CHAPTER 8

The z-Transform

The z-transform generalizes the discrete-time Fourier transform. It extends the domain of the DTFT of $x(n)$, the periodic analog signal $X(\omega)$, which is defined for $\omega \in \mathbb{R}$, to a function defined on $z \in \mathbb{C}$, the complex plane.

The motivations for introducing the z-transform are diverse:

- It puts the powerful tools of complex analysis at our disposal.
- There are possibilities for analyzing signals for which the DTFT analysis equation does not converge.
- It allows us to study linear, translation-invariant systems for which the frequency response does not exist.
- Some specialized operations such as signal subsampling and upsampling are amenable to z-transform techniques.
- It provides a compact notation, convenient for describing a variety of systems and their properties.

Having listed these z-transform benefits, we hasten to add that very often a series of z-transform manipulations concludes with a simple restriction of the transform to the DTFT. So we will not be forgetting the DTFT; on the contrary, it is the basic tool that we will be using for the spectral analysis of aperiodic discrete signals for the remainder of the book.

The development of z-transform theory proceeds along lines similar to those used in Chapter 7 for the DTFT. Many of the proofs of z-transform properties, for example, are very like the corresponding derivations for the DTFT. We often leave these results as exercises, and by now the reader should find them straightforward. This is a short chapter. It serves as a bridge between the previous chapter's theoretical treatment of discrete Fourier transforms and the diverse applications—especially filter design techniques—covered in Chapter 9.

Texts on systems theory introduce the z-transform and its analog world cousin, the Laplace transform [1, 2]. Books on digital signal processing [3–7] cover the z-transform in more detail. Explanations of the transform as a discrete filter design tool may also be found in treatments oriented to specific applications [8–10]. The

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z-transform was applied in control theory [11–13] long before it was considered for the design of digital filters [14]. Specialized treatises include Refs. 15 and 16.

8.1 CONCEPTUAL FOUNDATIONS

The z-transform has a very elegant, abstract definition as a power series in a complex variable. This power series is two-sided; it has both positive and negative powers of z, in general. Furthermore, there may be an infinite number of terms in the series in either the positive or negative direction. Like the DTFT, the theory of the z-transform begins with an investigation of when this doubly-infinite summation converges. Unlike the DTFT, however, the z-transform enlists a number of concepts from complex analysis in order to develop its existence and inversion results. This section introduces the z-transform, beginning with its abstract definition and then considering some simple examples.

Readers may find it helpful to review the complex variables tutorial in Chapter 1 (Section 1.7) before proceeding with the z-transform.

8.1.1 Definition and Basic Examples

The z-transform generalizes the DTFT to a function defined on complex numbers. To do this, we replace the complex exponential in the DTFT's definition with $z \in \mathbb{C}$. A simple change it is, but we shall nevertheless face some interesting convergence issues. For our effort, we will find that many of the properties of the DTFT carry through to the extended transform, and they provide us with tools for analyzing signals and systems for which the DTFT is not well-suited.

Definition (z-Transform). If $x(n)$ is a discrete signal and $z \in \mathbb{C}$, then its *z-transform,* $X(z)$, is defined by

$$X(z) = \sum_{n=-\infty}^{+\infty} x(n) z^{-n}. \tag{8.1}$$

To avoid some notation conflicts, the fancy-z notation, $X = \mathcal{Z}(x)$, is often convenient for writing the z-transform of $x(n)$. The signal $x(n)$ and the complex function $X(z)$ are called a z-transform pair. We also call (8.1) the z-transform *analysis equation*. Associated with a z-transform pair is a *region of convergence* (the standard acronym is ROC): $\mathrm{ROC}_X = \{z \in \mathbb{C}: X(z) \text{ exists}\}$. Sometimes as $|z|$ gets large, the value $X(z)$ approaches a limit. In this case, it is convenient to indicate that $\infty \in \mathrm{ROC}_X$. The notation $\mathbb{C}^+$ is useful for the so-called *extended complex plane*: $\mathbb{C}$ augmented with a special element, ∞.

Let us postpone, for a moment, convergence questions pertaining to the z-transform sum. Note that taking the restriction of complex variable z to the unit circle, $z = \exp(j\omega)$, and inserting this in (8.1), gives the DTFT. The DTFT is the restriction of the z-transform to the unit circle of the complex plane, $|z| = 1$: $X[\exp(j\omega)] = X(\omega)$, where the first "$X$" is the z-transform, and the second "X" is the DTFT of $x(n)$, respectively.

There is another form of the z-transform that uses only the causal portion of a signal.

Definition (One-sided z-Transform). If $x(n)$ is a discrete signal and $z \in \mathbb{C}$, then its *one sided z-transform,* $X^{+}(z)$, is defined by

$$X^{+}(z) = \sum_{n=0}^{+\infty} x(n) z^{-n}, \tag{8.2}$$

The one-sided, or unilateral, z-transform is important for the specialized problem of solving linear, constant-coefficient difference equations. Typically, one is given difference equations and initial conditions at certain time instants. The task is to find all the discrete signal solutions. The one-sided z-transform agrees with the standard two-sided transform on signals $x(n) = 0$ for $n < 0$. The linearity property is the same, but the shifting property differs. These ideas and an application are considered in the problems at the end of the chapter.

As with the DTFT, the infinite sum in the z-transform summation (8.1) poses convergence questions. Of course, the sum exists whenever the signal $x(n)$ has finite support; the corresponding z-transform $X(z)$ is a sum of powers (positive, zero, and negative) of the complex variable z. Let us consider some elementary examples of finding the z-transforms of discrete signals. Finding such z-transform pairs, $x(n)$ and $X(z)$, is typically a matter of finding the z-transform of a signal $y(n)$ which is similar to $x(n)$ and then applying the z-transform properties to arrive at $X(z)$ from $Y(z)$.

Example (Discrete Delta). Let us start simple by considering the discrete delta signal, $\delta(n)$. For any $z \in \mathbb{C}$, only the summand corresponding to $n = 0$ is nonzero in (8.1), and thus $\Delta(z) = \mathcal{Z}(\delta)(z) = 1$ for all $z \in \mathbb{C}$.

Example (Square Pulse). Again, let us consider the impulse response of the moving average system, H. It has impulse response $h(n) = [1, 1, \underline{1}, 1, 1]$; in other words, $h(n) = 1$ for $-2 \le n \le 2$, and $h(n) = 0$ otherwise. We write immediately

$$H(z) = \sum_{n=-\infty}^{+\infty} h(n) z^{-n} = \sum_{n=-2}^{+2} z^{-n} = z^{2n} + z^{n} + 1 + z^{-n} + z^{-2n}. \tag{8.3}$$

Note that $H(z)$ exists for all $z \in \mathbb{C}$, $z \neq 0$. Thus, a signal, $x(n)$, whose DTFT converges for all $\omega \in \mathbb{R}$ may have a z-transform, $X(z)$, which does not converge for all $z \in \mathbb{C}$. In general, a finitely supported signal, $x(n)$, that is nonzero for positive time instants will not have a z-transform, $X(z)$, which exists for $z = 0$.

Example (Causal Exponential Signal). Consider the signal $x(n) = a^{n}u(n)$. We calculate

$$X(z) = \sum_{n=-\infty}^{+\infty} x(n) z^{-n} = \sum_{n=0}^{\infty} a^{n} z^{-n} = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^{n} = \frac{1}{1-\frac{a}{z}} = \frac{z}{z-a}, \tag{8.4}$$

where the geometric series sums in (8.4) to $z/(z-a)$ provided that $|a/z| < 1$. Thus, we have $\mathrm{ROC}_X = \{z \in \mathbb{C}: |a/z| < 1\} = \{z \in \mathbb{C}: |a| < |z|\}$. In other words, the region of convergence of the z-transform of $x(n) = a^n u(n)$ is all complex numbers lying outside the circle $|z| = a$. In particular, the unit step signal, $u(n)$, has a z-transform, $U(z) = 1/(1 - z^{-1})$. We may take $a = 1$ above and find thereby that $\mathrm{ROC}_U = \{z \in \mathbb{C}: 1 < |z|\}$.

This example shows that a z-transform can exist for a signal that has no DTFT. If $|a| > 1$ in the previous example, for instance, then the analysis equation for $x(n)$ does not converge. But the z-transform, $X(z)$, does exist, as long as z lies outside the circle $|z| = a$ in the complex plane. Also, the ROC for this example was easy to discover, thanks to the geometric series form taken by the z-transform sum. There is a companion example, which we need to cover next. It illustrates the very important point that the ROC can be the only distinguishing feature between the z-transforms of two completely different signals.

Example (Anti-causal Exponential Signal). Consider the signal $y(n) = -a^n u(-n - 1)$. Now we find

$$Y(z) = \sum_{n=-\infty}^{+\infty} y(n) z^{-n} = \sum_{n=-\infty}^{-1} -a^n z^{-n} = -\sum_{n=1}^{\infty} \left(\frac{z}{a}\right)^n = -\frac{z}{a} \sum_{n=0}^{\infty} \left(\frac{z}{a}\right)^n$$
$$= -\left(\frac{\frac{z}{a}}{1 - \frac{z}{a}}\right) = \frac{z}{z - a}, \tag{8.5}$$

with the convergence criterion $|z/a| < 1$. In this case, we have $\mathrm{ROC}_Y = \{z \in \mathbb{C}: |z/a| < 1\} = \{z \in \mathbb{C}: |z| < |a|\}$. The region of convergence of the z-transform of $y(n) = -a^n u(-n - 1)$ is all complex numbers lying inside the circle $|z| = a$.

The upshot is that we must always be careful to specify the region of convergence of a signal's z-transform. In other words, given one algebraic expression for $X(z)$, there may be multiple signals for which it is the z-transform; the deciding factor then becomes the region of convergence.

8.1.2 Existence

Demonstrating the convergence of the z-transform for a particular signal makes use of complex variable theory. In particular, the z-transform is a Laurent series [17, 18].

