CHAPTER 6

Generalized Fourier Transforms of Analog Signals

This chapter extends Fourier analysis to common signals that lie outside of the spaces $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$.

The theory of $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$ Fourier transforms is broad enough to encompass a large body of signal processing and analysis. The foundation provided for the transform allows us to discover the frequency content of analog signals. One might be content with the situation as it stands, but several common and practical functions are neither absolutely integrable nor of finite energy. For example:

- The simple sinusoids $f(t) = \sin(\omega_0 t)$ and $f(t) = \cos(\omega_0 t)$. It is difficult to imagine functions for which the notion of frequency content is any more straightforward, yet the radial Fourier transform

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt = \mathcal{F}[f(t))](\omega) \tag{6.1}$$

  does not converge. Similar comments clearly apply to the complex exponential.
- The function $f(t) = c_0$, where c_0 is a positive or negative constant. Constant electrical signals are called direct current (DC) signals in engineering. Again, the notion of frequency content for this DC signal could hardly be more intuitive, but convergence of the Fourier integral fails.
- The unit step $u(t)$ and its close relative the signum, $\operatorname{sgn} t$ (see Figure 5.8), which clearly do not belong to the class of integrable or square-integrable functions.

Texts devoted to distributions and generalized Fourier transforms are Refs. 1–3. Mathematical analysis texts that also introduce the theory include Refs. 4–6.

6.1 DISTRIBUTION THEORY AND FOURIER TRANSFORMS

Our first encounter with useful integrals that defy solution using classical methods of calculus arose in Chapter 3, where integration of classically troublesome entities,

Signal Analysis: Time, Frequency, Scale, and Structure, by Ronald L. Allen and Duncan W. Mills
ISBN: 0-471-23441-9

such as the derivative of the unit step, were elegantly handled through distributions. The theoretical framework employed test functions of rapid descent which were classically well-behaved and generated a calculus of distributions simply because the classical notions of derivatives could be applied directly to the test functions themselves. These developments suggest that if the troublesome signals listed above are treated as distributions, and the test functions have traditional Fourier transforms, then a theory of generalized Fourier transforms, embracing the selected distributions, can be formulated.

Consider replacing the complex exponential with some function $\Phi(t)$ which is sufficiently well-behaved to allow the integral over time, namely

$$\int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt \to \int_{-\infty}^{\infty} f(t)\Phi(t)\,dt \tag{6.2}$$

to converge. Intuitively, one needs a $\Phi(t)$ which decays rapidly enough to counter the lack of integrability inherent in $f(t)$. Two key points follow:

- Each of the nonintegrable signals $f(t)$ under consideration is a function of slow growth (Chapter 3) and therefore represents a regular distribution of slow growth when set in the context of generalized integrals.
- The class of testing functions is Fourier transformable in the "regular" sense of Section 5.2; this is our link to frequency space.

The study of distributions in the time domain was based on the classical concept of integration by parts. Similarly, the classically derived Parseval relations extend the theory of distributions into the frequency domain. We propose the following:

Definition (Generalized Fourier Transform). Let $f(t)$ be a distribution of slow growth. Note that if $\phi(\alpha)$ is a testing function of rapid descent, we can define a Fourier transform,

$$\Phi(\beta) = \int_{-\infty}^{\infty} \phi(\alpha)e^{-j\beta\alpha}\,d\alpha. \tag{6.3}$$

By Parseval's theorem

$$\int_{-\infty}^{\infty} F(\omega)\phi(\omega)\,d\omega = \int_{-\infty}^{\infty} f(t)\Phi(t)\,dt. \tag{6.4}$$

The function $F(\omega)$ is the *generalized Fourier transform* of $f(t)$.

Remark. In the event that $f(t)$ is integrable, the generalized Fourier transform is merely an expression of Parseval's theorem for such functions. Consequently, $F(\omega)$ is a bona fide generalized Fourier transform encompassing both the integrable functions (which are also covered by the Paresval relation) and the distributions of slow growth.

In the generalized Fourier transform, note that ω and t within the integrals (6.4) are merely continuous variables; the simple form of the generalized Fourier transform may be extended so that

$$\int_{-\infty}^{\infty} F(\omega)\phi(\omega)\, d\omega = \int_{-\infty}^{\infty} f(t)\Phi(t)\, dt = \int_{-\infty}^{\infty} F(t)\phi(t)\, dt = \int_{-\infty}^{\infty} f(\omega)\Phi(\omega)\, d\omega. \quad (6.5)$$

This is of more than academic interest and allows for greater dexterity when deriving the properties of the generalized Fourier transform.

6.1.1 Examples

How does our formulation of the generalized Fourier transform perform for the important, simple signals? Let us investigate the case of constant (DC) signals and impulses.

Example (DC Waveform). Let $f(t) = 1$ for all $t \in \mathrm{R}$. This signal represents a constant DC level for all values of t and is a function of slow growth. The generalized Fourier transform takes the form

$$\int_{-\infty}^{\infty} F(\omega)\phi(\omega)\, d\omega = \int_{-\infty}^{\infty} \Phi(t)\, dt = F[\Phi(t)](\omega)\big|_{\omega = 0} = 2\pi\phi(-\omega)\big|_{\omega = 0}. \quad (6.6)$$

The quantity following the last equality is simply $2\pi\phi(0)$, which can be written in terms of the Dirac delta:

$$\int_{-\infty}^{\infty} F(\omega)\phi(\omega)\, d\omega = 2\pi \int_{-\infty}^{\infty} \delta(\omega)\phi(\omega)\, d\omega. \quad (6.7)$$

Comparing both sides of (6.7), it is readily apparent that

$$F(\omega) = 2\pi\delta(\omega) \quad (6.8)$$

represents the spectrum of the constant DC signal. This result supports the intuitively appealing notion that a constant DC level represents clusters its entire frequency content at the origin. We have already hinted at this in connection with the Fourier transform of the rectangular pulse in the limit of large width; in a sense, (6.8) is the ultimate expression of the scaling law for a rectangular pulse.

The time-dependent Dirac delta represents the converse:

Example (Impulse Function). Consider a Dirac delta impulse, $f(t) = \delta(t)$. The generalized Fourier transform now reads

$$\int_{-\infty}^{\infty} F(\omega)\phi(\omega)\,d\omega = \int_{-\infty}^{\infty} \delta(t)\Phi(t)\,dt = \mathrm{F}[\phi(t)](\omega)\big|_{\omega=0} = \int_{-\infty}^{\infty} \phi(t)\,dt$$

$$= \int_{-\infty}^{\infty} \phi(\omega)\,d\omega. \tag{6.9}$$

From a comparison both sides,

$$\mathcal{F}[\delta(t)](\omega) = 1 \tag{6.10}$$

for all $\omega \in \mathrm{R}$. The Dirac delta function's spectrum therefore contains equal contributions from all frequencies. Intuitively, this result is expected.

6.1.2 The Generalized Inverse Fourier Transform

The reciprocity in the time and frequency variables in (6.4) leads to a definition of a generalized inverse Fourier transform.

Definition (Generalized Inverse Fourier Transform). Let $F(\omega)$ be a distribution of slow growth. If $\Phi(\beta)$ is a testing function of rapid descent, then it generates an inverse Fourier transform:

$$\phi(\alpha) = \int_{-\infty}^{\infty} \Phi(\beta)e^{j\beta\alpha}d\beta. \tag{6.11}$$

Once again, by Parseval's theorem

$$\int_{-\infty}^{\infty} F(\omega)\phi(\omega)\,d\omega = \int_{-\infty}^{\infty} f(t)\Phi(t)\,dt, \tag{6.12}$$

and $f(t)$ is called the *generalized inverse Fourier transform* of $F(\omega)$. This definition is so intuitive it hardly needs to be written down.

No discussion of the generalized Fourier transform would be complete without tackling the remaining functions of slow growth which are central to many aspects of signal generation and analysis. These include the sinusoids and the appropriate piecewise continuous functions such as the unit step and signum functions. Their generalized spectra are most easily determined by judicious application of selected properties of the generalized Fourier transform. Prior to completing that discussion, it is useful to illustrate some general properties of the generalized Fourier transform.

6.1.3 Generalized Transform Properties

In the previous examples we have emphasized that the generalized Fourier transform is an umbrella which encompasses standard, classical Fourier-integrable as well as slow growth functions considered as regular distributions. The properties of the classically defined Fourier transform demonstrated in Chapter 5 apply with little or no modification to the generalized transform. Naturally, the methods for proving them involve the nuances specific to the use of generalized functions. A general strategy when considering properties of the generalized Fourier transform is to begin with integrals (6.5) and allow the desired parameter (scale, time shift) or operator (differential) to migrate to the classical Fourier transform of the test function, where its effects are easily quantified. The reader should study the following examples carefully. The generalized Fourier transform is elegant but seductive; a common pitfall is to rearrange the generalized transform so it resembles the familiar classical integral and then "declare" a transform when in fact the classical integral will not coverge because the integrand is not integrable.

Proposition (Linearity). Let $f(t)$ represent the linear combination of arbitrary distributions of slow growth,

$$f(t) = \sum_{k=1}^{N} a_k f_k(t). \tag{6.13}$$

Then,

$$\int_{-\infty}^{\infty} F(\omega)\phi(\omega)\, d\omega = \sum_{k=1}^{N} a_k \int_{-\infty}^{\infty} f_k(t)\Phi(t)\, dt = \int_{-\infty}^{\infty} \left\{ \sum_{k=1}^{N} a_k F_k(\omega) \right\} \phi(\omega)\, d\omega. \tag{6.14}$$

The expected result follows:

$$F(\omega) = \sum_{k=1}^{N} a_k F_k(\omega). \tag{6.15}$$

Proposition (Time Shift or Translation). Let $f(t)$ be a distribution of slow growth subjected to a time shift t_0 such that $f(t) \to f(t - t_0)$. Then,

$$\mathcal{F}[f(t - t_0)](\omega) = \mathcal{F}[f(t)]e^{-j\omega t_0}. \tag{6.16}$$

Use of the defining generalized Fourier transform relations leads to the following equalities (we reduce clutter in the integral by suppressing the (ω) suffix in $\mathcal{F}$):

$$\int_{-\infty}^{\infty} \mathcal{F}[f(t-t_0)]\phi(\omega)\,d\omega = \int_{-\infty}^{\infty} f(\alpha-t_0)\Phi(\alpha)\,d\alpha = \int_{-\infty}^{\infty} f(\gamma)\Phi(\gamma+t_0)\,d\gamma. \quad (6.17)$$

The change of variable, $\gamma \equiv \alpha - t_0$, in the last integral of (6.17) places the time shift conveniently within the classical Fourier transform of the test function. From here, matters are straightforward:

$$\begin{aligned}\int_{-\infty}^{\infty} f(\gamma)\Phi(\gamma+t_0)\,d\gamma &= \int_{-\infty}^{\infty} \mathcal{F}[f(t)]\,\mathcal{F}^{-1}[\Phi(\gamma+t_0)]\,d\omega \\ &= \int_{-\infty}^{\infty} \mathcal{F}[f(t)]\phi(\omega)e^{-j\omega t_0}\,d\omega \end{aligned} \quad (6.18)$$

so that

$$\int_{-\infty}^{\infty} \mathcal{F}[f(t-t_0)]\phi(\omega)\,d\omega = \int_{-\infty}^{\infty} \mathcal{F}[f(t)]\phi(\omega)e^{-j\omega t_0}\,d\omega, \quad (6.19)$$

and the property is proven:

$$\mathcal{F}[f(t-t_0)](\omega) = \mathcal{F}[f(t)]e^{-j\omega t_0}. \quad (6.20)$$

We leave the remaining significant properties of the generalized Fourier transform to the exercises. As we have noted, they are identical to the properties of the standard integral transform, and the proofs are straightforward.

