Functional Analysis III

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Contents

1 Continuous Transforms **2**
- 1.1 Short-Time Fourier Transform . . . 2
- 1.2 Continuous Wavelet Transform . . . 4

2 Frames **7**
- 2.1 What is a Frame? . . . 7
- 2.2 The Frame Operator . . . 8
- 2.3 Frame Decomposition . . . 10

3 Gabor Frames **11**
- 3.1 From the Short-Time Fourier Transform to Gabor Frames . . . 11
- 3.2 Necessary Conditions for Gabor Frames . . . 12
- 3.3 Sufficient Conditions for Gabor Frames . . . 14

4 Wavelet-Frames **16**
- 4.1 Form the Continuous Wavelet Transform to Wavelet-Frames . . . 16
- 4.2 Necessary and Sufficient Conditions for Wavelet-Frames . . . 16

5 Wavelet Orthonormal Bases **17**
- 5.1 Haar Wavelet . . . 17
- 5.2 Multiresolution Analysis . . . 20
- 5.3 Wavelets come into play . . . 25

6 Important classes of wavelets **27**
- 6.1 Spline Wavelets . . . 30

7 Compactly Supported Wavelets **30**
- 7.1 General Construction . . . 30
- 7.2 Smooth Wavelets . . . 33

8 Wavelet Decomposition **37**
- 8.1 Sampling . . . 37
- 8.2 The Fast Wavelet Transform . . . 38

9 Approximation Properties of Wavelets **40**

10 Systems in $L^2(\mathbb{R}^2)$ **41**
- 10.1 Wavelet Bases . . . 41
- 10.2 Shearlets . . . 43

11 Inverse Problems **45**
- 11.1 Getting started . . . 45
- 11.2 Basics of Linear Ill-Posed Problems . . . 45
- 11.3 Regularization of linear ill-posed problems . . . 48
- 11.4 Tikhonov Regularization . . . 50
- 11.5 Generalization of Tikhonov Regularization . . . 51

12 Compressed Sensing **52**
12.1 Main Idea of Compressed Sensing 52
12.2 Null Space Property 53
12.3 Restricted Isometry Property 53
12.4 Constructing RIP Matrices 54
12.5 Stability and Robustness 57
12.6 Optimality of bounds 58

Introduction

General concept: We consider for some system or family $(\varphi_i)_{i\in I} \in H$ the mapping

$$H \ni x \mapsto Tx := (\langle x, \varphi_i\rangle)_{i\in I}.$$

There we want to

- analyse x,
- thus moving from a continuous ($x \in H$) setting into a discrete one ($(\langle x, \varphi_i\rangle)_{i\in I}$).

Further we want to

- sample and
- recover x from Tx, where T is generally not invertible.

Considering the recovery of x, apart from studying methods of recovery, we want to quantify the quality of their result.

Subsequently this course covers the following *topics*:

(1) Applied harmonic analysis $\rightsquigarrow$ design of the system $(\varphi_i)_{i\in I}$, e.g. per wavelets or curvelets.

(2) Frame theory $\rightsquigarrow$ functional analytic extension of orthonormal bases.

(3) Compressed sensing $\rightsquigarrow$ search for minimal $\#I$ such that we are still able to recover x, provided that x allows the decomposition $x = \sum_{i\in I} c_i\widetilde{\varphi}_i$ with $(c_i)_{i\in I}$ fast decaying for some $(\widetilde{\varphi}_i)_{i\in I} \subset H$.

(4) Analysis of high-dimensional data & geometric functional analysis $\rightsquigarrow$ $H = \mathbb{R}^N$ with N large makes itnecessary to study the geometry of high-dimensional spaces.

(5) Applications $\rightsquigarrow$ relations to other areas of mathematics, and other research areas (imaging sciences)

1 Continuous Transforms

1.1 Short-Time Fourier Transform

(1.1) Remark. Recall: For $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ we have

$$\widehat{f}(\xi) = \mathcal{F}f(\xi) = \int\limits_{\mathbb{R}} f(x)e^{-i\xi\cdot x}\,\mathrm{d}x$$

for all $\xi \in \mathbb{R}$.

Problems:

- Local changes of f result in global changes of $\widehat{f}$.
- $\widehat{f}$ does not give information about the position of frequencies.

(1.2) Definition. Let $g \in L^2(\mathbb{R})$. Then the **short-time Fourier transform** of $f \in L^2(\mathbb{R})$ associated with some **window** g is defined by

$$\mathcal{S}_g f(t,\xi) = \int\limits_{\mathbb{R}} f(x)\overline{g(x-t)}e^{-i\xi\cdot x}\,\mathrm{d}x, \qquad t, \xi \in \mathbb{R}.$$

We define operators T_t and M_ξ with

$$T_t : L^2(\mathbb{R}) \to L^2(\mathbb{R}), g \mapsto g(\cdot - t),$$

the **translation operator** and

$$M_\xi : L^2(\mathbb{R}) \to L^2(\mathbb{R}), g \mapsto e^{i\xi\cdot} g(\cdot)$$

the **modulation operator**. This allows us to write

$$\mathcal{S}_g f(t,\xi) = \langle f, M_\xi T_t g\rangle_{L^2(\mathbb{R})} = \mathcal{F}\big[f \cdot \overline{T_t g}\big](\xi).$$

(1.3) Theorem (Orthogonoality Relations). *Let $g_1, g_2, f_1, f_2 \in L^2(\mathbb{R})$. Then*

$$\langle \mathcal{S}_{g_1} f_1, \mathcal{S}_{g_2} f_2\rangle_{L^2(\mathbb{R}^2)} = 2\pi \langle f_1, f_2\rangle_2 \overline{\langle g_1, g_2\rangle_2}.$$

In particular,

$$\|\mathcal{S}_g f\|^2_{L^2(\mathbb{R}^2)} = 2\pi \|f\|_2^2 \|g\|_2^2$$

for all $f, g \in L^2(\mathbb{R})$.

Proof. Assume that $g_1, g_2 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then, for a.e. $t \in \mathbb{R}$ we have $f_j \cdot \overline{T_t g_j} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ for $j \in \{1, 2\}$. Then

$$\begin{aligned}
\langle \mathcal{S}_{g_1} f_1, \mathcal{S}_{g_2} f_2\rangle_{L^2(\mathbb{R}^2)} &= \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} \mathcal{F}\big[f_1 \cdot \overline{T_t g_1}\big](\xi) \cdot \overline{\mathcal{F}\big[f_2 \cdot \overline{T_t g_2}\big](\xi)} \,\mathrm{d}\xi \,\mathrm{d}t \\
&= \int\limits_{\mathbb{R}} \left\langle \mathcal{F}\big[f_1 \cdot \overline{T_t g_1}\big], \mathcal{F}\big[f_2 \cdot \overline{T_t g_2}\big]\right\rangle_2 \mathrm{d}t \overset{\text{Plan-}}{\underset{\text{cherel}}{=}} 2\pi \int\limits_{\mathbb{R}} \left\langle f_1 \cdot \overline{T_t g_1}, f_2 \cdot \overline{T_t g_2}\right\rangle_2 \mathrm{d}t = \\
&= 2\pi \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} f_1(x) \overline{g_1(x-t)} \cdot \overline{f_2(x)} g_2(x-t) \,\mathrm{d}t \,\mathrm{d}\xi = 2\pi \langle f_1, f_2\rangle_2 \cdot \overline{\langle g_1, g_2\rangle_2}.
\end{aligned}$$

By standard density arguments, extend to $L^2(\mathbb{R})$. ☐

(1.4) Corollary. *Let $g \in L^2(\mathbb{R})$. Then $\mathcal{S}_g : L^2(\mathbb{R}) \to L^2(\mathbb{R}^2)$ is a bounded linear operator. (a multiple of an isometry).*

(1.5) Theorem (Inversion Formula). *Let $g, \gamma \in L^2(\mathbb{R})$ with $\langle g, \gamma\rangle \neq 0$. Then for all $f \in L^2(\mathbb{R})$ we find*

$$f(x) = \frac{1}{2\pi \langle \gamma, g\rangle_2} \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} \mathcal{S}_g f(t,\xi) M_\xi T_t \gamma(x) \,\mathrm{d}t \,\mathrm{d}\xi = \frac{\left\langle \mathcal{S}_g f, (M_{-\cdot} \overline{T_{\cdot} \gamma})(x)\right\rangle_{L^2(\mathbb{R}^2)}}{2\pi \langle g, \gamma\rangle_2}.$$

Proof. By Corollary 1.4, $\mathcal{S}_g f \in L^2(\mathbb{R}^2)$. Therefore

$$\widetilde{f} := \frac{1}{2\pi \langle \gamma, g\rangle_2} \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} \mathcal{S}_g f(t,\xi) M_\xi T_t \gamma \,\mathrm{d}t \,\mathrm{d}\xi.$$

is well defined.

Let $h \in L^2(\mathbb{R})$ be arbitrary. Then, by linearity of integrals and the scalar product we get

$$\begin{aligned}
\left\langle \widetilde{f}, h\right\rangle_2 &= \frac{1}{2\pi \langle \gamma, g\rangle_2} \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} \mathcal{S}_g f(t,\xi) \langle M_\xi T_t \gamma, h\rangle_2 \,\mathrm{d}t \,\mathrm{d}\xi. \\
&= \frac{1}{2\pi \langle \gamma, g\rangle_2} \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} \mathcal{S}_g f(t,\xi) \overline{\mathcal{S}_\gamma h(t,\xi)} \,\mathrm{d}t \,\mathrm{d}\xi \\
&\overset{1.3}{=} \frac{1}{2\pi \langle \gamma, g\rangle_2} \cdot 2\pi \langle f, h\rangle_2 \overline{\langle g, \gamma\rangle_2} = \langle f, h\rangle_2.
\end{aligned}$$

This shows $f = \widetilde{f}$ by the fundamental lemma of variational calculus. ☐

1.2 Continuous Wavelet Transform

Problem with $\mathcal{S}_g$: Assume $\operatorname{supp} g = [a,b]$. The intervall of interest has always the same size. Singularities (discontinuities) can not be detected precisely.

(1.6) Definition. Let $\psi \in L^2(\mathbb{R})$. Then the **continuous wavelet transform** of some $f \in L^2(\mathbb{R})$ associated with the **wavelet** ψ is defined for $(a,b) \in \mathbb{R}^+ \times \mathbb{R}$ by

$$\mathcal{W}_\psi f(a,b) = \int_{\mathbb{R}} f(x) a^{-\frac{1}{2}} \overline{\psi\left(\frac{x-b}{a}\right)} \,\mathrm{d}x = \langle f, T_b D_a \psi \rangle .$$

where

$$D_a : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \psi \mapsto a^{-\frac{1}{2}} \psi\left(\frac{\cdot}{a}\right)$$

is the **dilation operator**.

(1.7) Theorem (Orthogonality Relation). *Let $\psi \in L^2(\mathbb{R})$ satisfy the* ***admissibility condition***

$$C_\psi = \int_{\mathbb{R}} \frac{|\mathcal{F}\psi(\xi)|^2}{|\xi|} \,\mathrm{d}\xi < \infty.$$

Then, for $f, g \in L^2(\mathbb{R})$

$$\langle \mathcal{W}_\psi f, \mathcal{W}_\psi g \rangle_{L^2\left(\mathbb{R}^+ \times \mathbb{R}, \frac{\mathrm{d}a}{a^2}\,\mathrm{d}b\right)} = \int_{\mathbb{R}} \int_0^\infty \mathcal{W}_\psi f(a,b) \overline{\mathcal{W}_\psi g(a,b)} \frac{\mathrm{d}a}{a^2} \,\mathrm{d}b = C_\psi \langle f, g \rangle .$$

In particular,

$$\|\mathcal{W}_\psi f\|^2_{L^2\left(\mathbb{R}^+ \times \mathbb{R}, \frac{\mathrm{d}a}{a^2}\,\mathrm{d}b\right)} = \int_{\mathbb{R}} \int_0^\infty |\mathcal{W}_\psi f(a,b)|^2 \frac{\mathrm{d}a}{a^2} \,\mathrm{d}b = C_\psi \|f\|_2^2.$$ [1]

Proof. First, we recall, that $\mathcal{F}$ is linear,

$$\mathcal{F}\Big[f\left(\frac{\cdot}{c}\right)\Big](\xi) = c\mathcal{F}f(c\xi) \quad \text{and} \quad \mathcal{F}[T_b f] = M_{-b}\mathcal{F}f.$$

Using Plancherel twice,

$$\begin{aligned}
\int_{\mathbb{R}} \int_0^\infty \mathcal{W}_\psi f(a,b) \overline{\mathcal{W}_\psi g(a,b)} \frac{\mathrm{d}a}{a^2} \,\mathrm{d}b &= \int_{\mathbb{R}} \int_0^\infty \langle f, T_b D_a \psi \rangle_2 \overline{\langle g, T_b D_a \psi \rangle_2} \frac{\mathrm{d}a}{a^2} \,\mathrm{d}b \\
&= \frac{1}{(2\pi)^2} \int_{\mathbb{R}} \int_0^\infty \langle \mathcal{F}f, \mathcal{F}T_b D_a \psi \rangle_2 \overline{\langle \mathcal{F}g, \mathcal{F}T_b D_a \psi \rangle_2} \frac{\mathrm{d}a}{a^2} \,\mathrm{d}b \\
&= \frac{1}{(2\pi)^2} \int_{\mathbb{R}} \int_0^\infty \left\langle \mathcal{F}f, M_{-b}\left[a^{-\frac{1}{2}} \mathcal{F}\psi\left(\frac{\cdot}{a}\right)\right] \right\rangle_2 \cdot \\
&\qquad \overline{\left\langle \mathcal{F}g, M_{-b}\left[a^{-\frac{1}{2}} \mathcal{F}\psi\left(\frac{\cdot}{a}\right)\right] \right\rangle_2} \frac{\mathrm{d}a}{a^2} \,\mathrm{d}b \\
&= \frac{1}{(2\pi)^2} \int_{\mathbb{R}} \int_0^\infty \left[\int_{\mathbb{R}} \mathcal{F}f(\xi) \overline{a^{-\frac{1}{2}+1} \mathcal{F}\psi(a\xi) e^{-ib\cdot\xi}} \,\mathrm{d}\xi \right] \cdot \\
&\qquad \overline{\left[\int_{\mathbb{R}} a^{-\frac{1}{2}+1} \mathcal{F}g(\xi) \overline{\mathcal{F}\psi(a\xi) e^{-ib\cdot\xi}} \,\mathrm{d}\xi \right]} \frac{\mathrm{d}a}{a^2} \,\mathrm{d}b
\end{aligned}$$

[1]Theory for locally compact groups G and $L^2(G)$. The given ratio comes from the so called Haar-ratio. cf. [Louis, Maas, Rieder] for further info.

$$= \frac{1}{(2\pi)^2} \int_0^\infty \left\langle \mathcal{F}\left[a^{\frac{1}{2}} \mathcal{F}f \overline{\mathcal{F}\psi(a\cdot)}\right], \mathcal{F}\left[a^{\frac{1}{2}} \mathcal{F}g \overline{\mathcal{F}\psi(a\cdot))}\right]\right\rangle \frac{\mathrm{d}a}{a^2}$$

$$= \frac{1}{2\pi} \int_0^\infty \left\langle a^{\frac{1}{2}} \mathcal{F}f \overline{\mathcal{F}\psi(a\cdot)}, a^{\frac{1}{2}} \mathcal{F}g \overline{\mathcal{F}\psi(a\cdot))}\right\rangle \frac{\mathrm{d}a}{a^2}$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \mathcal{F}f(\xi)\overline{\mathcal{F}g(\xi)} \underbrace{\left[\int_0^\infty \frac{|\mathcal{F}\psi(a\xi)|^2}{a} \,\mathrm{d}a\right]}_{=: C_\psi} \mathrm{d}\xi = C_\psi \langle f, g\rangle . \square$$

(1.8) Corollary. *The operator $\mathcal{W}_\psi : L^2(\mathbb{R}) \to L^2\left(\mathbb{R}^+ \times \mathbb{R}, \frac{\mathrm{d}a}{a^2}\,\mathrm{d}b\right)$ is linear and bounded and a multiple of an isometry.*

(1.9) Remark. (i) If ψ satisfies $C_\psi < \infty$, then we call ψ **admissible**.

(ii) Let $\psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then $\mathcal{F}\psi$ is continuous and bounded. In this case, ψ can only be admissible, if $\mathcal{F}\psi(0) = 0$, which is equivalent to $\int_{\mathbb{R}} \psi(x)\,\mathrm{d}x = 0$. (This is related to vanishing moments)

(1.10) Theorem (Inversion Formula). *Let $\psi \in L^2(\mathbb{R})$ be admissible. Then, for $f \in L^2(\mathbb{R})$ we find*

$$f(x) = \frac{1}{C_\psi} \int_{\mathbb{R}} \int_0^\infty \mathcal{W}_\psi f(a,b) \overline{T_b D_a \psi(x)} \frac{\mathrm{d}a}{a^2}\,\mathrm{d}b.$$

Proof. Similar to the proof of Theorem 1.5. ☐

(1.11) Theorem. *Let $\psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ be admissible satisfying*

$$\int_{\mathbb{R}} (1 + |x|)\, |\psi(x)| \,\mathrm{d}x < \infty.$$

Let $f \in L^2(\mathbb{R})$ be bounded and Hölder-continuous with expontent $\alpha \in (0,1)$.[2] Then there exists $C > 0$ such that for all $(a,b) \in \mathbb{R}^+ \times \mathbb{R}$ we find

$$|\mathcal{W}_\psi f(a,b)| \leq C a^{\alpha + \frac{1}{2}}.$$

Proof. Let $(a,b) \in \mathbb{R}^+ \times \mathbb{R}$. By Remark 1.9 (ii), we know that $\int_{\mathbb{R}} \psi(x)\,\mathrm{d}x = 0$. We write

$$|\mathcal{W}_\psi f(a,b)| = \left| \int_{\mathbb{R}} f(x) a^{-\frac{1}{2}} \overline{\psi\left(\frac{x-b}{a}\right)} \mathrm{d}x - \int_{\mathbb{R}} f(b) a^{-\frac{1}{2}} \overline{\psi\left(\frac{x-b}{a}\right)} \mathrm{d}x \right|$$

$$\leq \int_{\mathbb{R}} |f(x) - f(b)| \cdot a^{-\frac{1}{2}} \left|\psi\left(\frac{x-b}{a}\right)\right| \mathrm{d}x \leq \int_{\mathbb{R}} L|x-b|^\alpha \cdot a^{-\frac{1}{2}} \left|\psi\left(\frac{x-b}{a}\right)\right| \mathrm{d}x$$

$$= L a^{-\frac{1}{2}} \int_{\mathbb{R}} |y|^\alpha \left|\psi\left(\frac{y}{a}\right)\right| \mathrm{d}y = L a^{-\frac{1}{2}} \int_{\mathbb{R}} a^\alpha |z|^\alpha |\psi(z)| \cdot a \,\mathrm{d}z = L a^{\alpha + \frac{1}{2}} \underbrace{\int_{\mathbb{R}} |z|^\alpha |\psi(z)| \,\mathrm{d}z}_{<\infty} . \quad \square$$

(1.12) Theorem. *Let $\psi \in L^2(\mathbb{R}) \cap C^1(\mathbb{R})$[3] be admissible and compactly supported. Further, let $f \in L^2(\mathbb{R}) \cap C(\mathbb{R})$ be bounded. Then, if there exists $C > 0$ with*

$$|\mathcal{W}_\psi f(a,b)| \leq C a^{\alpha + \frac{1}{2}} \qquad \text{for all } (a,b) \in \mathbb{R}^+ \times \mathbb{R}$$

for some $\alpha \in (0,1)$, then f is Hölder continuous with exponent α.

[2] A function $f : \mathbb{R} \to \mathbb{R}$ is said to be Hölder-continuous with exponent $\alpha \in (0,1)$ if there exists an $L \geq 0$ such that for all $x, y \in \mathbb{R}$ we find

$$|f(x) - f(y)| \leq L|x-y|^\alpha.$$

[3] In fact, $\psi \in C_c^1(\mathbb{R})$ already implies $\psi \in L^2(\mathbb{R})$.

Proof. By Theorem 1.10, we have

$$f = \underbrace{\frac{1}{C_\psi}\int\limits_{\mathbb{R}}\int\limits_{0}^{1} W_\psi f(a,b)\overline{T_b D_a \psi}\frac{\mathrm{d}a}{a^2}\,\mathrm{d}b}_{=:f_1} + \underbrace{\frac{1}{C_\psi}\int\limits_{\mathbb{R}}\int\limits_{1}^{\infty} W_\psi f(a,b)\overline{T_b D_a \psi}\frac{\mathrm{d}a}{a^2}\,\mathrm{d}b}_{=:f_2}.$$

First we note

$$|\mathcal{W}_\psi f(a,b)| = |\langle f, T_b D_a \psi\rangle| = a^{-\frac{1}{2}}\left|\left\langle f, \psi\left(\frac{\cdot - b}{a}\right)\right\rangle\right|$$
$$\leq a^{-\frac{1}{2}}\|f\|_2\left\|\psi\left(\frac{\cdot - b}{a}\right)\right\|_2 = a^{\frac{1}{2}}\|f\|_2\|\psi\|_2.$$

For f_2 we have

$$|f_2(x)| \leq \frac{1}{C_\psi}\int\limits_{\mathbb{R}}\int\limits_{1}^{\infty}|\mathcal{W}_\psi f(a,b)|\cdot a^{-\frac{1}{2}}\left|\psi\left(\frac{x-b}{a}\right)\right|\frac{\mathrm{d}a}{a^2}\,\mathrm{d}b$$
$$\leq \frac{\|f\|_2\|\psi\|_2}{C_\psi}\int\limits_{1}^{\infty}\int\limits_{\mathbb{R}}\left|\psi\left(\frac{x-b}{a}\right)\right|\mathrm{d}b\frac{\mathrm{d}a}{a^2} = \frac{\|f\|_2\|\psi\|_2}{C_\psi}\|\psi\|_1\int\limits_{1}^{\infty}a^{-1}\,\mathrm{d}a < \infty$$

For each $h > 0$ we have

$$|f_2(x) - f_2(x+h)| \leq$$
$$\frac{1}{C_\psi}\int\limits_{1}^{\infty}\int\limits_{\mathbb{R}}\left[\int\limits_{\mathbb{R}}|f(t)|a^{-\frac{1}{2}}\left|\psi\left(\frac{t-b}{a}\right)\right|\mathrm{d}t\right]a^{-\frac{1}{2}}\left|\psi\left(\frac{x-b}{a}\right) - \psi\left(\frac{x+h-b}{a}\right)\right|\mathrm{d}b\frac{\mathrm{d}a}{a^2}$$

We can now use the mean-value theorem to estimate small increments of ψ by $\|\psi'\|_\infty\cdot\left|\frac{h}{a}\right|$. Assume $|h| < 1$ and $\operatorname{supp}\psi \subset [-R, R]$ for some $R > 0$. Then

$$|f_2(x) - f_2(x+h)| \leq \frac{1}{C_\psi}\int\limits_{1}^{\infty}\int\limits_{|x-b|\leq aR+1}\left[\int\limits_{|t-b|\leq aR}\|f\|_\infty\|\psi\|_\infty\,\mathrm{d}t\right]\|\psi'\|_\infty\left|\frac{h}{a}\right|\mathrm{d}b\frac{\mathrm{d}a}{a^3}$$
$$\leq \frac{\|f\|_\infty\|\psi\|_\infty\|\psi'\|_\infty}{C_\psi}|h|\int\limits_{1}^{\infty}\frac{(4aR+2)^{\frac{1}{2}}}{a^4}\,\mathrm{d}a \leq \widetilde{C}|h| \leq \widetilde{C}|h|^\alpha$$

Now, for f_1, we have

$$|f_1(x)| \leq \frac{1}{C_\psi}\int\limits_{\mathbb{R}}\int\limits_{0}^{1}|\mathcal{W}_\psi f(a,b)|\cdot a^{-\frac{1}{2}}\left|\psi\left(\frac{x-b}{a}\right)\right|\frac{\mathrm{d}a}{a^2}\,\mathrm{d}b \leq \int\limits_{0}^{1}\int\limits_{\mathbb{R}}Ca^{\alpha+\frac{1}{2}}\cdot a^{\frac{1}{2}}\left|\psi\left(y\right)\right|\mathrm{d}y\frac{\mathrm{d}a}{a^2}$$
$$= C\|\psi\|_1\int\limits_{0}^{1}a^{\alpha-1}\,\mathrm{d}a < \infty.$$

Assume again $|h| < 1$ and $\operatorname{supp}\psi \subset [-R, R]$ for some $R > 0$. Then

$$|f_1(x) - f_1(x+h)|$$
$$\leq C\int\limits_{\mathbb{R}}\int\limits_{0}^{|h|}a^\alpha\left|\psi\left(\frac{x-b}{a}\right) - \psi\left(\frac{x+h-b}{a}\right)\right|\frac{\mathrm{d}a}{a^2}\,\mathrm{d}b + C\int\limits_{|h|}^{1}\int\limits_{|x-b|\leq aR+|h|}a^\alpha\left|\frac{h}{a}\right|\mathrm{d}b\frac{\mathrm{d}a}{a^2}$$
$$\leq C\int\limits_{0}^{|h|}\int\limits_{|x-b|\leq aR+1}2\|\psi\|_\infty a^{-1+\alpha}\,\mathrm{d}b\,\mathrm{d}a + C\int\limits_{|h|}^{1}|h|a^{\alpha-3}(aR+|h|)\,\mathrm{d}a \leq C_1|h|^\alpha.$$

Thus f is "locally Hölder-continuous" with exponent α for $|x-y| \leq 1$. Let now $h > 0$ be arbitrary. Then there exists some $m \in \mathbb{N}$ such that $\frac{h}{m} \leq 1$. We then write

$$
\begin{aligned}
|f(x) - f(x+h)| &= \left| \sum_{i=1}^{m} f\left(x + (k-1)\frac{h}{m}\right) - f\left(x + k\frac{h}{m}\right) \right| \\
&\leq \sum_{k=1}^{m} \left| f\left(x + (k-1)\frac{h}{m}\right) - f\left(x + k\frac{h}{m}\right) \right| \leq C \cdot m \left|\frac{h}{m}\right|^{\alpha} = Cm^{1-\alpha}|h|^{\alpha}.
\end{aligned}
$$

f is bounded, hence $|f(x)-f(x+h)| \leq |f(x)|+|f(x+h)| \leq 2K$. We consider two cases. First $|h|^{\alpha} \geq 2K$. Then we have trivially $|f(x) - f(x+h)| \leq |h|^{\alpha}$. If $|h|^{\alpha} < 2K$. There exists $m \in \mathbb{N}$ such that $\frac{h}{m} \leq 1$ and $m \leq |h| + 1$. Then

$$
|f(x) - f(x+h)| \leq C \cdot m^{1-\alpha}|h|^{\alpha} \leq C(|h|+1)^{1-\alpha}|h|^{\alpha} \leq C \cdot \left((2K)^{\frac{1}{\alpha}} + 1\right)^{1-\alpha} |h|^{\alpha}. \qquad \square
$$

2 Frames

2.1 What is a Frame?

(2.1) Remark. There are problems with the concept of orthonormal bases:

- When we have
$$
(\langle x, \varphi_i \rangle)_{i\in I} \rightsquigarrow x = \sum_{i \in I} \langle x, \varphi_i \rangle \varphi_i,
$$
but some $\langle x, \varphi_i \rangle$ are not available, x cannot be recovered.
- There is no flexibility to write x in terms of $(\varphi_i)_{i\in I}$. The coefficients $(c_i)_{i\in I}$ in the representation $x = \sum_{i\in I} c_i\varphi_i$ are fixed.

