CHAPTER 5

Fourier Transforms of Analog Signals

This chapter furnishes a detailed introduction to the theory and application of the Fourier transform—the first of several transforms we shall encounter in this book. Many readers, including engineers, scientists, and mathematicians, may already be familiar with this widely used transform. The Fourier transform analyzes the frequency content of a signal, and it has four variations, according to whether the time-domain signal is analog or discrete, periodic or aperiodic. The present chapter covers the two analog transforms: the Fourier series, for periodic signals, and the Fourier transform proper, for aperiodic signals.

Technology involving filtering, modulation, and wave propagation all rely heavily upon frequency analysis accomplished by the Fourier transform operation. But biological systems execute spectral analysis as well. Our senses, especially hearing and sight, are living examples of signal processors based on signal frequency spectra. The color response of the human eye is nothing more than the end result of optical signal processing designed to convert solar electromagnetic waves into the various hues of the visible electromagnetic spectrum. On a daily basis, we are exposed to sounds which are easily classified according to high and low pitch as well as purity—we are all too aware of a tenor or soprano who wobbles into a note. All instances of frequency-domain analysis, these life experiences beg the question of how engineered systems might achieve like results.

This chapter develops the first of several practical frequency-domain analysis tools. Indeed we already have practical motivations:

- Experiments in finding the period of apparently periodic phenomena, such as example of sunspot counts in the first chapter
- Attempts to characterize texture patterns in the previous chapter

Our actual theoretical development relies heavily upon the general notions of Hilbert space and orthogonal functions developed in Chapter 3. For the mathematician, who may already have a thorough understanding of the Fourier series as a complete orthonormal expansion, Chapters 5 and 6 present an opportunity to get

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down to the business of calculating the coefficients and functions which shed so much information about the physical world.

The transform consists of two complementary operations. The first is the *analysis*—that is, the breaking down of the signal into constituent parts. In the case of Fourier analysis, this involves generation and interpretation of coefficients whose magnitude and phase contain vital information pertaining to the frequency content of a signal. In the case of the continuous Fourier transform studied in this chapter, these coefficients are a continuous function of frequency as represented by the Fourier transform $F(\omega)$. The Fourier series, which is applicable to periodic waveforms, is actually a special case of this continuous Fourier transform, and it represents spectral data as a discrete set of coefficients at selected frequencies.

The second operation involves *synthesis*, a mathematical picking up of pieces, to reconstruct the original signal from $F(\omega)$ (or from the set of discrete Fourier coefficients, if appropriate), as faithfully as possible. Not all waveforms readily submit to Fourier operations, but a large set of practical signals lends itself quite readily to Fourier analysis and synthesis. Information obtained via Fourier analysis and synthesis remains by far the most popular vehicle for storing, transmitting, and analyzing signals. In some cases the analysis itself cannot be performed, leaving synthesis out of the question, while in others the physically valid analysis is available, but a reconstruction via Fourier synthesis may not converge. We will consider these issues in some detail as Chapter 5 develops. Some waveforms amenable to Fourier analysis may be better suited to more advanced transform methods such as time-frequency (windowed) Fourier transforms or time-scale (wavelet) transforms considered in later chapters. However, the basic notion of 'frequency content' derived from Fourier analysis remains an important foundation for each of these more advanced transforms.

Communication and data storage systems have a finite capacity, so the storage of an entire spectrum represented by a continuous function $F(\omega)$ is impractical. To accommodate the combined requirements of efficiency, flexibility, and economy, a discrete form of the Fourier transform is almost always used in practice. This discrete Fourier transform (DFT) is best known in the widely used fast Fourier transform (FFT) algorithm, whose development revolutionized data storage and communication. These algorithms are discussed in Chapter 7, but their foundations lie in the concepts developed in Chapters 5 and 6.

Introductory signal processing [1–5] and specialized mathematics texts [6–9] cover continuous domain Fourier analysis. Advanced texts include Refs. [10–12]. Indeed, the topic is almost ubiquitous in applied mathematics. Fourier himself developed the Fourier series, for analog periodic signals, in connection with his study of heat conduction.[1] This chapter presupposes some knowledge of Riemann integrals, ideas of continuity, and limit operations [13]. Familiarity with Lebesgue integration, covered briefly in Chapter 3, remains handy, but definitely not essential [14].

[1]Jean-Baptiste Joseph Fourier (1768–1830). The French mathematical physicist developed the idea without rigorous justification and amid harsh criticism, to solve the equation for the flow of heat along a wire [J. Fourier, *The Analytical Theory of Heat*, New York: Dover, 1955].

Essential indeed are the fundamentals of analog L^p and abstract function spaces [15, 16]. We use a few unrigorous arguments with the Dirac delta. Chapter 6 covers the generalized Fourier transform and distribution theory [17, 18]. Hopefully this addresses any misgivings the reader might harbor about informally applying Diracs in this chapter.

5.1 FOURIER SERIES

Consider the problem of constructing a synthesis operation for periodic signals based on complete orthonormal expansions considered in Chapter 3. More precisely, we seek a series

$$x_n(t) = \sum_{k=1}^{n} c_k \phi_k(t) \tag{5.1}$$

which converges to $x(t)$, a function with period T, as n approaches infinity. Equation (5.1) is a statement of the synthesis problem: Given a set of coefficients c_k and an appropriate set of orthonormal basis functions $\{\phi_1(t), \phi_2((t), \ldots, \phi_n(t))\}$, we expect a good facsimile of $x(t)$ to emerge when we include a sufficient number of terms in the series. Since the linear superposition (5.1) will represent a periodic function, it is not unreasonable to stipulate that the $\phi_k(t)$ exhibit periodicity; we will use simple sinusoids of various frequencies, whose relative contributions to $x(t)$ are determined by the phase and amplitude of the c_k. We will stipulate that the basis functions be orthonormal over some fundamental interval $[a, b]$; intuitively one might consider the period T of the original waveform $x(t)$ to be sufficiently "fundamental," and thus one might think that the length of this fundamental interval is $b - a = T$. At this point, it is not obvious where the interval should lie relative to the origin $t = 0$ (or whether it really matters). But let us designate an arbitrary point $a = t_0$, requiring that the set of $\{\phi_k(t)\}$ is a complete orthonormal basis in $L^2[t_0, t_0 + T]$:

$$\langle \phi_i, \phi_l \rangle = \int_{t_0}^{t_0+T} \phi_i(t)\phi_j(t)^* dt = \delta_{ij}, \tag{5.2}$$

where δ_{ij} is the Kronecker[2] delta.

We need to be more specific about the form of the basis functions. Since periodicity requires $x(t) = x(t + T)$, an examination of (5.1) suggests that it is desirable to select a basis with similar qualities: $\phi_k(t) = \phi_k(t + T)$. This affords us the prospect of a basis set which involves harmonics of the fundamental frequency $1/T$. Consider

$$\phi_k(t) = A_0 e^{jk2\pi Ft} = A_0 e^{jk\Omega t}, \tag{5.3}$$

[2]This simple δ function takes its name from Leopold Kronecker (1823–1891), mathematics professor at the University of Berlin. The German algebraist was an intransigent foe of infinitary mathematics—such as developed by his pupil, Georg Cantor—and is thus a precursor of the later *intuitionists* in mathematical philosophy.

where $F = 1/T$ cycles per second (the frequency common unit is the hertz, abbreviated Hz; one hertz is a single signal cycle per second). We select the constant A_0 so as to normalize the inner product as follows. Since

$$\langle \phi_l, \phi_m \rangle = A_0^{\;2}\int_{t_0}^{(t_0+T)} e^{jl\Omega t}e^{-jm\Omega t}p\,dt = \delta_{lm}, \tag{5.4a}$$

if $m = l$, then

$$\langle \phi_m, \phi_m \rangle = A_0^{\;2}\int_{t_0}^{t_0+T} dt = A_0^{\;2}T. \tag{5.4b}$$

Setting $A_0 = 1/\sqrt{T}$ then establishes normalization. Orthogonality is easily verified for $m \neq 1$, since

$$\begin{aligned}\langle \phi_l | \phi_m \rangle &= \frac{1}{T}\int_{t_0}^{t_0+T} e^{j(l-m)\Omega t}dt \\ &= \frac{1}{T}\int_{t_0}^{t_0+T}(\cos[(l-m)\Omega t] + j\sin[(l-(m))\Omega t])dt = 0.\end{aligned} \tag{5.5a}$$

This establishes orthonormality of the set

$$\left\{\frac{1}{\sqrt{T}}e^{jk\Omega t}\right\} \tag{5.5b}$$

for integer k.

When the set of complex exponentials is used as a basis, all negative and positive integer k must be included in the orthonormal expansion to ensure completeness and convergence to $x(t)$. (We can readily see that restricting ourselves to just positive or negative integers in the basis, for example, would leave a countably infinite set of functions which are orthogonal to each function in the basis, in gross violation of the notion of completeness.)

Relabeling of the basis functions provides the desired partial series expansion for both negative and positive integers k:

$$x_{2N+1}(t) = \sum_{k=-N}^{N} c_k \frac{1}{\sqrt{T}}e^{jk\Omega t}. \tag{5.6}$$

Completeness will be assured in the limit as $N \to \infty$:

$$\lim_{N\to\infty} x_{2N+1}(t) = \sum_{k=-\infty}^{\infty} c_k \frac{1}{\sqrt{T}}e^{jk\Omega t} = x(t), \tag{5.7}$$

where the expansion coefficients are determined by the inner product,

$$c_k = \langle x(t), \phi_k(t) \rangle = \int_{t_0}^{t_0+T} x(t)\frac{1}{\sqrt{T}}e^{-jk\Omega t}dt\ . \tag{5.8}$$

Remark. The c_k in (5.8) are in fact independent of t_0, which can be shown by the following heuristic argument. Note that all the constituent functions in (5.8)—namely $x(t)$, as well as $\cos(k\Omega t)$ and $\sin(k\Omega t)$, which make up the complex exponential—are (at least) T-periodic. As an exercise, we suggest the reader draw an arbitrary function which has period T: $f(t + T) = f(t)$. First, assume that $t_0 = 0$ and note the area under $f(t)$ in the interval $t \in [0, T]$; this is, of course, the integral of $f(t)$. Next, do the same for some nonzero t_0, noting that the area under $f(t)$ in the interval $t \in [t_0, t_0 + T]$ is unchanged from the previous result; the area over $[0, t_0]$ which was lost in the limit shift is compensated for by an equivalent gain between $[t_0, t_0 + T]$. This holds true for any finite t_0, either positive or negative, but is clearly a direct consequence of the periodicity of $x(t)$ and the orthogonal harmonics constituting the integral (5.8). Unless otherwise noted, we will set $t_0 = 0$, although there are some instances where another choice is more appropriate.

5.1.1 Exponential Fourier Series

We can now formalize these concepts. There are two forms of the Fourier series:

- For exponential basis functions of the form $Ae^{jk\Omega t}$
- For sinusoidal basis functions of the form $A\cos(k\Omega t)$ or $A\sin(k\Omega t)$

The exponential expansion is easiest to use in signal theory, so with it we begin our treatment.

5.1.1.1 Definition and Examples. The Fourier series attempts to analyze a signal in terms of exponentials. In the sequel we shall show that broad classes of signals can be expanded in such a series. We have the following definition.

Definition (Exponential Fourier Series). The *exponential Fourier series* for $x(t)$ is the expansion

$$x(t) = \sum_{k=-\infty}^{\infty} c_k \phi_k(t), \tag{5.9}$$

whose basis functions are the complete orthonormal set,

$$\phi_k(t) = \frac{1}{\sqrt{T}} e^{jk\Omega t}, \tag{5.10}$$

and whose expansion coefficients take the form (5.8).

According to the principles governing complete orthonormal expansions, (5.9) predicts that the right-hand side converges to $x(t)$, provided that the infinite summation is performed. In practice, of course, an infinite expansion is a theoretical ideal, and a cutoff must be imposed after a selected number of terms. This results in a partial series defined thusly:

Definition (Partial Series Expansion). A *partial Fourier series* for $x(t)$ is the expansion

$$x(t) = \sum_{k=-N}^{N} c_k \phi_k(t) \tag{5.11}$$

for some integer $0 < N < \infty$.

The quality of a synthesis always boils down to how many terms (5.11) should include. Typically, this judgment is based upon how much error can be tolerated in a particular application. In practice, every synthesis is a partial series expansion, since it is impossible to implement (in a finite time) an infinite summation.

Example (Sine Wave). Consider the pure sine wave $x(t) = \sin(\omega t)$. The analysis calculates the coefficients

$$c_k = \int_0^T \left(\sin\, \Omega t \frac{1}{\sqrt{T}} [\cos\, k\Omega t - j \sin\, k\Omega t] \right) dt\,. \tag{5.12}$$

Orthogonality of the sine and cosine functions dictates that all c_k vanish except for $k = \pm 1$:

$$c_{\pm 1} = \frac{\mp j}{\sqrt{T}} \left\{ \int_0^T [\sin(\Omega t)]^2 dt \right\} = (\mp j) \frac{\sqrt{T}}{2}\,. \tag{5.13}$$

Synthesis follows straightforwardly:

$$x(t) = (-j)\frac{\sqrt{T}}{2}\left(\frac{e^{j\Omega t}}{\sqrt{T}}\right) + j\frac{\sqrt{T}}{2}\left(\frac{e^{(-j)\Omega t}}{\sqrt{T}}\right) = \sin(\Omega t)\,. \tag{5.14}$$

Example (Cosine Wave). For $x(t) = \cos(\omega t)$ there are two equal nonzero Fourier coefficients:

$$c_{\pm 1} = \frac{1}{\sqrt{T}} \int_0^T [\cos(\Omega t)]^2 dt = \frac{\sqrt{T}}{2}\,. \tag{5.15}$$

Remark. Fourier analysis predicts that each simple sinusoid is composed of frequencies of magnitude $|\Omega|$, which corresponds to the intuitive notion of a pure oscillation. In these examples, the analysis and synthesis were almost trivial, which stems from the fact that $x(t)$ was projected along the real (in the case of a cosine) or imaginary (in the case of a sine) part of the complex exponentials comprising the orthonormal basis. This property—namely a tendency toward large coefficients when the signal $x(t)$ and the analyzing basis match—is a general property of orthonormal expansions. When data pertaining to a given signal is stored or transmitted, it is often in the form of these coefficients, so both disk space and bandwidth can be reduced by a judicious choice of analyzing basis. In this simple example of Fourier analysis applied to sines and cosines, only two coefficients are required to perform an exact synthesis of $x(\ddot{t})$. But Fourier methods do not always yield such economies, particularly in the neighborhood of transients (spikes) or jump discontinuities. We will demonstrate this shortly. Finally, note that the two Fourier coefficients are equal (and real) in the case of the cosine, but of opposite sign (and purely imaginary) in the case of the sine wave. This results directly from symmetries present in the sinusoids, a point we now address in more detail.

5.1.1.2 Symmetry Properties. The Fourier coefficients acquire special properties if $x(t)$ exhibits even or odd symmetry. Recall that if $x(t)$ is odd, $x(-t) = -x(t)$ for all t, and by extension it follows that the integral of an odd periodic function, over any time interval equal to the period T, is identically zero. The sine and cosine harmonics constituting the Fourier series are odd and even, respectively. If we expand the complex exponential in the integral for c_k,

$$c_k = \int_0^T \frac{x(t))}{\sqrt{T}}[\cos(k\Omega t) - j\sin(k\Omega t)]\ dt, \tag{5.16}$$

then some special properties are apparent:

- If $x(t)$ is real and even, then the c_k are also real and even, respectively, in k-space; that is, $c_k = c_{-k}$.
- If $x(t)$ is real and odd, then the coefficients are purely imaginary and odd in k-space: $c_{-k} = -c_k$.

The first property above follows since the second term in (5.16) vanishes identically and since $\cos(k\Omega t)$ is an even function of the discrete index k. If even–odd symmetries are present in the signal, they can be exploited in numerically intensive applications, since the number of independent calculations is effectively halved. Most practical $x(t)$ are real-valued functions, but certain filtering operations may transform a real-valued input into a complex function. In the exercises, we explore the implications of symmetry involving complex waveforms.

Example (Rectangular Pulse Train). Consider a series of rectangular pulses, each of width t and amplitude A_0, spaced at intervals T, as shown in Figure 5.1. This waveform is piecewise continuous according to the definition of Chapter 3, and in due course it will become clear this has enormous implications for synthesis. The inner product of this waveform with the discrete set of basis functions leads to a straightforward integral for the expansion coefficients:

$$c_k = \langle x(t), \phi_k(t)\rangle = \frac{A_0}{\sqrt{T}}\int_0^{\frac{\tau}{2}}(\cos k\Omega t - j\sin k\Omega t)\ dt + \frac{A_0}{\sqrt{T}}\int_{T-\frac{\tau}{2}}^{T}(\cos k\Omega t - j\sin k\Omega t)\ dt \tag{5.17}$$

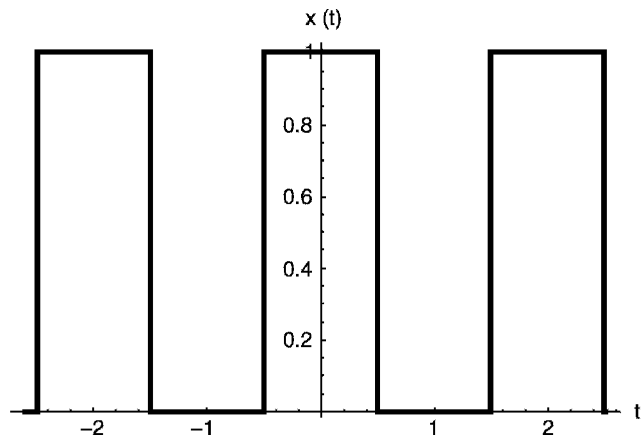

Fig. 5.1. A train of rectangular pulses. Shown for pulse width $\tau = 1$, amplitude $A_0 = 1$, and period $T = 2$.