Remark. In the case of regular functions $f(t)$ considered in Chapter 5, the validity of time differentiation property,

$$\mathcal{F}\left[\frac{d}{dt}f(t)\right](\omega) = j\omega F(\omega), \quad (6.21)$$

was conditioned upon $\lim_{|t|\to\infty} f(t) = 0$. No such restriction applies to distributions of slow growth, since the convergence of the generalized Fourier transform is assured by the decay of the testing functions of rapid descent.

Using the properties of the generalized transform, we can resume calculating spectra for the remaining functions of slow growth. These are central to much of signal analysis. We cover the signum function, the unit step, and the sinusoids.

Example (Signum). Consider the case $f(t) = \operatorname{sgn}(t)$. The differentiation property implies

$$\mathcal{F}\left[\frac{d}{dt}\operatorname{sgn}(t)\right](\omega) = j\omega F(\omega) = \mathcal{F}[2\delta(t)], \tag{6.22}$$

where we have used the recently derived transform of the Dirac delta function. From here, one is tempted to conclude that the desired spectrum $F(\omega) = \frac{2}{j\omega}$. However, a certain amount of care is required since in general $\omega F_1(\omega) = \omega F_2(\omega)$ does not imply $F_1(\omega) = F_2(\omega)$. This is another instance of unusual algebra resulting from the Dirac delta property, derived in Chapter 3: $\omega\delta(\omega) = 0$. Under the circumstances, this allows for the possibility of an additional term involving an impulse function, so that

$$\omega F_1(\omega) = \omega[F_2(\omega) + c_0\delta(\omega)], \tag{6.23}$$

where c_0 is a constant to be determined. Returning to the example, with $F_1(\omega) \equiv F(\omega)$ and $\omega F_2(\omega) \equiv 2$, we obtain a complete and correct solution:

$$F(\omega) = \frac{2}{j\omega} + c_0\delta(\omega). \tag{6.24}$$

A determination of c_0 can be made by appealing to symmetry. Since $\operatorname{sgn}(t)$ is a real, odd function of t, its transform must be purely imaginary and odd in the frequency variable ω. Hence, we conclude that $c_0 = 0$. The spectrum is shown in Figure 6.1.

Based on this result, we can proceed to the unit step.

Example (Unit Step). Let $f(t) = u(t)$. Then,

$$\mathcal{F}[u(t)](\omega) = \pi\delta(\omega) + \frac{1}{j\omega}. \tag{6.25}$$

This proof is left as an exercise. Note that the two terms above, $F_e(\omega)$ and $F_o(\omega)$, represent even and odd portions of the frequency spectrum. These may be obtained directly from even and odd components of $f(t)$, $f_e(t)$, and $f_o(t)$, respectively, in accordance with symmetries developed in Chapter 5. The Dirac delta impulse is a legacy of $f_e(t)$, which is not present in the signum function. The resulting spectrum Figure 6.1b is complex.

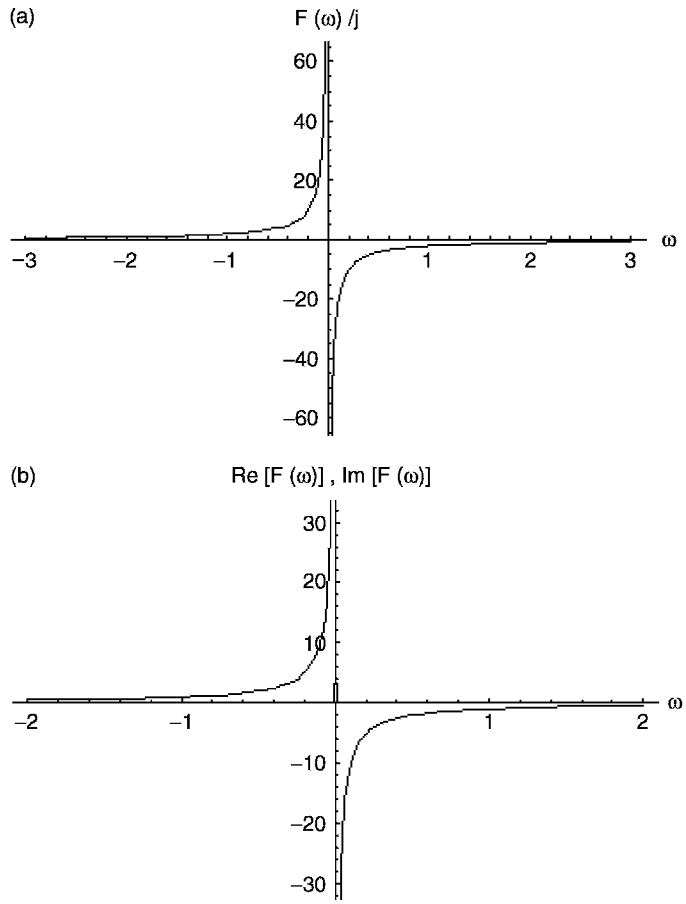

Fig. 6.1. (a) The Fourier transform of sgn(t) is purely imaginary and inversely proportional to ω. (b) The transform of the unit step consists of real Dirac delta function and an imaginary part, as shown.

Examples (Powers of t). The frequency differentiation property

$$\mathcal{F}[(-jt)^n f(t)](\omega) = \frac{d^n}{dt^n}F(\omega) \tag{6.26}$$

leads to several useful Fourier transforms involving powers of t and generalized functions. For integer $n \geq 0$

$$\mathcal{F}[t^n](\omega) = 2\pi j^n \cdot \frac{d^n\delta(\omega)}{d\omega^n}, \tag{6.27}$$

$$\mathcal{F}[t^n u(t)](\omega) = j^n \cdot \left[\pi \frac{d^n \delta(\omega)}{d\omega^n} + \frac{1}{j} \frac{(-1)^n n!}{\omega^{n+1}}\right], \tag{6.28}$$

$$\mathcal{F}[t^n \operatorname{sgn} t](\omega) = (-2)j^{n+1} \frac{(-1)^n n!}{\omega^{n+1}}. \tag{6.29}$$

The derivations are straightforward an left as an exercise. As expected, each of the above spectra contain singularities at $\omega = 0$ on account of the discontinuity in $f(t)$.

Integral powers of $|t|$ are easily handled. For even n,

$$\mathcal{F}[|t|^n](\omega) = \mathcal{F}[t^n](\omega), \tag{6.30}$$

so (6.27) applies. For odd n, note the convenient relation

$$\mathcal{F}[|t|^n](\omega) = \mathcal{F}[t^n \operatorname{sgn}(t)](\omega). \tag{6.31}$$

Remark. Treatment of fractional exponents $0 < n < 1$ and the theory of generalized Fourier transforms for $f(t)$ exhibiting logarithmic divergences is possible, but outside our scope.

Inverse integral powers of t, which are clearly neither integrable nor square integrable, readily yield generalized Fourier spectra. For example,

$$\mathcal{F}\left[\frac{1}{t}\right](\omega) = -j \cdot \pi \cdot \operatorname{sgn}(\omega) \tag{6.32}$$

follows from the application of the symmetry property to $\mathcal{F}[\operatorname{sgn}(t)](\omega)$. Repeated application of time differentiation leads to a more general result for integer $m > 0$:

$$\mathcal{F}\left[\frac{1}{t^m}\right](\omega) = \frac{-\pi}{(m-1)!} \cdot j^m \cdot \omega^{m-1} \cdot \operatorname{sgn}(\omega). \tag{6.33}$$

Example (Complex Exponential). Let $f(t) = e^{j\omega_0 t}$ represent a complex exponential with oscillations at a selected frequency ω_0. According to the frequency shift property of the generalized Fourier transform,

$$\mathcal{F}[g(t)e^{j\omega_0 t}](\omega) = G(\omega - \omega_0), \tag{6.34}$$

the simple substitution $g(t) = 1$— the constant DC signal—provides the desired result:

$$\mathcal{F}[e^{j\omega_0 t}](\omega) = 2\pi\delta(\omega - \omega_0). \tag{6.35}$$

An example of this transform is shown in Figure 6.2a. Not surprisingly, the spectrum consists of an oscillation at a single frequency. In the limit $\omega_0 \to 0$, the spectrum reverts to $2\pi\delta(\omega)$, as fully expected.

Proposition (General Periodic Signal). Let $f(t)$ represent a periodic distribution of slow growth with period *T*. Then

$$\mathcal{F}[f(t)](\omega) = \frac{2\pi}{\sqrt{T}} \sum_{n=-\infty}^{\infty} c_n \delta(\omega - n\omega_0), \tag{6.36}$$

where the c_n represent the exponential Fourier series coefficients for $f(t)$ and $\omega_0 = 2\pi T$.

This is almost trivial to prove using the linearity property as applied to an exponential Fourier series representation of the periodic signal:

$$f(t) = \frac{1}{\sqrt{T}} \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}. \tag{6.37}$$

This leads immediately to

$$\mathcal{F}[f(t)](\omega) = \frac{1}{\sqrt{T}} \sum_{n=-\infty}^{\infty} c_n \mathcal{F}[e^{jn\omega_0 t}] \tag{6.38}$$

from which the desired result (6.36) follows.

This is an important conclusion, demonstrating that the Fourier series is nothing more than a special case of the generalized Fourier transform. Furthermore, upon application of the Fourier inversion, the sifting property of the Dirac delta readily provides the desired synthesis of $f(t)$:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}[f(t)] e^{j\omega t} d\omega = \sum_{n=-\infty}^{\infty} \frac{c_n}{\sqrt{T}} \int_{-\infty}^{\infty} \delta(\omega - n\omega_0) e^{j\omega t} d\omega \tag{6.39}$$

which trivially reduces to the series representation of $f(t)$ as given in (6.37).

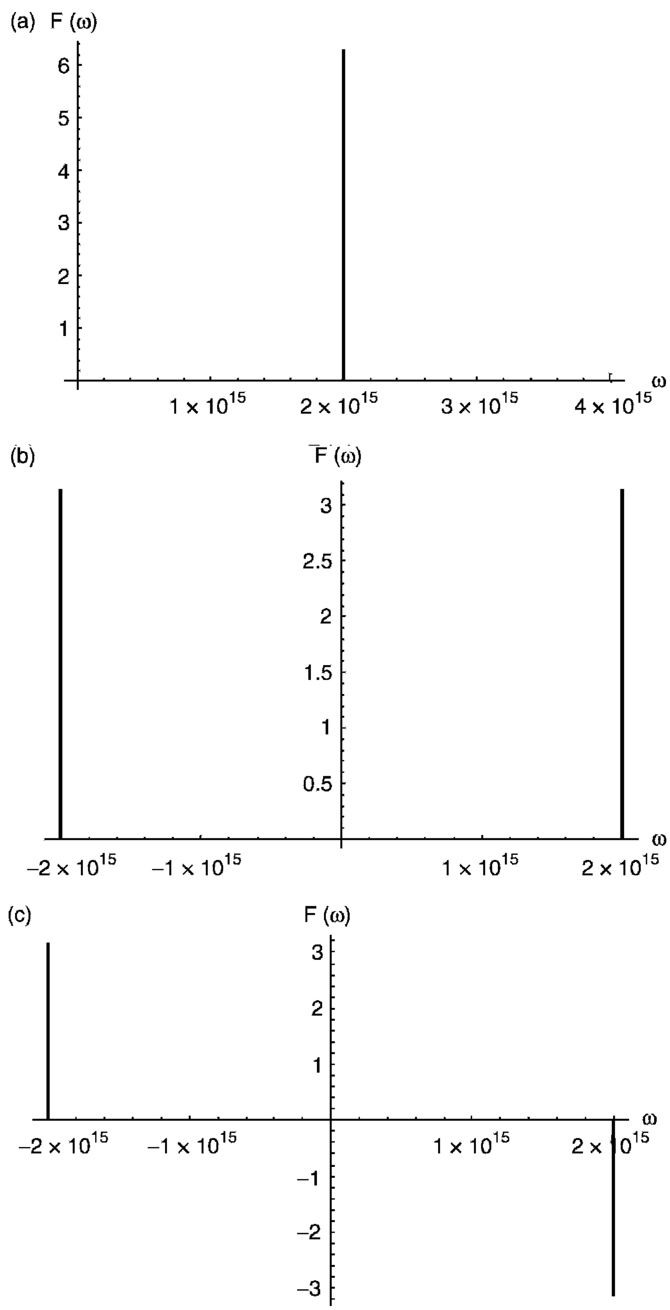

Fig. 6.2. Fourier transforms of (a) the complex exponential with fundamental frequency $\omega_0 = 2 \times 10^{15}$ rad/s, (b) a cosine of the same frequency, and (c) the corresponding sine wave.

