CHAPTER 2

Discrete Systems and Signal Spaces

The first chapter introduced many different sorts of signals—simple, complicated, interesting, boring, synthetic, natural, clean, noisy, analog, discrete—the initial mixed stock of fuel for the signal analysis machinery that we are going to build. But few signals are ends in themselves. In an audio system, for example, the end result is music to our ears. But that longitudinal, compressive signal is only the last product of many transformations of many representations of the sound on its path from compact disc to cochlea. It begins as a stretch of some six billion tiny pits on a compact disc. A laser light strobes the disc, with its reflection forming a pair of 16-bit sound magnitude values 44,000 times each second. This digital technique, known as *Pulse Code Modulation* (PCM), provides an extremely accurate musical tone rendition. Filtering circuits remove undesirable artifacts from the digital signal. The digital signal is converted into analog form, filtered again, amplified, bandpass filtered through the graphic equalizer, amplified again, and finally delivered strong and clean to the speakers. The superior sound quality of digital technology has the drawback of introducing some distortion at higher frequencies. Filtering circuits get rid of some distortion. Since we are human, though, we cannot hear the false notes; such interference occurs at frequencies above 22 kHz, essentially outside our audio range. Of the many forms the music signal takes through the stereo equipment, all but the last are transitional, intended for further conversion, correction, and enhancement. Indeed, the final output is pleasant only because the design of the system incorporates many special intermediate processing steps. The abstract notion of taking an input signal, performing an operation on it, and obtaining an output is called a *system*. Chapter 2 covers systems for discrete signals.

As a mathematical entity, a system is analogous to a vector-valued function on vectors, except that, of course, the "vectors" have an infinite extent. Signal processing systems may require a single signal, a pair of signals, or more for their inputs. We shall develop theory primarily for systems that input and output a single signal. Later (Chapter 4) we consider operations that accept a signal as an input but fundamentally change it or break it down somehow to produce an output that is not a signal. For example, the output could be a structural description, an interpretation, or

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just a number that indicates the type of the signal. We will call this a *signal analysis system* in order to distinguish it from the present case of *signal processing systems*.

Examples of systems abound in electronic communication technology: amplifiers, attenuators, modulators, demodulators, coders, decoders, and so on. A radio receiver is a system. It consists of a sequence of systems, each accepting an input from an earlier system, performing a particular operation on the signal, and passing its output on to a subsequent system. The entire cascade of processing steps converts the minute voltage induced at the antenna into sound waves at the loudspeaker. In modern audio technology, more and more of the processing stages operate on digital signal information. The compact disc player is a wonderful example, embodying many of the systems that we cover in this chapter. Its processed signals take many forms: discrete engravings on the disc, light pulses, digital encodings, analog voltages, vibration of the loudspeaker membrane, and finally sound waves.

Although systems that process on digital signals may not be the first to come to mind—and they are certainly not the first to have been developed in electronic signal conditioning applications—it turns out that their mathematical description is much simpler. We shall cover the two subjects separately, beginning here with the realm that is easier realm to conquer: discrete signal spaces and systems. Many signal processing treatments combine the introduction of discrete and continuous time systems [1–4]. Chapter 3 covers the subtler theory of analog systems.

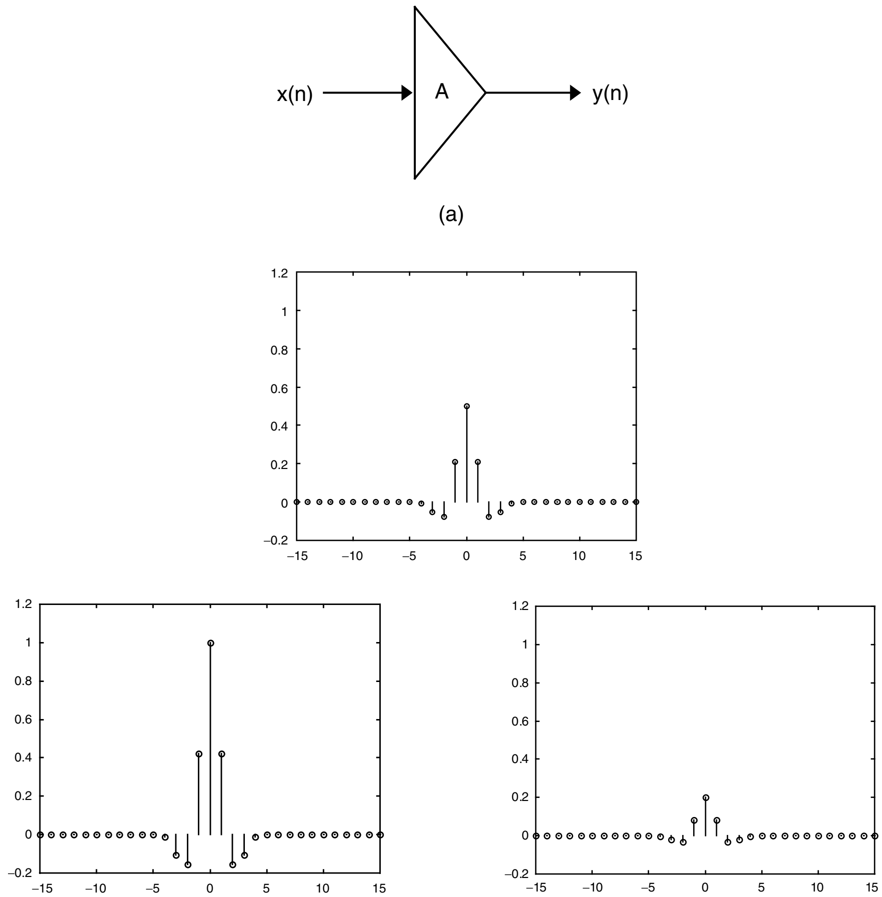
2.1 OPERATIONS ON SIGNALS

This section explores the idea of a discrete system, which performs an operation on signals. To help classify systems, we define special properties of systems, provide examples, and prove some basic theorems about them. The proofs at this level are straightforward.

A discrete system is a function that maps discrete signals to discrete signals. Systems may be defined by rules relating input signals to output signals. For example, the rule $y(n) = 2x(n)$ governs an *amplifier* system. This system multiplies each input value $x(n)$ by a constant $A \geq 1$. If H is a system and $x(n)$ is a signal, then $y = H(x)$ is the output of the system H, given the input signal x. More compact and common, when clarity permits, is $y = Hx$. To highlight the independent variable of the signals we can also say $y(n) = H(x(n))$. But there should be no misunderstanding: The system H operates on the whole signal x not its individual values $x(n)$, found at time instants n. Signal flow diagrams, with arrows and boxes, are good for visualizing signal processing operations (Figure 2.1).

Not every input signal to a given system produces a valid output signal. Recall that a function on the real numbers might not have all real numbers in its domain. An example is $f(t) = \sqrt{t}$ with $\mathrm{Domain}(f) = \{t \in \mathbb{R}: t \geq 0\}$. Consider now the *accumulator* system $y = Hx$ defined by the rule

$$y(n) = \sum_{k=-\infty}^{n} x(k)\,. \tag{2.1}$$



(a)

(b)

Fig. 2.1. (a) Amplifier system symbol. (b) A sinusoidal pulse, amplification by factor $A = 2$, and attenuation by factor $A = 0.4$.

With $x(n) = 1$ for all n, H has no output. So, as with functions on the real numbers, it is best to think of a system as a partial function on discrete signals.

2.1.1 Operations on Signals and Discrete Systems

There are many types of operations on signals and the particular cases that happen to be discrete systems. We list a variety of cases, some quite simple. But it will turn out that many types of discrete systems decompose into such simple system components, just as individual signals break down into sums of shifted impulses.

Definition (Discrete System). A *discrete system* H is a partial function from the set of all discrete signals to itself. If $y(n)$ is the signal output by H from the input $x(n)$, then $y = Hx$ or $y = H(x)$. It is common to call y the *response* of the system H to input

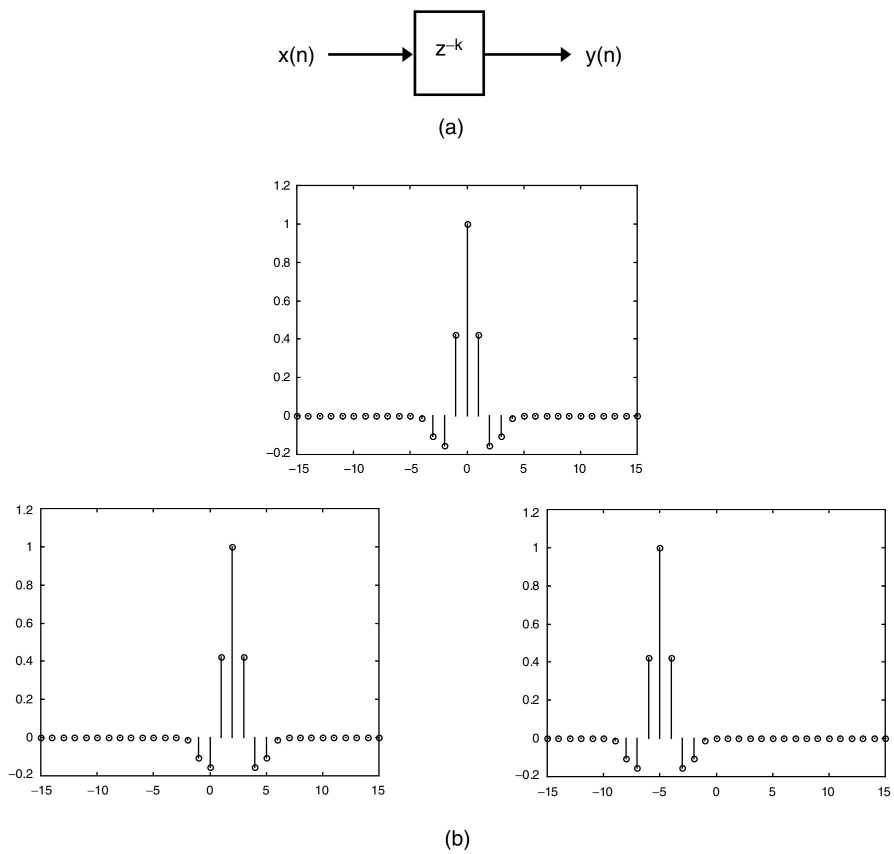


Fig. 2.2. Translation systems. (a) Diagram for a translation system, $y(n) = x(n-k)$. (b) discrete sinusoidal pulse within a Gaussian envelope, $x(n)$; $x(n-k)$, with $k = 2$; and $x(n-k)$ with $k = -5$.

x. The set of signals x for which some $y = Hx$ is the *domain* of the system H. The set of signals y for which $y = Hx$ for some signal x is the *range* of H.

One simple signal operation is to multiply each signal value by a constant (Figure 2.1). If H is the system and $y = Hx$, then the output values $y(n)$ are related to input values by $y(n) = Ax(n)$. This operation *inverts* the input signal when $A < 0$. When $|A| > 1$, the system *amplifies* the input. When $|A| < 1$, the system *attenuates* the input signal. This system is also referred to as a *scaling* system. (Unfortunately, another type of system, one that dilates a signal by distorting its independent variable, is also called a scaling system. Both appellations are widespread, but the two notions are so different that the context is usually enough to avoid confusion.)

The domain of an amplification system is all discrete signals. Except for the case $A = 0$, the range of all amplification systems is all discrete signals.

Another basic signal operation is to delay or advance its values (Figure 2.2). Thus, if $x(n)$ is an input signal, this system produces an output signal $y(n) = x(n-k)$

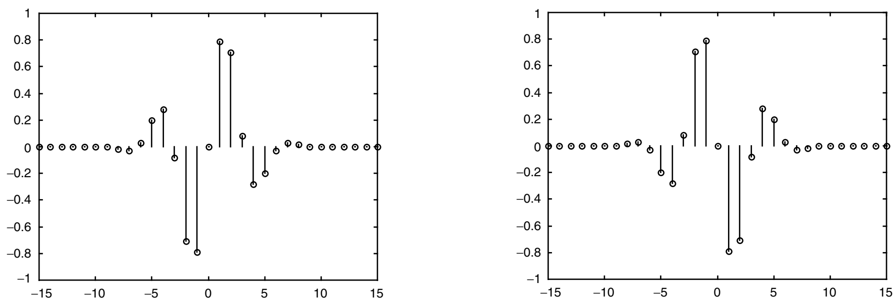


Fig. 2.3. Time reversal $y(n) = x(-n)$. There does not seem to be a conventional block diagram symbol for time reversal, perhaps owing to its physical impossibility for time-based signals.

for some integer k. This is called a *time shift* when the signal's independent variable indicates a time instant or a translation when the independent variable stands for a non-time quantity, such as distance. If $k > 0$, then the shift operation is a *delay.* If $k < 0$, then this system *advances* the signal. The diagram notation z^{-k} is inspired by the notions of the impulse response of the translation system, which we shall discover later in this chapter, and the z-transform (Chapter 8).

The set of all translates of a signal is closed under the translation operation. This system is also commutative; the order of two successive translations of a signal does not matter. Translations cause no domain and range problems. If T is a translation system, then $Domain(T)$ = Range(T) = $\{s(n)$: *s is a discrete signal*$\}$.

Signal *reflection* reverses the order of signal values: $y(n) = x(-n)$. For time signals, we will call this a *time reversal* system (Figure 2.3). It flips the signal values $x(n)$ around the instant $n = 0$. Note that the reflection and translation operations do not commute with one another. If H is a reflection system and G is a translation $y(n) = x(n-k)$, then $H(Gx) \neq G(Hx)$ for all x unless $k = 0$. Notice also that we are careful to say "for all x" in this property. In order for two systems to be identical, it is necessary that their outputs are identical when their inputs are identical. It is not enough that the system outputs coincide for a particular input signal.

Signal *addition* or *summation* adds a given signal to the input, $y(n) = x(n) + x_0(n)$, where $x_0(n)$ is a fixed signal associated with the system H (Figure 2.4). If we allow systems with two inputs, then we can define $y = H(v,w) = v + w$.

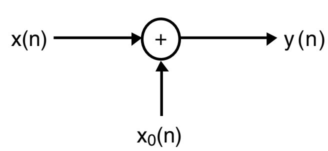


Fig. 2.4. System $y = Hx$ adds another (fixed) signal to the input.

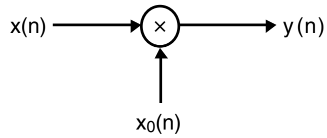


Fig. 2.5. System $y = Hx$ multiplies the input term-by-term with another (fixed) signal. This also called *modulation*, especially in communication theory.

Another type of system, called a *multiplier* or *modulator*, forms the termwise product of the input, $y(n) = x(n)x_0(n)$, where $x_0(n)$ is a fixed signal associated with the system H (Figure 2.5). Note carefully that the product system is *not* written with an asterisk operator $y = x*h$. This is the notation for convolution, a more important signal operation, which we will cover below.

The *correlation* of two signals is the sum of their termwise products:

$$C = \sum_{n=-\infty}^{\infty} x(n)\overline{y(n)}. \tag{2.2}$$

Signals $x(n)$ and $y(n)$ may be complex-valued; in this case, we take the complex conjugate of the second operand. The correlation of a signal with itself, the *autocorrelation*, will then always be a non-negative real number. In (2.2) the sum is infinite, so the limit may not exist. Also note that (2.2) does not define a system, because the output is a number, not a signal. When we study abstract signal spaces later, we will call correlation the *inner product* of the two signals. It is a challenge to find classes of signals, not necessarily having finite support, for which the inner product always exists.

The *cross-correlation system* is defined by the input–output relation

$$y(n) = (x \circ h)(n) = \sum_{k=-\infty}^{\infty} x(k)h(n+k). \tag{2.3}$$

In (2.3) the signal $h(n)$ is translated before the sum of products correlation is computed for each $y(n)$. If the signals are complex-valued, then we use the complex conjugate of $h(n)$:

$$y(n) = (x \circ h)(n) = \sum_{k=-\infty}^{\infty} x(k)\overline{h(n+k)}. \tag{2.4}$$

This makes the autocorrelation have a non-negative real value for $n = 0$; if $x = h$,

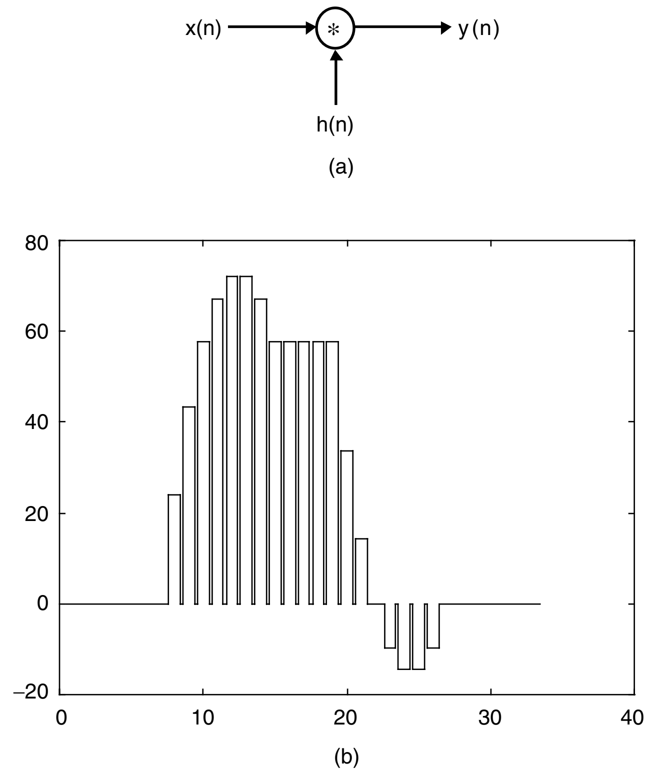


Fig. 2.6. Convolution. (a) The system $y = Hx = (h{*}x)(n)$. (b) An example convolving two finitely supported signals a square pulse $h(n) = 4.8[u(n-8) - u(n-20)]$ and a triangular pulse $x(n) = (6-n)[u(n-1) - u(n-9)]$.

$$y(0) = (x \circ h)(0) = \sum_{k=-\infty}^{\infty} x(k)\overline{x(nk)} = \|x\|_2^2, \tag{2.5}$$

the square of the l^2 norm of x, which will become quite important later.

Convolution seems strange at first glance—a combination of reflection, translation, and correlation:

$$y(n) = (x * h)(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k). \tag{2.6}$$

But this operation lies at the heart of linear translation invariant system theory, transforms, and filtering. As Figure 2.6 shows, in convolution one signal is flipped and then shifted relative to the other. At each shift position, a new $y(n)$ is calculated as the sum of products.

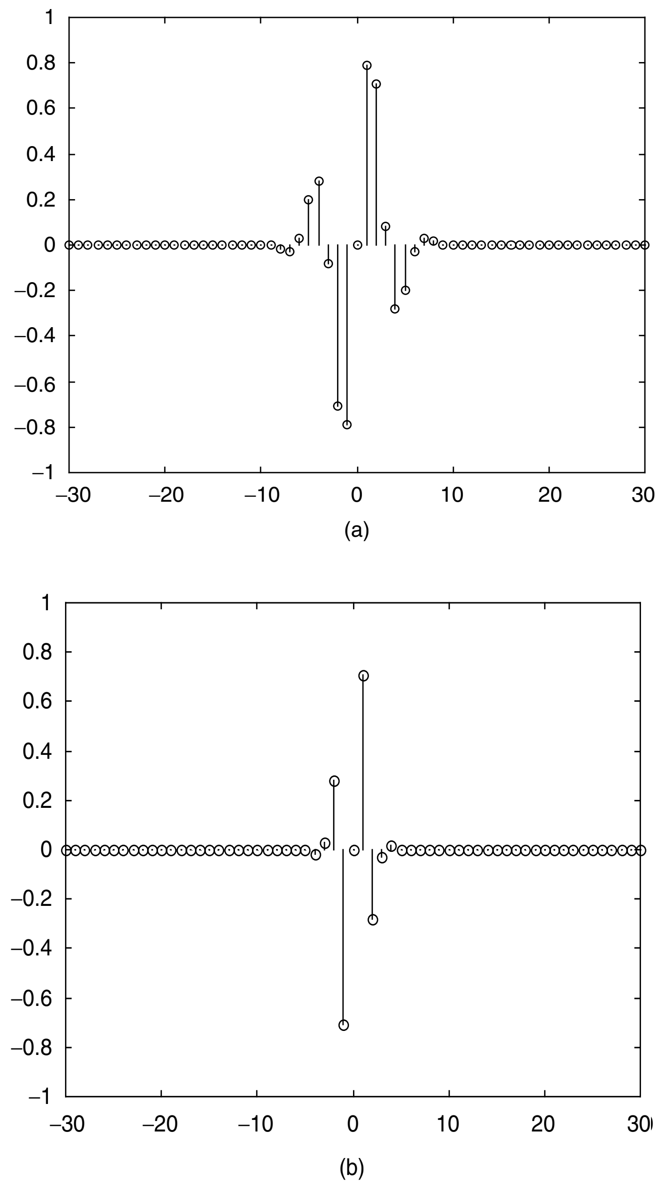


Fig. 2.7. Uniform sampling.

