{p}) = \begin{bmatrix} 0_{b\times b} & T_{b\times(|C|-b)} \\ 0_{(|C|-b)\times b} & I_{|C|-b} \end{bmatrix} \tag{31}$$

where the entries of T are negative as given in (22).

This confirms the intuitive result that only schools that are overdemanded at equilibrium can make use of excess capacity. In addition, observe that because $\mathbf{J}_q \hat{p}$ is upper triangular, adding capacity to a school whose competitiveness ratio is lower than that of c has no marginal effect on the equilibrium cutoff at c.

Next, it is interesting (although irrelevant to the incentive analysis) to consider the effect of a uniform marginal increase in the total student population $\eta(S)$, which nominally equals one. Since school capacities represent fractions of the total student population, this is equivalent to the effect of decreasing each school's capacity proportionally. In other words, when an attribute of the market is given by $f(q)$, it suffices to evaluate the derivative of $f(\alpha q)$ with respect to α at $\alpha = 1$ and flip the sign.

The change in cutoffs after a marginal increase in $\eta(S)$ is

$$\frac{d}{d\eta(S)}\hat{p}_c = -\left.\frac{d}{d\alpha}\hat{p}_c(\alpha q)\right|_{\alpha=1} = \begin{cases} 0, & \bar{p}_c < 0 \\ \text{undefined}, & \bar{p}_c = 0 \\ -A^{-1}_{c.}q, & \bar{p}_c > 0 \end{cases} \tag{32}$$

Since the entries of $A^{-1}q$ are strictly negative, any school with a positive cutoff will attain a higher cutoff after an increase in population. The change in demand and appeal can be derived similarly.

3.5 A numerical example

In this section, I offer a numerical demonstration of the cutoff sort order given by Theorem 8. I validate the interpretation of the equilibrium cutoffs as a stationary point of a tâtonnement process, as a market-clearing price vector, and as the limit point of stable assignments. (The demonstration of the latter two interpretations follows Azevedo and Leshno (2016).) Finally, I represent the incentive results above graphically.

Figure 1 demonstrates the relationship between the equilibrium cutoffs p_c^* and the competitiveness ratios γ_c/q_c in randomly generated markets. Some of the markets are overdemanded, yielding $p^* > 0$, and others are underdemanded; however, the equilibrium cutoffs are always ordered according to the competitiveness ratios. As the graph indicates, the precise relationship is nonlinear and highly sensitive to variance in the market parameters. This suggests that even in the highly stylized model under consideration, it is difficult to schools to predict the effect of small perturbation in a single γ_c or q_c value on the market as a whole myopically—that is, by looking only at their entering class. Instead, especially under a stable assignment paradigm, schools must model their demand curve in a way that accounts for the second-order effect of a change in cutoff on the location of the equilibrium.

Figures 2 through 4 consider a fictional admissions market called Pallet Town, which has

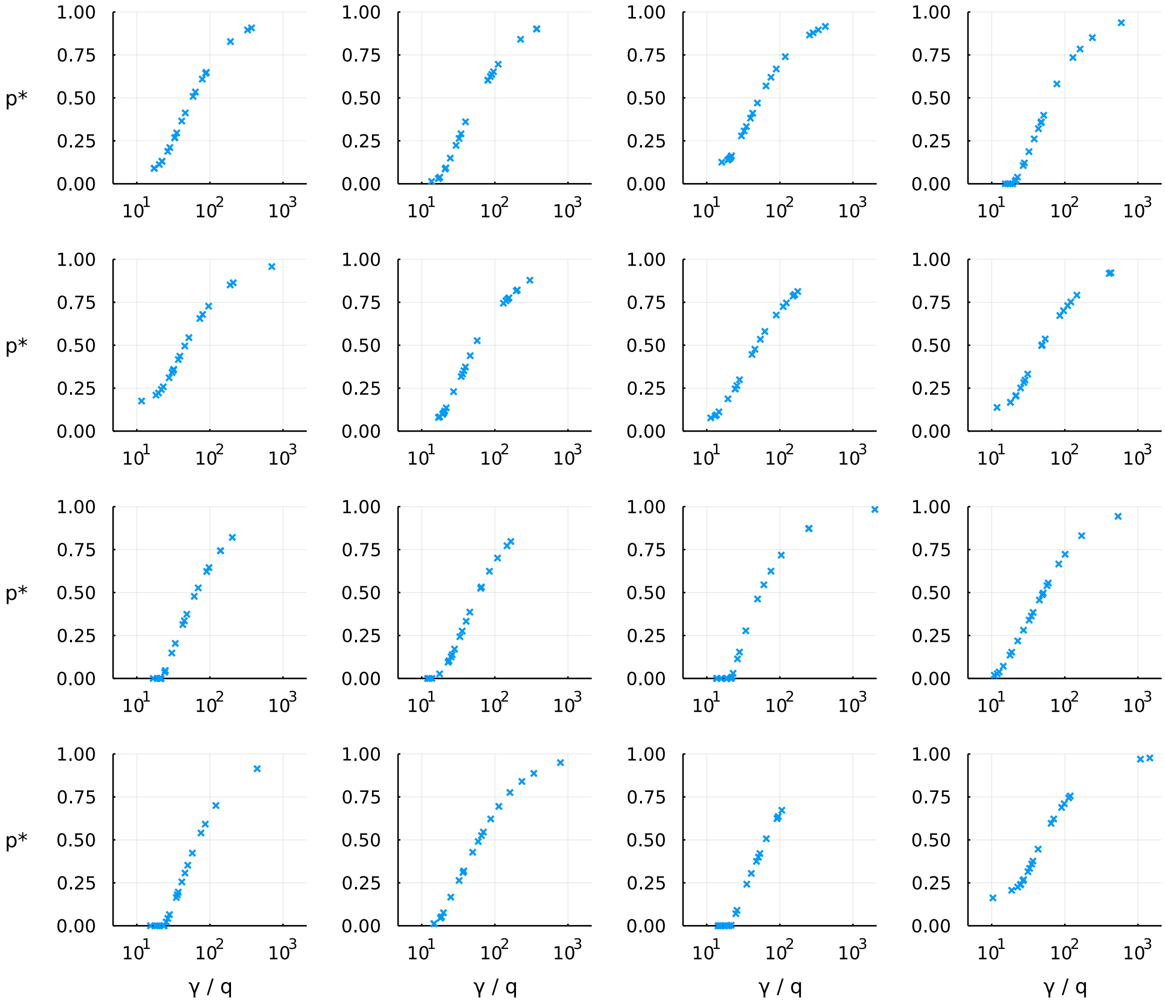

Figure 1: Competitiveness ratios γ_c/q_c and equilibrium cutoffs p_c^* in 16 randomly generated admissions markets, each containing 20 schools. The preferability parameters δ_c are drawn from $\mathrm{Uniform}(0,1)$, while the capacities are drawn from $\mathrm{Uniform}(0,1/10)$; hence, the market as a whole has a 50 percent chance of being over- or underdemanded. The figure suggests that the order of the equilibrium cutoffs is determined by the order of competitiveness ratios.

the following parameters. I have sorted the schools by their optimal cutoffs:

$$\begin{aligned} \gamma &= \left(\tfrac{2}{12}, \tfrac{1}{12}, \tfrac{3}{12}, \tfrac{6}{12}\right) \\ q &= (0.3, 0.1, 0.2, 0.2) \\ p^* &= (0.2, 0.3, 0.4, 0.6) \\ D(p^*, \gamma) &= (0.3, 0.1, 0.2, 0.2) \end{aligned}$$

As the total capacity is less than one, each school fills its capacity at equilibrium.