Our Goal is thus to generalize the notion of orthonormal bases. We will allow non-unique expensions $x = \sum_{i\in I} c_i\varphi_i$ (redundancy). In the mean time, we want to keep easy recovery of each x from $(\langle x, \varphi_i \rangle_{i\in I})$.

(2.2) Definition. Let $(\varphi_i)_{i\in I} \subset H$ be a family of vectors in a Hilbert space H. Then $(\varphi_i)_{i\in I}$ is called a **frame for H with frame bounds A and B**, if for $A, B \in \mathbb{R}^+$ with $A \leq B$ we have

$$
A\|x\|^2 \leq \sum_{i\in I} |\langle x, \varphi_i \rangle|^2 \leq B\|x\|^2, \qquad \forall\, x \in H.
$$

We call A the **upper frame bound** and B the **lower frame bound**.

If $A = B$ is possible, then $(\varphi_i)_{i\in I}$ is a **tight frame**. If $A = B = 1$ is possible, we have a **Parseval frame**.

(2.3) Example. (1) Consider in $\mathbb{R}^2$

$$
\varphi_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \varphi_2 = \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{pmatrix}, \quad \varphi_3 = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{pmatrix}.
$$

This is a tight frame with frame bound $\frac{3}{2}$.

(2) Let $(e_i)_{i\in\mathbb{N}}$ be an orthonormal basis for some Hilber space H. Then define

$$
\varphi_i = \begin{cases} e_{\frac{i}{2}}, & \text{if } i \text{ is even} \\ e_{\frac{i+1}{2}}, & \text{if } i \text{ is odd} \end{cases}
$$

for all $i \in \mathbb{N}$. Then

$$
\sum_{i=1}^{\infty} |\langle x, \varphi_i \rangle|^2 = 2\sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 = 2\|x\|^2
$$

holds for all $x \in H$. Thus $(\varphi_i)_{i\in\mathbb{N}}$ form a tight frame with frame bound 2.

(3) Now define

$$\varphi_i = \begin{cases} e_1, & \text{if } i = 1, \\ e_{i-1}, & \text{if } i > 1 \end{cases}$$

for all $i \in \mathbb{N}$. Then

$$\sum_{i=1}^{\infty} |\langle x, \varphi_i \rangle|^2 = |\langle x, e_1 \rangle|^2 + \|x\|^2.$$

By using Cauchy-Schwartz we see $|\langle x, e_1\rangle|^2 + \|x\|^2 \leq 2\|x\|^2$, which is fullfilled with equality for $x = e_1$. Thus $B = 2$ is the smallest upper frame bound.

Obviously we have $|\langle x, e_1\rangle|^2 + \|x\|^2 \geq \|x\|^2$. This is attained for any x orthogonal to e_1. Thus $A = 1$ is the largest lower frame bound.

(4) Consider

$$(\varphi_i)_{i\in\mathbb{N}} = \left(e_1, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, \dots \right).$$

Then for arbitrary $x \in H$ we find

$$\sum_{i=1}^{\infty} |\langle x, \varphi_i \rangle|^2 = \sum_{k=1}^{\infty} k \cdot \left| \left\langle x, \frac{1}{\sqrt{k}} e_k \right\rangle \right|^2 = \|x\|^2.$$

Thus $(\varphi_i)_{i\in\mathbb{N}}$ forms an Parseval frame, even thought the norms tend to zero.

(2.4) Remark. There exist frames (even tight ones), of which the elements have arbitrarily small norms, i.e. the can converge to zero. Also, a Parseval frame is not automatically an orthonormal basis

(2.5) Lemma. *Let H be a Hilbert space.*

(i) Each orthonormal basis is a Parseval frame

(ii) Each frame $(\varphi_i)_{i\in I}$ spans H, i.e. $\overline{\operatorname{span}\{\varphi_i | i \in I\}} = H$.

(iii) A Parseval frame $(\varphi_i)_{i\in I}$ with $\|\varphi_i\| = 1$ for all $i \in I$ is an orthonormal basis.

Proof. (i) is clear.

(ii): Assume there exists $x \in H \setminus \{0\}$ with $\langle x, \varphi_i \rangle = 0$ for all $i \in I$. Then we would have

$$\sum_{i\in I} |\langle x, \varphi_i \rangle|^2 = 0 < A\|x\|^2,$$

for any $A > 0$. This contradicts the definiton of a frame.

(iii): The elements φ_i, $i \in I$, are normalized and span H by (ii). Fix $i_0 \in I$ and consider for the Parseval frame $(\varphi_i)_{i\in I}$:

$$1 = \|\varphi_{i_0}\|^2 = \sum_{i\in I} |\langle \varphi_{i_0}, \varphi_i \rangle|^2 = \underbrace{\|\varphi_{i_0}\|^2}_{=1} + \sum_{i\in I\setminus\{i_0\}} |\langle \varphi_{i_0}, \varphi_i \rangle|^2.$$

This already shows $\langle \varphi_{i_0}, \varphi_i \rangle = 0$ for all $i \in I \setminus \{i_0\}$. □

2.2 The Frame Operator

(2.6) Definition. Let $\Phi = (\varphi_i)_{i\in I}$ be a frame for $\mathcal{H}$. Then the operator

$$T_\Phi : H \to l^2(I), x \mapsto (\langle x, \varphi_i \rangle)_{i\in I}$$

is called **analysis operator**. The **synthesis operator** is defined by

$$T_\Phi^* : l^2(I) \to H, (c_i)_{i\in I} \mapsto \sum_{i\in I} c_i \varphi_i.$$

(2.7) Lemma. *In the notation of Definition 2.6, the analysis operator T_Φ is linear and bounded. Further we have indeed $(T_\Phi)^* = T_\Phi^*$.*

Proof. *Linearity* is clear.

Boundedness: We have for all $x \in H$

$$\|T_\Phi x\|_{l^2}^2 = \sum_{i \in I} |\left\langle x, \varphi_i \right\rangle|^2 \leq B\|x\|^2.$$

Adjoint: W.l.o.g. we can assume $I = \mathbb{N}$. Let $(c_i)_{i \in I} \in l^2(I)$. Define $\left(c_i^{(n)}\right) \subset l^2(I)$ by

$$c_i^{(n)} := \begin{cases} c_i, & \text{if } i \leq n, \\ 0, & \text{else.} \end{cases}$$

Then $c^{(n)} \to c$ for $n \to \infty$ and

$$\left\langle (T_\Phi)^* c^{(n)}, x \right\rangle = \left\langle c^{(n)}, T_\Phi x \right\rangle = \left\langle c^{(n)}, (\left\langle x, \varphi_i \right\rangle)_{i \in I} \right\rangle = \sum_{i=1}^{n} c_i \overline{\left\langle x, \varphi_i \right\rangle}$$

$$= \left\langle x, \sum_{i=1}^{n} \overline{c_i \varphi_i} \right\rangle = \left\langle T_\Phi^* c, x \right\rangle$$

for all $x \in H$ and $n \in \mathbb{N}$. Letting $n \to \infty$ shows the claim. $\square$

(2.8) Definition. Let $\Phi = (\varphi_i)_{i \in I}$ be a frame for H. Then the **frame operator** $S_\Phi : H \to H$ is defined by

$$S_\Phi x = T_\Phi^* T_\Phi x = \sum_{i \in I} \left\langle x, \varphi_i \right\rangle \varphi_i.$$

(2.9) Theorem. *Let $\Phi = (\varphi_i)_{i \in I}$ be a frame for H. Then S_Φ is a self-adjoint operator and $\sigma(S_\Phi) \subset [A, B]$. In particular, S_Φ has a bounded inverse.*

Proof. For $x, y \in H$ we have

$$\left\langle S_\Phi x, y \right\rangle = \left\langle \sum_{i \in I} \left\langle x, \varphi_i \right\rangle \varphi_i, y \right\rangle = \sum_{i \in I} \left\langle x, \varphi_i \right\rangle \left\langle \varphi_i, y \right\rangle = \left\langle x, S_\Phi y \right\rangle.$$

Hence

$$\left\langle S_\Phi x, x \right\rangle = \sum_{i \in I} |\left\langle x, \varphi_i \right\rangle|^2.$$

By definition of a frame we get

$$A\|x\|^2 \leq \left\langle S_\Phi x, x \right\rangle \leq B\|x\|^2$$

and thus $\sigma(S_\Phi) \subset W(S_\Phi) \subset [A, B]$. $\square$

(2.10) Definition. Let $\Phi = (\varphi_i)_{i \in I} \subset H$ be a frame for H.

(i) A frame $(\psi_i)_{i \in I}$ satisfying

$$x = \sum_{i \in I} \left\langle x, \varphi_i \right\rangle \psi_i$$

for all $x \in H$ is called **dual frame** of Φ.

(ii) Then the system

$$(\widetilde{\varphi}_i)_{i \in I} := \left(S_\Phi^{-1} \varphi_i\right)_{i \in I}$$

is called **(canonical) dual frame**.

(2.11) Example. Let $(\varphi_i)_{i \in I}$ be an A-tight frame. Then $S_\Phi = AI$ and hence $\widetilde{\varphi}_i = \frac{1}{A}\varphi_i$ for all $i \in I$.

(2.12) Proposition. *Let $\Phi = (\varphi_i)_{i \in I}$ is a frame for H with frame bounds A and B. Then $(\widetilde{\varphi}_i)_{i \in I}$ is also a frame for H with frame bounds $\frac{1}{B}$ and $\frac{1}{A}$.*

Proof. For all $x \in H$ we have

$$\sum_{i\in I} \left|\left\langle x, S_\Phi^{-1}\varphi_i \right\rangle\right|^2 = \sum_{i\in I} \left|\left\langle S_\Phi^{-1}x, \varphi_i \right\rangle\right|^2 = \left\langle S_\Phi S_\Phi^{-1}x, S_\Phi^{-1}x \right\rangle = \left\langle x, S_\Phi^{-1}x \right\rangle = \left\langle S_\Phi^{-1}x, x \right\rangle.$$

This already implies

$$\frac{1}{B}\|x\|^2 \le \left\langle S_\Phi^{-1}x, x \right\rangle = \sum_{i\in I} \left|\left\langle x, S_\Phi^{-1}\varphi_i \right\rangle\right|^2 \le \frac{1}{A}\|x\|^2$$

for arbitrary $x \in H$. □

(2.13) Theorem. *Let $\Phi = (\varphi_i)_{i\in I}$ be a frame for H, and $(\widetilde{\varphi}_i)_{i\in I}$ be the associated canonical dual frame.*

(i) ***Reconstruction formula****: For any $x \in H$ we find*

$$x = \sum_{i\in I} \langle x, \varphi_i \rangle \, \widetilde{\varphi}_i.$$

(ii) ***Decomposition formula****: For any $x \in H$ we find*

$$x = \sum_{i\in I} \langle x, \widetilde{\varphi}_i \rangle \, \varphi_i.$$

Proof. For all $x \in H$ we find

$$x = S_\Phi^{-1}(S_\Phi x) = S_\Phi^{-1}\left(\sum_{i\in I} \langle x, \varphi_i \rangle \, \varphi_i\right) = \sum_{i\in I} \langle x, \varphi_i \rangle \, \widetilde{\varphi}_i.$$

$$x = S_\Phi(S_\Phi^{-1}x) = \left(\sum_{i\in I} \left\langle S_\Phi^{-1}x, \varphi_i \right\rangle \varphi_i\right) = \sum_{i\in I} \langle x, \widetilde{\varphi}_i \rangle \, \varphi_i.$$ □

2.3 Frame Decomposition

(2.14) Remark. If $(\varphi_i)_{i\in I}$ is a frame, but not a basis, there exist infinitely many $(c_i)_{i\in I}$ with $x = \sum_{i\in I} c_i\varphi_i$ for some x. Depending on the goal, one might want to take $(c_i)_{i\in I}$

- minimal in l^1-norm, which leads to compressed sensing
- minimal in l^2-norm,
- ...

(2.15) Theorem. *Let $\Phi = (\varphi_i)_{i\in I}$ be a frame for H, and let $(\widetilde{\varphi}_i)_{i\in I}$ the associated canonical dual frame. Also, let $x \in H$. Then*

$$\|(\langle x, \widetilde{\varphi}_i \rangle)_{i\in I}\|_{l^2} \le \|(c_i)_{i\in I}\|_{l^2}$$

for all $(c_i)_{i\in I} \in l^2(I)$ satisfying $x = \sum_{i\in I} c_i\varphi_i$.

Proof. Let $(c_i)_{i\in I} \in l^2(I)$ satisfy $x = \sum_{i\in I} c_i\varphi_i$. By Theorem 2.13 we obtain

$$0 = \sum_{i\in I} \left(c_i - \langle x, \widetilde{\varphi}_i \rangle\right) \varphi_i = S_\Phi^{-1}\left(\sum_{i\in I} \left(c_i - \langle x, \widetilde{\varphi}_i \rangle\right) \varphi_i\right),$$

which implies by testing with x

$$0 = \sum_{i\in I} \left(c_i - \langle x, \widetilde{\varphi}_i \rangle\right) \langle \widetilde{\varphi}_i, x \rangle .$$

Then we get with the just shown orthogonality

$$\|c\|_{l^2}^2 = \|(c_i - \langle x, \widetilde{\varphi}_i \rangle + \langle x, \widetilde{\varphi}_i \rangle)_{i\in I}\|_{l^2}^2 = \|(c_i - \langle x, \widetilde{\varphi}_i \rangle)_{i\in I}\| + \|(\langle x, \widetilde{\varphi}_i \rangle)_{i\in I}\|_{l^2}^2$$
$$\ge \|(\langle x, \widetilde{\varphi}_i \rangle)_{i\in I}\|_{l^2}^2$$ □

(2.16) Remark. It is often a numerical problem to compute S_Φ^{-1}. This rises the question whether there is a different way to reconstruct x from $(\langle x, \varphi_i\rangle)_{i\in I}$.

(2.17) Proposition (Frame algorithm). *Let $(\varphi_i)_{i\in\mathbb{N}}$ be a frame for H with frame bounds A and B and frame operator S_Φ. For some $x \in H$ define $(y_j)_{j\in\mathbb{N}}$ by $y_0 = 0$ and*

$$y_{j+1} := y_j + \frac{2}{A+B} S_\Phi(x - y_j) = y_j + \frac{2}{A+B} \sum_{i\in\mathbb{N}} \langle x - y_j, \varphi_i\rangle \varphi i$$

for all $j \in \mathbb{N}$. Then we have

$$\|x - y_j\| \leq \left(\frac{B-A}{B+A}\right)^j \|x\|$$

for any $j \in \mathbb{N}$.

Proof. First consider

$$\left\langle \left(I - \frac{2}{A+B} S_\Phi\right) x, x \right\rangle = \|x\|^2 - \frac{2}{A+B} \langle S_\Phi x, x\rangle = \|x\|^2 - \frac{2}{A+B} \sum_{i\in\mathbb{N}} |\langle x_i, \varphi_i\rangle|^2$$
$$\leq \left(1 - \frac{2A}{A+B}\right) \|x\|^2 = \frac{B-A}{A+B} \|x\|^2$$

Similarly one can show

$$-\frac{B-A}{A+B} \|x\|^2 \leq \left\langle \left(I - \frac{2}{A+B} S_\Phi\right) x, x \right\rangle.$$

This implies

$$\left\| I - \frac{2}{A+B} S_\Phi \right\| = \sup_{\|x\|=1} \left| \left\langle \left(I - \frac{2}{A+B} S_\Phi\right) x, x \right\rangle \right| \leq \frac{B-A}{B+A}.$$

By definition of $(y_j)_{j\in\mathbb{N}}$ we have

$$x - y_{j+1} = (x - y_j) - \frac{2}{A+B} S_\Phi(x - y_j) = \left(I - \frac{2}{A+B} S_\Phi\right)(x - y_j)$$
$$= \left(I - \frac{2}{A+B} S_\Phi\right)^2 (x - y_{j-1}) = \ldots = \left(I - \frac{2}{A+B} S_\Phi\right)^{j+1} (x - y_j).$$

Thus

$$\|x - y_{j+1}\| \leq \left\| I - \frac{2}{A+B} S_\Phi \right\|^{j+1} \|x\| \leq \left(\frac{B-A}{B+A}\right)^{j+1} \|x\|.$$

□

(2.18) Remark. The rate of convergence depends on $\frac{B}{A}$. Ideally, one might want $\frac{B}{A} = 1$, i.e. $A = B$.

(2.19) Remark (Recent Research Directions).

- Phase retrieval: What properties of x can one recover from the mapping $x \mapsto (|\langle x, \varphi_i\rangle|)_{i\in I}$. Treatment in frame theory came up in the last 5 years; applications include crystallography.
- Equiangular frames: When trying to recover x from a lossy mapping $x \mapsto (\langle x, \varphi_i\rangle)_{i\in I\setminus E}$, one can show that the optimal frames in this context are tight frames with equal norms which are equiangular, i.e. $\langle \varphi_i, \varphi_j\rangle = c$ for all $i, j \in I$. This is related to the question of equiangular lines and optimal packings.
- Scalable frames: For a not tight frame $(\varphi_i)_{i\in I}$ we search for $(c_i)_{i\in I} \subset \mathbb{C}$ such that $(c_i\varphi_i)_{i\in I}$ is a tight frame or an "almost" tight frame.

3 Gabor Frames

3.1 From the Short-Time Fourier Transform to Gabor Frames

(3.1) Remark. We introduced the transform

$$L^2(\mathbb{R}) \ni f \mapsto \mathcal{S}_g f(t, \xi) = \langle f, M_\xi T_t g\rangle, \qquad (t, \xi) \in \mathbb{R}^2.$$

We now want to discretize $\mathbb{R}^2$ to Λ so that $\{M_\xi T_t g : (t,\xi) \in \Lambda\}$ is a frame. For this we will choose the discretization $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$ with $a, b > 0$.

(3.2) Definition. Let $g \in L^2(\mathbb{R})$ and $a, b > 0$. Then the associated **Gabor system** $\mathcal{G}(g,a,b)$ is defined by

$$\mathcal{G}(g,a,b) := \{M_{bn} T_{am} g : m, n \in \mathbb{Z}\}.$$

We will abbreviate $g_{am,bn} := M_{bn} T_{am} g$.

(3.3) Theorem. *Let $g \in L^2(\mathbb{R})$, $a, b > 0$. Assume that $\mathcal{G}(g,a,b)$ forms a frame for $L^2(\mathbb{R})$, and let S be the associated Frame operator. Then the canonical dual frame*

$$(S^{-1} g_{am,bn})_{m,n\in\mathbb{Z}} = ((S^{-1}g)_{am,bn})_{m,n\in\mathbb{Z}}.$$

Then for all $f \in L^2(\mathbb{R})$ we have

$$f = \sum_{m,n\in\mathbb{Z}} \langle f, g_{am,bn} \rangle (S^{-1}g)_{am,bn}.$$

Proof. First note, that the second equation follows directly from the first one. Thus we just need to proof that S commutes with M_{bn} and T_{am}.

First

$$\langle T_{am_0} f, g_{am,bn} \rangle = \int_{\mathbb{R}} f(x - am_0)\overline{g(x-am)} e^{-ibnx} \, \mathrm{d}x = e^{-abm_0 n} \left\langle f, g_{a(m-m_0),bn} \right\rangle.$$

Then

$$\begin{aligned} S(T_{am_0} f)(x) &= \sum_{m,n\in\mathbb{Z}} e^{-iabm_0 n} \left\langle f, g_{a(m-m_0),bn} \right\rangle g_{am,bn}(x) \\ &= \sum_{m,n\in\mathbb{Z}} \langle f, g_{am,bn} \rangle e^{-iabm_0 n} \underbrace{g_{a(m+m_0),bn}(x)}_{=T_{am_0}(M_{bn}T_{am}g)(x)e^{iabm_0 n}} = T_{am_0}(Sf)(x). \end{aligned}$$

Similarly, $SM_{bn_0} = M_{bn_0} S$. □

3.2 Necessary Conditions for Gabor Frames

(3.4) Theorem. *Let $g \in L^2(\mathbb{R})$, $a, b > 0$. Assume that $\mathcal{G}(g,a,b)$ forms a frame for $L^2(\mathbb{R})$ with bounds A, B. Then*

$$A \le \frac{2\pi}{b} \sum_{m\in\mathbb{Z}} |g(x-am)|^2 \le B$$

for almost every $x \in \mathbb{R}$. If $\mathcal{G}(g,a,b)$ is a tight frame for $L^2(\mathbb{R})$ with bound A, then

$$\frac{2\pi}{b} \sum_{m\in\mathbb{Z}} |g(x-am)|^2 = A$$

for almost every $x \in \mathbb{R}$.

Proof. Towards a contradiction, assume the upper bound fails. Then there exists a set $\Delta \subset \mathbb{R}$ with positive measure such that

$$\sum_m |g(x-am)|^2 > B$$

for almost every $x \in \Delta$. W.l.o.g. assume $\Delta \subset I$ of length $\frac{2\pi}{b}$. Then define $G(x) := \sum_m |g(x-am)|^2$ and

$$\Delta_0 := \{x \in \mathbb{R} : G(x) > B + 1\}$$

$$\Delta_k := \left\{ x \in \mathbb{R} : B + \frac{1}{k+1} < G(x) < B + \frac{1}{k} \right\}, \qquad k \in \mathbb{N}.$$

Indeed, $\Delta = \bigcup_{k\in\mathbb{N}_0} \Delta_k$ and $\Delta_i \cap \Delta j = \emptyset$ if $i \neq j$. There exists $k' \in \mathbb{N}$, such that $\Delta_{k'}$ has positive measure. We define $f := \chi_{\Delta_{k'}}$.

Next, we will analyize

$$\sum_{m,n} |\langle f, g_{am,bn}\rangle|^2 = \sum_{m,n} \left| \int_{\mathbb{R}} f(t)g(t-am)e^{-ibnt}\,\mathrm{d}t \right|^2.$$

For $m \in \mathbb{Z}$ we have $\operatorname{supp}(fT_{am}g) \subset \Delta_{k'}$. Since $\Delta_{k'}$ is contained in an interval of length $\frac{2\pi}{b}$ we have

$$\sum_n |\langle f, g_{am,bn}\rangle|^4 = \frac{2\pi}{b} \int_{\mathbb{R}} |f(t)|^2 |g(t-am)|^2\,\mathrm{d}t.$$

Therefore

$$\sum_{m,n} |\langle f, g_{am,bn}\rangle|^2 = \sum_m \frac{2\pi}{b} \int |f(t)|^2 |g(t-am)|^2\,\mathrm{d}t = \frac{2\pi}{b} \int_{\Delta_{k'}} \sum_m |g(t-am)|^2\,\mathrm{d}t$$
$$= \frac{2\pi}{b} \int_{\Delta_{k'}} G(t)\,\mathrm{d}t > \frac{2\pi}{b} \int_{\Delta_{k'}} \int_{\Delta_{k'}} \frac{b}{2\pi} B + \frac{1}{k+1}\,\mathrm{d}t = \left(B + \frac{2\pi}{b(k'+1)}\right) \|f\|_2^2.$$

This is a contradiction to the upper frame bound of $\mathcal{G}(g,a,b)$.