A *subsampling* or *downsampling* system skips over samples of the input signal: $y(n) = x(kn)$, where $k > 0$ is the *sampling interval*. This is *uniform* sampling (Figure 2.7). In nonuniform sampling, the intervals between samples vary. Both forms of signal sampling have proven handy in applications.

Sampling is useful when the content of a signal contains less information than the density of the samples warrants. In uniform sampling, the relevant signal information is adequately conveyed by every kth sample of the input. Thus, subsampling is a preliminary signal compression step. Scientists and engineers, attempting to squeeze every bit of digital information through narrow bandwidth communication channels, have been seeking better ways to compress signals and images in recent years [5]. Technologies such as network video and cellular radio communications hinge on the efficiency and integrity of the compression operations. Also, in signal and image analysis applications, we can filter signals and subsample them at multiple rates for coarse-to-fine recognition. One outcome of all the research is that if the filtering and subsampling operations are judiciously chosen, then the sampled signals are adequate for exact reconstruction of the original signal. Progressive transmission is therefore feasible. Moreover, there is an unexpected, intimate connection between multirate signal sampling and the modern theory of time-scale transforms, or wavelet analysis [6]. Later chapters detail these aspects of signal subsampling operations. Nonuniform sampling is useful when some local regions of a signal must be more carefully preserved in the sampled output. Such systems have also become the focus of modern research efforts [7].

An *upsampling* operation (Figure 2.8) inserts extra values between input samples to produce an output signal. Let $k > 0$ be an integer. Then we form the upsampled output signal $y(n)$ from input signal $x(n)$ by

$$y(n) = \begin{cases} x\left(\frac{n}{k}\right) & \text{if } \frac{n}{k} \in \mathbb{Z}, \\ 0 & \text{if otherwise.} \end{cases} \tag{2.7}$$

A *multiplexer* merges two signals together:

$$y(n) = \begin{cases} x_0(n) & \text{if } n \text{ is even,} \\ x_1(n) & \text{if } n \text{ is odd.} \end{cases} \tag{2.8}$$

A related system, the *demultiplexer*, accepts a single signal $x(n)$ as an input and produces two signals on output, $y_0(n)$ and $y_1(n)$. It also is possible to multiplex and demultiplex more than two signals at a time. These are important systems for communications engineering.

Thresholding is an utterly simple operation, ubiquitous as well as notorious in the signal analysis literature. Given a threshold value T, this system segments a signal:

$$y(n) = \begin{cases} 1 & \text{if } x(n) \geq T, \\ 0 & \text{if } x(n) < T. \end{cases} \tag{2.9}$$

The threshold value T can be any real number; however, it is usually positive and a thresholding system usually takes non-negative real-valued signals as inputs. If the input signal takes on negative or complex values, then it may make sense to first produce its magnitude $y(n) = |x(n)|$ before thresholding.

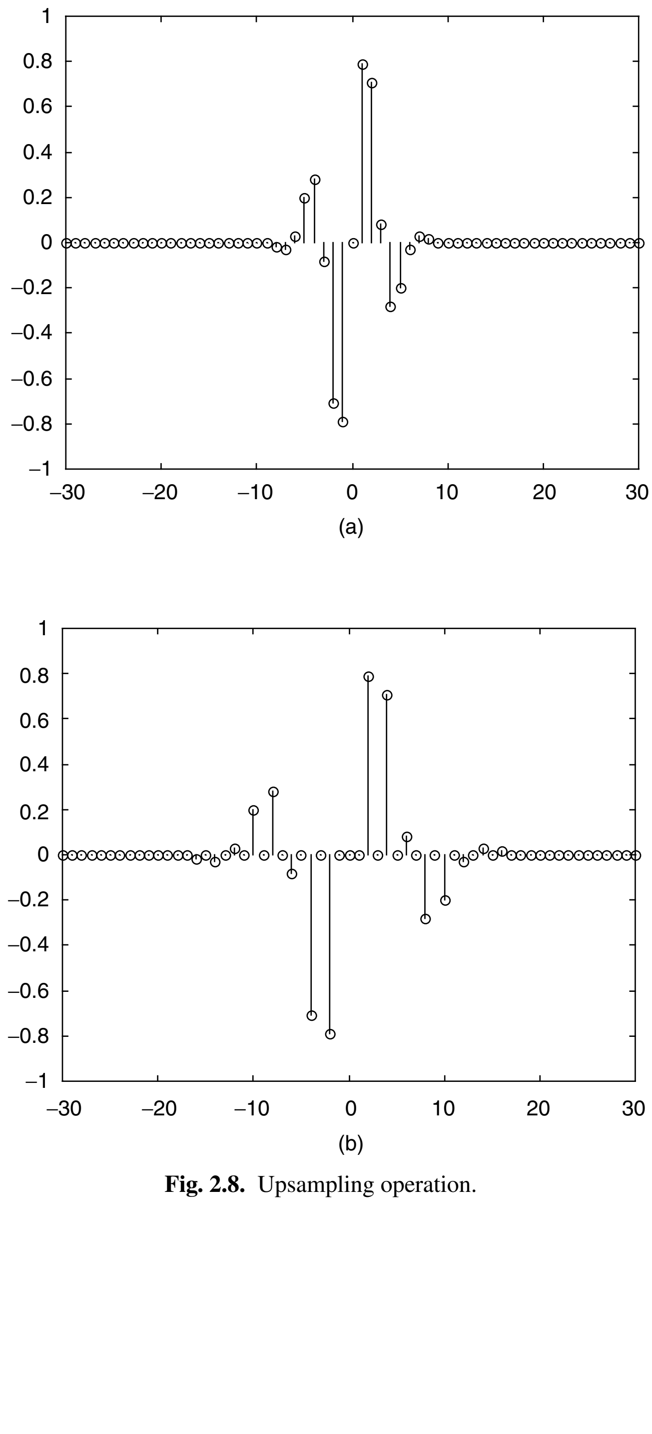


Fig. 2.8. Upsampling operation.

The thresholding operation of (2.9) is an elementary signal analysis system. Input signals $x(n)$ are labeled by the output signal $y(n)$ which takes on two values, 0 or 1. Thus, 1 indicates the presence of meaningful information, and 0 indicates the presence of nonmeaningful information in $x(n)$. When we say that the thresholding operation *segments* the signal into useful and nonuseful portions, we mean that the system partitions the input signal's domain. Typically, we use thresholding as a crude method for removing background noise from signals. A problem with brute-force thresholding is that signals may contain noise impulses that are as large in magnitude as some meaningful signal values. Furthermore, the noise magnitude may be as large as the interesting signal values and thresholding therefore fails completely. Chapter 4 elaborates on thresholding subtleties.

Threshold selection dramatically affects the segmentation. Thresholding usually follows the filtering or transforming of an input signal. The trick is choosing the proper threshold so that the output binary image correctly marks signal regions. To gain some appreciation of this, let's consider a thresholding problem on a two-dimensional discrete signal—that is an 8-bit, 256 gray scales image. Figure 2.9 shows the original image, a parcel delivery service's shipping label. A very thoroughly studied image analysis problem is to interpret the text on the label so that automated systems can handle and route the parcel to its destination. Optical character recognition (OCR) systems read the individual characters. Since almost all current OCR engines accept only binary image data, an essential first step is to convert this gray scale image to binary form [8, 9]. Although the image itself is fairly legible, only the most careful selection of a single threshold for the picture suffices to correctly binarize it.

The *accumulator* system, $y = Hx$, is given by

$$y(n) = \sum_{k=-\infty}^{n} x(k). \tag{2.10}$$

The accumulator outputs a value that is the sum of all input values to the present signal instant. Any signal with finite support is in the domain of the accumulator system. But, as already noted, not all signals are in the accumulator's domain. If a signal is absolutely summable (Section 1.6.2), then it is in the domain of the accumulator. Some finite-energy signals are not in the domain of the accumulator. An example that cannot be fed into an accumulator is the finite-energy signal

$$x(n) = \begin{cases} 0 & \text{if } n \geq 0, \\ -\dfrac{1}{n} & \text{if } n < 0. \end{cases} \tag{2.11}$$

A system may extract the nearest integer to a signal value. This called a *digitizer* because in principle a finite-length storage register can hold the integral values produced from a bounded input signal. There are two variants. One kind of digitizer produces a signal that contains the integral *ceiling* of the input,

$$y(n) = \lceil x(n) \rceil = \text{least integer} \geq x(n). \tag{2.12}$$

The other variant produces the integral *floor* of the input,

$$y(n) = \lfloor x(n) \rfloor = \text{greatest integer} \le x(n) . \tag{2.13}$$

The moving average system, $y = Hx$, is given by

Fig. 2.9. Thresholding. An 8-bit, 256 gray scales image of a shipping label (a). First threshold applied (b). Here the threshold results in an image that is too dark. The company logo is only marginally readable. With a different threshold (c), the logo improves, but the shipping code beneath begins to erode. It is difficult to pick an appropriate global threshold for the image.

$$y(n) = \frac{1}{2N+1}\sum_{k=-N}^{N} x(k). \tag{2.14}$$

This system outputs a signal that averages the $2N + 1$ values around each $x(n)$. This smoothes the input signal, and it is commonly used to improve the signal-to-noise ratio of raw signals in data acquisition systems.

2.1.2 Operations on Systems

We can build more complicated systems by combining the systems, taking the output of one system and using it as an input to another.

Definition (System Composition). Let H and G be systems. Then the *composite system* GH is the system defined by $y(n) = G(H(x(n)))$. GH is called the *composition* or the *cascade* of G with H.

Remark. In general, the order of system composition is important. Many signal operations—however intuitive and simple they may seem—may not commute with one another. In particular, note again that the shift operation does not commute with reflection. If G is a shift operation, $G(x(n)) = x(n - k)$, k is nonzero, and H is a reflection, then $GH \neq HG$.

2.1.3 Types of Systems

Now let us categorize the various systems we have considered. These characterizations will prove useful for understanding the behavior of a system in an application. Some systems, because of the form of their definition, may seem impossible to implement on a computer. For example, the accumulator (2.9) is defined with an infinite sum that uses every earlier input signal value. We have observed already that it is an example of a system in which not every input produces a meaningful output. Do we need an infinite set of memory locations to store input signal values? No, we can still implement an accumulator on a computer by exploiting the recurrence relation $y(n) = y(n-1) + x(n)$. It is only necessary to know an initial value of $y(n)$ at some past time instant $n = n_0$. The accumulator belongs to a class of systems—called *recursive* systems—that can be implemented by reusing output values from previous time instants.

Now we develop some basic categories for systems: static and dynamic systems, recursive systems, and causal systems.

We can start by distinguishing systems according to whether they require computer memory for signal values at other than the current instant for their implementation. Let H be a system $y = Hx$. H is *static* if $y(n)$ can always be found from the current value of input signal $x(n)$ and n itself. That is, H is *static* when $y(n) = F(x(n))$ for some defining function or rule F for H. H is *dynamic* when it must use values $x(m)$ or $y(m)$ for $m \neq n$ in order to specify $y(n)$.

A system that depends on future values of the input signal $x(n)$ is dynamic too. This seems impossible, if one thinks only of signals that have a time-related independent variable. But signals can independent variables of distance, for example. So the values of $x(n + k)$, $k > 0$, that need to be known to compute $y(n)$ are just values of the input signal in a different direction. Systems for two-dimensional signals (i.e., images) are in fact a very widely studied case of signals whose values may depend on "future" values of the input.

Example. The accumulator is a dynamic system because, for a general input $x(n)$, it cannot be implemented without knowing either

(i) $x(n)$ and all previous values of the input signal; or
(ii) for some $k > 1$, $y(n - k)$ and all $x(n - k + p)$ for $p = 0, ..., k$.

Dynamic systems require memory units to store previous values of the input signal. So static systems are also commonly called *memoryless* systems.

A concept related to the static versus dynamic distinction is that of a *recursive* system. Pondering the accumulator system once more, we note that this dynamic system cannot be implemented with a finite set of memory elements that only contain previous values of $x(n)$. Let H be a system, $y = Hx$. H is *nonrecursive* if there is an $M > 0$, such that $y(n)$ can always be found as a function of $x(n), x(n - 1), ..., x(n - M)$. If $y(n)$ depends on $y(n - 1), y(n - 2), ..., y(n - N)$, for some $N > 0$, and perhaps upon $x(n - 1), ..., x(n - M)$, for some M, then H is *recursive*.

A system $y = Hx$ is *causal* if $y(n)$ can always be computed from present and past inputs $x(n), x(n - 1), x(n - 2), ...$. Real, engineered systems for time-based signals must always be causal, and, if for no other reason, causality is important. Nevertheless, where the signals are not functions of discrete time variables, noncausal signals are acceptable. A nonrecursive system is causal, but the converse is not true.

Examples (System Causality)

(i) $y(n) = x(n) + x(n - 1) + x(n - 2)$ is causal and nonrecursive.
(ii) $y(n) = x(n) + x(n + 1)$ is not causal.
(iii) $y(n) = x(2 - n)$ is noncausal.
(iv) $y(n) = x(|n|)$ is noncausal.
(v) The accumulator (2.10) is causal.
(vi) The moving average system (2.14) is not causal.

2.2 LINEAR SYSTEMS

Linearity prevails in many signal processing systems. It is desirable in entertainment audio systems, for example. Underlying the concept of linearity are two ideas, and they are both straightforward:

(i) When the input signal to the system is larger (or smaller) in amplitude, then the output signal from the system produces is proportionally larger (or smaller). This is the *scaling* property of linearity. In other words, if a signal is amplified or attenuated and then input to a linear system, then the output is a signal that is amplified or attenuated by the same amount.

(ii) Furthermore, if two signals are added together before input, then the result is just the sum of the outputs that would occur if each input component were passed through the system independently. This is the *superposition* property.

Obviously, technology cannot produce a truly linear system; there is a range within which the linearity of a system can be assumed. Real systems add noise to any signal. When the input signal becomes too small, the output may disappear into the noise. The input could become so large intensity that the output is distorted. Worse, the system may fail if subjected to huge input signals. Practical, nearly linear systems are possible, however, and engineers have discovered some clever techniques to make signal amplification as linear as possible.

When amplification factors must be large, the nonlinearity of the circuit components—vacuum tubes, transistors, resistors, capacitors, inductors, and so on—becomes more and more of a factor affecting the output signal. The discrete components lose their linearity at higher power levels. A change in the output proportional to the change in the input becomes very difficult to maintain for large amplification ratios. Strong amplification is essential, however, if signals must travel long distances from transmitter to repeater to receiver. One way to lessen the distortion by amplification components is to feed back a tiny fraction of the output signal, invert it, and add it to the input signal. Subject to a sharp attenuation, the feedback signal remains much closer to true linearity. When the output varies from the amplification setting, the input biases in the opposite direction. This restores the amplified signal to the required magnitude. An illustration helps to clarify the concept (Figure 2.10).

From Figure 2.10 we can see that the output signal is $y(n) = A(x(n) - By(n-1))$. Assuming that the output is relatively stable, so that $y(n) \approx y(n-1)$, we can express

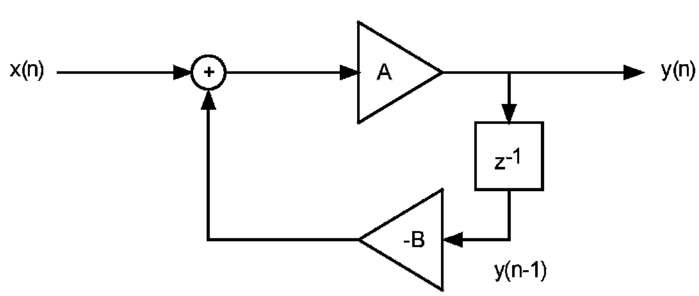


Fig. 2.10. Feedback amplifier.

the system gain as follows:

$$y(n) = \frac{A}{AB+1}x(n) = \frac{1}{B+A^{-1}}x(n)\,. \tag{2.15}$$

Evidently, if the amplification factor A is large, then A^{-1} is small, and the system gain depends mostly on the inverse of the attenuation factor in the feedback. It is approximately B^{-1}. Due to its low power levels, the feedback circuit is inherently more linear, and it improves the overall linearity of the system immensely. Achieving good linearity was an early stumbling block in the development of the telephone system.[1]

Since many important results and characterizations about systems follow from the assumption of linearity, it is a central object of our study in signal processing and analysis.

2.2.1 Properties

Let us formalize these ideas and explore the properties of linear systems.

Definition (Linear System). Let $y = Hx$, and let A be a scalar (real or complex number). The system H is *linear* if it obeys the scaling and superposition properties:

(i) *Scaling*:

$$H(Ax) = AH(x)\,. \tag{2.16}$$

(ii) *Superposition*:

$$H(x+y) = Hx + Hy\,. \tag{2.17}$$

There is a useful criterion for system linearity.

Proposition (Linearity Criterion). If system H is linear, $x(n) = 0$ for all n, and $y = Hx$, then $y(n) = 0$ for all n also.

Proof: Although it seems tricky to many who are not familiar with this type of argument, the proof is quite simple. If $x(n) = 0$ for all n, then $x(n) = x(n) + x(n)$. Hence, $y(n) = H(x(n) + x(n)) = H(x(n))$. But $H(x + x) = Hx + Hx$ by superposition. So $Hx + Hx = Hx$; the consequence, subtracting Hx from both sides of this equation, is that $0 = Hx = y$. That is, $y(n)$ is the zero signal, and the criterion is proven. Note,

[1]The big technical breakthrough came when a Bell Laboratories engineer, Harold Black, had a flash of insight on the ferry from Hoboken, New Jersey, to work in New York City on a summer morning in 1927 [J. R. Pierce and A. M. Noll, *Signals: The Science of Telecommunications*, New York: Scientific American Library, 1990].

by the way, that this last equality is an equality of signals and that the "0" in the equation is really the signal $y(n)$ that is zero for all time instants n. ■

Examples. To show a system is linear, we must check that it satisfies both the scaling and superposition properties. The linearity criterion is helpful for exposing nonlinear systems.

(i) The system $y(n) = 5 + x(n)$ is nonlinear, an easy application of the criterion.
(ii) The system $y(n) = x(n)x(n)$ is nonlinear. The criterion does not help with this example. However, the system violates both the scaling and superposition properties.
(iii) The reflection system $y(n) = x(-n)$ is linear.
(iv) The system $y(n) = \cos(n)x(n)$ is linear. Note this example. The idea of linearity is that systems are linear in their input signals. There may be nonlinear functions, such as $\cos(t)$, involved in the definition of the system relation. Nevertheless, as long as the overall relation between input and output signals obeys both scaling and superposition (and this one does indeed), then the system is linear.