Figure 2 shows fifty iterations of the simultaneous tâtonnement process (Algorithm 4) applied to Pallet Town. At each iteration, the demand is computed directly by evaluating the expression derived above (6). Then, the cutoff vector is adjusted in the direction of excess demand according to a predetermined, decreasing sequence of step size. The cutoffs converge smoothly toward p^*, which suggests the continuity of the demand function and the stability of the equilibrium.

Figures 3 and 4 consider discrete approximations of the Pallet Town admissions market with samples of 20, 200, and 2000 students. In Figure 3, schools admit students according to their equilibrium cutoffs, students choose their favorite school, and schools observe their demand. When there are many students, the demand at each school approximately equals its capacity. In Figure 4, a stable matching of the students is computed so that each school fills its capacity exactly. When there are many students, the implied cutoffs approximately equal p^*. Compare Figures 3 and 4 with Figure 4 of Azevedo and Leshno (2016), which demonstrates the asymptotic convergence of implicit cutoffs as the number of students increases using a numerical experiment in which scores are partially correlated between schools.

Figures 5 and 6 visualize the incentive analysis under two different assignment paradigms. Suppose that each school in Pallet Town's utility function is $\min\{D_c, \kappa p_c\}$ for some large constant κ. This means that each school's primary goal is to increase its cutoff, and at fixed cutoff, each school will try to obtain the highest demand it can.[3]

Suppose that for many years, the district has imposed a limit q_c on the number of students each school can accept, and thus schools have adopted lowest cutoff values p_c that they can without their demand exceeding this limit. Then, next year, the district announces that it will deregulate the school market and allow schools to accept as many students as they like. Because

In Figure 5, the market is decentralized. For the sake of comparison the schools' current cutoffs are fixed to the equilibrium cutoffs shown in the following figure, even though the decentralized market incorporates no notion of capacity. The four curves in the graph show how each school's demand would change if the school's γ_c-value changed. (Not shown are the decreases in the demand at the other schools.) The slope of each school's quality–demand curve is given by the diagonal of the Jacobian given in Equation (**??**). All schools have a positive

[3]A utility function with this shape is implied by the definition of stable assignment, which asserts that schools prefer *any* student to an empty seat, and that schools only increase their cutoffs from zero when doing so does not cause a decrease in their demand.

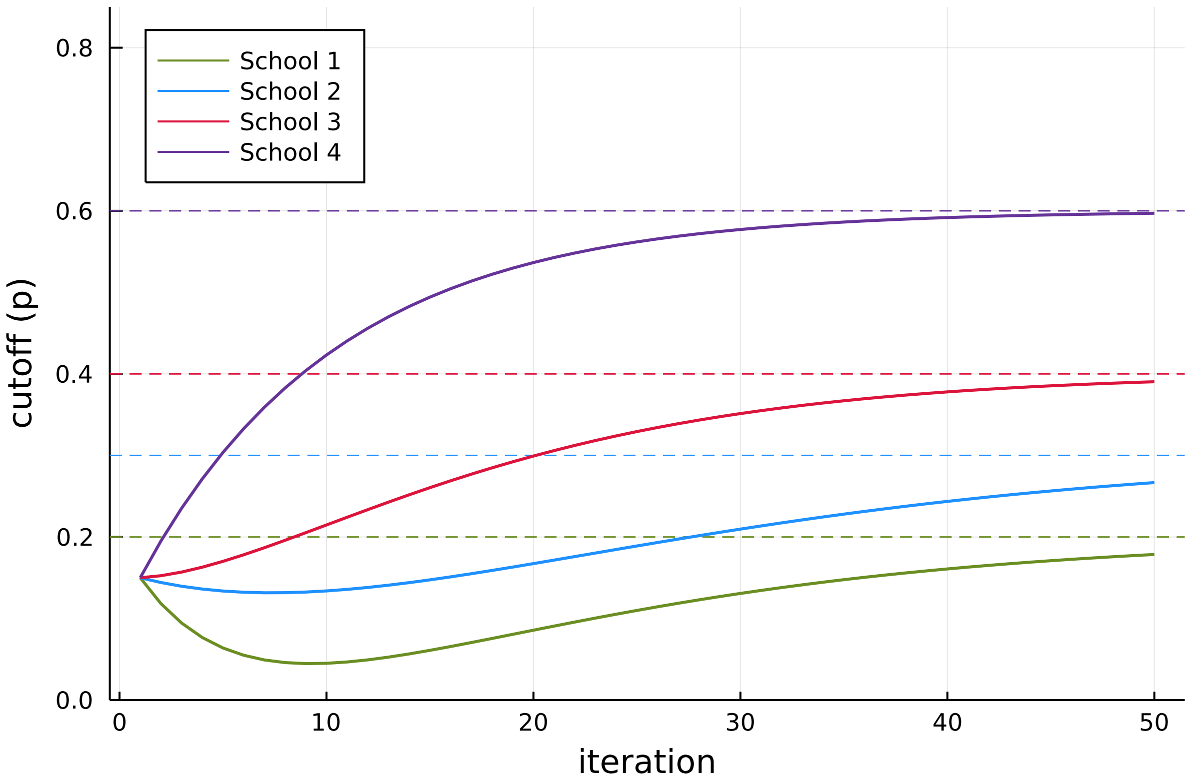

Figure 2: Convergence of fifty iterations of the simultaneous tâtonnement process (Algorithm 4) in the fictional nonatomic admissions market of Pallet Town (see text). The step size parameters are $\alpha = 0.2, \beta = 0.01$, and the initial cutoff vector is $p^{(0)} = (0.15, 0.15, 0.15, 0.15)$.

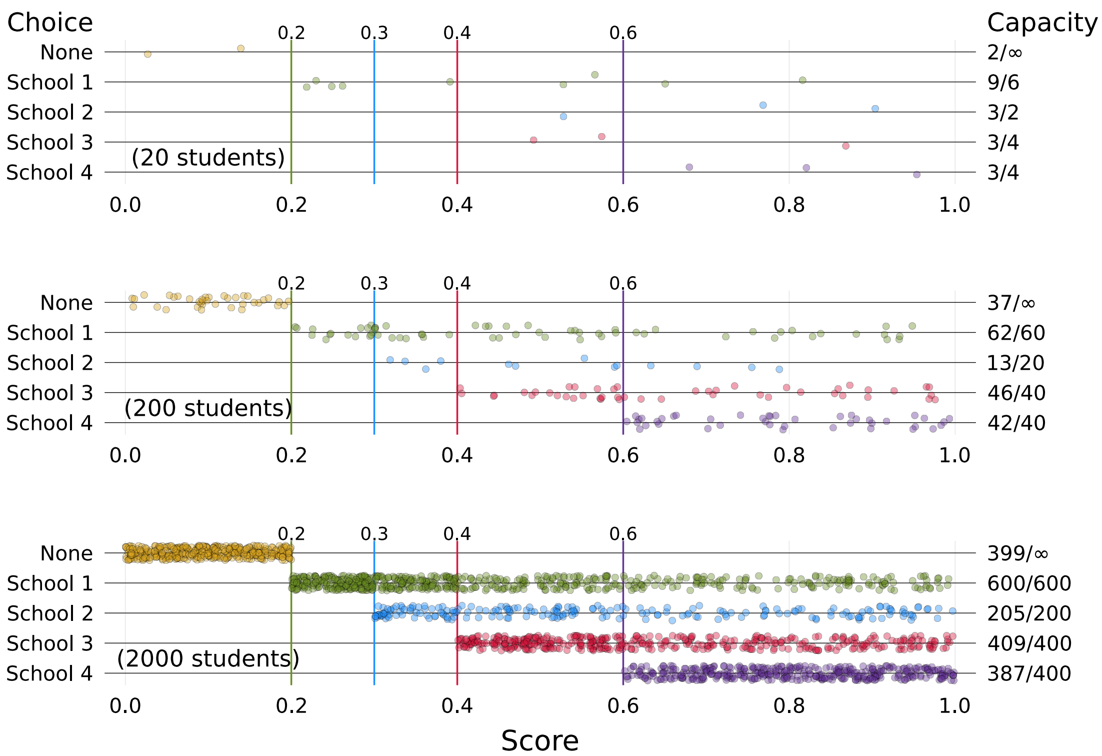

Figure 3: Simulation of a decentralized school-choice process in Pallet Town. A discrete sample of student preference lists and scores is drawn from η. Each school admits students whose score exceeds its equilibrium cutoff (shown as vertical lines), then each student chooses her favorite school from her consideration set. As the sample size increases, the demand at each school approximately equals its scaled capacity.