Some algebra reduces this to the succinct expression

$$c_k = \frac{A_0}{\sqrt{T}} \cdot \tau \cdot \frac{\sin\left(\frac{k\Omega\tau}{2}\right)}{\left(\frac{k\Omega\tau}{2}\right)}. \tag{5.18}$$

Example (Synthesis of Rectangular Pulse). In Figure 5.2 we illustrate the synthesis of periodic rectangular pulses for several partial series, using (5.10) and (5.16).

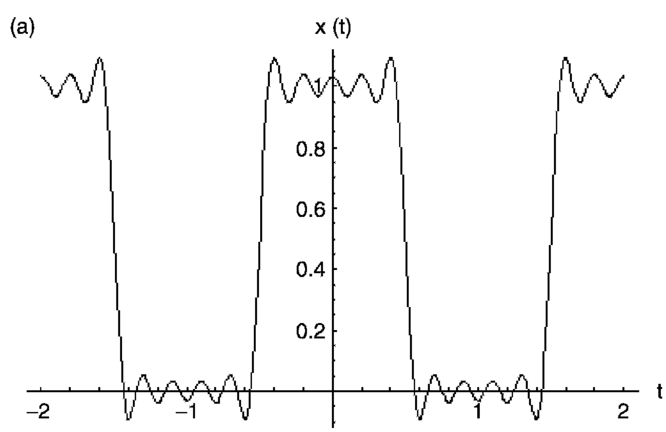

Fig. 5.2. Synthesis of the rectangular pulse train. (a) Partial series $N = 10$, (b) $N = 50$, (c) $N = 100$. The number of terms in the series is $2N + 1$.

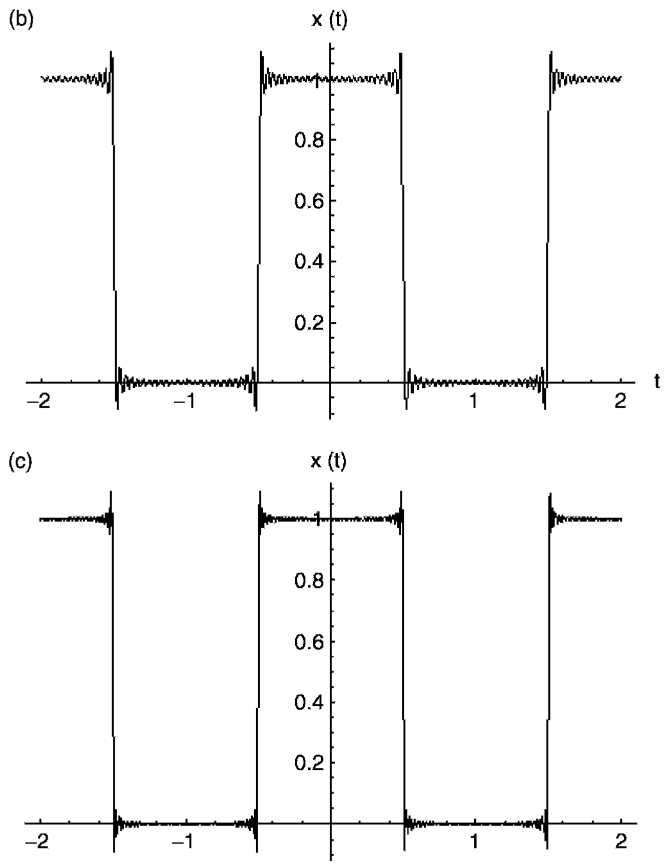

Fig. 5.2 (*Continued*)

5.1.2 Fourier Series Convergence

We are now in a position to prove the convergence of the exponential Fourier series for a signal $x(t)$. We shall consider two cases separately:

- At points where $x(t)$ is continuous;
- At points where $x(t)$ has a jump discontinuity.

5.1.2.1 Convergence at Points of Continuity. It turns out that the Fourier series does converge to the original signal at points of continuity. We have the following theorem.

Theorem (Fourier Series Convergence). Suppose $S_N(s)$ is a partial series summation of the form

$$S_N(s) = \sum_{k=-N}^{N} c_k \frac{1}{\sqrt{T}} e^{jk\Omega t}, \tag{5.19a}$$

where N is a positive integer. If $x(t)$ is continuous at s (including points of continuity within piecewise continuous functions), then

$$\lim_{N \to \infty} S_N(s) = x(s). \tag{5.19b}$$

Proof: Consider the partial series summation:

$$S_N(s) = \sum_{k=-N}^{N} \left\langle x(t), \frac{e^{jk\Omega t}}{\sqrt{T}} \right\rangle \frac{1}{\sqrt{T}} e^{jk\Omega t}. \tag{5.20}$$

Writing the inner product term (in brackets) as an explicit integral, we have

$$S_N(s) = \frac{1}{T} \sum_{k=-N} \int_0^T x(t) e^{jk\Omega(s-t)} dt = \frac{2}{T} \int_0^T x(t) \cdot K(s-t)\, dt, \tag{5.21}$$

where

$$K(s-t) = \frac{1}{2} + \sum_{k=1}^{N} \cos(k\Omega(s-t)). \tag{5.22}$$

The function $K(s-t)$ reduces—if we continue to exploit the algebraic properties of the exponential function for all they are worth—to the following:

$$K(s-t) = \mathrm{Re}\left[\left(1 - \frac{1}{2}\right) + \sum_{k=1}^{N} e^{jk\Omega(s-t)}\right]. \tag{5.23a}$$

This reduces to the more suggestive form,

$$K(s-t) = \frac{\sin\left[\left(N + \frac{1}{2}\right)(s-t)\right]}{2\sin\left[\frac{1}{2}(s-t)\right]}. \tag{5.23b}$$

Returning to the partial series expansion (5.21), the change of integration variable $u = s - t$ gives

$$S_N(s) = -\int_s^{s-T} x(s-u) \left[\frac{\sin\left(N + \frac{1}{2}\right)u}{T\sin\left(\frac{T}{2}\right)}\right] du. \tag{5.24}$$

The quantity in brackets is the *Dirichlet kernel*[3],

$$D_N(u) = \frac{\sin\left(N + \frac{1}{2}\right)u}{T\sin\left(\frac{u}{2}\right)}, \tag{5.25}$$

[3]P. G. Legeune Dirichlet (1805–1859) was Kronecker's professor at the University of Berlin and the first to rigorously justify the Fourier series expansion. His name is more properly pronounced "Dear-ah-klet."

whose integral exhibits the convenient property

$$-\int_{s}^{s-T} D_N(u)\ du = 1. \tag{5.26}$$

Equation (5.26) is easily demonstrated with a substitution of variables, $2v = u$, which brings the integral into a common tabular form:

$$-\int_{s}^{s-T} D_N(u)\ du = -\frac{1}{T}\int_{0}^{-\frac{T}{2}} 2 \cdot \frac{\sin\ 2N+1)v}{\sin\ v}\ dv = \frac{-2}{T}\left[2\sum_{m=1}^{N}\frac{\sin(mv)}{2m}+v\right]\Bigg|_{0}^{-\frac{T}{2}} = 1. \tag{5.27}$$

The beauty of this result lies in the fact that we can construct the identity

$$x(s) = -\int_{s}^{s-T} x(s)D_N(u)\ du \tag{5.28}$$

so that the difference between the partial summation $S_N(s)$ and the original signal $x(s)$ is an integral of the form

$$S_N(s)-x(s) = \frac{-1}{T}\int_{s}^{s-T} g(s,u)\cdot\sin\left[\left(N+\frac{1}{2}\right)u\right]\ du, \tag{5.29}$$

where

$$g(s,u) = \frac{x(s-u)-x(s)}{\sin\left(\frac{u}{s}\right)}. \tag{5.30}$$

In the limit of large N, (5.29) predicts that the partial series summation converges pointwise to $x(s)$ by simple application of the Riemann–Lebesgue lemma (Chapter 3):

$$\lim_{N\to\infty}[S_N(s)-x(s)] = \lim_{N\to\infty}\int_{s}^{s-T}\frac{-g(s,u)}{T}\sin\left[\left(N+\frac{1}{2}\right)u\right]\ du = 0, \tag{5.31}$$

thus concluding the proof. ■

The pointwise convergence of the Fourier series demonstrated in (5.31) is conceptually reassuring, but does not address the issue of how rapidly the partial series expansion actually approaches the original waveform. In practice, the Fourier series is slower to convergence in the vicinity of sharp peaks or spikes in $x(t)$. This aspect of the Fourier series summation is vividly illustrated in the vicinity of a step discontinuity—of the type exhibited by rectangular pulse trains and the family of sawtooth waves, for example. We now consider this problem in detail.

5.1.2.2 Convergence at a Step Discontinuity. It is possible to describe and quantify the quality of convergence at a jump discontinuity such as those exhibited by the class of piecewise continuous waveforms described in Chapter 3. We represent such an $x(t)$ as the sum of a continuous part $x_c(t)$ and a series of unit steps, each term of which represents a step discontinuity with amplitude $A_k = x(t_k^+) - x(t_k^-)$ located at $t = t_k$:

$$x(t) = x_c(t) + \sum_{k=1}^{M} A_k u(t - t_k). \tag{5.32}$$

In the previous section, convergence of the Fourier series was established for continuous waveforms and that result applies to the $x_c(t)$ constituting part of the piecewise continuous function in (5.32). Here we turn to the issue of convergence in the vicinity of the step discontinuities represented by the second term in that equation. We will demonstrate that

- The Fourier series converges pointwise at each t_k.
- The discontinuity imposes oscillations or ripples in the synthesis, which are most pronounced in the vicinity of each step. This artifact, known as the *Gibbs phenomenon*,[4] is present in all partial series syntheses of piecewise continuous $x(t)$; however, its effects can be minimized by taking a sufficiently large number of terms in the synthesis.

The issue of Gibbs oscillations might well be dismissed as a mere mathematical curiosity were it not for the fact that so many practical periodic waveforms are piecewise continuous. Furthermore, similar oscillations occur in other transforms as well as in filter design, where ripple or overshoot (which are typically detrimental) arise from similar mathematics.

Theorem (Fourier Series Convergence: Step Discontinuity). Suppose $x(t)$ exhibits a step discontinuity at some time t about which $x(t)$ and its derivative have well-behaved limits from the left and right, $t_{(l)}$ and $t_{(r)}$, respectively. Then $x(t)$ converges pointwise to the value

$$x(t) = \frac{[x(t_{(r)}) + x(t_{(l)})]}{2}. \tag{5.33}$$

Proof: For simplicity, we will consider a single-step discontinuity and examine the synthesis

$$x(t) = x_c(t) + A_s u(t - t_s), \tag{5.34}$$

[4]The Yale University chemist, Josiah Willard Gibbs (1839–1903), was the first American scientist of international renown.

where the step height is $A_s = x(t_s+) - x(t_s-)$. We begin by reconsidering (5.24):

$$S_N(s) = \int_0^T x(t)\left(\frac{\sin\left[\left(N+\frac{1}{2}\right)(s-t)\right]}{T\sin\left[\frac{1}{2}(s-t)\right]}\right) dt = \int_{-\frac{T}{2}}^{\frac{T}{2}} x(s-t)D_N(\Omega t)\ dt. \tag{5.35}$$

For convenience, we have elected to shift the limits of integration to a symmetric interval $[-T/2, T/2]$. Furthermore, let us assume that the discontinuity occurs at the point $t = t_s = 0$. (These assumptions simplify the calculations enormously and do not affect the final result. The general proof adds complexity which does not lead to any further insights into Fourier series convergence.) It is convenient to break up the integral into two segments along the t axis:

$$S_N(s) = \int_{-T/2}^{T/2} x_c(s-t)D_N(\Omega t)\Omega\ dt + A_0 \cdot \int_{-T/2}^{T/2} u(s-t)D_N(\Omega t)\Omega\ dt, \tag{5.36}$$

where $A_0 = x(0_{(r)}) - x(0_{(l)})$ is the magnitude of the jump at the origin. In the limit $N \to \infty$, the first term in (5.36) converges to $x_c(t)$, relegating the discontinuity's effect to the second integral, which we denote $e_N(s)$:

$$\varepsilon_N(s) = A_0 \cdot \int_{-\frac{T}{2}}^{\frac{T}{2}} u(t)D_N(\Omega(s-t))t\Omega\ dt = A_0 \cdot \int_0^{\frac{T}{2}} D_N(\Omega(s-t))\Omega\ dt. \tag{5.37}$$

The task at hand is to evaluate this integral. This can be done through several changes of variable. Substituting for the Dirichlet kernel provides

$$\varepsilon_N(s) = A_0 \cdot \int_0^{\frac{T}{2}} \left[\frac{\sin\left[\left(N+\frac{1}{2}\right)\Omega(s-t)\right]}{T\sin\left[\frac{\Omega(s-t)}{2}\right]}\right]\Omega\ dt. \tag{5.38}$$

Defining a new variable $u = \Omega(t-s)$ and expanding the sines brings the dependence on variable s into the upper limit of the integral:

$$\varepsilon_N(s) = A_0 \cdot \int_0^{-\Omega\left(s+\frac{T}{2}\right)} \left[\frac{\sin(nu)\cdot\cos\left(\frac{u}{2}\right)}{2\pi\sin\left[\frac{u}{2}\right]} + \cos(nu)\right] du. \tag{5.39}$$

Finally, we define a variable $2v = u$ which brings (5.39) into a more streamlined form

$$\varepsilon_N(s) = A_0 \cdot \int_0^{-\frac{\Omega}{2}\left(s+\frac{T}{2}\right)} \left[\frac{\sin(2nv)\cdot\cos(v)}{2\pi\sin(v)} + \cos(2nv)\right] dv . \quad (5.40)$$

This is as close as we can bring $e_N(s)$ to an analytic solution, but it contains a wealth of information. We emphasize that s appears explicitly as an upper limit in each integral. Tabulation of (5.40) produces an oscillatory function of s in the neighborhood of $s = 0$; this accounts for the ripple—Gibbs oscillations—in the partial series synthesis near the step discontinuity. As we approach the point of discontinuity at the origin, (5.40) can be evaluated analytically:

$$\varepsilon_N(s) = A_0 \cdot \int_0^{\frac{\pi}{2}} \left[\frac{\sin(2nv)\cdot\cos(v)}{2\pi\sin(v)} + \cos(2nv)\right] dv = A_0 \cdot \left[\left(\frac{1}{\pi}\cdot\frac{\pi}{2}\right)+0\right] = \frac{1}{2}A_0. \quad (5.41)$$

(Note that in going from (5.40) to (5.41), a sign change can be made in the upper limit, since the integrand is an even function of the variable v.) Accounting for both the continuous portion $x_c(t)$—which approaches $x(0_{(l)})$ as $N \to \infty$ and as $t \to 0$—and the discontinuity's effects described in (5.41), we find

$$x(0) = x_c(0) + \frac{1}{2}[x(0_{(r)}) - x(0_{(l)})] = \frac{1}{2}[x(0_{(r)}) + x(0_{(l)})] . \quad (5.42)$$

A similar argument works for a step located at an arbitrary t; this provides the general result

$$x(0) = \frac{1}{2}[x(t_{(r)}) + x(t_{(l)})] , \quad (5.43)$$

and the proof is complete. ■

Figure 5.3 illustrates the convergence of $e_N(s)$ near a step discontinuity in a rectangular pulse train. Note the smoothing of the Gibbs oscillations with increasing N.

As N approaches infinity and at points t where $x(t)$ is continuous, the Gibbs oscillations get infinitesimally small. At the point of discontinuity, they contribute an amount equal to one-half the difference between the left and right limits of $x(t)$, as dictated by (5.42).

When approaching this subject for the first time, it is easy to form misconceptions about the nature of the convergence of the Fourier series at step discontinuties, due to the manner in which the Gibbs oscillations (almost) literally cloud the issue. We complete this section by emphasizing the following points:

- The Gibbs oscillations do *not* imply a failure of the Fourier synthesis to converge. Rather, they describe how the convergence behaves.

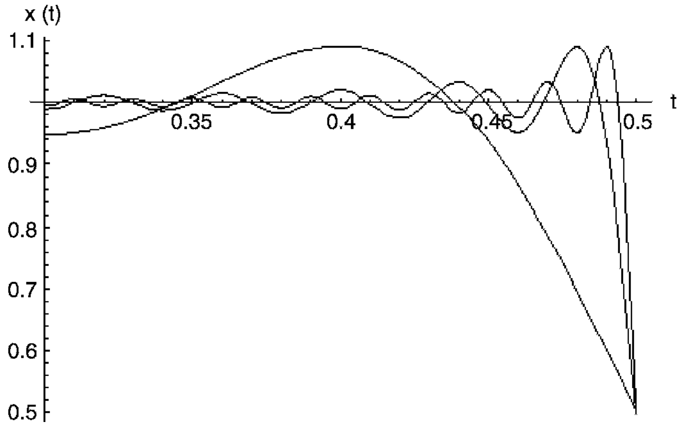

Fig. 5.3. Convergence of the Fourier series near a step, showing Gibbs oscillations for N = 10, 50, 100. For all N, the partial series expansion converges to 1/2 at the discontinuity.