2.2.2 Decomposition

A very useful mathematical technique is to resolve complicated entities into simpler components. For example, the Taylor series from calculus resolves differentiable functions into sums of polynomials. Linear systems can defy initial analysis because of their apparent complexity, and our goal is to break them down into elementary systems more amenable to study. There are two steps:

(i) We first break down signals into sums of scaled, shifted unit impulse signals.
(ii) This decomposition can it turn be used to resolve the outputs of linear systems into sums of simple component outputs.

Proposition (Decomposition). Let $x(n)$ be a discrete signal and define the constants $c_k = x(k)$. Then $x(n)$ can be decomposed into a sum of scaled, shifted impulse signals as follows:

$$x(n) = \sum_{k=-\infty}^{\infty} c_k \delta(n-k) . \tag{2.18}$$

Proof: Let $w(n)$ be the right-hand side of (2.18). Then $w(k) = c_k$, since all of the terms $\delta(n-k)$ are zero, unless $n = k$. But this is just $x(k)$ by the definition of the constants c_k. So $w(k) = x(k)$ for all k, completing the proof. ■

The next definition prepares us for a theorem that characterizes discrete linear systems. From the examples above, linear systems can come in quite a few varieties. Sometimes the system's linear nature is nonintuitive. However, any general linear system is completely known by its behavior on one type of input signal: a shifted impulse. (Our proof is informal, in that we assume that scaling and superposition apply to infinite sums and that a convergent series can be rearranged without affecting its limit.)

Definition (Output of Shifted Input). Let H be a linear system and $y = Hx$. Define $y(n, k) = H(x(n-k))$ and $h(n, k) = H(\delta(n-k))$, where δ is the discrete unit impulse signal.

Theorem (Linearity Characterization). Let H be a linear system, $x(n)$ a signal, $c_k = x(k)$, and $y = Hx$. Then

$$y(n) = \sum_{k=-\infty}^{\infty} c_k h(n, k). \tag{2.19}$$

Proof: By the decomposition of discrete signals into impulses, we know that $x(n) = \sum_{k=-\infty}^{\infty} c_k \delta(n-k)$. So with $y(n) = H(x(n))$, it follows from superposition that

$$\begin{aligned} y(n) &= H\left(\sum_{k=-\infty}^{\infty} c_k \delta(n-k)\right) \\ &= H\left(\sum_{k=1}^{\infty} c_k \delta(n-k)\right) + H(c_0\delta(n)) + H\left(\sum_{k=-\infty}^{-1} c_k \delta(n-k)\right). \end{aligned} \tag{2.20}$$

And then by the scaling property applied to the middle term of (2.20),

$$\begin{aligned} y(n) &= H\left(\sum_{k=-\infty}^{\infty} c_k \delta(n-k)\right) \\ &= H\left(\sum_{k=1}^{\infty} c_k \delta(n-k)\right) + c_0 h(n, 0) + H\left(\sum_{k=-\infty}^{-1} c_k \delta(n-k)\right). \end{aligned} \tag{2.21}$$

Repeatedly using the linearity properties to break out middle terms in (2.21) gives the desired result. ■

2.3 TRANSLATION INVARIANT SYSTEMS

In many real-life systems, the outputs do not depend on the absolute time at which the inputs were done. For example, a compact disc player is a reasonably good time-invariant system. The music played from a compact disc on Monday will not differ substantially when the same disc played on Tuesday. Loosely put, time-invariant systems produce identically shifted outputs from shifted inputs. Since not all systems, of course, are time-based, we prefer to call such systems *translation-invariant*.

To formalize this idea, we need to carefully distinguish between the following:

- The signal $y(n, k)$ is the output of a system, given a delay (or advance, depending on the sign of k) of the input signal $x(n)$ by k time units.
- On the other hand, $y(n - k)$ is the signal obtained by finding $y = Hx$ and then shifting the resulting output signal, $y(n)$, by k.

These two results may not be the same. An example is the system $y(n) = x(n) + n$. It is easy to find input signals $x(n)$ which for this system give $y(n-k) \neq y(n,k)$. Hence, the important definition:

Definition (Translation-Invariant Systems). Let H be the system $y = Hx$. Then, if for all signals $x(n)$ in *Domain*(H), if $y(n, k) = H(x(n - k)) = y(n - k)$, then H is *translation-invariant*. Another term is *shift-invariant*. For time-based signals, it is common to say *time-invariant*.

To show a system is translation-invariant, we must compare the shifted output, $y(n - k)$, with the output from the shifted input, $H(x(n - k))$. If these are equal for all signals in the system's domain, then H is indeed translation-invariant. It is very easy to confuse these two situations, especially for readers new to signal processing. But there is a simple, effective technique for showing translation-invariance: we rename the input signal after it is shifted, hiding the shift amount.

Examples (Translation-Invariance). Let us try this technique on the systems we checked for linearity earlier.

(i) The system $y(n) = 5 + x(n)$ is translation-invariant. If $y = Hx$, then the output shifted by k is $y(n - k) = 5 + x(n - k)$. We just substitute "$n - k$" for "n" at each instance in the defining rule for the system's input–output relation. What if we translate the input by k units also? This produces $x(n - k)$. We rename it $w(n) = x(n - k)$. This hides the shift amount within the new name for the input signal. It is easy to see what H does to any signal, be it named x, w, u, v, or whatever. H adds the constant 5 to each signal value. Hence we see $y(n, k) = H(w(n)) = 5 + w(n)$. Now we put the expression for x in terms of w back into the expression: $y(n, k) = 5 + x(n - k)$. So $y(n - k) = y(n, k)$, and the system is translation-invariant.

(ii) The system $y(n) = x(n)x(n)$ is translation-invariant. The output shifted by k is $y(n - k) = x(n - k)x(n - k)$. We rename once more the shifted input $w(n) = x(n - k)$. Then $y(n, k) = H(w(n)) = w(n)w(n) = x(n - k)x(n - k)$. Again, the system is translation-invariant.

(iii) The reflection system $y(n) = x(-n)$ is not translation-invariant. The shifted output in this case is $y(n - k) = x(-(n - k)) = x(k - n)$. We make a new name for the shifted input, $w(n) = x(n - k)$. Then $y(n, k) = H(w(n)) = w(-n)$. But $w(-n) = x(-n - k)$. Thus, $y(n - k) \neq y(n, k)$ in general. In particular, we can take $x(n) = \delta(n)$, the unit impulse signal, and $k = 1$. Then $y(n - k) = \delta(1 - n)$, although $y(n, k) = \delta(-1 - n)$.

(iv) The system $y(n) = \cos(n)x(n)$ is not translation-invariant. The shifted output in this case is $y(n - k) = \cos(n - k)x(n - k)$. Once more, we rename the shifted input $w(n) = x(n - k)$. Then $y(n, k) = H(w(n)) = \cos(n)w(n) = \cos(n)x(n - k)$. Again, $y(n - k) \neq y(n, k)$ in general.

2.4 CONVOLUTIONAL SYSTEMS

The most important systems obey both the linearity and translation-invariance properties. Many physical systems are practically linear and practically translation-invariant. If these special properties can be assumed for a physical system, then the analysis of that system simplifies tremendously. The key relationship, it turns out, is that linear translation-invariant systems are fully characterized by one signal associated with the system, the impulse response. By way of contrast, think again of the characterization of linear systems that was given in Section 2.2. For a linear system, there is a characterization of the system's outputs as sums of scaled signals $h(n, k)$. However, there are in general an infinite number of $h(n, k)$ components. This infinite set reduces to just one signal if the system is translation-invariant too.

2.4.1 Linear, Translation-Invariant Systems

This is the most important type of system in basic signal processing theory.

Definition (LTI System). A system H that is both linear and translation-invariant is called an *LTI system*. When signals are functions of an independent time variable, we may say *linear, time-invariant*, but the abbreviation is the same. Some authors use the term *shift-invariant* and refer to *LSI* systems.

Definition (Impulse Response). Let H be an LTI system and $y = Hx$. Then we define the impulse response of the system as $h(n) = H(\delta(n))$, where δ is the discrete unit impulse signal.

Theorem (Convolution). Let H be an LTI system, $y = Hx$, and $h = H\delta$. Then

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)\,. \tag{2.22}$$

Proof: With H linear, there follows the decomposition

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n,k) = \sum_{k=-\infty}^{\infty} x(k)H(\delta(n-k))\,. \tag{2.23}$$

But since H is translation-invariant, $h(n-k) = H(\delta(n-k))$. Inserting this into (2.23) proves the theorem. ■

Remarks. Note that (2.22) is the convolution of the input signal $x(n)$ and the impulse response $\delta(n)$ of the LTI system H: $y(n) = (x*h)(n)$. For this reason, we call LTI systems *convolutional*. Although it is a simple result, the importance of the theorem cannot be overemphasized. In order to understand a system, we must know how it operates on its input signals. This could be extremely complicated. But for LTI systems, all we need to do is find one signal—the system's impulse response. Then, for any input, the output can be computed by convolving the input signal and the impulse response. There is some more theory to cover, but let us wait and give an application of the theorem.

Example (System Determination). Consider the LTI system $y = Hx$ for which only two test cases of input–output signal pairs are known: $y_1 = Hx_1$ and $y_2 = Hx_2$, where $y_1 = [-2, 2, \underline{0}, 2, -2]$, $x_1 = [2, \underline{2}, 2]$, $y_2 = [1, -2, \underline{2}, -2, 1]$, $x_2 = [-1, \underline{0}, -1]$. (Recall from Chapter 1 the square brackets notation: $x = [n_{-M}, ..., n_{-1}, \underline{n_0}, n_1, ..., n_N]$ is the finitely supported signal on the interval $[M, N]$ with $x(0) = n_0$, $x(-1) = n_{-1}$, $x(1) = n_1$, and so on. The underscored value indicates the zero time instant value.) The problem is to find the output $y = Hx$ when x is the ramp signal $x = [1, \underline{2}, 3]$. An inspection of the signal pairs reveals that $2\delta(n) = x_1 + 2x_2$. Thus, $2h = Hx_1 + 2Hx_2 = y_1 + 2y_2$ by the linearity of H. The impulse response of H must be the signal $(y_1 + 2y_2)/2 = h = [-1, \underline{2}, -1]$. Finding the impulse response is the key. Now the convolution theorem implies that the response of the system to $x(n)$ is the convolution $x*h$. So $y = [-1, 0, \underline{0}, 4, -3]$.

Incidentally, this example previews some of the methods we develop extensively in Chapter 4. Note that the output of the system takes zero values in the middle of the ramp input and large values at or near the edges of the ramp. In fact, the impulse response is known as a *discrete Laplacian operator*. It is an elementary example of a signal *edge detector*, a type of signal analysis system. They produce low magnitude outputs where the input signal is constant or uniformly changing, and they produce large magnitude outputs where the signal changes abruptly. We could proceed further to find a threshold value T for a signal analysis system that would mark the signal edges with nonzero values. In fact, even this easy example gives us a taste of the problems with picking threshold values. If we make T large, we will only detect the large step edge of the ramp input signal above. If we make T small enough to detect the beginning edge of the ramp, then there are two instants at which the final edge is detected.

2.4.2 Systems Defined by Difference Equations

A discrete difference equation can specify a system. Difference equations are the discrete equivalent of differential equations in continuous-time mathematical analysis, science, and engineering. Consider the LTI system $y = Hx$ where the input and output signals always satisfy a linear, constant-coefficient difference equation:

$$y(n) = \sum_{k=1}^{K} a_k y(n-k) + \sum_{m=0}^{M} b_m x(n-m). \tag{2.24}$$

The output $y(n)$ can be determined from the current input value, recent input values, and recent output values.

Example. Suppose the inputs and outputs of the LTI system H are related by

$$y(n) = ay(n-1) + x(n), \tag{2.25}$$

where $a \neq 0$. In order to characterize inputs and outputs of this system, it suffices to find the impulse response. There are, however, many signals that may be the impulse response of a system satisfying (2.25). If $h = H\delta$, is an input–output pair that satisfies (2.25), with $x = \delta$ and $y = h$, then a single known value of $h(n_0)$ determines all of $h(n)$. Let us say that $h(0) = c$. Then repeatedly applying (2.25) we can find $h(1) = ac + \delta(1) = ac$; $h(2) = a^2c$; and, for non-negative k, $h(k) = a^kc$. Furthermore, by writing the equation for $y(n-1)$ in terms of $y(n)$ and $x(n)$, we have

$$y(n-1) = \frac{y(n) - x(n)}{a}. \tag{2.26}$$

Working with (2.26) from the initial known value $h(0) = c$ gives $h(k) = a^k(c-1)$, for $k < 0$. So

$$h(n) = \begin{cases} a^n c & \text{if } n \geq 0, \\ a^n(c-1) & \text{if } n < 0. \end{cases} \tag{2.27}$$

From the convolution relation, we now know exactly what LTI systems satisfy the difference equation (2.25). It can be shown that if the signal pair (δ, h) satisfies the difference equation (2.24), then the pair $(x, h*x)$ also satisfies (2.24).

Example. Not all solutions (x, y) of (2.24) are an input–output pair for an LTI system. To see this, let $a \neq 0$ and consider the *homogeneous* linear difference equation

$$y(n) - ay(n-1) = 0. \tag{2.28}$$

Clearly, any signal of the form $y(n) = da^n$, where d is a constant, satisfies the homogeneous equation. If (x, y) is a solution pair for (2.25), and y_h is a solution of the homogeneous equation, then $(x, y + y_h)$ is yet another solution of (2.25). We know by the linearity criterion (Section 2.2) that for an LTI system, the only possible input–output pair (x, y) when $x(n) = 0$ for all n is $y(n) = 0$ for all n. In particular, $(0, da^n)$ is a solution of (2.25), but not an input–output pair on any LTI system.

2.4.3 Convolution Properties

Although it seems at first glance to be a very odd operation on signals, convolution is closely related to two quite natural conditions on systems: linearity and translation-invariance. Convolution in fact enjoys a number of algebraic properties: associativity, commutativity, and distributivity.

Proposition (Convolution Properties). Let x, y, and z be discrete signals. Then

(i) (*Associativity*) $x*(y*z) = x*(y*z)$.
(ii) (*Commutativity*) $x*y = y*x$.
(iii) (*Distributivity*) $x*(y+z) = x*y + x*z$.

Proof: We begin with associativity. Let $w = y*z$. Then, by the definition of convolution, we have $[x*(y*z)](n) = (x*w)(n)$. But, x convolved with w is

$$\begin{aligned}\sum_{k=-\infty}^{\infty} x(k)w(n-k) &= \sum_{k=-\infty}^{\infty} x(k) \sum_{l=-\infty}^{\infty} y(l)z((n-k)-l) \\ &= \sum_{l=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} x(k)y(l)z((n-k)-l). \end{aligned} \tag{2.29}$$

Let $p = k + l$ so that $l = p - k$, and note that $p \to \pm\infty$ as $l \to \pm\infty$. Then make the change of summation in (2.29):

$$\begin{aligned}(x*w)(n) = (x*(y*z))(n) &= \sum_{p=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} x(k)y(p-k)z(n-p) \\ &= \sum_{p=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty} x(k)y(p-k)\right)z(n-p) \\ &= \sum_{p=-\infty}^{\infty}(x*y)(p)z(n-p) = ((x*y)*z)(n). \end{aligned} \tag{2.30}$$

Commutativity is likewise a matter of juggling summations:

$$(x * y)(n) = \sum_{k=-\infty}^{\infty} x(k)y(n-k) = \sum_{k=-\infty}^{\infty} y(n-k)x(k)$$
$$= \sum_{k=\infty}^{-\infty} y(n-k)x(k) = \sum_{p=-\infty}^{\infty} y(p)x(n-p) = (y * x)(n). \tag{2.31}$$

Finally, distributivity is the easiest property, since

$$(x * (y+z))(n) = \sum_{k=-\infty}^{\infty} x(k)(y+z)(n-k) = \sum_{k=-\infty}^{\infty} [x(k)y(n-k) + x(k)z(n-k)]$$
$$= \sum_{k=-\infty}^{\infty} x(k)y(n-k) + \sum_{k=-\infty}^{\infty} x(k)z(n-k)$$
$$= (x * y)(n) + (x * z)(n). \tag{2.32}$$

This completes the theorem. ■

The convolution theorem has a converse, which means that convolution with the impulse response characterizes LTI systems.

Theorem (Convolution Converse). Let $h(n)$ be a discrete signal and H be the system defined by $y = Hx = x*h$. Then H is LTI and $h = H\delta$.

Proof: Superposition follows from the distributive property. The scaling property of linearity is straightforward (exercise). To see that H is translation-invariant, note that the shifted output $y(n-l)$ is given by

$$y(n-l) = \sum_{k=-\infty}^{\infty} x(k)h((n-l)-k). \tag{2.33}$$

We compare this to the response of the system to the shifted input, $w(n) = x(n-l)$:

$$H(x(n-l)) = (Hw)(n) = \sum_{k=-\infty}^{\infty} w(k)h(n-k) = \sum_{k=-\infty}^{\infty} x(k-l)h(n-k)$$
$$= \sum_{p=-\infty}^{\infty} x(p)h(n-(p+l)) = \sum_{p=-\infty}^{\infty} x(p)h((n-l)-p) = y(n-l). \tag{2.34}$$

So H is indeed translation-invariant. It is left to the reader to verify that $h = H\delta$. ■

Definition (FIR and IIR Systems). An LTI system H having an impulse response $h = H\delta$ with finite support is called a *finite impulse response* (*FIR*) system. If the LTI system H is not FIR, then it is called *infinite impulse response* (*IIR*).

Remarks. FIR systems are defined by a convolution sum that may be computed for any input signal. IIR systems will have some signals that are not in their domain. IIR systems may nevertheless have particularly simple implementations. This is the case when the system can be implemented via a difference equation, where previous known output values are stored in memory for the computation of current response values. Indeed, an entire theory of signal processing with IIR systems and their compact implementation on digital computers has been developed and is covered in signal processing texts [10].

2.4.4 Application: Echo Cancellation in Digital Telephony

In telephone systems, especially those that include digital links (almost all intermediate- and long-distance circuits in a modern system), *echo* is a persistent problem. Without some special equipment—either echo suppressors or echo cancellers—a speaker can hear a replica of his or her own voice. This section looks at an important application of the theory of convolution and LTI systems for constructing effective echo cancellers on digital telephone circuits.

Echo arises at the connection between two-wire telephone circuits, such as found in a typical residential system, and four-wire circuits, that are used in long-haul circuits. The telephone circuits at the subscriber's site rely on two wires to carry the near-end and far-end speakers' voices and an earth ground as the common conductor between the two circuit paths. The earth ground is noisy, however, and for long-distance circuits, quite unacceptable. Good noise immunity requires a four-wire circuit. It contains separate two-wire paths for the far-end and near-end voices. A device called a *hybrid transformer*, or simply a *hybrid*, effects the transition between the two systems [11]. Were it an ideal device, the hybrid would convert all of the energy in a signal from the far-end speaker into energy on the near-end two-wire circuit. Instead, some of the energy leaks through the hybrid (Figure 2.11) into the circuit that carries the near-end voice outbound to the far-end speaker. The result

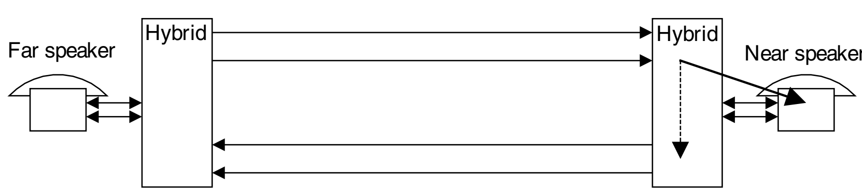


Fig. 2.11. Impedance mismatches in the four-wire to two-wire hybrid transformer allow an echo signal to pass into the speech signal from the near-end speaker. The result is that the far-end speaker hears an echo.

is that far-end speakers hear echoes of their own voices. Since the system design is often symmetrical, the echo problem is symmetrical too, and near-end speakers also suffer annoying echoes. The solution is to employ an *echo suppression* or *echo cancellation* device.