Example (Cosine and Sine Oscillations). Let $f(t) = \cos(\omega_0 t)$. From the Euler relation—linking the sinusoids to the complex exponential—and the linearity property, we obtain

$$\mathcal{F}[\cos(\omega_0 t)](\omega) = \pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]. \tag{6.40}$$

Hence, the cosine exhibits a two-sided spectrum with contributions at $\omega = \pm\omega_0$, as Figure 6.2b illustrates. Note that in the process of forming a sinusoid, the spectral amplitude 2π inherent in the complex exponential has been redistributed equally amongst the positive and negative frequencies.

In the case of the sinusoid $f(t) = \sin(\omega_0 t)$, similar arguments demonstrate that

$$\mathcal{F}[\sin(\omega_0 t)](\omega) = j\pi(\delta(\omega + \omega_0) - \delta(\omega - \omega_0)). \tag{6.41}$$

The sine spectrum, shown in Figure 6.2c, is similar in form to the cosine but is, according to symmetry arguments, an odd function in frequency space. We will make use of both of the previous examples in the sequel. In particular, when we develop transforms that combine time- and frequency-domain information, the calculations of sinusoidal spectra will play an important role.

Generalized functions, particularly the Dirac delta function, arise repeatedly in applications and theoretical development of signal analysis tools. Far from being fringe elements in our mathematical lexicon, generalized functions provide the only mathematically consistent avenue for addressing the Fourier transform of several important waveforms. And, as we have just demonstrated, they link two analysis tools (the discrete Fourier series and the continuous Fourier transform) which initially appeared to be fundamentally distinct.

6.2 GENERALIZED FUNCTIONS AND FOURIER SERIES COEFFICIENTS

In this section, we apply generalized functions to develop an alternative technique for evaluating the Fourier coefficients of selected piecewise continuous periodic signals. We have encountered a number of such waveforms in earlier chapters, including the sawtooth wave and the train of rectangular pulses. In Chapter 5 we analyzed such waveforms by application of the the Fourier series expansion of periodic signals in terms of a sinusoidal orthonormal basis. There are no calculations performed in this section which could not, in principle, be performed using the well-established methods previously covered in this chapter and in Chapter 5, so casual readers can safely skip this section without a loss of the essentials. Nonetheless, readers who master this section will emerge with the following:

- A method that affords the determination of Fourier series coefficients—for a certain class of periodic functions—without the use of integrals;
- Further experience applying generalized functions to Fourier analysis, including the Fourier expansion of a periodic train of impulse functions known as the *Dirac comb*; and
- An introduction to linear differential equations as they apply to Fourier analysis.

The central theme of these developments is the Dirac delta function and its role as the derivative of a step discontinuity. This discontinuity may appear in one or more of the derivatives of $f(t)$ (including the zeroth-order derivative), and this is the tie-in to differential equations. Our discussion is heuristic and begins with the Fourier series expansion of an impulse train. This forms the analytical basis for the other piecewise continuous functions considered in this section.

6.2.1 Dirac Comb: A Fourier Series Expansion

The term "Dirac comb" is a picturesque moniker for a periodic train of Dirac delta functions (Figure 6.3). The Dirac comb is a periodic generalized function, and it is natural to inquire into its Fourier series representations. The discussion had been deliberately slanted to emphasize the role of differential equations in selected problems where the Dirac comb is applicable. We derive the trigonometric and exponential Fourier series representations of the Dirac comb prior to examining some practical problems in the next section.

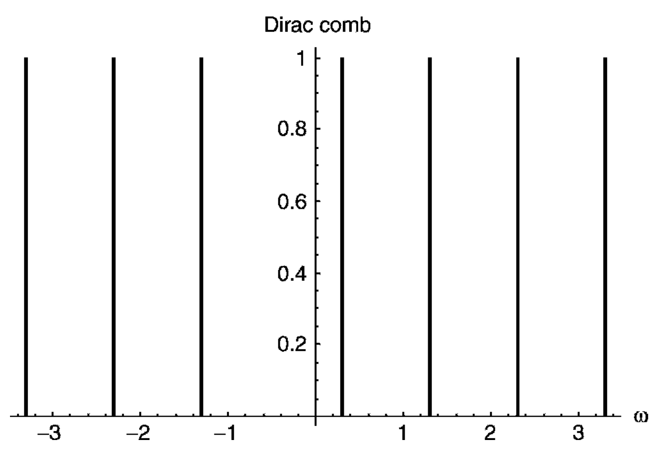

Fig. 6.3. A Dirac comb. By definition, the comb has unit amplitude. The version illustrated here has a unit period and is phase-shifted relative to the origin by an increment of 0.3.

6.2.1.1 Dirac Comb: Trigonometric Fourier Series. Let us revisit the periodic sawtooth wave discussed in Chapter 5. There is nothing sacred about our selection of the sawtooth wave to demonstrate the desired results other than the fact that its step transitions are such that a Dirac comb structure appear in the derivatives of the sawtooth wave. The Dirac comb is an odd function of period T with a sine Fourier series expansion:

$$x(t) = \sum_{k=1}^{\infty} b_k \sin(k\Omega t), \tag{6.42}$$

where $\Omega = (2\pi)/T$, and

$$b_k = -\frac{4h}{2\pi k}(-1)^k. \tag{6.43}$$

The signal $x(t)$ consists of a continuous portion $f(t)$ separated by periodically spaced steps of magnitude $-2h$:

$$x(t) = f(t) - 2h \sum_{m=-\infty}^{\infty} u\left(t - T\left(m + \frac{1}{2}\right)\right), \tag{6.44}$$

whose derivative is

$$x'(t) = \frac{2h}{T} - 2h \sum_{m=-\infty}^{\infty} \delta\left(t - \left(m + \frac{1}{2}\right)T\right). \tag{6.45}$$

Substituting the sine Fourier series representation for $x(t)$ into the left-hand side of (6.45) gives

$$\sum_{k=1}^{\infty} b_k \cdot k\Omega \cdot \cos(k\Omega t) = \frac{2h}{T} - 2h \sum_{m=-\infty}^{\infty} \delta\left(t - \left(m + \frac{1}{2}\right)T\right). \tag{6.46}$$

Therefore,

$$\sum_{m=-\infty}^{\infty} \delta\left(t - \left(m + \frac{1}{2}\right)T\right) = \frac{1}{T} + \frac{2}{T}\sum_{k=1}^{\infty} (-1)^k \cos(k\Omega t). \tag{6.47}$$

This is one form of the Dirac comb whose teeth are arranged along the t axis according to the sawtooth wave used in the derivation. A cleaner and more general form of the Dirac comb may be obtained through a time shift of T/2, giving the basic series representation for a canonical Dirac comb with impulses placed at integer values of T:

$$\sum_{m=-\infty}^{\infty} \delta(t-mT) = \frac{1}{T}+\frac{2}{T}\sum_{k=1}^{\infty}(-1)^k\cos(k\Omega(t+T/2))$$
$$= \frac{1}{T}+\frac{2}{T}\sum_{k=1}^{\infty}\cos(k\Omega t). \quad (6.48)$$

Note that the series representation for a Dirac comb of arbitrary phase shift relative to the origin can always be obtained from the canonical representation in (6.48) .

6.2.1.2 Dirac Comb: Exponential Fourier Series. The exponential Fourier series representation,

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\Omega t}, \quad (6.49a)$$

can be derived directly from first principles or from the trigonometric form using the conversion derived in Chapter 5. The result is elegant:

$$c_n = c_{-n} = \frac{1}{\sqrt{T}} \quad (6.49b)$$

for all integer n. Therefore,

$$\sum_{m=-\infty}^{\infty} \delta(t-mT) = \frac{1}{\sqrt{T}}\sum_{k=-\infty}^{\infty} e^{jk\Omega t}. \quad (6.50)$$

6.2.2 Evaluating the Fourier Coefficients: Examples

The problem of finding Fourier series expansion coefficients for piecewise continuous functions from first principles, using the Fourier basis and integration, can be tedious. The application of a Dirac comb (particularly its Fourier series representations), to this class of functions replaces the integration operation with simpler differentiation.

We will proceed by example, considering first the case of a rectified sine wave and selected classes of rectangular pulse waveforms. In each case, we develop a differential equation that can then be solved for the Fourier expansion coefficients. As we proceed, the convenience as well as the limitations of the method will become apparent. Mastery of these two examples will provide the reader with sufficient understanding to apply the method to other piecewise continuous waveforms.

6.2.2.1 Rectified Sine Wave. Consider a signal

$$x(t) = \left|A_0 \sin \omega_0 t\right|, \tag{6.51}$$

where $\omega_0 = (2\pi)/T$, as in Figure 6.4. Now, $x(t)$ is piecewise continuous with discontinuities in its derivative at intervals of $T/2$ (*not* T). The derivative consists of continuous portions equal to the first derivative of the rectified sine wave, separated by step discontinuities of magnitude $\omega_0 A_0$:

$$x'(t) = \frac{d}{dt}\left|A_0 \sin \omega_0 t\right| + 2\omega_0 A_0 \sum_{n=-\infty}^{\infty} u(t - n\tau), \tag{6.52}$$

where $\tau = T/2$. Taking a further derivative,

$$x''(t) = -\omega_0^2 x(t) + 2\omega_0 A_0 \sum_{n=-\infty}^{\infty} \delta(t - n\tau), \tag{6.53}$$

brings in a train of impulses and—equally important—a term proportional to the original waveform. Substituting the trigonometric series representation of the impulse train and rearranging terms, we have the differential equation

$$x''(t) + \omega_0^2 x(t) = 2\omega_0 A_0 \left[\frac{1}{\tau} + \frac{2}{\tau} \sum_{n=1}^{\infty} \cos(n\Omega t)\right], \tag{6.54}$$

where $\Omega = ((2\pi)/T)$.

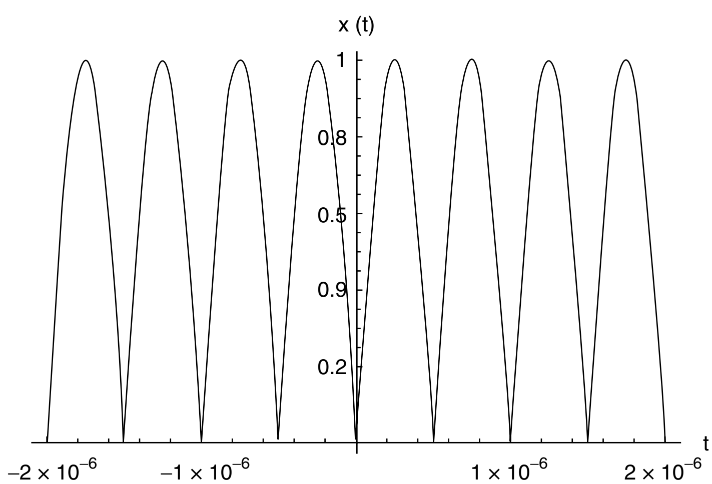

Fig. 6.4. A rectified sine wave. There is a discontinuity in the first derivative.