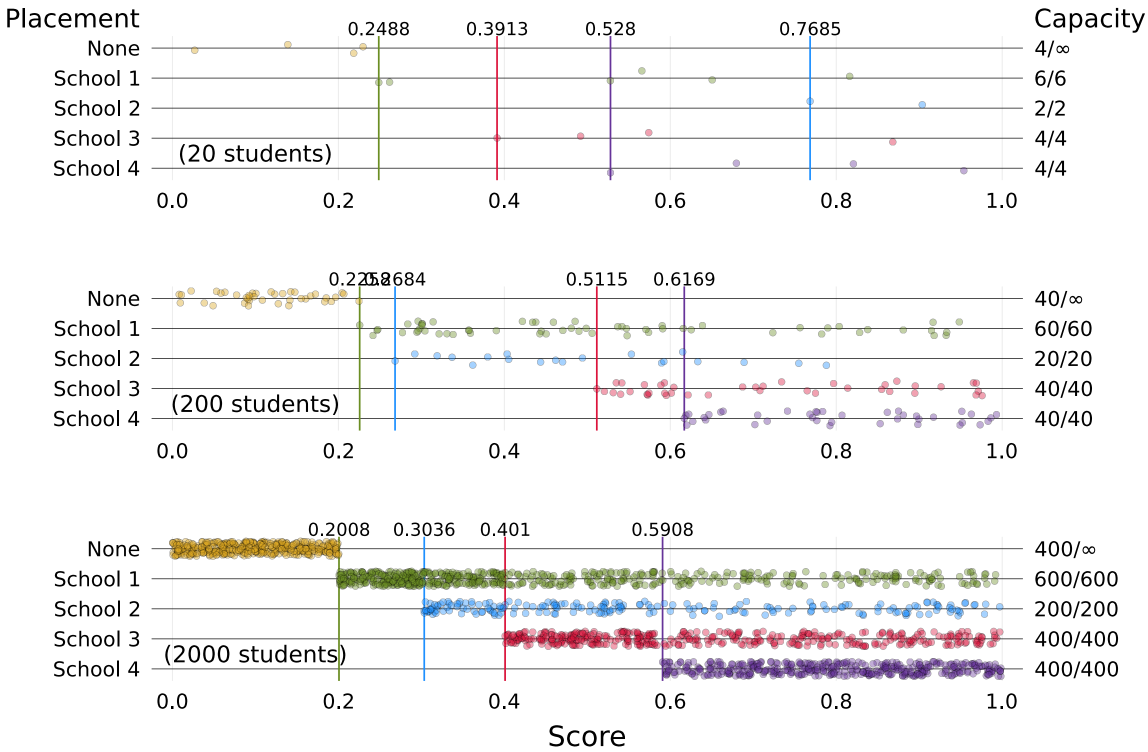

Figure 4: Simulation of a deferred acceptance process in Pallet Town. A discrete sample of student preference lists and scores is drawn from η. The student-proposing deferred acceptance algorithm (Algorithm 1) is used to compute a stable matching. The score of the least-qualified admit at each school, or the school's implied score cutoff, is computed and represented as a vertical line. Regardless of the sample size, each school fills its scaled capacity, and as the sample size increases, the implied cutoffs approach the market equilibrium. Comparison with Figures 2 and 3 suggests the equivalence among stationary points of a tâtonnement process, market clearing cutoffs, and stable assignments in nonatomic admissions markets.

incentive to improve in quality, and school 2's incentive is the strongest. The figure shows the incentive for schools to increase their quality when they are not allowed to change their cutoffs. Another strategy to increase the demand, of course, is simply to lower the cutoffs. But for the sake of argument we may assume that lowering the cutoffs is unacceptable, perhaps because they represent the minimal proficiency needed for a student to succeed in its curriculum.

Figures 5 considers a market which is confined to equilibrium, which can be interpreted as a centralized market that uses a DA mechanism or as the equilibrium of a competitive market in which schools' capacities represent hard constraints.

3.6 Inverse optimization of single-score admissions markets

In this section, I consider the inverse optimization task, in which the demand of the market and the cutoff vector is provided and we attempt to compute the quality parameters. I provide an analytic solution, discussion its usefulness to admissions planners, and offer a proof-of-concept demonstration of the inverse optimization task using admissions data from 677 American universities.

3.6.1 The inverse optimization task and an analytic solution

We must solve the following system for γ:

$$D_c = \sum_{d=c}^{|C|} \frac{\gamma_c}{\sum_{i=1}^{d} \gamma_i} \left(p_{d+1} - p_d\right), \quad \forall c \in C \tag{33}$$

Assume the schools are sorted in ascending order of cutoffs, and by homogeneity, let $\sum_{c \in C} \gamma_c \equiv 1$.

Consider the demand for $|C|$, the school with the highest index and therefore highest cutoff. Students who get into this school necessarily get into every school, so the outer sum of the system (33) has only one term, and the equation becomes

$$D_{|C|} = \frac{\gamma_{|C|}}{\sum_{i=1}^{|C|} \gamma_i} \left(1 - p_{|C|}\right) \tag{34}$$

$$\implies \gamma_{|C|} = \frac{D_{|C|}}{1 - p_{|C|}} \tag{35}$$

Now suppose that $\gamma_{c+1}, \gamma_{c+2}, \ldots, \gamma_{|C|}$ are known. Then γ_c can also be calculated from the observation that

$$\sum_{i=1}^{d} \gamma_i = 1 - \sum_{j=d+1}^{|C|} \gamma_j$$

where I take $\sum_{j=|C|+1}^{|C|} \gamma_j \equiv 0$.