Echo suppression is the older of the two approaches. Long-haul telephone circuits are typically digital and, in the case of the common North American T1 standard, multiplex upwards of 24 digital signals on a single circuit. Since the echo removal is most economically viable at telephone service provider central offices, the echo removal equipment must also be digital in nature. A digital echo suppressor extracts samples from the far-end and near-end digital voice circuits. It compares the magnitudes of the two signals, generally using a threshold on the difference in signal amplitudes. It opens the near-end to far-end voice path when there is sufficient far-end speech detected to cause an echo through the hybrid, but insufficient near-end speech to warrant maintaining the circuit so that the two speakers talk at once. (This situation, called *double-talk* in telephone engineering parlance, occurs some 30% of the time during a typical conversation.) Thus, let $T > 0$ be a threshold parameter, and suppose that M far-end samples and N near-end samples are compared to decide a suppression action. Let $x(n)$ and $s(n)$ be the far- and near-end digital signals, respectively. If $|x(n)| + |x(n-1)| + \cdots + |x(n-M+1)| > T(|s(n)| + |s(n-1)| + \cdots + |s(n-N+1)|)$ at time instant n, then the suppressor mutes the near-end speaker's voice. Now, this may seem crude.

And echo suppression truly is crude. When both people speak, there is echo, but it is less noticeable. The suppressor activates only when the near-end speaker stops talking. Hence, unless the threshold T, the far-end window size M, and the near-end window size N are carefully chosen, the suppressor haphazardly interrupts the near-end signal and makes the resultant voice at the far-end sound choppy. Even granting that these parameters are correct for given circuit conditions, there is no guarantee that system component performances will not drift, or that one speaker will be unusually loud, or that the other will be unusually soft-voiced. Moreover, the suppressor design ought to provide a noise matching capability, whereby it substitutes *comfort noise* during periods of voice interruption; otherwise, from the utter silence, the far-end listener also gets the disturbing impression that the circuit is being repeatedly broken and reestablished. Chapter 4 will study some methods for updating such parameters to maintain good echo suppressor performance. For now, however, we wish to turn to the echo canceller, a more reliable and also more modern alternative.

An echo canceller builds a signal model of the echo which slips through the hybrid. It then subtracts the model signal from the near-end speaker's outbound voice signal (Figure 2.12). How can this work? Note that the hybrid is an approximately linear device. If the far-end speaker's signal $x(n)$ is louder, then the echo gets proportionally louder. Also, since telephone circuit transmission characteristics do not change much over the time of a telephone call, the echo that the far-end signal produces at one moment is approximately the same as it produces at another moment. That is, the hybrid is very nearly translation-invariant. This provides an opportunity to invoke the convolution theorem for LTI systems.

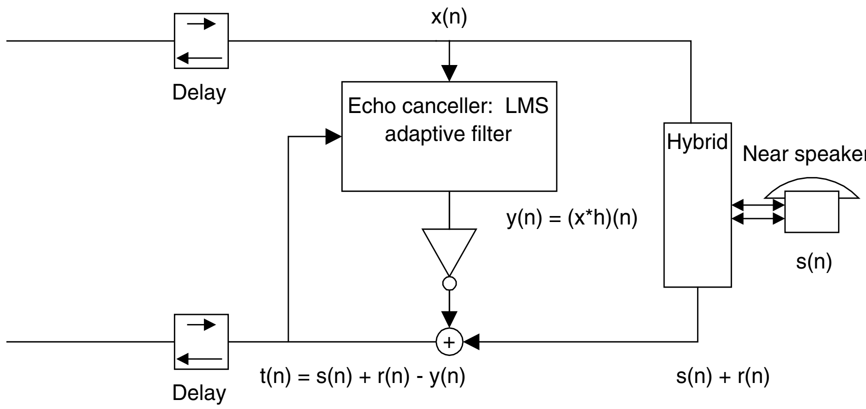


Fig. 2.12. An echo canceller holds a set of coefficients from which it can approximately model the echo signal $r(n)$. Assuming that the hybrid leakage is both linear and translation-invariant, we can apply the convolution theorem for LTI systems. The canceller stores the impulse response of the inbound far-end voice signal $h(n)$ and computes the echo as $y(n) = (x*h)(n)$. Subtracting $y(n)$ from the near-end speech and echo signal, $s(n) + r(n)$, gives the transmitted digital voice signal $t(n) = s(n) + r(n) - y(n)$.

Let H be the hybrid system, so that $r = Hx$ is LTI, where $x(n)$ is the inbound signal from the far-end speaker, and $r(n)$ is the echo through the hybrid. Suppose that the canceller stores the impulse response of the inbound far-end voice signal $h(n)$. Then it can approximate the echo as $y(n) = (x*h)(n)$, and this is the crux of the design. Subtracting $y(n)$ from the near-end speech and echo signal, $s(n) + r(n)$, gives the transmitted digital voice signal $t(n) = s(n) + r(n) - y(n)$ with echo largely removed. Digital signal processors can perform these convolution and subtraction steps in real time on digital telephony circuits [12]. Alternatively, echo cancellation can be implemented in application-specific integrated circuits [13].

An intriguing problem is how to establish the echo impulse response coefficients $h(k)$. The echo may change based on the connection and disconnection of equipment on the near-end circuit, including the two-wire drop to the subscriber. Thus, it is useful to allow $h(k)$ to change slowly over time. We can allow $h(k)$ to adapt so as to minimize the residual error signal, $e(n) = r(n) - y(n)$, that occurs when the near-end speaker is silent, $s(n) = 0$. Suppose that discrete values up to time instant n have been received and the coefficients must be adjusted so that the energy of the error $e(n)$ is a minimum. The energy of the error signal's last sample is $e^2(n)$. Viewing $e^2(n)$ as a function of the coefficients $h(k)$, the maximum decrease in the error is in the direction of the negative gradient, given by the vector with components

$$-\nabla e^2(n)_k = -\frac{\partial}{\partial h(k)}(e^2(n)) = 2r(n)x(n-k). \tag{2.35}$$

To smoothly converge on a good set of $h(k)$ values, it is best to use some proportion parameter η; thus, we adjust $h(k)$ by adding $2\eta r(n)x(n-k)$. This is an adaptive

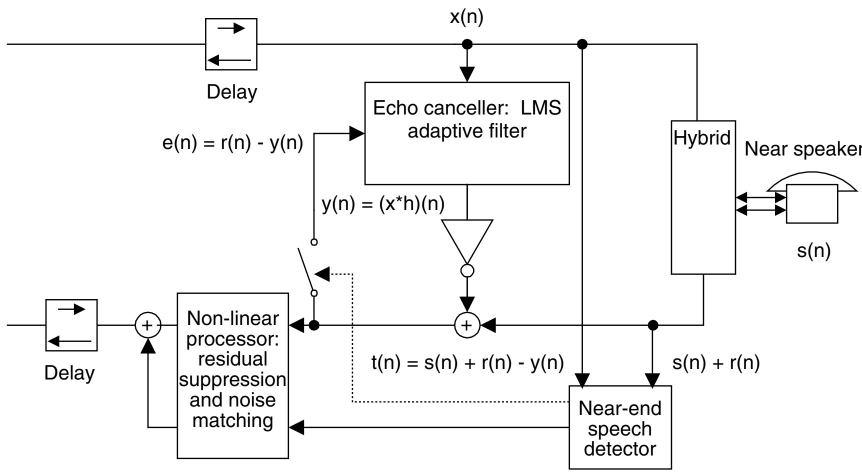


Fig. 2.13. An advanced echo canceller implements some of the design ideas from echo suppressors. The nonlinear processor mutes residual echo (the signal $t(n)$ after cancellation) when it exceeds a threshold. Noise matching supplies comfort noise to the outbound circuit so that the far-end listener does not suspect a dead line.

filtering algorithm known in the research literature as the *least mean squares* (*LMS*) *algorithm* [14].

An improvement in echo cancellation results from implementing some of the ideas from echo suppressor designs. A component commonly called the nonlinear processor (NLP) performs muting and noise insertion in the advanced canceller. Referring to Figure 2.13, note that the NLP mutes residual echo $t(n)$ when it falls below a small threshold and there is no near-end speech. NLP noise matching supplies comfort noise to the outbound circuit so the far-end listener does not suspect a dead line. Finally, the adaptation of the coefficients $h(n)$ should cease during periods of double-talk; otherwise, just as does genuine echo, the large magnitude near-end speech will be seen as echo by the canceller. The upshot is that the $h(n)$ coefficients will diverge from their proper settings, and a burst of echo will occur when the near speaker stops talking.

2.5 THE l^p SIGNAL SPACES

The remainder of the chapter develops the mathematical foundation for discrete signal processing theory. Vector calculus relies on such a foundation. But it is so intuitive, however, that many students do not even realize that the finite-dimensional vector spaces, $\mathbb{R}^n$ and $\mathbb{C}^n$, underlie multivariable calculus. The theories of differentiation, integration, limits, and series easily generalize to work with vectors instead of just scalars. Our problems are much more complicated now, since the objects of our theory—signals—are like vectors that are infinitely long and extend in two directions. We cannot, for example, carelessly generalize the dot product from finite dimensional vector spaces to signals; the resulting sum may not converge.

2.5.1 l^p Signals

What signal operations must our formal framework support? We have discovered that convolution operations play a pivotal role in the study of LTI systems. Since the summation in this important operation is infinite, we may ask under what circumstances one can compute the limit which it represents. It is clear that if the system is FIR, then the convolution sum is computable for any input signal. From a practical perspective, every signal encountered in engineering is finitely supported and cannot continue with nonzero values forever. However, it is reassuring to know that this is not a fundamental limitation of the theory we develop for signals. By looking a little deeper into the mathematics of signals and operations on them, we can in fact find classes of signals, not finitely supported, that allow us to compute convolutions for IIR systems.

Definition (l^p Signal Spaces). Let $p \geq 1$ be a real number. Then l^p is the set of all real-valued (or complex-valued) signals $x(n)$ such that

$$\sum_{n=-\infty}^{\infty} |x(n)|^p < \infty . \tag{2.36}$$

We sometimes call the l^p signals *p-summable*. If $x(n)$ is in l^p, then its l^p*-norm* is

$$\|x(n)\|_p = \left(\sum_{n=-\infty}^{\infty} |x(n)|^p \right)^{\frac{1}{p}} . \tag{2.37}$$

There is an l^p *distance measure* as well: $d_p(x, y) = \|x-y\|_p$. There is a special case of l^p signals, the bounded signals. A discrete signal $x(n)$ is *bounded* if there is a positive real number M_x such that for all n,

$$|x(n)| \leq M_x . \tag{2.38}$$

It is customary to denote the class of bounded signals as l^∞. (This notation allows an elegant formulation of an upcoming theorem.) Finally, we define the l^∞ *norm* of a signal to be its least upper bound:

$$\|x(n)\|_\infty = \min \{M: x(n) \leq M \text{ for all } n\}. \tag{2.39}$$

The l^p signal classes share a number of important properties that we need in order to do signal processing within them. This and the next section comprise a tutorial on the mathematical discipline known as *functional analysis* [15], which studies the properties of mathematical functions from a geometric standpoint. It is a generalization of finite-dimensional real and complex vector space theory. In fact, we already know some examples of l^p spaces.

Example (Finite-Energy Signals). The l^2 signals are precisely the finite energy, or square-summable discrete signals. The l^2 norm is also the square root of the energy of the signal. We will find that l^2 is a standout l^p space, since it supports an inner product relation that generalizes the dot product on signals. There are signal spaces for finite energy analog signals too (Chapter 3). Furthermore, we will find in Chapter 7 that it is possible to build a complete theory for the frequency analysis of l^2 signals [16]. Chapter 11 explains the recent theory of multiresolution analysis for finite energy signals—a recent advance in signal analysis and one kind of signal decomposition based on wavelets [17].

Example (Absolutely Summable Signals). The l^1 signal space is the set of absolutely summable signals. It too enjoys a complete signal frequency theory. Interestingly, for any p, $1 \le p \le 2$, there is an associated frequency analysis theory for the space of l^p signals [16]. Since signal processing and analysis uses mainly l^1 and l^2, however, we will not elaborate the Fourier transform theory for l^p, $1 < p < 2$.

2.5.2 Stable Systems

The notion of the l^p signal spaces applies readily to study of stable systems.

Definition (Stability). The system H is *stable* if $y = Hx$ is bounded whenever x is bounded. Another term for stable is *bounded input–bounded output* (*BIBO*).

Thus, the response of a stable system to an l^∞ input signal is still an l^∞ signal. The next theorem is useful for discovering whether or not an LTI system is stable. This result depends on an important property of the real number system: Every sequence that has a upper bound has a least upper bound [18].

Theorem (Stability Characterization). An LTI system H is stable if and only if its impulse response, $h = H\delta$, is in l^1 (absolutely summable).

Proof: First, suppose $h = H\delta$ is in l^1, and $x(n)$ is an l^∞ input signal with $|x(n)| \le M$. Let $y = Hx$. Then,

$$|y(n)| = \left|\sum_{k=-\infty}^{\infty} h(k)x(n-k)\right| \le \sum_{k=-\infty}^{\infty} |h(k)||x(n-k)| \le \sum_{k=-\infty}^{\infty} |h(k)|M$$
$$= M\sum_{k=-\infty}^{\infty} |h(k)| = M\|h\|_1 < \infty. \tag{2.40}$$

So $y(n)$ is bounded, proving that H is stable. Conversely, suppose the system H is stable, but the impulse response $h(n)$ is not in l^1. Now, the expression (2.37) for the l^1-norm of h is in fact a limit operation on a monotonically increasing sequence of sums. This sequence is either bounded or it is not. If it is bounded, then it would have a least upper bound, which must be the limit of the infinite sum (2.37). But we are assuming this limit does not exist. Thus, the sum must in fact be unbounded. And the only way that the limit cannot exist is if it diverges to infinity. That is,

$$\|h\|_1 = \sum_{k=-\infty}^{\infty} |h(k)| = \infty . \tag{2.41}$$

Let us consider the bounded input signal $x(n)$ defined

$$x(n) = \begin{cases} \dfrac{h(-n)}{|h(-n)|} & \text{if } h(-n) \neq 0, \\ 0 & \text{if } h(-n) = 0. \end{cases} \tag{2.42}$$

What is the response of our supposedly stable system to the signal $x(n)$? The convolution theorem for LTI systems tells us that we can find, for example, $y(0)$ to be

$$y(0) = \sum_{k=-\infty}^{\infty} h(k)x(0-k) = \sum_{k=-\infty}^{\infty} h(k)\frac{h(k)}{|h(k)|} = \sum_{k=-\infty}^{\infty} |h(k)| = \|h\|_1 = \infty. \tag{2.43}$$

This shows that y is unbounded, so that H is not stable, contradicting the assumption. Consequently, it must be the case that h is in l^1. ■

2.5.3 Toward Abstract Signal Spaces

One of the first things to verify is the closure of the l^p spaces under certain arithmetic or algebraic signal operations. This involves two steps:

(i) Verifying that the result of the operation is still a signal; that is, we can compute the value of the result at any time instant (if one of the signal values becomes infinite, then it is not a signal).
(ii) Verifying that the resulting signal is in the signal space of the operands.

2.5.3.1 Closure Properties. The closure property for a signal operation shows that we can process l^p space signals through systems that are defined by the given operation. For example, the proof of the following closure proposition is easy and left as an exercise. What it shows is that an l^p signal can be fed into an amplifier system and yet it remains an l^p signal. Similarly, a delay or advance system preserves the l^p nature of its input signals.

Proposition (Closure of l^p Spaces). Let $x(n)$ be a signal in l^p, $1 \le p \le \infty$, let c be a real (or complex) number, and let k be an integer. Then:

(i) $cx(n)$ is in l^p and $\|cx\|_p = |c|\ \|x\|_p$.
(ii) $x(k-n)$ is in l^p and $\|x(k-n)\|_p = \|x(n)\|_p$.

Proof: Exercise. ■

Proposition (Closure of l^1 Spaces). Let $x(n)$ and $y(n)$ be l^1 signals. Then $w(n) = x(n) + y(n)$ is in l^1 also.

Proof: To show that $\|w(n)\|_1$ is finite, we only need to generalize the *triangle inequality* from arithmetic, $|x+y| \le |x| + |y|$, to infinite sums, and this is straightforward. ■

Proposition (Closure of l^∞ Spaces). Let $x(n)$ and $y(n)$ be l^∞ signals. Then $w(n) = x(n) + y(n)$ is in l^∞ also.

Proof: $\|w(n)\|_\infty$ is the least upper bound of $\{|w(n)|: n \textit{ an integer}\}$. The arithmetic triangle inequality extends to infinite sets for upper bounds as well. ■

2.5.3.2 Vector Spaces. Before getting into the thick of abstract signal spaces, let us review the properties of a vector V space over the real numbers $\mathbb{R}$. There is a zero vector. Vectors may be added, and this addition is commutative. Vectors have additive inverses. Vectors can be multiplied by scalars, that is, elements of $\mathbb{R}$. Also, there are distributive and associative rules for scalar multiplication of vectors. One vector space is the real numbers over the real numbers: not very provocative. More interesting is the space with vectors taken from $\mathbb{R} \times \mathbb{R} = \{(a, b): a, b \in \mathbb{R}\}$. This set of real ordered pairs is called the *Cartesian product*, after Descartes,[2] but the concept was also pioneered by Fermat.[3] In general, we can take the Cartesian

[2]Forming a set of ordered pairs into a structure that combines algebraic and geometric concepts originates with Rene Descartes (1596–1650) and Pierre Fermat (1601–1665) [D. Struik, ed., *A Source Book in Mathematics,* 1200–1800, Princeton, NJ: Princeton University Press, 1986]. Descartes, among other things, invented the current notation for algebraic equations; developed coordinate geometry; inaugurated, in his *Meditations*, the epoch of modern philosophy; and was subject to a pointed critique by his pupil, Queen Christina of Sweden.

[3]Fermat is famous for notes he jotted down while perusing Diophantus's *Arithmetic*. One was a claim to the discovery of a proof for his Last Theorem: There are no nonzero whole number solutions of $x^n + y^n = z^n$ for $n > 1$. Historians of science, mathematicians, and the lay public for that matter have come to doubt that the Toulouse lawyer had a "marvelous demonstration," easily inserted but for the tight margins left by the printer. Three centuries after Fermat's teasing marginalia, A. Wiles of Princeton University followed an unexpected series of deep results from mathematicians around the world with his own six-year assault and produced a convincing proof of Fermat's Last Theorem [K. Devlin, *Mathematics: The Science of Patterns*, New York: Scientific American Library, 1994.]

product any positive number of times. The vector space exemplar is in fact the set of ordered n-tuples of real numbers $(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$. This space is called *Euclidean n-space*.[4]

Ideas of linear combination, span, linear independence, basis, and dimension are important; we will generalize these notions as we continue to lay our foundation for signal analysis. When a set of vectors $S = \{\boldsymbol{u}_i: i \in I\}$ *spans* V, then each vector $\boldsymbol{v} \in V$ is a *linear combination* of some of the $\boldsymbol{u}_i$: there are real numbers a_i such that $\boldsymbol{v} = a_1\boldsymbol{u}_1 + \cdots + a_N\boldsymbol{u}_N$. If S is finite and spans V, then there is a *linearly independent* subset of S that spans V. In other words, there is some $B \subseteq S$, $B = \{\boldsymbol{b}_i: 1 \le i \le N\}$, B spans V, and no nontrivial linear combination of the $\mathbf{b_i}$ is the zero vector: $\mathbf{0} = a_1\mathbf{u}_1 + \cdots + a_N\mathbf{u}_N$ implies $a_i = 0$. A spanning, linearly independent set is called a *basis* for V. If V has a finite basis, then every basis for V contains the same number of vectors—the *dimension* of V. A vector in Euclidean n-space has a *norm*, or *length*, which is the square root of the sum of the squares of its elements.

There is also an *inner product*, or *dot product*, for $\mathbb{R}^n$: $\langle \boldsymbol{a}, \boldsymbol{b} \rangle = \sum_{i=1}^{n} a_i b_i = \boldsymbol{a} \cdot \boldsymbol{b}$.