Definition (Power and Laurent Series). A complex power series is a sum of scaled powers of the complex variable z:

$$\sum_{n=0}^{+\infty} a_n z^n. \tag{8.6}$$

A Laurent series is a two-sided series of the form

$$\sum_{n=-\infty}^{+\infty} a_n z^{-n} = \sum_{n=1}^{+\infty} a_n z^{-n} + \sum_{n=-\infty}^{0} a_n z^{-n}, \tag{8.7}$$

where the a_n are (possibly complex) coefficients. (Most mathematics texts do not use the negative exponent in the definition—their term z^n has coefficient a_n; our definition goes against the tradition so that its form more closely follows the definition of the z-transform.) One portion of the series consists of negative powers of z, and the other part consists of non-negative powers of z. We say the Laurent series (8.7) converges for some $z \in \mathbb{C}$ if both parts of the series converge.

Obviously, we are interested in the situation were the Laurent series coefficients are the values of a discrete signal, $x(n) = a_n$. From complex variable theory, which we introduced in the first chapter, the following results are relevant to Laurent series convergence. We prove the first result for the special case where the z-transform of $x(n)$ contains only non-negative powers of z; that is, it is a conventional power series in z. This will happen when $x(n) = 0$ for $n > 0$.

The next definition identifies upper and lower limit points within a sequence.

Definition (lim sup and lim inf). Let $A = \{a_n: 0 \le n < \infty\}$. Define $A_N = A \setminus \{a_n: 0 \le n < N\}$, which is the set A after removing the first N elements. Let κ_N be the least upper bound of A_N. Then the limit, κ, of the sequence $\{\kappa_N: N > 0\}$ is called the *lim sup* of A, written

$$\kappa = \lim_{N\to\infty} \{\kappa_N: N > 0\} = \lim_{n\to\infty} \sup A = \lim_{n\to\infty} \sup\{a_n: 0 \le n < \infty\}. \tag{8.8}$$

Similarly, if we let λ_N be the greatest lower bound of A_N, then the limit of the sequence $\{\lambda_N: N > 0\}$ is called the *lim inf*[1] of A:

$$\lambda = \lim_{N\to\infty} \{\lambda_N: N > 0\} = \lim_{n\to\infty} \inf A = \lim_{n\to\infty} \inf\{a_n: 0 \le n < \infty\}. \tag{8.9}$$

Sometimes a sequence of numbers does not have a limit, but there are convergent subsequences within it. The lim sup is the largest limit of a convergent subsequence, and the lim inf is the smallest limit of a convergent subsequence, respectively. The sequence has a limit if the lim sup and the lim inf are equal. Readers with advanced calculus and mathematical analysis background will find these ideas familiar [19–21]. We need the concept of the lim sup to state the next theorem. Offering a convergence criterion for a power series, it is a step toward finding the ROC of a z-transform.

Theorem (Power Series Absolute Convergence). Suppose $x(n)$ is a discrete signal; its z-transform, $X(z)$, has only non-negative powers of z,

$$X(z) = \sum_{n=0}^{+\infty} a_n z^n; \tag{8.10}$$

[1]These are indeed the mathematical community's standard terms. The lim sup of a sequence is pronounced "lim soup," and lim inf sounds just like its spelling.

and

$$\kappa = \lim_{n \to \infty} \sup\{|a_n|^{1/n} : 1 \le n < \infty\}. \tag{8.11}$$

Then $X(z)$ converges absolutely for all z with $|z| < \kappa^{-1}$ and diverges for all $|z| > \kappa^{-1}$. (This allows A to be unbounded, in which case $\kappa = \infty$ and, loosely speaking, $\kappa^{-1} = 0$.)

Proof: Consider some z such that $|z| < \kappa^{-1}$, and choose $\lambda > \kappa$ so that λ^{-1} lies between these two values: $|z| < \lambda^{-1} < \kappa^{-1}$. Because $\kappa = \lim\sup A$, there is an N such that $|a_n|^{1/n} < \lambda$ for all $n > N$. But this implies that $|a_n z^n| < |z\lambda|^n$ for $n > N$. Since $|z\lambda| < 1$ by the choice of λ, the power series (8.10) is bounded above by a convergent geometric series. The series must therefore converge absolutely. We leave the divergence case as an exercise. ∎

Definition (Radius of Convergence). Let $\rho = \kappa^{-1}$, where κ is given by (8.11) in the Power Series Absolute Convergence Theorem. Then ρ is called the *radius of convergence* of the complex power series (8.10).

Corollary (Power Series Uniform Convergence). Suppose $x(n)$ is a discrete signal; its z-transform, $X(z)$, has only non-negative powers of z as in the theorem (8.10); and κ is defined as in (8.11). Then for any $0 < R < \rho = \kappa^{-1}$, $X(z)$ converges uniformly in the complex disk $\{z \in \mathbb{C}: |z| < R < \rho = \kappa^{-1}\}$.

Proof: For any disk of radius R, $0 < R < \rho$, the proof of the theorem implies that there is a convergent geometric series that bounds the power series (8.10). Since the convergence of the dominating geometric series does not depend on z, the sum of the series in z (8.10) can be made arbitrarily close to its limit independent of z. The convergence is therefore uniform. ∎

Corollary (Analyticity of the Power Series Limit). Again, if $x(n)$ is a discrete signal; its z-transform, $X(z)$, has the form (8.10); and $\rho = \kappa^{-1}$ is given by (8.11), then $X(z)$ is an analytic function, the derivative $X'(z) = dX(z)/dz$ can be obtained by termwise differentiation of the power series (8.10), and $\mathrm{ROC}_X = \mathrm{ROC}_{dX/dz}$.

Proof: This proof was given already in Section 1.7, where we assumed the uniform convergence of the power series. From the Uniform Convergence Corollary, we know this to be the case within the radius of convergence, ρ; the result follows. ∎

Finally, we consider the situation that most interests us, the Laurent series. The z-transform assumes the form of a Laurent series. We have, in fact, already developed the machinery we need to discover the region of convergence of a z-transform. We apply the Power Series Convergence Theorems above for both parts of the Laurent series: the negative and non-negative powers of z.

Theorem (z-Transform Region of Convergence). Let $x(n)$ be a discrete signal and let $X(z) = X_1(z) + X_2(z)$ be its z-transform (which may be two-sided). Suppose

$X(z) = X_1(z) + X_2(z)$, where $X_2(z)$ consists on non-negative powers of z, and $X_1(z)$ contains only negative powers of z. Then $X(z)$ converges absolutely within an annulus of the complex plane, $\mathrm{ROC}_X = \{z \in \mathbb{C}: \rho_1 < |z| < \rho_2\}$, where ρ_1 and ρ_2 are the radii of convergence of $X_1(z)$ and $X_2(z)$, respectively. The convergence of the z-transform Laurent series is uniform within any closed annulus contained in ROC_X, and its limit, $X(z)$ is analytic within this same closed annulus.

Proof: $X(z)$'s true power series portion, $X_2(z)$, converges inside some circle $|z| = \rho_2$, where ρ_2 is the radius of convergence. The $X_1(z)$ portion of $X(z)$ converts to power series form by setting $w = z^{-1}$. Then the radius of convergence may be found for the power series $Y(w) = X_1(w^{-1}) = X_1(z)$. $Y(w)$ converges inside some circle of radius R_1, say, which means $X_1(z)$ converges outside the circle $\rho_1 = 1/R_1$. The region formed by intersecting the exterior of the circle $|z| = \rho_1$ and the interior of the circle $|z| = \rho_2$ is the annulus we seek. ■

Example. Suppose $x(n)$ is given as follows:

$$x(n) = \begin{cases} 3^{-n} & \text{for } n \geq 0, \\ -4^{n} & \text{for } n < 0. \end{cases} \tag{8.12}$$

Let $x_1(n) = a^n u(n)$, where $a = 1/3$, and $x_2(n) = -b^n u(-n-1)$, where $b = 4$. Then $x(n) = x_1(n) + x_2(n)$. We have computed the z-transforms of signals of this form in earlier examples. We have $X(z) = X_1(z) + X_2(z)$, where $X_1(z)$ converges outside the circle $|z| = 1/3$, and $X_2(z)$ converges inside the circle $|z| = 4$ (Figure 8.1).

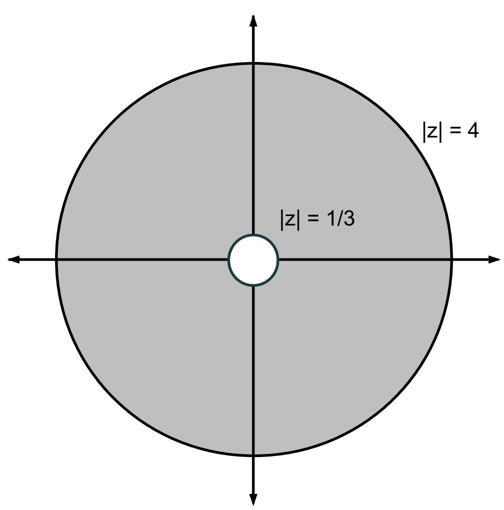

Fig. 8.1. The region of convergence of the z-transform of $x(n) = 3^{-n}u(n) - 4^n u(-n-1)$ is an open annulus in the complex plane. The causal portion of $x(n)$ produces a power series in z^{-1}, which converges outside the circle $|z| = 1/3$. The anti-causal part of $x(n)$ produces a power series in z, which converges inside the circle $|z| = 4$.

Let us now turn to the basic properties of the z-transform. The above example made tacit use of the linearity property. The z-transform properties we cover resemble very closely those of our previously studied transforms, especially the DTFT. One subtlety is the region of convergence, which we must account for during algebraic and analytic operations on the transformed signals.

8.1.3 Properties

The properties of the z-transform closely resemble those of the discrete transform that it generalizes—the discrete-time Fourier transform. With the z-transform, there is a twist, however; now the region of convergence of the transform figures prominently in validating the properties. We divide this section's results into two categories: basic properties and those that rely on the principles of contour integration in the complex plane.

8.1.3.1 Basic Properties. This theorems herein are much like those we developed for the DTFT. Their proofs are also similar, involve familiar methods, and we leave them as exercises for the most part. One caveat in dealing with the z-transform is the region of convergence; one must always be careful to specify this annulus and to consider the special cases of $z = 0$ and $z = \infty$.

Proposition (Linearity, Time Shift, Frequency Shift, and Time Reversal). Let $x(n)$ and $y(n)$ be discrete signals and let $X(z)$ and $Y(z)$ be their z-transforms, respectively. Then

(a) (Linearity) The z-transform of $ax(n) + by(n)$ is $aX(z) + bY(z)$, and its region of convergence contains $\text{ROC}_X \cap \text{ROC}_Y$.