Hence, the following recursive relation allows us to compute all the γ_c values in reverse

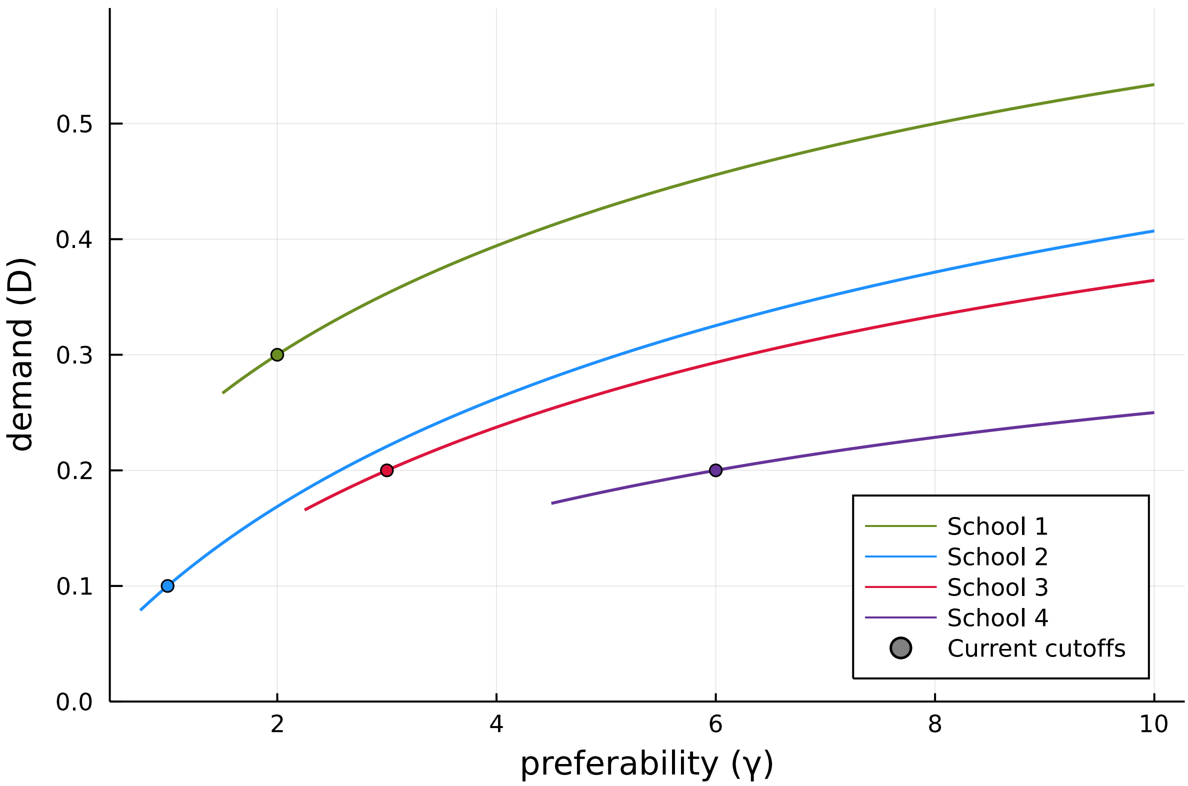

Figure 5: Quality effects in Pallet Town under a decentralized admissions process like that considered in Figure 3. Each line shows the change in each school's demand when it changes its quality γ_c while holding the cutoff vector and other schools' quality fixed. To the extent that each school's goal is to increase its demand, the slope of the curve represents the strength the incentive to improve in quality. At the current quality vector, the Jacobian of the demand is

$$\mathbf{J}_\gamma D = A = \frac{1}{360}\begin{bmatrix} 22 & -14 & -6 & -2 \\ -7 & 29 & -3 & -1 \\ -9 & -9 & 15 & -3 \\ -6 & -6 & -6 & 6 \end{bmatrix}$$

according to the expression provided in §3.3.2.

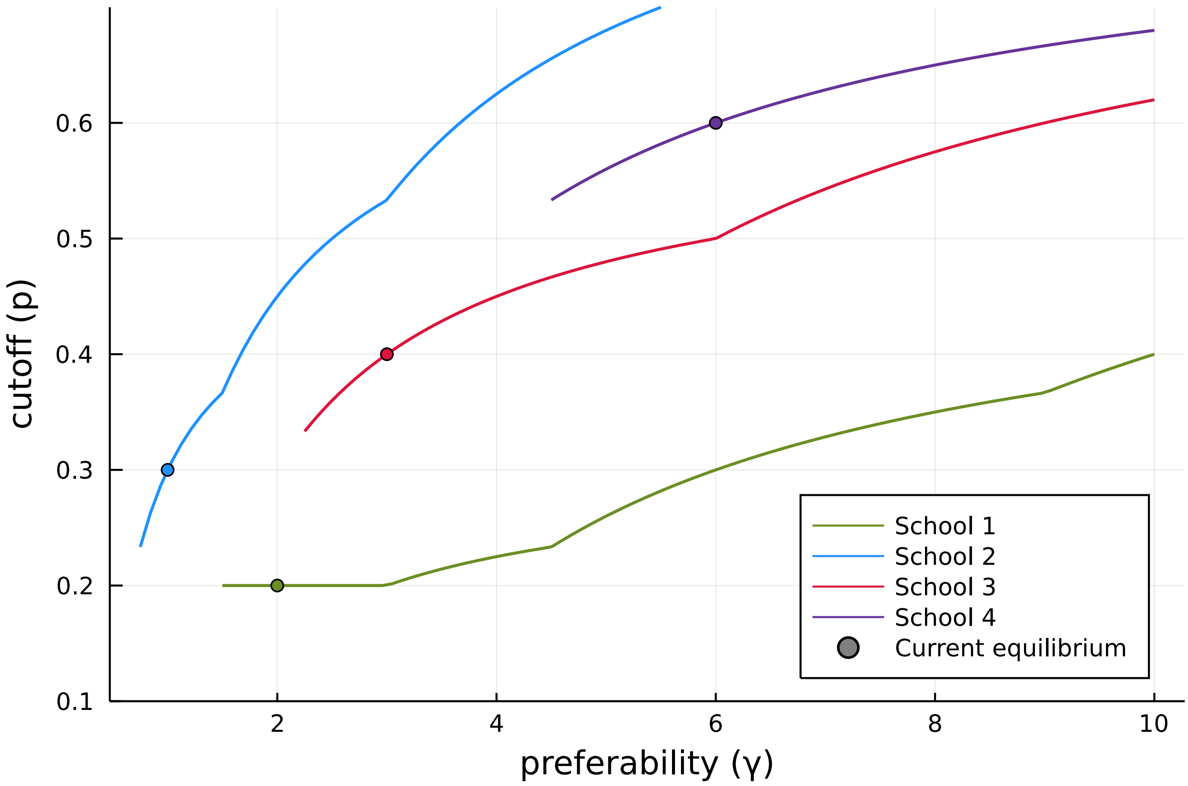

Figure 6: Quality effects in Pallet Town under a centralized admissions process such as the deferred acceptance process shown in Figure 4. Each line shows the change in each school's equilibrium cutoff when it changes its quality γ_c while holding other schools' quality fixed. To the extent that each school's objective is increase its cutoff, the slope of the curve represents the strength of the incentive to improve in quality. At the current quality vector, the Jacobian of the equilibrium cutoffs is

$$\mathbf{J}_\gamma \hat{p} = \frac{1}{30}\begin{bmatrix} 0 & & & \\ -3 & 6 & & \\ -2 & -2 & 2 & \\ -1 & -1 & -1 & 1 \end{bmatrix}$$

according to the expression derived in §3.4.1. The elbows in the graph correspond to ties among the competitiveness ratios.

order, starting with $\gamma_{|C|}$ and moving down.[4]

$$\gamma_{|C|} = \frac{D_{|C|}}{1 - p_{|C|}} \tag{36}$$

$$\gamma_c = \frac{D_c}{\displaystyle\sum_{d=c}^{|C|} \frac{p_{d+1} - p_d}{1 - \sum_{j=d+1}^{|C|} \gamma_j}}, \quad \forall c \in \{|C| - 1, |C| - 2, \ldots, 1\} \tag{37}$$

3.6.2 A demonstration using admissions data from American universities

By way of demonstrating the inverse optimization process, I present an example using a public-domain admissions dataset from a large set of American universities (Qian 2018).