Remark. This is a second-order, linear differential equation for $x(t)$ whose presence in physical science is almost ubiquitous. The order refers to the highest derivative in the equation. Linearity implies no powers of $x(t)$ (or its derivative), greater than one. Due to the oscillatory nature of its solutions $x(t)$, it is termed the *wave equation*. In the form above, it contains a separate time-dependent term (in this case, representing the Dirac comb) on the right-hand side of (6.54). Depending on the physical context, this is referred to as the source term (in electromagnetic theory) or a forcing function (in circuit analysis and the study of dynamical systems). This equation is a major player in a number of disciplines. When no forcing function is present, the right-hand side vanishes, leaving the *homogeneous wave equation*

$$x''(t) + \omega_0^2 x(t) = 0 \tag{6.55}$$

whose solutions are the simple sinusoids of period *T*: $\sin(\Omega t)$ and $\cos(\Omega t)$, or linear combinations thereof.

Returning to the problem at hand, we can obtain expressions for the trigonometric Fourier series coefficients of $x(t)$ by substituting a trigonometric series representation,

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\Omega t)] + b_n \sin(n\Omega t), \tag{6.56}$$

and the second derivative

$$x''(t) = \sum_{n=1}^{\infty} -(n\Omega)^2 [a_n \cos(n\Omega t)] + b_n \sin(n\Omega t) \tag{6.57}$$

into (6.54). Then we solve the two resulting equations for a_n and b_n by equating the $\cos(n\Omega t)$ and $\sin(n\Omega t)$ components. So,

$$\omega_0^2 \cdot \frac{1}{2} a_0 = 2\omega_0 A_0 \frac{1}{\tau}, \tag{6.58}$$

giving

$$a_0 = \frac{4A_0}{\omega_0 \tau}, \tag{6.59}$$

and for $n \neq 0$,

$$-(n\Omega)^2 a_n + \omega_0^2 a_n = \frac{2}{\tau}. \tag{6.60}$$

Thus,

$$a_n = \frac{4A_0}{\pi}\left(\frac{1}{1-4n^2}\right). \tag{6.61}$$

The b_n vanish identically, as would be expected considering the even symmetry of $x(t)$, and this is confirmed by the governing equation:

$$(-(n\Omega)^2 + \omega_0^2) \cdot b_n = 0 . \tag{6.62}$$

The exponential Fourier series representation can be obtained through these trigonometric coefficients or by direct substitution of the exponential Fourier series representation of the Dirac comb (6.50) into (6.54).

6.2.2.2 Periodic- "Shaped" Rectangular Pulse. Another problem we investigated in Section 5.1 is the periodic rectangular pulse train. Now let us consider a more general version of this waveform, consisting of a piecewise continuous portion denoted $p(t)$, with steps of magnitude A and B:

$$x(t) = p(t) + A_0 \sum_{n=-\infty}^{\infty} u[t-(p+nT)] - B_0 \sum_{n=-\infty}^{\infty} u[t-(q+nT)] . \tag{6.63}$$

In this notation, the pulse width is $d = q - p$. For the moment, we defer specification of a particular form for $p(t)$, but our experience with the previous example suggests that some restrictions will apply if we are to solve for the Fourier coefficients via a linear differential equation. Experience also suggests that the differential equation governing this situation will be of first order, since the Dirac comb appears when the first derivative is taken:

$$x'(t) = p'(t) + A_0 \sum_{n=-\infty}^{\infty} \delta[t-(p+nT)] - B_0 \sum_{n=-\infty}^{\infty} \delta[t-(q+nT)] . \tag{6.64}$$

Substituting the appropriate trigonometric Fourier series (6.48) for the impulse trains and expanding the cosines within the series leads to

$$x'(t) = p'(t) + \frac{(A_0 - B_0)}{T} + C(t) + S(t) , \tag{6.65}$$

where

$$C(t) = \frac{2}{T}\sum_{n=1}^{\infty}\cos(n\omega_0 t)[A_0\cos(n\omega_0 p) - B_0\cos(n\omega_0 q)] \tag{6.66}$$

and

$$S(t) = \frac{2}{T}\sum_{n=1}^{\infty}\sin(n\omega_0 t)[A_0\sin(n\omega_0 p) - B_0\sin(n\omega_0 q)]\,. \tag{6.67}$$

Notice that for arbitrary choice of p and q, $x(t)$ is neither even nor odd; the expansion involves both sine and cosine components, as it should.

To complete this problem, $p(t)$ needs to be specified. If we are going to apply this technique successfully, we will have to restrict $p(t)$ so that the first-order differential equation governing $x(t)$ will be linear. One reasonable option is to specify that $p(t)$ is a linear function of t. Thus,

$$p'(t) = \frac{(B_0 - A_0)}{T}\,. \tag{6.68}$$

In this instance, the differential equation (6.65) is linear because $p(t)$ returns a constant upon differentiation. (From the previous example involving the rectified sine wave, it is obvious that $p(t)$ itself does not have to be a linear function of its independent variable in order to generate a linear governing differential equation for $x(t)$, but a recursion—as exhibited by the sinusoids—is necessary.) Returning to the problem at hand, we substitute the general Fourier series expansion for the derivative of $x(t)$ into (6.65) and solve for the Fourier coefficients,

$$a_n = \frac{-2}{n\omega_0 T}[A_0\sin(n\omega_0 p) - B_0\sin(n\omega_0 q)] \tag{6.69}$$

and

$$b_n = \frac{2}{n\omega_0 T}[A_0\cos(n\omega_0 p) - B_0\cos(n\omega_0 q)]\,. \tag{6.70}$$

As a check, notice that in the limit $A_0 = B_0$, $p = -q$ we generate the special case of a flat (zero-slope) rectangular pulse train of even symmetry, which was treated in Chapter 5. In this case, (6.69) predicts $b_n = 0$, as the even symmetry of $x(t)$ would dictate, and

$$a_n = \frac{4A_0}{n\omega_0 T}\sin[n\pi(d/T)]\,. \tag{6.71}$$

This is consistent with our previous derivation in Chapter 5 using the standard inner product with the Fourier basis.

Remarks. The problems associated with applying Fourier analysis to nonlinear differential equations can be appreciated in this example. Instead of equaling a constant, suppose that the derivative of $p(t)$ is proportional to some power, $p^n(t)$. Substituting the Fourier series for $p(t)$ would result in multiple powers of the Fourier coefficients, in various combinations, whose determination would be difficult, if not impossible. Furthermore, the example involving the rectified sine wave highlights the convenience associated with the fact that derivatives of the sinusoids are recursive: Up to multiplicative constants, one returns to the original function upon differentiating twice. These observations illustrate why the sinusoids (and their close relatives) figure so prominently in the solution of second-order linear differential equations.

6.3 LINEAR SYSTEMS IN THE FREQUENCY DOMAIN

Communication is the business of passing information from a source to a receiver as faithfully as possible. This entails preparation or encoding of the message, which is then impressed upon a waveform suitable for transmission across a channel to the receiver. At the receiver end, the signal must be decoded and distributed to the intended recipients. If all has gone well, they are provided with an accurate reproduction of the original information. The technical ramifications of each step are vast and involve the questions of analog versus digital encoding, the suitability of the transmission channel, and the design of suitable decoding apparatus—all of which are impacted in some way by the techniques described throughout this book.

This section is intended to provide a basic introduction to *filtering* and *modulation*, with an emphasis on the time and frequency domains implied by Fourier analysis. Filtering implies conditioning in the frequency domain; typically a given filter is designed to highlight or suppress portions of the spectrum. Filtering, in its ideal form, is conceptually simple, but in practice involves nuances and tradeoff due to restrictions imposed by the real world.

Modulation is an operation that inhabits the time domain; it is here that we connect the information-bearing message and a carrier signal, whose role is to aid in transporting the information across the designated channel. From the standpoint of our present knowledge base, the details of modulation are quite user-friendly, and we will present a somewhat detailed account of amplitude and frequency modulation—AM and FM—whose practical role in communications needs no introduction [7–9].

Since filtering and modulation involve the interaction of waveforms with linear systems, we rely extensively on the linear systems principles introduced in Chapter 3.

Two relatively simple theorems involving the Fourier transform establish the foundations of filtering and modulation. These are the *convolution theorem* and the *modulation theorem*, which we prove below. There are few electronic communication devices that do not, in some way, make use of the analytical mileage they provide.

6.3.1 Convolution Theorem

A *filter* is a linear system designed to suppress or enhance selected portions of a signal spectrum. In Chapter 3 we established an input–output relation for a linear, time-invariant system based on the system impulse response, denoted $h(t)$, and the input signal $f(t)$:

$$g(t)=\int_{-\infty}^{\infty} f(\tau)h(t-\tau)\,d\tau. \tag{6.72}$$

The convolution theorem relates the spectrum of the output $g(t)$ to those of the input and the system impulse response:

Theorem (Convolution). Let $f_1(t)$ and $f_2(t)$ be two functions for which radial Fourier transforms $F_1(\omega)$ and $F_1(\omega)$ exist and let $f(t) = (f_1 * f_2)(t)$. Then the Fourier spectrum of $f(t)$ is

$$F(\omega) = F_1(\omega)F_2(\omega). \tag{6.73}$$

Proof: By definition,

$$F(\omega) = \int_{-\infty}^{\infty} e^{-j\omega t}\left[\int_{-\infty}^{\infty} f_1(\tau)f_2(t-\tau)\,d\tau\right] dt. \tag{6.74}$$

Interchanging the order of integration gives

$$F(\omega) = \int_{-\infty}^{\infty} f_1(\tau)\left[\int_{-\infty}^{\infty} e^{-j\omega t}f_2(t-\tau)\,dt\right] d\tau. \tag{6.75}$$

The time-shift property of the Fourier transform takes care of the integral with respect to t, so that

$$F(\omega) = \int_{-\infty}^{\infty} f_1(\tau)e^{-j\omega\tau}F_2(\omega)\,d\tau = F_1(\omega)F_2(\omega), \tag{6.76}$$

completing the proof. ■

It is hard to imagine a simpler relationship between spectra. Set in the context of linear systems, the input and output spectra are linked:

$$G(\omega) = F(\omega)H(\omega), \tag{6.77}$$

so that $G(\omega)$ can be shaped or modified by an appropriately designed and implemented system transfer function $H(\omega)$. This forms the backbone of filter design. It will be considered in more detail following a proof of the modulation theorem, which is effectively a converse to the convolution theorem.

6.3.2 Modulation Theorem

Modulation is an operation whereby two or more waveforms, typically an information-bearing modulating signal $m(t)$ and a sinusoidal carrier $c(t)$, are multiplied to form a composite. The termwise product signal $f(t)$ is appropriate for transmission across a communication channel:

$$f(t) = m(t)c(t). \tag{6.78}$$

The modulation theorem relates the spectrum of the composite to those of the constituent modulating wave and carrier:

Theorem (Modulation). Let $f_1(t)$ and $f_2(t)$ be two functions for which Fourier transforms $F_1(\omega)$ and $F_2(\omega)$ exist. Let $f(t) = f_1(t)f_2(t)$. Then the Fourier transform of $f(t)$ is a convolution in the frequency domain:

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(\alpha)F_2(\omega-\alpha)\, d\alpha. \tag{6.79}$$

Proof: The Fourier transform of the time product,

$$F(\omega) = \int_{-\infty}^{\infty} e^{-j\omega t}f_1(t)f_2(t)\, dt, \tag{6.80}$$

can be rearranged by substitution of the inverse Fourier transform of $f_2(t)$:

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f_1(t)\left[\int_{-\infty}^{\infty} F_2(\gamma)e^{j(\gamma-\omega)t}d\gamma\right] dt. \tag{6.81}$$

A change of variables, $\alpha = \omega - \gamma$, gives (noting carefully the signs and integration limits),

$$F(\omega) = \frac{-1}{2\pi}\int_{-\infty}^{\infty} f_1(t)\left[\int_{\infty}^{(-\infty)} F_2(\omega-\alpha)e^{-j\alpha t}\, d\alpha\right] dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(\alpha)F_2(\omega-\alpha)\, d\alpha, \tag{6.82}$$

and the proof is complete. ■

The exact form of this spectrum depends heavily upon the nature of $f_1(t)$ and $f_2(t)$ and, in the framework of a modulated carrier signal, gives

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} M(\alpha)C(\omega-\alpha)\, d\alpha = \frac{1}{2\pi}\int_{-\infty}^{\infty} C(\alpha)M(\omega-\alpha)\, d\alpha. \tag{6.83}$$

The *productive* $\leftrightarrow$ *convolution* relationship is one of the most elegant and useful aspects of the Fourier transform. It forms the basis for the design and application of linear filters considered in the next section.