- The Fourier series synthesis converges to an exact, predictable value at the point of discontinuity, namely the arithmetic mean of the left and right limits of $x(t)$, as dictated by (5.43).
- In the vicinity of the discontinuity, at points for which $x(t)$ is indeed continuous, the Gibbs oscillations disappear in the limit as N becomes infinite. That is, the synthesis converges to $x(t)$ with no residual error. It is exact.

5.1.3 Trigonometric Fourier Series

Calculations with the exponential basis functions make such liberal use of the orthogonality properties of the constituent sine and cosine waves that one is tempted to reformulate the entire Fourier series in a set of sine and cosine functions. Such a development results in the trigonometric Fourier series, an alternative to the exponential form considered in the previous section.

Expanding the complex exponential basis functions leads to a synthesis of the form

$$x(t) = \sum_{k=-\infty}^{\infty} c_k \phi_k(t) = \frac{1}{\sqrt{T}} \sum_{k=-\infty}^{\infty} c_k [\cos(k\Omega t) + j\sin(k\Omega t)]\,. \tag{5.44}$$

Since $\cos(k\Omega t)$ and $\sin(k\Omega t)$ are even and odd, respectively, in the variable k, and since for $k = 0$ there is no contribution from the sine wave, we can rearrange the summation and regroup the coefficients. Note that the summations now involve only the positive integers:

$$x(t) = \frac{1}{\sqrt{T}}\left\{c_0 + \sum_{k=1}^{\infty} (c_k + c_{-k})\cos(k\Omega t)\right\} + \sum_{k=1}^{\infty} (c_k - c_{-k})\sin(k\Omega t). \tag{5.45}$$

The zeroth coefficient has the particularly simple form:

$$c_0 = \langle x(t), 1\rangle = \int_0^T x(t)\, dt, \tag{5.46}$$

where 1 is the unit constant signal on $[0, T]$. Regrouping terms gives an expansion in sine and cosine:

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\Omega t) + \sum_{k=1}^{\infty} b_k \sin(k\Omega t), \tag{5.47}$$

where

$$a_k = \frac{c_k + c_{-k}}{\sqrt{T}}, \tag{5.48a}$$

$$b_k = j \cdot \frac{c_k - c_{-k}}{\sqrt{T}}, \tag{5.48b}$$

and

$$a_0 = \frac{1}{\sqrt{T}} \int_0^T x(t)\, dt. \tag{5.48c}$$

Under circumstances where we have a set of exponential Fourier series coefficients c_k at our disposal, (5.47) is a valid definition of the trigonometric Fourier series. In general, this luxury will not be available. Then a more general definition gives explicit intergrals for the expansion coefficients, a_k and b_k, based on the inner products $\langle x(t), f_m(t)\rangle$, where $f_m(t) = C_m \cos(m\Omega t)$ or $S_m \sin(m\Omega t)$ and C_m and S_m are normalization constants.

The C_m are determined by expanding the cosine inner product:

$$\begin{aligned}\langle x(t), C_m \cos(m\Omega t)\rangle &= \langle a_0, C_m \cos(m\Omega t)\rangle + \sum_{k=1}^{\infty} a_k \langle C_k \cos(k\Omega t), C_m \cos(m\Omega t)\rangle \\ &+ \sum_{k=1}^{\infty} a_k \langle S_k \sin(k\Omega t), C_m \cos(m\Omega t)\rangle \end{aligned} \tag{5.49}$$

Consider each term above. The first one vanishes for all m, since integrating cosine over one period $[0, T]$ gives zero. The third term also vanishes for all k, due to the orthogonality of sine and cosine. The summands of the second term are zero, except for the bracket

$$\langle C_m \cos(m\Omega t), C_m \cos(m\Omega t)\rangle = \frac{T}{2} C_m^{\ 2}. \tag{5.50}$$

To normalize this inner product, we set

$$C_m = \sqrt{\frac{2}{T}} \tag{5.51}$$

for all m. Consequently, the inner product defining the cosine Fourier expansion coefficients a_k is

$$a_k = \left\langle x(t), \sqrt{\frac{2}{T}}\cos(k\Omega t) \right\rangle. \tag{5.52}$$

The sine-related coefficients are derived from a similar chain of reasoning:

$$b_k = \left\langle x(t), \sqrt{\frac{2}{T}}\sin(k\Omega t) \right\rangle. \tag{5.53}$$

Taking stock of the above leads us to define a Fourier series based on sinusoids:

Definition (Trigonometric Fourier Series). The *trigonometric Fourier series* for $x(t)$ is the orthonormal expansion

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \phi_k(t) + \sum_{k=1}^{\infty} b_k \Psi_k(t), \tag{5.54}$$

where

$$\phi_k(t) = \sqrt{\frac{2}{T}}\cos(k\Omega t) \tag{5.55a}$$

and

$$\Psi_k(t) = \sqrt{\frac{2}{T}}\sin(k\Omega t). \tag{5.55b}$$

Remark. Having established both the exponential and trigonometric forms of the Fourier series, note that it is a simple matter to transform from one coefficient space to the other. Beginning in (5.48a), we derived expressions for the trigonometric series coefficients in terms of their exponential series counterparts. But these relations are easy to invert. For $k > 0$, we have

$$c_k = \frac{\sqrt{T}}{2}(a_k - jb_k) \tag{5.56a}$$

and

$$c_{-k} = \frac{\sqrt{T}}{2}(a_k + jb_k). \tag{5.56b}$$

Finally, for $k = 0$, we see

$$c_0 = a_0\sqrt{T}. \tag{5.57}$$

5.1.3.1 Symmetry and the Fourier Coefficients. As in the case of the exponential Fourier coefficients, the a_k and b_k acquire special properties if $x(t)$ exhibits even or odd symmetry in the time variable. These follow directly from (5.52) and (5.53), or by the application of the previously derived c_k symmetries to (5.45). Indeed, we see that

- If $x(t)$ is real and odd, then the a_k vanish identically, and the b_k are purely imaginary.
- On the other hand, if $x(t)$ is real and even, the b_k vanish and the a_k are real quantities.

The even/odd coefficient symmetry with respect to k is not an issue with the trigonometric Fourier series, since the index k is restricted to the positive integers.

5.1.3.2 Example: Sawtooth Wave. We conclude with a study of the trigonometric Fourier series for the case of a sawtooth signal. Consider the piecewise continuous function shown in Figure 5.4a. In the fundamental interval $[0, T]$, $x(t)$ consists of two segments, each of slope μ. For $t \in [0, T/2]$:

$$x(t) = \mu t, \tag{5.58a}$$

and for $t \in [T/2, T]$:

$$x(t) = \mu(t - T). \tag{5.58b}$$

The coefficients follow straightforwardly. We have

$$b_n = \frac{T}{2}\langle x(t), \sin(n\Omega t)\rangle = \frac{2\mu}{T}\int_0^T t\sin(n\Omega t)\,dt + \left(-\frac{4h}{T}\right)\int_{T/2}^{T}\sin(n\Omega t)\,dt. \tag{5.59}$$

The first integral on the right in (5.59) is evaluated through integration by parts:

$$\frac{2\mu}{T}\int_0^T t\sin(n\Omega t)\,dt = \frac{-2h}{\pi n}. \tag{5.60}$$

The second integral is nonzero only for $n = 1, 3, 5, \ldots$,

$$\frac{-4h}{T}\int_{T/2}^{T}\sin(n\Omega t)\,dt = \frac{4h}{\pi n}. \tag{5.61}$$

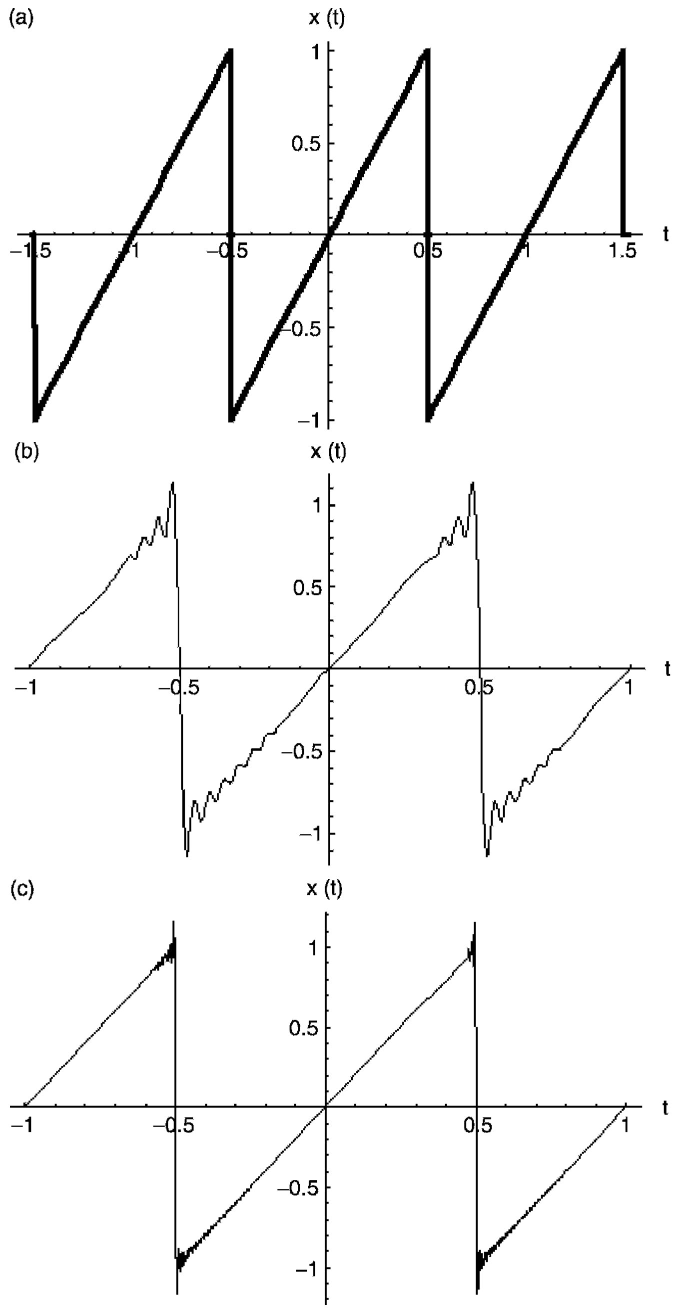

Fig. 5.4. Synthesis of the sawtooth wave using the trigonometric Fourier series. (a) The original waveform. (b) Partial series, $N = 20$. (c) For $N = 100$. (d) For $N = 200$. There are $N + 1$ terms in the partial series. (e) Details illustrating Gibbs oscillation near a discontinuity, for $N =$ 20, 100, and 200. Note that all partial series converge to $x_N(t) = 0$ at the discontinuity.

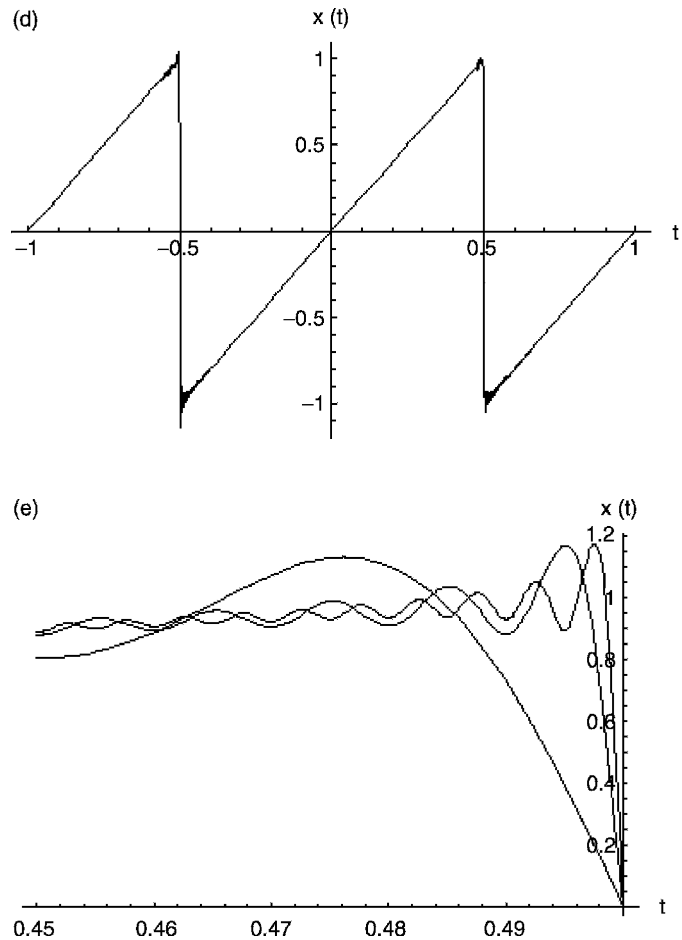

Fig. 5.4 (*Continued*)

Therefore, for $n = 1, 3, 5, \ldots,$

$$b_n = \frac{2h}{\pi n}, \tag{5.62a}$$

while for $n = 2, 4, 6, \ldots,$

$$b_n = \frac{-2h}{\pi n}. \tag{5.62b}$$

Since $x(t)$ exhibits odd symmetry in t, the coefficients for the cosine basis are identically zero for all n:

$$a_n = 0. \tag{5.63}$$

Example (Sawtooth Wave Synthesis). Figure 5.4 illustrates several partial series syntheses of this signal using the coefficients (5.62a). The Gibbs oscillations are clearly in evidence. The convergence properties follow the principles outlined earlier and illustrated in connection with the rectangular pulse train.

Remark. From the standpoint of physical correctness, the exponential and trigonometric series are equally valid. Even and odd symmetries—if they exist—are more easily visualized for the trigonometric series, but mathematically inclined analysts find appeal in the exponential Fourier series. The latter's formalism more closely relates to the Fourier transform operation considered in the next section, and it forms the basis for common numerical algorithms such as the fast Fourier transform (FFT) discussed in Chapter 7.

5.2 FOURIER TRANSFORM

In the case of periodic waveforms considered in the previous section, the notion of "frequency content" is relatively intuitive. However, many signals of practical importance exhibit no periodicity whatsoever. An isolated pulse or disturbance, or an exponentially damped sinusoid, such as that produced by a resistor–capacitor (RC) circuit, would defy analysis using the Fourier series expansion. In many practical systems, the waveform consists of a periodic sinusoidal carrier wave whose envelope is modulated in some manner; the result is a composite signal having an underlying sinusoidal structure, but without overall periodicity. Since the information content or the "message," which could range from a simple analog sound signal to a stream of digital pulses, is represented by the modulation, an effective means of signal analysis for such waves is of enormous practical value. Furthermore, all communications systems are subject to random fluctuations in the form of noise, which is rarely obliging enough to be periodic.

5.2.1 Motivation and Definition

In this section, we develop a form of Fourier analysis applicable to many practical aperiodic signals. In fact, we will eventually demonstrate that the Fourier series is a special case of the theory we are about to develop; we will need to equip ourselves, however, with a mathematical arsenal appropriate to the task. Many notions will carry over from the Fourier series. The transformation to frequency space—resulting in an analysis of the waveform in terms of its frequency content—will remain intact. Similarly, the synthesis, whereby the original signal is reconstructed based on the frequency spectrum, will be examined in detail. We will develop the criteria by which a given waveform will admit a transform to frequency space and by which the resulting spectra will admit a viable *synthesis*, or *inverse transform*.

Since our nascent Fourier transform involves integrals, analysis and synthesis relations lean heavily on the notion of absolute integrability. Not surprisingly, the analog L^p signal spaces—in particular, $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$—will figure prominently.

We recall these abstract function spaces from Chapter 3: $L^p(\mathbb{K}) = \{x(t) \mid \|x\|_p < \infty\}$. Here

$$\|x\|_p = \left[\int_{-\infty}^{\infty} |x(t)|^p dt\right]^{\frac{1}{p}} \tag{5.64}$$

is the L^p norm of $x(t)$ and $\mathbb{K}$ is either the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$. We define the set of bounded signals to be L^∞. These signal classes turn out to be Banach spaces, since Cauchy sequences of signals in L^p converge to a limit signal also in L^p. L^1 is also called the space of *absolutely integrable* signals, and L^2 is called the space of *square-integrable* signals. The case of $p = 2$ is special: L^2 is a Hilbert space. That is, there is an inner product relation on square-integrable signals $\langle x, y\rangle \in \mathbb{K}$, which extends the idea of the vector space dot product to analog signals.