(b) (Time Shift) The z-transform of $x(n - m)$ is $z^{-m}X(z)$, and its region of convergence is ROC_X, except, perhaps, that it may include or exclude the origin, $z = 0$, or the point at infinity, $z = \infty$;

(c) (Frequency Shift, or Time Modulation) Let $a \in \mathbb{C}$. Then the z-transform of $a^n x(n)$ is $Y(z) = X(z/a)$ with $\text{ROC}_Y = |a|\text{ROC}_X$.

(d) (Time Reversal) If $\mathcal{Z}[x(n)] = X(z)$, and $y(n) = x(-n)$, then $\mathcal{Z}[y(n)] = Y(z) = X(z^{-1})$, with $\text{ROC}_Y = \{z \in \mathbb{C}: z^{-1} \in \text{ROC}_X\}$.

Proof: In (a), the formal linearity is clear, but it is only valid where both transforms exist. If $a \neq 0$, then the ROC of $aX(z)$ is ROC_X, and a similar condition applies to $Y(z)$. However, when the two transforms are added, cancellations of their respective terms may occur. This expands the sum's ROC beyond the simple intersection of ROC_X and ROC_Y. Also in (b), multiplication of $X(z)$ by z^k for $k > 0$ may remove from ROC_X. However, if $X(z)$ contains only powers $z^{-|n|}$, for $n > k$, then multiplication by z^k will have no effect on ROC_X. Similarly, multiplication of $X(z)$ by z^k for $k < 0$ may remove 0 from ROC_X, and so on. The other cases are just as straightforward to list. Toward proving (c), let us remark that the power series expansion for $X(z/a)$ will have a new region of convergence that is scaled by $|a|$ as follows: If $\text{ROC}_X = \{z: r_1 < |z| < r_2\}$, then $\text{ROC}_Y = \{z: |a|r_1 < |z| < |a|r_2\}$. Time reversal is an exercise. ∎

Proposition (Frequency Differentiation). Suppose $x(n)$ is a discrete signal, and $X(z)$ is its z-transform. Then the z-transform of $nx(n)$ is $-zdX(z)/dz$. The region of convergence remains the same, except that ∞ may be deleted or 0 may be inserted.

Proof: Similar to the Frequency Differentiation Property of the DTFT. Note that within the region of convergence, the z-transform Laurent series is differentiable. Since we multiply by a positive power of z, will be deleted from the ROC if the highest power of z in $X(z)$ is z^0. Similarly, 0 may be inserted into the ROC if the lowest power of z in $X(z)$ is z^{-1}. ■

Example. Suppose $x(n) = -na^n u(-n - 1)$. Find $X(z)$. We already know from the example of the anti-causal exponential signal in Section 8.1.1 that the z-transform of $-a^n u(-n-1)$ is $(1 - a/z)^{-1}$, with ROC = $\{z \in \mathbb{C}: |z| < |a|\}$. Thus, the frequency differentiation property applies and we have

$$X(z) = -z\frac{d}{dz}\left(1-\frac{a}{z}\right)^{-1} = z\left(1-\frac{a}{z}\right)^{-2}\frac{d}{dz}\left(1-\frac{a}{z}\right) = \frac{az}{(z-a)(z-a)}. \tag{8.13}$$

Also, $\mathrm{ROC}_X = \{z \in \mathbb{C}: |z| < |a|\}$. The properties are useful in finding the z-transforms of new signals.

The z-transform is not without a convolution theorem. Sometimes signal processing systems must deal with signals or system impulse responses for which the DTFT does not converge. A useful tool in this instance is the z-transform. And (not unexpectedly by this point!) there is a convolution result for the z-transform; it finds use in studying LTI systems, questions of stability, subsampling, and interpolation operations for discrete signals.

Theorem (Convolution in Time). Let $x(n)$ and $y(n)$ be signals; let $X(z)$ and $Y(z)$ their z-transforms, respectively; and let $w = x * y$. Then the z-transform of $w(n)$ is $W(z) = X(z)Y(z)$, and $\mathrm{ROC}_W \supseteq \mathrm{ROC}_X \cap \mathrm{ROC}_Y$.

Proof: The proof of the DTFT Convolution-in-Time Theorem extends readily to the z-transform:

$$\begin{aligned} W(z) &= \sum_{n=-\infty}^{\infty} w(n)z^{-n} = \sum_{n=-\infty}^{\infty} (x*y)(n)z^{-n} = \sum_{n=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty} x(k)y(n-k)\right)z^{-n} \\ &= \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} x(k)y(n-k)z^{-(n-k)}z^{-k} = \sum_{k=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} x(k)y(n-k)z^{-(n-k)}z^{-k} \\ &= \sum_{k=-\infty}^{\infty} x(k)z^{-k}\sum_{n=-\infty}^{\infty} y(n-k)z^{-(n-k)} = X(z)Y(z). \end{aligned} \tag{8.14}$$

If $x \in \mathrm{ROC}_X \cap \mathrm{ROC}_Y$, then the z-transform's Laurent series converges absolutely. This justifies the step from an iterated summation to a double summation as well as the interchange in the order of summation in (8.14). Hence, $\mathrm{ROC}_W \supseteq \mathrm{ROC}_X \cap \mathrm{ROC}_Y$. ■

Corollary (LTI System Function). Suppose H is a discrete LTI system $y = Hx$; its impulse response is $h = H\delta$; and $X(x)$, $Y(z)$, and $H(z)$ are the z-transforms of $x(n)$, $y(n)$, and $h(n)$, respectively. Then, $Y(z) = H(z)X(z)$.

Proof: The output signal $y = h * x$ by the Convolution Theorem for LTI Systems, and the result follows from the theorem. ■

Definition (System or Transfer Function). Let H be a discrete LTI system $y = Hx$ and $h = H\delta$ its impulse response. Then the z-transform of $h(n)$, $H(z)$ is called the system function or the transfer function for the system H.

Remark. For our multiple uses of the uppercase "H," we once again ask the reader's indulgence. Here, both the system itself and its impulse response, a complex-valued function of a complex variable, are both denoted "H."

8.1.3.2 Properties Involving Contour Integration. With the DTFT synthesis equation, we can identify a time-domain signal $x(n)$ with the analog Fourier Series coefficients of its DTFT, $X(\omega)$. The interconnection is at once elegant and revealing. The z-transform is like the DTFT, in that it is a discrete transform with an inversion relation which involves a continuous domain integration operation. However, because the domain of definition of the z-transform is a region in the complex plane, the inversion formula becomes quite exotic: It depends on a complex contour integral.

Readers who skimmed the material in Section 1.7 may wish to review it more carefully before proceeding with the next several theorems.

Theorem (Inversion). Suppose $x(n)$ is a discrete signal and $X(z)$ is its z-transform. If C is any simple, counterclockwise, closed contour of the complex plane; the origin is in the interior of C; and $C \subseteq \mathrm{ROC}_X$, then

$$x(n) = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz. \tag{8.15}$$

Proof: The integrand in (8.15) contains a power of z, and the contour is closed; this suggests the Cauchy integral theorem from Section 1.7.3.

$$\frac{1}{2\pi j}\oint_C X(z)z^{n-1}\,dz = \frac{1}{2\pi j}\oint_C\left[\sum_{k=-\infty}^{k=+\infty} x(k)z^{-k}\right]z^{n-1}dz = \frac{1}{2\pi j}\sum_{k=-\infty}^{k=+\infty} x(k)\oint_C z^{n-k-1}\,dz. \tag{8.16}$$

Once again, inserting the analysis equation directly into the integral and then interchanging the order of summation and integration pays off. Since the z-transform is absolutely and uniformly convergent within its ROC, the order of summation and integration is unimportant in (8.16). Recall, from Section 1.7.3, the Cauchy integral theorem:

$$\frac{1}{2\pi j}\oint_C z^m dz = \begin{cases} 0 & \text{if} \quad m \neq -1, \\ 1 & \text{if} \quad m = -1. \end{cases} \tag{8.17}$$

As a consequence, all terms in the summation of are zero except for the one where $n = k$. This implies that

$$x(n) = \frac{1}{2\pi j}\oint_C X(z) z^{n-1} dz, \tag{8.18}$$

as desired. ■

Equation (8.18) is the z-transform synthesis equation.

Example (Unit Circle Contour). Suppose now that $\mathcal{Z}(x(n)) = X(z)$ and that ROC_X contains the unit circle: $C = \{z: |z| = 1\} \subseteq \mathrm{ROC}_X$. Then, $z = \exp(j\omega)$ on C, $\omega \in [-\pi, +\pi]$; $dz = j\exp(j\omega)\, d\omega$; and evaluating the inverse z-transform contour integral gives

$$\begin{aligned}\frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz &= \frac{1}{2\pi j}\oint_{|z|=1} X(z)z^{n-1}dz = \frac{1}{2\pi j}\int_{-\pi}^{+\pi} X(e^{j\omega})(e^{j\omega})^{n-1}\, je^{j\omega} d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{+\pi} X(e^{j\omega})(e^{j\omega})^n\, d\omega = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X(\omega)\exp(jn\omega)\, d\omega. \end{aligned} \tag{8.19}$$

In (8.19), $X(e^{j\omega})$ is the z-transform evaluated at $z = \exp(j\omega)$, and $X(\omega)$ is the DTFT of $x(n)$ evaluated at ω, $-\pi \le \omega \le \pi$. This example thus shows that if C is the unit circle and it lies within ROC_X, then the inverse z-transform relation reduces to the IDTFT.

Prominent among the z-transform's basic properties in the previous section, the Convolution-in-Time Theorem linked convolution in time to simple multiplication in the z-domain. The next theorem is its counterpart for z-domain convolutions. Although this results lacks the aesthetic appeal of the DTFT's Convolution-in-Frequency Theorem, it will nevertheless prove useful for discrete filter design applications in the next chapter.