The dot product is a means of comparing two vectors via the relation $\langle \boldsymbol{a}, \boldsymbol{b} \rangle = \|\boldsymbol{a}\|\|\boldsymbol{b}\|\cos\theta$, where θ is the angle between the two vectors.

The vector space may also be defined over the complex numbers $\mathbb{C}$ in which casewe call it a *complex vector space*. The set of n-tuples of complex numbers $\mathbb{C}^n$ is called *unitary n-space*, an n-dimensional complex vector space. All of the vector space definitions and properties carry directly over from real to complex vector spaces, except for those associated with the inner product. We have to define $\langle \boldsymbol{a}, \boldsymbol{b} \rangle = \sum_{i=1}^{n} a_i \overline{b_i} = \boldsymbol{a} \cdot \boldsymbol{b}$, where $\overline{c}$ is the complex conjugate. A classic reference on vector spaces (and modern algebra all the way up to the unsolvability of quintic polynomials by radicals $\sqrt[n]{r}$, roots of order n) is Ref. 19.

2.5.3.3 Metric Spaces. A *metric space* is an abstract space that incorporates into its definition only the notion of a distance measure between elements.

Definition (Metric Space). Suppose that M is a set and d maps pairs of elements of M into the real numbers, $d: M \times M \to \mathbb{R}$. Then M is a *metric space* with *metric*, or *distance measure d,* if:

(i) $d(u, v) \ge 0$ for all u, v in M.
(ii) $d(u, v) = 0$ if and only if $u = v$.
(iii) $d(u, v) = d(v, u)$ for all u, v in M.
(iv) For any u, v, and w in M, $d(u, v) \le d(u, w) + d(w, v)$.

[4]The ancient Greek mathematician Euclid (ca. 300 B.C.) was (probably) educated at Plato's Academy in Athens, compiled the *Elements*, and founded a school at Alexandria, Egypt.

Items (i) and (ii) are known as the *positive-definiteness* conditions; (iii) is a *symmetry* condition; and (iv) is the *triangle inequality*, analogous to the distances along sides of a triangle.

Example (Euclidean, Unitary Metric Spaces). Clearly the Euclidean and unitary spaces are metric spaces. We can simply take the metric to be the Euclidean norm: $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$.

Example (City Block Metric). But other distance measures are possible, too. For example, if we set $d((u_1, v_1), (u_2, v_2)) = |u_1 - u_2| + |v_1 - v_2|$, then this is a metric on $\mathbb{R}^2$, called the *city block distance*. This is easy to check and left as an exercise. Note that the same set of elements can underlie a different metric space, depending upon the particular distance measure chosen, as the city block distance shows. Thus, it is common to write a metric space as an ordered pair (M, d), where M is the set of elements of the space, and d is the distance measure.

The triangle inequality is a crucial property. It allows us to form groups of metric space elements, all centered around a single element. Thus, it is the mathematical foundation for the notion of the proximity of one element to another. It makes the notion of distance make sense: You can jump from u to v directly, or you can jump twice, once to w and thence to v. But since you end up in the same place, namely at element v, a double jump should not be a shorter overall trip.

We would like to compare two signals for similarity—for instance, to match one signal against another (Chapter 4). One way to do this is to subtract the signal values from each other and calculate the size or magnitude of their difference. We can't easily adapt our inner product and norm definitions from Euclidean and unitary spaces to signals because signals contain an infinite number of components. Intuitively, a finitely supported signal could have some norm like a finite-dimensional vector. But what about other signals? The inner product sum we are tempted to write for a discrete signal becomes an infinite sum if the signal is not finitely supported. When does this sum converge? What about bases and dimension? Can the span, basis, and dimension notions extend to encompass discrete signals too?

2.5.4 Normed Spaces

This section introduces the *normed space*, which combines the ideas of vector spaces and metric spaces.

Definition (Normed Space). A *normed space*, or *normed linear space*, is a vector space V with a norm $\|v\|$ such that for any u and v in V,

(i) $\|v\|$ is real and $\|v\| \geq 0$.
(ii) $\|v\| = 0$ if and only if $v = \mathbf{0}$, the zero vector in V.

(iii) $\|av\| = |a|\ \|v\|$, for all real numbers a.
(iv) $\|u + v\| \le \|u\| + \|v\|$.

If $S \subseteq V$ is a vector subspace of V, then we may define a norm on S by just taking the norm of V restricted to S. Then S becomes a normed space too. S is called a *normed subspace* of V.

We adopt Euclidean and unitary vector spaces as our inspiration. The goal is to take the abstract properties we need for signals from them and define a special type of vector space that has at least a norm. Now, we may not be able to make the class of all signals into a normed space; there does not appear to be any sensible way to define a norm for a general signal with infinite support. One might try to define a norm on a restricted set of signals in the same way as we define the norm of a vector in $\mathbb{R}^3$. The l^2 signals have such a norm defined for them. The problem is that we do not yet know whether the l^2 signals form a vector space. In particular, we must show that a sum of l^2 signals is still an l^2 signal (additive closure). Now, we have already shown that the l^1 signals with the norm $\|x\|_1$ do form a normed space. Thus, our strategy is to work out the specific properties we need for signal theory, specify an abstract space with these traits, and then discover those concrete classes of signals that fulfill our axiom system's requirements. This strategy has proven quite successful in applied mathematics.[5] The discipline of *functional analysis* provides the tools we need [15, 20–23]. There is a complete history as well [24]. We start with a lemma about conjugate exponents [15].

Definition (Conjugate Exponents). Let $p > 1$. If $p^{-1} + q^{-1} = 1$, then q is a *conjugate exponent* of p. For $p = 1$, the conjugate exponent of p is $q = \infty$.

Let us collect a few simple facts about conjugate exponents.

Proposition (Conjugate Exponent Properties). Let p and q be conjugate exponents. Then

(i) $(p + q)/pq = 1$.
(ii) $pq = p + q$.
(iii) $(p - 1)(q - 1) = 1$.
(iv) $(p - 1)^{-1} = q - 1$.
(v) If $u = tp^{-1}$, then $t = uq^{-1}$.

[5]Several mathematicians—among them Wiener, Hahn, and Banach—simultaneously and independently worked out the concept and properties for a normed space in the 1920s. The discipline of functional analysis grew quickly. It incorporated the previous results of Hölder and Minkowski, found applications in quantum mechanics, helped unify the study of differential equations, and, within a decade, was the subject of general treatises and reviews.

Lemma (Conjugate Exponents). Let $a > 0$ and $b > 0$ be real numbers. Let p and q be conjugate exponents. Then $ab \le p^{-1}a^p + q^{-1}b^q$.

Proof: The trick is to see that the inequality statement of the lemma reduces to a geometric argument about the areas of regions bounded by the curve $u = tp^{-1}$. Note that

$$\frac{a^p}{p} = \int_0^a t^{p-1}\,dt, \tag{2.44}$$

$$\frac{b^q}{q} = \int_0^b u^{q-1}\,du. \tag{2.45}$$

Definite integrals (2.44) and (2.45) are areas bounded by the curve and the t- and u-axis, respectively (Figure 2.14). The sum of these areas is not smaller than the area of the rectangle defined by (0,0) and (a,b) in any case. ■

Our first nontrivial closure result on l^p spaces, Hölder's[6] inequality [15], shows that it is possible to form the product of signals from conjugate l^p spaces.

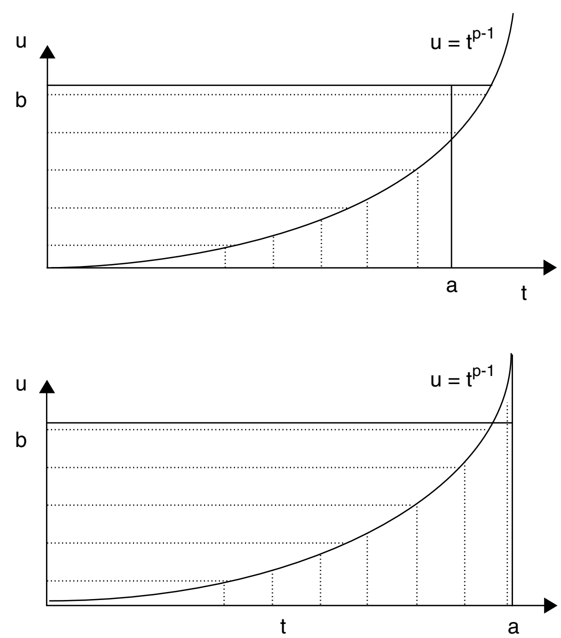


Fig. 2.14. There are two cases: Either point (a, b) is below the curve (bottom) or above the curve (top). In either case the area of the rectangle determined by the origin and (a, b) has a smaller area than the total area of regions bounded by the curve and the t- and u-axes.

[6]German mathematician Otto Ludwig Hölder (1859–1937) discovered the relation in 1884.

Theorem (Hölder's Inequality). Let $x(n)$ be in l^p and let $h(n)$ in l^q, where p and q are conjugate exponents. Then $y(n) = x(n)h(n)$ is in l^1 and $\|y\|_1 \le \|x\|_p \|h\|_q$.

Proof: Since the inequality clearly holds if $\|x\|_p = 0$ or $\|h\|_q = 0$, we assume that these signals are not identically 0. Next, let $x'(n) = \frac{x(n)}{\|x\|_p}$ and $h'(n) = \frac{h(n)}{\|h\|_q}$, and set $y'(n) = x'(n)h'(n)$. By the lemma,

$$|x'(n)||h'(n)| \le \frac{|x'(n)|^p}{p} + \frac{|h(n)|^q}{q}. \tag{2.46}$$

Putting (2.46) into the expression for $\|y'\|_1$, we find that

$$\begin{aligned} \|y'\|_1 &= \sum_{n=-\infty}^{\infty} |x'(n)||h'(n)| \le \sum_{n=-\infty}^{\infty} \frac{|x'(n)|^p}{p} + \sum_{n=-\infty}^{\infty} \frac{|h'(n)|^q}{q} \\ &= \frac{1}{p} \sum_{n=-\infty}^{\infty} \frac{|x(n)|^p}{\|x\|_p} + \frac{1}{q} \sum_{n=-\infty}^{\infty} \frac{|h(n)|^q}{\|h\|_q} = \frac{1}{p} + \frac{1}{q} = 1. \end{aligned} \tag{2.47}$$

Hence,

$$\|y'\|_1 = \frac{\|xh\|_1}{\|x\|_p \|h\|_q} = \frac{\|y\|_1}{\|x\|_p \|h\|_q} \le 1 \tag{2.48}$$

and the Hölder inequality follows. ■

Remarks. From the Hölder inequality, it is easy to show that there are IIR systems that have large classes of infinitely supported signals in their domains. This shows that our theory of signals can cope with more than FIR systems. In particular, the space of finite-energy signals is contained within the domain of any LTI system with a finite-energy impulse response.

Theorem (Domain of l^q Impulse Response Systems). Suppose H is an LTI system and $h = H\delta$ is in l^q. Then any $x(n)$ in l^p, where p and q are conjugate exponents, is in *Domain*(H).

Proof: By the convolution theorem for LTI systems, the response of H to input x is $y = x * h$. Thus,

$$|y(n)| = \left| \sum_{k=-\infty}^{\infty} x(k)h(n-k) \right| \le \sum_{k=-\infty}^{\infty} |x(k)||h(n-k)|. \tag{2.49}$$

Since the q-summable spaces are closed under translation and reflection, $h(n-k)$ is in l^q for all n. So the Hölder inequality implies that the product signal $w_n(k) = x(k)h(n-k)$ is in l^1. Computing $\|w_n\|_1$, we see that it is just the right-hand side of (2.49). Therefore, $|y(n)| \le \|w_n\|_1$ for all n. The convolution sum (2.49) converges. ■

Corollary (Cauchy–Schwarz[7] Inequality). Suppose $x(n)$ and $h(n)$ are in l^2. Then the product $y(n) = x(n)h(n)$ is in l^1 and

$$\|y\|_1 = \sum_{n=-\infty}^{\infty} |x(n)||h(n)| \le \sqrt{\sum_{n=-\infty}^{\infty} |x(n)|^2}\sqrt{\sum_{n=-\infty}^{\infty} |h(n)|^2} = \|x\|_2\|h\|_2 . \quad (2.50)$$

Proof: Take $p = q = 2$ in the Hölder inequality. ■

Theorem (Minkowski[8] Inequality). If x and y are in l^p, then $w = x + y$ is in l^p, and $\|w\|_p \le \|x\|_p + \|y\|_p$.

Proof: Assume $p > 1$. Then,

$$|w(n)|^p = |w(n)|^{p-1}|x(n) + y(n)| \le |w(n)|^{p-1}(|x(n)| + |y(n)|) . \quad (2.51)$$

Summing over all n gives

$$\sum_{n=-\infty}^{\infty} |w(n)|^p \le \sum_{n=-\infty}^{\infty} |x(n)||w(n)|^{p-1} + \sum_{n=-\infty}^{\infty} |y(n)||w(n)|^{p-1} . \quad (2.52)$$

The Hölder inequality applies to the first sum on the right-hand side of (2.52) with $q = p/(p-1)$ as follows:

$$\sum_{n=-\infty}^{\infty} |x(n)||w(n)|^{p-1} \le \left(\sum_{n=-\infty}^{\infty} |x(n)|^p\right)^{\frac{1}{p}}\left(\sum_{n=-\infty}^{\infty} [|w(n)|^{p-1}]^q\right)^{\frac{1}{q}}$$

$$= \left(\sum_{n=-\infty}^{\infty} |x(n)|^p\right)^{\frac{1}{p}}\left(\sum_{n=-\infty}^{\infty} |w(n)|^p\right)^{\frac{1}{q}}. \quad (2.53)$$

Similarly, for the second sum of the right-hand side of (2.52), we have

$$\sum_{n=-\infty}^{\infty} |y(n)||w(n)|^{p-1} \le \left(\sum_{n=-\infty}^{\infty} |y(n)|^p\right)^{\frac{1}{p}}\left(\sum_{n=-\infty}^{\infty} |w(n)|^p\right)^{\frac{1}{q}}. \quad (2.54)$$

Putting (2.52)–(2.54) together gives

$$\sum_{n=-\infty}^{\infty} |w(n)|^p \le \left[\left(\sum_{n=-\infty}^{\infty} |x(n)|^p\right)^{\frac{1}{p}} + \left(\sum_{n=-\infty}^{\infty} |y(n)|^p\right)^{\frac{1}{p}}\right]\left(\sum_{n=-\infty}^{\infty} |w(n)|^p\right)^{\frac{1}{q}}. \quad (2.55)$$

[7]After French mathematician Augustin-Louis Cauchy (1789–1857) and German mathematician Hermann Amandus Schwarz (1843–1921).

[8]Although he was born in what is now Lithuania, Hermann Minkowski (1864–1909) spent his academic career at German universities. He studied physics as well as pure mathematics. He was one of the first to propose a space-time continuum for relativity theory.

Finally, (2.55) entails

$$\frac{\sum_{n=-\infty}^{\infty}|w(n)|^p}{\left(\sum_{n=-\infty}^{\infty}|w(n)|^p\right)^{\frac{1}{q}}} = \left(\sum_{n=-\infty}^{\infty}|w(n)|^p\right)^{1-\frac{1}{q}} = \left(\sum_{n=-\infty}^{\infty}|w(n)|^p\right)^{\frac{1}{p}}$$

$$= \|w\|_p \le \left(\sum_{n=-\infty}^{\infty}|x(n)|^p\right)^{\frac{1}{p}} + \left(\sum_{n=-\infty}^{\infty}|y(n)|^p\right)^{\frac{1}{p}} = \|x\|_p + \|y\|_p, \tag{2.56}$$

completing the proof. ■

Now, it's easy to check that the norms associated with the l^p signal spaces, $\|x\|_p$, are in fact norms under the above abstract definition of a normed linear space:

Theorem (l^p Spaces Characterization). The l^p spaces are normed spaces for $1 \le p \le \infty$. ■

So far we find that the l^p spaces support several needed signal operations: addition, scalar multiplication, and convolution. Sometimes the result is not in the same class as the operands, but it is still in another related class; this is not ideal, but at least we can work with the result. Now let us try to incorporate the idea of signal convergence—limits of sequences of signals—into our formal signal theory.

2.5.5 Banach Spaces

In signal processing and analysis, we often consider sequences of signals $\{x_k(n)\}$. For example, the sequence could be a series of transformations of a source signal. It is of interest to know whether the sequence of signals converges to another limiting signal. In particular, we are concerned with the convergence of sequences of signals in l^p spaces, since we have already shown them to obey special closure properties, and they have a close connection with such signal processing ideas as stability. We have also shown that the l^p signal spaces are normed spaces. We cannot expect every sequence of signals to converge; after all, not every sequence of real numbers converges. However, we recall from calculus the Cauchy condition for convergence: A sequence of real (or complex) numbers $\{a_k\}$ is a *Cauchy sequence* if for every $\varepsilon > 0$, there is an $N > 0$ such that for $k, l > N$, $|a_k - a_l| < \varepsilon$. Informally, this means that if we wait long enough, the numbers in the sequence will remain arbitrarily close together. An essential property of the real line is that every Cauchy sequence of converges to a limit [18]. If we have an analogous property for signals in a normed space, then we call the signal space a Banach[9] space [15].

Definition (Banach Space). A Banach space B is a normed space that is *complete*. That is, any sequence $\{x_k(n)\}$ of signals in B that is a Cauchy sequence converges in B to a signal $x(n)$ also in B. Note that $\{x_k(n)\}$ is a *Cauchy sequence* if for every $\varepsilon > 0$, there is an $N > 0$ so that whenever $k, l > N$, we have $\|x_k - x_l\| < \varepsilon$. If $S \subseteq B$ is a complete normed subspace of B, then we call S a *Banach subspace* of B.

Theorem (Completeness of l^p Spaces). The l^p spaces are complete, $1 \le p \le \infty$.

Proof: The exercises sketch the proofs. ■

Banach spaces have historically proven difficult to analyze, evidently due to the lack of an inner product relation. Breakthrough research has of late cleared up some of the mysteries of these abstract spaces and revealed surprising structure. The area remains one of intense mathematical research activity. For signal theory, we need more analytical power than what Banach spaces furnish. In particular, we need some theoretical framework for establishing the similarity or dissimilarity of two signals—we need to augment our abstract signal theory space with an inner product relation.

Examples

(i) An example of a Banach subspace is l^1, which is a subspace of all l^p $1 \le p \le \infty$.

(ii) The set of signals in l^p that are zero on a nonempty subset $Y \subseteq \mathbb{Z}$ is easily shown to be a Banach space. This is a proper subspace of l^p for all p.

(iii) The normed subspace of l^p that consists of all finitely supported p-summable signals is not a Banach subspace. There is a sequence of finitely supported signals that is a Cauchy sequence (and therefore converges inside l^p) but does not converge to a finitely supported signal.

Recall from calculus the ideas of open and closed subsets of the real line. A set $S \subseteq \mathbb{R}$ is *open* if for every point p in S, there is $\varepsilon > 0$ such that $Ball(p, \varepsilon) = \{x \in \mathbb{R}: |x - p| < \varepsilon\} \subseteq S$. That is, every point p of S is contained in an open ball that is contained in S. A set $S \subseteq \mathbb{R}$ is *closed* if its complement is open. Let V be a normed space. Then a set $S \subseteq V$ is *open* if for every point p in S, there is $\varepsilon > 0$ such that $Ball(p, \varepsilon) = \{x \in V: \|x - p\| < \varepsilon\} \subseteq S$. That is, every point p of S is contained in an open ball that is contained in S.

Theorem (Banach Subspace Characterization). Let B be a Banach space and S a normed subspace of B. Then S is a Banach subspace if and only if S is closed in B.

[9]The Polish mathematician S. Banach (1892–1945) developed so much of the initial theory of complete normed spaces that the structure is named after him. Banach published one of the first texts on functional analysis in the early 1930s.