In the case of the Fourier series, the frequency content was represented by a set of discrete coefficients, culled from the signal by means of an inner product involving the signal and a discrete orthonormal basis:

$$c_k = \langle x(t), \phi_k(t)\rangle = \int_{t_0}^{t_0+T} x(t)\frac{1}{\sqrt{T}}e^{-jk\Omega t}dt. \tag{5.65}$$

One might well ask whether a similar integral can be constructed to handle nonperiodic signals $f(t)$. A few required modifications are readily apparent. Without the convenience of a fundamental frequency or period, let us replace the discrete harmonics $k\omega$ with a continuous angular frequency variable ω, in radians per second. Furthermore, all values of the time variable t potentially contain information regarding the frequency content; this suggests integrating over the entire time axis, $t \in [-\infty, \infty]$. The issue of multiplicative constants, such as the normalization constant $1/\sqrt{T}$, appears in a different guise as well. Taking all of these issues into account, we propose the following definition of the Fourier transform:

Definition (Radial Fourier Transform). The radial Fourier transform of a signal $f(t)$ is defined by the integral,

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt. \tag{5.66a}$$

It is common to write a signal with a lowercase letter and its Fourier transform with the corresponding uppercase letter. Where there may be confusion, we also write $F(\omega) = \mathcal{F}[f(t)](\omega)$, with a “fancy F” notation.

Remark. Note that the Fourier transform operation $\mathcal{F}$ is an analog system that accepts time domain signals $f(t)$ as inputs and produces frequency-domain signals

$F(\omega)$ as outputs. One must be cautious while reading the signal processing literature, because two other definitions for $\mathcal{F}$ frequently appear:

- The *normalized* radial Fourier transform;
- The *Hertz* Fourier transform.

Each one has its convenient aspects. Some authors express a strong preference for one form. Other signal analysts slip casually among them. We will mainly use the radial Fourier transform, but we want to provide clear definitions and introduce special names that distinguish the alternatives, even if our terminology is not standard. When we change definitional forms to suit some particular analytical endeavor, example, or application, we can then alert the reader to the switch.

Definition (Normalized Radial Fourier Transform). The *normalized radial Fourier transform* of a signal $f(t)$ is defined by the integral,

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt . \tag{5.66b}$$

The $(2\pi)^{-1}$ factor plays the role of a normalization constant for the Fourier transform much as the factor $1/\sqrt{T}$ did for the Fourier series development. Finally, we have the Hertz Fourier transform:

Definition (Hertz Fourier Transform). The *Hertz Fourier transform* of a signal $x(t)$ is defined by the integral

$$x(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt . \tag{5.66c}$$

Remark. The units of ω in both the radial and normalized Fourier transforms are in radians per second, assuming that the time variable t is counted in seconds. The units of the Hertz Fourier transform are in hertz (units of inverse seconds or cycles per second). A laboratory spectrum analyzer displays the Hertz Fourier transform—or, at least, it shows a reasonably close approximation. So this form is most convenient when dealing with signal processing equipment. The other two forms are more convenient for analytical work. It is common practice to use ω (or Ω) as a radians per second frequency variable and use f (or F) for a Hertz frequency variable. But we dare to emphasize once again that Greek or Latin letters do no more than hint of the frequency measurement units; it is rather the particular form of the Fourier transform definition in use that tells us what the frequency units must be.

The value of the Fourier transform at $\omega = 0$, $F(0)$, is often called, in accord with electrical engineering parlance, the *direct current* or *DC* term. It represents that portion of the signal which contains no oscillatory, or *alternating current* (*AC*), component.

Now if we inspect the radial Fourier transform's definition (5.66a), it is tempting to write it as the inner product $\langle x(t), e^{j\omega t}\rangle$. Indeed, the Fourier integral has precisely this form. However, we have not indicated the signal space to which $x(t)$ may belong. Suppose we were to assume that $x(t) \in L^2(\mathbb{R})$. This space supports an inner product, but that will not guarantee the existence of the inner product, because, quite plainly, the exponential signal, $e^{j\omega t}$ is not square-integrable. Thus, we immediately confront a theoretical question of the Fourier transform's existence. Assuming that we can justify this integration for a wide class of analog signals, the Fourier transform does appear to provide a measure of the amount of radial frequency ω in signal $x(t)$. According to this definition, the frequency content of $x(t)$ is represented by a function $X(\omega)$ which is clearly analogous to the discrete set of Fourier series coefficients, but is—as we will show— a continuous function of angular frequency ω.

Example (Rectangular Pulse). We illustrate the radial Fourier transform with a rectangular pulse of width $2a > 0$, where

$$f(t) = 1 \tag{5.67}$$

for $-a \le t < a$, and vanishes elsewhere. This function has compact support on this interval and its properties under integration are straightforward when the Fourier transform is applied:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt = \int_{-a}^{a} e^{-j\omega t}dt = 2a\left[\frac{\sin(\omega a)}{\omega a}\right]. \tag{5.68}$$

The most noteworthy feature of the illustrated frequency spectrum, Figure 5.5, is that the pulse width depends upon the parameter a.

Note that most of the spectrum concentrates in the region $\omega \in [-\pi/a, \pi/a]$. For small values of a, this region is relatively broad, and the maximum at $\omega = 0$ (i.e.,

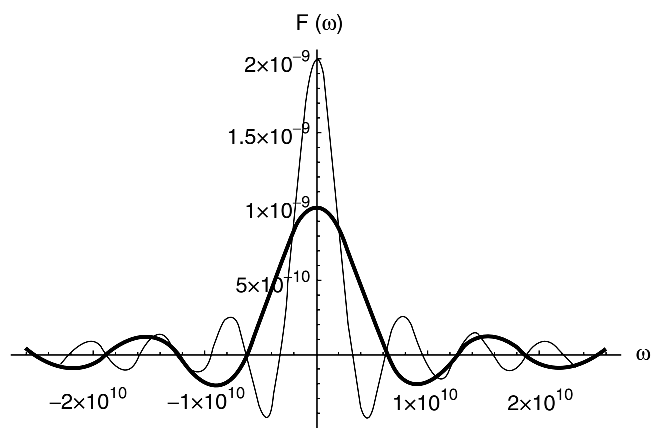

Fig. 5.5. The spectrum for a 1-ns rectangular pulse (solid line), and a 2-ns pulse. Note the inverse relationship between pulse width in time and the spread of the frequency spectrum.

the DC contribution) is, relatively speaking, low. This is an indication that a larger proportion of higher frequencies are needed to account for the relatively rapid jumps in the rectangular pulse. Conversely, as the pulse width increases, a larger proportion of the spectrum resides near the DC frequency. In fact, as the width of the pulse approaches infinity, its spectrum approaches the Dirac delta function $\delta(\omega)$, the generalized function introduced in Chapter 3. This scaling feature generalizes to all Fourier spectra, and the inverse relationship between the spread in time and the spread in frequency can be formalized in one of several uncertainty relations, the most famous of which is attributed to Heisenberg. This topic is covered in Chapter 10.

Example (Decaying Exponential). By their very nature, transient phenomena are short-lived and often associated with exponential decay. Let $\alpha > 0$ and consider

$$f(t) = e^{-\alpha t}u(t), \tag{5.69}$$

which represents a damped exponential for all $t > 0$. This signal is integrable, and the spectrum is easily calculated:

$$F(\omega) = \int_0^{\infty} e^{-\alpha t}e^{-j\omega t}dt = \frac{-1}{\alpha + j\omega}e^{-(\alpha - j\omega)t}\Big|_0^{\infty} = \frac{1}{\alpha + j\omega}. \tag{5.70}$$

Remark. $F(\omega)$ is characterized by a singularity at $\omega = j\alpha$. This pole is purely imaginary—which is typical of an exponentially decaying (but nonoscillatory) response $f(t)$. In the event of decaying oscillations, the pole has both real and imaginary parts. This situation is discussed in Chapter 6 in connection with the modulation theorem. In the limit $\alpha \to 0$, note that $f(t) \to u(t)$, but (it turns out) $\mathcal{F}[u(t)](\omega)$ does *not* approach $1/j\omega$. In this limit, $f(t)$ is no longer integrable, and the Fourier transform as developed so far does not apply. We will rectify this situation with a generalized Fourier transform developed in Chapter 6.

TABLE 5.1. Radial Fourier Transforms of Elementary Signals

Signal Expression	Radial Fourier Transform
$f(t)$	$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$
Square pulse: $u(t+a) - u(t-a)$	$2a\left[\frac{\sin(\omega a)}{\omega a}\right] = 2a\,\mathrm{sinc}(\omega a)$
Decaying exponential: $e^{-\alpha t}u(t)$, $\alpha > 0$	$\frac{1}{\alpha + j\omega}$
Gaussian: $e^{-\alpha t^2}$, $\alpha > 0$	$\sqrt{\frac{\pi}{\alpha}}e^{\frac{-\omega^2}{4\alpha}}$

5.2.2 Inverse Fourier Transform

The integral Fourier transform admits an inversion formula, analogous to the synthesis for Fourier series. One might propose a Fourier synthesis analogous to the discrete series (5.9):

$$f(t) \approx \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega. \tag{5.71}$$

In fact, this is complete up to a factor, encapsulated in the following definition:

Definition (Inverse Radial Fourier Transform). The *inverse radial Fourier transform* of $F(\omega)$ is defined by the integral

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega. \tag{5.72}$$

The Fourier transform and its inverse are referred to as a *Fourier transform pair.*

The inverses for the normalized and Hertz variants take slightly different forms.

Definition (Inverse Normalized Fourier Transform). If $F(\omega)$ is the normalized Fourier transform of $f(t)$, then the *inverse normalized Fourier transform* of $F(\omega)$ is the integral

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega. \tag{5.73a}$$

Definition (Inverse Hertz Fourier Transform). If $X(f)$ is the Hertz Fourier transform of $x(t)$, then the *inverse Hertz Fourier transform* of $X(f)$ is

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df. \tag{5.73b}$$

Naturally, the utility of this pair is constrained by our ability to carry out the integrals defining the forward and inverse transforms. At this point in the development one might consider the following:

- Does the radial Fourier transform $F(\omega)$ exist for all continuous or piecewise continuous functions?
- If $F(\omega)$ exists for some $f(t)$, is it always possible to invert the resulting spectrum to synthesize $f(t)$?

The answer to both these questions is *no*, but the set of functions which are suitable is vast enough to have made the Fourier transform the stock and trade of signal analysis. It should come as no surprise that the integrability of $f(t)$, and of its spectrum, can be a deciding factor. On the other hand, a small but very important set of common signals do not meet the integrability criteria we are about to develop, and for these we will have to extend the definition of the Fourier transform to include a class of generalized Fourier transform, treated in Chapter 6.

We state and prove the following theorem for the radial Fourier transforms; proofs for the normalized and Hertz cases are similar.

Theorem (Existence). If $f(t)$ is absolutely integrable—that is, if $f(t) \in L^1(\mathbb{R})$—then the Fourier transform $F(\omega)$ exists.

Proof: This follows directly from the transform's definition. Note that

$$|F(\omega)| = \left|\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt\right| \le \int_{-\infty}^{\infty} |f(t)|\left|e^{-j\omega t}\right|dt = \int_{-\infty}^{\infty} |f(t)|\ dt. \tag{5.74}$$

So $F(\omega)$ exists if

$$\int_{-\infty}^{\infty} |f(t)|dt < \infty; \tag{5.75}$$

that is, $f(t) \in L^1(\mathbb{R})$. ■

Theorem (Existence of Inverse). If $F(\omega)$ is absolutely integrable, then the inverse Fourier transform $\mathcal{F}^{-1}[F(\omega)](t)$ exists.

Proof: The proof is similar and is left as an exercise. ■

Taken together, these existence theorems imply that if $f(t)$ and its Fourier spectrum $F(\omega)$ belong to $L^1(\mathbb{R})$, then both the analysis and synthesis of $f(t)$ can be performed. Unfortunately, if $f(t)$ is integrable, there is no guarantee that $F(\omega)$ follows suit. Of course, it *can* and often *does*. In those cases where synthesis (inversion) is impossible because $F(\omega)$ not integrable, the spectrum is still a physically valid representation of frequency content and can be subjected to many of the common operations (filtering, band-limiting, and frequency translation) employed in practical systems. In order to guarantee both analysis and synthesis, we need a stronger condition on $f(t)$, which we will explore in due course. For the time being, we will further investigate the convergence of the Fourier transform and its inverse, as applied to continuous and piecewise continuous functions.

Theorem (Convergence of Inverse). Suppose $f(t)$ and $F(\omega)$ are absolutely integrable and continuous. Then the inverse Fourier transform exists and converges to $f(t)$.

Proof: Define a band-limited inverse Fourier transform as follows:

$$f_{\Omega}(t) = \frac{1}{2\pi}\int_{-\Omega}^{\Omega} F(\omega)e^{j\omega t}\ d\omega. \tag{5.76}$$

In the limit $\Omega \to \infty$, (5.76) should approximate $f(t)$. (There is an obvious analogy between the band-limited Fourier transform and the partial Fourier series expansion.) Replacing $F(\omega)$ with its Fourier integral representation (5.66a) and interchanging the limits of integration, (5.76) becomes

$$f_\Omega(t) = \frac{1}{2\pi}\int_{-\Omega}^{\Omega}\left[\int_{-\infty}^{\infty} f(\tau)e^{-j\omega t}d\tau\right]e^{j\omega t}d\omega = \int_{-\infty}^{\infty} f(\tau)K_\Omega(t-\tau)\,d\tau, \tag{5.77a}$$

where

$$K_\Omega(t-\tau) = \frac{\sin\Omega(t-\tau)}{\pi(t-\tau)} \equiv \int_{-\Omega}^{\Omega} e^{j\omega(t-\tau)}d\omega. \tag{5.77b}$$

There are subtle aspects involved in the interchange of integration limits carried out in the preceding equations. We apply Fubini's theorem [13, 14] and the assumption that both $f(t)$ and $F(\omega)$ are in $L^1(\mathbb{R})$. This theorem, which we reviewed in Chapter 3, states that if a function of two variables is absolutely integrable over a region, then its iterated integrals and its double integral over the region are all equal. In other words, if $\|x(t,\omega)\|_1 < \infty$, then:

- For all $t \in \mathbb{R}$, the function $x_t(\omega) = x(t,\omega)$ is absolutely integrable (except—if we are stepping up to Lebesgue integration—on a set of measure zero).
- For all $\omega \in \mathbb{R}$, the function $x_\omega(t) = x(t,\omega) \in L^1(\mathbb{R})$ (again, except perhaps on a measure zero set).
- And we may freely interchange the order of integration:

$$\int_{-\infty}^{\infty}\left[\int_{\infty}^{\infty} x(t,\omega)dt\right]d\omega = \int_{-\infty}^{\infty}\left[\int_{\infty}^{\infty} x(t,\omega)d\omega\right]dt. \tag{5.78}$$

So we apply Fubini here to the function of two variables, $x(\tau,\omega) = f(\tau)e^{-j\omega\tau}e^{j\omega t}$, with t fixed, which appears in the first iterated integral in (5.77a). Now, the function

$$K_\Omega(x) = \frac{\sin\Omega x}{\pi x} \tag{5.79}$$

is the Fourier kernel. In Chapter 3 we showed that it is one of a class of generalized functions which approximates a Dirac delta function in the limit of large Ω. Thus,

$$\lim_{\Omega\to\infty} f_\Omega(t) = \lim_{\Omega\to\infty}\int_{-\Omega}^{\Omega} f(\tau)\frac{\sin\Omega(t-\tau)}{\pi(t-\tau)}d\tau = \int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)\,d\tau = f(t). \tag{5.80}$$

completing the proof. ■

5.2.3 Properties

In this section we consider the convergence and algebraic properties of the Fourier transform. Many of these results correspond closely to those we developed for the Fourier series. We will apply these properties often—for instance, in developing analog *filters*, or frequency-selective convolutional systems, in Chapter 9.

5.2.3.1 Convergence and Discontinuities. Let us first investigate how well the Fourier transform's synthesis relation reproduces the original time-domain signal. Our first result concerns time-domain discontinuities, and the result is quite reminiscent of the case of the Fourier series.

Theorem (Convergence at Step Discontinuities). Suppose $f(t) \in L^1(\mathbb{R})$ has a step discontinuity at some time *t*. Let $F(\omega) = \mathcal{F}[f(t)](\omega)$ be the radial Fourier transform of $f(t)$ with $F(\omega) \in L^1(\mathbb{R})$. Assume that, in some neighborhood of $t, f(t)$ and its derivative have well-defined limits from the left and from the right: $f(t_{(l)})$ and $f(t_{(r)})$, respectively. Then the inverse Fourier transform, $\mathcal{F}^{-1}[F(\omega)](t)$, converges pointwise to the value,

$$\mathcal{F}^{-1}[F(\omega)](t) = \frac{[f(t_{(r)}) + f(t_{(l)})]}{2}. \tag{5.81}$$

Proof: The situation is clearly analogous to Fourier series convergence at a step discontinuity. We leave it as an exercise to show that the step discontinuity (assumed to lie at $t = 0$ for simplicity) gives a residual Gibbs oscillation described by

$$\varepsilon_N(t) = A_0\left[\int_{-\infty}^{0} \frac{\sin v}{\pi v}\,dv + \int_{0}^{\Omega t} \frac{\sin v}{\pi v}\,dv\right] = \frac{1}{2}A_0 + A_0\int_{0}^{\Omega t} \frac{\sin v}{\pi v}\,dv, \tag{5.82a}$$

where the amplitude of the step is

$$A_0 = [f(0_{(r)}) - f(0_{(l)})]. \tag{5.82b}$$

Therefore in the limit as $\Omega \to \infty$,

$$\varepsilon_N(0) = \frac{1}{2}A_0. \tag{5.83}$$

Hence the inverse Fourier transform converges to the average of the left- and right-hand limits at the origin,

$$f(0) = f_c(0) + \frac{1}{2}[f(0_{(r)}) - f(0_{(l)})] = \frac{1}{2}[f(0_{(r)}) + f(0_{(l)})]. \tag{5.84}$$

For a step located at an arbitrary $t = t_s$ the result generalizes so that

$$f(0) = f_c(t_s) + \frac{1}{2}[f(t_{s(r)}) - f(t_{s(l)})] = \frac{1}{2}[f(t_{s(r)}) + f(t_{s(l)})], \tag{5.85}$$

and the proof is complete. ■

The Gibbs oscillations are an important consideration when evaluating Fourier transforms numerically, since numerical integration over an infinite interval always involves approximating infinity with a suitably large number. Effectively, they are band-limited Fourier transforms; and in analogy to the Fourier series, the Gibbs oscillations are an artifact of truncating the integration.