Example (Damped Oscillations). When pure sinusoids are applied to a damped exponential (whose spectrum contained a pole along the imaginary axis; see Chapter 5), the pole acquires a real part. Consider

$$f(t) = e^{-\alpha t}\sin(\omega_0 t)u(t), \tag{6.84}$$

where α is a positive definite constant. Designating $f_1(t) = \sin(\omega_0 t)$ and $f_2(t) = e^{-\alpha t}u(t)$ then (6.79) gives

$$\mathcal{F}[f_1(t)f_2(t)](\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{\pi}{j}\{\delta(\gamma-\omega_0)-\delta(\gamma+\omega_0)\}\right]\frac{1}{\alpha+j(\omega_0-\gamma)}\,d\gamma. \tag{6.85}$$

This reduces, after some algebra, to

$$\frac{1}{2j}\left[\frac{1}{\alpha+j(\omega-\omega_0)}-\frac{1}{\alpha+j(\omega+\omega_0)}\right] = \frac{\omega_0}{(\alpha+j\omega)^2+\omega_0^2}. \tag{6.86}$$

There are poles in the complex plane located at

$$\omega = \pm\omega_0 + j\alpha \tag{6.87}$$

whose real parts are proportional to the frequency of oscillation. The imaginary part remains proportional to the decay. For $\cos(\omega_0 t)$, the spectrum is similar,

$$F(\omega) = \frac{\alpha+j\omega}{(\alpha+j\omega)^2+\omega_0^2} \tag{6.88}$$

but exhibits a zero at $\omega = j\alpha$. Note that it is impossible to distinguish the spectra of the sine and cosine on the basis of the poles alone.

6.4 INTRODUCTION TO FILTERS

To design a filter, it is necessary to specify a system transfer function $H(\omega)$ that will pass frequencies in a selected range while suppressing other portions of the input spectrum. A *filter design* is a specification of $H(\omega)$, including the frequency bands to be *passed*, those to be *stopped*, and the nature of the transition between these regions. In general, $H(\omega)$ is a complex-valued function of frequency,

$$H_0(\omega) = He^{-j\omega\Theta(\omega)}, \tag{6.89}$$

composed of a real-valued *amplitude spectrum* $H_0(\omega)$ and a *phase spectrum* $\Theta(\omega)$. In this chapter, we will work with the class of so-called *ideal filters,* whose transitions between the *stop bands* and the *pass bands* are unit steps:

Definition (Ideal Filter). An *ideal filter* is a linear, translation-invariant system with a transfer function of the form

$$H_0(\omega) = \sum_{n=1}^{N} a_n u(\omega - \omega_n) \tag{6.90}$$

and a zero phase spectrum across all frequencies: $\Theta(\omega) = 0$.

The amplitude spectrum of an ideal filter is characterized by an integer number N transitions, each of which is a unit step of amplitude an at specified frequencies ω_n. The idealization is twofold:

- The unit step transitions are abrupt and perfectly clean. In practice, the transition exhibits rolloff—that is, it is gradual—and overshoot, which is signal processing parlance for oscillations or ripple near the corners of the step transitions, similar to Gibbs's oscillations.
- It is impossible to design and implement a filter whose phase spectrum is identically zero across all frequencies.

The nuisance imposed by a nonzero phase spectrum can be readily appreciated by the following simple illustration. Suppose an audio waveform *f*(*t*) acts as an input to a linear system representing a filter with a transfer function $H(\omega)$; for the purposes of illustration we will assume that the amplitude spectrum is unity across all frequencies. The output signal $g(t)$ is characterized by a spectrum,

$$G(\omega) = e^{-j\omega\Theta(\omega)}F(\omega) \tag{6.91}$$

so that when $G(\omega)$ is inverted back to the time domain, the nonzero phase introduces time shifts in $g(t)$. If the input $f(t)$ were an audio signal, for example, $g(t)$ would sound distorted, because each nonzero phase would introduce a time shift that is a function of $\Theta(\omega)$. (Such *phasing* introduces a reverberation and was deliberately applied to audio entertainment during the so-called psychedelic era in the late 1960s. In more serious communication systems, such effects are not conducive to faithful and accurate data transmission.) In practice, there are ways to minimize phase distortion, but for the present discussion we will continue to inhabit the ideal world with zero phase.

Filter types are classified according to the frequency bands they pass, and the user makes a selection based upon the spectral characteristics of the signal he intends to modify via application of the filter. A signal $m(t)$ whose spectrum is clustered around $\omega = 0$ is termed *baseband*. In audio signals, for example, the power resides in the under-20-kHz range, and visual inspection of the spectrum shows a spread in the frequency domain whose nominal width is termed the *bandwidth*. The precise measure of bandwidth may depend upon the context, although a common measure of spectral spread is the 3-dB bandwidth:

Definition (3-dB Bandwidth). The *3-dB bandwidth* occupied by a spectrum $F(\omega)$ is the frequency range occupied by the signal as measured at the point at which the squared magnitude $|F(\omega)|^2$ is equal to one-half its maximum value.

The use of the squared magnitude allows the definition to encompass complex-valued spectra and eliminates any issues with +/– signs in the amplitude, which have no bearing on frequency spread. This definition of bandwidth applies equally well to baseband and bandpass spectra, but will be illustrated here with a Gaussian at baseband.

Example (3-dB Bandwidth of a Gaussian). In the previous chapter we noted that the spectrum of a Gaussian pulse $f(t) = e^{-\alpha t^2}$ was a Gaussian of the form

$$F(\omega) = \sqrt{\frac{\pi}{\alpha}} e^{-\omega^2/4\alpha}. \tag{6.92}$$

According to the definition, the 3-dB bandwidth is the spread in frequency between the points defined by the condition

$$\frac{\pi}{\alpha} e^{-\omega^2/2\alpha} = \frac{1}{2}\frac{\pi}{\alpha} \tag{6.93}$$

or

$$\omega^2 = -2\alpha \ln\frac{1}{2} = 2\alpha \ln 2. \tag{6.94}$$

These points are $\omega = \pm\sqrt{2\ln 2}\sqrt{\alpha}$ so that the total 3-dB bandwidth is

$$\Delta\omega = 2\sqrt{2\ln 2}\sqrt{\alpha}. \tag{6.95}$$

As expected, large values of α result in a greater spectral spread. In communications systems it is common to describe performance in Hz (cycles/s), which scales the bandwidth accordingly,

$$\Delta f = \Delta\omega/2\pi. \tag{6.96}$$

The typical baseband audio signal is not exactly Gaussian, but occupies approximately 40 kHz (i.e., 2 × 20 kHz), a relatively small increment in (Hertz) frequency space. Television picture signals carry more information—including audio and visual signals—and occupy approximately 9 MHz.

There are other definitions of frequency spread which will be introduced when appropriate.

Much of analog and digital communication involves translating baseband signals in frequency space and filtering to suit the needs of a given system. Frequency translation will be discussed in the next subsection. Prior to that, we turn to an illustration of three common filter types and their uses.

6.4.1 Ideal Low-pass Filter

A low-pass filter is characterized by transitions $\omega_1 = -\omega_t$ and $\omega_2 = \omega_t$ with associated transition amplitudes $a_{1,2} = \pm 1$, as illustrated in Figure 6.5a. The ideal filter has created a passband in the interval $[-\omega_t, \omega_t]$, while suppressing all other frequencies by creating a stopband in those regions. The effect of low-pass filtering is to rid a signal of unwanted high frequencies, which can occur in several contexts. If we plan to sample and digitize a baseband signal, for example, frequencies above a certain limit will end up contaminating the reconstructed waveform since information from the high frequencies will be spuriously thrown into the lower frequency range. This phenomenon is known as aliasing—high frequencies are falsely identified with the lower—and the best course of action is to rid the signal of the offending spectral content prior to sampling.

Low-pass filters are useful when a baseband signal needs to be isolated from other signals present in the received waveform. In selected modulation schemes, the process in which a baseband signal is recovered at the receiver introduces an additional waveform residing near a higher frequency. This waveform is useless and the baseband signal can be isolated from it with a suitable low-pass filter.

6.4.2 Ideal High-pass Filter

A high-pass filter passes all frequencies $|\omega| > \omega_t$. As with the low-pass filter, we locate transitions at $\omega_1 = -\omega_t$ and $\omega_2 = \omega_t$ but the associated transition amplitudes are $a_1 = -1$, $a_2 = -1$, as illustrated in Figure 6.5b. A primary application of high-pass filtering involves cutting out redundant portions of a signal spectrum to reduce overhead associated with bandwidth. In the forthcoming discussion on modulation, we will consider this in further detail.

6.4.3 Ideal Bandpass Filter

A bandpass filter is characterized by four transitions $\omega_1 = \omega_{t1}$, $\omega_1 = \omega_{t2}$, $\omega_1 = \omega_{t3}$, and $\omega_1 = \omega_{t4}$, with associated transition amplitudes $a_{1,2} = \pm 1$, $a_{3,4} = \pm 1$. As illustrated in Figure 6.5c, two passbands have been created which effectively isolate a band in the middle region of the spectrum.