Theorem (Convolution in the z-Domain). Let $s(n) = x(n)y(n)$ be the termwise product of $x(n)$ and $y(n)$; $\mathcal{Z}(x(n)) = X(z)$, with $\mathrm{ROC}_X = \{z \in \mathbb{C}: r_X < |z| < R_X\}$; and $\mathcal{Z}(y(n)) = Y(z)$, where $\mathrm{ROC}_Y = \{z \in \mathbb{C}: r_Y < |z| < R_Y\}$. Then, $\mathrm{ROC}_S \supseteq \{z \in \mathbb{C}: r_X r_Y < |z| < R_X R_Y\}$. Furthermore, let C be a simple, closed contour of the complex plane whose interior contains the origin. If $\mathrm{ROC}_{X(w)} \cap \mathrm{ROC}_{Y(z/w)}$ contains C, then

$$S(z) = \frac{1}{2\pi j}\oint_C X(w) Y\left(\frac{z}{w}\right) w^{-1} dw. \tag{8.20}$$

Proof: The z-transform analysis equation for $S(z)$ is

$$S(z) = \sum_{n=-\infty}^{+\infty} s(n)z^{-n} = \sum_{n=-\infty}^{+\infty} x(n)y(n)z^{-n} = \sum_{n=-\infty}^{+\infty} \left[\frac{1}{2\pi j}\oint_C X(w)w^{n-1}\right] y(n)z^{-n}, \tag{8.21}$$

where the z-transform synthesis equation (8.18), with dummy variable of integration w, replaces $x(n)$ inside the sum. This substitution is valid for any simple, closed path C, when $C \subseteq \mathrm{ROC}_X$. Summation and integration may change order in , so long as $z \in \mathrm{ROC}_S$, where uniform convergence reigns:

$$S(z) = \frac{1}{2\pi j}\oint_C X(w) \sum_{n=-\infty}^{+\infty} y(n)\left(\frac{z}{w}\right)^{-n} w^{-1}dw = \frac{1}{2\pi j}\oint_C X(w)Y\left(\frac{z}{w}\right)w^{-1}dw. \tag{8.22}$$

When does $S(z)$ exist? We need $C \subseteq \mathrm{ROC}_X = \{w \in \mathbb{C}\colon r_X < |w| < R_X\}$ and $z/w \in \mathrm{ROC}_Y = \{z \in \mathbb{C}\colon r_Y < |z| < R_Y\}$. The latter occurs if and only if $r_Y < |z/w| < R_Y\}$; this will be the case if $|w|r_Y < |z| < |w|R_Y$ for $w \in \mathrm{ROC}_X$. Hence ROC_S includes $\{z \in \mathbb{C}\colon r_X r_Y < |z| < R_X R_Y\}$. The contour integral in (8.22) will exist whenever $w \in C$ and $z/w \in \mathrm{ROC}_Y$; in other words, $C \subseteq \mathrm{ROC}_X \cap \mathrm{ROC}_{Y(z/w)}$, as stated. ■

Corollary (Parseval's Theorem). Suppose that $x(n), y(n) \in l^2$, $\mathcal{Z}(x(n)) = X(z)$, and $\mathcal{Z}(y(n)) = Y(z)$. If C is a simple, closed contour whose interior contains the origin, and $C \subseteq \mathrm{ROC}_X \cap \mathrm{ROC}_{Y^*(1/w^*)}$, then

$$\langle x, y\rangle = \sum_{n=-\infty}^{+\infty} x(n)y^*(n) = \frac{1}{2\pi j}\oint_C X(w)Y^*\left(\frac{1}{w^*}\right)w^{-1}dw. \tag{8.23}$$

Proof: The inner product $\langle x, y\rangle$ exists, since x and y are square-summable, and if $s(n) = x(n)y^*(n)$, then $s(n) \in l^1$ (Cauchy–Schwarz). If $\mathcal{Z}(s(n)) = S(z)$, then

$$S(z) = \sum_{n=-\infty}^{+\infty} x(n)y^*(n)z^{-n}, \tag{8.24}$$

so that

$$\langle x, y\rangle = \sum_{n=-\infty}^{+\infty} x(n)y^*(n) = S(1). \tag{8.25}$$

It is an easy exercise to show that $\mathcal{Z}(y^*(n)) = Y^*(z^*)$; and together with the Convolution in z-Domain Theorem (8.22), this entails

$$\langle x, y\rangle = S(1) = \frac{1}{2\pi j}\oint_C X(w)Y^*\left(\frac{1}{w^*}\right)w^{-1}dw, \tag{8.26}$$

completing the proof. ■

8.2 INVERSION METHODS

Given a complex function of a complex variable, $X(z)$, there are three methods for finding the time domain signal, $x(n)$, such that $Z[x(n)] = X(z)$. These approaches are:

- Via the inverse z-transform relation, given by the contour integral (8.15).
- Through an expansion of $X(z)$ into a Laurent series; then the $x(n)$ values read directly from the expansion's coefficient of z^{-n}.
- By way of algebraic manipulation (especially using partial fractions) of $X(z)$ into a form in which its various parts are readily identified as the z-transforms of known discrete signals. This approach relies heavily on the z-transform's basic properties (Section 8.2.1).

Actually, the second two methods are the most useful, because the contour integrals prove to be analytically awkward. This section considers some examples that illustrate each of these z-transform inversion tactics.

8.2.1 Contour Integration

Let us look first at the easiest nontrivial example using contour integration in the complex plane to discover the discrete signal $x(n)$ whose z-transform is the given complex function $X(z)$. On first reading, this section can be reviewed casually. But those readers who accepted our invitation—several chapters back—to skip the complex variables tutorial should note that those ideas are key to this approach for z-transform inversion.

Example (Inversion by Contour Integration). Suppose $X(z) = z/(z - a)$, $a \neq 0$, with $\mathrm{ROC}_X = \{z \in \mathbb{C}: |a| < |z|\}$. Of course, we already know the signal whose z-transform is $X(z)$; it is the causal exponential signal $a^n u(n)$ from the example in Section 8.1.1. But, in order to learn the technique, let us proceed with pretenses toward discovery. From the z-transform synthesis equation (8.15), we may choose the contour C to be a circle outside $z = |a|$ and immediately write

$$x(n) = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz = \frac{1}{2\pi j}\oint_C \frac{z^n}{(z-a)}\,dz. \tag{8.27}$$

Is the contour integral (8.27) easy to evaluate? To the fore, from complex analysis, comes a powerful tool: the Cauchy residue theorem (Section 1.7.3). Assume that C is a simple, closed curve; $a_m \notin C$, $1 < m < M$; $f(z)$ is a complex function, which is analytic (has a derivative df/dz) on and within C, except for poles ($|f(z)| \to \infty$ near a pole) at each of the a_m. The residue theorem then states that

$$\frac{1}{2\pi j}\oint_C f(z)\,dz = \sum_{m=1}^{M} \mathrm{Res}(f(z), a_m). \tag{8.28a}$$

Recall that the residue of $f(z)$ at the pole $z = p$ is given by

$$\mathrm{Res}(f(z), p) = \begin{cases} \dfrac{1}{(k-1)!} g^{(k-1)}(p) & \text{if } p \in \mathrm{Interior}(C), \\ 0 & \text{otherwise,} \end{cases} \tag{8.28b}$$

where k is the order of the pole, $g(z)$ is the nonsingular part of $f(z)$ near p, and $g^{(k-1)}(p)$ is the $(k-1)$th derivative of $g(z)$ evaluated at $z = p$. (Complex function $f(z)$ has a pole of order $(k-1)$ at $z = 0$ if there exists $g(z)$ such that $f(z) = g(z)/(z-p)^k$, $g(z)$ is analytic near $z = p$, and $g(p) \neq 0$.) Let us continue the example. To find $x(n)$, $n \geq 0$, we set $f(z) = z^n/(z-a)$, as in (8.27). Note that $f(z)$ is analytic within C, except for a first-order pole at $z = a$. Therefore, $g(z) = z^n$, and we have $\mathrm{Res}(f(z), a) = g^{(0)}(a) = x(n) = a^n$. For non-negative values of n, computing the contour integral (8.27) is generally quite tractable. Elated by this result, we might hope to deduce values $x(n)$, $n < 0$, so easily. When $n < 0$, however, there are multiple poles inside C: a pole at $z = a$ and a pole of order n at $z = 0$. Consider the case $n = -1$. We set $f(z) = z^{-1}/(z-a)$. Thus, the pole at $z = p = 0$ is of order $k = 1$, since $g(z) = (z-a)^{-1}$ is analytic around the origin. Therefore, $\mathrm{Res}(f(z), z = 0) = g^{(0)}(p) = g^{(0)}(0) = (-a)^{-1}$. There is another residue, and we must sum the two, according to (8.28b). We must select a different analytic part of $f(z)$ near the pole at $z = a$; we thus choose $g(z) = z^{-1}$, so that $g(z)$ is analytic near $z = a$ with $f(z) = g(z)/(z-a)^1$. Consequently, this pole is also first order. Since $g^{(0)}(a) = a^{-1} = \mathrm{Res}(f(z), z = a)$, we have $x(-1) = \mathrm{Res}(f(z), z = 0) + \mathrm{Res}(f(z), z = a) = (-a)^{-1} + a^{-1} = 0$. Now let us turn our attention to the case $n = -2$. Now $f(z) = z^{-2}/(z-a)$, whence the pole at $z = 0$ is of order 2. Still, by (8.28b), $x(-2) = \mathrm{Res}(f(z), z = 0) + \mathrm{Res}(f(z), z = a)$, but now $f(z)$ has a pole of order 2 at $z = 0$. First, we set $g(z) = (z-a)^{-1}$ as before, but now we find $\mathrm{Res}(f(z), z = 0) = g^{(1)}(0) = -1(0-a)^{-2} = -a^{-2}$. For the pole at $z = a$, we put $g(z) = z^{-2}$ and verify that $\mathrm{Res}(f(z), z = a) = a^{-2}$. Thus, coincidentally, $x(-2) = -a^{-2} + a^{-2} = 0$. It is possible to show that, indeed, $x(n) = 0$ for $n < 0$. Therefore, $x(n) = a^n u(n)$ for all n.

The lesson of the example is that z-transform inversion by complex contour integration is sophisticated, easy, and fun for $n > 0$ where the integrand in (8.27) has first order poles, but tedious when there are higher-order poles. We seek simpler methods.