Similar computations can be done for the lower bound. $\square$

(3.5) Corollary. *Let $g \in L^2(\mathbb{R})$, $a, b > 0$. If $\mathcal{G}(g,a,b)$ is a frame with the frame bounds A and B, then*

$$A \leq \frac{2\pi}{a \cdot b} \|g\|_2^2 \leq B.$$

In particular, if $\mathcal{G}(g,a,b)$ is a tight frame with bound A, then

$$A = \frac{2\pi}{a \cdot b} \|g\|^2.$$

Proof. By Theorem 3.4 we have

$$aA \leq \frac{2\pi}{b} \sum_m \int_0^a |g(t-am)|^2\,\mathrm{d}t \leq Ba.$$

By integration using substitution and deviding by a gives the result. $\square$

(3.6) Theorem. *Let $g \in L^2(\mathbb{R})$ Then the following holds*

(i) If $\mathcal{G}(g,a,b)$ forms an orthonormal basis, then $a \cdot b = 2\pi$.

(ii) If $\mathcal{G}(g,a,b)$ constitutes a frame for $L^2(\mathbb{R})$, then $a \cdot b \leq 2\pi$.

Proof. (i) Use Corollary 3.5 with $A = B = 1$ and $\|g\| = 1$.

(ii): Ten lectures on wavelets, I. Daubechies. $\square$

(3.7) Remark.

(3.8) Theorem (Balian-Low). *Let $g \in L^2(\mathbb{R})$, $a, b > 0$. If $\mathcal{G}(g,a,b)$ is an ONB, then*

$$\left(\int |x|^2 |g(x)|^2\,\mathrm{d}x\right)\left(\int |\xi|^2 |\mathcal{F}g(\xi)|^2\,\mathrm{d}\xi\right) = \infty.$$

Proof. Define $(Xf)(x) := xf(x)$ and $(Pf)(x) := -if'(x)$. Then we claim that either Xg or $\mathcal{F}[Pg]$ is not in $L^2(\mathbb{R})$. Assume $a = 1$, $b = 2\pi$, and g is differentiable and towards a contradiction $Xg, Pg \in L^2(\mathbb{R})$. Clearly

$$\langle Xg, \mathbb{P}g\rangle = \sum_{m,n} \langle Xg, g_{m,2\pi n}\rangle \langle g_{m,2\pi n}, Pg\rangle .$$

We rewrite the first term by computing

$$\langle Xg, g_{m,2\pi n}\rangle = \langle Xg, M_{2\pi n}T_m g\rangle = \langle g, XM_{2\pi n}T_m g\rangle = m \underbrace{\langle g, g_{m,2\pi n}\rangle}_{=0,\text{ b/c of ONB}} + \langle g, (Xg)_{m,2\pi n}\rangle .$$

Te second term can be written as

$$\langle g_{m,2\pi n}, Pg\rangle = \ldots = \langle (Pg)_{m,2\pi n}, g\rangle .$$

Therefore

$$\begin{aligned}\langle Xg, Pg\rangle &= \sum_{m,n} \langle g, (Xg)_{m,2\pi n}\rangle \langle (Pg)_{m,2\pi n}, g\rangle \\ &= \sum_{m,n} \langle g, M_{2\pi n}T_m Xg\rangle \langle M_{2\pi n}T_m Pg, g\rangle \\ &= \sum_{m,n} \langle Pg, M_{-2\pi n}T_{-m} g\rangle \langle M_{-2\pi n}T_{-m} g, Xg\rangle = \langle Pg, Xg\rangle .\end{aligned}$$

Therefore we obtain $\langle (PX - XP)g, g\rangle = 0$. Since $PX - XP = -iI$ and $g \neq 0$ we get $0 = -i\langle g, g\rangle = -i\|g\|^2 \neq 0$. So we cannot have $Xg, Pg \in L^2(\mathbb{R})$. ☐

3.3 Sufficient Conditions for Gabor Frames

(3.9) Theorem. *Let $a, b > 0$ and $g \in L^2(\mathbb{R})$ and set*

$$A = \frac{2\pi}{b}\left(\inf_{0<|x|<a} \sum_m |g(x-am)|^2 - \sum_{k\in\mathbb{Z}\setminus\{0\}} \left[\beta\left(\frac{2\pi}{b}k\right)\beta\left(-\frac{2\pi}{b}k\right)\right]^{\frac{1}{2}}\right),$$

$$B = \frac{2\pi}{b}\left(\sup_{0<|x|<a} \sum_m |g(x-am)|^2 + \sum_{k\in\mathbb{Z}\setminus\{0\}} \left[\beta\left(\frac{2\pi}{b}k\right)\beta\left(-\frac{2\pi}{b}k\right)\right]^{\frac{1}{2}}\right),$$

where

$$\beta(y) = \sup_{0<|x|<a} \sum_m |g(x-am)||g(x-am+y)|.$$

If $A > 0$ and $B < \infty$, then $\mathcal{G}(g, a, b)$ is a frame with frame bounds A and B.

Proof. Let $f \in L^2(\mathbb{R})$. Then

$$\begin{aligned}\sum_{m,n\in\mathbb{Z}} |\langle f, g_{am,bn}\rangle|^2 &= \sum_{m,n\in\mathbb{Z}} \left|\int_{\mathbb{R}} f(x)\overline{g(x-am)}e^{-ibnx}\,\mathrm{d}x\right|^2 \\ &= \sum_{m,n\in\mathbb{Z}} \left|\sum_{k\in\mathbb{Z}} \int_0^{\frac{2\pi}{b}} f\left(x+\frac{2\pi}{b}k\right)\overline{g\left(x+\frac{2\pi}{k}-am\right)}e^{-ibnx}\,\mathrm{d}x\right|^2 \\ &\overset{\text{Plancherel}}{=} \sum_{m\in\mathbb{Z}} \frac{2\pi}{b}\int_0^{\frac{2\pi}{b}} \left|\sum_{k\in\mathbb{Z}} f\left(x+\frac{2\pi}{b}k\right) g\left(x+\frac{2\pi}{b}k-am\right)\right|^2 \mathrm{d}x \\ &= \frac{2\pi}{b}\sum_{m,k,l}\int^{\frac{2\pi}{b}} |2f\left(x+\frac{2\pi}{b}k\right) g\left(x+\frac{2\pi}{b}k-am\right)\overline{f\left(x+\frac{2\pi}{b}l\right)g\left(x+\frac{2\pi}{b}l-am\right)}\,\mathrm{d}x \\ &= \frac{2\pi}{b}\sum_{m,k}\int_{\mathbb{R}} \overline{f(x)}f\left(x+\frac{2\pi}{b}k\right)\overline{g(x-am)}g\left(x+\frac{2\pi}{b}k-am\right)\mathrm{d}x\end{aligned}$$

$$= \frac{2\pi}{b} \int_{\mathbb{R}} |f(x)|^2 \sum_m |g(x-am)|^2 \,\mathrm{d}x$$

$$+ \underbrace{\frac{2\pi}{b} \sum_m \sum_{k \neq 0} \int_{\mathbb{R}} \overline{f(x)} f\left(x + \frac{2\pi}{b}k\right) \overline{g(x-am)} g\left(x + \frac{2\pi}{b}k - am\right) \mathrm{d}x}_{=: R(x)} .$$

Next we estimate $R(x)$ by

$$\begin{aligned}
|R(x)| &= \frac{2\pi}{b} \left| \sum_m \sum_{k \neq 0} \int_{\mathbb{R}} \overline{f(x)} f\left(x + \frac{2\pi}{b}k\right) \overline{g(x-am)} g\left(x + \frac{2\pi}{b}k - am\right) \mathrm{d}x \right| \\
&\overset{\text{CS}}{\leq} \frac{2\pi}{b} \sum_m \sum_{k \neq 0} \left(\int_{\mathbb{R}} |f(x)|^2 |g(x-am)| \left| g\left(x + \frac{2\pi}{b}k - am\right) \right| \mathrm{d}x \right)^{\frac{1}{2}} \\
&\qquad \cdot \left(\int_{\mathbb{R}} \left| f\left(y + \frac{2\pi}{b}k\right) \right|^2 |g(y-am)| \left| g\left(y + \frac{2\pi}{b}k - am\right) \right| \mathrm{d}y \right)^{\frac{1}{2}} \\
&\overset{\text{CS}}{\leq} \frac{2\pi}{b} \sum_{k \neq 0} \left(\int_{\mathbb{R}} |f(x)|^2 \sum_m |g(x-am)| \left| g\left(x + \frac{2\pi}{b}k - am\right) \right| \mathrm{d}x \right)^{\frac{1}{2}} \\
&\qquad \cdot \left(\int_{\mathbb{R}} |f\,(y)|^2 \sum_m |g(y-am)| \left| g\left(y - \frac{2\pi}{b}k - am\right) \right| \mathrm{d}y \right)^{\frac{1}{2}} \\
&\leq \frac{2\pi}{b} \sum_{k \neq 0} \beta\left(\frac{2\pi}{b}k\right)^{\frac{1}{2}} \beta\left(-\frac{2\pi}{b}k\right)^{\frac{1}{2}} \|f\|^2.
\end{aligned}$$

The lower frame bound follows from

$$\sum_{m,n} |\langle f, g_{am,bn}\rangle|^2 \geq \frac{2\pi}{b} \|f\|_2^2 \cdot \inf_{0<|x|<a} \sum_m |g(x-am)|^2 - \frac{2\pi}{b} \|f\|_2^2 \sum_k \left(\beta\left(\frac{2\pi}{b}k\right) \beta\left(\frac{2\pi}{b}k\right) \right)^{\frac{1}{2}} = A\|f\|_2^2.$$

Proving the upper bound works similarly. □

(3.10) Examples. (1) Choose $g := \chi_{[0,1)}$, $a = 1$ and $b = 2\pi$. Then

$$\mathcal{G}(g,a,b) = \left\{ \chi_{[0,1)}(x-m) e^{2\pi i n x} : m,n \in \mathbb{Z} \right\} = \left\{ \chi_{[m,m+1)}(x) e^{2\pi i n x} : m,n \in \mathbb{Z} \right\}.$$

Since $\{e^{2\pi i n x} : n \in \mathbb{Z}\}$ is an ONB for $L^2([0,1))$, hence also for $L^2([m,m+1))$, $m \in \mathbb{Z}$. Using $\mathbb{R} = \bigcup_{m \in \mathbb{Z}} [m,m+1)$, we obtain that $\mathcal{G}(g,a,b)$ is an ONB for $L^2(\mathbb{R})$

(2) The uncertainty principle is minimized by the Gaussians. Consider $g(x) = e^{-\frac{x^2}{2}}$. A theorem by Seip and Wallenstein (1992) proves that $\mathcal{G}(g,a,b)$ forms a frame for $L^2(\mathbb{R})$ if and only if $a \cdot b < 2\pi$. The proof uses a lot of complex analysis.

(3) How to circumvent the problem that there are no Gabor ONBs with “good” time-frequency localization?

- Work with frames with $\frac{B}{A} \approx 1$.
- Wilson bases, which consist of two Garbor system elemnts.

4 Wavelet-Frames

4.1 Form the Continuous Wavelet Transform to Wavelet-Frames

(4.1) Remark. Recall the continuous wavelet transform

$$L^2(\mathbb{R}) \ni f \mapsto W_\psi f(a,b) = \int\limits_{\mathbb{R}} f(x) a^{-\frac{1}{2}} \overline{\psi\left(\frac{x-b}{a}\right)} \,\mathrm{d}x, \qquad (a,b) \in \mathbb{R}^+ \times \mathbb{R}, \psi \in L^2(\mathbb{R}).$$

We discretize $\mathbb{R}^+ \times \mathbb{R}$ by $\Lambda = \{(a^j, a^j bm) : j,m \in \mathbb{Z}\}$ for fixed $a, b > 0$. On Λ the wavelets are $a^{-\frac{j}{2}} \psi\left(a^{-j} \cdot -bm\right)$.

(4.2) Definition. Let $a, b > 0$, $\psi \in L^2(\mathbb{R})$. Then the associated **wavelet system** $\mathcal{W}(\psi, a, b)$ is defined by

$$\mathcal{W}(\psi,a,b) = \{\underbrace{a^{-\frac{j}{2}} \psi(a^{-j}x - bm)}_{=:\,\psi_{j,m}(x)} : j,m \in \mathbb{Z}\}.$$

(4.3) Theorem. *If $\mathcal{W}(\psi,a,b)$ is a frame for $L^2(\mathbb{R})$ for $a,b>0$, $\psi \in L^2(\mathbb{R})$, then*

$$f = \sum_{j,m} \langle f, \psi_{j,m} \rangle \, S^{-1} \psi_{j,m}, \qquad \forall\, f \in L^2(\mathbb{R})$$

with S the associated frame operator.

Proof. See Section 2.2. $\square$

(4.4) Remark. (i) In this case $S^{-1}\psi_{j,m} = (S^{-1}\psi)_{j,m}$ is in general not true. It is most of the time not even clear, whether there exists a dual frame of the form of a wavelet system.

(ii) This is not a huge problem, since there exist wavelet ONBs with excellent time-frequency localization.

4.2 Necessary and Sufficient Conditions for Wavelet-Frames

(4.5) Theorem. *Let $a, b > 0$, $\psi \in L^2(\mathbb{R})$. Let $\mathcal{W}(\psi,a,b)$ form a frame for $L^2(\mathbb{R})$ with frame bounds A and B. Then*

$$A \le \frac{2\pi}{b} \sum_j \left|\mathcal{F}\psi(a^j \xi)\right|^2 \le B, \quad \textit{for a.e. } \xi \in \mathbb{R}.$$

If $\mathcal{W}(\psi,a,b)$ is tight, then

$$\frac{2\pi}{b} \sum_j \left|\mathcal{F}\psi(a^j \xi)\right|^2 = A, \quad \textit{for a.e. } \xi \in \mathbb{R}.$$

Proof. Somehow similar to the Gabor case. $\square$

(4.6) Corollary. *With the same assumptions as in Theorem 4.5, we have*

$$A \le \frac{2\pi}{b \cdot \ln a} \int\limits_0^\infty \frac{|\mathcal{F}f(\xi)|^2}{\xi} \,\mathrm{d}\xi \le B.$$

For a tight frame we have

$$A = \frac{2\pi}{b \cdot \ln a} \int\limits_0^\infty \frac{|\mathcal{F}f(\xi)|^2}{\xi} \,\mathrm{d}\xi.$$

Proof. In Theorem 4.5, multiply with $\frac{1}{\xi}$ and then integrate over $(\min\{1,a\}, \max\{1,a\})$. $\square$

(4.7) Remark. (i) For $\mathcal{W}(\psi,a,b)$ to be a frame, ψ needs to be admissible.

(ii) There is no "Nyquist density" for the discretization like for Gabor systems.

(4.8) Theorem. *Let $a, b > 0$, $\psi \in L^2(\mathbb{R})$, and*

$$A = \frac{2\pi}{b}\left(\inf_{1\le|\xi|\le a}\sum_j |\mathcal{F}\psi(a^j\xi)|^2 - \sum_{k\neq 0}\left(\beta\left(\frac{2\pi}{b}k\right)\beta\left(-\frac{2\pi}{b}k\right)\right)^{\frac{1}{2}}\right)$$

$$B = \frac{2\pi}{b}\left(\sup_{1\le|\xi|\le a}\sum_j |\mathcal{F}\psi(a^j\xi)|^2 + \sum_{k\neq 0}\left(\beta\left(\frac{2\pi}{b}k\right)\beta\left(-\frac{2\pi}{b}k\right)\right)^{\frac{1}{2}}\right)$$

with

$$\beta(y) = \sup_{1\le|\xi|\le a}\sum_j |\mathcal{F}\psi(a^j\xi)||\mathcal{F}\psi(a^j\beta + y)|.$$

If $0 < A \le B < \infty$, then $\mathcal{W}(\psi, a, b)$ forms a frame for $L^2(\mathbb{R})$.

Proof. Works similar as in the Gabor case. □

5 Wavelet Orthonormal Bases

5.1 Haar Wavelet

(5.1) Definition. Let $\psi \in L^2(\mathbb{R})$ be defined by

$$\psi(x) = \chi_{[0,\frac{1}{2})} - \chi_{[\frac{1}{2},1)}.$$

Then ψ is called **Haar wavelet**. Further,

$$\mathcal{W}\left(\psi, \frac{1}{2}, 1\right) = \left\{\psi_{j,m} := 2^{\frac{j}{2}}\psi(2^j\cdot - m) : j, m \in \mathbb{Z}\right\}$$

is called the **Haar wavelet system**.

(5.2) Remark. The Haar wavelet has a bad time-frequency localization. But it is still used even for some engineering applications.

(5.3) Theorem (Haar, 1910). *The Haar wavelet system forms an orthonormal basis of $L^2(\mathbb{R})$.*

Proof. We first proof that $\mathcal{W}\left(\psi, \frac{1}{2}, 1\right)$ forms an orthonormal system. After that we will show that it is indeed a basis.

We set $I_{j,m} = \left[2^{-j}m, 2^{-j}(m+1)\right)$. Obviously, we have $\operatorname{supp}\psi_{j,m} = I_{j,m}$. We note that $\|\psi_{j,m}\|_2 = \|\psi\|_2 = 1$ for all $j, m \in \mathbb{Z}$. Next we proof, that $\langle\psi_{j,m}, \psi_{j',m'}\rangle = 0$ if $(j,m) \neq (j',m')$. For that we consider several cases:

(i) $j = j'$: We have $\bigcup_{m\in\mathbb{Z}} I_{j,m} = \mathbb{R}$ disjointly, hence $\operatorname{supp}\psi_{j,m} \cap \operatorname{supp}\psi_{j',m'} = \emptyset$.

(ii) $j > j'$ and $I_{j,m} \cap I_{j',m'} = \emptyset$: trivial.

(iii) $j > j'$ and $I_{j,m} \subseteq I_{j',m'}$: Towards a contradiction, assume that

$$2^{-j}(m+1) > 2^{-j'+1}(2m'+1), \text{ but } 2^{-j}m < 2^{-j'+1}(2m'+1).$$

Set $j = j' + \widetilde{j}$ with $\widetilde{j} \ge 1$. Then

$$2^{-(j-\widetilde{j}+1)}(2m'+1) - 2^{-j} < 2^{-j}m < 2^{-(j-\widetilde{j}+1)}(2m'+1),$$

which is equivalent to

$$2^{\widetilde{j}-1}(2m'+1) - 1 < m < 2^{\widetilde{j}-1}(2m'+1),$$

leading up to a contradiction, since all terms in the above inequality must be integers.

W.l.o.g. we can assume, that $I_{j,m} \subset [2^{-j'}m', 2^{-j'-1}(2m'+1)]$, then

$$\langle \psi_{j,m}, \psi_{j',m'} \rangle = \int_{\mathbb{R}} \psi_{j,m}(x)\,\mathrm{d}x = 0.$$

(iv) $j > j'$ and not (ii) or (iii). Towards a contradiction, assume that

$$2^{-j}(m+1) > 2^{-j'}m', \text{ and } 2^{-j}m < 2^{-j'}m'.$$

Let $\widetilde{j}$ be as before. Then

$$2^{-j}m < 2^{-(j-\widetilde{j})}m' < 2^{-j}(m+1).$$

This leads again to a contradiction, which shows that this case does not apply.

Now we show that the system spans $L^2(\mathbb{R})$. Let $\varepsilon > 0$ and $f \in L^2(\mathbb{R})$. Then there exists $M > 0$ and $j \in \mathbb{Z}$ as well as a sequence $(c_m)_{m\in\mathbb{Z}}$ with

$$\left\| f - \int_{|m|\le M} c_m \chi_{I_{j,m}} \right\|_2 < \varepsilon.$$

W.l.o.g. we can assume that $\operatorname{supp} f \subset \left[-2^{j_1}, 2^{j_1}\right)$ and

$$f = \sum_{m=-2^{j_0+j_1}}^{2^{j_0+j_1}-1} c_m \chi_{I_{j_0,m}}.$$

First we decompose $f = f^1 + g^1$, where

(i) f^1 is a step function with step-size 2^{-j_0+1},

(ii) g^1 shall be represented by Haar wavelets.

We set

$$f^1\big|_{I_{j_0,2m}} = f^1\big|_{I_{j_0,2m+1}} := \frac{c_{2m}+c_{2m+1}}{2} = c_m^1, \quad g^1\big|_{I_{j_0,2m}} = -g^1\big|_{I_{j_0,2m+1}} := \frac{c_{2m}-c_{2m+1}}{2} =: d_m^1$$

This ensures that

$$c_{2m}\chi_{I_{j_0,2m}} + c_{2m+1}\chi_{I_{j_0,2m+1}} = c_m^1 \chi_{I_{j_0-1,m}} + d_m^1 \psi_{j_0-1,m}.$$

Thus

$$f = f^1 + g^1 = \sum_m c_m^1 \chi_{I_{j_0-1,m}} + \sum_m d_m^1 \psi_{j_0-1,m}$$

We continue splitting f^1 into $f^2 + g^2$ by the same procedure and continue until in the $j_0 + j_1$-st step we have

$$f = f^{j_0+j_1} + \sum_{j=j_0-1}^{-j_1} \sum_m d_{j,m}\psi_{j,m}.$$

Now continue by $f = f^{j_0+j_1+k} + \sum_{j=j_0-1}^{-j_1-k} \sum_m d_{j,m}\psi_{j,m}$, with $\operatorname{supp} f^{j_0+j_1+k} \subset [-2^{j_1+k}, 2^{j_1+k}]$ and

$$f^{j_0+j_1+k}\big|_{[-2^{j_1+k},0)} = -\frac{1}{2^k} c_{-1}^{j_0+j_1}$$

$$f^{j_0+j_1+k}\big|_{[0,2^{j_1+k})} = \frac{1}{2^k} c_0^{j_0+j_1}$$

This leads to

$$\left\| f - \sum_{j=j_0-1}^{-j_1-k} \sum_m d_{j,m}\psi_{j,m} \right\|_2^2 = \|f^{j_0+j_1+k}\|_2^2 = 2^{j_1+k}2^{-k}|c_{-1}^{j_0+j_1}|^2 + 2^{j_1+k}2^{-2k}|c_0^{j_0+j_1}|^2$$

$$= 2^{j_1-k}\left(|c_{-1}^{j_0+j_1}|^2 + |c_0^{j_0+j_1}|^2\right) \overset{k\to\infty}{\longrightarrow} 0. \quad \square$$

(5.4) Definition. (i) We call $\varphi = \chi_{[0,1)}$ the **Haar scaling function** and $\mathcal{W}\left(\varphi, \frac{1}{2}, 1\right)$ the **Haar scaling system**.

(ii) We call $V_j = \overline{\operatorname{span}\{\varphi_{j,m} : m \in \mathbb{Z}\}}$, $j \in \mathbb{Z}$, **scaling spaces**.

(iii) We call $W_j = \overline{\operatorname{span}\{\psi_{j,m} : m \in \mathbb{Z}\}}$, $j \in \mathbb{Z}$, with ψ the Haar wavelet **wavelet spaces**.

(5.5) Theorem. *Let ψ be a Haar wavelet, φ a Haar scaling function, $(V_j)_{j\in\mathbb{Z}}$, $(W_j)_{j\in\mathbb{Z}}$ the spaces as in Definition 5.4. Then*

(i) $(V_j)_{j\in\mathbb{Z}}$, $(W_j)_{j\in\mathbb{Z}}$ *are closed subspaces of* $L^2(\mathbb{R})$.

(ii) $\{0\} \subset \ldots \subset V_{-1} \subset V_0 \subset V_1 \subset \ldots \subset L^2(\mathbb{R})$.

(iii)
$$\bigcup_{j\in\mathbb{Z}} V_j = \{0\}, \quad \textit{and} \quad \overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}).$$

(iv) $f \in V_j$ *if and only if* $f\left(2^{-j}\cdot\right) \in V_0$.

(v) $\{\varphi_{0,m} : m \in \mathbb{Z}\}$ *forms an orthonormal basis of* V_0.

(vi) $V_{j+1} = V_j \oplus W_j$ *for all* $j \in \mathbb{Z}$.

Proof. (i) is clear.