Proof: First suppose that S is a Banach subspace. We need to show that S is closed in B. Let $p \in B$, and $p \notin S$. We claim that there is an $\varepsilon > 0$ such that the open ball $Ball(p, \varepsilon) \cap S = \varnothing$. If not, then for any integer $n > 0$, there is a point $s_n \in S$ that is within the ball $Ball(p, 1/n)$. The sequence $\{s_n: n > 0\}$ is a Cauchy sequence in S. Since S is Banach, this sequence converges to $s \in S$. However, this means we must have $s = p$, showing that $p \in S$, a contradiction.

Conversely, suppose that S is closed and $\{s_n: n > 0\}$ is a Cauchy sequence in S. We need to show that $\{s_n\}$ converges to an element in S. The sequence is still a Cauchy sequence in all of B; the sequence converges to $p \in B$. We claim $p \in S$. If not, then since p is in the complement of S and S is closed, there must be an $\varepsilon > 0$ and an open ball $Ball(p, \varepsilon) \subseteq S'$. This contradicts the fact that $d(s_n, s) \to 0$, proving the claim and the theorem. ■

2.6 INNER PRODUCT SPACES

Inner product spaces have a binary operator for measuring the similarity of two elements. Remember that in linear algebra and vector calculus over the normed spaces $\mathbb{R}^n$, the inner (or dot) product operation has a distinct geometric interpretation. We use it to find the angle between two vectors in $\mathbb{R}^n$: $\langle u, v \rangle = \|u\|\|v\|\cos(\theta)$, where θ is the angle between the vectors. From the inner product, we define the notion of orthogonality and of an orthogonal set of basis elements. Orthogonal bases are important because they furnish a very easy set of computations for decomposing general elements of the space.

2.6.1 Definitions and Examples

Once again, we abstract the desired properties from the Euclidean and unitary spaces.

Definition (Inner Product Space). An *inner product space I* is a vector space with an inner product defined on it. The inner product, written with brackets notation $\langle x, y \rangle$, can be real- or complex-valued, according to whether I is a real or complex vector space, respectively. The inner product satisfies these five rules:

(i) $0 \le \langle x, x \rangle$ for all $x \in I$.

(ii) For all $x \in I, 0 = \langle x, x \rangle$ if and only if $x = 0$ (that is, x is the zero vector).

(iii) For all $x, y \in I$, $\langle x, y \rangle = \overline{\langle y, x \rangle}$, where $\bar{c}$ is the complex conjugate of c.

(iv) For all $c \in \mathbb{C}$ (or just $\mathbb{R}$, if I is a real inner product space) and all $x, y \in I$, $\langle cx, y \rangle = c\langle x, y \rangle$.

(v) For all $w, x, y \in I$, $\langle w + x, y \rangle = \langle w, y \rangle + \langle x, y \rangle$.

If $S \subseteq I$, then S becomes an inner product space by taking its inner product to be the inner product of I restricted to S. We call S an *inner product subspace* of I.

Remarks. Note that the inner product is linear in the first component, but, when the inner product spaces are complex, it is *conjugate linear* in the second component. When the definition speaks of "vectors," these are understood to be abstract elements; they could, for example, be infinitely long or functions from the integers to the real numbers (discrete signals).

Examples (Inner Product Spaces)

(i) The normed space R^n, Euclidean n-space, with inner product defined $\langle x, y\rangle = \sum_{k=1}^{n} x_k y_k$ is a real inner product space. This space is familiar from linear algebra and vector calculus.
(ii) The normed space $\mathbb{C}^n$ with inner product defined $\langle x, y\rangle = \sum_{k=1}^{n} x_k \overline{y_k}$ is a complex inner product space. (We take complex conjugates in the definition so that we can define $\|x\|$ on $\mathbb{C}^n$ from the inner product.)
(iii) The signal space l^2 is an inner product space when we define its inner product

$$\langle x, y\rangle = \sum_{n=-\infty}^{\infty} x(n)\overline{y(n)}. \tag{2.57}$$

Remarks. Notice that the Cauchy–Schwarz result (2.50) implies convergence of (2.57). Furthermore, since we know that the l^p spaces are Banach spaces and therefore complete, l^2 is our first example of a *Hilbert space*[10] (Figure 2.15).

The ideas underlying Banach and Hilbert spaces are central to understanding the latest developments in signal analysis: time-frequency transforms, time-scale transforms, and frames (Chapters 10–12).

[10] D. Hilbert studied the special case of l^2 around 1900. Later, in a landmark 1910 paper, F. Riesz generalized the concept and defined the l^p spaces we know today. The Cauchy–Schwarz inequality was known for discrete finite sums (i.e., discrete signals with finite support) by A. Cauchy in the early nineteenth century. H. Schwarz proved the analagous result for continuous signals (we see this in the next chapter, when the summations become integrals) and used it well in a prominent 1885 paper on minimal surfaces. V. Buniakowski had in fact already discovered Schwarz's integral form of the inequality in 1859, but his result drew little attention. O. Hölder published a paper containing his inequality in 1889. H. Minkowski disclosed the inequality that now bears his name as late as 1896; it was, however, restricted to finite sums. Riesz's 1910 paper would extend both the Hölder and Minkowski results to analog signals, for which integrals replace the discrete sums.

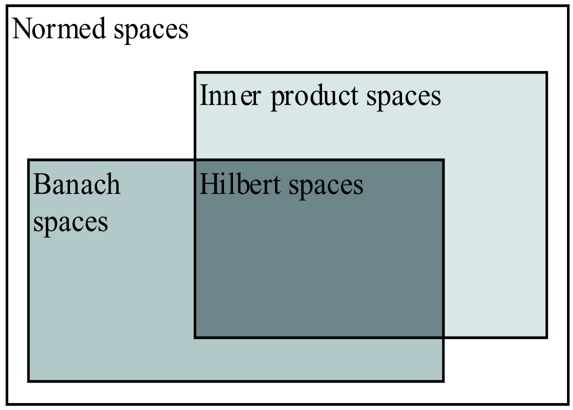


Fig. 2.15. Relationships between signal spaces. Hilbert spaces are precisely the inner prodct spaces that are also Banach spaces.

2.6.2 Norm and Metric

An inner product space I has a natural *norm* associated with its inner product: $\|x\| = \sqrt{\langle x, x\rangle}$. With this definition, the triangle inequality follows from the Cauchy–Schwarz relation, and the other normed space properties follow easily. There is also a natural distance measure: $d(x,y) = \|x - y\|$.

Theorem (Cauchy–Schwarz Inequality for Inner Product Spaces). Let I be an inner product space and $u, v \in I$. Then

$$|\langle u, v\rangle| \le \|u\|\|v\| \tag{2.58}$$

Furthermore, $|\langle u, v\rangle| = \|u\|\|v\|$ if and only if u and v are linearly dependent.

Proof: First, suppose that u and v are linearly dependent. If, say, $v = 0$, then both sides of (2.58) are zero. Also, if $u = cv$ for some scalar (real or complex number) c, then $|\langle u, v\rangle| = c\|v\|^2 = \|u\|\|v\|$, proving (2.58) with equality. Next, let us show that there is strict inequality in (2.58) if u and v are linearly independent. We resort to a standard trick. By linear independence, $u + cv \neq 0$ for any scalar c. Hence,

$$0 < \|u - cv\|^2 = \langle u, u\rangle - \bar{c}\langle u, v\rangle - c\langle v, u\rangle - |c|^2\langle v, v\rangle \tag{2.59}$$

for any c. In particular, by taking $c = \dfrac{\overline{\langle v, u\rangle}}{\langle v, v\rangle}$ in (2.59),

$$0 < \|u\|^2 - \frac{|\langle u, v\rangle|^2}{\|v\|^2}. \tag{2.60}$$

The Cauchy–Schwarz inequality follows. ■

We can now show that with $\|x\| = \sqrt{\langle x, x\rangle}$, I is a normed space. All of the properties of a normed space are simple, except for the triangle inequality. By expanding the inner products in $\|u + v\|^2$, it turns out that

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2 + 2Real(\langle u, v\rangle) \le \|u\|^2 + \|v\|^2 + 2|\langle u, v\rangle|, \tag{2.61}$$

where $Real(c)$ is the real part of $c \in \mathbb{C}$. Applying Cauchy–Schwarz to (2.61),

$$\|u+v\|^2 \le \|u\|^2 + \|v\|^2 + 2\|u\|\|v\| = (\|u\| + \|v\|)^2, \tag{2.62}$$

and we have shown the triangle inequality for the norm.

The natural distance measure $d(x,y) = \|x - y\|$ is indeed a metric. That is, for all $x, y \in I$ we have $d(x, y) \ge 0$. Also, $d(x, y) = 0$ if and only if $x = y$. Symmetry exists: $d(x, y) = d(y, x)$. And the triangle inequality holds: Given $z \in I$, $d(x, y) \le d(x, z) + d(z, y)$. We use the distance metric to define convergent sequences in I: If $\{x_n\}$ is a sequence in I, and $d(x_n, x) \to 0$ as $n \to \infty$, then we say $x_n \to x$. The inner product in I is continuous (exercise), another corollary of the Cauchy–Schwarz inequality.

Proposition (Parallelogram Rule). Let I be an inner product space, let x and y be elements of I, and let $\|x\| = \sqrt{\langle x, x\rangle}$ be the the inner product space norm. Then $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$.

Proof: Expanding the norms in terms of their definition by the inner product gives

$$\|x+y\|^2 = \|x\|^2 + \langle x, y\rangle + \langle y, x\rangle + \|y\|^2, \tag{2.63}$$

$$\|x-y\|^2 = \|x\|^2 - \langle x, y\rangle - \langle y, x\rangle + \|y\|^2. \tag{2.64}$$

Adding (2.63) and (2.64) together gives the rule. ∎

The reason for the rule's name lies in a nice geometric interpretation (Figure 2.16). The parallelogram rule is an abstract signal space equivalent of an elementary

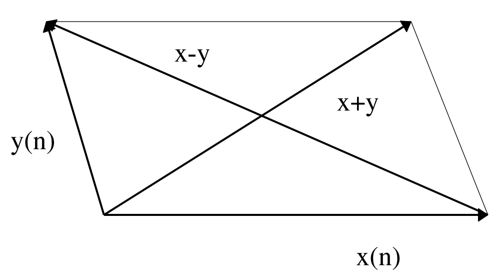


Fig. 2.16. Parallelogram rule. This simple geometric relationship between signals in an inner product space imposes a severe constraint on the l^p spaces. Only subspaces of l^2 support an inner product relation for our signal theory.

property from plane geometry. This shows the formal and conceptual power of the function space approach to signal theory. We can derive the algebraic rule from blind manipulations, and we can also resort to geometric constructs for insights into the relationships between the abstractions.

There is a negative consequence of the parallelogram rule: Except for l^2, none of the l^p spaces support an inner product definition that is related to the norm, $\| \|_p$. To verify this, consider the signals $x = [\underline{0}, 1]$, $y = [\underline{1}, 0]$, and compute $\|x\|_p^2 = \|y\|_p^2 = 1$ and $\|x+y\|_p^2 = \|x-y\|_p^2 = 2^{2/p}$. By the rule, we must have $2 = 2^{2/p}$, so that $p = 2$. The consequence is that although we have developed a certain body of theory for p-summable signals, found closure rules for basic operations like signal summation, amplification, and convolution, and shown that Cauchy sequences of p-summable signals converge, we cannot have an inner product relation for comparing signals unless $p = 2$. One might worry that l^2 is a signal analysis monoculture—it lacks the diversity of examples, counterexamples, classes, and conditions that we need to run and maintain the signal analysis economy. The good news is that the square-summable signal space is quite rich and that we can find in its recesses the exceptional signals we happen to need.

2.6.3 Orthogonality

The inner product relation is the key to decomposing a signal into a set of simpler components and for characterizing subspaces of signals. The pivotal concept for inner product signal spaces, as with simple Euclidean and unitary vector spaces, is orthogonality.

2.6.3.1 Definition and Examples. If we consider inner product spaces of discrete signals, then the inner product is a measure of the similarity of two signals. Signals are similar to themselves, and so nonzero signals have a positive inner product with themselves: $\langle u, u\rangle > 0$. Two signals that are not at all alike, following this intuition, have zero inner product with each other: $\langle u, u\rangle = 0$.

Definition (Orthogonality). In an inner product space *I,* two elements—*u* and *v*— are *orthogonal* if $\langle u, v\rangle = 0$; in this case, we write $u \perp v$. If *u* is orthogonal to every element of a set *S*, we write $u \perp S$. The set of all *u* in *I* such that for all *s* in *S*, we have $u \perp s$ is called the *orthogonal complement* of *S*; it is written $S^{\perp}$. A set S of nonzero elements of *I* is an *orthogonal* set if $u \perp v$ for *u, v* in *S*, $u \neq v$. If *S* is an orthogonal set such that $\|u\| = 1$ for all *u* in *S*, then *S* is an *orthonormal* set.

Example (Unit Impulses). The shifted unit impulse signals $S = \{\delta(n-k)\}$ form an orthonormal set in the inner product space l^2.

Notice that we do not say that the unit impulse signals are a *basis* for the square-summable signals. In finite-dimensional vector spaces, bases span the entire space; thus, every vector is a linear combination of a finite number of elements from the basis set. This is not true for general l^2 signals and the shifted unit impulses. No finite set of shifted unit impulses can span the whole Hilbert space. Moreover, some signals—those without finite support—cannot be a linear combination of a finite number of shifted unit impulses. While we were able to smoothly migrate most of the ideas from finite-dimensional vector spaces to inner product spaces, we now find that the concept of a basis did not fare so well. The problem is that general inner product spaces (among them our remaining l^p space, l^2) may have an "infinite" dimension.

We must loosen up the old idea of basis considerably. Let us reflect for a moment on the shifted unit impulses and a signal $x \in l^2$. Since $\|x\|_2 < \infty$, given $\varepsilon > 0$, we can find $N > 0$, so that the energy of the outer fringes of the signal—$x(n)$ values for $n > N$—have energy less than ε. This means that some linear combination of shifted impulses comes arbitrarily close to x in l^2 norm. Here then is where we loosen the basis concept so that it works for inner product spaces. We allow that a linear combination come arbitrarily close to signal x. This is the key idea of *completeness*: A set of elements S is *complete* or *total* if every element of the space I is the limit of a sequence of elements from the linear span of S. We also say that S is *dense* in I. The term "complete" for this idea is terrible, but standard. In a space with a distance measure, "complete" means that every Cauchy sequence converges to an element in the space. For example, the real number line is complete this sense. Now we use the same term for something different, namely the existence of a Cauchy sequence of linear combinations. The better term is "total" [15]. The only good advice is to pay close attention to the context; if the discussion is about bases, then "complete" probably signifies this new sense. So we could stipulate that a *basis* for I is a linearly independent set of signals whose linear span is dense in I.

Should we also assert that the shifted unit samples are a basis for the square-integrable signals? Notice that we are talking here about convergence in a general inner product space. Some inner product spaces are not complete. So, we will postpone the proper definition of completeness and the generalization of the basis concept that we need until we develop more inner product space theory.

Example (Rademacher[11] Signals). Consider the signals $e_0 = [\underline{1}, -1]$, $e_1 = [\underline{1}, 1, -1, -1]$, $e_2 = [\underline{1}, 1, 1, 1, -1, -1\ -1, -1]$, and so on. These signals are orthogonal. Notice that we can shift e_0 by multiples of 2, and the result is still orthogonal to all of the e_i. And we can delay e_1 by $4k$ for some integer k, and it is still orthogonal to all of the e_i and all of the shifted versions of e_0. Let us continue to use $e_{i,k}$ to denote the signal e_i delayed by amount k. Then the set $\{e_{i,k}$: *i is a natural number and* $k = m^{2i+1}$, *for some* $m\}$ is orthogonal. Notice that the sum of signal values of a linear combination

[11]Although not Jewish, but rather a pacifist, German number theorist and analyst Hans Rademacher (1892–1969) was forced from his professorship at Breslau in 1934 by the Nazi regime and took refuge in the United States.

of Rademacher signals is always zero. Hence, the Rademacher signals are not complete. The signal $\delta(n)$, for instance, is not in the closure of the linear span of the Rademacher signals. Note too that the Rademacher signals are not an orthonormal set. But we can orthonormalize them by dividing each $e_{i,k}$ by its norm, 2^{i+1}.

We will use the notion of orthogonality extensively. Note first of all that this formal definition of orthogonality conforms to our geometric intuition. The Pythagoras[12] relation for a right triangle states that the square of the length of the hypoteneuse is equal to the sum of the squares of the other two sides. It is easy to show that $u \perp v$ in an inner product space entails $\|u\|^2 + \|v\|^2 = \|u+v\|^2$. This also generalizes to an arbitrary finite orthogonal set. Other familiar inner product properties from the realm of vector spaces reappear as well:

(i) If $x \perp u_i$ for $i = 1, ..., n$, then x is orthogonal to any linear combination of the u_i.
(ii) If $x \perp u_i$ for i in the natural numbers, and $u_i \to u$, then $x \perp u$.
(iii) If $x \perp S$ (where S is an orthogonal set), u_i is in S, and $u_i \to u$, then $x \perp u$.

Orthonormal sets can be found that span the inner product signal spaces I that we commonly use to model our discrete (and later, analog) signals. The idea of spanning an inner product space generalizes the same notion for finite-dimensional vector spaces. We are often interested in decomposing a signal $x(n)$ into linear combination of simpler signals $\{u_0, u_1, ...\}$. That is, we seek scalars c_k such that $x(n) = \sum c_k u_k(n)$. If the family $\{c_k\}$ is finite, we say that $x(n)$ is in the *linear span* of $\{u_0, u_1, ...\}$. Orthonormal sets are handy for the decomposition because the scalars c_k are particularly easy to find. If $x(n) = \sum c_k u_k(n)$ and the u_k are orthonormal, then $c_k = \langle x, u_k \rangle$. For example, the shifted unit impulse signals $S = \{\delta(n - k)\}$ form an orthonormal set in the inner product space l^2. Decomposing $x(n)$ on the orthonormal shifted unit impulses is trivial: $c_k = x(k)$. The unit impulses are not a very informative decomposition of a discrete signal, however, because they do not provide any more information about the signal than its values contain at time instants. The problem of signal decomposition becomes much more interesting and useful when the composition elements become complicated. Each u_k then encapsulates more elaborate information about $x(n)$ within the decomposition. We may also interpret $|c_k|$ as a measure of how much alike are $x(n)$ and $u_k(n)$.

Example (Discrete Fourier Transform). To illustrate a nontrivial orthonormal decomposition, let $N > 0$ and consider the windowed exponential signals

$$u(n) = \frac{\exp\left(2\pi jk\frac{n}{N}\right)}{\sqrt{N}}[u(n) - u(n-N)], \tag{2.65}$$

[12]This property of right triangles was known to the Babylonians long before the mystic Greek number theorist Pythagoras (fl. ca. 510 B.C.) [B. L. van der Waerden, *Science Awakening*, translator. A. Dresden, Groningen, Holland: Nordhoff, 1954].

where $u(n)$ is the unit step signal. They form an orthonormal set on $[0, N-1]$. Suppose $x(n)$ also has support in $[0, N-1]$. We will show later that $x(n)$ is in the linear span of $\{u_k(n)\colon 0 \le k \le N-1\}$ (Chapter 7). Since x, u_1, ..., u_{N-1} are in l^2, we have

$$x(n) = \sum_{k=0}^{N-1} c_k u_k = \sum_{k=0}^{N-1} \langle x, u_k \rangle u_k. \tag{2.66}$$

This decomposes $x(n)$ into a sum of scaled frequency components; we have, in fact, quite easily discovered the *discrete Fourier transform* (DFT) using a bit of inner product space theory. Fourier transforms in signal processing are a class of signal operations that resolve an analog or discrete signal into its frequency components—sinusoids or exponentials. The components may be called "Fourier components" when the underlying orthonormal set is not made up of sinusoidal or exponential components. Thus we have the definition: If $\{u_k(n)\colon 0 \le k \le N-1\}$ is an orthonormal set and x is a signal in an inner product space, then $c_k = \langle x, u_k \rangle$ is the kth Fourier coefficient of $x(n)$ with respect to the $\{u_k\}$. If $x(n) = \sum c_k u_k$, then we say that x is represented by a *Fourier series* in the $\{u_k\}$.