8.2.2 Direct Laurent Series Computation

Let us now try to exploit the idea that the z-transform analysis equation is a two-sided power, or Laurent, series in the complex variable z. Given by the z-transform inversion problem are a complex function of z, $X(z)$, and a region of convergence, ROC_X. The solution is to find $x(n)$ so that $\mathcal{Z}(x(n)) = X(z)$. Direct Laurent series computation solves the inversion problem by algebraically manipulating $X(z)$ into a form that resembles the z-transform analysis equation (8.1). Then the $x(n)$ values read off directly as the coefficients of the term z^{-n}. Not just any algebraic fiddling will do; the method can go awry if the algebraic manipulations do not stay in consonance with the information furnished by ROC_X.

Once again, we consider a well-known example to learn the technique.

Example (Laurent Series Computation for Causal Discrete Signal). Suppose $X(z) = z/(z-a)$, $a \neq 0$, with $\mathrm{ROC}_X = \{z \in \mathbb{C}: |a| < |z|\}$. Of course, we already know the signal whose z-transform is $X(z)$; it is the causal exponential signal $a^n u(n)$. Performing long division on $X(z)$ produces a Laurent series:

$$X(z) = \frac{z}{z-a} = \begin{array}{r@{}l} & 1 + az^{-1} + a^2z^{-2} + a^2z^{-2} + \cdots \\ \hline z - a \,\big) & z \end{array}. \tag{8.29}$$

Using the z term in $z - a$ as the principal divisor produces a quotient that is a Laurent expansion. Since ROC_X is the region outside the circle $|z| = a$, we see by inspection that $x(n) = u(n)a^n$.

Now we consider the same $X(z)$, but allow that ROC_X is inside the circle $|z| = a$.

Example (Laurent Series Computation for Anti-causal Discrete Signal). Suppose $X(z) = z/(z-a)$, $a \neq 0$, with $\mathrm{ROC}_X = \{z \in \mathbb{C}: |z| < |a|\}$. Performing long division again, but this time using the $-a$ term in $z - a$ as the principal divisor, produces a different Laurent series:

$$X(z) = \frac{z}{z-a} = \begin{array}{r@{}l} & -a^{-1}z - a^{-2}z^2 - a^{-3}z^3 - a^{-4}z^4 - \cdots \\ \hline -a + z \,\big) & z \end{array}. \tag{8.30}$$

The algebraic manipulation takes into account the fact that ROC_X is the region inside the circle $|z| = a$, and the expansion is in positive powers of z. This means that $x(n)$ is anti-causal: $x(n) = -u(-n-1)a^n$.

The next example complicates matters a bit more.

Example (Quadratic Denominator). Suppose $X(z) = z(z-2)^{-1}(z-1)^{-1}$, with $\mathrm{ROC}_X = \{z \in \mathbb{C}: |2| < |z|\}$. Attacking the problem directly with long division gives

$$X(z) = \frac{z}{(z-2)(z-1)} = \begin{array}{r@{}l} & z^{-1} + 3z^{-2} + 7z^{-3} + 15z^{-4} - \cdots \\ \hline z^2 - 3z + 2 \,\big) & z \end{array}. \tag{8.31}$$

We observe that $x(n) = u(n)(2^n - 1)$ from the derived form of the Laurent series (8.31). We can check this result by using linearity. Note that $X(z) = z(z-2)^{-1} - z(z-1)^{-1}$. The first term is the z-transform of $u(n)2^n$, and its radius of convergence is $\{z \in \mathbb{C}: 2 < |z|\}$. The second term is the z-transform of $u(n)1^n$, with $\mathrm{ROC} = \{z \in \mathbb{C}: 1 < |z|\}$. Therefore, their difference, $u(n)(2^n - 1^n)$, has z-transform $z(z-2)^{-1} - z(z-1)^{-1}$, whose radius of convergence equals $\{z \in \mathbb{C}: 2 < |z|\} \cap \{z \in \mathbb{C}: 1 < |z|\} = \{z \in \mathbb{C}: 2 < |z|\} = \mathrm{ROC}_X$.

Our method of checking this last example leads to the table lookup technique of the section.

8.2.3 Properties and *z*-Transform Table Lookup

The last method for computing the inverse z-transform is perhaps the most common. We tabulate a variety of z-transforms for standard signals and use the various properties of the transform to manipulate a given $X(z)$ into a form whose components are z-transforms of the known signals. Typically, then, $x(n)$ is a linear combination of these component signals. One standard trick that is useful here is to break up a complicated rational function in z, $X(x) = P(z)/Q(z)$, where P and Q are polynomials, into a sum of simpler fractions that allows table lookup. This is called the partial fractions method, and we will consider some examples of its use as well.

Example. Suppose $X(z) = (1 - az)^{-1}$, with $\text{ROC}_X = \{z \in \mathbb{C}: |z| < |a|^{-1}\}$. From a direct computation of the z-transform, we know that $\mathcal{Z}[a^n u(n)] = z/(z - a)$, with $\text{ROC} = \{z \in \mathbb{C}: |z| > |a|\}$. Let $y(n) = a^n u(n)$ and $x(n) = y(-n)$. The time-reversal property implies

$$X(z) = \frac{z^{-1}}{(z^{-1} - a)} = \frac{1}{(1 - az)}, \tag{8.32}$$

with $\text{ROC}_X = \{z \in \mathbb{C}: z^{-1} \in \text{ROC}_Y\} = \{z \in \mathbb{C}: |z| < |a|^{-1}\}$, as desired.

Table 8.1 provides a list of common signals, their z-transforms, and the associated regions of convergence. These pairs derive from

TABLE 8.1. Signals, Their *z*-Transforms, and the Region of Convergence of the *z*-Transform

$x(n)$	$X(z)$	ROC_X
$\delta(n - k)$	z^{-k}	$k > 0 : \{z \in \mathbb{C}^+ : z \neq 0\}$ $k < 0 : \{z \in \mathbb{C}^+ : z \neq \infty\}$
$a^n u(n)$	$z/(z - a)$	$\{z \in \mathbb{C} : \lvert a\rvert < \lvert z\rvert\}$
$-a^n u(-n-1)$	$z/(z - a)$	$\{z \in \mathbb{C} : \lvert z\rvert < \lvert a\rvert\}$
$a^{-n} u(-n)$	$\dfrac{1}{(1 - az)}$	$\{z \in \mathbb{C} : \lvert z\rvert < \lvert a\rvert^{-1}\}$
$-a^{-n} u(n-1)$	$\dfrac{1}{(1 - az)}$	$\{z \in \mathbb{C}^+ : \lvert z\rvert > \lvert a\rvert^{-1}\}$
$na^n u(n)$	$az/(z^2 - 2az + a^2)$	$\{z \in \mathbb{C}^+ : \lvert a\rvert < \lvert z\rvert\}$
$-na^n u(-n - 1)$	$az/(z^2 - 2az + a^2)$	$\{z \in \mathbb{C} : \lvert z\rvert < \lvert a\rvert\}$
$\cos(an)u(n)$	$\dfrac{z^2 - \cos(a)z}{z^2 - 2\cos(a)z + 1}$	$\{z \in \mathbb{C} : 1 < \lvert z\rvert\}$
$\sin(an)u(n)$	$\dfrac{\sin(a)z}{z^2 - 2\cos(a)z + 1}$	$\{z \in \mathbb{C} : 1 < \lvert z\rvert\}$
$u(n)/(n!)$	$\exp(z)$	$\{z \in \mathbb{C}\}$
$n^{-1}u(n-1)(-1)^{n+1}a^n$	$\log(1 + az^{-1})$	$\{z \in \mathbb{C}^+ : \lvert a\rvert < \lvert z\rvert\}$

- Basic computation using the z-transform analysis equation;
- Application of transform properties;
- Standard power series expansion from complex analysis.

Example. Suppose that

$$X(z)=\frac{z^2-\left(\frac{\sqrt{2}}{2}\right)z}{(z^2-\sqrt{2}z+1)},\tag{8.33}$$

with $\mathrm{ROC}_X = \{z \in \mathbb{C}\colon |z| > 1\}$. The table entry, $\cos(an)u(n)$, applies immediately. Taking $a = \pi/4$ gives $x(n) = \cos(\pi n/4)u(n)$. Variants of a z-transform pair from Table 8.1 can be handled using the transform properties. Thus, if

$$Y(z)=\frac{1-\left(\frac{\sqrt{2}}{2}\right)z^{-1}}{(z^2-\sqrt{2}z+1)},\tag{8.34}$$

then $Y(z) = z^{-2}X(z)$, so that $y(n) = x(n + 2) = \cos[\pi(n + 2)/4]u(n + 2)$ by the time shift property.

Example (Partial Fractions Method). Suppose that we are given a rational function in the complex variable z,

$$X(z)=\frac{2z^2}{2z^2-3z+1},\tag{8.35}$$

where $\mathrm{ROC}_X = \{z \in \mathbb{C}\colon |z| > 1\}$. The partial fraction technique factors the denominator of (8.35), $2z^2 - 3z + 1 = (2z - 1)(z - 1)$, with an eye toward expressing $X(z)$ in the form

$$X(z)=2z\frac{z}{2z^2-3z+1}=2z\left[\frac{A}{2z-1}+\frac{B}{z-1}\right],\tag{8.36}$$

where A and B are constants. Let us concentrate on finding the inverse z-transform of $Y(z) = X(z)/(2z)$, the bracketed expression in (8.36). Table 8.1 covers both of its terms: they are of the form $(1 - az)^{-1}$. The sum of these fractions must equal $z(2z^2 - 3z + 1)^{-1}$, so $A(z - 1) + B(2z - 1) = z$. Grouping terms involving like powers of z produces two equations: $A + 2B = 1$, $A + B = 0$. Hence,

$$\frac{X(z)}{2z}=\frac{z}{2z^2-3z+1}=\left[\frac{-1}{2z-1}+\frac{1}{z-1}\right]=\left[\frac{1}{1-2z}-\frac{1}{1-z}\right]=Y(z).\tag{8.37}$$

Now, $y(n) = -2^{-n}u(n - 1) + u(n - 1) = (1 - 2^{-n})u(n - 1)$ by linearity and Table 8.1. Also, $\mathrm{ROC}_Y = \{z \in \mathbb{C}^+\colon |z| > 2^{-1}\} \cap \{z \in \mathbb{C}^+\colon |z| > 1\} = \{z \in \mathbb{C}^+\colon |z| > 1\}$.