(ii) Let $f \in V_j$. W.l.o.g. we can assume $f = \sum_{|m|\leq M} c_m\varphi_{j,m}$. We write $I_{j,m} := \operatorname{supp}(\varphi_{j,m} = 2^{-j}[m, m+1)$. We can consder

$$f = \sum_{|m|\leq M} c_m\varphi_{j+1,2m} + \sum_{|m|\leq M} c_m\varphi_{j+1,2m+1} \in V_{j+1} + V_{j+1} = V_{j+1}.$$

(iii) Consider $f = \sum_m c_m\varphi_{j,m}$. Then we have $\|f\|_2^2 = \sum_m |c_m|^2 2^j$. Then for $f \in \cap_{j\in\mathbb{Z}} V_j$ we need to have $\sum_m |c_m|^2 = 0$, and thus $f = 0$.

$\overline{\bigcup_j V_j} = L^2(\mathbb{R})$ holds, since every $f \in L^2(\mathbb{R})$ can be arbitrarily well approximated by step functions.

(iv) If $f = \sum_{m\in\mathbb{Z}} c_m\varphi_{j,m}$, then

$$f(2^{-j}x) = \sum_m c_m 2^{\frac{j}{4}}\varphi\left(2^j\left(2^{-j}x\right) - m\right) \in V_0.$$

(vi) Let $f \in V_{j+1}$. By the proof technique of Theorem 5.3, we can write $f \in V_j + W_j$, which shows $V_{j+1} \leq V_j + W_j$. By (ii) we have $V_j \leq V_{j+1}$. Let $f \in W_j$, hence f is constant on $I_{j+1,m}$, $m \in \mathbb{Z}$. Then $f \in V_{j+1}$. Let now $f \in V_j \cap W_j$. Then $f \equiv 0$. ☐

(5.6) Remark. For $f \in L^2(\mathbb{R})$ and $P_{V_{j+1}}(f)$ the orthogonal projection of f onto V_{j+1}, so

$$P_{V_{j+1}}(f) = P_{V_j}(f) + \sum_{m\in\mathbb{Z}} \langle f, \psi_{j,m}\rangle\, \psi_{j,m}.$$

(5.7) Example. We study how wavelets react to discontinuities. Let $f \in L^2(\mathbb{R})$, $x_0 \in (0,1)$, $f \in C^2(-\infty, x_0) \cup C^2(x_0, \infty)$, and f is discontinuous in x_0. Consider $(\langle f, \psi_{j,m}\rangle)_{j,m}$ with the Haar-wavelet ψ.

First, it is clear, that for $x_0 \notin I_{j,m}$ we have $|\langle f, \psi_{j,m}\rangle| = \mathcal{O}(2^{-\frac{3}{2}j}) \to 0$ as $j \to \infty$. For that use the taylor expansion of f in the middle point $x_{j,m}$ of $I_{j,m}$:

$$f(x) = f(x_{j,m}) + f'(x_{j,m})(x - x_{j,m} + \frac{1}{2}f''(\xi)(x - x_{j,m})^2.$$

If now $x_0 \in I_{j,m}$, we have

$$|\langle f, \psi_{j,m}\rangle| \approx \frac{1}{2} \cdot 2^{-\frac{j}{2}}|f(x_0^-) - f(x_0^+)|$$

Where $f(x_0^-)$ and $f(x_0^+)$ are the limits of $f(t)$ for $t \to x$ from below and above respectively.

Thus the discontinuity in x_0 can be detected by the rate of decay of the wavelet coefficients as $j \to \infty$.

5.2 Multiresolution Analysis

(5.8) Definition. A family $(e_k)_{k\in\mathbb{N}}$ is a **Riesz basis** of a Hilbert space H, if it spans H and if there exist $0 < c_1 \le c_2 < \infty$ such that for all finitely supportet sequences $(x_k)_{k\in\mathbb{N}}$ of coefficients we have

$$c_1 \sum_{k\in\mathbb{N}} |x_k|^2 \le \left\| \sum_{k\in\mathbb{N}} x_k e_k \right\|_H^2 \le C_2 \sum_{k\in\mathbb{N}} |x_k|^2.$$

(5.9) Lemma. *Let $(e_k)_{k\in\mathbb{N}}$ be a Riesz basis of H.*

(i) *$(e_k)_{k\in\mathbb{N}}$ is a frame for H.*

(ii) *The associated synthesis operator $T^* : l^2(\mathbb{N}) \to H, (x_k)_{k\in\mathbb{N}} \mapsto \sum_{k\in\mathbb{N}} x_k e_k$ is an isomorphism.*

(iii) *The series $\sum_{k\in\mathbb{N}} x_k e_k$ converges unconditionally in L^2, i.e. permutation of coefficients does not affect convergence, if and only if $(x_k) \in l^2(\mathbb{N})$.*

(iv) *For each $x \in H$, there exists a unique expansion $x = \sum_{k\in\mathbb{N}} x_k e_k$.*

(v) *There exists a unique biorthogonal Riesz basis $(e_k)_{k\in\mathbb{N}}$ such that $\langle e_k, e_l\rangle = \delta_{kl}$ and the coefficients of x in the basis are $\langle x, \widetilde{e}_k\rangle$, $k \in \mathbb{N}$, i.e.*

$$x = \sum_{k\in\mathbb{N}} \langle x, \widetilde{e}_k\rangle\, e_k = \sum_{k\in\mathbb{N}} \langle x, e_k\rangle\, \widetilde{e}_k \quad \forall\, x \in H.$$

(vi) *(e_k) is an orthonormal basis if $\widetilde{e}_k = e_k$ for all $k \in \mathbb{N}$ and $c_1 = c_2 = 1$.*

(5.10) Definition. A **multiresolution analysis** (MRA) is a sequence of closed subspaces $V_j \subset L^2(\mathbb{R})$, $j \in \mathbb{Z}$, such that

(i) $\{0\} \subset \ldots \subset V_{-1} \subset V_0 \subset V_1 \subset \ldots \subset L^2(\mathbb{R})$,

(ii) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$ or equivalently $\lim_{j\to\infty} \|f - P_{V_j} f\| = 0$ for all $f \in L^2(\mathbb{R})$,

(iii) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$ or equivalenly $\lim_{j\to-\infty} \|P_{V_j} f\| = 0$ for all $f \in L^2(\mathbb{R})$,

(iv) $f \in V_j$ if and only if $f(2^{-j}\cdot) \in V_0$ for all $j \in \mathbb{Z}$,

(v) There exists a scaling function $\varphi \in L^2(\mathbb{R})$ such that $\{T_m\varphi =: \varphi_{0,m} : m \in \mathbb{Z}\}$ is a Riesz basis for V_0.

(5.11) Remark. (1) $\{\varphi_{j,m} : m \in \mathbb{Z}\}$ is a Riesz basis for V_j, $j \in \mathbb{Z}$ (this follows from (iv) and (v)).

(2) V_0 is a **shift-invariant space**, i.e. $f \in V_0$ if and only if $T_m f \in V_0$, $m \in \mathbb{Z}$.[4]

(5.12) Lemma. *Let φ be a scaling function with respect to a MRA. Then φ satisfies a* ***scaling equation*** *or refinement equation, i.e. there exists $(h_m)_{m\in\mathbb{Z}} \in l^2(\mathbb{Z})$ with*

$$\varphi(x) = \sum_{m\in\mathbb{Z}} h_m \varphi(2x - m), \qquad (x \in \mathbb{R}).$$

Proof. Let $\varphi \in V_0 \subset V_1$. By (iv) we get $\varphi(2^{-1}\cdot) \in V_0$. Hence $\varphi(2^{-1}\cdot) \in \overline{\operatorname{span}\{\varphi(\cdot - m) : m \in \mathbb{Z}\}}$. ☐

(5.13) Example. (1) $\varphi = \chi_{[0,1)}$. Then $\varphi(x) = \varphi(2x) + \varphi(2x - 1)$. We can choose $h_0 = h_1 = 1$ and $h_m = 0$ for $m \ne 0, 1$.

(2) The space of band-limited functions

$$V_j := \{f \in L^2(\mathbb{R}) : \operatorname{supp}(\mathcal{F}f) \subset [-2^j\pi, 2^j\pi]\}$$

constitues an MRA. Proving (i)-(iv) is easy. For $\mathcal{F}V_0$ we have the orthonormal basis $e_n(x) = (2\pi)^{-\frac{1}{2}} e^{inx}\chi_{[-\pi,\pi]}$. Thus $\{(\mathcal{F}^{-1}e_n) = (T_n\varphi) : n \in \mathbb{Z}\}$ with $\varphi(x) = \frac{\sin(\pi x)}{\pi x}$.

(5.14) Definition. (i) A scaling function $\varphi \in L^2(\mathbb{R}) \cap C_c(\mathbb{R})$ is called **interpolatory**, if $\varphi(k) = \delta_{0,k}$ for all $k \in \mathbb{Z}$.

[4] $f = \sum_{m\in\mathbb{Z}} c_m T_m \varphi$, thus $T_l f = \sum_{m\in\mathbb{Z}} c_m T_l T_m \varphi = \sum_{k\in\mathbb{Z}} c_{k-l} T_k \varphi$.

(ii) It is called **orthonormal**, if $\{T_m\varphi : m \in \mathbb{Z}\}$ is an orthonormal system.

(5.15) Proposition. *Let $\varphi \in L^2(\mathbb{R}) \cap C_c(\mathbb{R})$ be an interpolatory scaling function. Then*

$$f = \sum_m f(2^{-j}m)\varphi(2^j \cdot -m) = \sum_m 2^{-\frac{j}{2}} f(2^{-j}m)\varphi_{j,m}$$

for all $j \in \mathbb{Z}$ and $f \in V_j$.

Proof. We have $f = \sum_m c_m \varphi_{j,m}$. But

$$f(2^{-j}l) = \sum_m c_m 2^{\frac{j}{2}} \varphi\left(2^j(2^{-j}l) - m\right) = c_l 2^{\frac{j}{2}}.$$

Solving for c_l shows the claim. □

(5.16) Example (5.13 (2)). The scaling function $\varphi(x) = \frac{\sin(\pi x)}{\pi x}$ is interpolatory. By Proposition 5.15 we do have (with $j = 1$)

$$\varphi(x) = \sum_{m\in\mathbb{Z}} \frac{2\sin\left(\frac{\pi m}{2}\right)}{\pi m}\varphi(2x - m), \qquad x \in \mathbb{R}.$$

This is the scaling equation for φ.

(5.17) Remark. There are several ways to construct an MRA:

(1) Start with $(V_j)_{j\in\mathbb{Z}}$. Checking (i)-(iv) is often easy. Finding φ is much more difficult.

(2) Start with φ and define $V_0 = \overline{\operatorname{span}\{T_m\varphi : m \in \mathbb{Z}\}}$ and V_j accordingly. Then (i)-(iii) need to be checked and the Riesz property.

(3) Start with a scaling equation. Construct φ accordingly, as well as $(V_j)_{j\in\mathbb{Z}}$.

(5.18) Theorem. *(i) Let $\varphi \in L^2(\mathbb{R})$. Then*

$$\sum_{m\in\mathbb{Z}} |\mathcal{F}\varphi(\xi + 2\pi m)|^2, \qquad \xi \in \mathbb{R}$$

converges in $L^1(I)$ for any compact I to a 2π-periodic function in L^1_{loc}.

(ii) $\varphi \in L^2$ is L^2-stable[5], i.e. $\{T_m\varphi : m \in \mathbb{Z}\}$ form a Riesz basis for their L^2-span, if and only if

$$0 < C_1 \le \sum_{m\in\mathbb{Z}} |\mathcal{F}\varphi(\xi + 2\pi m)|^2 \le C_2 < \infty, \qquad \text{for a.e. } \xi.$$

(iii) φ is an orthonormal scaling function if and only if

$$\sum_{m\in\mathbb{Z}} |\mathcal{F}\varphi(\xi + 2\pi m)|^2 = 1 \qquad \text{for a.e. } \xi.$$

(iv) If, for some $\varepsilon > 0$, $|\mathcal{F}\varphi(\xi)| \lesssim (1 + |\xi|)^{-1-\varepsilon}$, then φ is interpolatory if and only if

$$\sum_m \mathcal{F}\varphi(\xi + 2\pi m) = 1 \qquad \text{for a.e. } \xi.$$

Proof. (i): We have $\mathcal{F}\varphi \in L^2(\mathbb{R})$. Hence

$$S_\varphi(\xi) := \sum_{m\in\mathbb{Z}} |\mathcal{F}\varphi(\xi + 2\pi m)|^2$$

convergens in $L^2([-k\pi, k\pi])$ for all $k \in \mathbb{Z}$.

[5] It is then called **Riesz sequence.**

(ii): Let $f = \sum_{|m|<N} c_m \varphi(\cdot - m)$. Then

$$\begin{aligned}
\|f\|^2 = \frac{1}{2\pi}\|\mathcal{F}f\|^2 &= \frac{1}{2\pi}\int_{\mathbb{R}} \left| \sum_{|m|<N} c_m e^{im\xi} \mathcal{F}\varphi(\xi) \right|^2 \mathrm{d}\xi \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{k\in\mathbb{Z}} \left| \sum_{|m|<N} c_m e^{-im(\xi+2\pi k)} \varphi(\xi+2\pi k) \right|^2 \mathrm{d}\xi \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{\left| \sum_{|m|<N} c_m e^{-im\xi} \right|^2}_{=:h(\xi)} S_\varphi(\xi)\,\mathrm{d}\xi.
\end{aligned}$$

Also

$$\sum_{|m|<N} |c_m|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |h(\xi)|^2\,\mathrm{d}\xi.$$

By a density argument, (ii) folows

(iii): We have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} S_\varphi(\xi) e^{-ik\xi}\,\mathrm{d}\xi = \frac{1}{2\pi}\int_{\mathbb{R}} |\mathcal{F}\varphi(\xi)|^2 e^{-ik\xi}\,\mathrm{d}\xi = \langle \varphi, T_k\varphi\rangle = \frac{1}{2\pi}\int_{-pi}^{\pi} e^{-ik\pi}\,\mathrm{d}\xi = \delta_{0,k}.$$

Thus we get orthonormality if and only if $S_\varphi \equiv 1$.

(iv): Set $R_\varphi(x) = \sum_m \mathcal{F}\varphi(\xi + 2\pi m)$. We have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} R_\varphi(\xi) e^{-ik\xi}\,\mathrm{d}\xi = \frac{1}{2\pi}\int_{\mathbb{R}} \mathcal{F}\varphi(\xi) e^{-ik\xi}\,\mathrm{d}\xi = \varphi(-k) \overset{!}{=} \delta_{0,k}.$$

As in (iii), we have $\varphi(k) = \delta_{0,k}$ for all $k \in \mathbb{Z}$ if and only if $R_\varphi \equiv 1$. □

(5.19) Remark. As we saw, sometimes the Fourier-viewpoint is better. We hence also consider $\mathcal{F}V_0$: For every $f(x) = \sum_m c_m \varphi(x-m) \in V_0$, then

$$\mathcal{F}f(\xi) = \left[\sum_m c_m e^{-i\xi m}\right] \mathcal{F}\varphi(\xi), \xi \in \mathbb{R}.$$

This shows that

$$\mathcal{F}V_0 = \{h \cdot \mathcal{F}\varphi : h \text{ is } 2\pi\text{-periodic and in } L^2([0,2\pi])\}.$$

(5.20) Corollary. *Let φ be a scaling function.*

(i) The function φ^o defined by

$$\mathcal{F}\varphi^o(\xi) = (S_\varphi(\xi))^{-\frac{1}{2}} \mathcal{F}\varphi(\xi)$$

is in V_0 and orthonormal.

(ii) The function φ^d defined by

$$\mathcal{F}\varphi^d(\xi) = (S_\varphi(\xi))^{-1} \mathcal{F}\varphi(\xi)$$

is in V_0 and generates a dual Riesz basis, that is $\langle T_k\varphi, T_k\varphi^d\rangle = \delta_{k,l}$ for all $k,l \in \mathbb{Z}$.

(iii) If $\varphi \in L^2$, $|\mathcal{F}\varphi(\xi)| \lesssim (1+|\xi|)^{-1-\varepsilon}$, $\varepsilon > 0$, and $R_\varphi(\xi) \geq c > 0$, $\xi \in \mathbb{R}$, then the function φ^i defined by

$$\mathcal{F}\varphi^i(\xi) = (R_\varphi(\xi))^{-1} \mathcal{F}\varphi(\xi)$$

is in V_0 and interpolary

Proof. By Theorem 5.18, S_φ and R_φ are bounded from below. By Remark 5.19 we have $\varphi^o, \varphi^d, \varphi^i \in V_0$. Using Theorem 5.18, (i) and (iii) can be directly proven.

(ii): Consider $U_{\varphi,\varphi^d}(\xi) = \sum_m \overline{\mathcal{F}\varphi}\mathcal{F}\varphi^d(\xi + 2\pi m)$. Then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} U_{\varphi,\varphi^d}(\xi)e^{-ik\xi}\,\mathrm{d}\xi = \ldots = \left\langle T_k\varphi^d, \varphi\right\rangle.$$

By definition of φ^d, we get $\left\langle T_k\varphi^d, \varphi\right\rangle = \delta_{0,k}$. □

(5.21) Definition. Let $\varphi \in L^2(\mathbb{R})$ satisfy

$$\varphi(x) = \sum_{m\in\mathbb{Z}} h_m\varphi(2x - m)$$

for $(h_m) \in l^2(\mathbb{Z})$. Then φ is often called **refinable function** and the equation **refinement equation**. Then the function

$$m(\xi) = \frac{1}{2}\sum_{n\in\mathbb{Z}} h_n e^{-in\xi}$$

is called **symbol** of φ.

(5.22) Lemma. *Let φ be a refinable function and m its symbol. Then*

$$\mathcal{F}\varphi(\xi) = m\left(\frac{\xi}{2}\right)\mathcal{F}\varphi\left(\frac{\xi}{2}\right), \qquad \xi \in \mathbb{R}.$$

Also, if φ is orthonormal, then $|m(\xi)|^2 + |m(\xi+\pi)|^2 = 1$ a.e.

Proof. By Theorem 5.18, we have

$$\begin{aligned}
1 = \sum_{l\in\mathbb{Z}} |\mathcal{F}\varphi(\xi + 2\pi l)|^2 &= \sum_{l\in\mathbb{Z}} \left|m\left(\frac{\xi}{2} + \pi l\right)\right|^2 \left|\mathcal{F}\varphi\left(\frac{\xi}{2} + \pi l\right)\right|^2 \\
&= \sum_{k\in\mathbb{Z}} \left|m\left(\frac{\xi}{2} + 2\pi k\right)\right|^2 \left|\mathcal{F}\varphi\left(\frac{\xi}{2} + 2\pi k\right)\right|^2 + \sum_{k\in\mathbb{Z}} \left|m\left(\frac{\xi}{2} + \pi + 2\pi k\right)\right|^2 \left|\mathcal{F}\varphi\left(\frac{\xi}{2} + \pi + 2\pi k\right)\right|^2 \\
&= \left|m\left(\frac{\xi}{2}\right)\right|^2 \underbrace{\sum_k \left|\mathcal{F}\varphi\left(\frac{\xi}{2} + 2\pi k\right)\right|^2}_{=1} + \left|m\left(\frac{\xi}{2} + \pi\right)\right|^2 \underbrace{\sum_k \left|\mathcal{F}\varphi\left(\frac{\xi}{2} + \pi + 2\pi k\right)\right|^2}_{=1}.
\end{aligned}$$

□

(5.23) Remark. Let's assume we have a refinable function φ and a symbol m. This would make a good candidate for an MRA. The following questions arise:

(1) How to get such a φ?

(2) Which properties of m imply which properties of φ?

These questions (in particular the first one) are related to so-called subdivision schemes.

Consider $f(x) = \sum_{y\sim x} h_y f(y)$. Then $(h_y)_y$ are functional analytic properties of the limit function $f : \mathbb{R} \to \mathbb{R}$. φ can be constructed as limit of such a subdevision scheme.

(5.24) Theorem. *Let $\varphi \in L^2(\mathbb{R})$ with:*

(a) $\{T_m\varphi : m \in \mathbb{Z}\}$ form a Riesz sequence in $L^2(\mathbb{R})$,

(b) $\varphi(x) = \sum_m h_m\varphi(2x - m)$ converges in $L^2(\mathbb{R})$[6],

(c) $\mathcal{F}\varphi$ is continuous in 0 and $\mathcal{F}\varphi(0) \neq 0$.

Then $V_j = \overline{\operatorname{span}\{\varphi_{j,m} : m \in \mathbb{Z}\}}$, $j \in Z$, form a MRA.

(5.25) Proposition. *In this situation, we do have $\bigcup_j V_j = \{0\}$.*

[6] The mode of convergence is $\|\varphi - \sum_{|m|\leq N} h_m\varphi(2\cdot - m)\|_{L^2} \to 0$ as $N \to 0$

Proof. It suffices to prove

$$\lim_{j\to-\infty}\|P_{V_j}g\| = 0 \text{ for } g\in L^2(\mathbb{R}) \text{ with } \operatorname{supp} g\subset[-R,R].$$

We have

$$\begin{aligned}
\|P_{V_j}g\|_2^2 &\le C\cdot\sum_m \left|\left\langle P_{V_j}g,\varphi_{j,m}\right\rangle\right|^2 = C\cdot\sum_m |\langle g,\varphi_{j,m}\rangle|^2\\
&= C\cdot\sum_m \left|\int_{-R}^{R} g(x)2^{\frac{j}{2}}\overline{\varphi(2^jx-m)}\,\mathrm{d}x\right|^2 \le C\cdot\sum_m\int_{-R}^{R}|g(x)|^2\,\mathrm{d}x\cdot\int_{-R}^{R}2^j|\varphi(2^jx-m)|^2\,\mathrm{d}x\\
&= C\cdot\|g\|_2^2\cdot\sum_m\int_{-2^jR-m}^{2^jR-m}|\varphi(y)|^2\,\mathrm{d}y
\end{aligned}$$

There exists $j\in\mathbb{Z}$ with $2^jR<\frac{1}{2}$ for all $j\le j_0$. For all $j\le j_0$ we have

$$[-2^jR-m,2^jR-m]\cap[-2^jR-m',2_jR-m']=\emptyset$$

for all $m\neq m'$.
Thus $\|P_{V_i}f\|_2^2\le C\cdot\|g\|_2^2\cdot\int_{U_j}|\varphi(y)|^2\,\mathrm{d}y$, $U_j=\bigcup_m[-2^jR-m,2^jR-m]$ disjointly.
By dominated convergence theorem we get

$$\int_{U_j}|\varphi(y)|^2\,\mathrm{d}y \xrightarrow{j\to-\infty} 0.$$

Thus $\|P_{V_j}g\|_2\to 0$ as $j\to-\infty$. ☐

(5.26) Proposition. *In this situation, we do have* $\overline{\bigcup_j V_j}=L^2(\mathbb{R})$.

Proof. Let $f\perp\bigcup_j V_j$. Further, let $\varepsilon>0$, and let $g\in L^2(\mathbb{R})$ with $\mathcal{F}g=\mathcal{F}f\cdot\chi_{[-R,R]}$ such that $\|f-g\|_2<\varepsilon$. We have $\|P_{V_j}f\|_2=0$ for all $j\in\mathbb{Z}$, hence $\|P_{V_j}g\|_2<\varepsilon$ for all $j\in\mathbb{Z}$. Thus we have

$$\begin{aligned}
\|P_{V_j}g\|_2^2 &\ge C\cdot\sum_m\left|\left\langle P_{V_j}g,\varphi_{j,m}\right\rangle\right|^2 = C'\sum_m|\langle\mathcal{F}g,\mathcal{F}\varphi_{j,m}\rangle|^2\\
&= C'\sum_m\left|\int_{-R}^{\mathbb{R}}\mathcal{F}g(\xi)2^{-\frac{j}{2}}e^{im2^{-j}\xi}\overline{\mathcal{F}\varphi(2^{-j}\xi)}\,\mathrm{d}\xi\right|^2
\end{aligned}$$

Assume now j to be such that $2^j\pi>R$. Then

$$\|P_{V_j}g\|_2^2\le C'\sum_m\left|\int_{-2^j\pi}^{2^j\pi}\mathcal{F}g(\xi)\underbrace{2^{-\frac{j}{2}}e^{im2^{-j}\xi}}_{\text{ONB for}L^2[-2^j\pi,2^j\pi]}\overline{\mathcal{F}\varphi(2^{-j}\xi)}\,\mathrm{d}\xi\right|^2 = C\int_{-2^j\pi}^{2^j\pi}\left|\mathcal{F}g(\xi)\overline{\mathcal{F}\varphi(2^{-j}\xi)}\right|^2\mathrm{d}\xi$$

Thus $|\mathcal{F}\varphi(2^{-j}\xi)|^2\to|\mathcal{F}\varphi(0)|^2$ uniformly on $[-R,R]$ as $j\to\infty$. Thus

$$\varepsilon^2>\|P_{V_j}g\|^2\ge C'\int_{-R}^{R}|\}(\xi)|^2\cdot|\mathcal{F}\varphi(0)|^2\,\mathrm{d}\xi = C''\cdot\|g\|_2^2|\mathcal{F}\varphi(0)|^2.$$

Thus $\|f\|_0=0$. ☐

Proof of Theorem 5.25. Follows from Proposition 5.26 and 5.27 and some further calculations. ☐

(5.27) Proposition. *Let φ be a scaling function of an MRA. Also, let $\mathcal{F}\varphi$ be continuous in 0. Then $|\mathcal{F}\varphi(0)| = 1$. In particular,*

$$\left| \int_{\mathbb{R}} \varphi(x)\,\mathrm{d}x \right| = 1.$$

Proof. Let $g \in L^2(\mathbb{R})$ with $g \neq 0$ and $\operatorname{supp}(\mathcal{F}g) \subset [-1, 1]$. Then

$$\|P_{V_j} g\|_2^2 = \sum_{m\in\mathbb{Z}} |\langle g, \varphi_{j,m}\rangle|^2 \underset{\text{before}}{\overset{\text{as}}{=}} \frac{1}{2\pi} \int_{-1}^{1} \left| \mathcal{F}g(\xi)\overline{\mathcal{F}\varphi\left(2^{-j}\xi\right)}\right|^2 \mathrm{d}\xi \xrightarrow{j\to\infty} \frac{1}{2\pi}\|\mathcal{F}g\|_2^2 \cdot |\mathcal{F}\varphi(0)|^2 = \|g\|_2^2 |\mathcal{F}\varphi(0)|^2.$$

On the other hand we have $\|P_{V_j} g\|_2^2 \xrightarrow{j\to\infty} \|g\|_2^2$, since (V_j) forms a MRA. Thus $|\mathcal{F}\varphi(0)| = 1$. □

(5.28) Corollary. *Let $\varphi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ be a orthonormal scaling function.*

(i) $|\mathcal{F}\varphi(2\pi l)| = 0$ for all $l \in \mathbb{Z} \setminus \{0\}$.