The results and discussion below should be taken only as a proof of concept, for four reasons: First, this model makes the unrealistic assumption that all colleges have the same preference order, which is derived from students' standardized test scores. Second, I did not attempt to account for the fact that many students do not bother applying to schools for which they are are overqualified; as a result, the cutoffs are systematic underestimates. Third, the dataset is not adequately sourced, and appears to mix data across years. I used data from the 2018 ACT and 2015 SAT examinations to derive school cutoffs (ACT 2018; College Board 2015). Finally, I excluded from consideration schools for which test statistics were not listed, reducing the dataset from 1534 schools to 677. Thus, whereas the inverse optimization procedure provided above estimates schools' *preferability* when given perfect knowledge of their *selectivity*, in the present analysis, both quantities had to be estimated.

The dataset contains information regarding four test scores: the critical reading, math, and writing SAT subscores, and ACT composite scores. For each score, the dataset shows the 25th and 75th percentile of scores among students admitted to each school. By comparing these percentiles to percentile tables released by the testing agencies, I derived eight different estimates of each school's cutoff value. For example, at the University of Alabama, the 75th-percentile ACT composite score among admitted students is a 30 (out of 36). Relative to the population of ACT examinees, a student who earns a 30 is at the 90th percentile overall; hence, if the University of Alabama looks only at ACT composite scores, it follows that the minimum score among Alabama's admits is at the 60th percentile among all test-takers. In the occasional case where this calculation yielded a negative value, I cropped it to zero. Explicitly, letting p_{rel} (here 0.75) denote the percentile under consideration, and p_{abs} (here 0.90) denote the percentile score of a student with that score relative to the whole student population, the implied school cutoff is

$$p_{\text{impl}} = \max\left\{0, 1 - \frac{1 - p_{\text{abs}}}{1 - p_{\text{rel}}}\right\}$$

[4] Because this expression requires repeated division by small numbers, it is numerically unstable when the number of schools is large or there are many schools whose cutoffs are close together or equal. More over, the error accumulates with each iteration. Thus, it is sometimes more effective to solve the system (33) using a generic root finder; this spreads the numerical error out over all the schools.

To compute a composite estimate of each school's cutoff, I first averaged the cutoffs implied by the ACT and SAT data separately. Then, I took the average of these two weighted by the percentage of applicants who submitted each test score (which is also included in the dataset) and treated this as the school's p_c. Schools missing data for any of the eight data points mentioned above were excluded from consideration.

To compute each school's demand, I divided the number of students enrolled at each school by the sum of the same, which is 752,987. Thus, this model assumes that every student in the market can get into at least one college, and that the dataset includes all the college-like options that students in this market would consider. The first assumption is mild, because there are several colleges in the dataset whose estimated cutoff is zero. The second assumption is much more restrictive, as there are many colleges that do not report test scores, not to mention international schools, that are not represented in the data. (In the tabular results and graphs, I report demand as a number of students instead of the proportion D_c for legibility.)

The results are shown in Figure 7 and Table 1. The list of the schools with the highest preferability is a predictable list of prestigious universities. This outcome is remarkable, because the input data contains no explicit indicators of "prestige" (such as data on endowment size and employment outcomes), nor any survey data akin to the questionnaires that some newspapers send to college executives to generate their rankings. The model also does not require observations of individual student choices, as used to estimate MNL parameters under traditional statistical paradigms. The preferability parameters simply emerge organically from the selectivity of each school, each school's total demand, and the assumptions about the distribution of student talent. For comparison, Table 1 provides other metrics that might be used to assess school preferability such as the demand, cutoff, yield (proportion of admitted students who choose to attend), and *true yield*. The last is a contrived term for the proportion of *qualified* students, regardless of whether they applied, who chose to attend the school; this can be computed from the cutoff and demand.

Again, due to the low quality of the underlying data, I caution against reading deep meaning into this model's estimates. However, it is worthwhile to take the results at face value for a moment to demonstrate the sort of descriptive analysis enabled by the inverse optimization procedure.

First, consider the top-ranked schools, whose figures are shown in Table 1. In this market, the top two, Harvard and Yale, are similarly selective (they have almost the same cutoff). However, Harvard achieves a higher preferability parameter because at the same admissions standards, it attracts a larger student body (1659 students, versus Yale's 1356). Conspiciously absent from the list is the school with the highest cutoff, California Institute of Technology ($p_c = 0.9590$, $\gamma_c = 0.0081$, rank 34). While Caltech is a little more selective than Harvard, its entering class size is much smaller, at 249 students. Since, in this model, the set of students admitted to Caltech is only a slight subset of those admitted to Harvard, Harvard must be about $1659/249 = 6.7$ times as preferable. Perhaps an admissions director at Caltech would argue that the school's small entering class size is a key element of its appeal, and this is certainly true for many students. However, in this model, Harvard could easily achieve a similar class size to Caltech at a much higher cutoff, eliciting the same conclusion.

It is worth considering schools in other parts of the preferability distribution. Because this model considers not only selectivity but also entering class size as an indication of market power, compared to conventional college rankings, it grants an elevated position to public flagships like the University of Michigan at Ann Arbor, which draw large entering classes while maintaining fairly high admissions standards. Although tuition price has not figured into any stage of the analysis, this example suggests that the computed preferability parameters also incorporate a notion of *value,* whereas traditional college rankings arrange schools as if their tuition prices are the same.

While conventional wisdom posits small liberal-arts colleges and large public universities as incommensurable, administrators at both types of school share a common goal of recruiting an entering class that is both "large" relative to the physical size of the campus and "highly qualified" relative to competitor schools. γ_c offers us a way to compare the effectiveness of two schools' marketing efforts even when their recruitment strategies diverge. For example, the University of Vermont and Whitman College (a liberal-arts college in Eastern Washington) are ranked 106th and 107th, with preferability parameters 1.079×10^{-3} and 1.044×10^{-3}, respectively. Vermont has a large class size and a middling cutoff, while Whitman has a small class size and a cutoff close to that of UM Ann Arbor. Looking only at conventional statistics, it is hard to predict the decision of a student choosing between the two schools, but comparing γ-values (which, by definition, are choice probability weights) reveals that in this case it is a nearly even coin flip. Indeed, the two school's demand curves (shown in Figure 8) are all but identical.

The inverse optimization task makes no equilibrium assumption, and indeed invokes no notion of capacity or target class size. It simply reports the status of the market with respect to the current allocation of students. Thus, a possible application of this model is for an admissions director to use in modeling her school's demand curve. Figure 8 shows the predicted demand curves for Vermont, Whitman, and Caltech. In the future, suppose that Caltech decides that a larger cohort of 350 students better suits its goals. By how much should it relax its admissions standards in order to achieve this class size? One way to answer this question is to assume that Caltech's true yield remains approximately fixed. Then, Caltech should try to become $\frac{350}{249}$ as selective, by updating its cutoff to $1 - \frac{350}{249}(1 - 0.9590) = 0.9424$. A simple calculation shows that this way of estimating the demand curve is equivalent to fitting a linear model to the observed demand $(p_c, D_c) = (.9590, 249)$ and the implicit x-intercept $(1, 0)$.

However, the predicted cutoff associated with the higher target demand will be a slight underestimate, because Caltech's true yield also varies as a function of p. Under the new cutoff, students admitted with scores of (say) 0.95 are *not* qualified for Harvard and Yale, so Caltech will not have to compete as fiercely to recruit them as it does to recruit its current enrollees. The model presented here accounts for this change in the consideration set of marginal students, and thus calls for a more modest reduction of Caltech's cutoff, to the value of 0.9437.

Figure 9 presents a detailed view of Caltech's demand curve as predicted by this model alongside the linear model. The curve has a slightly bowed shape, which confirms the intuitive argument for a gentler relaxation of admissions standards in order to achieve lower target enrollment. In fact, every school's demand curve in this model is piecewise linear convex, meaning that linear regression will always underestimate the demand at cutoffs distant from those used to fit the line.[5] Of course, a more accurate regression could be constructed using demand observations from multiple years and by adding a quadratic term. But even then, if the observations used to fit the curve remain in the same "piece" of the piecewise linear demand function, then the expected regression curve will be a straight line.