Example (Shannon Function). The Shannon function

$$f(t) = \frac{\sin(2\pi t) - \sin(\pi t)}{\pi t} \tag{6.97}$$

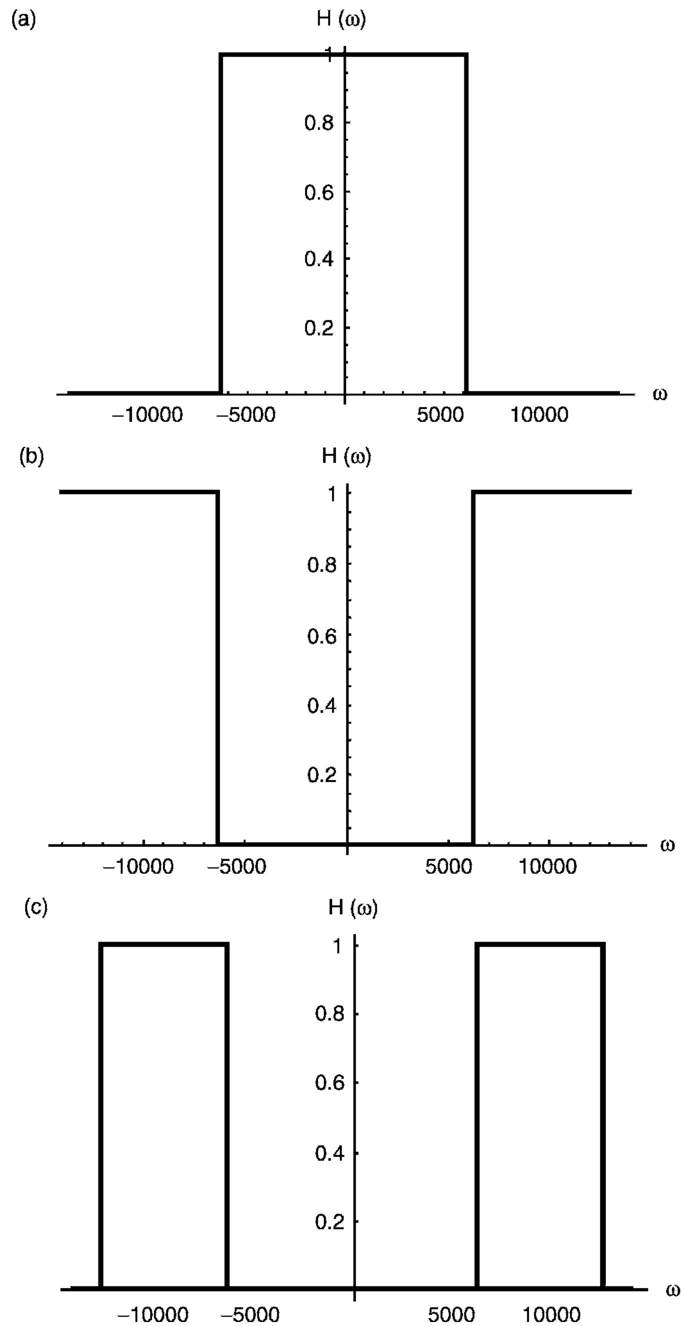

Fig. 6.5. (a) Ideal filter types. (a) Ideal low-pass filter, shown with a transition frequency f_t = 1 kHz. (b) Ideal high-pass filter. (c) Ideal bandpass filter, illustrated with passband widths of 1 kHz.

has the remarkable property of exhibiting and ideal two-sided bandpass spectrum. This is easily demonstrated. Note that the Shannon function is the difference of two sinc terms,

$$f_1(t) = \frac{\sin(2\pi t)}{\pi t} = 2\frac{\sin(2\pi t)}{2\pi t} \tag{6.98}$$

and

$$f_2(t) = \frac{\sin(\pi t)}{\pi t}, \tag{6.99}$$

each of which is integrable, so that one can evaluate the Fourier integral directly. Alternatively, we can apply the symmetry property

$$\mathcal{F}[F(t)](\omega) = 2\pi f(-\omega) \tag{6.100}$$

to the problem of the unit rectangular pulse supported on the interval $t \in [-a, a]$, whose spectrum was (see Chapter 5)

$$F(\omega) = 2a\frac{\sin(a\omega)}{a\omega}. \tag{6.101}$$

It follows immediately that the spectra of $f_1(t)$ and $f_2(t)$ are unit amplitude rectangular pulses of width 4π and 2π, respectively:

$$F_1(\omega) = u(\omega + 2\pi) - u(\omega - 2\pi), \tag{6.102}$$

$$F_2(\omega) = u(\omega + \pi) - u(\omega - \pi). \tag{6.103}$$

The composite spectrum of the Shannon function is the difference $F(\omega) = (F_2(\omega) - F_1(\omega))$ of two nested rectangular pulses, forming a perfect two-sided bandpass spectrum with transition frequencies $\omega_{t1} = -2\pi, \omega_{t2} = -\pi, \omega_{t3} = \pi$, $\omega_{t4} = 2\pi$. In terms of the unit step function,

$$F(\omega) = u(t + 2\pi) - u(t + \pi) + u(t - \pi) + u(t - 2\pi). \tag{6.104}$$

As expected given the properties of the Shannon function, the spectrum is a real function of even symmetry.

In general, bandpass filters are useful for isolating non-baseband spectra. For example, consider a multiuser communication link in which several operators are simultaneously transmitting information over several channels, each allocated to a given frequency range. Tuning in to a particular user typically involves some form of bandpass filter to isolate the desired channel.

Example (Derivative of a Gaussian). The Gaussian

$$g(t) = e^{-\alpha t^2} \tag{6.105}$$

exhibited a low-pass spectrum

$$\mathcal{F}[g(t)](\omega) = \sqrt{\frac{\pi}{\alpha}} e^{-\omega^2/\alpha}. \tag{6.106}$$

Multiplying the time-domain Gaussian by a pure sinusoid is one method of translating the bulk of the signal energy to higher frequencies to create a spectrum that approximates a bandpass filter (as we consider in the next section). Alternatively, one can induce the necessary waviness by taking the second derivative,

$$f(t) = -\frac{d^2}{dt^2} g(t) = 2\alpha[1 - 2\alpha t^2] e^{-\alpha t^2}. \tag{6.107}$$

Its spectrum,

$$F(\omega) = -(j\omega)(j\omega)\mathcal{F}[g(t)](\omega) = \omega^2 \sqrt{\frac{\pi}{\alpha}} e^{-\omega^2/(4\alpha)} \tag{6.108}$$

demonstrates bandpass characteristics in the form of two quasi-Gaussian pass bands centered about

$$\omega = \pm 2\sqrt{\alpha}. \tag{6.109}$$

The characteristic is hardly ideal, because it passes portions of all finite frequencies except at DC ($\omega = 0$), but as such could be used to eliminate any unwanted DC portion of a waveform. Note that the lobes are not perfect Gaussians due to the effect of the quadratic factor; thus the use of the term "centered" in connection with (6.109) is only approximate. This also complicates the calculation of 3-dB bandwidth, a matter that is taken up in the exercises.

Remark. Both the Shannon function and the second derivative of the Gaussian are localized *atoms* in the time domain and make suitable *wavelets* (Chapter 11). In wavelet applications, their bandpass characteristics are used to advantage to select out features in the neighborhood of specific frequency.

6.5 MODULATION

The implications of the modulation theorem are far-reaching and quite useful. Most audio and video information begins as a baseband signal $m(t)$ whose frequency range is typically inappropriate for long-distance radio, TV, satellite, and optical fiber links. (Most often, a basic problem is attenuation in the channel, due to absorption in the transmission medium, at frequencies in the kHz regime.) There is also the question of multiple users. Whatever the medium, hundreds of audio and video programs are communicated simultaneously and must be so transferred without interference. Since the bandwidth of an individual audio or video signal is relatively small compared to the total bandwidth available in a given transmission medium, it is convenient to allocate a slot in frequency space for each baseband signal. This allocation is called *frequency division multiplexing* (FDM). The modulation theorem makes this multichannel scheme possible.

Modulation theory is treated in standard communications theory books [7–9].

6.5.1 Frequency Translation and Amplitude Modulation

Let us reconsider the notion of modulation, where by our baseband signal $m(t)$ is multiplied by an auxiliary signal $c(t)$, to form a composite waveform $f(t)$. The composite is intended for transmission and eventual demodulation at the receiver end. Thus,

$$f(t) = m(t)c(t), \tag{6.110}$$

where $c(t)$ is a sinusoidal *carrier* wave,

$$c(t) = A_c \cos(\omega_c t) = A_c \cos(2\pi f_c t). \tag{6.111}$$

As this scheme unfolds, we will find that the carrier effectively translates the baseband information to a spectral region centered around the carrier frequency f_c. Multiplication by a sinusoid is quite common in various technologies. In various parts of the literature, the carrier signal is also referred to as the *local oscillator* signal, *mixing* signal, or *heterodyning* signal, depending upon the context.

The modulation theorem describes the Fourier transform of the composite signal. Let $m(t) = f_1(t)$ and $c(t) = f_2(t)$. Then

$$\begin{aligned} F_2(\omega) &= \frac{2\pi}{\sqrt{T}}\left[\frac{\sqrt{T}}{2}\delta(\omega+\omega_c) + \frac{\sqrt{T}}{2}\delta(\omega+\omega_c)\right] \\ &= \pi[\delta(\omega+\omega_c) + \delta(\omega+\omega_c)], \end{aligned} \tag{6.112}$$

where we used the exponential Fourier series with $c_1 = c_{-1} = \sqrt{T}/2$. Designating the Fourier transform of $m(t)$ by $M(\omega)$, the spectrum of the composite signal is, according to the modulation theorem,

$$F(\omega) = \frac{1}{2}\int_{-\infty}^{\infty} M(\alpha)\delta(\omega + (\omega_c - \alpha))\, d\alpha + \frac{1}{2}\int_{-\infty}^{\infty} M(\alpha)\delta(\omega - (\omega_c - \alpha))\, d\alpha. \tag{6.113}$$

Using simple algebra to rearrange the arguments of the delta functionals and making use of their even symmetry, we can reduce the above to straightforward integrals:

$$F(\omega) = \frac{1}{2}\int_{-\infty}^{\infty} M(\alpha)\delta(\alpha - (\omega + \omega_c))\, d\alpha + \frac{1}{2}\int_{-\infty}^{\infty} M(\alpha)\delta(\alpha - (\omega - \omega_c))\, d\alpha \tag{6.114}$$

Equation (6.114) evaluates easily, resulting in

$$F(\omega) = \frac{1}{2}M(\omega+\omega_c) + \frac{1}{2}M(\omega-\omega_c). \tag{6.115}$$

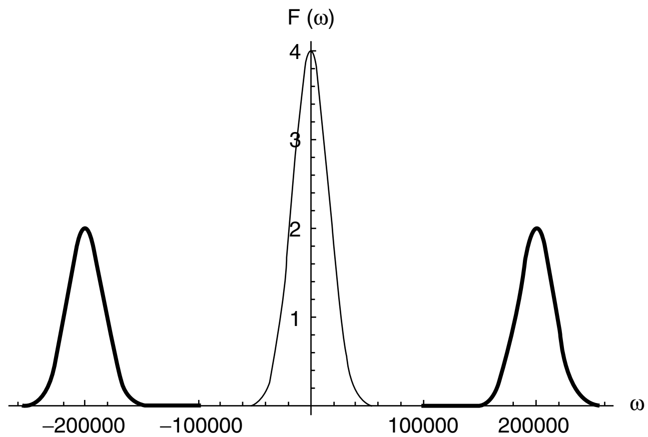

Fig. 6.6. Frequency translation resulting from carrier modulation of a baseband Gaussian spectrum of amplitude 4. For purposes of illustration, f_c was set at 100 kHz, but most broadcast systems utilize carrier frequencies up to several orders of magnitude higher.

The composite spectrum consists of two facsimiles of the baseband spectra, translated in frequency so that they are centered around $\omega = \pm\omega_c$. The power in the original signal has been equally split between the two portions of the frequency-translated spectrum. The situation is illustrated in Figure 6.6 for a hypothetical Gaussian baseband spectrum.

6.5.2 Baseband Signal Recovery

Our emerging picture of multiuser communications systems consists of multiple baseband signals. Each baseband signal centers around a given carrier frequency, which is broadcast and available to end users. Each end user, in turn, can recover the desired information-bearing baseband signal in a number of ways.

One method of recovery requires the receiver to multiply the incoming composite signal by a local oscillator with frequency ω_s, giving

$$\begin{aligned} s(t) &= [m(t)\cos(\omega_c t)]\cos(\omega_s t) \\ &= \frac{m(t)}{2}\{\cos((\omega_c+\omega_s)t+\cos(\omega_c-\omega_s)t)\}. \end{aligned} \tag{6.116}$$

Let us assume that the carrier and local oscillator frequencies differ by some amount $2\pi\Delta f$:

$$\omega_s = \omega_c + 2\pi\Delta f. \tag{6.117}$$

Then,

$$s(t) = \frac{m(t)}{2}[\cos\{2\omega_c + 2\pi\Delta f\}t + \cos(2\pi\Delta f t)]. \tag{6.118}$$

If the frequency deviation Δf is identically zero, this reduces to

$$s(t) = \frac{m(t)}{2}\cos(2\omega_c t) + \frac{m(t)}{2}. \tag{6.119}$$

This is the sum of a half-amplitude baseband signal centered around $2\omega_c$ and a similar contribution residing at baseband.