5.2.3.2 Continuity and High- and Low-Frequency Behavior of Fourier Spectra. The continuity of the Fourier spectrum is one of its most remarkable properties. While Fourier analysis can be applied to both uniform and piecewise continuous signals, the resulting spectrum is *always* uniformly continuous, as we now demonstrate.

Theorem (Continuity). Let $f(t) \in L^1(\mathbb{R})$. Then $F(\omega)$ is a uniformly continuous function of ω.

Proof: We need to show that for any $\varepsilon > 0$, there is a $\delta > 0$, such that $|\omega - \theta| < \delta|$ implies that $|F(\omega) - F(\theta)| < \varepsilon|$. This follows by noting

$$|F(\omega + \delta) - F(\omega)| = \int_{-\infty}^{\infty} f(t)(e^{-j\delta t} - 1)e^{-j\omega t}\,dt \le 2\|f\|_1. \tag{5.86}$$

Since $|F(\omega + \delta) - F(\omega)|$ is bounded above by $2\|f\|_1$, we may apply the Lebesgu-dominated convergence theorem (Chapter 3). We take the limit, as $\delta \to 0$, of $|F(\omega + \delta) - F(\omega)|$ and the last integral in (5.86). But since $e^{-j\delta t} \to 1$ as $\delta \to 0$, this limit is zero:

$$\lim_{\delta \to 0} |F(\omega + \delta) - F(\omega)| = 0 \tag{5.87}$$

and $F(\omega)$ is continuous. Inspecting this argument carefully, we see that the limit of the last integrand of (5.86) does not depend on ω, establishing uniform continuity as well. ■

Remark. This theorem shows that absolutely integrable signals—which includes every practical signal available to a real-world processing and analysis system—can have no sudden jumps in their frequency content. That is, we cannot have $|F(\omega)|$ very near one value as ω increases toward ω_0, and $|F(\omega)|$ approaches a different value as ω decreases toward ω_0. If a signal is in $L^1(\mathbb{R})$, then its spectra are smooth. This is an interesting situation, given the abundance of piecewise continuous waveforms (such as the rectangular pulse) which are clearly in $L^1(\mathbb{R})$ and, according to this theorem, exhibit continuous (note, not piecewise continuous) spectra. Moreover, the uniform continuity assures us that we should find no cusps in our plots of $|F(\omega)|$ versus ω.

Now let us consider the high-frequency behavior of the Fourier spectrum. In the limit of infinite frequency, we shall show $F(\omega) \to 0$. This general result is easily demonstrated by the Riemann–Lebesgue lemma, a form of which was examined in Chapter 3 in connection with the high-frequency behavior of simple sinusoids (as distributions generated by the space of testing functions with compact support). Here the lemma assumes a form that suits the Fourier transform.

Proposition (Riemann–Lebesgue Lemma, Revisited). If $f(t)$ is integrable, then

$$\lim_{|\omega| \to \infty} |F(\omega)| = 0. \tag{5.88}$$

Proof: The proof follows easily from a convenient trick. Note that

$$e^{-j\omega t} = -e^{-j\omega t - j\pi} = -e^{-j\omega\left(t + \frac{\pi}{\omega}\right)}. \tag{5.89}$$

Thus, the Fourier integral can be written

$$F(\omega) = -\int_{-\infty}^{\infty} f(t) e^{-j\omega\left(t + \frac{\pi}{\omega}\right)} dt = -\int_{-\infty}^{\infty} f\left(t - \frac{\pi}{\omega}\right) e^{-j\omega t} dt. \tag{5.90}$$

Expressing the fact that $F(\omega) \equiv \frac{1}{2}[F(\omega) + F(\omega)]$ by utilizing the standard and revised representations (as given in (5.90)) of $F(\omega)$, we have

$$F(\omega) = \frac{1}{2}\int_{-\infty}^{\infty} \left[f(t) - f\left(t - \frac{\pi}{\omega}\right)\right] e^{-j\omega t} d\omega, \tag{5.91}$$

so that

$$|F(\omega)| \le \int_{-\infty}^{\infty} \left|\left[f(t) - f\left(t - \frac{\pi}{\omega}\right)\right]\right| d\omega. \tag{5.92}$$

Taking the high-frequency limit, we have

$$\lim_{\omega \to \infty} |F(\omega)| \le \lim_{\omega \to \infty} \int_{-\infty}^{\infty} \left|\left[f(t) - f\left(t - \frac{\pi}{\omega}\right)\right]\right| d\omega = 0, \tag{5.93}$$

and the lemma is proven. ■

Taken in conjunction with continuity, the Riemann–Lebesgue lemma indicates that the spectra associated with integrable functions are well-behaved across all frequencies. But we emphasize that despite the decay to zero indicated by (5.93), this does not guarantee that the spectra decay rapidly enough to be integrable. Note that the Fourier transform $F(\omega)$ of $f(t) \in L^1(\mathbb{R})$ is bounded. In fact, we can easily estimate that $\|F\|_\infty \le \|f\|_1$ (exercise).

5.2.3.3 Algebraic Properties. These properties concern the behavior of the Fourier transform integral under certain algebraic operations on the transformed signals. The Fourier transform is an analog system, mapping (some) time-domain signals to frequency-domain signals. Thus, these algebraic properties include such operations that we are familiar with from Chapter 3: scaling (amplification and attenuation), summation, time shifting, and time dilation.

Proposition (Linearity). The integral Fourier transform is linear; that is,

$$\mathcal{F}\left[\sum_{k=1}^{N} a_k f_k(t)\right](\omega) = \sum_{k=1}^{N} a_k F_k(\omega)\,. \tag{5.94}$$

Proof: This follows from the linearity of the integral. ■

From a practical standpoint, the result is of enormous value in analyzing composite signals and signals plus noise, indicating that the spectra of the individual components can be analyzed and (very often) processed separately.

Proposition (Time Shift). $\mathcal{F}[f(t-t_0)](\omega) = e^{-j\omega t_0}F(\omega)$.

Proof: A simple substitution of variables, $v = t - t_0$, applied to the definition of the Fourier transform leads to

$$\mathcal{F}[f(t-t_0)](\omega) = \int_{-\infty}^{\infty} f(t-t_0)e^{-j\omega t}dt = e^{-j\omega t_0}\int_{-\infty}^{\infty} f(v)e^{-j\omega v}dv = e^{-j\omega t_0}F(\omega) \tag{5.95}$$

completing the proof. ■

Remark. Linear systems often impose a time shift of this type. Implicit in this property is the physically reasonable notion that a change in the time origin of a signal $f(t)$ does not affect the magnitude spectrum $|F(\omega)|$. If the same signal arises earlier or later, then the relative strengths of its frequency components remain the same since the energy $\|F(\omega)\|_2$ is invariant.

Proposition (Frequency Shift). $\mathcal{F}[f(t)e^{j\omega_0 t}](\omega) = F(\omega-\omega_0)$.

Proof: Writing out the Fourier transform explicitly, we find

$$\mathcal{F}[f(t)e^{j\omega t_0}](\omega) = \int_{-\infty}^{\infty} f(t)e^{-j(\omega-\omega_0)t}\,dt = F(\omega-\omega_0) \tag{5.96}$$

completing the proof. ■

Remark. This is another result which is central to spectral analysis and linear systems. Note the ease with which spectra can be translated throughout the frequency domain by simple multiplication with a complex sinusoidal phase factor in the time domain. Indeed, (5.96) illustrates exactly how radio communication and broadcast frequency bands can be established [19–21]. Note that the Fourier transform itself is not translation invariant. The effect of a frequency shift is shown in Figure 5.6. Note that ω_0 can be positive or negative.

The simplicity of the proof belies the enormous practical value of this result. Fundamentally, it implies that by multiplying a waveform $f(t)$ by a sinusoid of known frequency, the spectrum can be shifted to another frequency range. This idea makes multichannel communication and broadcasting possible and will be explored more fully in Chapter 6.

Proposition (Scaling). Suppose $a \neq 0$. Then

$$F[f(at)](\omega) = \frac{1}{|a|}F\left(\frac{\omega}{a}\right). \tag{5.97}$$

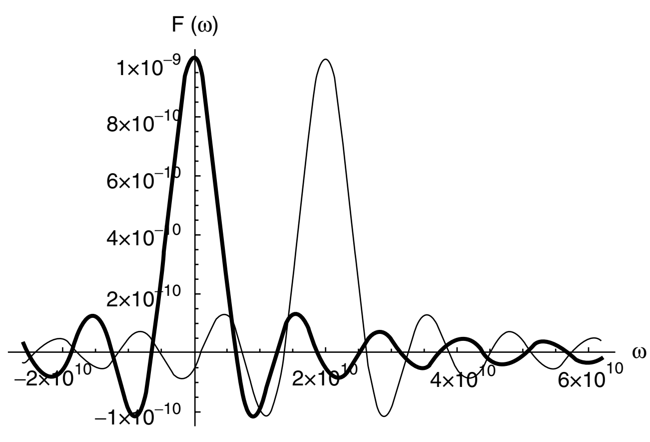

Fig. 5.6. Frequency translation. A "sinc" spectrum (solid line) and same spectrum shifted in frequency space by an increment $\omega_0 = 2 \times 10^{10}$.

Proof: Consider the cases of the scale parameter, $a > 0$ and $a < 0$, separately. First, suppose $a > 0$. With the substitution, $v = at$, it follows that

$$\mathcal{F}[f(at)](\omega) = \frac{1}{a}\int_{-\infty}^{\infty} f(v)e^{-j(\omega/a)v}dv = \frac{1}{a}F\left(\frac{\omega}{a}\right). \tag{5.98}$$

Following a similar argument for $a < 0$, carefully noting the signs on variables and the limits of integration, we find

$$\mathcal{F}[f(at)](\omega) = \frac{-1}{a}\int_{\infty}^{-\infty} f(v)e^{-j(\omega/a)v}dv = \frac{-1}{a}F\left(\frac{\omega}{a}\right). \tag{5.99}$$

In either case, the desired result (5.97) follows. ∎

Scaling in the time domain is a central feature of the wavelet transform, which we develop in Chapter 11. For example, (5.97) can be used to describe the spectral properties of the crucial ‘mother’ wavelet, affording the proper normalization and calculation of wavelet coefficients (analogous to Fourier coefficients).

The qualitative properties previously observed in connection with the rectangular pulse and its spectrum are made manifest by these relations: The multiplicative scale a in the time-domain scales as a^{-1} in the spectrum. A Gaussian pulse serves as an excellent illustration of scaling.

Example (Gaussian). The Gaussian function

$$f(t) = e^{-\alpha t^2} \tag{5.100}$$

and its Fourier transform are shown in Figure 5.7. Of course, we assume $\alpha > 0$. Panel (a) shows Gaussian pulses in the time domain for $\alpha = 10^{11}$ and $\alpha = 11^{11}$. Panel (b) shows the corresponding Fourier transforms.

We will turn to the Gaussian quite frequently when the effects of noise on signal transmission are considered. Although noise is a nondeterministic process, its statistics—the spread of noise amplitude—often take the form of a Gaussian. Noise is considered a corruption whose effects are deleterious, so an understanding of its spectrum, and how to process noise so as to minimize its effects, plays an important role in signal analysis. Let us work out the calculations. In this example, we have built in a time scale, $a = \sqrt{\alpha}$, and we will trace its effects in frequency space:

$$F(\omega) = \int_{-\infty}^{\infty} e^{-\alpha t^2}[\cos(\omega t) - j\sin(\omega t)]\, dt. \tag{5.101}$$

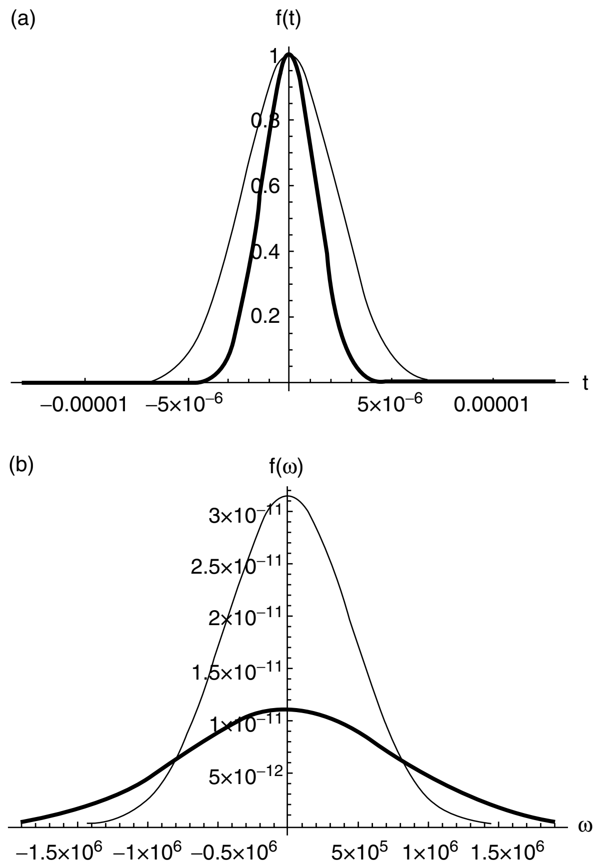

Fig. 5.7. (a) Gaussian pulses in the time domain, for $\alpha = 10^{11}$ and $\alpha = 11^{11}$ (solid lines). (b) Corresponding Fourier transforms.

If we informally assume that the integration limits in (5.101) approach infinity in perfect symmetry, then—since sine is an odd signal—we can argue that the contribution from $\sin(\omega t)$ vanishes identically. This leaves a common tabulated integral,

$$F(\omega) = \int_{-\infty}^{\infty} e^{-\alpha t^2} \cos(\omega t) dt = \sqrt{\frac{\pi}{\alpha}} e^{\frac{-\omega^2}{4\alpha}} . \tag{5.102}$$

Note several features:

- The Fourier transform of a Gaussian is again a Gaussian.
- Furthermore, the general effects of scaling, quantified in (5.97), are clearly in evidence in Figure 5.7 where the Gaussian spectra are illustrated.

Corollary (Time Reversal)

$$\mathcal{F}[f(-t)](\omega) = F(-\omega). \tag{5.103}$$

Proof: This is an instance of the scaling property. ■

Proposition (Symmetry)

$$\mathcal{F}[F(t)](\omega) = 2\pi f(-\omega). \tag{5.104}$$

Proof: A relationship of this sort is hardly surprising, given the symmetric nature of the Fourier transform pair. Since

$$\mathcal{F}^{-1}[F(\omega)](t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega, \tag{5.105a}$$

it follows that

$$f(-t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{-j\omega t}d\omega. \tag{5.105b}$$

With a simple change of variables, we obtain

$$f(-\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(t)e^{-j\omega t}d\omega, \tag{5.106}$$

concluding the proof. ■

From a practical standpoint, this symmetry property is a convenient trick, allowing a list of Fourier transform pairs to be doubled in size without evaluating a single integral.

5.2.3.4 Calculus Properties. Several straightforward but useful properties are exhibited by the Fourier transform pair under differentiation. These are easily proven.