(ii) $\sum_{m\in\mathbb{Z}} \varphi(x + m) = \alpha$ for a.e. x, $|\alpha| = 1$.

Proof. (i): By Proposition 5.27 we get $|\mathcal{F}\varphi(0)| = 1$. We have $\sum_{k\in\mathbb{Z}} |\mathcal{F}\varphi(\xi + 2\pi k)|^2 = 1$. With $\xi = 0$ and $|\mathcal{F}\varphi(0)| = 1$ we see that $|\mathcal{F}\varphi(2\pi k)|$ must already be zero for all $k \in \mathbb{Z} \setminus \{0\}$.

(ii) follows directly from the Poisson summation formula. □

5.3 Wavelets come into play

(5.29) Definition. Let $(V_j)_{j\in\mathbb{Z}} \subset L^2(\mathbb{R})$ be an MRA. The associated **wavelet spaes** $(W_j)_{j\in\mathbb{Z}}$ are definied by $W_j \perp V_j$ and $V_{j+1} = V_j \oplus W_j$, $j \in \mathbb{Z}$.

(5.30) Remark. (1) We now have the following decomposition:

$$V_{j+1} = V_j \oplus W_j = (V_{j-1} \oplus W_{j-1}) \oplus W_j = \dots.$$

Hence $V_{j+1} = \bigoplus_{k\le j} W_k$.[7] In particular $L^2(\mathbb{R}) = \bigoplus_{j\in\mathbb{Z}} W_j$.

(2) The wavelet spaces inherit property (iv) from the MRA, i.e. $f \in W_j$ if and only if $f(2^{-j}\cdot) \in W_0$ for all $j \in \mathbb{Z}$.

(3) We can use the decomposition from (1) for decomposing functions $f \in L^2(\mathbb{R})$; more precisely a projected version $P_{V_j} f$. Then

$$P_{V_j} f = P_{V_{j-1}} f + P_{W_{j-1}} f = \dots = P_{V_{j_0}} f + \sum_{k=j_0}^{j-1} P_{W_k} f.$$

(5.31) Proposition. *Let $f \in L^2(\mathbb{R})$. Then the following are equivalent*

(i) $f \in W_0$ (for φ orthonormal)

(ii) $\mathcal{F}f(\xi) = e^{-i\frac{\xi}{2}} v(\xi) \overline{m_\varphi\left(\frac{\xi}{2} + \pi\right)} \mathcal{F}\varphi\left(\frac{\xi}{2}\right)$, where m_φ is the symbol of φ and v is a 2π-periodic function.

In particular, we have $\|f\|_2^2 = \frac{1}{2\pi}\int_0^{2\pi} |v(\xi)|^2\,\mathrm{d}\xi$.

Proof. We have $f \in W_0$ if and only if $f \in V_1$ and $f \perp V_0$, i.e. if $\langle f, \varphi_{0,m}\rangle\, 0$ for all $m \in \mathbb{Z}$. First, we note

$$\langle f, \varphi_{0,m}\rangle = \frac{1}{2\pi} \int_{\mathbb{R}} \mathcal{F}f(\xi) \cdot e^{im\xi} \cdot \overline{\mathcal{F}\varphi(\xi)}\,\mathrm{d}\xi = \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{k\in\mathbb{Z}} \mathcal{F}f(\xi + 2\pi k)\overline{\mathcal{F}\varphi(\xi + 2\pi k)} \right) e^{im\xi}\,\mathrm{d}\xi.$$

[7] We set symbolically $V_{-\infty} = \{0\}$.

Hence, $f \in W_0$ if and only if $f \in V_0$ and

$$\sum_{k\in\mathbb{Z}} \mathcal{F}f(\xi + 2\pi k)\overline{\mathcal{F}\varphi(x + 2\pi k)} = 0, \qquad \text{for a.e. } \xi.$$

Recall, that $\mathcal{F}\varphi(\xi) = m_\varphi\left(\frac{\xi}{2}\right) \cdot \mathcal{F}\varphi\left(\frac{\xi}{2}\right)$ and also $\mathcal{F}f(\xi) = m_f\left(\frac{\xi}{2}\right)\mathcal{F}\varphi\left(\frac{\xi}{2}\right)$ for $f \in V_1$.

Then

$$\begin{aligned}
0 &= \sum_k \mathcal{F}f(\xi + 2\pi x)\overline{\mathcal{F}\varphi(\xi + 2\pi k)} \\
&= \sum_k m_f\left(\frac{\xi}{2} + \pi k\right)\mathcal{F}\varphi\left(\frac{\xi}{2} + \pi x\right)\overline{m_\varphi\left(\frac{\xi}{2} + \pi k\right)\mathcal{F}\varphi\left(\frac{\xi}{2} + \pi k\right)} \\
&\overset{\eta:=\frac{\xi}{2}}{=} \sum_k m_f\left(\eta + 2\pi k\right)\overline{m_\varphi\left(\eta + 2\pi k\right)}\left|\mathcal{F}\varphi\left(\eta + 2\pi k\right)\right|^2 \\
&\qquad + \sum_k m_f\left(\eta + \pi + 2\pi k\right)\overline{m_\varphi\left(\eta + \pi + 2\pi k\right)}\left|\mathcal{F}\varphi\left(\eta + \pi + 2\pi k\right)\right|^2 \\
&\overset{8}{=} m_f(\eta)\overline{m_\varphi(\eta)} + m_f(\eta + \pi)\overline{m_\varphi(\eta + \pi)}.
\end{aligned}$$

Thus $(m_f(\eta), m_f(\eta+\pi) \perp \left(\overline{m_\varphi(\eta)}, \overline{m_\varphi(\eta+\pi)}\right)$ in $\mathbb{R}^2$ for a.e. η. We have $(m_\varphi(\eta), m_\varphi(\eta+\pi)) \neq 0$, since $|m_\varphi(eta)|^2 + |m_\varphi(\eta+\pi)|^2 = 1$.

Let α be a 2π-periodic complex valued function such that

$$(m_f(\eta), m_f(\eta + \pi)) = \alpha(\eta) \cdot \left(\overline{m_\varphi(\eta + \pi)}, \overline{-m_\varphi(\eta)}\right).$$

By replacing η by $\eta + \pi$ and 2π-periodicity, we get

$$(m_f(\eta + \pi), m_f(\eta)) = \alpha(\eta + \pi) \cdot \left(\overline{m_\varphi(\eta)}, \overline{-m_\varphi(\eta + \pi)}\right).$$

By these two relations, we get $m_f(\eta) = \alpha(\eta)\overline{m_\varphi(\eta + \pi)}$ and $\alpha(\eta) = -\alpha(\eta + \pi)$.

Then $f \in W_0$ if and only if $\mathcal{F}f(\xi) = m_f\left(\frac{\xi}{2}\right)\mathcal{F}\varphi\left(\frac{\xi}{2}\right)$, with $m_f(\eta) = \alpha(\eta) \cdot \overline{m_\varphi(\eta + \pi)}$ and α is a 2π-periodic function with $\alpha(\eta) = -\alpha(\eta + \pi)$, which is the same as requireing $h(\eta) = e^{-i\eta}\alpha(\eta)$ to be a π-periodic function. Write $v(\xi) := h\left(\frac{\xi}{2}\right)$. Then

$$f \in W_0 \quad \Leftrightarrow \quad \mathcal{F}f(\xi) = \alpha\left(\frac{\xi}{2}\right)\overline{m_\varphi\left(\frac{\xi}{2} + \pi\right)}\mathcal{F}\varphi\left(\frac{\xi}{2}\right) = e^{i\frac{\xi}{2}}h\left(\frac{\xi}{2}\right) = e^{i\frac{\xi}{2}}v(\xi). \qquad \square$$

(5.32) Lemma. *Let φ be orthonormal and let $f \in W_0$ and v, m_φ be as in Proposition 5.31. Then the following are equivalent:*

(i) $\{f_{0,m} : m \in \mathbb{Z}\}$ *is an ONB of* W_0.

(ii) $|v(\xi)| = 1$ *a.e.*

[8] 2π-periodicity and orthonormality of φ.

Proof. By Proposition 5.31,

$$\begin{aligned}
\sum_k |\mathcal{F}f(\xi+2\pi k)|^2 &= \sum_k |v(\xi)|^2 \cdot \left| m_\varphi\left(\frac{\xi}{2}+\pi k+\pi\right)\right|^2 \cdot \left|\mathcal{F}\varphi\left(\frac{\xi}{2}+\pi k\right)\right|^2 \\
&= |v(\xi)|^2 \Bigg(\left| m_\varphi\left(\frac{\xi}{2}+\pi\right)\right|^2 \underbrace{\sum_k \left|\mathcal{F}\varphi\left(\frac{\xi}{2}+2\pi k\right)\right|^2}_{=1} \\
&\qquad + \left| m_\varphi\left(\frac{\xi}{2}\right)\right|^2 \underbrace{\sum_k \left|\mathcal{F}\varphi\left(\frac{\xi}{2}+\pi+2\pi k\right)\right|^2}_{=1} \Bigg) \\
&= |v(\xi)|^2 \underbrace{\left(\left| m_\varphi\left(\frac{\xi}{2}+\pi\right)\right|^2 + \left| m_\varphi\left(\frac{\xi}{2}\right)\right|^2 \right)}_{=1} = |v(\xi)|^2.
\end{aligned}$$

Thus to really have a orthonormal basis, we need $|v(\xi)|^2 = 1$ a.e. ☐

(5.33) Theorem. *Let $(V_j)_{j\in\mathbb{Z}}$ be an MRA with an orthonormal scaling function φ. Then the following are equivalent:*

(i) $\{\varphi_{0,m} : m \in \mathbb{Z}\}$ is an ONB of W_0 for some $\psi \in W_0$.

(ii) $\mathcal{F}\psi(\xi) = e^{i\frac{\xi}{2}} v(\xi) \cdot \overline{m_\varphi\left(\frac{\xi}{2}+\pi\right)} \cdot \mathcal{F}\varphi\left(\frac{\xi}{2}\right)$ with $|v(\xi)| = 1$ a.e. and v is a 2π-periodic function.

Proof. Follows from Proposition 5.31 and Lemma 5.32. ☐

(5.34) Remark. An obvious choice for v is $v \equiv 1$. Then $\mathcal{F}\psi(\xi) = e^{i\frac{\xi}{2}} \overline{m_\varphi\left(\frac{\xi}{2}+\pi\right)} \mathcal{F}\phi\left(\frac{\xi}{2}\right)$. We write

$$\begin{aligned}
m_\varphi(\xi) &= \frac{1}{2}\sum_k h_k e^{-ik\xi} \\
\mathcal{F}\psi(\xi) &= \sum_k \overline{h}_k (-1)^k \cdot \frac{1}{2} \cdot e^{i(k+1)\frac{\xi}{2}} \cdot \mathcal{F}\varphi\left(\frac{\xi}{2}\right).^{9}
\end{aligned}$$

This can be used, to obtain the Haar-wavelet from the Haar-scaling function $\varphi = \chi_{[0,1)}$.

6 Important classes of wavelets

(6.1) Remark (Crucial Properties of Wavelets). Some cruicial properties of Wavelets are

(1) decay,

(2) smoothness,

(3) vanishing moments.

Intuitive Reasoning:

(1) For $(\varphi_{j,m})_{j,m}$ the scalar product $\langle \varphi_{j,m}, \varphi_{j,m'} \rangle$ shoulds be small, if $m - m'$ is large

(2) Testing the PDE $\Delta u = f$ with wavelets gives $\langle \Delta, \varphi_\lambda \rangle = \langle f, \varphi_\lambda \rangle$, $\lambda \in \Lambda$, which translates (by using a Wavelet system as Galerkin scheme) to

$$\sum_{i=1}^d u_i \langle \Delta\varphi_i, \varphi_\lambda \rangle = \langle f, \varphi_\lambda \rangle, \qquad (\lambda \in \Lambda).$$

This, we can interprete as $\Phi u = b$. (Here smoothness comes into play)

[9] Note that then $\psi(x) = \sum_k \overline{h}_k (-1)^k \cdot \frac{1}{2}\varphi(2x + (k+1))$.

(3) For detecting point discontinuities, the property $\int \varphi = 0$ was useful. Vanishing moments require the wavelet to be orthogonal to higher order polynomials, i.e.

$$\int x^r \varphi = 0, \quad r \in \{0, 1, \dots, n\}.$$

Then $\langle f, \varphi_{j,m} \rangle$ decays fast for $j \to \infty$, if f is smooth.

(6.2) Definition. A wavelet $\varphi \in L^2(\mathbb{R})$ has k**-vanishing moments**, if

$$\int_{\mathbb{R}} x^r \varphi(x) = 0, \qquad \text{for all } r \in \{0, \dots, k\}.$$

(6.3) Proposition. *Let $\psi \in L^2(\mathbb{R})$ be a wavelet such that $(\varphi_{j,m})_{j,m}$ is an ONB for $L^2(\mathbb{R})$. Fruther suppose, that there exists $k > 0$ such that*

(i) $\varphi \in C^k(\mathbb{R})$,

(ii) $\varphi^{(r)}$ *is bounded for all* $r \in \{0, \dots, k\}$,

(iii) there exists $\alpha > k + 1$ and a constant $c \in \mathbb{R}$ such that for a.e. $x \in \mathbb{R}$ we have

$$|\psi(x)| \leq \frac{C}{(1 + |x|)^\alpha}.$$

Proof. Towards a contradiction assume s is the smallest integer in $\{1, \dots, k\}$ such that

$$\int_{\mathbb{R}} x_s \psi(x) \, \mathrm{d}x \neq 0.$$

By (iii) we know, that ψ is not polynomial.[10] Hence $\psi^{(s)} \not\equiv 0$. Let $m, J \in \mathbb{Z}$ be such thath $\psi^{(s)}(m2^{-J}) \neq 0$. We abbreviate $a := m2^{-J}$ and use Taylor

$$\psi(x) = \sum_{n=0}^{s} c_n (x - a)^n + R(x).$$

where for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|x - a| < \delta \quad \Rightarrow \quad |R(x)| \leq \varepsilon |x - a|^s.$$

Also we have $|R(x)| \leq c|x - a|^s$ for all $x \in \mathbb{R}$. Since $\psi^{(s)}(a) \neq 0$ and by the Taylor expansion we get $c_s \neq 0$. For $j > J$, let $m_j := 2^j a = 2^{j-J} m \in \mathbb{Z}$. By the orthogonality of $(\psi_{j,m})_{j,m}$ we get $\langle \psi, \psi_{j,m_j} \rangle = 0$. Using $y := x - a$ in the Taylor expansion, we get

$$0 = \int_{\mathbb{R}} \left(\sum_{n=0}^{s} c_n y^n + R(y + a) \right) \overline{\psi(2^j y)} \, \mathrm{d}y. \tag{$*$}$$

By the choice of s we have

$$\int_{\mathbb{R}} y^n \psi(2^j y) \, \mathrm{d}y = 0.$$

Hence, by $(*)$ we get

$$-cs \int_{\mathbb{R}} y^s \overline{\psi(2^j p)} \, \mathrm{d}y = \int R(y + a) \overline{\psi(2^j y)} \, \mathrm{d}y.$$

Setting $x = 2^j y$, we obtain

$$\int_{\mathbb{R}} x^s \overline{\psi(x)} = -\frac{1}{c_s} 2^{j(s+1)} \int R(y + a) \overline{\psi(2^j y)} \, \mathrm{d}y. \tag{$**$}$$

[10] It cannot be equal to zero everywhere to be an ONB.

By the above bounds on $|R(x)|$ we get

$$\left| 2^{j(s+1)} \int\limits_{\mathbb{R}} R(y+a)\overline{\psi(2^j y)}\,\mathrm{d}y \right| \leq 2^{j(s+1)} \int\limits_{-\delta}^{\delta} \varepsilon |y|^s \frac{1}{(1+|2^j y|)^\alpha} + 2 \cdot 2^{j(s+1)} \int\limits_{\delta}^{\infty} c|y|^s \frac{1}{(1+|2^j y|)^\alpha}\,\mathrm{d}y$$

$$= \varepsilon \int\limits_{-2^j\delta}^{2^j\delta} \frac{|x|^s}{(1+|x|)^\alpha}\,\mathrm{d}x + 2 \int\limits_{2^j\delta}^{\infty} \frac{|x|^s}{(1+|x|)^\alpha}\,\mathrm{d}x.$$

Sending $\varepsilon \to 0$ and $j \to \infty$ will let the RHS of $(**)$ tend to zero. Thus $\int_{\mathbb{R}} x^s \psi(x)\,\mathrm{d}x$. $\square$

(6.4) Definition. Let $f \in L^2(\mathbb{R})$ satisfy

(i) $0 \leq f(\xi) \leq 1$ for all $\xi \in \mathbb{R}$,

(ii) $f(\xi) = f(-\xi)$ for all $\xi \in \mathbb{R}$,

(iii) $f(\xi) = 1$, if $|\xi| < \frac{2}{3}\pi$ and $f(\xi) = 0$, if $|\xi| > \frac{4}{3}\pi$.

(iv) $f^2(\xi) + f^2(\xi - 2\pi) = 1$ for all $\xi \in (0, 2\pi]$.

Then $\mathcal{F}^{-1} f$ is called **Meyer scaling function**.

(6.5) Proposition. *Let $\varphi \in L^2(\mathbb{R})$ be a Meyer scaling function. Then it is a scaling function. The corresponding symbol m_φ is 2π-periodic and equals $\mathcal{F}\varphi(2\xi)$ on $[-\pi, \pi]$.*

Proof. We want to use Theorem 5.24. For (c) we have alread $\mathcal{F}\varphi(0) = 1$ and continuity. For (a) we want to show that $\sum_l |\mathcal{F}\varphi(\xi + 2\pi l)|^2 = 1$ to use Corollary 5.18. Let ψ be a 2π-periodic function such that $\psi|_{[-\pi,\pi]} = \mathcal{F}\varphi(2\cdot)$. Since $\operatorname{supp} \widehat{\varphi}(2\cdot) \subset \left[-\frac{3}{2}\pi, \frac{3}{2}, \pi\right]$, (iii) implien $\mathcal{F}\varphi(2\xi) = \psi(\xi)\varphi(\xi)$, hence $m_\varphi = \phi$. $\square$

(6.6) Proposition. *Let $\varphi \in L^2(\mathbb{R})$ be a Meyer scaling function. Then a function $\psi \in L^2(\mathbb{R})$ with*

$$\mathcal{F}\psi(\xi) = e^{-i\frac{\xi}{2}} m_\varphi\left(\frac{\xi}{2} + \pi\right) \mathcal{F}\varphi\left(\frac{\xi}{2}\right)$$

is an associated wavelet such that

(i) $\operatorname{supp} \mathcal{F}\psi \subset \left[-\frac{8}{3}\pi, -\frac{3}{2}\pi\right] \cup \left[\frac{2}{3}\pi, \frac{8}{3}\pi\right]$,

(ii) ψ is a real-valued C^∞-function,

(iii) $\psi\left(-\frac{1}{2} - x\right) = \psi\left(-\frac{1}{2} + x\right)$.

Proof. The fact that ψ is an associated wavelet follows from Theorem 5.33. (??)

(i): Using Proposition 6.5 we have $\operatorname{supp} \varphi(2\cdot) \subset \left[-\frac{2}{3}\pi, \frac{2}{3}\pi\right]$ and

$$\operatorname{supp} m_\varphi \subset \bigcup_k \left[2k\pi - \frac{2}{3}\pi, 2k\pi + \frac{2}{3}\pi\right]$$

and hence by (iii) from Definition 6.4 we obtain the claim.

(ii):

$$m_\varphi\left(-\frac{\xi}{2} + \pi\right) \mathcal{F}\varphi\left(-\frac{\xi}{2}\right) = m_\varphi\left(-\frac{\xi}{2} - \pi\right) \mathcal{F}\varphi\left(-\frac{\xi}{2}\right) = m_\varphi\left(-\frac{\xi}{2} + \pi\right) \mathcal{F}\varphi\left(\frac{\xi}{2}\right).$$

We have

$$\begin{aligned}
\overline{\psi(x)} &= \frac{1}{2\pi} \int \overline{e^{ix\xi} e^{i\frac{\xi}{2}} m_\varphi\left(\frac{\xi}{2} + \pi\right) \mathcal{F}\varphi\left(\frac{\xi}{2}\right)}\,\mathrm{d}\xi \\
&= \frac{1}{2\pi} \int e^{-ix\xi} e^{-i\frac{\xi}{2}} m_\varphi\left(\frac{\xi}{2} + \pi\right) \mathcal{F}\varphi\left(\frac{\xi}{2}\right)\,\mathrm{d}\xi \\
&= \frac{1}{2\pi} \int e^{ix\xi} e^{i\frac{\xi}{2}} m_\varphi\left(-\frac{\xi}{2} + \pi\right) \mathcal{F}\varphi\left(-\frac{\xi}{2}\right)\,\mathrm{d}\xi = \psi(x).
\end{aligned}$$

(iii):

$$
\begin{aligned}
\psi\left(-\frac{1}{2}-x\right) &= \frac{1}{2\pi}\int e^{i(-\frac{1}{2}-x)\xi}e^{i\frac{\xi}{2}}m_\varphi\left(\frac{\xi}{2}+\pi\right)\mathcal{F}\varphi\left(\frac{\xi}{2}\right)\mathrm{d}\xi \\
&= \frac{1}{2\pi}\int e^{i\xi x}m_\varphi\left(\frac{\xi}{2}+\pi\right)\mathcal{F}\varphi\left(\frac{\xi}{2}\right)\mathrm{d}\xi \\
&= \frac{1}{2\pi}\int e^{i\xi x}m_\varphi\left(\frac{\xi}{2}+\pi\right)\mathcal{F}\varphi\left(\frac{\xi}{2}\right) = \psi\left(x-\frac{1}{2}\right).\square
\end{aligned}
$$

(6.7) Definition. The wavelet of Proposition 6.6 is called **Meyer wavelet**.

(6.8) Theorem. *There exists a real valued wavelet $\psi \in L^2(\mathbb{R})$ such that*

(i) $\psi \in \mathcal{S}(\mathbb{R})$,

(ii) $\psi\left(-\frac{1}{2}+x\right) = \varphi\left(-\frac{1}{2}-x\right)$,

(iii) $\operatorname{supp}\mathcal{F}\varphi \subset \left[-\frac{8}{3}\pi, -\frac{2}{3}\pi\right] \cup \left[\frac{2}{3}\pi, \frac{8}{3}\pi\right]$.

Proof. Choose a Meyer scaling function such that $\mathcal{F}\varphi \in C^\infty(\mathbb{R})$. $\square$

6.1 Spline Wavelets

(6.9) Remark. Wavelets based on spline functions are easy to construct and have exponentially fast decay, i.e. there exists $\alpha > 0$ such that

$$|\psi(x)| \leq c e^{-\alpha|x|}.$$

For more details cf. [2, sect. 3.3].