Since systems are designed so that the carrier frequency is much larger than the highest frequency present in the baseband signal, these two contributions are well-separated in the frequency domain. The simple application of a low-pass filter to output described by (6.119) allows the user to eliminate the unwanted power near the frequency $2\omega_c$, leaving the half-amplitude baseband waveform intact. In the case of a multiuser channel, all of these double-frequency waves can be eliminated by a low-pass filter.

When $\Delta f = 0$, the local oscillator is said to be *synchornized* with the carrier. In practice, such precise tuning is not always possible, and a familiar problem with this technique is signal distortion and fading. This occurs, for instance, when the local oscillator drifts from the desired frequency. The source of this fading is evident in (6.118). A small amount of mistuning has little effect on the first term, since it is usually easy to maintain $\Delta f < f_c$; filtering removes this term. On the other hand, the second term is more sensitive to frequency adjustment. Any frequency deviation is going to cause distortion and undesired fading as $\cos(2\pi\Delta f t)$ periodically nears zero. Naturally, as Δf increases, the recovered baseband signal fades with greater frequency. Furthermore, the second term in (6.118) is effectively a baseband signal translated to an carrier frequency Δf. If this value gets too large—even a fraction of the typical baseband frequencies in $m(t)$—there is the possibility of translating a portion of the spectrum outside the passband of the low-pass filter used to retrieve $\frac{1}{2}m(t)$. This is not a trivial matter, because frequency deviations are usually specified as a percentage of the carrier frequency; so even a few percent can be a problem if the carrier frequency is relatively large.

6.5.3 Angle Modulation

Angle modulation is a method whereby the phase of the carrier wave is modulated by the baseband signal $m(t)$. That is,

$$f(t) = A_c \cos(\omega_c t + \phi_c(t)), \tag{6.120}$$

where the *phase deviation*

$$\phi_c(t) = \phi_c[m(t)] \tag{6.121}$$

is a function to be specified according to the application. The term angle modulation refers to the time-varying angle between the fixed carrier oscillation and the added phase $\phi_c(t)$. In practice, two functional relationships (6.121) are common. The first is a direct proportionality between the phase and the baseband modulation:

$$\phi_c(t) = \text{const} \times m(t), \tag{6.122}$$

which is referred to simply as *phase modulation* (PM). Another common arrangement makes the phase offset proportional to the integral of the baseband signal:

$$\phi_c(t) = \text{const} \times \int_{-\infty}^{t} m(\tau)\, d\tau, \tag{6.123}$$

which is called *frequency modulation* (FM), for reasons that will emerge as we proceed.

This classification scheme can seem confusing at first glance. Bear in mind that both phase modulation and frequency modulation do, in their respective ways, modulate the phase of the carrier signal. Furthermore, the astute reader has probably wondered how it is possible to distinguish between a PM and an FM waveform by inspection. And, as a matter of fact, you cannot distinguish between them visually. Indeed, for most purposes in this book, the distinction is academic, since in either case $\phi_c(t)$ is simply some function of t. The distinction between PM and FM arises in the implementation. Without explicit knowledge as to how the phase offset $\phi_c(t)$ was constructed, FM and PM are effectively identical. For this reason, it is often sufficient to lapse into the generic label *angle modulation* to describe these waveforms. Of course, the end user, whose task it is to extract information (i.e., the baseband signal $m(t)$) from a given signal will find it of inestimable value to know whether a PM- or FM-style implementation was actually used in the transmission.

Example (Angle Modulation). Much of this can be clarified by looking at a typical angle modulated signal. Consider a quadratic phase offset of the form

$$\phi_c(t) = \Omega t^2. \tag{6.124}$$

We illustrate the resulting angle modulated waveform (6.120) in Figure 6.7

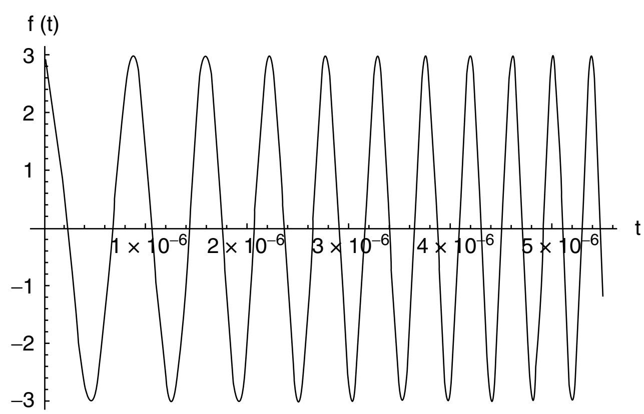

Fig. 6.7. Angle modulation. The illustrated waveform has a carrier frequency f_c = 1 MHz, amplitude A_c = 3, and $\Omega = 10^{12}$. The chirp induced by time-varying frequency is clearly in evidence.

Note the constant envelope (equal to A_c), which makes the class of *single-frequency* angle-modulated signals readily distinguishable from their amplitude-modulated counterparts (on the other hand, the superposition of multiple carriers can result in a time-varying envelope, as we will see). Furthermore, observe the apparent variation in frequency over time. This phenomenon, known as chirp, for its resemblance to the sound made by inhabitants of the avian world, is the most distinctive feature of angle modulated waveforms.

This motivates the following definition:

Definition (Instantaneous Frequency). Intuitively, the *instantaneous frequency* of a cosine-based angle modulated waveform (6.120) is defined

$$\omega(t) = \frac{d}{dt}[\omega_c t + \phi_c(t)] = \omega_c + \frac{d}{dt}\phi_c(t). \tag{6.125}$$

From this perspective, in which both carrier and time-varying phase effects are lumped into a general phase offset, the term *frequency* modulation makes sense. In the limit of vanishing or constant phase, the instantaneous frequency defaults to that of the carrier, as expected. According to (6.123), when employing an FM system, the baseband signal $m(t)$ is proportional to the second term in the instantaneous frequency defined in (6.125).

For the example in (6.124), the instantaneous frequency is a linear function of time, equal to $\omega_c + \Omega t$. This *linear chirp* is one of several common modulation schemes that involve higher-order polynomial or inverse powers of t and that are considered in the exercises.

More complicated signals, which may involve multiple frequency components, require the extra degree of freedom afforded by the complex exponential representation

$$f(t) = A(t)e^{j\Phi(t)}. \tag{6.126}$$

Taking the real part gives

$$f(t) = A(t)\cos\Phi(t), \tag{6.127}$$

leading to a general definition of instantaneous frequency:

$$\omega(t) \equiv \frac{d\Phi(t)}{dt}. \tag{6.128}$$

6.5.3.1 Multiple Frequencies. The time-varying amplitude $A(t)$ is a natural occurrence in signals that consist of multiple oscillations. For example, consider a simple composite signal consisting of two equal-amplitude pure oscillations represented by $\cos(\omega_1 t)$ and $\cos(\omega_2 t)$. Representing the composite as a sum of complex exponentials, it is easy to show that

$$f(t) = Ae^{j\omega_1 t} + Ae^{j\omega_2 t} = A\cos(\Delta t)e^{j\Sigma t}, \tag{6.129}$$

where Δ is the difference,

$$\Delta \equiv \frac{\omega_1 - \omega_2}{2}, \tag{6.130}$$

and the sum

$$\Sigma \equiv \frac{\omega_1 + \omega_2}{2} \tag{6.131}$$

is the average of the two pure oscillations. For this simple case, it is also the instantaneous frequency according to (6.128).

6.5.3.2 Instantaneous Frequency versus the Fourier Spectrum.

The instantaneous frequency should not be confused with the Fourier spectrum. This is general, but we will illustrate the point by examining the instantaneous and Fourier spectra of FM signals with sinusoidal phase deviation,

$$\phi_c(t) = k \cdot \sin(\omega_m t). \tag{6.132}$$

This is a useful analysis, since arbitrary information $m(t)$ can be decomposed into a spectrum of sinusoids. The spread of instantaneous frequencies is quantified by the *frequency deviation* defined as the maximum difference between the carrier frequency and the instantaneous frequency (6.128),

$$\Delta\omega \equiv sup[|\omega_c - \omega(t)|] = k\omega_m. \tag{6.133}$$

The derivation of (6.133) is straightforward and left as an exercise. It is common to express the amplitude k as the ratio $\Delta\omega / \omega_m$ and call it the *modulation index*. Equation (6.133) implies that the range of instantaneous frequency occupies a range $\omega_c \pm \Delta\omega$ implying a nominal bandwidth of instantaneous frequencies $2\Delta\omega$. This is intuitively appealing since it is directly proportional to the amplitude and frequency of the phase deviation. In the limit that either of these vanish, the signal reverts to a pure carrier wave.

We now turn to the Fourier spectrum. It can be shown that

$$f(t) = A_c \cos(\omega_c t + k \cdot \sin(\omega_m t)) \tag{6.134}$$

can be elegantly expressed as a superposition a carrier wave and an infinite set of discrete oscillations in multiples (harmonics) of ω_m:

$$f(t) = J_0(k)\cos(\omega_c t) - \sigma_{\text{odd}} + \sigma_{\text{even}}, \tag{6.135}$$

where

$$\sigma_{\text{odd}} \equiv \sum_{n=1}^{\infty} J_{2n-1}(k)[\cos(\omega_c - (2n-1)\omega_m)t - \cos(\omega_c + (2n-1)\omega_m)t] \tag{6.136}$$

and

$$\sigma_{\text{even}} \equiv \sum_{n=1}^{\infty} J_{2n}(k)[\cos(\omega_c - 2n\omega_m)t - \cos(\omega_c + 2n\omega_m)t]. \tag{6.137}$$

In the exercises we lead the reader through the steps necessary to arrive at (6.135). The coefficients $J_p(k)$ are pth-order *Bessel functions of the first kind* and arise in a number of disciplines, notably the analysis of optical fibers, where good engineering treatments of the Bessel functions may be found [10]. Most scripted analysis tools make also them available as predefined functions. In general, they take the form of damped oscillations along the k axis [11]; by inspection of (6.135), they act as weights for the various discrete spectral components present in the signal. The carrier is weighted by the zeroth-order Bessel function, which is unity at the origin and, for k much smaller than unity, can be approximated by the polynomial,

$$J_0(k) \cong 1 - \frac{k^2}{4}. \tag{6.138}$$

The first *sideband* is represented by

$$J_1(k) \cong \frac{k}{2} - \frac{k^3}{16}. \tag{6.139}$$

The remaining sidebands, weighted by the higher-order Bessel functions for which $p > 1$, can be approximated as a single term,

$$J_p(k) \cong \frac{1}{p!}\left(\frac{k}{2}\right)^p \tag{6.140}$$

for small k. In the limit of zero phase deviation, the representation (6.135) reverts to a pure carrier wave $f(t) = \cos(\omega_c t)$, as expected. For k sufficiently small, but nonzero, the signal power consists mainly of the carrier wave and a single pair of sidebands oscillating at $\pm\omega_m$. Operation in this regime is termed *narrowband* FM.

In closing, we highlight the salient difference between Fourier spectral components and the instantaneous frequency: The spectrum of an FM signal with sinusoidal phase deviation is a superposition of Dirac impulses $\delta(\omega - (\omega_c \pm p\omega_m))$ for all integers p. On the other hand, the instantaneous frequency is continuous and oscillates through the range $\omega_c \pm \Delta\omega$.

The general case, in which information $m(t)$ is applied to the phase deviation, as given by (6.121), leads to analytically complex spectra that are beyond the scope of this discussion. But an appropriately behaved $m(t)$ can be naturally decomposed into pure Fourier oscillations similar to the sinusoids. So the simple case presented here is a foundation for the more general problem.