Proposition (Time Differentiation). Let $f(t)$ and $\frac{d^k f}{dt^k}$ be integrable functions, and suppose $\lim_{|t| \to \infty} \frac{d^k f}{dt^k} = 0$ for all $k = 0, 1, \ldots, n$. Then

$$\mathcal{F}\left[\frac{d^n f}{dt^n}\right](\omega) = (j\omega)^n F(\omega). \tag{5.107}$$

Proof: We establish this result for the first derivative; higher orders follow by induction. Representing the Fourier integral by parts gives

$$\int_{-\infty}^{\infty} \frac{d}{dt} f(t) e^{-j\omega t} dt = f(t) e^{-j\omega t}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(t)(-j\omega) e^{-j\omega t} dt. \tag{5.108}$$

Under the expressed conditions, the result for the first derivative follows immediately:

$$\mathcal{F}\left[\frac{df}{dt}\right](\omega) = j\omega F(\omega). \tag{5.109}$$

Repeated application of this process lead to (5.107). ■

Proposition (Frequency Differentiation)

$$\mathcal{F}[(-jt)^n f(t)](\omega) = \frac{d^n}{d\omega^n} F(\omega). \tag{5.110}$$

Proof: The proof is similar to time differentiation. Note that the derivatives of the spectrum must exist in order to make sense of this propostion. ■

The differentiation theorems are useful when established $f(t)$ or spectra are multiplied by polynomials in their respective domains. For example, consider the case of a Gaussian time signal as in (5.100), multiplied by an arbitrary time-dependent polynomial. According to the frequency differentiation property,

$$\mathcal{F}\left[(a_0 + a_1 t + \cdots + a_k t^k) e^{-\alpha t^2}\right](\omega) = \left[a_0 + \frac{-a_1}{j} \frac{d}{d\omega} + \cdots + \frac{a_k}{(-j)^k} \frac{d^k}{d\omega^k}\right] \sqrt{\frac{\pi}{\alpha}} e^{-\omega^2/(4\alpha)} \tag{5.111}$$

so that the act of taking a Fourier transform has been reduced to the application of a simple differential operator. The treatment of spectra corresponding to pure polynomials defined over all time or activated at some time t_0 will be deferred until Chapter 6, where the generalized Fourier transform of unity and $u(t - t_0)$ are developed.

Now let us study the low-frequency behavior of Fourier spectra. The Riemann–Lebesgue lemma made some specific predictions about Fourier spectra in the limit of infinite frequency. At low frequencies, in the limit as $\omega \to 0$, we can formally expand $F(\omega)$ in a Maclaurin series,

$$F(\omega) = \sum_{k=0}^{\infty} \frac{\omega^k}{k!} \frac{d^k F(0)}{d\omega^k}. \tag{5.112}$$

The Fourier integral representation of $F(\omega)$ can be subjected to a Maclaurin series for the frequency-dependent exponential:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\left[\sum_{k=0}^{\infty} \frac{(-j\omega t)^k}{k!}\right] dt = \sum_{k=0}^{\infty} (-j)^k \frac{\omega^k}{k!} \int_{-\infty}^{\infty} t^k f(t)\, dt. \tag{5.113}$$

The last integral on the right is the kth moment of $f(t)$, defined as

$$m_k = \int_{-\infty}^{\infty} t^k f(t)\, dt \tag{5.114}$$

If the moments of a function are finite, we then have the following proposition.

Proposition (Moments)

$$\frac{d^k F(0)}{d\omega^k} = (-j)^k m_k, \tag{5.115}$$

which follows directly on comparing (5.112) and (5.113).

The moment theorem allows one to predict the low-frequency behavior of $f(t)$ from an integrability condition in time. This is often useful, particularly in the case of the wavelet transform. In order to qualify as a wavelet, there are necessary conditions on certain moments of a signal. This matter is taken up in Chapter 11.

Table 5.2 lists radial Fourier transformation properties. Some of these will be shown in the sequel.

5.2.4 Symmetry Properties

The even and odd symmetry of a signal $f(t)$ can have a profound effect on the nature of its frequency spectrum. Naturally, the impact of even–odd symmetry in transform analysis comes about through its effect on integrals. If $f(t)$ is odd, then its integral over symmetric limits $t \in [-L, L]$ vanishes identically; if $f(t)$ is even, this integral may be nonzero. In fact, this property was already put to use when discussing the Gaussian and its spectrum.

Not all functions $f(t)$ exhibit even or odd symmetry. But an arbitrary $f(t)$ may be expressed as the sum of even and odd parts: $f(t) = f_e(t) + f_o(t)$, where

$$f_e(t) = \frac{1}{2}[f(t) + f(-t))] \tag{5.116a}$$

and

$$f_o(t) = \frac{1}{2}[f(t) - f(-t)]\,. \tag{5.116b}$$

For example, in the case of the unit step,

$$f_e(t) = \frac{1}{2}[u(t) + u(-t))] = \frac{1}{2} \tag{5.117a}$$

and

$$f_o(t) = \frac{1}{2}[u(t) - u(-t)] = \frac{1}{2}\operatorname{sgn} t. \tag{5.117b}$$

where $\operatorname{sgn} t$ is the signum function. These are illustrated in Figure 5.8 using the unit step as an example.

TABLE 5.2. Summary of Radial Fourier Transform Properties

Signal Expression	Radial Fourier Transform or Property
$f(t)$	$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$ (Analysis equation)
$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$	$F(\omega)$ (Inverse, synthesis equation)
$af(t) + bg(t)$	$aF(\omega) + bG(\omega)$ (Linearity)
$f(t-a)$	$e^{-j\omega a}F(\omega)$ (Time shift)
$f(t)\exp(j\theta t)$	$F(\omega - \theta)$ (Frequency shift, modulation)
$f(at)$, $a \neq 0$	$\frac{1}{\lvert a\rvert}F\left(\frac{\omega}{a}\right)$ (Scaling, dilation)
$f(-t)$	$F(-\omega)$ (Time reversal)
$\left[\frac{d^n f(t)}{dt^n}\right]$	$(j\omega)^n F(\omega)$ (Time differentiation)
$(-jt)^n f(t)$	$\frac{d^n}{d\omega^n}F(\omega)$ (Frequency differentiation)
$\lVert x\rVert_2 = \frac{\lVert X(\omega)\rVert_2}{\sqrt{2\pi}}$	Plancherel's theorem
$\langle f, g\rangle = \frac{1}{2\pi}\langle F, G\rangle$ $f, g \in L^2(\mathbb{R})$	Parseval's theorem
$f * h$, where $f, h \in L^2(\mathbb{R})$	$F(\omega)H(\omega)$
$f(t)h(t)$	$(2\pi)^{-1}F(\omega) * H(\omega)$

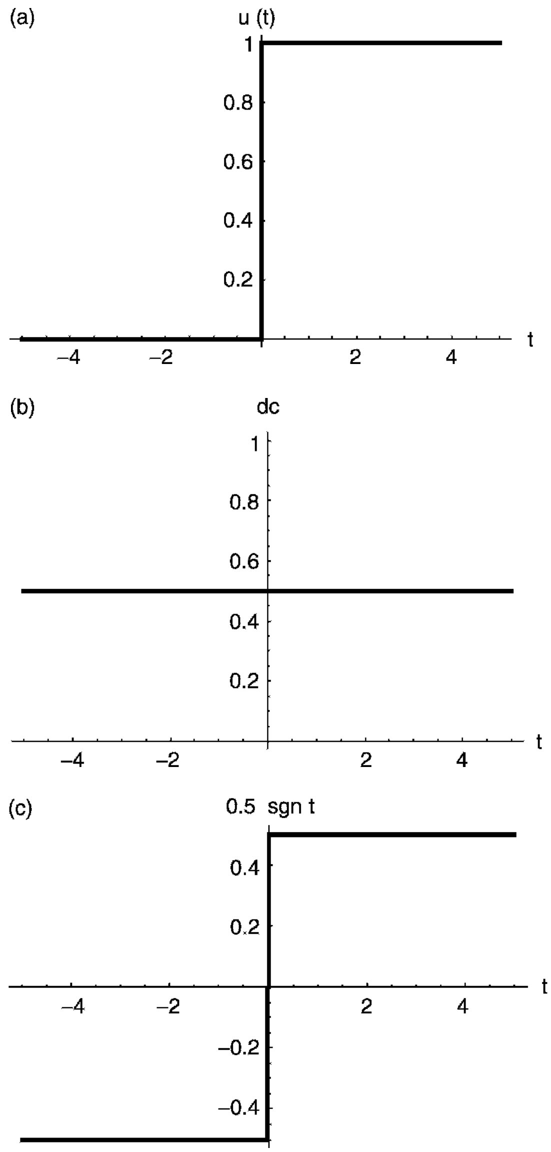

Fig. 5.8. (a) The unit step. (b) Its even-symmetry portion, a DC level of amplitude 1/2. (c) Its odd-symmetry portion, a signum of amplitude 1/2.

Whether the symmetry is endemic to the $f(t)$ at hand, or imposed by breaking it into even and odd parts, an awareness of its effects on the Fourier transform can often simplify calculations or serve as a check. Consider the spectrum of an arbitrary $f(t)$ written as the sum of even and odd constituents, $f(t) = f_e(t) + f_o(t)$:

$$F(\omega) = \int_{-\infty}^{\infty} [f_e(t) + f_o(t)] e^{-j\omega t} dt$$
$$= \int_{-\infty}^{\infty} [f_e(t) + f_o(t)] \cos(\omega t)\ dt - j \int_{-\infty}^{\infty} [f_e(t) + f_o(t)] \sin(\omega t)\ dt. \tag{5.118}$$

Elimination of the integrals with odd integrands reduces (5.110) to the elegant form

$$F(\omega) = \int_{-\infty}^{\infty} f_e(t) \cos(\omega t) dt - j \int_{-\infty}^{\infty} f_o(t) \sin(\omega t)\ dt. \tag{5.119}$$

This is a general result, applicable to an arbitrary $f(t)$ which may be real, complex, or purely imaginary. We consider these in turn.

5.2.4.1 Real f(t). A Fourier spectrum will, in general, have real and imaginary parts:

$$\mathrm{Re}[F(\omega)] = \int_{-\infty}^{\infty} f_e(t) \cos(\omega t)\ dt, \tag{5.120a}$$

which is an even function of ω (since $\cos(\omega t)$ is even in this variable) and

$$\mathrm{Im}[F(\omega)] = \int_{-\infty}^{\infty} f_o(t) \sin(\omega t)\ dt, \tag{5.120b}$$

which inherits the odd ω symmetry of $\sin(\omega t)$.

According to (5.120a), if $f(t)$ is even in addition to being real, then $F(\omega)$ is also real and even in ω. The Gaussian is a prime example, and many “mother wavelets” considered in Chapter 11 are real-valued, even functions of time. On the other hand, (5.120b) implies that if $f(t)$ is real but of odd symmetry, its spectrum is real and odd in ω.

5.2.4.2 Complex f(t). An arbitrary complex $f(t)$ can be broken into complex even and odd constituents $f_e(t)$ and $f_o(t)$ in a manner similar to the real case. When an expansion similar to (5.118) is carried out, it becomes apparent that $F(\omega)$ is, in general, complex, and it will consist of even and odd parts, which we denote $F_e(\omega)$ and $F_o(\omega)$. It is straightforward to show that the transforms break down as follows:

$$\mathrm{Re}[F_e(\omega)] = \int_{-\infty}^{\infty} \mathrm{Re}[f_e(t)] \cos(\omega t)\ dt, \tag{5.121a}$$

$$\mathrm{Re}[F_o(\omega)] = \int_{-\infty}^{\infty} \mathrm{Im}[f_o(t)] \sin(\omega t)\ dt, \tag{5.121b}$$

$$\mathrm{Im}[F_e(\omega)] = \int_{-\infty}^{\infty} \mathrm{Im}[f_e(t)]\cos(\omega t)\, dt, \tag{5.121c}$$

and

$$\mathrm{Im}[F_o(\omega)] = -\int_{-\infty}^{\infty} \mathrm{Re}[f_o(t)]\sin(\omega t)\, dt. \tag{5.121d}$$

The reader may easily verify that the earlier results for real $f(t)$ can be derived as special cases of (5.121a)–(5.121d).

5.2.4.3 Imaginary f(t). This is also a special case of the above, derived by setting $\mathrm{Re}[f_e(t)] = \mathrm{Re}[f_o(t)] = 0$. In particular, note that if $f(t)$ is imaginary and odd, then $F(\omega)$ is odd but real. If $f(t)$ is imaginary and even, then the spectrum is also even but imaginary.

Most signals $f(t)$ are real-valued, but there are notable cases where a signal may be modified, intentionally or as a by-product of transmission and processing, to become complex or even purely imaginary. Examples include exponential carrier modulation and filtering. Either operation may impose a phase shift that is not present in the original signal.

5.2.4.4 Summary. Familiarity with the symmetry properties of the Fourier transform can reduce unnecessary calculations and serve as a check of the final results. In the event that a waveform is not intrinsically odd or even, it is not always necessary, or even advisable, to break it into even and odd constituents, but doing so may be helpful when one is calculating transforms by hand and has access to a limited set of tabulated integrals. In most practical situations, numerical implementation of the Fourier transform, such as the fast Fourier transform (FFT) considered in Chapter 7, will handle the symmetries automatically.

5.3 EXTENSION TO $L^2(\mathbb{R})$

This section extends the Fourier transform to square-integrable signals. The formal definition of the Fourier transform resembles an inner-product integral of a signal $f(t)$ with the exponential $\exp(j\omega t)$. The inner product in $L^2(\mathbb{R})$ works as a measure of similarity between two signals, so $F(\omega)$, when it exists, indicates how much of radial frequency ω we find in $f(t)$. This is intuitive, simple, and attractive.

There are some fundamental difficulties, however, with this quite informal reasoning. We have shown that the Fourier transform of a signal $f(t)$ exists when $f(t)$ is absolutely integrable, but $L^1(\mathbb{R})$ signals are not the best realm for signal theorizing. In Chapter 3, for example, we found that they do not comprise an inner product space; that alone immediately breaks our intuitive concept of the Fourier integral as an inner product. Moreover, the Fourier transform of an $L^1(\mathbb{R})$ signal is not necessarily integrable, so we cannot assume to take its inverse transform. An example is the square pulse, $x(t) = u(t+1) - u(t-1)$ whose radial Fourier transform is a sinc function, $X(\omega) = 2a\mathrm{sinc}(\omega a)$.

Several approaches exist for developing a Fourier transform for $f(t) \in L^2(\mathbb{R})$. These methods include:

(i) Defining $F(\omega)$ as an infinite series expansion using special Hermite functions [22];

(ii) Writing square-integrable signals as limits of elements of the Schwarz class S of infinitely differentiable, rapidly decreasing signals (Chapter 3) [10];

(iii) Using the familiar L^p spaces, in particular the intersection of L^1 and L^2, which is dense in L^2, as the seed for a general Fourier transform for square-integrable signals [23, 24].

We follow the last approach above.

5.3.1 Fourier Transforms in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$

Toward defining a Fourier transform for finite-energy signals, the main ideas are to:

- Show the validity of the forward transforms for a narrow signal class: $L^1 \cap L^2$
- Argue that this is a dense set within L^2, so we can write any general square-integrable $f(t)$ as a limit of integrable, finite-energy signals: $f(t) = \lim_{n\to\infty} f_n(t)$ where $\{f_n(t) \in L^1 \cap L^2 \mid n \in \mathbb{N}\}$;
- Then extend the transform to square-integrable by defining the transform as a limit $F(\omega) = \lim_{n\to\infty} F_n(\omega)$, where $F_n(\omega) = \mathcal{F}[f_n(t)]$.

Theorem. If $f(t) \in L^1 \cap L^2$, then its Fourier transform $F(\omega)$ is square-integrable.

Proof: Consider a rectangular pulse of width 2α in the frequency domain. For $-\alpha < \omega < \alpha$

$$P_\alpha(\omega) = 1 \tag{5.122}$$

and vanishes outside this interval. Our strategy is to incorporate one such pulse inside the spectral energy integral and consider the limit as the pulse width becomes infinite. We claim

$$\lim_{\alpha\to\infty} \int_{-\infty}^{\infty} P_\alpha(\omega)|F(\omega)|^2 d\omega = \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega = \|F(\omega)\|_2 . \tag{5.123}$$

Let us verify that this integral is finite: $F \in L^2(\mathbb{R})$. Inserting the explicit integrals for $F(\omega)$ and its complex conjugate into (5.123) gives

$$\int_{-\infty}^{\infty} P_\alpha(\omega)F(\omega)F^*(\omega)\, d\omega = \int_{-\infty}^{\infty} f(t) \int_{-\infty}^{\infty} f^*(\tau)\left\{\int_{-\infty}^{\infty} e^{j\omega(\tau - t)} P_\alpha(\omega) d\omega\right\} d\tau\, dt. \tag{5.124}$$

The integral in curly brackets is proportional to the Fourier kernel,

$$\int_{-\infty}^{\infty} e^{j\omega(\tau - t)} P_\alpha(\omega) d\omega = 2\pi K_\alpha(\tau - t) = 2\pi \frac{\sin \alpha(\tau - t)}{\pi(\tau - t)}, \tag{5.125}$$

so thereby takes the streamlined form,

$$\int_{-\infty}^{\infty} P_\alpha(\omega)|F(\omega)|^2 d\omega = \int_{-\infty}^{\infty} f(t) \int_{-\infty}^{\infty} f^*(\tau) K_\alpha(\tau - t)\, d\tau\, dt. \tag{5.126}$$

A substitution of variables, $v = t - \tau$, provides

$$\int_{-\infty}^{\infty} P_\alpha(\omega)|F(\omega)|^2 d\omega = 2\pi \int_{-\infty}^{\infty} C_f(v) K_\alpha(v)\, dv, \tag{5.127}$$

where

$$C_f(v) = \int_{-\infty}^{\infty} f(v + \tau) f(\tau)\, d\tau. \tag{5.128}$$

Taking the limit as $\alpha \to \infty$, the kernel behaves like a Dirac:

$$\begin{aligned}\lim_{\alpha \to \infty} \int_{-\infty}^{\infty} P_\alpha(\omega)|F(\omega)|^2 d\omega &= \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega = 2\pi \int_{-\infty}^{\infty} C_f(v)\delta(v)\, dv \\ &= 2\pi C_f(0) = 2\pi \int_{-\infty}^{\infty} |f(t)|^2 dt \end{aligned} \tag{5.129}$$

Since $\|f\|_2 < \infty$, so too is $\|F\|_2 < \infty$. ■

An interesting and valuable corollary from the above proof is the following result showing the proportionality of the energy of a signal and its Fourier transform.