Therefore, the z-transform of $x(n) = 2y(n + 1) = (2 - 2^{-n})u(n)$ is $2zY(z)$ by the time shift property.

Example (Partial Fractions, Multiple Roots in Denominator). Now suppose that the denominator of $X(z)$ has multiple roots:

$$X(z) = \frac{z}{(z-1)(z+2)^2}. \tag{8.38}$$

It turns out that a partial fractions expansion of $X(z)/z$ into, say,

$$\frac{X(z)}{z} = \frac{1}{(z-1)(z+2)^2} = \frac{A}{(z-1)} + \frac{B}{(z+2)^2} \tag{8.39}$$

does not work, in general. Rather, the partial fractions arithmetic is satisfactory when $X(z)/z$ is broken down as follows:

$$\frac{X(z)}{z} = \frac{1}{(z-1)(z+2)^2} = \frac{A}{(z-1)} + \frac{B}{(z+2)^2} + \frac{C}{(z+2)}. \tag{8.40}$$

The solutions are $A = 1/7$, $B = 2/7$, and $C = -1/7$. Applying linearity, time shift, and Table 8.1 completes the example. This is left as an exercise.

8.2.4 Application: Systems Governed by Difference Equations

The above theory applies directly to the study of linear, translation-invariant systems where a difference equation defines the input–output relation. Chapter 2 introduced this kind of system (Sections 2.4.2 and 2.10). We shall see here and in Chapter 9 that:

- For such systems, the transfer function $H(z)$ is a quotient of complex polynomials.
- Difference equations govern a wide variety of important signal processing systems.
- Recursive algorithms very efficiently implement these systems on digital computers.
- The filters that arise from difference equations can be derived straightforwardly from equivalent analog systems.
- For almost any type of filter—low-pass, high-pass, bandpass, or band-reject—a difference equation governed system can be devised that very well approximates the required frequency selection behavior.

In fact, the filters within almost all signal analysis systems derive from difference equations, and we describe them by the z-transform of their impulse response.

Suppose that the difference equation for a system H is

$$\begin{aligned} y(n) &= b_0 x(n) + b_1 x(n-1) + \cdots + b_M x(n-M) \\ &\quad - a_1 y(n-1) - a_2 y(n-2) - \cdots - a_N y(n-N). \end{aligned} \tag{8.41}$$

We see that $y(n)$ can be computed from its past N values, $\{y(n) \mid 1 \le n \le N\}$, the current input value $x(n)$, and the past M values of input signal $\{x(n) \mid 1 \le n \le M\}$. Collecting output terms on the left-hand side and input terms on the right-hand side of (8.41), taking the z-transform of both sides, and finally applying the shift property, we have

$$Y(z)\left[1 + \sum_{k=1}^{N} a_k z^{-k}\right] = X(z)\left[\sum_{m=0}^{M} b_m z^{-m}\right]. \tag{8.42}$$

Hence,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\left[\sum_{m=0}^{M} b_m z^{-m}\right]}{\left[1 + \sum_{k=1}^{N} a_k z^{-k}\right]}, \tag{8.43}$$

confirming that the system function for H is a rational function of a single complex variable.

Now the methods of z-transform inversion come into play. The partial fractions technique converts the rational function (8.43) into a sum of simpler terms to which table lookup applies. Thus, we can find the impulse response $h(n)$ of the LTI system H. Finally, we can compute the response of H to an input signal $x(n)$ by convolution $y(n) = (h * x)(n)$.

Example (Smoothing System). The system H smoothes input signals by weighting the previous output value and adding it to the weighted input value as follows:

$$y(n) = Ay(n-1) + Bx(n). \tag{8.44}$$

By z-transforming both sides of (8.44), we get

$$Y(z) = Az^{-1}Y(z) + BX(z), \tag{8.45}$$

so that

$$H(z) = \frac{Y(z)}{X(z)} = \frac{B}{1 - Az^{-1}}. \tag{8.46}$$

Assuming that the system is causal, so that $h(n) = 0$ for $n < 0$, we have

$$h(n) = BA^n u(n), \tag{8.47}$$

by Table 8.1.

8.3 RELATED TRANSFORMS

This section introduces two other transforms: the chirp z-transform (CZT) and the Zak transform (ZT). A short introduction to them gives the reader insight into recent research efforts using combined analog and discrete signal transformation tools.

8.3.1 Chirp *z*-Transform

The CZT samples the z-transform on a spiral contour of the complex plane [7, 22]. The CZT transform has a number of applications [23]:

- It can efficiently compute the discrete Fourier transform (DFT) for a prime number of points.
- It can be used to increase the frequency resolution of the DFT, zooming in on frequency components (Chapter 9).
- It has been applied in speech [8, 24], sonar, and radar signal analysis [6, 25], where chirp signals prevail and estimations of their parameters—starting frequency, stopping frequency, and rate of frequency change—are crucial.

8.3.1.1 Definition. Recall that evaluating the z-transform $X(z)$ of $x(n)$ on the unit circle $z = \exp(j\omega)$ gives the discrete-time Fourier transform: $X(\omega) = [\mathcal{Z}(x)](e^{j\omega})$. If $N > 0$ and $x(n)$ is finitely supported on the discrete interval $[0, N-1]$, then $X(z)$ becomes

$$X(z) = \sum_{n=0}^{N-1} x(n)z^{-n}. \tag{8.48}$$

Furthermore, if $\omega = 2\pi k/N$, $0 \le k < N$, so that $z = \exp(2\pi jk/N)$, then the DTFT analysis equation (8.48) becomes a discrete Fourier transform of $x(n)$. So we are evaluating the z-transform $X(z)$ on a discrete circular contour of the complex plane. The idea behind the CZT is evaluate the z-transform on a discrete spiral—as distinct from purely circular—contour. We use the notation and generally follow the presentation of Ref. 7.

Definition (Chirp *z*-Transform). Let $A = A_0\exp(2\pi j\theta_0)$; $W = W_0\exp(2\pi j\phi_0)$; M, $N > 0$ be natural numbers; $x(n) = 0$ outside $[0, N-1]$; and set $z_k = AW^{-k}$ for $0 \le k < M$. The chirp z-transform of $x(n)$ with respect to A and W is

$$X_{A,W}(k) = \sum_{n=0}^{N-1} x(n){z_k}^{-n} = \sum_{n=0}^{N-1} x(n)A^{-n}W^{nk}. \tag{8.49}$$

If $A = 1$, $M = N$, and $W = \exp(-2\pi j/N)$, then the CZT gives the DFT of order N for the signal $x(n)$ (exercise). Figure 8.2 shows a typical discrete spiral contour for a CZT.

Further note that if $W_0 > 1$, then the contour spirals inward, whereas $W_0 < 1$ means the contour winds outward (Figure 8.2).

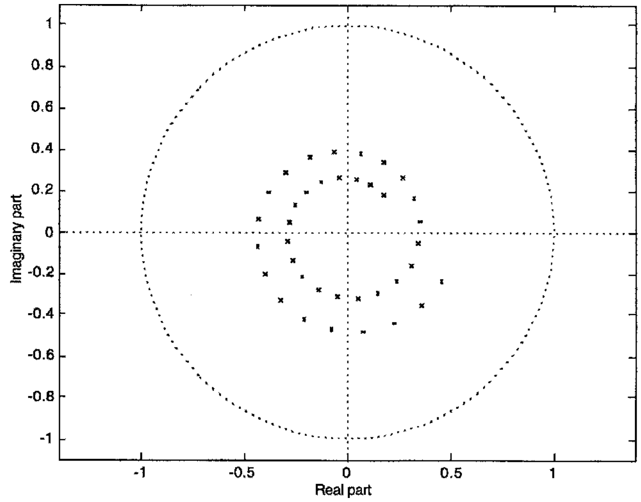

Fig. 8.2. Discrete spiral path for a CZT. Innermost point is $A = (.25)\exp(j\pi/4)$. Ratio between the M = 37 contour samples is W = (0.98)exp($-j\pi/10$). Unit circle $|z| = 1$ is shown by dots.

8.3.1.2 Algorithm. An efficient implementation of the CZT, as a weighted convolution of two special functions, is possible as follows. Using our earlier notation, define the discrete signals $v(n)$ and $y(n)$ by

$$v(n) = W^{\frac{n^2}{2}}, \tag{8.50a}$$

$$y(n) = x(n)v(n)A^{-n}. \tag{8.50b}$$

Since

$$nk = \frac{n^2 + k^2 - (k-n)^2}{2}, \tag{8.51}$$

we calculate

$$X_{A,W}(k) = \sum_{n=0}^{N-1} x(n)A^{-n}W^{\frac{n^2+k^2-(k-n)^2}{2}} = W^{\frac{k^2}{2}} \sum_{n=0}^{N-1} y(n)v(k-n). \tag{8.52}$$

Equation (8.52) gives $X_{A,W}(k)$ as the convolution of $y(n)$ and $v(n)$, but weighted by the factor $v(k)$. Now, thanks to the convolution theorem for the DFT, we can compute discrete convolutions by Fourier transforming both signals, taking the frequency-domain product term-by-term, and then inverse transforming the result. Hence, if we have an efficient fast Fourier transform algorithm available, then a CZT may much more efficiently compute the DFT for a problematic—even prime—order N.

Here are the steps in the CZT algorithm [7]:

(i) First, we define $y(n)$:

$$y(n) = \begin{cases} A^{-n}W^{2/2}x(n) & \text{if } n \in [0, N-1], \\ 0 & \text{otherwise.} \end{cases} \tag{8.53}$$

(ii) Next, we determine the size of the FFT operation to perform. Inspecting the convolution equation (8.52), where k ranges between 0 and $M - 1$, we see that we need $v(n)$ values for $-N + 1 \le n \le M - N$, for a total of $M - N - (-N + 1) + 1 = M$ samples. Since $y(n)$ is supported on $[0, N - 1]$, the convolution result will be supported on $[0, (M - 1) + (N - 1)] = [0, M + N - 2]$. So the full $y{*}v$ requires $(M + N - 2) + 1 = M + N - 1$ samples (of which, for the CZT, we only care about M of them). Thus, we pick a power 2^P (or another FFT-suitable composite integer L), so that $M + N - 1 \le L$. This will be the order of the fast forward transforms and of the inverse transform after pointwise multiplication in the frequency domain.