7 Compactly Supported Wavelets

7.1 General Construction

(7.1) Theorem. *Let $m(\xi) = \sum_{k=T}^{S} h_k e^{-ik\xi}$ be a trigonometric polynomial such that*

(i) $|m(\xi)|^2 + |m(\xi+\pi)|^2 = 1$ *for all* $\xi \in \mathbb{R}$,

(ii) $m(0) = 1$,

(iii) $m(\xi) \neq 0$ *for all* $\xi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Then $\prod_{j=1}^{\infty} m(2^{-j}\cdot)$ converges in the L^∞-sense. Let $\varphi \in L^2(\mathbb{R})$ be defined by $\mathcal{F}\varphi(\xi) = \prod_{j=1}^{\infty} m(2^{-j}\xi)$. Then $\mathcal{F}\varphi$ is continuous and $\operatorname{supp}\varphi \subset [T,S]$. Moreover φ is a scaling function of an MRA and

$$\psi(x) := 2\sum_{k=T}^{S} \overline{h_k}(-1)^k \varphi(2x+k+1)$$

is an associated wavelet

$$\operatorname{supp}\psi \subset \left[\frac{T-S-1}{2}, \frac{S-T-1}{2}\right].$$

(7.2) Remark. The Theorem 7.1 gives a new approach to construct a scaling function. Suppose $(h_k)_k$ is given and that we want to find φ. Then $\varphi(x) = \sum h_k \varphi(2x-k)$, which is equivalent to

$$\mathcal{F}\varphi(x) = m\left(\frac{\xi}{2}\right)\mathcal{F}\varphi\left(\frac{\xi}{2}\right).$$

here $m(\xi) = \sum_k h_k e^{-ik\xi}$. But $\mathcal{F}\varphi \in C$ and $\mathcal{F}\varphi(0) = 1$, which implies

$$\mathcal{F}\varphi(\xi) = \prod_{j=1}^{\infty} m(2^{-j}\xi).$$

(7.3) Lemma. *Let m be a trigonometrical polynomial such that (i) and (ii) from Theorem 7.1 hold. Then*

$$\mathcal{F}\varphi(\xi) = \prod_{j=1}^{\infty} m(2^{-j}\xi)$$

converges in the L^∞-sense and

$$\int \left| \prod_{j=1}^{\infty} m(2^{j}\xi) \right|^2 \mathrm{d}\xi \leq 2\pi.$$

In particular, $\mathcal{F}\varphi \in C(\mathbb{R})$ and $\mathcal{F}\varphi(0) = 1$. If also (iii) holds, then

$$\int \left| \prod_{j=1}^{\infty} m(2^{-j}\xi) \right|^2 e^{-2\pi i k\xi}\, \mathrm{d}\xi = \begin{cases} 2\pi, & \textit{if } k = 0, \\ 0, & \textit{else.} \end{cases}$$

Proof. m is a trigonometric polynomial and thus Lipschitz-continuous. Thus there exists a constant $C > 0$ such that for all $\xi \in \mathbb{R}$ we have

$$|m(\xi) - 1| \leq C|\xi|.$$

This implies

$$|m(2^{-j}\xi) - 1| \leq C2^{-j}|\xi|.$$

Now define

$$\Pi_N(\xi) = \prod_{j=1}^{N} m(2^{-j}\xi),$$

$$g_N(\xi) = \Pi_N(\xi)\mathbb{1}_{[-2^N\pi, 2^N\pi]}(\xi) = \Pi_N(\xi)\mathbb{1}_{[-\pi,\pi]}(2^{-j}\xi),$$

$$I_N^k(\xi) = \int_{-2^N\pi}^{2^N\pi} |\Pi_N(\xi)|^2\, e^{-2\pi i k\xi}\, \mathrm{d}\xi.$$

Note that Π_N is $2^N \cdot 2\pi$-periodic for each N. Using 7.1 (i)

$$\begin{aligned} I_N^k(\xi) &= \int_0^{2^{N+1}\pi} |\Pi_N(\xi)|^2\, e^{-2\pi i k\xi}\, \mathrm{d}\xi \\ &= \int_0^{2^N\pi} |\Pi_N(\xi)|^2\, e^{-2\pi i k\xi}\, \mathrm{d}\xi + \int_{2^N\pi}^{2^{N+1}\pi} |\Pi_N(\xi)|^2\, e^{-2\pi i k\xi}\, \mathrm{d}\xi \\ &= \int_{2^N\pi}^{0} |\Pi_{N-1}(\xi)|^2\, e^{-2\pi i k\xi} |m(2^{-N}\xi)|^2\, \mathrm{d}\xi + \int_{2^N\pi}^{0} |\Pi_{N-1}(\xi)|^2\, e^{-2\pi i k\xi} |m(2^{-N}\xi + \pi)|^2\, \mathrm{d}\xi \\ &= \int_0^{2^N\pi} |\Pi_{N-1}(\xi)|^2 e^{-2\pi i k\xi}\, \mathrm{d}\xi = I_{N-1}^k. \end{aligned}$$

Thus

$$I_N^k = \ldots = I_1^k = \int_{-2\pi}^{2\pi} \left| m\left(\frac{\xi}{2}\right) \right|^2 e^{-2\pi i k\xi}\, \mathrm{d}\xi = 2\int_{-\pi}^{0} \left| m\left(\frac{\xi}{2}\right) \right|^2 + \left| m\left(\frac{\xi}{2} + m\right) \right|^2 e^{4\pi i k\xi}\, \mathrm{d}\xi = \begin{cases} 2\pi, & \text{if } k = 0, \\ 0, & \text{else.} \end{cases}$$

7.1 (i) implies $|m(\xi)| \leq 1$ for all $\xi \in \mathbb{R}$, which in turn implies

$$\int_{-2^N\pi}^{2^N\pi} \left| \prod_{j=1}^{\infty} m(2^{-j}\xi) \right|^2 \mathrm{d}\xi \leq \int_{-2^N\pi}^{2^N\pi} |\Pi_N(\xi)|^2\, \mathrm{d}\xi = I_N^0 \leq 2\pi.$$

7.1 (iii) implies that there exists $c > 0$:

$$|m(\xi)| > c \quad \forall\, \xi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

Since $\prod_{j=1}^{\infty} m(2^{-j}\cdot)$ converges uniformly on $[-\pi, \pi]$ there exists $M \in \mathbb{Z}$ with

$$\prod_{j\geq N}^{\infty} m(2^{-j}\xi) > \frac{1}{2} \quad \forall\, \xi \in [-\pi, \pi].$$

This implies for all $\xi \in [-\pi, \pi]$

$$|\mathcal{F}\varphi(\xi)| = \prod_{j=1}^{M-1} |m(2^{-j}\xi)| \prod_{j>M}^{\infty} |m(2^{-j}\xi)| \geq c^{M-1}\frac{1}{2} > 0.$$

Since $\mathcal{F}\varphi(\xi) = \Pi_N(\xi) \cdot \mathcal{F}\varphi(2^{-N}\xi)$ we conclude

$$|\Pi_N(\xi)| \leq \frac{1}{c'}|\mathcal{F}\varphi(\xi)| \quad \forall\, \xi \in [-2^N\pi, 2^N\pi].$$

This implies

$$|g_N(\xi)| \leq \frac{1}{c'}|\mathcal{F}\varphi(\xi)| \quad \forall\, x \in \mathbb{R}.$$

Obviously $g_N \to \mathcal{F}\varphi$ as $N \to \infty$ pointwise. By the dominated convergence theorem we get

$$\int_{\mathbb{R}} |\mathcal{F}\varphi(\xi)|^2 e^{-2\pi i\xi k}\,\mathrm{d}\xi = \int_{\mathbb{R}} \lim_{N\to\infty} |g_N(\xi)|^2 e^{-2\pi ik\xi}\,\mathrm{d}\xi = \lim_{N\to\infty} \int_{\mathbb{R}} |g_N(\xi)|^2 e^{-2\pi ik\xi}\,\mathrm{d}\xi = \lim_{N\to\infty} I_N^k.$$

By the equality $I_N^k = I_0^k$ we obtain the last claim. □

(7.4) Lemma. *Let $m(\xi) = \sum_{k=T}^{S} h_k e^{-ik\xi}$ be a trigonometric polynomial which satisfies (i) and (ii) of Theorem 7.1 and let $\varphi \in L^2$ be defined by*

$$\mathcal{F}\varphi(\xi) = \prod_{j=1}^{\infty} m(2^{-j}\xi).$$

Then $\operatorname{supp}\varphi \subset [T, S]$ *and if $h_k \in \mathbb{R}$ for all $k \in [[T, S]]$, then* $\operatorname{Im}\varphi = 0$.

Proof. Consider a measure μ of bounded variation. Then the Fourier transform of μ is defined as

$$\mathcal{F}\mu(\xi) := \int_{\mathbb{R}} e^{-ix\xi}\,\mathrm{d}\mu(x).$$

We have as for functions, that $\mathcal{F}(\mu * \nu) = \mathcal{F}\mu \cdot \mathcal{F}\nu$. Define

$$\mu_j := \sum_{k=T}^{S} h_k \delta(2^{-j}k).$$

Then $\mathcal{F}\mu_j = m(2^{-j}\xi)$. By the convolutional formula we get

$$\mathcal{F}[\mu_1 * \ldots * \mu_N] = \prod_{j=1}^{N} m(2^{-j}\cdot).$$

Also, using linearity, we get, with appropriate scalars $H_{k_1,\ldots,k_N}$

$$\mu_1 * \ldots * \mu_N = \sum_{\substack{k_1,\ldots,k_N \\ T\leq k_i\leq S}} H_{k_1,\ldots,k_N} \underbrace{\delta(2^{-1}k_1) * \ldots * \delta(2^{-N}k_N)}_{=\delta(2^{-1}k_1+2^{-2}k_2+\ldots+2^{-N}k_N}.$$

In particular, with $C_{S,T}$ appropriate

$$\operatorname{supp}\mu_1 * \ldots * \mu_N \subset [T,S] - 2^{-N-1}C_{S,T}.$$

Let now $f \in C^\infty$ be compalctly supported with $\operatorname{supp} f \cap [T,S] = \emptyset$. By Plancherel

$$\int \varphi(t)\overline{f(t)}\,\mathrm{d}t = \frac{1}{2\pi}\int_{\mathbb{R}} \mathcal{F}\varphi\overline{\mathcal{F}f}\,\mathrm{d}\xi = \lim_{N\to\infty}\frac{1}{2\pi}\int_{\mathbb{R}}\prod_{j=1}^{N} m(2^{-j}\xi)\overline{f(\xi)}\,\mathrm{d}\xi$$
$$= \lim_{N\to\infty}\frac{1}{2\pi}\int_{\mathbb{R}}\mathcal{F}[\mu_1 * \ldots * \mu_N]\overline{\mathcal{F}f}\,\mathrm{d}\xi = \lim_{N\to\infty}\frac{1}{2\pi}\int_{\mathbb{R}}\overline{f(s)}\,\mathrm{d}\mu_1 * \ldots * \mu_N = 0.$$

Thus $\operatorname{supp} \subset [T,S]$. $\square$

Proof of Theorem 7.1. In Lemma 7.4 we showed that $\operatorname{supp}\mathcal{F}\varphi \subset [T,S]$. For showing that it is a scaling function, we use Theorem 5.21. By Lemma 7.3 $\mathcal{F}\varphi$ is continuous and nonzero in zero. By the definition of $\mathcal{F}\varphi$, we have $\varphi(\xi) = m\left(\frac{\xi}{2}\right)\mathcal{F}\varphi\left(\frac{\xi}{2}\right)$. This shows 5.24 (ii) and (iii). For showing (i), i.e. that $\{\varphi_{0,m} : m \in \mathbb{Z}\}$ forms an orthonormal system, we use form Lemma 7.3:

$$\int_0^{2\pi}\sum_{l\in\mathbb{Z}}|\mathcal{F}\varphi(\xi+2\pi l)|^2 e^{-2\pi ik\xi}\,\mathrm{d}\xi = \begin{cases}2\pi, & \text{if } k=0,\\ 0, & \text{else.}\end{cases}$$

From $k=0$, we obtain $\sum_l |\mathcal{F}\varphi(\xi+2\pi l)|^2 = 1$. By 5.24, φ is a scaling function. The formula for ψ comes from 5.31. The support property comes from the form of ψ and the support of φ. $\square$

(7.5) Remark. (ii) and (iii) are easy to accomodate, (i) is more challenging (cf. upcoming section)

7.2 Smooth Wavelets

(7.6) Lemma. *If $g(\xi) = \sum_{k=-T}^{T}\gamma_k e^{-ik\xi}$ is a non-negative trigonometric polynomial with $\gamma_k \in \mathbb{R}$, $k \in [[-T,T]]$, then there exists a polynomial m with*

$$m(\xi) = \sum_{k=-T}^{T} h_k e^{ik\xi}$$

with $h_k \in \mathbb{R}$ and

$$|m(\xi)|^2 = g(\xi), \qquad \forall\,\xi \in \mathbb{R}.$$

Proof. $g(\xi) \in \mathbb{R}$, $\gamma_k \in \mathbb{R}$, hence $\gamma_k = \gamma_{-k}$ for all $k \in [[-T,T]]$. Let's assume $\gamma_T \neq 0$. Then consider

$$G(z) = \sum^{T} k = -T\gamma_k z^k = z^{-T}(\gamma_{-T} + \gamma_{-T+1}z + \ldots + \gamma_T z^{2T}) =: z^{-T}f(z), \quad \forall\, z \in \mathbb{C}.$$

By the fundamental theorem of algebra, we know there exists a representation

$$G(z) = z^{-T}\prod_{j=1}^{2T}(z - c_j).$$

Since $\gamma_T \neq 0$, we have also $\gamma_{-T} \neq 0$. Hence zero is not a root of fi, i.e. $c_j \neq 0$ for all $j \in [[2T]]$. By symmetry we get $G(\overline{z}) = \overline{G(z)}$ and $G(z^{-1}) = G(z)$ for any $z \in \mathbb{C}$.

If $G(z_0) = 0$, then also $G(\overline{z_0}) = G(z_0^{-1}) = G(\overline{z_0}^{-1}) = 0$. Also, if $|z_0| = 1$ with $G(z_0) = 0$, then z_0 is a zero with even multiplicity. Indeed, then $z_0 = e^{i\xi_0}$ and $g(\xi_0) = 0$. Thus $g(\xi) = 0$ for all ξ, i.e. ξ_0 is a local minimum of g and g has a zero of even multiplicity in ξ_0.

Thus

$$f(z) = \prod_s g_s(z),$$

where either $g_s(z) = (z-c)(z-\overline{c})(z-\overline{c}^{-1})(z-c^{-1})$ for $c \in \mathbb{C} \setminus \mathbb{R}$, or $g_s(z) = (z-c)(z-c^{-1})$ for $c \in \mathbb{R}$. For $|z| = 1$ and $c \in \mathbb{C} \setminus \{0\}$, we have

$$|(z-c)(z-\overline{c}^{-1}) = \ldots = \frac{1}{|c|}|z-c|^2$$

Thus for complex c and $|z| = 1$ we have

$$|g_s(z)| = \left| \frac{1}{|c|}(z-c)(z-\overline{c}) \right|^2 \text{[11]}$$

and for real c we have

$$|g_s(z)| = \frac{1}{|c|}|z-c|^2 = \left| \frac{1}{\sqrt{|c|}}(z-c) \right|^2.$$

This shows, that for each s, there exists a polynomia p_s with real coefficients with

$$|g_s(z)| = |p_s(z)|^2, \quad \forall\, z \in \{c \in \mathbb{C} : |c| = 1\}.$$

We have $|z| = 1$, i.e. $z = e^{i\xi}$ and thus

$$g(\xi) = G(z) = z^{-T} f(z),$$

which shows the claim by the aforementioned decomposition of f into the g_s. □

(7.7) Remark. (1) The polynomial m in Lemma 7.6 is not unique.

(2) For constructing m such that Theorem 7.1 (i) holds, it is now sufficient to construct some trigonometric polynomial g with

$$g(\xi) + g(\xi + \pi) = 1, \quad \forall\, \xi \in \mathbb{R}.$$

This is easy to check. First

$$g(\xi) = \sum_{k=-T}^{T} a_k e^{ik\xi}$$

iss a real-valued trigonometric polynomial if and only if $a_k = \overline{a}_k$ for all $k \in [[N]]$. Then, if the condition on g holds,

$$\sum_{k=-T}^{T} a_k e^{ik\xi} + \sum_{k=-T}^{T} a_k (-1)^k e^{ik\xi} = 1, \quad \xi \in \mathbb{R}.$$

Thus $a_0 = \frac{1}{2}$ and $a_k = 0$ for all $k \in [[2, T, 2]]$.

(7.8) Theorem (Daubechies, 1991). *There exists a constant $C > 0$ such that for every $r \in \mathbb{N}$ there exsits a MRA in $L^2(\mathbb{R})$ with scaling function φ and associated wavelet ψ such that the following holds:*

(i) $\varphi, \psi \in C^r(\mathbb{R})$

(ii) *φ and ψ are compactly supported with* $\operatorname{supp}\varphi, \operatorname{supp}\psi \subset [-Cr, Cr]$.

(7.9) Remark. For $k \in \mathbb{N}$ define

$$g_k(\xi) = 1 - c_k \int_0^{\xi} (\sin t)^{2k+1}\,\mathrm{d}t, \quad \text{with} \quad c_k = \left(\int_0^{\pi} (\sin t)^{2k+1}\,\mathrm{d}t \right)^{-1}.$$

Then

- g_k is a trigonometric polynomial of degree $2k+1$

[11] Here the second order polynomial has real coefficients.

- We can write

$$g_k(\xi) = c_k \int\limits_{\xi}^{\pi} (\sin t)^{2k+1}\,\mathrm{d}t = c_k \int\limits_{\xi}^{\pi} (1-\cos^2 t)^k \sin t\,\mathrm{d}t$$

  Appling $u = \cos t$, we have $g_k(\xi) = p_k(\cos t)$ with

$$p_k(x) = c_k \int\limits_{-1}^{x} (1-u^2)^k\,\mathrm{d}u.$$

(7.10) Lemma. *Let g_k, p_k be as before. Then*

(i) $0 \le g_k(\xi) \le 1$ *and* $g_k(\xi) = g_k(-\xi)$ *for all k and ξ*

(ii) $g_k(\xi) \neq 0$ *for all* $\xi \in (-\pi, \pi)$

(iii) $g_k(0) = 1$ *for all k*

(iv) $1 = g_k(\xi) + g_k(\xi + \pi)$ *for all ξ*

(v) $c_k \le 3\sqrt{k}$ *and g_k can be factored as* $g_k \left(\frac{1+\cos\xi}{2}\right)^{k+1} \varphi_k(\xi)$, *with φ_k a trigonometric polynomial.*

Proof. (i)-(iii) follows from the definition. (iv) can be proven by simple computation.
(v): Estimate of c_k uses stated estimates of sin and cos.
By 7.9 $x-1$ i a zero of p_k of order $k+1$. Hence

$$p_k(x) = (x+1)^{k+1}\widetilde{p}_k(x)$$

$\square$

(7.11) Lemma. *Set $m = \frac{k}{2}$ and write the factorization of Lemma 7.10 (v) as*

$$g_k(\xi) = \left(\frac{1+\cos\xi}{2}\right)^m M_k(\xi).$$

Then there exists some integer N and $\alpha < 1$ such that

$$\sup_{\xi} |M_k(\xi)| \le 2^{\alpha k} \qquad \forall\, k \ge N.$$

Proof. We have $M_k(\xi) = 2^m g_k(\xi)(1+\cos\xi)^{-m}$. Since $g_k(\xi) = p_k(\cos\xi)$, we get

$$\sup_{\xi} |M_k(\xi)| = 2^m \sup_{x\in[-1,1]} p_k(x)(1+x)^{-m}.$$

Then

$$p_k(x)(1+x)^{-m} \overset{7.9}{=} c_k \int\limits_{1}^{x} \frac{(1-u^2)^k}{(1+x)^m}\,\mathrm{d}u = c_k \int\limits_{1}^{x} \left(\frac{(1+u)}{(1+x)}\right)^m (1+u)^m(1-u)^k\,\mathrm{d}u$$

$$\le c_k \int\limits_{1}^{x} (1+u)^m(1-u)^k\,\mathrm{d}u = c_k \int\limits_{1}^{x} \left(\sqrt{1+u}(1-u)\right)^k \mathrm{d}u$$

Since the integrand has a maximum in $[-1,1]$ of size $\frac{4}{3}\sqrt{\frac{2}{3}}$, we get, using Lemma 7.10 (v),

$$\sup_{\xi} |M_k(\xi)| \le 2^k 3\sqrt{k} \cdot \left(\frac{4}{3}\sqrt{\frac{2}{3}}\right)^k \le 2^{\alpha k}.$$

$\square$

(7.12) Proposition. *Let g_k, $k \in \mathbb{N}$, be as before, and set*

$$G_k(\xi) = \prod_{j=1}^{d} g_k(2^{-j}\xi)$$

Then, for $|\xi| > 1$ and $k \geq N$, we have

$$|G_k(\xi)| \leq C_k \cdot |\xi|^{(\alpha-1)k},$$

where α, N as in Lemma 7.11.

Proof. Use M_k from Lemma 7.11, then

$$G_k(\xi) = \left(\prod_{j=1}^{\infty} \frac{1+\cos(2^{-j}\xi)}{2} \right)^m \left(\prod_{j=1}^{\infty} M_k\left(2^{-j}\xi\right) \right).$$

First, observe that

$$\prod_{j=1}^{m} \cos(2^{-j}\xi) = \prod_{j=1}^{m} \frac{\sin\left(2^{1-j}\xi\right)}{2 \cdot \sin(2^{-j}\xi)} = \frac{\sin(\xi)}{2^m \sin(2^{-m}\xi)}.$$

Thus $\prod_{j=1}^{\infty} \cos(2^{-j}\xi) = \frac{1}{\xi}\sin\xi$. Thus by $\frac{1}{2}(1+\cos x) = \cos^2\left(\frac{x}{2}\right)$, we obtain

$$\prod_{j=1}^{\infty} \frac{1+\cos(2^{-j}\xi)}{2} = \frac{4}{\xi^2}\sin^2\left(\frac{\xi}{2}\right).$$

Now in a second step we have

$$g_k(\xi) = \left(\frac{1+\cos\xi}{2}\right)^m M_k(\xi) = \left(\frac{1+\cos\xi}{2}\right)^{k+1} \varphi_k(\xi).$$

M_k is continuous and satisfies $|M_k(\xi) - 1| \leq C|\xi|$ for all ξ. As in the proof of Lemma 7.3, we have uniform convergence to a constant function and

$$\sup_{|\xi|\leq 1} \left| \prod_{j=1}^{\infty} M_k(2^{-j}\xi) \right| \leq C_k.$$

For $|xi| > 1$, fix an integer r with $2^{r-1} \leq |\xi| < 2^r$, and

$$\left| \prod_{j=1}^{\infty} M_k(2^{-j}\xi) \right| = \prod_{j=1}^{r} |M_k(2^{-j}\xi)| \cdot \prod_{j=1}^{\infty} |M_k(2^{-j}2^{-r}\xi)| \leq 2^{\alpha k r \cdot C_k} \leq 2C_k|\xi|^{\alpha k}.$$

Combining the above statements proves the claim. □

Proof of Theorem 7.3. Let g_k be the trigonometric polynomial from Remark 7.9. Since $g_k(\xi) \geq 0$ for all $\xi \in \mathbb{R}$ by Proposition 7.10, by Lemma 7.6, there exists a polynomial m_k of degree $2k+1$ with

$$|m_k(\xi)|^2 = g_k(\xi), \qquad \forall\, \xi \in \mathbb{R}.$$

By Proposition 7.10 (ii)-(iv), the conditions of Theorem 7.1 are fulfilled. By Theorem 7.1, there exists a scaling function φ_k and a wavelet ψ_k supported in $[-2k-1, 2k+1]$.

Apply Proposition 7.12 to obtain

$$|\psi_k(\xi)| \leq C_k|\xi|^{\frac{(\alpha-1)k}{2}}$$

for all ξ with $|\xi| \geq 1$. Thus $\varphi_k \in C^r$ for $r < \frac{1-\alpha}{2}k - 1$. By the form of ψ_k from Theorem 7.1, we obtain that also $\psi_k \in C^r$. □

8 Wavelet Decomposition

8.1 Sampling

(8.1) Definition. (1) Let $f \in L^1(\mathbb{R})$ or $f \in L^2(\mathbb{R})$. Then f is called **band-limited**, if there exists $\Omega > 0$ with $\operatorname{supp} \mathcal{F}f \subset [-\Omega, \Omega]$.

(2) The sinc function is defined by

$$\operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x}, \quad \forall\, \xi \in \mathbb{R} \setminus \{0\}.$$

It can be continuously extended to $\mathbb{R}$.