6.6 SUMMARY

This chapter has provided tools for studying the frequency content of important signals—sinusoids, constants, the unit step, and the like—for which the standard

Fourier analysis in Chapter 5 does not work. This generalized Fourier transform rests on the theory of distributions, which we covered in Chapter 3. The properties of the new transform are direct generalizations of those for the standard Fourier transform. An inverse transform exists as well. Further more, the generalized transform is equal to the standard transform for signals in the spaces $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$.

The transform theory for generalized functions draws the links between the Fourier series and transform.

We applied the generalized transform to the study of communication systems. Understanding modulation schemes, for example, depends on spectral analysis of sinusoidal signals, and for this purpose the generalized transform makes the calculations particularly simple. We also considered the design of basic frequency selective linear, translation-invariant systems—filters. Chapter 9 will delve deeper into the theoretical and practical aspects of filter design using traditional Fourier analysis techniques.

Our next step is to develop the frequency theory task for the realm of discrete signals. Now, as we observed in the interplay between the ideas in Chapters 2 and 3, it is easier to justify a discrete summation (even if it has infinite limits) than an integration. Therefore, Chapter 7's mathematical work turns out to be much more concrete. With a discrete signal Fourier theory, computer implementations of frequency domain signal analysis becomes possible. We shall also build a link between analog and discrete signals through the famous Sampling Theorem, so our continuous domain results will appear once again.

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PROBLEMS

1. Assume that $f(t)$ is a distribution of slow growth and prove the Fourier transform properties listed in Chapter 5.

2. Show that

(a)

$$F\left[\frac{d}{dt}\delta(t)\right](\omega) = j\omega, \tag{6.141}$$

(b)

$$F[t](\omega) = j2\pi \cdot \frac{d}{d\omega}(\delta(\omega)). \tag{6.142}$$

3. Show that

(a)

$$F[\cos(\omega_0 t)u(t)](\omega) = \frac{\pi}{2} \cdot [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] + j \cdot \frac{\omega_0}{(\omega_0^2 - \omega^2)}, \tag{6.143}$$

(b)

$$F[\sin(\omega_0 t)u(t)](\omega) = -j\frac{\pi}{2} \cdot [\delta(\omega - \omega_0) - \delta(\omega + \omega_0)] + \frac{\omega_0}{(\omega_0^2 - \omega^2)}. \tag{6.144}$$

4. Demonstrate the following generalized Fourier transforms, where $u(t)$ is the unit step.

(a)

$$F[u(t)](\omega) = \pi\delta(\omega) + \frac{1}{j\omega}, \tag{6.145}$$

(b)

$$F[(-jt)^n f(t)](\omega) = \frac{d^n}{dt^n}F(\omega), \tag{6.146}$$

(c)

$$F[t^n](\omega) = 2\pi j^n \cdot \frac{d^n\delta(\omega)}{d\omega^n}, \tag{6.147}$$

(d)

$$F[t^n u(t)](\omega) = j^n \cdot \left[\pi\frac{d^n\delta(\omega)}{d\omega^n} + \frac{1}{j}\frac{(-1)^n n!}{\omega^{n+1}}\right], \tag{6.148}$$

(e)

$$F[t^n \operatorname{sgn}(t)](\omega) = (-2)j^{n+1}\frac{(-1)^n n!}{\omega^{n+1}}. \tag{6.149}$$

5. A cosine-modulated signal $s(t) = m(t)\cos(\omega_c t)$ is recovered by multiplying it by $\cos(\omega_c t + \theta)$, where θ is a constant. (a) If this product is subjected to a low-pass filter designed to reject the contribution at $2\omega_c$, write the expression for the resulting waveform. (b) If the baseband signal $m(t)$ occupies a range $f \in [300, 3000]$ Hz, what is the minimum value of the carrier frequency ω_c for which it is possible to recover $m(t)$ according to the scheme in part a)? (c) What is the maximum value of the phase θ if the recovered signal is to be 95% of the maximum possible value?

6. The AM cosine-modulated signal $s(t) = m(t)\cos(\omega_c t)$ is recovered by multiplying by a periodic signal $\rho(t)$ with period k/f_c, where k is an integer. (a) Show that $m(t)$ may be recovered by suitably filtering the product $s(t)\rho(t)$. (b) What is the largest value of k for which it is possible to recover $m(t)$ if the baseband signal $m(t)$ occupies a range $f \in [0, 9000]$ Hz and the carrier frequency f_c is 1 MHz?

7. Consider two signals with *quadratic chirp*:

$$f_1(t) = a_1\cos(bt^2 + ct), \tag{6.150}$$

$$f_2(t) = a_2\cos(bt^2). \tag{6.151}$$

(a) Derive expressions for the instantaneous frequency of each. Comparing these, what purpose is served by the constant c?

(b) For convenience let $a_1 = b = c = 1$ and plot $f_1(t)$ over a reasonable interval (say, 10 to 30 s). Qualitatively, how is this plot consistent with the instantaneous frequency derived in part (a)?

(c) Let $a_2 = 1$ and plot the composite signal $f(t) = f_1(t) + f_2(t)$ over a 30-s interval. Are the effects of the instantaneous frequencies from part (a) still evident? Compared to the single waveform of part (b), what is the most noteworthy feature induced by superposing $f_1(t)$ and $f_2(t)$?

8. A signal exhibits *hyperbolic chirp*:

$$f(t) = a\cos\left(\frac{\alpha}{\beta - t}\right). \tag{6.152}$$

(a) Let $\alpha_1 = 1000$ and $\beta_1 = 900$, and plot $f(t)$ over a sufficiently large interval, say $t \in [0,3500]$.

(b) What qualitative effects are controlled by the parameters α and β? Let $\alpha_1 = 500$ and $\beta_1 = 740$ and replot $f(t)$.

(c) Derive an expression for the instantaneous frequency of $f(t)$ and plot your result using the signal parameters in part (a).

9. Consider an FM signal modulated by multiple cosines,

$$f(t) = \cos\left[\omega_c t + \sum_{n=1}^{N} k_n \cdot \cos(n \cdot \omega_m t)\right]. \tag{6.153}$$

For $N = 1, 2$, derive expressions for

(a) The instantaneous frequency of $f(t)$.

(b) The frequency deviation.

10. Derive the Bessel function expansion for the sinusoidally modulated FM signal, (6.135). *Hint*: Make use of the identities

$$\cos(k \cdot \sin\omega_m t) = J_0(k) + 2\sum_{n=1}^{\infty} J_{2n}(k)\cos(2n \cdot \omega_m t), \tag{6.154}$$

$$\sin(k \cdot \sin\omega_m t) = 2\sum_{n=1}^{\infty} J_{2n-1}(k)\sin((2n-1) \cdot \omega_m t) \tag{6.155}$$

and the trigonometric relations

$$\cos x \cdot \cos y = \frac{1}{2}\cos(x-y) + \frac{1}{2}\cos(x+y), \tag{6.156}$$

$$\sin x \cdot \sin y = \frac{1}{2}\cos(x-y) - \frac{1}{2}\cos(x+y). \tag{6.157}$$

11. A carrier wave is angle modulated sinusoidally.

(a) In principle, how many values of the modulation index k result in a completely suppressed (i.e., zero) carrier wave? List the first five of these values.

(b) On the basis of your answer in (a), is the zeroth-order Bessel function perfectly periodic?

(c) Let $k = 0.1$ and plot the ratio $J_p(k)/J_0(k)$ as a function of the order p, for $p \in [0, 10]$. Qualitatively, what is the effect of increasing p? (Of course, for our purposes, only integer values of p are relevant.)

(d) For $k \in [0, 0.1]$, plot the ratios $J_1(k)/J_0(k)$ and $J_2(k)/J_0(k)$. What is happening to the amplitudes of the sidebands relative to the carrier as k is increased? Is this true for the remaining sidebands as well? Outside of the narrowband FM operating regime, can such behavior be expected for arbitrary k?

12. Consider a unit-amplitude FM carrier signal which is modulated by two sinusoids,

$$f(t) = \cos(\omega_c t + k_1 \cdot \sin\omega_1 t + k_2 \cdot \sin\omega_2 t). \tag{6.158}$$

(a) Demonstrate the existence of sidebands at harmonics of the sum and difference frequencies $\omega_1 + \omega_2$ and $\omega_1 - \omega_2$ as well as at harmonics of ω_1 and ω_2.

(b) Show that in the limit of small k_1 and k_2 we may approximate $f(t)$ as a linear superposition of cosine and sine carrier waves,

$$f(t) \approx \cos\omega_c t - (k_1 \cdot \sin\omega_1 t + k_2 \cdot \sin\omega_2 t)\sin\omega_c t \tag{6.159}$$

Hint: Apply the approximations, valid for small x:

$$\cos x \cong 1\,, \tag{6.160}$$

$$\sin x \cong x\,. \tag{6.161}$$

13. As an application of Fourier transforms and their generalized extensions, this problem develops part of Schoenberg's Theorem on spline functions [12, 13]. We stated the theorem in Chapter 3: If $x(t)$ is a spline function of degree n having integral knots $K = \{m = k_m : m \in \mathbb{Z}\}$, then there are constants c_m such that

$$s(t) = \sum_{m=-\infty}^{\infty} c_m \beta_n(t-m)\,. \tag{6.162}$$

In (6.162) the B-spline of order zero is

$$\beta_0(t) = \begin{cases} 1 & \text{if } -\frac{1}{2} < t < \frac{1}{2} \\ \frac{1}{2} & \text{if } \quad |t| = \frac{1}{2} \\ 0 & \text{if otherwise} \end{cases} \tag{6.163}$$

and higher-order B-splines are defined as

$$\beta_n(t) = \underbrace{\beta_0(t)*\beta_0(t)*\ldots\beta_0(t)}_{n+1 \text{ times}}\,. \tag{6.164}$$

(a) Explain why $\beta_n(t)$ has a Fourier transform.

(b) Let $B_n(\omega) = \mathcal{F}(\beta_n)(\omega)$ be the Fourier transform of $\beta_n(t)$. Following Ref. 13, show that

$$B_n(\omega) = \left[\frac{\sin\left(\frac{\omega}{2}\right)}{\frac{\omega}{2}}\right]^{n+1} = \left[\frac{e^{j\omega/2} - e^{-j\omega/2}}{j\omega}\right]^{n+1}. \tag{6.165}$$

(c) Let $y_n(t) = u(t)t^n$ be the one-sided polynomial of degree n. Show that

$$\frac{d^{n+1}}{dt^{n+1}} y_n(t) = n!\delta(t), \tag{6.166}$$

where $\delta(t)$ is the Dirac delta.

(d) Conclude that $Y_n(\omega) = n!/(j\omega)^{n+1}$.

(e) Next, show that

$$B_n(\omega) = \frac{Y_n(\omega)[e^{j\omega/2} - e^{-j\omega/2}]^{n+1}}{n!}. \tag{6.167}$$

(f) Use the binomial expansion from high-school algebra to get

$$B_n(\omega) = \frac{1}{n!}\sum_{k=0}^{n+1} \binom{n+1}{k} (-1)^k e^{-j\omega\left(k - \frac{n+1}{2}\right)} Y(\omega). \tag{6.168}$$

(g) Infer that

$$\beta_n(t) = \frac{1}{n!}\sum_{k=0}^{n+1} \binom{n+1}{k} (-1)^k y\left(t - k + \frac{n+1}{2}\right) \tag{6.169}$$

and that $\beta_n(t)$ is a piecewise polynomial of degree n.

(h) Show that $\frac{d^{n+1}}{dt^{n+1}}\beta_n(t) = n!\delta(t)$ is a sum of shifted Diracs.

(i) Show that an nth-order spline function on uniform knots is a sum of scaled, shifted B-splines.