(iii) We set $v(n)$ to be L-periodic such that

$$v(n) = \begin{cases} W^{-n^2/2} & \text{if } n \in [0, M-1], \\ W^{-(L-n)^2/2} & \text{if } n \in [L-N+1, L-1], \\ 0 & \text{otherwise.} \end{cases} \tag{8.54}$$

(iv) Compute the FFTs, $Y(k)$ and $V(k)$, of $y(n)$ and $v(n)$, respectively.

(v) Compute $G(k) = Y(k)V(k)$, for $0 \le k \le L - 1$.

(vi) Compute the inverse FFT of $G(k)$: $g(n)$.

(vii) Set

$$X_{A,W}(k) = W^{\frac{k^2}{2}} g(k) \qquad \text{for} \qquad 0 \le k \le M-1. \tag{8.55}$$

Evidently, the computational burden within the algorithm remains the three fast transforms [7]. Each of these requires on the order of $L\log_2(L)$ operations, depending, of course, on the particular FFT available. So we favor the CZT when $L\log_2(L)$ is much less than the cost of a full-convolution MN operation.

8.3.2 Zak Transform

The Zak transform (ZT) is an important tool in Chapter 10 (time-frequency analysis). The transform and its applications to signal theory are covered in Refs. 26 and 27.

8.3.2.1 Definition and Basic Properties. The Zak transform maps an analog signal $x(t)$ to a two-dimensional function having independent variables in both time and frequency. We know that restricting the z-transform to the unit circle $|z| = 1$

gives the discrete-time Fourier transform. The idea behind the Zak transform is that discrete signals generally come from sampling analog signals $x(n) = x_a(nT)$, for some $T > 0$, and that we can compute a DTFT for a continuum of such sampled analog signals.

Definition (Zak Transform). Let $a > 0$ and $x(t)$ be an analog signal. Then the Zak transform with parameter a of $x(t)$ is

$$X_a(s,\omega) = \sqrt{a}\sum_{k=-\infty}^{\infty} x(as - ak)\exp(2\pi j\omega k). \tag{8.56}$$

Remark. We use a fancy $\mathcal{Z}_a$ for the map taking an analog signal to its Zak transform: $(\mathcal{Z}_a x)(s, \omega) = X_a(s, \omega)$. Our transform notation uses the sign notation $x(s - k)$ following [28]. Generally, we take $a = 1$ and omit it from the notation: $(\mathcal{Z}x)(s, \omega) = X(s, \omega)$; this is the form of the definition we use later in several parts of Chapter 10.

Proposition (Periodicity). If $X_a(s, \omega)$ is the ZT of $x(t)$, then

$$X_a(s+1,\omega) = \exp(2\pi j\omega)X_a(s,\omega), \tag{8.57a}$$

$$X_a(s,\omega+1) = X_a(s,\omega). \tag{8.57b}$$

Proof: Exercise. ∎

Observe that with parameter s fixed $y(-k) = x(s - k)$ is a discrete signal. If we further set $z = \exp(2\pi j\omega)$, then the ZT summation with $a = 1$ becomes

$$X(s,\omega) = \sum_{k=-\infty}^{\infty} y(-k)\exp(2\pi j\omega k) = \sum_{k=-\infty}^{\infty} y(k)\exp(-2\pi j\omega k) = \sum_{k=-\infty}^{\infty} y(k)z^{-k}. \tag{8.58}$$

Equation (8.58) is the z-transform of $y(k)$ evaluated on the unit circle. More precisely, it is the DTFT of $y(k)$ with 2π frequency dilation.

8.3.2.2 An Isomorphism. We now show an interesting isomorphism between the Hilbert space of finite-energy analog signals $L^2(\mathbb{R})$ and the square-integrable two-dimensional analog signals on the unit square $S = [0, 1] \times [0, 1]$.

The Zak transform is in fact a unitary map from $L^2(\mathbb{R})$ to $L^2(S)$; that is, $\mathcal{Z}$ is a Hilbert space map that takes an $L^2(\mathbb{R})$ orthonormal basis to an $L^2(S)$ orthonormal basis in a one-to-one and onto fashion [27, 28].

Lemma. Let $b(t) = u(t) - u(t - 1)$ be the unit square pulse, where $u(t)$ is the analog step function. Then $\{b_{m,n}(t) = \exp(2\pi jmt)b(t - n) \mid m, n \in \mathbb{Z}\}$ is a basis for $L^2(\mathbb{R})$. Moreover, $\{e_{m,n}(s, t) = \exp(2\pi jms)\exp(2\pi jnt) \mid m, n \in \mathbb{Z}\}$ is a basis for $L^2(S)$.

Proof: Apply Fourier series arguments to the unit intervals on $\mathbb{R}$. The extension of Fourier series to functions of two variables is outside our one-dimensional perspective, but is straightforward, and can be found in advanced Fourier analysis texts (e.g., Ref. 29.) ■

Theorem (Zak Isomorphism). The Zak transform $\mathcal{Z}$: $L^2(\mathbb{R}) \to L^2(S)$ is unitary.

Proof: Let's apply the ZT to the Fourier basis on $L^2(\mathbb{R})$, $\{b_{m,n}(t) \mid m, n \in \mathbb{Z}\}$ of the lemma:

$$
\begin{aligned}
(\mathcal{Z}b_{m,n})(s,\omega) &= \sum_{k=-\infty}^{\infty} b_{m,n}(s-k)\exp(2\pi j\omega k) \\
&= \exp(2\pi jms)\exp(-2\pi jn\omega)\sum_{k=-\infty}^{\infty} b(s-n-k)\exp(2\pi j\omega(n-k)) \\
&= \exp(2\pi jms)\exp(-2\pi jn\omega)(\mathcal{Z}b)(s,\omega) = e_{m,-n}(s,\omega)(\mathcal{Z}b)(s,\omega). \qquad (8.59)
\end{aligned}
$$

Let us reflect on the last term, $(\mathcal{Z}b)(s, \omega)$ for $(s, \omega) \in S$. We know that

$$(\mathcal{Z}b)(s,\omega) = \sum_{k=-\infty}^{\infty} b(s-k)\exp(2\pi j\omega k). \qquad (8.60)$$

On the interior of S, $(0, 1) \times (0, 1)$, we have $b(s - k) = 0$ for all $k \neq 0$. So only one term counts in the infinite sum (8.60), namely $k = 0$, and this means $(\mathcal{Z}b)(s, \omega) = 1$ on the unit square's interior. On the boundary of S, we do not care what happens to the ZT sum, because the boundary has (two-dimensional) Lebesgue measure zero; it does not affect the $L^2(S)$ norm. Thus, $\mathcal{Z}$ sends $L^2(\mathbb{R})$ basis elements $b_{m,n}$ to $L^2(S)$ basis elements $e_{m,-n}$, and is thus unitary. ■

8.4 SUMMARY

The z-transform extends the discrete-time Fourier transform from the unit circle to annular regions complex plane, called regions of convergence. For signal frequency, the DTFT is the right inspection tool, but system properties such as stability can be investigated with the z-transform. Readers may recall the Laplace transform from system theory and differential equations work; it bears precisely such a relationship to the analog Fourier transform (Chapter 5). The Laplace transform extends the definition of the Fourier transform, whose domain is the real numbers, to regions of the complex plane.

The next chapter covers frequency-domain signal analysis, including both analog and digital filter design. It most assuredly explores further z-transform techniques.

This chapter closed with an introduction to two related tools: the chirp z-transform and the Zak transform. The CZT is a discretized z-transform computed on a custom contour. If the contour follows the unit circle, then the CZT can be used to save some computational steps that we would ordinarily suffer when computing a DFT of difficult (prime) order. Or, careful contour selection with the CZT gives more frequency coefficients in a narrow application range than the Fourier transform. The Zak transform's isomorphism property effectively converts questions about $L^2(\mathbb{R})$ analog signals into questions about finite-energy signals on the unit square. Analytically, the unit square, even though it is two-dimensional, is often easier to deal with. This benefit of the ZT makes it especially powerful when we study frames based on windowed Fourier atoms in Chapter 10.

8.4.1 Historical Notes

Kaiser [14] introduced the z-transform into the signal processing discipline from control theory only in the early 1960s. At the time, digital computer applications had stimulated interest in discrete transforms, filtering, and speech processing. Filters are systems that pass some frequency ranges while suppressing others, and they are common at the front end of a signal analysis system that must interpret oscillatory data streams. It turns out—as we shall see in the next chapter—that very good filters can be built out of simple recursive structures based on difference equations. The z-transform readily gives the system function for such difference equations as a rational function of a single complex variable: $H(z) = B(z)/A(z)$. We have developed straightforward algebraic methods for inverting such rational functions, which in turn reveals the system impulse response and allows us to calculate the system response to various inputs.

In the late 1960s, Bluestein [30] first showed how to compute the DFT using a chirped linear filtering operation. The formalization of CZT algorithm and many of its original applications are due to Rabiner, Schafer, and Rader [22, 23].

The ZT arrives relatively late to signal theory from physics [31], where Zak developed it independently for solid-state applications. Janssen introduced it into the mainstream signal analysis literature [26]. The transform has been many places—indeed, Gauss himself may have known of it [28].

8.4.2 Guide to Problems

Readers should find most problems straightforward. Problems 2 and 3 explore some of the limit ideas and radius of converge concepts used in the chapter. There is a z-transform characterization of stable systems, which is developed in the later problems. Finally, some computer programming tasks are suggested.

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PROBLEMS

1. Find the z-transform and ROC for each of the following signals:

(a) $x(n) = u(n-5) - u(n+2)$, where $u(n)$ is the discrete unit step signal. Can one simply apply the linearity and shift properties to $x(n)$ for the right answer?

(b) $\delta(n-4) + u(n)$, where $\delta(n)$ is the discrete impulse.

(c) $x(n) = 3^n u(-n) + n2^{-n}u(n)$.

(d) $x(n) = u(n)[n2^{n-1}]$.

(e) $x(n) = u(n)[n2^{n-1} + n]$.

(f) $x(n) = 1/n!$

(g) $x(n) = u(-n-1)(1/3)^n$.