(8.2) Theorem (Shannon Sampling Theorem). *Let $f \in L^1(\mathbb{R})$ be band-limited with $\operatorname{supp} \mathcal{F}f \subset [-\Omega, \Omega]$. If $h < h^* := \frac{\pi}{\Omega}$, then f can be recovered exactly from $(f(hk))_{k\in\mathbb{Z}}$ by*

$$f(x) = \sum_{k\in\mathbb{Z}} f(hk) \cdot \operatorname{sinc}\left(\frac{x}{h} - k\right), \quad \forall\, x \in \mathbb{R}.$$

Proof. Set $\widetilde{D}_h f(x) := f(hx)$, hence $(f(hk))_{k\in\mathbb{Z}} = (\widetilde{D}_h f(k))_{k\in\mathbb{Z}}$.
First we rewrite

$$\widetilde{D}_h f(k) = \mathcal{F}^{-1}\mathcal{F}\widetilde{D}_h f(k) = \frac{1}{2\pi}\int_{\mathbb{T}} \mathcal{F}\widetilde{D}_h f(\xi) e^{i\xi k}\,\mathrm{d}\xi = \int_{\mathbb{T}} \sum_{l\in\mathbb{Z}} \widetilde{D}_h f(l) e^{-il\xi} e^{i\xi k}\,\mathrm{d}\xi.$$

Now

$$\begin{aligned}
\widetilde{D}_h f(k) = f(hk) &= \frac{1}{2\pi}\int_{\mathbb{R}} \mathcal{F}f(\xi) e^{i\xi hk}\,\mathrm{d}\xi = \frac{1}{2\pi}\sum_{j\in\mathbb{Z}} \int_{h^{-1}[-\pi+2\pi j, \pi+2\pi j]} \mathcal{F}f(\xi) e^{i\xi hk}\,\mathrm{d}\xi \\
&= \frac{1}{2\pi h}\sum_{j\in\mathbb{Z}} \int_{-\pi+2\pi j}^{\pi+2\pi j} \mathcal{F}f(h^{-1}\xi) e^{ik\xi}\,\mathrm{d}\xi = \frac{1}{2\pi h}\sum_{j\in\mathbb{Z}} \int_{-\pi}^{\pi} \mathcal{F}f(h^{-1}(\xi + 2\pi j)) e^{ik\xi}\,\mathrm{d}\xi \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{h}\sum_{j\in\mathbb{Z}} \mathcal{F}f(h^{-1}(\xi + 2\pi j)) e^{ik\xi}\,\mathrm{d}\xi.
\end{aligned}$$

Since $(e^{ik\xi})_{k\in\mathbb{Z}}$ is an orthonormal Basis for $L^2([-\pi,\pi])$ we obtain

$$\sum_{k\in\mathbb{Z}} f(hk) e^{-i\xi k} = \frac{1}{h}\sum_{j\in\mathbb{Z}} \mathcal{F}f(h^{-1}(\xi + 2\pi j)).$$

If $h^{-1}[-\pi,\pi] \supset [-\Omega,\Omega]$, which is equivalent to $\pi > h\Omega$ or $h < \frac{\pi}{\Omega}$, then for $j > 0$ we have

$$h^{-1}(\xi + 2\pi j) > \frac{\Omega}{\pi}(\xi + 2\pi j) \geq \frac{\Omega}{\pi}(-\pi + 2\pi j) \geq \Omega(2j-1) \geq \Omega.$$

Similarly, for $j < 0$ we can prove

$$h^{-1}(\xi + 2\pi j) < -\Omega.$$

Therefore

$$\mathcal{F}f(h^{-1}\xi) = h\sum_{m} f(hm) e^{-im\xi}, \qquad (\xi \in [-\pi,\pi]),$$

which implies

$$\mathcal{F}f(\omega) = h\sum_{m} f(hm) e^{-imh\omega}, \qquad (\omega = h^{-1}\xi \in h^{-1}[-\pi,\pi]).$$

Now, we consider for each $x \in \mathbb{R}$:

$$f(x) = \frac{1}{2\pi} \int \mathcal{F}f(\omega) e^{ix\omega} \,\mathrm{d}\omega = \frac{1}{2\pi} \int\limits_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \mathcal{F}f(\omega) e^{ix\omega} \,\mathrm{d}\omega = \frac{h}{2\pi} \int\limits_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \sum_m f(hm) e^{-inh\omega} e^{ix\omega} \,\mathrm{d}\omega.$$

For all $m \in \mathbb{Z}$ we have

$$\frac{h}{2\pi} \int\limits_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{i\omega(x-hm)} \,\mathrm{d}\omega = \operatorname{sinc}\left(\frac{x}{h} - m\right)$$

and thus

$$f(x) = \sum_{m\in\mathbb{Z}} f(hm) \operatorname{sinc}\left(\frac{x}{h} - m\right). \qquad \square$$

(8.3) Remark. (1) The value $(h^*)^{-1}$ is called **Nyquist-rate**.

(2) The sinc-function is not optimal, because it has slow decay. This becomes a problem if only finitely many are used for the reconstruction.

(3) Sometimes $\frac{h^*}{m}$, $m \in \mathbb{Z}\setminus\{0\}$ is used, which means that we use "m-times the necessary sampling rate". This is called **m-times oversampled**.

(4) The main idea in the proof is the equality

$$\sum_k \mathcal{F}f\left(h^{-1}(\xi + 2\pi k)\right) = h \sum_k f(hk) e^{-i\xi k}, \qquad (\xi \in [-\pi, \pi]).$$

Paraphrased into words, this means that the periodization of $\mathcal{F}f$ is equal to the Fourier transform of a sampled sequence.

If h is so large that $\operatorname{supp} \mathcal{F}f(h^{-1}\cdot) \supset [-\pi, \pi]$, then the periodization smears the Fourier transform at the tails.

Aliasing: By overlapping of frequencies, which are unrelated and are now cosidered modulo 2π, the reconstruction causes artifacts which are called aliasing.

8.2 The Fast Wavelet Transform

(8.4) Remark. Let $f \in L^2(\mathbb{R})$. Assume we already have the inner products $(\langle f, \varphi_{j,m}\rangle)_m$ for some $j \in \mathbb{Z}$. First, we note that $\psi = \sum_k \varphi_{1,k}$ holds with $(g_k)_k = (\langle \psi, \varphi_{1,k}\rangle)_k$, since $(\varphi_{1,k})_k$ is a basis for V_1. This implies

$$\begin{aligned}\psi_{j,m}(x) = 2^{\frac{j}{2}} \psi(2^j x - m) = 2^{\frac{j}{2}} \sum_k g_k \varphi_{1,k}(2^j x - m) &= 2^{\frac{j+1}{2}} \sum_k g_k \varphi\left(2^{j+1}x - 2m - k\right) \\ = \sum_k g_k \varphi_{j+1,2m+k}(x) &= \sum_k g_{k-2m} \varphi_{j+1,k}(x) = (g_{-\cdot} * \varphi_{j+1,\cdot}(x))(2m).\end{aligned}$$

Therefore

$$\langle f, \psi_{j,m}\rangle = \left\langle f, \sum_k g_{k-2m}\varphi_{j+1,k} \right\rangle = \sum \overline{g_{k-2m}} \langle f, \varphi_{j+1,k}\rangle = \left(\overline{g_{-\cdot}} * \langle f, \varphi_{j+1,\cdot}\rangle\right)(2m).$$

This shows that we can compute $(\langle f, \psi_{j-1,m}\rangle)$ by just using $(g_k)_k$ and convolution. Also we can compute $(\langle f, \varphi_{j+1,m}\rangle)$ as follows: From the scaling equation

$$\varphi(x) = \sqrt{2} \sum_k h_k \varphi(2x - k),$$

we get

$$\begin{aligned}\varphi_{j-1,m}(x) = 2^{\frac{j-1}{2}} \varphi(2^{j-1}x - m) = 2^{\frac{j}{2}} \sum_k h_k \varphi(2^j x - 2m - k) &= \sum_k h_k \varphi_{j,2m+k}(x) \\ &= \sum_k h_{k-2m} \varphi_{j,k}(x) = (h_{-\cdot} * \varphi_{j,\cdot}(x))(2m).\end{aligned}$$

Therefore

$$\langle f, \varphi_{j+1,m}\rangle = \left\langle f, \sum_k h_{k-2m}\varphi_{j,k}\right\rangle = \sum_k \overline{h_{k-2m}}\,\langle f, \varphi_{j,k}\rangle = \left(\overline{h_{-\cdot}} * \langle f, \varphi_{j,k}\rangle\right)(2m).$$

Schematically, we can compute

$$(\langle f, \varphi_{0,m}\rangle) \to (\langle f, \varphi_{-1,m}\rangle) \to (\langle f, \varphi_{-1,m}\rangle)$$

Algorithm: Let $f \in L^2(\mathbb{R})$. Consider $f^0 = P_{V_0} f \in V_0$, where P_{V_0} is the orthogonoal projection onto V_0. Compute $f^0 \to f^{-1} \in V_{-1} \to f^{-2} \in V_{-2}$ Now set $f^j = \sum_m c_m^j \varphi_{j,m}$, $g^j = \sum_m d_m^j \psi_{j,m}$.

We proved that

$$c_m^{j-1} = (\overline{h_{-\cdot}} * c^j)(2m) = \sum_k \overline{h_{k-2m}} c_k^j$$

and

$$c_m^{j-1} = (\overline{g_{-\cdot}} * c^j)(2m) = \sum_k \overline{g_{k-2m}} c_k^j.$$

Now use the notation $a = (a_m)_m$, $\overline{a} = (\overline{a_m})$,

$$(Ab)_m = \sum_k a_{2m-k} b_k$$

and $d^{j-1} = \overline{G}c^j$.

We recall $g_k = (-1)^k h_{-k+1}$.

(8.5) Proposition (Algorithm). *Input:* c^0, J, G, H.

Compute:

Output: $(d^{-j})_{j\in[\![J]\!]}$, c^{-J}.

(8.6) Remark. Given $(d^{-j})_{j\in[\![J]\!]}$ and c^{-J}, we aim to compute c^0.

For arbitrary $-j \in [\![0; J-1]\!]$, we have

$$\begin{aligned}\sum_m c_m^j \varphi_{j,m} &= \sum_m c_m^{j-1}\varphi_{j-1,m} + \sum_m d_m^{j-1}\psi_{j-1,m} \\ &= \sum_m c_m^{j-1}\left(\sum_k h_{k-2m}\varphi_{j,m}\right) + \sum_m d_m^{j-1}\left(\sum_k g_{k-2m}\varphi_{j,m}\right) \\ &= \sum_m \left[\sum_k h_{k-2m} c_m^{j-1} + \sum_k g_{k-2m} d_m^{j-1}\right]\varphi_{j,m}.\end{aligned}$$

Since $(\varphi_{j,m})$ is an orthonormal basis, we have

$$c_m^j = \sum_k h_{k-2m} c_m^{j-1} + \sum_m g_{k-2m} d_m^{j-1},$$

i.e., using $\overline{H}$ and $\overline{G}$, we have

$$c^j = \overline{H}^* c^{j-1} + \overline{G} d^{j-1}.$$

(8.7) Proposition (Algorithm). *Input:* $(d^{-j})_{j\in[\![J]\!]}$, c^{-J}, J, $\overline{H}$, $\overline{G}$.

Compute:

Output: c^0.

(8.8) Remark. A convolution $a * b$ can be efficiently computed by $\mathbf{F}^{-1}[(\mathbf{F}a) * (\mathbf{F}b)]$, where $\mathbf{F}$ represents the fast Fourier transform (FFT), since $\mathcal{F}a \cdot \mathcal{F}b = \mathcal{F}[a * b]$.

(8.9) Corollary. *Let $(V_j)_j$, $(W_j)_j$, φ and ψ be as usual. Denote the orthogonal projection of some $f \in L^2(\mathbb{R})$ on V_j or W_j by $P_{V_j} f$ or $P_{W_j} f$ respectively. If $(c^{-j})_{j\in[\![0;J]\!]}$ and $(d^{-j})_{j\in[\![0;J]\!]}$ come from the wavelet decomposition, then $P_{V_j} f = \sum_m c_m^j \varphi_{i,m}$, and $P_{W_j} f = \sum_m d_m^j \psi_{i,m}$.*

9 Approximation Properties of Wavelets

(9.1) Remark. Define Haar wavelets on $[0,1]$: Let ψ be the Haar wavelet and ψ the Haar scaling function. Then define $\Lambda = \{(j,m) : j \geq 0, m \in [\![0; 2^j - 1]\!]\}$ and

$$\psi_{0,0} := \psi, \qquad \psi_{j,m} = 2^{\frac{j}{2}} \psi\left(2^j \cdot -m\right), \qquad j \geq 1$$

as functions in $L^2([0,1])$. Set $\Psi = \{\psi_{j,m} : (j,m) \in \Lambda\}$.

(9.2) Theorem. *Ψ is an ONB for $L^2([0,1])$.*

(9.3) Definition. Let $\Phi = (\phi_i)_{i \in I}$ be a frame for a Hilbertspace $\mathcal{H}$. The set

$$\Sigma_N(\Phi) = \{\sum_{i \in I_n} c_i \varphi_i : I_N \subset I, \# I_N \leq N\} \subset \mathcal{H}$$

is the **nonlinear N-term approximation manifold**. The **best N-term approximation** of some $f \in \mathcal{H}$ by Φ is $g \in \Sigma_N(\Phi)$ with

$$\|f - g\| \leq \|f - h\|, \qquad \forall\, h \in \Sigma_N(\Phi).$$

The **error of the best N-term approximation** is

$$\sigma_N(f, \Phi) := \inf_{g \in \Sigma_N(\Phi)} \|f - g\|.$$

(9.4) Theorem. *Let s be the saw tooth function with singularity (discontinuity) at x_0, i.e.*

$$s : [0,1] \to \mathbb{R}, x \mapsto \begin{cases} x, & \text{if } x \in [0, x_0], \\ x - x_0, & \text{if } x \in (x_0, 1]. \end{cases}$$

Let Ψ be the Haar basis on $[0,1]$, then

$$\sigma_N(s, \Phi) = \mathcal{O}(N^{-1}), \qquad \text{for } N \to \infty,$$

i.e. there exists $C > 0$, independent of N, such that $\sigma_N(s, \Psi) \leq C N^{-1}$ for all $N > N_0$ fixed.

“Wavelets give the optimal decay of the error of the best N-term approximation for function, wich are smooth apart from finitely many point discontinuities.”

Proof. Split Λ into two parts:

$$\Lambda^R := \{(j,m) = \lambda \in \Lambda : x_0 \notin \operatorname{supp} \psi_\lambda\},$$
$$\Lambda^S := \Lambda \setminus \Lambda^R.$$

Consider $\Lambda^R(\varepsilon) := \{\lambda \in \Lambda^R : |\langle s, \psi_\lambda \rangle| > \varepsilon\}$. Let $\lambda \in \Lambda^R$. Then on $\operatorname{supp} \psi_\lambda$, the function s has the form $x - c$. Then

$$|\langle s, \varphi_\lambda \rangle| = \left| \int_0^1 s(x) \psi_\lambda(x) \,\mathrm{d}x \right| = \left| \int_{\frac{m}{2^j}}^{\frac{m+1}{2^j}} (x - c) 2^{\frac{j}{2}} \psi(2^j x - m) \,\mathrm{d}x \right|.$$

We rewrite $x - c = \left(x - \frac{m + \frac{1}{2}}{2^j}\right) + \left(\frac{m + \frac{1}{2}}{2^j} - c\right)$, which yield

$$\begin{aligned}
|\langle s, \varphi_\lambda \rangle| &= \left| \int_{\frac{m}{2^j}}^{\frac{m+1}{2^j}} \left(x - \frac{m + \frac{1}{2}}{2^j}\right) 2^{\frac{j}{2}} \psi(2^j x - m) \,\mathrm{d}x \right| \\
&\leq \left| \int_{\frac{m}{2^j}}^{\frac{m+1}{2^j}} \left(\frac{m+1}{2^j} - \frac{m + \frac{1}{2}}{2^j}\right) 2^{\frac{j}{2}} \psi(2^j x - m) \,\mathrm{d}x \right| \\
&\leq 2^{\frac{j}{2}} \int_{\frac{m}{2^j}}^{\frac{m+1}{2^j}} \frac{1}{2^j} \underbrace{|\psi(2^j x - m)|}_{=1} \,\mathrm{d}x = \frac{1}{2} 2^{-\frac{3}{2} j}.
\end{aligned}$$

For $\varepsilon > 0$, this implies that for all scales

$$j > \frac{2}{3}\frac{\log(\varepsilon^{-1})}{\log 2} =: j_\varepsilon,$$

we have $|\langle s, \psi_\lambda\rangle| < \varepsilon$.

Thus there are at most $2^{j_\varepsilon+1}$ indices with $j \le j_\varepsilon$. Thus

$$\#\Lambda^R(\varepsilon) \le 2^{j_\varepsilon+1} = 2\cdot\varepsilon^{-\frac{2}{3}}.$$

Now consider $\Lambda^S(\varepsilon) := \{\lambda \in \Lambda^S : |\langle s, \psi_\lambda\rangle| > \varepsilon\}$. Let $l \in \Lambda^S$. Then

$$|\langle s, \psi_\lambda\rangle| = \left|\int_{\frac{m}{2^j}}^{\frac{m+1}{2^j}} s(x)2^{\frac{j}{2}}\psi(2^j x - m)\,\mathrm{d}x\right| = 2^{\frac{j}{2}}\int_{\frac{m}{2^j}}^{\frac{m+1}{2^j}} 1\,\mathrm{d}x = 2^{-\frac{j}{2}}.$$

Hence, for all

$$j > \frac{2\log(\varepsilon^{-1})}{\log(2)} =: j'_\varepsilon$$

we have $|\langle s, \psi_\lambda\rangle| < \varepsilon$.

We observe, that for each scale j, only there is only one m such that $(j,m) \in \Lambda^S$. Thus there are only j'_ε indices in Λ^S with $j \le j'_\varepsilon$. This implies

$$\#\Lambda^S \le j'_\varepsilon \le \varepsilon^{-\frac{2}{3}}.$$

This yields

$$\#\Lambda(\varepsilon) := \#\{\lambda \in \Lambda : |\langle s, \psi_\lambda\rangle| > \varepsilon\} = \#\Lambda^R(\varepsilon) + \#\Lambda^S(\varepsilon) \le C\cdot\varepsilon^{\frac{2}{3}}.$$

By Lemma 9.5 one can now conclude, that $(\langle s, \psi_\lambda\rangle)_\lambda \in l_{\frac{2}{3},w}$. This implies (Exercise) $\sigma_N(s,\Psi) \lesssim N^{-1}$. □

(9.5) Lemma. *For a sequence* $(c_\lambda)_{\lambda\in\Lambda}$ *and* $\varepsilon > 0$ *denote*

$$\Lambda(\varepsilon) := \{\lambda \in \Lambda : |c_\lambda| > \varepsilon\}.$$

Then

$$\|(c_\lambda)_{\lambda\in\Lambda}\|_{l_{p,w}} := \inf\{d : |c^*_\lambda| \le d\cdot n^{-\frac{1}{p}}\} \lesssim \inf\{C : \#\Lambda(\varepsilon) \le C\cdot\varepsilon^{-p}\,\forall\,\varepsilon > 0\},$$ [12]

where (c^*_λ) *represents the ordered sequence of* (c_λ).

Proof. Let $C > 0$ be such that

$$\#\Lambda(\varepsilon) \le C\varepsilon^{-p} \qquad \forall\,\varepsilon > 0.$$

Denote by $(c^*_n)_{n\in\mathbb{N}}$ the non-increasing rearangement of $(c_\lambda)_{\lambda\in\Lambda}$.

Let $N > 0$, then

$$N \le \#\{\lambda \in \Lambda : |c_\lambda| \ge c^*_N\} \le C\cdot(c^*_N)^{-p}.$$

Thus $c^*_N \le C^{\frac{1}{p}}N^{-\frac{1}{p}}$. □

10 Systems in $L^2(\mathbb{R}^2)$

10.1 Wavelet Bases

(10.1) Remark. Let H_1 and H_2 be Hilbert spaces, $v_1 \in H_1$ and $v_2 \in H_2$. A **tensor product** $v_1 \otimes v_2$ satisfies

(i) $\lambda(v_1\otimes v_2) = (\lambda v_1)\otimes v_2 = v_1 \otimes(\lambda v_2)$.

[12] $l_{p,w}$ denotes the **weak l_p-space**.

(ii) For all $w_1 \in H_1$ and $w_2 \in H_2$ we have

$$(v_1 + w_2) \otimes (v_2 + w_2) = (v_1 \otimes v_2) + (v_1 \otimes w_2) + (w_1 \otimes v_2) + (w_1 \otimes w_2).$$

Taking all tensor products and linear combinations gives a new Hilbert space $H_1 \otimes H_2$ with inner product

$$\langle v_1 \otimes v_2, w_1 \otimes w_2 \rangle_{H_1 \otimes H_2} = \langle v_1, w_1 \rangle_{H_1} + \langle v_2, w_2 \rangle_{H_2}.$$

(10.2) Theorem. *Let H_1, H_2 be Hilbert spaces and $(e_\lambda)_{\lambda \in \Lambda}$, $(f_\mu)_{\mu \in \Delta}$ be ONB of H_1 and H_2 respectively. Then, $(e_\lambda \otimes f_\mu)_{(\lambda,\mu) \in \Lambda \times \Delta}$ is an ONB for $H_1 \otimes H_2$.*

(10.3) Example. Let $H_1 = H_2 = L^2(\mathbb{R})$ and $f, g \in L^2(\mathbb{R})$, then

$$(f \otimes g)(x_1, x_2) := f(x_1) \cdot g(x_2)$$

and $H_1 \otimes H_2 = L^2(\mathbb{R}^2)$.

(10.4) Proposition. *Let $(V_j)_j$ be a MRA for $L^2(\mathbb{R})$ and φ be an associated scaling function. Then define*

$$V_j^{(2)} := V_j \otimes V_j \subset L^2(\mathbb{R}^2), \qquad j \in \mathbb{Z}.$$

Then $(V_j^{(2)})_j$ satisfies the following

(i) $\{0\} \subset \ldots \subset V_{-1}^{(2)} \subset V_0^{(2)} \subset V_1^{(2)} \subset \ldots \subset L^2(\mathbb{R}^2)$ and the $V_j^{(2)}$ are closed.

(ii) $\bigcap_j V_j^{(2)} = \{0\}$, $\overset{\bigcup_j V_j^{(2)}}{=} L^2(\mathbb{R}^2)$,

(iii) $f \in V_j^{(2)}$ if and only if $f(2\cdot, 2\cdot) \in V_{j+1}^{(2)}$ for all $j \in \mathbb{Z}$.

(iv) The system

$$\{\varphi^{(2)}(x_1 - m_1, x_2 - m_2) := \varphi(x_1 - m_1) \cdot \varphi(x_2 - m_2) : m_1, m_2 \in \mathbb{Z}\}$$

is an ONB (Riesz basis) for $V_0^{(2)}$.

Proof. Use Theorem 10.2, then it is straight forward. □

(10.5) Example. Let $\varphi := \chi_{[0,1)}$. Then $\varphi^{(2)} = \chi_{[0,1)^2}$.

(10.6) Definition. Let $(V_j^{(2)})_j$ be as in Proposition 10.4. Then define the associated wavelet spaces $W_j^{(2)}$ by

$$V_{j+1}^{(2)} = V_j^{(2)} \oplus W_j^{(2)}, \qquad j \in \mathbb{Z}.$$

(10.7) Theorem. *Let (V_j) be an MRA for $L^2(\mathbb{R})$ with scaling function φ and wavelet ψ. For $(x_1, x_2) \in \mathbb{R}^2$ define*

$$\psi^{(2),1}(x_1, x_2) = \varphi(x_1)\psi(x_2)$$
$$\psi^{(2),2}(x_1, x_2) = \psi(x_1)\varphi(x_2)$$
$$\psi^{(2),3}(x_1, x_2) = \psi(x_1)\psi(x_2)$$

Set $\psi_{j,m}^{(2),\iota}(x_1, x_2) := 2^{-j}\psi^{(2),\iota}(2^{-j}x_1 - m_1, 2^{-j}x_2 - m_2)$. Then $\{\psi_{j,m}^{(2),\iota} : m \in \mathbb{Z}^2, \iota \in [\![3]\!]\}$ is an ONB for $W_j^{(2)}$. Also $\{\psi_{j,m}^{(2),\iota} : j \in \mathbb{Z}, m \in \mathbb{Z}^2, \iota \in [\![3]\!]\}$ is an ONB for $L^2(\mathbb{R}^2)$.

Proof. Since$V_{j+1}^{(2)} = V_j^{(2)} \oplus W_j^{(2)}$, we also have $V_{j+1} \otimes V_{j+1} = (V_j \otimes V_j) \oplus W_j^{(2)}$. Also

$$V_{j+1} \otimes V_{j+1} = (V_j \oplus W_j) \otimes (V_j \oplus W_j) = (V_j \otimes V_j) \oplus (V_j \otimes W_j) \oplus (W_j \otimes V_j) \oplus (W_j \otimes W_j).$$

Thus

$$W_j^{(2)} = (V_j \otimes W_j) \oplus (W_j \otimes V_j) \oplus (W_j \otimes W_j).$$

The first claim follows from Theorem 10.2 and ONBs for V_j and W_j. The second claim follows from $V_{j+1}^{(2)} = V_j^{(2)} \oplus W_j^{(2)}$. □

(10.8) Example. Consider the Haar system.

$$\varphi = \chi_{[0,1)}, \qquad \psi = \chi_{[0,\frac{1}{2})} - \chi_{[\frac{1}{2},1)}.$$

For $(W_j^{(2)})_j$, we have the following generators

The first wavelet reacts to horizontal singularities, the second to vertical and the last to diagonal singularities.