Note that the DFT system is linear, but, owing to the fixed decomposition window, not translation-invariant. There are a great many other properties and applications of the DFT (Chapter 7).

2.6.3.2 Bessel's Inequality. Geometric intuition about inner product spaces can tell us how we might use Fourier coefficients to characterize a signal. From (2.66) we can see that each Fourier coefficient indicates how much of each $u_k(n)$ there is in the signal $x(n)$. If $0 = \langle x, u_k \rangle$, then there is nothing like $u_k(n)$ in $x(n)$; if $|\langle x, u_k \rangle|$ is large, it means that there is an important u_k-like element in $x(n)$; and when $|\langle x, u_k \rangle|$ is large and the rest of the Fourier coefficients are small, it means that as a signal $x(n)$ has a significant similarity to $u_k(n)$. Given an orthonormal set $S = \{u_k\}$ and a signal $x(n)$, what do the Fourier coefficients of $x(n)$ with respect to S look like? It is clear from (2.66) that when $\|x(n)\|$ is large, then the norms of the Fourier coefficients also become large. How large can the Fourier coefficients be with respect to $\|x(n)\|$? If the set S is infinite, are most of the Fourier coefficients zero? Is it possible for the Fourier coefficients of $x(n)$ to be arbitrarily large? Perhaps the $\langle x, u_k \rangle \to 0$ as $k \to \infty$? If not, then there is an $\varepsilon > 0$, such that for any $N > 0$, there is a $k > N$ with $|\langle x, u_k \rangle| > \varepsilon$. In other words, can the significant Fourier components in $x(n)$ with respect to S go on forever? The next set of results, leading to Bessel's theorem for inner product spaces, helps to answer these questions.

Consider a signal $x(n)$ and an orthonormal set $S = \{u_k(n)\colon 0 \le k < N\}$ in an inner product space I. $x(n)$ may be in the linear span of S, in which case the Fourier coefficients tell us the degree of similarity of $x(n)$ to each of the elements u_k. But if $x(n)$ is not in the linear span of S, then we might try to find the $y(n)$ in the linear span of S that is the closest signal to $x(n)$. In other words, if S cannot give us an exact breakdown of $x(n)$, what is the best model of $x(n)$ that S can provide? Let c_k be complex numbers, and let $y = \sum c_k u_k$. $y(n)$ is a linear combination of elements of S which we want to be close to $x(n)$. Then, after some inner product algebra and help from Pythagoras,

$$\|x-y\|^2 = \left\langle x - \sum c_k u_k, x - \sum c_k u_k \right\rangle = \|x\|^2 - \sum \left|c_k - \langle u_k, x\rangle\right|^2 - \sum \left|\langle u_k, x\rangle\right|^2. \tag{2.67}$$

By varying the c_k, $y(n)$ becomes any general signal in the linear span of S. The minimum distance between $x(n)$ and $y(n)$ occurs when the middle term of (2.67) is zero: $c_k = \langle u_k, x\rangle$. With this choice of $\{c_k\}$, $y(n)$ is the best model S can provide for $x(n)$.

We can apply this last result to answer the question about the magnitude of Fourier coefficients for a signal $x(n)$ in an inner product space. We would like to find an orthogonal set S so that its linear span contains every element of the inner product space. Then, we might characterize a signal $x(n)$ by its Fourier coefficients with respect to S. Unfortunately, as with the Rademacher signals, the linear span of an orthonormal set may not include the entire inner product space. Nevertheless, it is still possible to derive Bessel's inequality for inner product spaces.

Theorem (Inner Product Space Bessel[13] Inequality). Let I be an inner product space, let $S = \{u_k\}$ an orthonormal family of signals in I, and let $x \in I$. Then

$$\|x\|^2 \geq \sum \left|\langle u_k, x\rangle\right|^2. \tag{2.68}$$

Proof: Proceeding from (2.67), we set $c_k = \langle u_k, x\rangle$. Then, $\sum \left|c_k - \langle u_k, x\rangle\right|^2 = 0$, and $\|x-y\|^2 = \|x\|^2 - \sum \left|\langle u_k, x\rangle\right|^2$. But $\|x-y\|^2 \geq 0$. Thus, $\|x\|^2 - \sum \left|\langle u_k, x\rangle\right|^2 \geq 0$ and (2.68) follows. ■

2.6.3.3 Summary. We use the inner product relation as a measure of the similarity of signals, just as we do with finite-dimensional vectors. Orthonormal families of signals $S = \{u^k\}$ are especially convenient for decomposing signals $x(n)$, since the coefficients of the decomposition sum are readily computed as the inner product $c_k = \langle u_k, x\rangle$. It may well be that $x(n)$ cannot be expressed as a sum (possibly infinite) of elements of S; nevertheless, we can find the coefficients c_k that give us the closest signal to $x(n)$. From Bessel's relation, we see that the Fourier coefficients for a general signal $x(n)$ with respect to an orthonormal family S are bounded by $\|x\|$. One does not find a monstrous Fourier coefficient unless the original signal itself is monstrous. Moreover, the sum in (2.68) could involve an infinite number of nonzero terms. Then the fact that the sum converges indicates that the Fourier coefficients must eventually become arbitrarily small. No signal has Fourier coefficients with respect to S that get arbitrarily large. When S is infinite, every signal has Fourier coefficients $c_k = \langle u_k, x\rangle$ such that $c_k \to 0$ as $k \to \infty$. Bessel's inequality guarantees that the Fourier coefficients for $x(n)$ are well-behaved.

[13]After German mathematician and astronomer Friedrich Wilhelm Bessel (1794–1846).

Under what conditions does $x = \sum c_k u_k$, with $c_k = \langle u_k, x\rangle$? That is, we are interested in whether $x(n)$ has a Fourier series representation with respect to the $\{u_k\}$. If the orthonormal set S is finite and x is in the linear span of S, then this is true. If, on the other hand, S is infinite, then the sum becomes (possibly) infinite and the problem becomes whether the limit that formally defines this summation exists. Recall that calculus explains convergence of infinite sums in terms of Cauchy sequences of partial series sums. Thus, $\sum a_k = a$, if $\lim A_n = a$, where $A_n = \sum_{k=1}^{n} a_k$. If every Cauchy sequence has a limit, then the abstract space is called *complete*. Banach spaces are normed spaces that are complete. If we add completeness to the required properties of an inner product space, then what we get is the abstract structure known as a *Hilbert space*—one of the most important tools in applied mathematics, signal analysis, and physics.

2.7 HILBERT SPACES

In addition to the many excellent treatments of inner product and Hilbert spaces in functional analysis treatises, Hilbert space theory is found in specialized, introductory texts [25, 26].

2.7.1 Definitions and Examples

Definition (Hilbert Space). A *Hilbert space* is a complete inner product space. If $S \subseteq H$ is an inner product subspace of H and every Cauchy sequence of elements of S converges to an element of S, then S is a *Hilbert subspace* of H.

Recall that $\{x_k(n)\}$ is a Cauchy sequence if for every $\varepsilon > 0$, there is an $N > 0$ so that whenever $k, l > N$, we have $d(x_k, x_l) = \|x_k - x_l\| < \varepsilon$. Completeness means that every Cauchy sequence of signals in the space converges to a signal also in that space. Since we are working with inner product spaces, this norm must be interpreted as the inner product space norm. The least special of all the spaces is the normed space. Within its class, and distinct from one another, are the inner product and Banach spaces. The Banach spaces that are blessed with an inner product are the Hilbert spaces (Figure 2.15).

Examples (Hilbert Spaces). The following are Hilbert spaces:

(i) l^2 with the inner product defined

$$\langle x, y\rangle = \sum_{n=-\infty}^{\infty} x(n)\overline{y(n)}. \tag{2.69}$$

(ii) The set of signals in l^2 that are zero on a nonempty subset $Y \subseteq \mathbb{Z}$.

(iii) The inner product space $\mathbb{R}^n$, Euclidean n-space, with the standard dot product.

(iv) Similarly, the unitary space $\mathbb{R}^n$ with the standard inner product is a complex Hilbert space.

(v) Consider some subset of the shifted unit impulse signals $S = \{\delta(n-k)\}$. The linear span of S is an inner product subspace of l^2. If we take the set of limit points of Cauchy sequences of the linear span of S, then we get a Hilbert subspace of l^2. These subspaces are identical to those of (ii).

2.7.2 Decomposition and Direct Sums

The notions of orthogonality, basis, and subspace are interlinked within Hilbert space theory. The results in this section will show that l^2 Hilbert space looks very much like an "infinite-dimensional" extension of our finite-dimensional Euclidean and unitary n-spaces.

2.7.2.1 Subspace Decomposition. The following theorem is basic.

Theorem (Hilbert Space Decomposition). Let H be a Hilbert space, let X be a Hilbert subspace of H, and let $Y = X^{\perp}$ be the orthogonal complement of X in H. Then for any h in H, $h = x + y$, where $x \in X$ and $y \in Y$.

Proof: Let $h \in H$ and consider the distance from the subspace X to h. This number, call it $\delta = d(h, X)$, is the greatest lower bound of $\{\|x-h\|: x \text{ in } X\}$. Since we can find elements $x \in X$ whose distance to h differs by an arbitrarily small value from δ, there must be a sequence $\{x_n: x_n \text{ in } X, n > 0\}$ with $\|x_n - h\| < 1/n + \delta$.

We claim that $\{x_n\}$ is a Cauchy sequence in X. By applying the parallelogram rule to $x_n - h$ and $x_m - h$, we have

$$\|(x_n - h) + (x_m - h)\|^2 + \|x_n - x_m\|^2 = 2\|(x_n - h)\|^2 + 2\|(x_m - h)\|^2. \tag{2.70}$$

Since X is closed under addition and scalar multiplication, we have $\frac{x_n + x_m}{2} \in X$, and therefore

$$\|(x_n - h) + (x_m - h)\| = \|x_n + x_m - 2h\| = 2\left\|\frac{x_n + x_m}{2} - 2\right\| \geq 2\delta. \tag{2.71}$$

Putting the inequality (2.71) into (2.70) and rearranging gives

$$\|x_n - x_m\|^2 \leq 2\left(\delta + \frac{1}{n}\right) + 2\left(\delta + \frac{1}{m}\right) - \|x_n - h + x_m - h\|^2. \tag{2.72}$$

Consequently,

$$\|x_n - x_m\|^2 \le 4\left(\frac{\delta}{n}\right) + \frac{2}{n^2} + 4\left(\frac{\delta}{m}\right) + \frac{2}{m^2}, \tag{2.73}$$

which shows $\{x_n\}$ is Cauchy, as claimed. Since X is complete and contains this sequence, there must be a limit point: $x_n \to x \in X$. Let $y = h - x$. We claim that $y \in X^{\perp} = Y$. To prove this claim, let $0 \ne w \in X$. We must show that $\langle y, w\rangle = 0$. First note that since $\delta = d(h, X)$ and $x \in X$, we have

$$\delta \le \|y\| = \|h - x\| = \left\|h - \lim_{k \to \infty} x_k\right\| \le \frac{1}{n} + \delta \tag{2.74}$$

for all n. Thus, $\delta = \|y\| = \langle y, y\rangle^{1/2}$. Next, let a be a scalar. Closure properties of X imply that $x + aw \in X$ so that $\|y - aw\| = \|h - (x + aw)\| \ge \delta$. Expanding this last inequality in terms of the inner product on H gives

$$\langle y, y\rangle - \bar{a}\langle y, w\rangle - a\langle w, y\rangle + |a|^2\langle w, w\rangle \ge \delta^2. \tag{2.75}$$

Because $\langle y, y\rangle = \delta^2$, we can simplify (2.75) and then take $a = \frac{\langle y, w\rangle}{\langle w, w\rangle}$, which greatly simplifies to produce $|\langle y, w\rangle|^2 \le 0$. This must mean that $0 = \langle y, w\rangle$, $y \perp w$, and $y \in X^{\perp}$. This proves the last claim and completes the proof. ■

Let us list a few facts that follow from the theorem:

(i) The norm of element y is precisely the distance from h to the subspace X, $\|y\| = \delta = d(h, X)$. This was shown in the course of the proof.

(ii) Also, the decomposition of $h = x + y$ is unique (exercise).

(iii) One last corollary is that a set S of elements in H is complete (that is, the closure of its linear span is all of H) if and only if the only element that is orthogonal to all elements of S is the zero element.

Definition (Direct Sum, Projection). Suppose that H is a Hilbert space with $X, Y \subseteq H$. Then H is the *direct sum* of X and Y if every $h \in H$ is a unique sum of a signal in X and a signal in Y: $h = x + y$. We write $H = X \oplus Y$ and, in this case, if $h = x + y$ with $x \in X$ and $y \in Y$, we say that x is the *projection* of h onto X and y is the projection of h onto Y.

The decomposition theorem tells us that a Hilbert space is the direct sum of any Hilbert subspace and its orthogonal complement. The direct sum decomposition of a Hilbert space leads naturally to a linear system.

Definition (Projection System). Let H be a Hilbert space and X a Hilbert subspace of H. The *projection from H to X* is the mapping $T\colon H \to X$ defined by $T(h) = x$, where $h = x + y$, with $y \in X^{\perp}$.

Remark. This definition makes sense (in other words, the mapping is *well-defined*) because there is a unique $x \in X$ that can be associated with any $h \in H$.

2.7.2.2 Convergence Criterion. Combining the decomposition system with orthogonality gives Hilbert space theory much of the power it has for application in signal analysis. Consider an orthonormal set of signals $\{u_k(n)\}$ in a Hilbert space H, a set of scalars $\{a_k\}$, and the sum

$$x(n) = \sum_{k=-\infty}^{\infty} a_k u_k(n). \tag{2.76}$$

This sum may or may not converge in H. If the sequence of partial sums $\{s_N(n)\}$

$$s_N(n) = \sum_{k=-N}^{N} a_k u_k(n). \tag{2.77}$$

is a Cauchy sequence, then (2.76) has a limit.

Theorem (Series Convergence Criterion). The sum (2.76) converges in Hilbert space H if and only if the signal $a(n) = a_n$ is in l^2.

Proof: Let $N > M$ and take the difference $s_N - s_M$ in (2.77). Then,

$$\|s_N - s_M\|^2 = |a_{-N}|^2 + |a_{-N+1}|^2 + \cdots + |a_{-M-1}|^2 + |a_{M+1}|^2 + \cdots + |a_N|^2, \tag{2.78}$$

because of the orthonormality of the $\{u_k(n)\}$ signal family. Thus, $d(s_N, s_M)$ tends to zero if and only if the sums of squares of the $|a_n|$ tend to zero. ■

Note too that if (2.76) converges, then the a_n are the Fourier coefficients of x with respect to the orthonormal family $\{u_k(n)\}$. This follows from taking the inner product of x with a typical u_k:

$$\langle x, u_k\rangle = \left\langle \sum_{i=-\infty}^{\infty} a_i u_i(n), u_k \right\rangle = \left\langle \lim_{N\to\infty} \sum_{i=-N}^{N} a_i u_i(n), u_k \right\rangle = a_k \langle u_k, u_k\rangle = a_k. \tag{2.79}$$

Therefore,

$$x(n) = \sum_{k=-\infty}^{\infty} \langle x, u_k\rangle u_k(n). \tag{2.80}$$

Thus, if the orthonormal family $\{u_k(n)\}$ is complete, then any $x(n)$ in H can be written as a limit of partial sums, and the representation (2.80) holds.

The theorem shows that there is a surprisingly close relationship between a general Hilbert space and the square-summable sequences l^2.

Orthonormal families and inner products are powerful tools for finding the significant components within signals. When does a Hilbert space have a complete orthonormal family? It turns out that every Hilbert space has a complete orthonormal family, a result that we will explain in a moment. There is also a method whereby any linearly independent set of signals in an inner product space can be converted into an orthonormal family.

2.7.2.3 Orthogonalization. Let us begin by showing that there is an algorithm, called *Gram–Schmidt*[14] *orthogonalization*, for converting a linearly independent set of signals into an orthormal family. Many readers will recognize the procedure from linear algebra.

Theorem (Gram–Schmidt Orthogonalization). Let H be a Hilbert space containing a linearly independent family $\{u_n\}$. Then there is an orthonormal family $\{v_n\}$ with each v_n in the linear span of $\{u_k: 0 \le k \le n\}$.

Proof: The proof is by induction on n. For $n = 0$, we can take $v_0 = \frac{u_0}{\|u_0\|}$. Now suppose that the algorithm works for $n = 0, 1, ..., k$. We want to show that the orthonormal elements can be expanded one more time, for $n = k + 1$. Let U be the subspace of H that consists of the linear span of $\{u_0, u_1, ..., u_k\}$. This is a Hilbert subspace; for instance, it is closed and therefore complete. Let $V = U^{\perp}$. By linear independence, u_{k+1} is not in U. This means that in the unique decomposition $u_{k+1} = u + v$, with u in U and v in V, we must have $v \ne 0$, the zero signal. If we set $v_{k+1} = \frac{v}{\|v\|}$, then $\|v_{k+1}\| = 1$; $v_{k+1} \in U^{\perp}$; and, because $v = u_{k+1} - u$, v_{k+1} is in the linear span of $\{u_i: 0 \le i \le k + 1\}$. ■

It is easier to find linearly independent than fully orthogonal signal families. So the Gram–Schmidt method is useful. The Gram–Schmidt procedure shows that if the linearly independent family is complete, then the algorithm converts it into a complete, orthonomal family.

[14]Erhard Schmidt (1876–1959), to whom the algorithm had been attributed, was Hilbert's student. Schmidt specified the algorithm in 1907. But it was discovered later that Jorgen Pedersen Gram (1850–1916) of Denmark had resorted to the same technique during his groundbreaking 1883 study on least squares approximation problems.

2.7.3 Orthonormal Bases

We now show how to build complete orthonormal families of signals in Hilbert space. That is, we want every element in the space to be approximated arbitrarily well by some linear combination of signals from the orthonormal family. Euclidean and unitary n-dimensional vector spaces all have orthonormal bases. This is a central idea in linear algebra. We are close to having shown the existence of orthonormal bases for general Hilbert spaces, too. But to get there with the Gram–Schmidt algorithm, we need to start with a complete (total) linearly independent family of signals. At this point, it is not clear that a general Hilbert space should even have a total linearly independent set.

Definition (Orthonormal Basis). In a Hilbert space, a complete orthonormal set is called an *orthonormal basis*.

We have already observed that the shifted unit sample signals are an orthonormal basis for the Hilbert space l^2. Remember the important distinction between this looser concept of basis and that for the finite-dimensional Euclidean and unitary spaces. In the cases of $\mathbb{R}^n$ and $\mathbb{C}^n$, the bases span the entire space. For some Hilbert spaces, however—and l^2 is a fine example—the linear combinations of the orthonormal basis signals only come arbitrarily close in norm to some signals.

2.7.3.1 Set Theoretic Preliminaries. There are some mathematical subtleties involved in showing that every Hilbert space has an orthonormal basis. The notions we need hinge on some fundamental results from mathematical set theory. A very readable introduction to these ideas is [27]. Most readers are probably aware that there are different orders of infinity in mathematics. (Those that are not may be in for a shock.) The number of points on a line (i.e., the set of real numbers) is a larger infinity than the natural numbers, because $\mathbb{R}$ cannot be placed in a one-to-one correspondence with $\mathbb{N}$. We say that two sets between which a one-to-one map exists have the same *cardinality.* The notation for the cardinality of a set X is $|X|$. In fact, the natural numbers, the integers, the rational numbers, and even all the real numbers which are roots of rational polynomials have the same cardinality, $|\mathbb{N}|$. They are called *countable* sets, because there is a one-to-one and onto map from $\mathbb{N}$, the counting set, to each of them. The real numbers are an *uncountable* set. Also uncountable is the set of subsets of the natural numbers, called the *power set* of $\mathbb{N}$, written $\mathcal{P}(\mathbb{N})$. It turns out that $|\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$. The discovery of different orders of infinity—different cardinalities—is due to Cantor.[15]

[15] Georg Cantor (1845–1918) worked himself to the point of physical, emotional, and mental exhaustion trying to demonstrate the *continuum hypothesis*: there is no cardinality of sets in between $|\mathcal{N}|$ and $|\mathcal{R}|$. He retreated from set theory to an asylum, but never proved or disproved the continuum hypothesis. It is a good thing, too. In 1963, Paul Cohen proved that the continuum hypothesis is independent of the usual axioms of set theory; it can be neither proved nor disproved! [K. Devlin, *Mathematics: The Science of Patterns*, New York: Scientific American Library, 1994.]