(h) $x(n) = u(n)(-1/5)^n + u(-n-1)(1/2)^n$.

2. Consider the lim sup and lim inf of a sequence, $A = \{a_n: 0 \le n < \infty\}$. Suppose we have defined elements of the sequence as follows: $a_0 = 0$; $a_n = 1 + 1/n$, if n is even; and $a_n = -1 - 1/n$, if n is odd.

(a) Show that the sequence A has no limit.

(b) Show that the lim sup A is 1.

(c) Show that the lim inf of A is –1.

(d) Let $A_N = A \setminus \{a_n: 0 \le n < N\}$ and κ_N be the least upper bound of A_N. Show that $\kappa_N \le \kappa_M$ if $M < N$.

(e) Show that a sequence $B = \{b_n: 0 \le n < \infty\}$ has a limit if and only if its lim inf and its lim sup are equal. What about the cases where the limit is $\pm\infty$?

(f) Show that

$$\liminf_{n\to\infty}\{b_n: 0 \le n < \infty\} = -\limsup_{n\to\infty}\{-b_n: 0 \le n < \infty\}. \tag{8.61}$$

3. Suppose $\mathcal{Z}(x(n)) = X(z)$ has only non-negative powers of z:

$$X(z) = \sum_{n=0}^{+\infty} a_n z^n. \tag{8.62}$$

Let

$$\kappa = \limsup_{n\to\infty}\{|a_n|^{1/n}: 1 \le n < \infty\}, \tag{8.63}$$

so that $\rho = \kappa^{-1}$ is the radius of convergence of $X(z)$. Show that the radius of convergence for the derivative,

$$X'(z) = \frac{dX(z)}{dz} = \sum_{n=0}^{+\infty} n a_n z^{n-1}, \tag{8.64}$$

is also ρ.

4. Let $Z(x(n)) = X(z)$. Show the following z-transform symmetry properties:

(a) $Z[x^*(n)] = X^*(z^*)$, where z^* is the complex conjugate of z.

(b) (Time Reversal) If $y(n) = x(-n)$, then $Z[y(n)] = Y(z) = X(z^{-1})$, and $\text{ROC}_Y = \{z \in \mathbb{C}\colon z^{-1} \in \text{ROC}_X\}$.

(c) If $y(n) = \text{Real}[x(n)]$, then $Y(z) = [X(z) + X^*(z^*)]/2$.

(d) If $y(n) = \text{Imag}[x(n)]$, then $Y(z) = j[X^*(z^*) - X(z)]/2$.

(e) Find the z-transform of $x_e(n)$, the even part of $x(n)$.

(f) Find the z-transform of $x_o(n)$, the odd part of $x(n)$.

5. Suppose $X(z) = z/(z - a)$, $a \neq 0$, with $\text{ROC}_X = \{z \in \mathbb{C}\colon |a| < |z|\}$. In the first example of Section 8.1.1, we found that $x(-1) = x(-2) = 0$ and claimed that $x(n) = 0$ for $n < -2$. For the last case, $n < -2$, verify that

(a) $\text{Res}(f(z), z = 0) = -a^{-n}$.

(b) $\text{Res}(f(z), z = a) = a^{-n}$.

(c) $x(n) = 0$ for $n < -2$.

6. Suppose $X(z) = z/(z - a)$, $a \neq 0$, with $\text{ROC}_X = \{z \in \mathbb{C}\colon |z| < |a|\}$. Using the method of contour integration, find $x(n)$ for all $n \in \mathbb{Z}$.

7. Suppose $X(z) = z(z-2)^{-1}(z-1)^{-1}$.

(a) Let $\text{ROC}_X = \{z \in \mathbb{C}\colon |z| < 1\}$. With the method of inverse z-transformation by computation of the Laurent series, find $x(n)$.

(b) Suppose now that $\text{ROC}_X = \{z \in C\colon 2 > |z| > 1\}$. Is it possible to use the long division method to find the Laurent series form of $X(z)$ and thence find $x(n)$? Explain.

8. Suppose that

$$X(z) = \frac{z}{(z-1)(z+2)^2}, \tag{8.65}$$

and $\text{ROC}_X = \{z \in \mathbb{C}\colon 2 < |z|\}$.

(a) Find A, B, and C to derive the expansion of $z^{-1}X(z)$ into partial fractions:

$$\frac{X(z)}{z} = \frac{1}{(z-1)(z+2)^2} = \frac{A}{(z-1)} + \frac{B}{(z+2)^2} + \frac{C}{(z+2)}. \tag{8.66}$$

(b) Find the discrete signal whose z-transform is $z^{-1}X(z)$.

(c) Find the discrete signal whose z-transform is $X(z)$.

9. Again suppose

$$X(z) = \frac{z}{(z-1)(z+2)^2}. \tag{8.67}$$

Find all of the discrete signals whose z-transforms are equal to $X(z)$. For each such signal,

(a) State the region of convergence.

(b) Sketch the region of convergence.

(c) State whether the signal is causal, anticausal, or neither.

10. Signal $x(n)$ has z-transform $X(z)/z = 1 / (z^2 - 3z/2 - 1)$. Find three different possibilities for $x(n)$ and give the ROC of $X(z)$ for each.

11. If $x(n)$ has z-transform $X(z) = z / (z^2 - 5z - 14)$, then find three different possibilities for $x(n)$ and give the ROC of $X(z)$ for each.

12. Let $X^+(z)$ be the one-sided z-transform for $x(n)$.

(a) Show that the one-sided z-transform is linear.

(b) Show that the one-sided z-transform is not invertible by giving examples of different signals that have the same transform.

(c) Show that if $x(n) = 0$ for $n < 0$, then $X^+(z) = X(z)$.

(d) Let $y(n) = x(n - k)$. If $k > 0$, show that the shift property becomes

$$Y^+(z) = x(-k) + x(-k+1)z^{-1} + \cdots + x(-1)z^{-m+1} + z^{-m}X^+(z). \tag{8.68}$$

13. A simple difference equation,

$$y(n) = ay(n-1) + x(n), \tag{8.69}$$

describes a signal processing system. Some signed fraction $0 < |a| < 1$ of the last filtered value is added to the current input value $x(n)$. One application of the one-sided z-transform is to solve the difference equation associated with this system [4, 6]. Find the unit step response of this system, given the initial condition $y(-1) = 1$, as follows.

(a) Take the one-sided z-transforms of both sides of (8.69):

$$Y^+(z) = a\left[Y^+(z)z^{-1} + y(-1)\right] + X^+(z). \tag{8.70}$$

(b) Use the initial condition to get

$$Y^+(z) = \frac{a}{(1-az^{-1})} + \frac{1}{(1-z^{-1})(1-az^{-1})}. \tag{8.71}$$

(c) Apply the partial fractions method to get the inverse z-transform:

$$y(n) = \frac{(1-a^{n+2})}{(1-a)}u(n). \tag{8.72}$$

14. The Fibonacci[2] sequence is defined by $f(-2) = 1$, $f(-1) = 0$, and

$$f(n) = f(n-1) + f(n-2). \tag{8.73}$$

(a) Show that $f(0) = f(1) = 1$.

(b) Using the one-sided z-transform [4], show

$$y(n) = \frac{u(n)}{2^{n+1}\sqrt{5}}\left[(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}\right]. \tag{8.74}$$

15. Consider the system H given by the following difference equation:

$$y(n) = 0.25 * y(n-2) + 0.25 * y(n-1) + 0.5 * x(n). \tag{8.75}$$

(a) Find the system function $H(z)$.

(b) Assuming that H is a causal system, find $h(n)$.

(c) Give ROC_H for the causal system.

(d) What are the poles for the system function?

(e) Is the system stable? Explain.

16. Show that a discrete LTI system H is causal, $h(n) = 0$ for $n < 0$, if and only if ROC_H is $\{z: |z| > r\}$ for some $r > 0$.

17. Show that a discrete LTI system H is stable (bounded input implies bounded output signal) if and only if its z-transform ROC includes the unit circle $|z| = 1$.

18. Show that a causal LTI system H is stable if and only if all of the poles of $H(z)$ lie inside the unit circle $|z| = 1$.

19. Consider the causal system H given by the following difference equation:

$$y(n) = Ay(n-1) + Bx(n). \tag{8.76}$$

(a) Find necessary and sufficient conditions on constants A and B so that H is stable.

(b) Find the unit step response $y = Hu$, where $u(n)$ is the unit step signal.

(c) Show that if A and B satisfy the stability criteria in (a), then the unit step response in (b) is bounded.

(d) Find the poles and zeros of the system function $H(z)$.

20. Assume the notation for chirp z-transform of Section (8.31).

(a) Show that if $A = 1$, $M = N$, and $W = \exp(-2\pi j/N)$ in (8.49), then $X_{A,W}(k) = X(k)$, where $X(k)$ is the DFT of order N for the signal $x(n)$.

[2]Leonardo of Pisa (c. 1175–1250) is known from his father's name. The algebraist and number theorist asked a question about rabbits: If an adult pair produces a pair of offspring, which mature in one month, reproduce just as their parents, and so on, then how many adult rabbits are there after N months? The answer is F_N, the Nth Fibonacci number.

(b) Show that $W_0 > 1$ implies an inward spiral and $W_0 < 1$ produces an outward spiral path.

21. Derive the periodicity relations for the Zak transform (8.57a, 8.57b).

The next few problems are computer programming projects.

22. As a programming project, implement the CZT algorithm of Section 8.3.1.2. Compare the fast CZT algorithm performance to the brute-force convolution in (8.52). Use the algorithm to compute some DFTs for large prime orders. Compare the CZT-based algorithm to straight DFT computations.

23. Implement the z-transform for finitely supported discrete signals in a computer program. Verify the convolution property of the z-transform by calculating the z-transforms, $X(z)$ and $Y(z)$, of two nontrivial signals, $x(n)$ and $y(n)$, respectively; their convolution $z(n)$; and the z-transform $Z(z)$. Finally, confirm that $Z(z) = X(z)Y(z)$ with negligible numerical error.

24. Consider the implementation of the inverse z-transform on a digital computer. Which approach might be easiest to implement? Which is the most general? Develop an application that handles some of possible forms of $X(z)$. Explain the strengths and limitations of the application.