This analysis has not accounted for the hypothesis that Caltech's appeal depends on a small class size, in which case looser admissions standards also reduce the school's preferability, necessitating a lower cutoff after all. Thus, in practical decisionmaking a model like mine is unlikely to be competitive with colleges' in-house models, which incorporate specific observations of students who applied to the school and chose to attend another. However, the current model produces an informative approximation of the demand curve without requiring students' personal data.

The code used to produce these results, as well as more details regarding how the cutoffs were estimated from available SAT and ACT data, is available on GitHub (Kapur 2021).

3.6.3 The informational quality of γ

The question of how to measure aggregate college preferability is not a simple one, and newspaper university rankings attract regular controversy for their imprecise methodology (Jaschik 2021). Typically, such measures are based on a combination of survey data and performance measures of a university's quality, for example, its research output or data on alumni salaries. However, conducting accurate surveys is costly and difficult, and while performance measures may correlate with college preferability, they offer a normative indication of which colleges "should" be popular without accounting for the decisions students actually make—decisions that may depend less on hard facts than on intangible notions of fitness and gut feelings. The latter may seem beyond the purview of economic analysis, but college administrators often want to know in quantitative terms how preferable their school is from a marketing point of view. That is, to a graduating high school senior who is deciding among a subset of schools, what is the probability that she will pick c?

A few candidate indicators of school popularity are as follows.

- *Total enrollment* is a function of how selective the school is. For example, Stanford and CSU Chico both enroll about 17,000 students and serve a global market.

[5] If the model is constrained to contain the point $(1, 0)$, then it forms a chord of the convex curve, and it will *overestimate* the demand at p_c-values *higher* than those used to fit the model. On the other hand, if this constraint is dropped and the model is fitted to two or more local observations, then it will *underestimate* the demand at both higher and lower cutoffs.

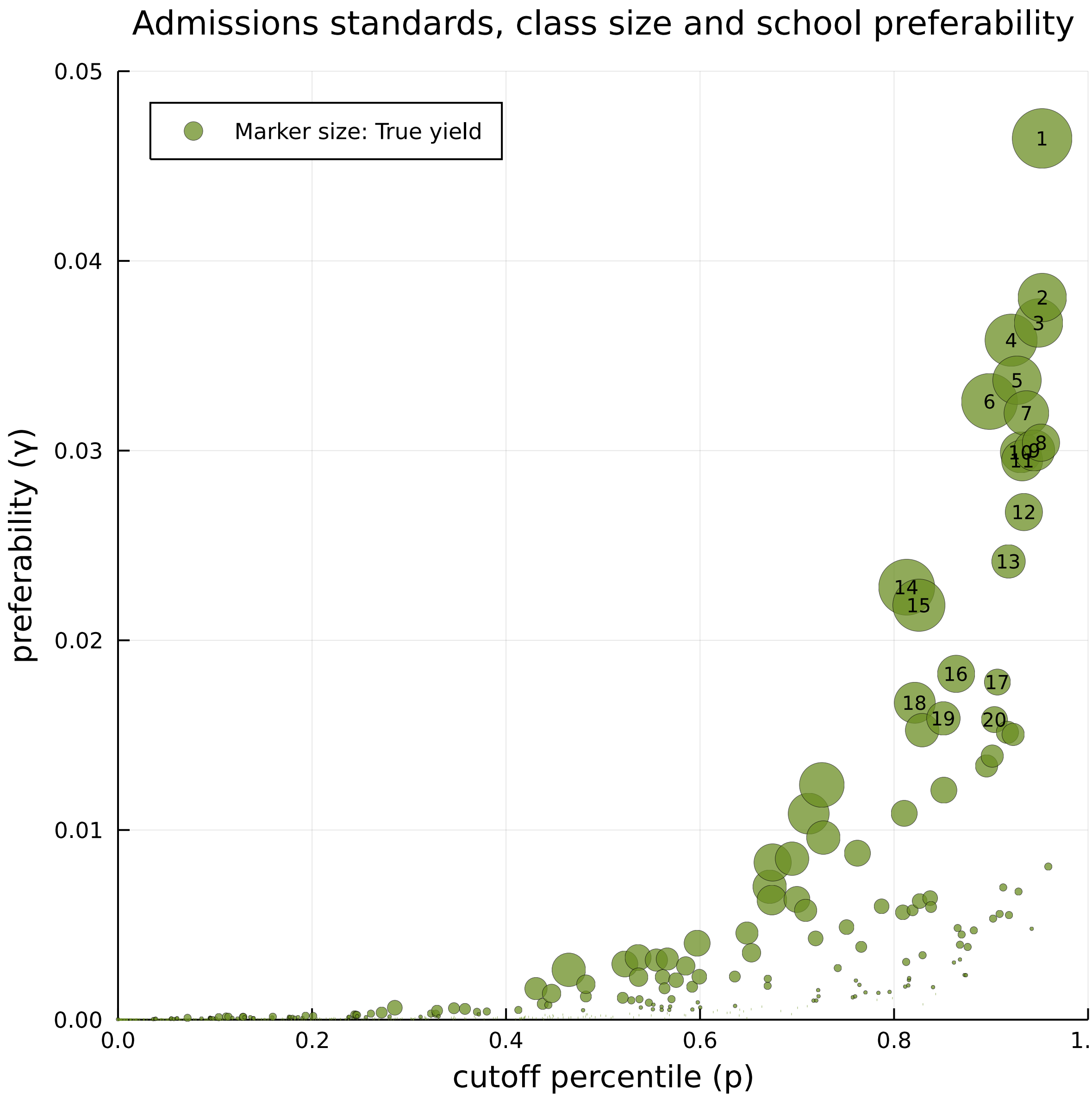

Figure 7: Inverse optimization procedure applied to a dataset of 677 American universities. School cutoffs were determined using a weighted average of cutoffs inferred from admitted students' SAT and ACT scores at the 25th and 75th percentiles. The marker size indicates the true yield, or percentage of qualified students who choose to attend each school. Details for the top twenty schools, as ranked by γ_c, are given in Table 1.

Rank	University	Demand	Cutoff (p_c)	Yield	True yield	Preferability (γ_c)
1	Harvard U	1659	0.9526	0.81	0.0465	0.0465
2	Yale U	1356	0.9527	0.66	0.0381	0.0381
3	U of Chicago	1426	0.949	0.53	0.0371	0.0367
4	U of Pennsylvania	2421	0.9207	0.63	0.0405	0.0358
5	Northwestern U	2037	0.9268	0.41	0.0369	0.0337
6	Cornell U	3223	0.8984	0.52	0.0421	0.0326
7	Washington U in St Louis	1610	0.9364	0.34	0.0336	0.032
8	Mass. Institute of Technology	1115	0.9515	0.72	0.0305	0.0304
9	Princeton U	1285	0.9445	0.65	0.0308	0.03
10	Stanford U	1677	0.9307	0.76	0.0321	0.0299
11	Vanderbilt U	1613	0.932	0.41	0.0315	0.0295
12	Columbia U	1415	0.9338	0.6	0.0284	0.0268
13	Duke U	1714	0.918	0.42	0.0278	0.0242
14	U of Michigan-Ann Arbor	6200	0.813	0.4	0.044	0.0228
15	New York U	5207	0.8256	0.35	0.0397	0.0218
16	Northeastern U	2891	0.864	0.19	0.0282	0.0182
17	Brown U	1543	0.9065	0.58	0.0219	0.0178
18	U of California-Berkeley	4162	0.8214	0.37	0.0309	0.0167
19	U of Southern California	2922	0.8509	0.31	0.026	0.0159
20	Carnegie Mellon U	1442	0.9035	0.3	0.0198	0.0158

Table 1: The top twenty schools by preferability γ_c, as determined by applying the inverse optimization process to a dataset of 677 American universities. Each school's demand is given as the number of students in the entering class; to compute D_c, divide by the total number of students, 752,987. The school's yield is as reported by the admissions office, while the true yield, which represents the proportion of qualified students who chose to attend the school, was computed by comparing the size of the entering class to the computed cutoff p_c.