Some basic facts about countable sets are as follows (exercises):

(i) The Cartesian product $X \times Y$ of two countable sets is countable.
(ii) The Cartesian product $X_1 \times X_2 \times \cdots \times X_n$ of a finite number of countable sets is countable.
(iii) A countable union of countable sets is countable.
(iv) The set that consists of all finite subsets of a countable set is countable.
(v) The set of all subsets of a set X always has a larger cardinality than X; in other words, $|X| < |\mathcal{P}(X)|$.

Observe carefully that indexing notation presupposes a one-to-one, onto map from the indexing set to the indexed set. Suppose X is a countable set—for example, the set of shifted impulses, $X = \{\delta(n-k)\text{: } k \text{ an integer}\}$. We can index X by $\mathbb{N}$ with the map $f(k) = \delta(n-k)$. Trivially, f is a one-to-one and onto map of $\mathbb{N}$ to X. Now let $Y = \{a\delta(n)\text{: } a \text{ is a real number}\}$ be the set of amplified unit impulse signals. It is impossible to index Y with the natural numbers, because Y has the same cardinality as the real line. Instead, if it is necessary to index such a collection, we must pick an indexing set that has the same cardinality as Y.

2.7.3.2 Separability. We draw upon these set theoretic ideas in order to show that every Hilbert space has an orthonormal basis. In particular, we need to bring the notion of cardinality into the discussion of Hilbert space and to invoke another concept from set theory—the Axiom of Choice.

Definition (Separable Hilbert Space). A Hilbert space is *separable* if it contains a countable dense set.

Notice that l^2 is a separable Hilbert space. The set of shifted impulse signals is an orthonormal basis for l^2. Now the set of all scalar multiples of linear combinations of the shifted impulses is not countable, because there are an uncountable number of magnitude values possible. However, we can get arbitrarily close to a linear combination of shifted impulses with a linear combination that has rational coefficients. There are a countable number of rationals. The set of finite sequences of rationals is therefore countable. Thus, the set of linear combinations of shifted impulses with rational coefficients is a countable dense subset of l^2.

Let's continue this line of reasoning and assume that we have a countable dense subset S of a Hilbert space H. We wish to fashion S into an orthonormal basis. We may write the dense family using the natural numbers as an indexing set: $S = \{s_n\text{: } n \text{ is in } \mathbb{N}\}$. If S is a linearly independent family, then the Gram–Schmidt procedure applies, and we can construct from S an orthonormal family that is still dense in H. Thus, in this case, H has a countable basis. If S is has some linear dependency, we can pick the first element of S, call it d_0, that is a linear combination of the previous ones. We delete d_0 from S to form S_0, which still has the same linear span, and hence is just as dense as S in H. Continue this process for all natural numbers, finding d_{n+1} and cutting it from S_n to produce S_{n+1}. The result is a linearly independent set, S_ω. If S_ω is not linearly independent, then there is an element that is a linear combination of the others; call it t. We see immediately a contradiction, because t had to be chosen from the orthogonal complement of the elements that we chose before it and because the elements that were chosen later had to be orthogonal—and therefore linearly independent—to t. We note as well that S_ω has a linear span which is dense in H; and, using the Gram–Schmidt algorithm, it can be sculpted into an orthonormal basis for H.

Without separability of the Hilbert space H, the above argument breaks down. We could begin an attack on the problem by assuming a dense subset $S \subseteq H$. But what subsets, other than H itself, can we assume for a general, abstract Hilbert space? Examining the separable case's argument more closely, we see that we really built up a linearly independent basis incrementally, beginning from the bottom $s_0 \in S$. Here we can begin with some nonzero element of H, call it s_a, where we index by some other set A that has sufficient cardinality to completely index the orthonormal set we construct. If the linear span of $\{s_a\}$ includes all of H, then we are done; otherwise, there is an element in the orthogonal complement of the Hilbert subspace spanned by $\{s_a\}$. Call this element s_b. Then $\{s_a, s_b\}$ is a linearly independent set in H. Continue the process: Check whether the current set of linearly independent elements has a dense linear span; if not, select a vector from the orthogonal complement, and add this vector to the linearly independent family. In the induction procedure for the case of a separable H, the ultimate completion of the construction was evident. Without completion a contradiction arises. For if our "continuation" on the natural numbers does not work, can we find a least element that is a linear combination of the others, leading to a contradiction. But how can we find a "least" element of the index set A in the nonseparable case? We do not even know of an ordering for A. Thus there is a stumbling block in showing the existence of an orthonormal basis for a nonseparable Hilbert space.

2.7.3.3 Existence. The key is an axiom from set theory, called the *Axiom of Choice*, and one of its related formulations, called *Zorn's lemma.*[16] The Axiom of Choice states that the Cartesian product of a family of sets $\{S_a : a \in A\}$ is not empty. That is, $P = \{(s_a, s_b, s_c, \ldots) : s_a \in S_a, s_b \in S_b, \ldots\}$ has at least one element. The existence of an element in P means that there is a way to simultaneously choose one element from each of the sets S_a of the collection S. Zorn's lemma seems to say nothing like this. The lemma states that if a family of sets $S = \{S_a : a \in A\}$ has the property that for every chain $S_a \subseteq S_b \subseteq \ldots$ of sets in S, there is a T in S that is a superset of each of the chain elements, then S itself has an element that is contained properly in no other element of S; that is, S has a *maximal* set. Most people are inclined to think that the Axiom of Choice is obviously true and that Zorn's lemma is very suspicious, if not an outright fiction. On the contrary: Zorn's lemma is true if and only if the Axiom of Choice is true [27].

[16]Algebraist Max Zorn (1906–1993) used his *maximal set principle* in a 1935 paper.

Let us return now to our problem of constructing a dense linearly independent set in a Hilbert space H and apply the Zorn's lemma formulation of the Axiom of Choice. In a Hilbert space, the union of any chain of linearly independent subsets is also linearly independent. Thus, H must have, by Zorn, a maximal linearly independent set S. We claim that K, the linear span of S, is dense. Suppose not. Now K is a Hilbert subspace. So there is a vector v in the orthogonal complement to K. Contradiction is imminent. The set $S \cup \{v\}$ is linearly independent and properly includes S; this is impossible since S was selected to be maximal. So S must be complete (total). Its linear span is dense in H. Now we apply the Gram–Schmidt procedure to S. One final obstacle remains. We showed the Gram–Schmidt algorithm while using the natural numbers as an index set, and thus implicitly assumed a countable collection! We must not assume this now. Instead we apply Zorn's lemma to the Gram–Schmidt procedure, finding a maximal orthonormal set with same span as S. We have, with the aid of some set theory, finally shown the following.

Theorem (Existence of Orthonormal Bases). Every Hilbert space contains an orthonormal basis. ■

If the Hilbert space is spanned by a finite set of signals, then the orthonormal basis has a finite number of elements. Examples of finite-dimensional Hilbert spaces are the familiar Euclidean and unitary spaces. If the Hilbert space is separable, but is not spanned by a finite set, then it has a countably infinite orthonormal basis. Lastly, there are cases of Hilbert spaces which are not separable.

2.7.3.4 Fourier Series. Let us complete this chapter with a theorem that wraps up many of the ideas of discrete signal spaces: orthonormal bases, Fourier coefficients, and completeness.

Theorem (Fourier Series Representation). Let H be a Hilbert space and let $S = \{u_a : a \in A\}$ be an orthonormal family in H. Then,

(i) Any $x \in H$ has at most countably many nonzero Fourier coefficients with respect to the u_a.

(ii) S is complete (its linear span is dense in H) if and only if for all signals $x \in H$ we have

$$\|x\|^2 = \sum_{a \in A} |\langle x, u_a \rangle|^2, \tag{2.81}$$

where the sum is taken over all a, such that the Fourier coefficient of x with respect to u_a is not zero.

(iii) (Riesz–Fischer Theorem[17]) If $\{c_a : a \in A\}$ is a set of scalars such that

[17]Hungarian Frigyes Riesz (1880–1956) and Austrian Ernst Sigismund Fischer (1875–1954) arrived at this result independently in 1907 [22].

$$\sum_{a \in A} |c_a|^2 < \infty , \tag{2.82}$$

then there is a unique x in H such that $\langle x, u_a \rangle = c_a$, and

$$x = \sum_{a \in A} c_a u_a . \tag{2.83}$$

Proof: We have already used most of the proof ideas in previous results.

(i) The set of nonzero Fourier coefficients of x with respect to the u_a is the same as the set of Fourier coefficients that are greater than $1/n$ for some integer n. Since there can only be finitely many Fourier coefficients that are greater than $1/n$, we must have a countable union of finite sets, which is still countable. Therefore, there may only be a countable number of $\langle x, u_a \rangle \neq 0$.

(ii) Suppose first that S is complete and $x \in H$. Since there can be at most a countably infinite number of nonzero Fourier coefficients, it is possible to form the series sum,

$$s = \sum_{a \in A} \langle x, u_a \rangle u_a . \tag{2.84}$$

This sum converges by the Bessel inequality for inner product spaces. Consider $t = s - x$. It is easy to see that $t \in S^{\perp}$ by taking the inner product of t with each $u_a \in S$. But since S is complete, this means that there can be no nonzero element in its orthogonal complement; in other words, $t = 0$ and $s = x$. Now, since $\langle u_a, u_b \rangle \neq 0$ when $a \neq b$, we see that

$$\|x\|^2 = \langle x, x \rangle = \left\langle \sum_{a \in A} \langle x, u_a \rangle u_a, \sum_{a \in A} \langle x, u_a \rangle u_a \right\rangle = \sum_{a \in A} \langle x, u_a \rangle \overline{\langle x, u_a \rangle}. \tag{2.85}$$

Next, suppose that the relation (2.81) holds for all x. Assume for the sake of contradiction that S is not complete. Then by the Hilbert space decomposition theorem, we know that there is some nonzero $x \in S^{\perp}$. This means that $\langle x, u_a \rangle = 0$ for all u_a and that the sum (2.81) is zero. The contradiction is that now we must have $x = 0$, the zero signal.

(iii) If $\{c_a : a \in A\}$ is a set of scalars such that (2.82) holds, then at most a countable number of them can be nonzero. This follows from an argument similar to the proof of (i). Since we have a countable collection in (2.82), we may use the Hilbert space series convergence criterion, which was stated (implicitly at that point in the text) for a countable collection. ■

An extremely powerful technique for specifying discrete systems follows from these results. Given a Hilbert space, we can find an orthonormal basis for it. In the case of a separable Hilbert space, there is an iterative procedure to find a linearly independent family and orthogonalize it using the Gram–Schmidt algorithm. If the

Hilbert space is not separable, then we do not have such a construction. But the existence of the orthonormal basis $U = \{u_a : a \in A\}$ is still guaranteed by Zorn's lemma. Now suppose we use the orthonormal basis to analyze a signal. Certain of the basis elements, $V = \{v_b : b \in B \subseteq A\}$, have features we seek in general signals, x. We form the linear system $T(x) = y$, defined by

$$Tx = y = \sum_{b \in B} \langle x, v_b \rangle v_b . \tag{2.86}$$

Now the signal y is that part of x that resembles the critical basis elements. Since the theorem guarantees that we can expand any general element in terms of the orthonormal basis U, we know that the sum (2.86) converges. We can tune our linear system to provide precisely the characteristics we wish to preserve in or remove from signal x by selecting the appropriate orthonormal basis elements. Once the output $y = Tx$ is found, we can find the features we desire in x more easily in y. Also, y may prove that x is desirable in some way because it has a large norm; that is $\|x\| \approx \|y\|$. And, continuing this reasoning, y may prove that x is quite undesirable because $\|y\|$ is small.

In its many guises, we will be pursuing this idea for the remainder of the book.

2.8 SUMMARY

This chapter began with a practical—perhaps even naïve—exploration of the types of operations that one can perform on signals. Many of these simple systems will arise again and again as we develop methods for processing and interpreting discrete and continuous signals. The later chapters will demonstrate that the most important type of system we have identified so far is the linear, time-invariant system. In fact, the importance of the characterization result, the convolution theorem for LTI systems, cannot be overemphasized. This simple result underlies almost all of our subsequent work. Some of the most important concepts in signal filtering and frequency analysis depend directly on this result.

Our explorations acquire quite a bit of mathematical sophistication, however, when we investigate the closure properties of our naively formulated signal processing systems. We needed some good answers for what types of signals can be used with certain operations. It seems obvious enough that we would like to be able to sum any two signals that we consider, and this is clearly feasible for finitely supported signals. For other signals, however, this simple summing problem is not so swiftly answered. We need a formal mathematical framework for signal processing and analysis. Inspired by basic vector space properties, we began a search for the mathematical underpinnings of signal theory with the idea of a normed space. The l^p Banach spaces conveniently generalize some natural signal families that we first encountered in Chapter 1. Moreover, these spaces are an adequate realm for developing the theory of signals, stable systems, closure, convolution, and convergence of signals.

Unfortunately, except for l^2, none of the l^p spaces support an inner product definition that is related to the norm, $\| \|_p$. This is a profoundly negative result. But it once again shows the unique nature of the l^2 space. Of the l^p Banach spaces, only l^2 can be equipped with an inner product that makes it into a Hilbert space. This explains why finite-energy signals are so often the focus of signal theory. Only the l^2 Hilbert space, or one of its closed subspaces, has all of the features from Euclidean vector spaces that we find so essential for studying signals and systems.

We see that all Hilbert spaces have orthonormal bases, whether they are finite, countable, or uncountable. Furthermore, a close link exists between orthonormal bases for Hilbert spaces and linear systems that map one signal to another yet retain only desirable properties of the input. We will see in the sequel that it is possible to find special orthonormal bases that provide for the efficient extraction of special characteristics of signals, help us to find certain frequency and scale components of a signal, and, finally, allow us to discover the structure and analyze a signal.

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PROBLEMS

1. Find the domain and range of the following systems:

(a) The amplifier system: $y(n) = Ax(n)$.

(b) A translation system: $y(n) = x(n - k)$.

(c) The discrete system on real-valued signals, $y(n) = x(n)^{1/2}$.

(d) The discrete system on complex-valued signals, $y(n) = x(n)^{1/2}$.

(e) An adder: $y(n) = x(n) + x_0(n)$.

(f) Termwise multiplication (modulation): $y(n) = x(n)x_0(n)$.

(g) Convolution: $y(n) = x(n)*h(n)$.

(h) Accumulator: $y(n) = x(n) + y(n-1)$.

2. Consider the LTI system $y = Hx$ that satisfies a linear, constant-coefficient difference equation

$$y(n) = \sum_{k=1}^{K} a_k y(n-k) + \sum_{m=0}^{M} b_m x(n-m). \tag{2.87}$$

Show that any K successive values of the output $h = H\delta$ are sufficient to characterize the system.

3. Consider an LTI system $y = Hx$ that satisfies the difference equation (2.87).

(a) Give the homogeneous equation corresponding to (2.87).

(b) Show that if (x, y) is a solution pair for (2.87) and y_h is a solution of its homogeneous equation, then $(x, y + y_h)$ is a solution of the difference equation.

(c) Show that if (x, y_1) and (x, y_2) are solution pairs for (2.87), then $y_1 - y_2$ is a solution to the homogeneous equation in (a).

4. Consider the LTI system $y = Hx$ that satisfies a linear, constant-coefficient difference equation (2.87). Prove that if the signal pair (δ, h) satisfies the difference equation and $y = x*h$, then the pair (x, y) also satisfies the difference equation.

5. Prove the converse of the convolution theorem for LTI Systems: Let $h(n)$ be a discrete signal and H be the system defined by $y = Hx = x*h$. Then H is LTI and $h = H\delta$.

6. Suppose $x(n)$ is in l^p, $1 \le p \le \infty$. Let c be a scalar (real or complex number) and let k be an integer. Show the following closure rules:

(a) $cx(n)$ is in l^p and $\|cx\|_p = |c|\ \|x\|_p$.

(b) $x(k - n)$ is in l^p and $\|x(k - n)\|_p = \|x(n)\|_p$.

7. Show that the signal space l^p is a normed space. The triangle inequality of the norm is proven by Minkowski's inequality. It remains to show the following:

(a) $\|x\|_p \ge 0$ for all x.

(b) $\|x\|p = 0$ if and only if $x(n) = 0$ for all n.

(c) $\|ax\|_p = |a|\ \|x\|_p$ for all scalars a and all signals $x(n)$.

8. Let p and q be conjugate exponents. Show the following:

(a) $(p + q)/pq = 1$.

(b) $pq = p + q$.

(c) $(p - 1)(q - 1) = 1$.

(d) $(p - 1)^{-1} = q - 1$.

(e) If $u = t^{p-1}$, then $t = uq^{-1}$.

9. Show that the l^p spaces are complete, $1 \le p < \infty$. Let $\{x_k(n)\}$ be a Cauchy sequence of signals in l^p.

(a) Show that for any integer n, the values of the signals in the sequence at time instant n are a Cauchy sequence. That is, with n fixed, the sequence of scalars $\{x_k(n)$: k an integer$\}$ is a Cauchy sequence.

(b) Since the real (complex) numbers are complete, we can fix n, and take the limit

$$c_n = \lim_{k \to \infty} x_k(n)\,. \tag{2.88}$$

Show that the signal defined by $x(n) = c_n$ is in l^p.

(c) Show that

$$\lim_{k \to \infty} \|x_k - x\|_p = 0, \tag{2.89}$$

so that the signals x_k converge to x in the l^p distance measure d_p.

10. Show that the l^∞ signal space is complete.

11. Let I be an inner product space. Show that the inner product is continuous in I; that is if $x_n \to x$ and $y_n \to y$, then $\langle x_n, y_n \rangle \to \langle x, y \rangle$.

12. Show that orthogonal signals in an inner product space are linearly independent.

13. Let I be an inner product space and $d(u, v) = \|u - v\|$ be its distance measure. Show that with the distance measure $d(u, v)$, I is a metric space:

(a) $d(u, v) \geq 0$ for all u, v.

(b) $d(u, v) = 0$ if and only if $u = v$.

(c) $d(u, v) = d(v, u)$ for all u, v.

(d) For any w, $d(u, v) \leq d(u, w) + d(w, v)$.

14. Show that the *discrete Euclidean* space $\mathbb{Z}^n = \{(k_1, k_2, \cdots, k_n) \mid k_i \text{ is in } \mathbb{Z}\}$ is a metric space. Is it a normed linear space? Explain.

15. Show that if for the Euclidean space $\mathbb{R}^n$, we define the metric $d((u_1, u_2, ..., u_n), (v_1, v_2, \cdots, v_n)) = |u_1 - v_1| + |u_2 - v_2| + \cdots + |u_n - v_n|$, then $(\mathbb{R}^n, d)$ is a metric space.

16. Show that the following sets are countable.

(a) The integers $\mathbb{Z}$ by arranging them in two rows:

0, 2, 4, 6, ...

1, 3, 5, 7, ...

and enumerating them with a zigzag traversal.

(b) All ordered pairs in the Cartesian product $\mathbb{N} \times \mathbb{N}$.

(c) The rational numbers $\mathbb{Q}$.

(d) All ordered k-tuples of a countable set X.

(e) Any countable union of countable sets.