Corollary (Plancherel's Theorem). If $f(t) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then $\|F\|_2 = (2\pi)^{1/2} \|f\|_2$.

Proof: Evident from the theorem's proof (5.129). ■

Corollary. The radial Fourier transform $\mathcal{F}\colon L^1 \cap L^2 \to L^2$ is a bounded linear operator with $\|\mathcal{F}\| = (2\pi)^{1/2}$.

Proof: Recall from Chapter 3 that a bounded linear operator T is a linear map T: $N \to K$ of normed spaces when there is a constant $0 \le M$ such that $\|x\|_N \le M\|Tx\|_K$ for all $x \in N$. The Fourier transform is linear, which follows from the linearity of the integral. Also, the norm of T is $\|T\| = \sup\{\|x\|_N / \|Tx\|_K \mid x \in N, x \neq 0\}$; as long as T is bounded, this set is nonempty, and so it must have a least upper bound. From Plancherel's theorem, the ratio between the norms in N and K is always $(2\pi)^{1/2}$, giving the bound condition. ■

Corollary. If $\{f_n(t) \mid n \in \mathbb{N}\}$ is a Cauchy sequence in $L^1 \cap L^2$, then the sequence of Fourier transforms $\{F_n(\omega) \mid n \in \mathbb{N}\}$ is also Cauchy in L^2.

Proof: The Fourier transform is linear, so Plancherel's theorem implies $\|F_m - F_n\|_2 = (2\pi)^{1/2}\|f_m - f_n\|_2$. ■

5.3.2 Definition

Now we are in a position to define the Fourier transform for $L^2(\mathbb{R})$. We can write any general square-integrable f(t) as a limit of integrable, finite-energy signals: $f(t) = \lim_{n\to\infty} f_n(t)$. It is easy to find the requisite sequence by setting $f_n(t) \in L^1 \cap L^2$ to be $f(t)$ restricted to $[-n, n]$ and zero otherwise. In Chapter 3, we noted that $L^1 \cap L^2$ is dense in L^2, and by the last corollary the Fourier transforms $\{F_n(\omega) \mid n \in \mathbb{N}\}$ also comprise a Cauchy sequence in L^2.

Definition (Fourier Transform for L^2(R)). If $f(t) \in L^2(\mathbb{R})$, then we define the *Fourier transform* of $f(t)$ by $F(\omega) = \lim_{n\to\infty} F_n(\omega) = \mathcal{F}+[f(t)](\omega)$, where $\{f_n(\mathrm{t}) \mid n \in \mathbb{N}\}$ is any Cauchy sequence in $L^1 \cap L^2$ that converges to $f(t)$, and $F_n(\omega) = \mathcal{F}[f_n(t)](\omega)$.

Remark. $F(\omega)$ must exist because $L^2(\mathbb{R})$ is complete and $\{F_n(\omega) \mid n \in \mathbb{N}\}$ is Cauchy. In order for the definition to make sense, we need to show the following:

- The designation of $\lim_{n\to\infty} F_n(\omega)$ to be the Fourier transform of $f(t)$ must be shown independent of what particular sequence $\{f_n(t) \mid n \in \mathbb{N}\}$ is taken as having $f(t)$ as its limit.
- The definition of $F(\omega)$ should match the conventional definition in terms of the Fourier transform analysis equation when $f(t) \in L^1(\mathbb{R})$ too.

We introduce a very temporary notation $\mathcal{F}+$ for the extension. Once we show that the extension of the Fourier transform from $L^1 \cap L^2$ to all of L^2 makes mathematical sense, then we can forget the superscript "+" sign. This next proposition shows that the Fourier transform on $L^2(\mathbb{R})$ is in fact well-defined and agrees with our previous definition for absolutely integrable signals.

Proposition (Well-Defined). The Fourier transform of $f(t) \in L^2(\mathbb{R})$, $F(\omega) = \lim_{n\to\infty} F_n(\omega)$, where $F_n(\omega) = \mathcal{F}[f_n(t)](\omega)$ and $\lim_{n\to\infty} f_n(t) = f(t)$. Then:

(i) $F(\omega)$ is well-defined; that is, it does not depend on the choise of limit sequence.

(ii) If $f(t) \in L^1(\mathbb{R})$, and $F(\omega)$ is given by the radial Fourier transform analysis equation (5.66a), then $F(\omega) = \lim_{n\to\infty} F_n(\omega)$.

(iii) $\mathcal{F}+: L^2 \to L^2$ is a norm-preserving extension of the map $\mathcal{F}: L^1 \cap L^2 \to L^2$ defined in Section 5.3.1.

Proof: That the limit $F(\omega)$ does not depend on the particular sequence whose limit is $f(t)$ follows from Plancherel's theorem. For the second claim, let $f_n(t)$ be $f(t)$ restricted to $[-n, n]$. Note that the $F_n(\omega) = \mathcal{F}[f_n(t)](\omega)$ in fact converge pointwise to $F(\omega)$ given by (5.66a), and any other Cauchy sequence $\{g_n(t) \mid n \in \mathbb{N}\}$ in $L^1 \cap L^2$ which converges to $f(t)$ must converge to $F(\omega)$ almost everywhere as well [24]. The third point follows immediately. ■

The next result shows inner products are preserved by $\mathcal{F}+$.

Corollary (Parseval's Theorem). If $f(t),\ g(t) \in L^2(\mathbb{R})$ with radial Fourier transforms $F(\omega)$ and $G(\omega)$, respectively, then $\langle f, g\rangle = (2\pi)^{-1}\langle F, G\rangle$.

Proof: This follows because we can define the inner product in terms of the norm by the polarization identity (Chapter 2) for inner product spaces [15]:

$$4\langle f, g\rangle = \|f+g\|_2^2 - \|f-g\|_2^2 + \frac{\|f-jg\|_2^2}{j} - \frac{\|f+jg\|_2^2}{j}. \tag{5.130}$$

■

Corollary (Plancherel's Theorem). If $f(t) \in L^2(\mathbb{R})$ with radial Fourier transform $F(\omega)$, then $\|f\|_2 = (2\pi)^{-1/2}\|F\|_2$.

Proof: By Parseval's relation for $L^2(\mathbb{R})$ signals above. ■

Now that we have successfully extended the Fourier transform to all finite-energy signals, let us agree to drop the special notation $\mathcal{F}+$ for the extension and consider Domain($\mathcal{F}$) = $L^2(\mathbb{R})$. Now that we have enough machinery, we can build a theory of analog signal frequency quite rapidly. For example, signals with almost everywhere identical spectra must themselves be identical almost everywhere.

Corollary (Uniqueness). Let $f(t), g(t) \in L^2(\mathbb{R})$ with radial Fourier transforms $F(\omega)$ and $G(\omega)$, respectively. Suppose $F(\omega) = G(\omega)$ for almost all $\omega \in \mathbb{R}$. Then $f(t) = g(t)$ for almost all $t \in \mathbb{R}$.

Proof: If $F(\omega) = G(\omega)$ for almost all $\omega \in \mathbb{R}$, then $\|F - G\|_2 = 0$. But by Plancherel's theorem for $L^2(\mathbb{R})$, we then know $\|f - g\|_2 = 0$, whence $f(t) = g(t)$ for almost all $t \in \mathbb{R}$ by the properties of the Lebesgue integral (Chapter 3). ■

Theorem (Convolution). Let $f(t), h(t) \in L^2(\mathbb{R})$ with radial Fourier transforms $F(\omega)$ and $H(\omega)$, respectively, and let $g(t) = (f * h)(t)$ be the convolution of f and h. Then $G(\omega) = F(\omega)H(\omega)$.

Proof: By the Schwarz inequality, $g = (f * h) \in L^1(\mathbb{R})$, and it has a Fourier transform $G(\omega)$. Let us expand the convolution integral inside the Fourier transform analysis equation for $g(t)$:

$$\int_{-\infty}^{\infty} (f * h)(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(s) h(t-s) ds e^{-j\omega t} dt. \qquad (5.131)$$

Since $g \in L^1(\mathbb{R})$, we can apply Fubini's theorem (Section 3.4.2.4) to the integrand of the Fourier analysis equation for $g(t)$. Interchanging the order of integration gives

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(s) h(t-s) ds e^{-j\omega t} dt = \int_{-\infty}^{\infty} f(s) e^{-j\omega s} ds \int_{-\infty}^{\infty} h(t-s) e^{-j\omega(t-s)} dt. \qquad (5.132)$$

The iterated integrals on the right we recognize as $F(\omega)H(\omega)$. ■

The convolution theorem lies at the heart of analog filter design, which we cover in Chapter 9.

Finally, we observe that the normalized and Hertz Fourier transforms, which are a scaling and a dilation of the radial transform, respectively, can also be extended precisely as above.

5.3.3 Isometry

The normalized radial Fourier transform, extended as above to finite enery signals, in fact constitutes an isometry of $L^2(\mathbb{R})$ with itself. We recall that an isometry T between Hilbert spaces, H and K, is a linear map that is one-to-one and onto and preserves inner products. Since $\langle x, y\rangle_H = \langle Tx, Ty\rangle_J$, T also preserves norms, and so it must be bounded; in fact, $\|T\| = 1$. Conceptually, if two Hilbert spaces are isometric, then they are essentially identical.We continue working out the special properties of the radial Fourier transform $\mathcal{F}$ and, as the last step, scale it to $(2\pi)^{-1/2}\mathcal{F}$, and thereby get the isometry.

The following result is a variant of our previous Plancherel and Parseval formulas.

Theorem. Let $f(t)$, $g(t) \in L^2(\mathbb{R})$ with radial Fourier transforms $F(\omega)$ and $G(\omega)$, respectively. Then,

$$\int_{-\infty}^{\infty} F(\omega) g(\omega) d\omega = \int_{-\infty}^{\infty} f(t) G(t)\, dt. \qquad (5.133)$$

Proof: We prove the result in two steps:

(i) First for $f(t)$, $g(t) \in (L^1 \cap L^2)(\mathbb{R})$;
(ii) For all of $L^2(\mathbb{R})$, again using the density of $L^1 \cap L^2$ within L^2.

Let us make the assumption (i) and note that this stronger condition implies that $F(\omega)$ and $G(\omega)$ are bounded (exercises). Then, since $F(\omega) \in L^{\infty}(\mathbb{R})$, the Hölder inequality gives $\|Fg\|_1 \le \|F\|_{\infty}\|g\|_1$. Because the integrand $F(\omega)g(\omega)$ is absolutely integrable, Fubini's theorem allows us to interchange the order of integration:

$$\int_{-\infty}^{\infty} F(\omega)g(\omega)\, d\omega = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt\right)g(\omega)\, d\omega = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} g(\omega)e^{-j\omega t}d\omega\right)f(t)\, dt. \tag{5.134}$$

Notice that the integral in parentheses on the right-hand side of (5.134) is precisely $G(t)$, from which the result for the special case of $L^1 \cap L^2$ follows.

For (ii), let us assume that $\lim_{n\to\infty} f_n(t) = f(t)$ and $\lim_{n\to\infty} g_n(t) = g(t)$, where f_n, $g_n \in (L^1 \cap L^2)(\mathbb{R})$. Then $F(\omega) = \lim_{n\to\infty} F_n(\omega)$, where $\mathrm{F_n}(\omega) = \mathcal{F}[f_n(t)](\omega)$ and $G(\omega) = \lim_{n\to\infty} G_n(\omega)$, where $G_n(\omega) = \mathcal{F}[g_n(t)](\omega)$. Then, f_n, g_n, F_n, and $G_n \in L^2(\mathbb{R})$, so that by the Schwarz inequality, $F_n g_n$ and $f_n G_n \in \mathrm{L}^1(\mathbb{R})$. The Lebesgue Dominated Convergence theorem applies (Section 3.4.2.3). By part (i) of the proof, for all $n \in \mathbb{N}$, $\int F_n g_n = \int f_n G_n$. Taking limits of both sides gives

$$\lim_{n\to\infty}\int F_n g_n = \int \lim_{n\to\infty} F_n g_n = \int Fg = \lim_{n\to\infty}\int G_n f_n = \int \lim_{n\to\infty} G_n f_n = \int Gf. \tag{5.135}$$

as required. ■

We know that every $f \in L^2(\mathbb{R})$ has a radial Fourier transform $F \in L^2(\mathbb{R})$ and that signals with (almost everywhere) equal Fourier transforms are themselves (almost everywhere) equal. Now we can show another result—an essential condition for the isometry, in fact—that the Fourier transform is onto.

Theorem. If $G \in L^2(\mathbb{R})$, then there is a $g \in L^2(\mathbb{R})$ such that $F(g)(\omega) = G(\omega)$ for almost all $\omega \in \mathbb{R}$.

Proof: If $G(\omega) \in L^2(\mathbb{R})$, we might well guess that the synthesis formula for the case $G(\omega) \in (L^1 \cap L^2)(\mathbb{R})$ will give us a definition of $g(t)$:

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{j\omega t}d\omega\,. \tag{5.136}$$

We need to show that the above integral is defined for a general $G(\omega)$, however. If we let $H(\omega) = G(-\omega)$ be the reflection of $G(\omega)$, then $H \in L^2(\mathbb{R})$. We can take its radial Fourier transform, $\mathcal{F}[H]$:

$$p\frac{1}{2\pi}\mathcal{F}[H](t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)e^{-j\omega t}d\omega\,. \tag{5.137}$$

A change of integration variable in (5.137) shows that $(2\pi)^{-1}\mathcal{F}[H](t)$ has precisely the form of the radial Fourier synthesis equation (5.136). We therefore propose $g(t) = (2\pi)^{-1}\mathcal{F}[H](t) \in L^2(\mathbb{R})$. Now we need to show that the Fourier transform of $g(t)$ is equal to $G(\omega)$ almost everywhere. We calculate

$$\|\mathcal{F}g - G\|_2^2 = \langle \mathcal{F}g, \mathcal{F}g\rangle - 2\mathrm{Real}\langle \mathcal{F}g, G\rangle + \langle G, G\rangle . \tag{5.138}$$

We manipulate the middle inner product in (5.138),

$$\mathrm{Real}\langle \mathcal{F}g, G\rangle = \mathrm{Real}\langle G, \mathcal{F}g\rangle = \mathrm{Real}\int G(\overline{\mathcal{F}g}) = \mathrm{Real}\int \mathcal{F}G(\overline{g(-t)}) , \tag{5.139a}$$

applying the previous Parseval result to obtain the last equality above. Using the definition of $g(t)$, we find

$$\mathrm{Real}\langle \mathcal{F}G, \frac{1}{2\pi}[\mathcal{F}H](-t)\rangle = \mathrm{Real}\left\langle \mathcal{F}G, \frac{1}{2\pi}[\mathcal{F}H](-t)\right\rangle = \frac{1}{2\pi}\mathrm{Real}\langle \mathcal{F}G, \mathcal{F}G\rangle . \tag{5.139b}$$

By Parseval's theorem, $\langle \mathcal{F}g, \mathcal{F}g\rangle = 2\pi\langle g, g\rangle$ and $\langle \mathcal{F}G, \mathcal{F}G\rangle = 2\pi\langle G, G\rangle$, which is real. Thus, putting (5.138), (5.139a), and together implies

$$\begin{aligned}\|\mathcal{F}g - G\|_2^2 &= 2\pi\langle g, g\rangle - \frac{2}{2\pi}\langle \mathcal{F}G, \mathcal{F}G\rangle + \frac{1}{2\pi}\langle \mathcal{F}G, \mathcal{F}G\rangle = 2\pi\langle g, g\rangle - \frac{1}{2\pi}\langle \mathcal{F}G, \mathcal{F}G\rangle \\ &= \frac{2\pi}{(2\pi)^2}\langle \mathcal{F}[G(-\omega)], \mathcal{F}[G(-\omega)]\rangle - \frac{1}{2\pi}\langle \mathcal{F}G, \mathcal{F}G\rangle .\end{aligned} \tag{5.140}$$

But $\|\mathcal{F}G\|_2 = \|\mathcal{F}[G(-\omega)]\|_2$, so the last term in (5.140) is zero: $\|\mathrm{F}g - G\|_2^2 = 0$ almost everywhere, and the theorem is proven. ■

Corollary (Isometry of Time and Frequency Domains). The normalized radial Fourier transform $(2\pi)^{-1/2}\mathcal{F}$, where $\mathcal{F}$ is the radial Fourier transform on $L^2(\mathbb{R})$, is an isometry from $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$.