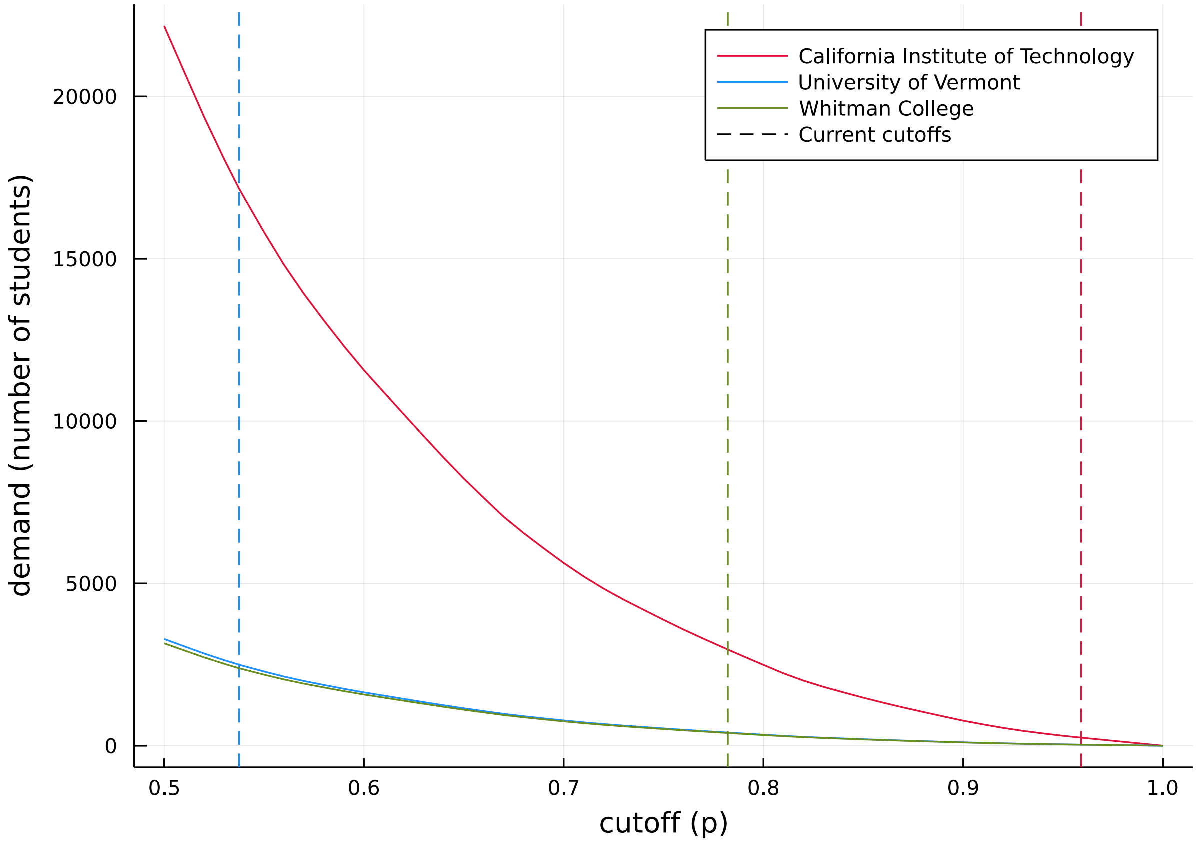

Figure 8: Three demand curves derived via the inverse optimization process. The University of Vermont and Whitman College have similar preferability parameters, and thus similar demand curves. However, each school has chosen a different selectivity threshold, reflecting its distinct admissions priorities. A detailed view of Caltech's demand curve appears in Figure 9.

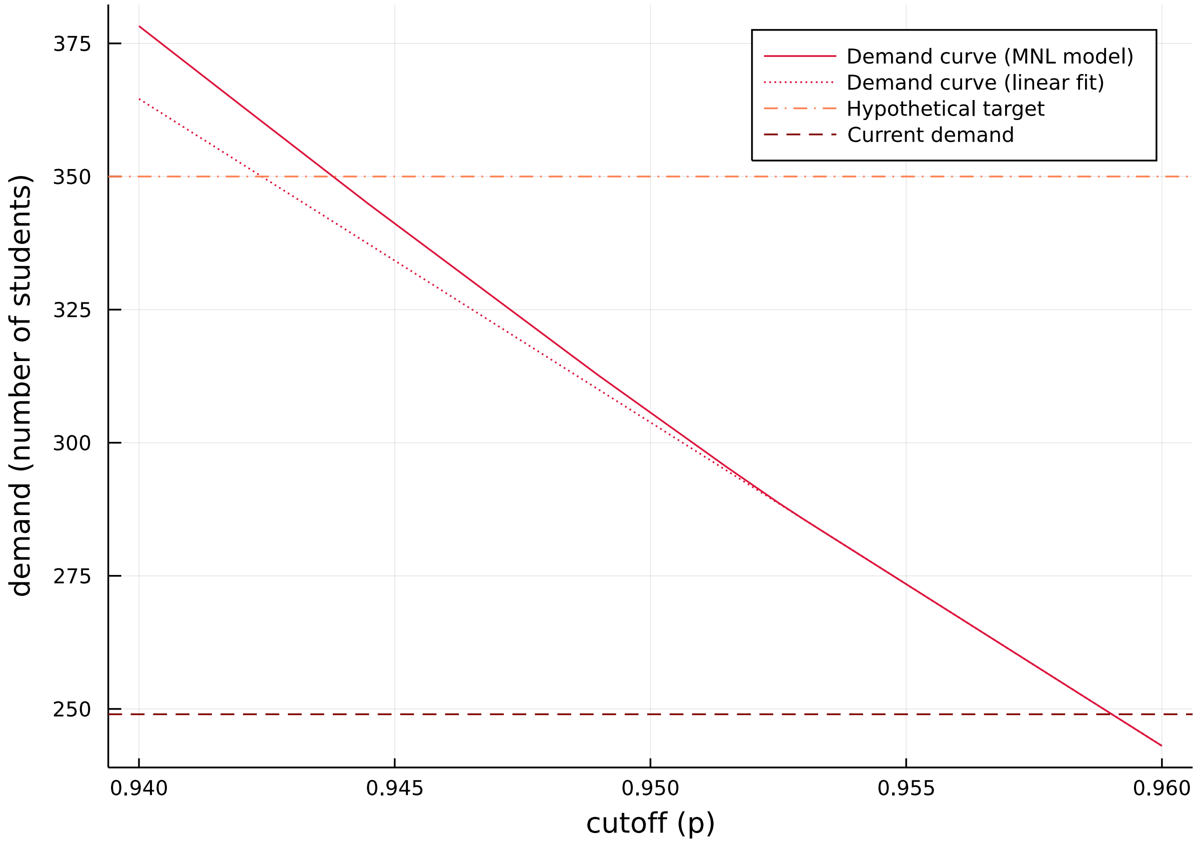

Figure 9: Detailed view of Caltech's demand curve near its current cutoff. Assuming preferability and other schools' cutoffs remain fixed, if Caltech wishes to increase its class size to 350, it should update its cutoff to the value indicated by the intersection of the demand curve and the coral line. A linear model prescribes decrease the cutoff to $p_c = 0.9424$, whereas the model provided here accounts for the fact that there is less competition for marginal students at the lower cutoff and prescribes a more conservative decrease to 0.9437.