Proof: Linearity follows from the properties of the integral. We have shown that $\mathcal{F}$ on $L^2(\mathbb{R})$ is one-to-one; this is a consequence of the Parseval relation. The map $\mathcal{F}$ is also onto, as shown in the previous theorem. Since $\langle \mathrm{F}x, \mathrm{F}y\rangle = 2\pi\langle x, y\rangle$, we now see clearly that $(2\pi)^{-1/2}\mathcal{F}$ preserves inner products and constitutes an isometry. ■

Remark. The uniqueness implied by this relationship assures that a given Fourier spectrum is a true signature of a given time-domain signal $f(t)$. This property is a valuable asset, but we emphasize that two signals are equivalent if their spectra are identical across the *entire* frequency spectrum. Deviations between spectra, even if they are small in magnitude or restricted to a small range of frequencies, can result in large discrepancies between the respective $f(t)$. This is a legacy of the complex exponentials which form a basis of the Fourier transform. They are defined across the entire line (both in frequency and time), so small perturbations run the risk of infecting the entire Fourier synthesis. In practical situations, where spectral information is stored and transmitted in the form of discrete Fourier coefficients, errors or glitches can wreak havoc on the reconstruction of $f(t)$. This inherent sensitivity to error is one of the least appealing attributes of Fourier analysis.

The time-domain and frequency-domain representations of a square-integrable signal are equivalent. Neither provides more information. And Fourier transformation, for all its complexities, serves only to reveal some aspects of a signal at the possible risk of concealing others.

5.4 SUMMARY

The Fourier series and transform apply to analog periodic and aperiodic signals, respectively.

The Fourier series finds a set of discrete coefficients associated with a periodic analog signal. These coefficients represent the expansion of the signal on the exponential or sinusoidal basis sets for the Hilbert space $L^2[0, T]$. We shall have more to say about the Fourier series in Chapter 7, which is on discrete Fourier transforms.

For absolutely integrable or square-integrable aperiodic signals, we can find a frequency-domain representation, but it is an analog, not discrete, signal. We have had to consider three different Banach spaces in our quest for a frequency-domain description of a such a signal: $L^1(\mathbb{R})$, $(L^1 \cap L^2)(\mathbb{R})$, and $L^2(\mathbb{R})$. We began by defining the Fourier transform over the space of absolutely integrable signals. Then we considered the restricted transform on $L^1 \cap L^2$, but noted that this transform's range is in L^2. Applying the limit theorems available with the modern (Lebesgue) integral to this restricted signal class, we were able to extend the transform to the full space of square-integrable signals. Ultimately, we found an isometry between the time and frequency domain representations of a finite energy signal.

5.4.1 Historical Notes

Prior to Fourier, there were a number of attempts by other mathematicians to formulate a decomposition of general waves into trigonometric functions. D'Alembert, Euler, Lagrange, and Daniel Bernoulli used sinusoidal expansions to account for the vibrations of a string [4]. Evidently, ancient Babylonian astronomers based their predictions on a rudimetary Fourier series [10]. Fourier applied trigonometric series to the heat equation, presented his results to the French Academy of Sciences, and published his result in a book [25]. Criticism was severe, however, and the method was regarded with suspicion until Poisson, Cauchy, and especially Dirichlet (1829) provided theoretical substantiation of the Fourier series.

Plancherel proved that L^2 signals have L^2 Fourier transforms in 1910. The basis for so doing, as we have seen in Section 5.3, is the modern Lebesgue integral and the powerful limit properties which it supports. The L^2 theory of the Fourier integral is often called the Plancherel theory.

5.4.2 Looking Forward

The next chapter generalizes the Fourier transform to include even signals that are neither absolutely integrable nor square-integrable. This so-called generalized Fourier transform encompasses the theory of the Dirac delta, which we introduced in Chapter 3. Chapters 7 and 8 consider the frequency-domain representation for discrete signals. Chapter 9 covers applications of analog and discrete Fourier transforms.

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PROBLEMS

1. Find the exponential Fourier series coefficients (5.9) for the following signals.
  (a) $x(t) = \cos(2\pi t)$.
  (b) $y(t) = \sin(2\pi t)$.
  (c) $s(t) = \cos(2\pi t) + \sin(2\pi t)$.
  (d) $z(t) = x(t - \pi/4)$.
  (e) $w(t) = 5y(-2t)$.

2. Find the exponential Fourier series coefficients for the following signals:
  (a) Signal $b(t)$ has period $T = 4$ and for $0 \le t < 4$, $b(t) = u(t) - u(t-2)$, where $u(t)$ is the analog unit step signal.
  (b) $r(t) = tb(t)$, where $b(t)$ is given in (a).

3. Let $x(t) = 7\sin(1600t - 300)$, where t is a (real) time value in seconds. Give:
  (a) The amplitude of x.
  (b) The phase of x.
  (c) The frequency of x in radians/second.
  (d) The frequency of x in Hz (cycles/second).
  (e) The period of x.
  (f) Find the exponential Fourier series coefficients for $x(t)$.

4. Suppose $x(t)$ has period $T = 1$ and $x(t) = t^2$ for $0 \le t < 1$.
  (a) Find the exponential Fourier series coefficients for $x(t)$.
  (b) Sketch and label the signal $y(t)$ to which x's Fourier series synthesis equation converges.

5. Find the exponential Fourier series coefficients for the periodic sawtooth signal $x(t)$ (Figure 5.9).

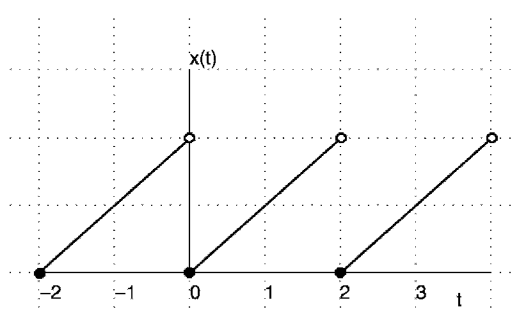

Fig. 5.9. Sawtooth signal $x(t)$.

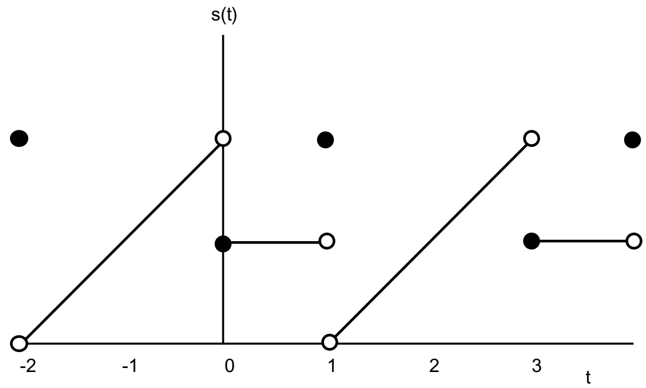

Fig. 5.10. Another sawtooth signal $s(t)$.

6. Consider the signal $s(t)$ shown in Figure 5.10.

(a) Find the exponential Fourier series for the signal $s(t)$.

(b) Consider the signal $y(t)$ to which the Fourier series synthesis equation for $s(t)$ converges. Sketch $y(t)$ and label the graph to show the exact values of $y(t)$ at places where it is the same and where it differs from $s(t)$.

7. The impulse response of an analog linear time invariant (LTI) system H is $h(t) = u(t + 50) - u(t)$, where $u(t)$ is the unit step signal.

(a) What is the response of the system to the signal $x(t) = u(t)e^{-t}$?

(b) Find the radial Fourier transform of $h(t)$.

(c) Does $x(t)$ have a radial Fourier transform? If so, find it; otherwise, give a reason why $X(\omega)$ does not exist; and, in any case, explain your answer.

8. Consider the analog signal $x(t) = [u(t + 1) - u(t - 1)]$, where $u(t)$ is the unit step signal.

(a) Find the radial Fourier transform of $x(t)$, $X(\omega) = \mathcal{F}[x(t)](\omega)$.

(b) Let $y(t) = x(t - 2)$; find $Y(\omega)$.

(c) Find $\mathcal{F}[x(2t)]$.

(d) Find $\mathcal{F}[x(t/5)]$.

(e) Find $\mathcal{F}[\sin(t)x(t)]$.

9. Let H be a linear time-invariant (LTI) analog system; $y = Hx$; $h = H\delta$; and $X(\omega)$, $Y(\omega)$, and $H(\omega)$ are their respective radial Fourier transforms. Which of the following are true? Explain.

(a) $Y(\omega)/X(\omega)$ is the Fourier transform of h.

(b) $y(t)/h(t) = x(t)$.

(c) $y(t) = x(t)*h(t)$, where * is the analog convolution operation.

10. Prove or disprove the following statement: If an periodic analog signal $x(t)$ is represented by a Fourier series, but this series does not converge to $x(t)$ for all t, then $x(t)$ is not continuous.

11. Prove or disprove the following statement: If $x(t) \in L^2(\mathbb{R})$ is an odd signal (i.e., $x(t) = -x(-t)$), and $X(\omega)$ is the radial Fourier transform of x, then $X(0) = 0$.

12. Suppose the analog signals x(t) and h(t) have radial Fourier transforms $X(\omega) = u(\omega + 1) - u(\omega - 1)$ and $H(\omega) = \exp(-\omega^2)$, respectively. Let the signal $y = x * h$.

(a) Find $x(t)$.

(b) Find $h(t)$.

(c) Find $Y(\omega)$.

(d) Find $y(t)$.

13. Suppose that $X(\omega)$ and $Y(\omega)$ are the radial Fourier transforms of $x(t)$ and $y(t)$, respectively, and let $h(n)$ be a discrete signal with

$$x(t) = \sum_{n=-\infty}^{\infty} h(n)y(t-n). \tag{5.141}$$

(a) Find an expression for $X(\omega)$.

(b) What kind of conditions should be imposed upon the discrete signal $h(n)$ so that your answer in (a) is mathematically justifiable? Explain.

14. Show that the radial Fourier transform for analog signals is a linear operation. Is it also translation invariant? Explain.

15. Show that if $F(\omega)$ is absolutely integrable, then the inverse Fourier transform $\mathcal{F}^{-1}[F(\omega)](t)$ exists.

16. Suppose that analog periodic signal x(t) has exponential Fourier series coefficients c_k:

$$c_k = \langle x(t), \phi_k(t) \rangle = \int_{t_0}^{t_0+T} x(t)\frac{1}{\sqrt{T}}e^{-jk\Omega t}dt. \tag{5.142}$$

Prove the following symmetry properties:

(a) If $x(t)$ is real-valued and even, then the c_k are also real and even: $c_k = c_{-k}$.

(b) If $x(t)$ is real and odd, then the c_k are purely imaginary and odd: $c_{-k} = _{-}c_k$.

17. For analog signals $x(t)$ and $y(t) = x(t - a)$, show that the magnitudes of their radial Fourier transforms are equal, $|X(\omega)| = |Y(\omega)|$.

18. Prove or disprove: For all analog signals $x(t) \in L^2(\mathbb{R})$, if $x(t)$ is real-valued, then $X(\omega)$ is real-valued.

19. Let $x(t) \in L^1(\mathbb{R})$ be a real-valued analog signal and let $X(\omega)$ be its radial Fourier transform. Which of the following are true? Explain.

(a) $X(\omega)$ is bounded: $X(\omega) \in L^{\infty}(R)$.

(b) $|X(\omega)| \to 0$ as $|\omega| \to \infty$.

(c) $X(\omega)$ is unbounded.

(d) $X(0) = 0$.

(e) $X(\omega)$ has a Fourier transform.

(f) $X(\omega)$ has an inverse Fourier transform.

(g) $X(\omega)$ has an inverse Fourier transform and it is identical to x(t).

(h) $X(\omega) \in L^1(\mathbb{R})$ also.

20. Let $x(t) \in L^2(\mathbb{R})$ be a real-valued analog signal and let $X(\omega)$ be its radial Fourier transform. Which of the following are true? Explain.

(a) $X(\omega)$ is bounded: $X(\omega) \in L^\infty(R)$.

(b) $|X(\omega)| \to 0$ as $|\omega| \to \infty$.

(c) $X(\omega)$ is unbounded.

(d) $X(0) = 0$.

(e) $X(\omega)$ has a Fourier transform.

(f) $X(\omega)$ has an inverse Fourier transform.

(g) $X(\omega)$ has an inverse Fourier transform and it is identical to $x(t)$.

(h) $X(\omega) \in L^2(\mathbb{R})$ also.

21. Let $x(t) \in L^1(\mathbb{R})$. Show that:

(a) Fourier transform $X(\omega)$ of $x(t)$ is bounded.

(b) $\|X\|_\infty \le \|x\|_1$.

22. Loosely speaking, an analog *low-pass filter H* is a linear, translation-invariant system that passes low frequencies and suppresses high frequencies. We can specify such a system more precisely with the aid of the Fourier transform. Let $h(t)$ be the impulse response of H and let $H(\omega) = \mathcal{F}(h(t))(\omega)$ be its Fourier transform. For a low-pass filter we require $|H(0)| = 1$ and $|H(\omega)| \to 0$ as $|\omega| \to \infty$.

(a) Show that if $|H(0)| \ne 1$ but still $|H(0)| \ne 0$ and $|H(\omega)| \to 0$ as $|\omega| \to \infty$, then we can convert H into a low-pass filter by a simple normalization;

(b) Show that, with an appropriate normalization, the Gaussian signal is a low-pass filter (sometimes we say that any system whose impulse response can be so normalized is a low-pass filter).

23. An analog *high-pass filter H* is a linear, translation-invariant system that suppresses low frequencies and passes high frequencies. Again, if $h(t)$ is the impulse response of H and $H(\omega) = \mathcal{F}(h(t))(\omega)$, then we stipulate that $|H(0)| = 0$ and $|H(\omega)| \to 1$ as $|\omega| \to \infty$. Explain why our Fourier theory might not accept analog high-pass filters.

24. An analog *bandpass filter H* passes a range of frequencies, suppressing both low and high frequencies.

(a) Formalize the idea of a bandpass filter using the spectrum of the impulse response of H.

(b) Give an example of a finite-energy bandpass filter.

(c) Let $h(t)$ be an analog low-pass filter and let $g(t)$ be an analog bandpass filter. What kind of filters are $h * h$, $g * h$, and $g * g$? Explain your answer.

(d) Why would we not consider the case that $g(t)$ is an analog high-pass filter? Explain this too.

(e) Formulate and formalize the concept of an analog *bandstop* (also *bandreject* or *notch*) filter. Can these be absolutely integrable signals? Finite-energy signals? Explain.

25. Suppose $h(t)$ is the impulse response for an analog filter H that happens to be a *perfect low-pass filter*. That is, for some $\omega_c > 0$, $|H(\omega)| = 1$ for $|\omega| \le \omega_c$, and $|H(\omega)| = 0$ for all $|\omega| > \omega_c$.

(a) Show that as a time-domain filter, H is noncausal.

(b) Explain why, in some sense, H is impossible to implement.

(c) Show that $h(t) \in L^2(\mathbb{R})$, but $h(t) \notin L^1(\mathbb{R})$.

26. Suppose an analog signal, $x(t)$, has radial Fourier transform, $X(\omega)$, given by Figure 5.11

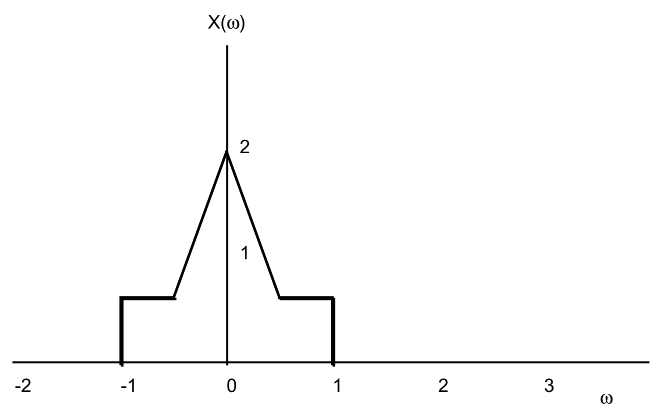

Fig. 5.11. Radial Fourier transform of signal $x(t)$.

Without attempting to compute $x(t)$, sketch $Y(\omega)$ for the following signals $y(t)$:

(a) $y(t) = x(2t)$.

(b) $y(t) = x(t/3)$.

(c) $y(t) = (x * g)(t)$, where $g(t) = \exp(-t^2)$.

(d) $y(t) = \cos(t)x(t)$.

(e) $y(t) = \sin(t)x(t)$.

(f) $y(t) = \exp(j5t)x(t)$.

27. Let $f(t), h(t) \in L^2(\mathbb{R})$ and $F(\omega)$, $H(\omega)$ be their radial Fourier transforms.

(a) If $g(t) = f(t)h(t)$, show that $g(t) \in L^2(\mathbb{R})$.

(b) Show that $G(\omega) = (2\pi)^{-1}F(\omega) * H(\omega)$.

28. Develop an alternative approach to showing that signals with identical spectra must be the same. Let $f_1(t)$ and $f_2(t)$ have identical Fourier transforms; that is, $\mathcal{F}[f_1(t)] = \mathcal{F}[f_2(t)] = G(\omega)$.

(a) For an arbitrary $h(t)$ show that an alternative formulation of Parseval's relation holds:

$$2\pi \int_{-\infty}^{\infty} f_2(t)h(t)\, dt = \int_{-\infty}^{\infty} G(-\omega)H(\omega)\, d\omega. \tag{5.143}$$

(b) Show that the above leads to

$$2\pi \int_{-\infty}^{\infty} [f_1(t) - f_2(t)]h(t)\, dt = 0. \tag{5.144}$$

(c) Explain why it follows that $f_1(t) = f_2(t)$ for almost all t.