- *Class statistics* like average GPA or cutoff scores are also a function of selectivity. It is easy for a school to admit 100 students with perfect grades, and it only has to recruit one of them to obtain a high class statistic.

- *Acceptance rate* overlooks that not all applicant pools are the same. High-achieving students typically don't bother applying to more than one or two safety schools, because they know they will be able to get in someone better. At the other end, many students remove themselves from the applicant pool at selective schools because they know they don't have a chance of getting in, or know they will not be able to afford to attend.

- *Yield,* the percentage of admitted students who choose to enroll, is commonly used as one factor in rankings of colleges in the US. However, this measure unfairly punishes safety schools, and is sensitive to applicant behavior and recruiting practices. If widely adopted, it also creates a perverse incentive for a midtier school to reject highly qualified applicants whom it expects to enroll elsewhere (so-called yield protection).

- *True yield* (a contrived term), the percentage of qualified students, whether they apply or not, who choose to enroll at a given school. This measure is better than yield, but it systematically overestimates the popularity of lenient schools, which face less competition for students. For example, consider a market with only two schools, Antarctic University and Oxvard University. Oxvard-tier students can choose between two schools, while Antarctic University–tier students have only one choice. Both schools may have similar true yield, but clearly Oxvard is globally preferable when controlling for selectivity.

Each one has certain flaws, and I argue that γ is a better indicator of the underlying notion of preferability that the measures above approximate. Because γ is free of bias at the extreme ends of the distribution, so it allows us to compare the preferability of Oxvard and Antarctic University even though very few students actually make this binary choice.

4 Discussion

Complexity of equilibrium in general.

Extend to mixed MNL and other choice models.

Need to consider assignment paradigms other than centralized and decentralized. For example, in South Korea. In order to ensure an even geographic distribution of students, the government places a firm limit q_c on the number of students who can attend each university. At the beginning of the admissions cycle, colleges are given the profiles of all the students interested in attending. Each college makes admissions offers over the course of several rounds, beginning with the highest-qualified students, and at each round a subset of the admitted students tentatively commit to attending one of the colleges that admitted them. Continuing in this manner, colleges continue lowering their cutoffs and offering admission to new students until either they fill their capacity, or lowering the cutoff fails to elicit an increase in demand. This is a distributed approximation of school-proposing DA, and it would be interesting to quantify its inefficiency as well as the cost of limited the number of schools to which each student can apply.

Computable instances include mine and iid scores.

Admissions coalitions and clusters.

Bibliography

Abdulkadiroğlu, Atila, Yeon-Koo Che, and Yosuke Yasuda. 2015. "Expanding 'Choice' in School Choice." *American Economic Journal: Microeconomics* 7, no. 1 (Feb.): 1–42.

ACT. 2018. *The ACT Profile Report: National.* https://www.act.org/content/dam/act/unsecured/documents/cccr2018/P_99_999999_N_S_N00_ACT-GCPR_National.pdf.

Ashlagi, Itai, and Afshin Nikzad. 2020. "What Matters in School Choice Tie-Breaking? How Competition Guides Design." *Journal of Economic Theory* 190:article no. 105120.

Azevedo, Eduardo M., and Jacob D. Leshno. 2016. "A Supply and Demand Framework for Two-Sided Matching Markets." *Journal of Political Economy* 124, no. 5 (Sept.): 1235–68.

Bancroft, Amanda. 2015. "How Do High-Status Parents Choose Schools?" Rice University, Houston, TX.

Bodoh-Creed, Aaron L. 2020. "Optimizing for Distributional Goals in School Choice Problems." *Management Science* 66, no. 8 (Sept.): 3657–76.

Codenotti, Bruno, and Kasturi Varadarajan. 2007. "Computation of Market Equilibria by Convex Programming." Edited by Noam Nisan, Tim Roughgarden, Éva Tardos, and Vijay V. Vazirani. Chap. 6 in *Algorithmic Game Theory*. Cambridge: Cambridge University Press.

College Board. 2015. *SAT Percentile Ranks.* https://web.archive.org/web/20160331125512/https://secure-media.collegeboard.org/digitalServices/pdf/sat/sat-percentile-ranks-crit-reading-math-writing-2014.pdf.

Dobbie, Will, and Jr. Roland G. Fryer. 2014. "The Impact of Attending a School with High-Achieving Peers: Evidence from the New York City Exam Schools." *American Economic Journal: Applied Economics 2014* 6 (3).

Gale, David, and Lloyd Shapley. 1962. "College Admissions and the Stability of Marriage." *American Mathematics Monthly* 69 (1): 9–15.

Ginsburg, Aaron. 2017. *School Choice, Information, and Applicant Distributions in NYC Public Middle Schools*. New York City.

Intriligator, Michael D. 1971. *Mathematical Optimization and Economic Theory*. Englewood Cliffs, NJ: Prentice-Hall.

Irving, Robert W., Paul Leather, and Dan Gusfield. 1987. "An Efficient Algorithm for the 'Optimal' Stable Marriage." *Journal of the Association for Computing Machinery* 34, no. 3 (July): 532–43.

Jaschik, Scott. 2021. "International Rankings and Conflicts of Interest." *Inside Higher Ed*, May 3, 2021. https://www.insidehighered.com/admissions/article/2021/05/03/international-rankings-and-conflicts-interest.

Kapur, Max. 2021. StudentPrefsRevOpt. https://github.com/maxkapur/StudentPrefsRevOpt

Lucey, Catherine Reinis, and Aaron Saguil. 2020. “The Consequences of Structural Racism on MCAT Scores and Medical School Admissions: The Past Is Prologue.” *Academic Medicine* 95, no. 3 (March): 351–56.

Oden, J. T., and N. Kikuchi. 1980. “Theory of Variational Inequalities with Applications to Problems of Flow Through Porous Media.” *International Journal of Engineering Science* 18 (10–A): 1173–1284.

Primary Research Group. 2014. *Estimating Applications, Transfers and Accepted Offers to the College*.

Qian, Samson. 2018. “College Admissions: Admission/Class Demographics by University,” 2018. https://www.kaggle.com/samsonqian/college-admissions.

Robbins, Herbert, and Sutton Monro. 1951. “A Stochastic Approximation Method.” *The Annals of Mathematical Statistics* 22, no. 3 (Sept.): 400–407.

Roth, Alvin E. 1982. “The Economics of Matching: Stability and Incentives.” *Mathematics of Operations Research* 7, no. 4 (Nov.): 617–28.

Sackett, Paul R., Nathan R. Kuncel, Adam S. Beatty, Jana L. Rigdon, Winny Shen, and Thomas B. Kiger. 2012. “The Role of Socioeconomic Status in SAT-Grade Relationships and in College Admissions Decisions.” *Psychological Science* 23 (9): 1000–1007.

Steimle, Lauren N., Yuming Sun, Lauren Johnson, Tibor Besedeš, Patricia Mokhtarian, and Dima Nazzal. 2020. “Students’ Preferences for Returning to Colleges and Universities During the COVID-19 Pandemic: A Discrete Choice Experiment.” Preprint, submitted December 20, 2020. https://osf.io/mzxs6/.

Uzawa, H. 1960. “Walras’ Tâtonnement in the Theory of Exchange.” *The Review of Economic Studies* 27, no. 3 (June): 182–194.