(10.9) Remark. The fast wavelet transform (generalized to two dimensions) gives:

This is often visualized as

10.2 Shearlets

(10.10) Remark. *Question*: If a function $f \in L^2(\mathbb{R}^2)$ has not only a point singularit, but also curvelike singularities (e.g. a discontinuity curve), is there something better than wavelets?

Consider: $f_s : [0,1]^2 \to \mathbb{R}, (x_1, x_2) \mapsto \mathbb{1}_{\{x_1+sx_2>0\}}$. Let Ψ be the Haar wavelet system on $\mathbb{R}^2$. We index Ψ by Λ, i.e. $\Psi = (\psi_\lambda)_{\lambda\in\Lambda}$, and define $c_\lambda := \langle f_s, \psi_\lambda\rangle$, $\lambda \in \Lambda$. There are two cases:

(1) $\operatorname{supp} \psi_\lambda \cap \{(x_1, x_2) : x_1 + sx_2 = 0\} =: T = \emptyset$. Then

$$c_\lambda = \int f_s(x) 2^j \overline{\psi(2^j x - m)} \, \mathrm{d}x = 0,$$

which follows from the vanishing moments.

(2) $T \neq \emptyset$. Then,

$$c_\lambda = \int f_s(x) 2^j \overline{\psi(2^j x - m)} \, \mathrm{d}x = \int\limits_{[0,1]^2 \setminus \{x_1+sx_2 \leq 0\}} 2^j \overline{\psi(2^j x - m)} \, \mathrm{d}x \leq c \cdot 2^j.$$

Also for each scale j there are approximately 2^j such wavelets. We have $2^{-j} < \varepsilon$, or written differently $j > \log_2(\varepsilon)$. Thus,

$$\#\Lambda(\varepsilon) \approx \sum_{j=0}^{\log_2(\varepsilon)} 2^j \approx \varepsilon^{-1}.$$

Thus $\sigma_n(f_s, \Psi) \leq N^{-\frac{1}{2}}$. (But we can do better!)???

Idea: Use a different type of scaling $A_j = \begin{pmatrix} 2^j & 0 \\ 0 2^{\frac{j}{2}} & \end{pmatrix}$ and get

????

(10.11) Definition. The class $\mathcal{E}^2(\mathbb{R}^2)$ of cartoon-like functions is the set of functions $f : \mathbb{R}^2 \to \mathbb{C}$ of the form $f = f_0 + \chi_B f_1$, where $B \subset [0,1]^2$ is simply connected and ∂B is a C^2-curve with bounded curvature and $\operatorname{supp} f_i \subset [0,1]^2$, $f_i \in C^2(\mathbb{R}^2)$, $\|f_i\|_{C^2} \leq 1$, $i = 1, 2$.

(10.12) Theorem (Donoho, 2001). *Let $\Psi = (\psi_\lambda)_\lambda$ be a frame for $L^2(\mathbb{R}^2)$, the there exists $f \in \mathcal{E}^2(\mathbb{R}^2)$ with $\sigma_N(f, \Psi) \gtrsim N^{-1}$.*

Proof. “Sparse Components . . . ”, Constr. Approx. 17 (2001), 353-382. □

(10.13) Definition. We define the **parabolic scaling matrix** $A_j = \begin{pmatrix} 2^j & 0 \\ 0 & 2^{\frac{j}{2}} \end{pmatrix}$ and the **shearing matrix** $S_k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$. $(j, k \in \mathbb{Z})$

Let $\psi \in L^2(\mathbb{R}^2)$. Then the associated **shearlet system** is defined by

$$\mathcal{SH} = \{2^{\frac{3}{4} \cdot j} \psi \left(S_k A_j \cdot - m\right), j, k \in \mathbb{Z}, m \in \mathbb{Z}^2\}.$$

(10.14) Remark. (1) Each element of a shearlet system is associated with three parameter values (i) the scale j, (ii) the orientation k and (iii) the position m.

(2) The advantage of using a shearing matrix (as opposed to rotation) is the fact that $S_k\mathbb{Z}^2 \subset \mathbb{Z}^2$. Thus faithful implementations are possible.

(10.15) Definition. Choose $\psi \in L^2(\mathbb{R}^2)$ as follows: $\mathcal{F}\psi(\xi_1, \xi_2) := \mathcal{F}\psi_1(\xi_1) \cdot \mathcal{F}\psi_2\left(\frac{\xi_2}{\xi_1}\right)$, where

(a) (i) $\sum_j |\mathcal{F}\psi_1(2^{-j}\xi)|^2 = 1$,

(ii) $\mathcal{F}\psi_1 \in C^\infty$,

(iii) $\operatorname{supp} \mathcal{F}\psi_1 \subset \left[-\frac{1}{2}, -\frac{1}{16}\right] \cup \left[\frac{1}{16}, \frac{1}{2}\right]$.

(b) (i) $\sum_{k=1}^n |\mathcal{F}\psi_2(\xi + k)|^2 = 1$, ???

(ii) $\mathcal{F}\psi_2 \in C^\infty$,

(iii) $\operatorname{supp} \mathcal{F}\psi_1 \subset [-1, 1]$.

Then ψ is called **classical shearlet**.

(10.16) Remark. Let ψ be a classical shearlet.

Problem: There does not exist a shearlet which is vertically aligned (in Fourier domain).

Solution: We need to redefine classical shearlets.

(10.17) Definition. Let $\varphi, \psi, \widetilde{\psi} \in L^2(\mathbb{R}^2)$. Then the associated **cone-adapted shearlet system** is defined by

$$\begin{aligned}\mathcal{SH}(\varphi, \psi, \widetilde{\psi}) = \{\varphi(\cdot - m) : m \in \mathbb{Z}\} \\ \cup \{2^{\frac{1}{4}j}\psi(S_k A_j \cdot - m) : j \geq 0, |k| \leq \lceil 2^{\frac{j}{4}} \rceil, m \in \mathbb{Z}^2\} \\ \cup \{2^{\frac{3}{4}j}\widetilde{\psi}(S_k A_j R \cdot - m) : j \geq 0, |k| \leq \lceil 2^{\frac{j}{4}} \rceil, m \in \mathbb{Z}^2\},\end{aligned}$$

where $R(x_1, x_2) = (x_2, x_1)$.

(10.18) Theorem. *Let $\psi \in L^2(\mathbb{R}^2)$ be a classical shearlet.*

(i) $\mathcal{SH}(\psi)$ is a Parseval frame for $L^2(\mathbb{R}^2)$,

(ii) $\{2^{\frac{1}{4}j}\psi(S_k A_j \cdot - m) : j \geq 0, |k| \leq \lceil 2^{\frac{j}{2}} \rceil, m \in \mathbb{Z}^2\}$ is a Parseval frame for $\{f \in L^2(\mathbb{R}^2) : \operatorname{supp} \mathcal{F}f \subset$ cone$\}$.

(10.19) Theorem.

- *There exist decay conditions on $\mathcal{F}\varphi, \mathcal{F}\psi, \mathcal{F}\widetilde{\psi}$ such that $\mathcal{SH}(\varphi, \psi, \widetilde{\psi})$ forms a frame, also including compactly supported systems.*
- *Under certain decay conditions on $\mathcal{F}\varphi, \mathcal{F}\psi, \mathcal{F}\widetilde{\psi}$, and if $\mathcal{SH}(\varphi, \psi, \widetilde{\psi})$ forms a frame for $L^2(\mathbb{R}^2)$, we have $\sigma_n(f, \mathcal{SH}(\varphi, \psi, \widetilde{\psi}) \lesssim N^{-1} \cdot (\log N)^{\frac{3}{2}}$ for all $f \in \mathcal{E}^2(\mathbb{R}^2)$.*

(10.20) Remark. Note that this includes compactly supported systems / generators.

(10.21) Remark. There exists different systems based on parabolic which was introduced by Candès & Donoho called curvelets. It is based on rotation and also satisfies the optimal sparse approximation of cartoon-like impages (up to a log factor). But

- those are not affine systems, i.e. they are not generated by a single function)
- they don't provide uniform treatment of the continuous and discrete setting (due to the rotation)
- there is no compactly supported curvelet system available.

There is a more general concept called 'Parabolic-Molecules' of which curvelets and shearlets are a special case.

11 Inverse Problems

11.1 Getting started

(11.1) Remark. In an inverse problem one is interested to determine the cause of an observed / desired effect. A prototype is:

For a given function F and an element $y \in Y$ find $x \in X$ such that $F(x) = y$.

Most inverse problems are **ill-posed**. In the sense of Hadamard a mathematical problem is called **well-posed**, if:

- Existance: For any $y \in Y$ there exists some $x \in X$ such that $F(x) = y$;
- Uniqueness: For any $y \in Y$ there exists at most one solution $x \in X$ such that $F(x) = y$;
- Stability: The solution x depends continuously on the data y

are provided for F.

(11.2) Example. (i) Recover of missing data: Let $X = Y = \mathcal{H}$ and $\mathcal{H} = \mathcal{H}_m \oplus \mathcal{H}_k$ the missing and the known part. Further, let $F = P_{\mathcal{H}_k} : \mathcal{H} \to \mathcal{H}$ be the orthogonal projection onto $\mathcal{H}_k$.

Then the problem reads as follows. Then the problem reads as follows

$$F(x) = F(x_m + x_k) = F(x_k) = y.$$

(ii) Magnetic resonance imaging: $X = L^2([0,1]^2)$, $S \subset [0,1]^2$ and $y = \mathbb{R}^{\#S}$. Further, let

$$F : X \to Y, g \mapsto F(g) = (\mathcal{F}g(\xi))_{\xi \in S}\,.$$

(iii) Inverse scattering problem (non-linear): Let $f \in L^2(\mathbb{R}^2)$ moderthe scatterer. We assume that we emit time harmonic waves of the form $U(x,t) = e^{ikt}u(x)$, $k \in \mathbb{R}$. The observed waves are $u = u^s + u^{inc} \in H^2_{\text{loc}}(\mathbb{R}^2)$. Then we have to consider the Helmholtz equation (PDE)

$$\Delta u + k^2(1-f)u = 0.$$

Let $F : L^2(\mathbb{R}^2) \to H^2_{\text{loc}}(\mathbb{R}^2)$ be the operator that maps f to u satisfying the above PDE.

11.2 Basics of Linear Ill-Posed Problems

(11.3) Remark. Let's start with the finite-dimensional case: Consider

$$Ax = y$$

with $A \in \mathbb{R}^{n\times n}$[13] being symmetric and positive-definite. Then there exist eigenvalues $0 < \lambda_1 \leq \ldots \leq \lambda_n$ and corresponding eigenvectors $u_i \in \mathbb{R}^n$, $\|u_i\| = 1$, $i \in [\![n]\!]$ such that we can write

$$A = \sum_{i=1}^n \lambda_i u_i u_i^T.$$

Assume $\lambda_n = 1$. Then we get $\operatorname{cond}(A) = \frac{\lambda_n}{\lambda_1} = \lambda_1^{-1}$.

Let y^δ be the noisy data / version of y, i.e. let it suffice $\|y^\delta - y\| \leq \delta$ and let x^δ be the sollution of $Ax^\delta = y^\delta$. Then

$$\|x - x^\delta\|^2 = \left\|\sum_{i=1}^n \lambda_i^{-1} u_i u_i^T (y - y^\delta)\right\| = \sum_{i=1}^n |\lambda_i^{-1} u_i^T (y-y^\delta)|^2 \leq \lambda_1^{-1}\|y - y^\delta\|^2 = \operatorname{cond}(A)\cdot\|y-y^\delta\|^2.$$

This shows how the conditioning of A is directly responsible for the stability of solving the problem. The bound is sharp for $y - y^\delta = \delta u_1$.

[13]This can be generalized to rectangular matricies $A \in \mathbb{R}^{n\times m}$ by considering $A^TAx = A^Ty$ instead.

Further, not all possible versions of noise are equally bad, e.g. if $y^\delta - y = \delta u_n$, then $\|x - x^\delta\| = \delta$.

Of course, a main issue is the case where λ_1 is really small. The idea to work around this is to stabilize ('regularize') the problem. Shift λ_1 from zero away by considering for some $\alpha > 0$ the Matrix $A_\alpha := A + \alpha I$. The eigenvalues of A_α are $\lambda_i + \alpha$, $i \in [\![n]\!]$ and the eigenvectors stay the same.

Then

$$\|x - x_\alpha\| = \left\| \sum_{i=1}^n \left(\lambda_i^{-1} - (\lambda_i + \alpha)^{-1}\right) u_i u_i^T y \right\| = \left\| \sum_{i=1}^n \frac{\alpha}{\lambda_i(\lambda_i + \alpha)} u_i u_i^T y \right\|$$
$$\leq \frac{\alpha}{\lambda_1(\lambda_1 + \alpha)} \|y\| \xrightarrow{\alpha \to 0} 0.$$

For noisy data we get then

$$\|x - x_\alpha^\delta\| \leq \frac{\alpha}{\lambda_1(\lambda_1 + \alpha)} \left(\|y^\delta\| + \delta\right) + \frac{\delta}{\lambda_1 + \alpha}.$$

(11.4) Remark. Let X, Y be Hilbert spaces and $A : X \to Y$ be a linear bounded operator.

If $\operatorname{ran} A \subsetneqq Y$ (maybe not even dense), then $Ax = y$ is not solvable for some $y \in Y$. In this situation some redemption must be made.

(11.5) Definition. Let X, Y be Hilbert spaces, $A : X \to Y$ be a linear bounded operator. Then $x \in X$ is called

(i) **least squares solution** of the inverse problem, if

$$\|Ax - y\| = \inf\{\|Az - y\| : z \in X\},$$

(ii) **best-approximation solution** or **minimal norm solution** of the inverse problem if

$$\|x\| = \inf\{\|z\| : z \text{ is a least squares solution of the inverse problem.}\}.$$

(11.6) Remark. Least squares solution or minimal norm solutions might not exists. If it exists, then we can characterize them using the generalized inverse.

(11.7) Definition. Let $A \in L(X, Y)$ and $\widetilde{A} : (\ker A)^\perp \to \operatorname{ran} A$. be the restriction of A. Then the **(Moore-Penrose) generalized inverse** or **pseudo inverse** A^+ is defined as the unique extension of $\widetilde{A}^{-1}$ to $\mathcal{D}(A^+) := \operatorname{ran} A \oplus (\operatorname{ran} A)^\perp$ with $\ker A^+ = (\operatorname{ran} A)^\perp$.

(11.8) Lemma. *A^+ is well-defined.*

Proof. By construction $\widetilde{A}$ is bijective and hence $\widetilde{A}^{-1}$ exists. Therefore A^+ is well defined on $\operatorname{ran} A$. For $y \in \mathcal{D}(A^+)$, there exists a unique $y = y_1 + y_2 \in \operatorname{ran} A \oplus (\operatorname{ran} A)^\perp$. We get

$$A^+ y = \widetilde{A}^{-1} y_1.$$

□

(11.9) Theorem. *Let $y \in \mathcal{D}(A^+)$.*

(i) The inverse problem has a minimal norm solution given by $x^+ = A^+ y$. The set o all minimal norm solutions is $x^+ + \ker A^+$.

*(ii) $x \in X$ is a least-squares solution of the inverse problem if and only if x satisfies the **Gaussian normal equation***

$$A^* A x = A^* y.$$

Proof. (i) is an exercise.

(ii): By definition x is a least squares solution if and only if $Ax = P_{\operatorname{ran} A} y$ which is equivaent to $(Ax - y) \in (\operatorname{ran} A)^\perp$. Since $(\operatorname{ran} A)^\perp = \ker A^*$ this is equivalent to the Gaussian normal equation. □

(11.10) Theorem. *Let X, Y be infinite dimensional Hilbert spaces and $A : X \to Y$ be a linear compact with infinite dimensional range. Then the inverse problem is ill-posed, in particular, A^+ is discontinuous.*

Proof. Since $\dim(\operatorname{ran} A) \le \dim(\ker A)^\perp$ we have, that $(\ker A)^\perp$ is infinite-dimensional.

Hence there exists a sequence $(x_n)_n \subset (\ker A)^\perp$ which is normalized and pairwise orthogonal. Since A is compact, $(y_n) = (Ax_n)$ has a convergent subsequence. Thus for each $\varepsilon > 0$, we find m, n such that

$$\|y_n - y_m\| < \varepsilon,$$

but

$$\|A^+ y_n - A^+ y_m\| = \|x_n - x_m\|^2 = \|x_n\|^2 + \|x_m\|^2 = 2 > 0.$$

Hence A^+ is not continuous. $\square$

(11.11) Remark. Let A be linear, compact. Since $B := A^*A$ and $C := AA^*$ are self-adjoint, we have by spectral decomposition

$$Bx = \sum \sigma_n^2 \langle x, u_n \rangle u_n.$$

$$Cy = \sum \widetilde{\sigma}_n \langle y, v_n \rangle v_n.$$

for al $x, y \in X$. Then consider

$$\widetilde{\sigma}_n^2 A^* v_n = A^* C v_n = A^* A A^* v_n = B A^* v_n.$$

Further, we assume

$$\widetilde{\sigma}_n = \sigma_n \quad \text{and} \quad v_n = \frac{Au_n}{\|Au_n\|}.$$

(11.12) Definition. Let A be a linear compact operator. Then $(\sigma_n, u_n, v_n) \subset \mathbb{R}^+ \times X \times X$ is called a **singular system** (as constructed above) and

$$Ax = \sum_i \sigma_i \langle x, u_i \rangle v_i \qquad \forall\, x \in X$$

the **singular-value decomposition**.

(11.13) Remark. We have

$$A^* y = \sum_i \sigma_i \langle y, v_i \rangle u_i \qquad \forall\, y \in Y.$$

Regarding the convergence of the series we note that for any $x \in X$

$$\left\| \sum_{i=1}^n \sigma_i \langle x, u_i \rangle v_i \right\|^2 = \sum_{i=1}^n \sigma_i^2 \langle x, u_i \rangle^2 \le \sigma_1^2 \|x\|^2$$

after ordering the σ_i decreasingly. This is a uniform bound in n.

(11.14) Theorem. *Let A be a linear compact operator and A^+ its generalized inverse. Further let (σ_n, u_n, v_n) be a singular system. Then*

$$A^+ y = \sum \frac{1}{\sigma_n} \langle y, v_i \rangle u_i \quad \forall\, y \in Y.$$

Proof. Set $x^+ := A^+ y$. Then we have

$$\sum \sigma_n^2 \left\langle x^+, u_n \right\rangle u_n = A^* A x^+ = A^* y = \sum_n \sigma_n \langle y, v_n \rangle u_n.$$

Therefore $\langle x^+, u_n \rangle = \frac{1}{\sigma_n} \langle y, v_n \rangle$ for all n. Thus, the claim follows. $\square$

11.3 Regularization of linear ill-posed problems

(11.15) Remark. Regularization shall give a stable approximation to the sought solution.

Idea: Choose $R_\alpha \subset L(Y,X)$, $\alpha \in I \subset (0,\alpha_0)$ such that $R_\alpha \to A^+$ as $\alpha \to 0$ on $\mathcal{D}(A^+)$. For $y \in Y \setminus \mathcal{D}(A^+)$ we expect $\|R_\alpha y\| \to \infty$ as $\alpha \to 0$. If $\|y - y^\delta\| \leq \delta$, we aim for a sequence $(\alpha(\delta, y^\delta))$ such that $R_{\alpha(\delta,y^\delta)} y^\delta \to A^+ y$ as $\delta \to 0$ for $y \in \mathcal{D}(A^+)$.

(11.16) Definition. (i) A sequence $\{\mathbb{R}_\alpha\}_{\alpha \in I} \subset L(Y,X)$ is called a **regularization scheme** for A^+ if for all $y \in \mathcal{D}(A^+)$ there exists $\alpha : \mathbb{R}^+ \times Y \to I$, a so-called **parameter choice rule**, such that

$$\limsup_{\delta \to 0} \{\|R_{\alpha(\delta,y^\delta)} y^\delta - A^+ y\| : y^\delta \in Y, \|y - y^\delta\| \leq \delta\} = 0$$

and

$$\limsup_{\delta \to 0} \{\alpha(\delta, y^\delta) : y^\delta \in Y, \|y - y^\delta\| \leq \delta\} = 0.$$

For a specific $y \in \mathcal{D}(A^+)$, the pair $(\mathbb{R}_a, \alpha)$ is called **(convergent) regularization method** of the inverse problem if the two conditions are fulfilled.

(ii) The rule $\alpha = \alpha(\delta, y^\delta)$ is called **a-priori**, if it does not depend on y^δ and **a-posteriori** otherwise.

(11.17) Theorem. *Let $A \in L(X,Y)$ and $(\mathbb{R}_\alpha)_\alpha$ be a regularization of A^+ such that it converges for every $y \in \mathcal{D}(A^+)$ and such that the parameter choice rule depends only on y^δ (and not directly on δ). Then A^+ can be extended to a continuous operator from Y to X.*

In particular, this means that such a strategy only works for well-posed problems.

Proof. For $\alpha = \alpha(y^\delta)$ we have by the first equation of Definition 11.16 (i), that

$$\limsup_{\delta \to 0} \{\|R_{\alpha(y^\delta)} y^\delta - A^+ y\| : y^\delta \in Y, \|y - y^\delta\| \leq \delta\} = 0$$

and in particular $R_{\alpha(y)} y = A^+ y$ for all $y \in \mathcal{D}(A^+)$. Thus, for any $(y_n)_n \subset \mathcal{D}(A^+)$ with $y_n \to y$ we have

$$A^+ y_n = R_{\alpha(y_n)} y_n \to R_{\alpha(y)} y = A^+ y.$$

Since $\mathcal{D}(A^+)$ is dense in Y we can extend A^+ to a continuous operator on Y. $\square$

(11.18) Proposition. *Let $A \in L(X,Y)$ and $(R_\alpha)_\alpha \subset L(Y,X)$. Then R_α is a regularization for A^+ if $R_\alpha \to A^+$ pointwise on $\mathcal{D}(A^+)$ for $\alpha \to 0$.*

Sketch of the proof. There exists $\sigma : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $\varepsilon > 0$ we have

$$\|R_{\sigma(\varepsilon)} y - A^+ y\| \leq \frac{\varepsilon}{2}$$

for all $y \in \mathcal{D}(A^+)$. The operator $R_{\sigma(\varepsilon)}$ is continuous for fixed ε, and thus there exists $\rho(\varepsilon) \in I$ such that

$$\|R_{\sigma(\varepsilon)} z - R_{\sigma(\varepsilon)} y\| \leq \frac{\varepsilon}{2}$$

if $\|z - y\| \leq \rho(\varepsilon)$. W.l.o.g. we can assume ρ to be monotonically increasing, continuous and $\lim_{\varepsilon \to 0} \rho(\varepsilon) = 0$. Then there exists ρ^{-1} on $\rho(\mathbb{R}^+)$ which can be extended to $\mathbb{R}^+$. Define

$$\alpha : \mathbb{R}^+ \to \mathbb{R}^+, \delta \mapsto \sigma(\rho^{-1}(\delta)).$$

Now it is left to check whether (R_α, α) is a regularization method. $\square$

(11.19) Proposition. *Let $A \in L(X,Y)$, $R_\alpha \subset L(Y,X)$ a regularization operator with a-priori parameter choice rule $\alpha = \alpha(\delta)$. Then the following two statements are equivalent:*

(i) $(\mathbb{R}_\alpha, \alpha)$ *is a convergent regularization method;*

(ii) $\lim_{\delta \to 0} \alpha(\delta) = 0$ *and* $\lim_{\delta \to 0} \delta \cdot \|R_{\alpha(\delta)}\| = 0$.

Proof. "(ii) ⇒ (i)": For $y \in Y$ with $\|y - y^\delta\| \le \delta$ we have

$$\|R_{\alpha(\delta)}y^\delta - A^+y\| \le \|x_\alpha^\delta - A^+y\| + \|x_\alpha^\delta - R_{\alpha(\delta)}y^\delta\| \le \|x_\alpha^\delta - A^+y\| + \delta\|R_{\alpha(\delta)}\| \to 0,$$

due to pointwise convergence (cf. Proposition 11.18) and (ii).

"(i) ⇒ (ii)": Assume there exists $\delta_n \to 0$ with $\delta_n\|R_{\alpha(\delta_n)}\| \ge \frac{C}{2} > 0$ for some fixed C. Then there exists a normalized sequence (z_n) with $\delta_n\|R_{\alpha(\delta_n)}z_n\| \ge \frac{c}{2}$.

Moreover, for all $y \in \mathcal{D}(A^+)$ and $y_n = y + \delta_n z_n$ we have $\|y_n - y\| \le \delta_n$ but

$$R_{\alpha(y_n)}y_n - A^+y = (R_{\alpha(y_n)}y - A^+y) + \delta_n R_{\alpha(y_n)}z_n$$

which does not converge to zero. This gives a contradiction. □

(11.20) Remark. An ill-posed problem can be translated into a problem satisfying the first two properties in Remark 11.1 by considering A^+. For stability the problem s that the spectrum of A is not necessarily bounded away from zero. The solution lies in modifying the smallest singular value by regularization.

We now construct R_α with

$$R_\alpha y := \int_{n=1}^{\infty} g_\alpha(\sigma_n)\,\langle y, v_n\rangle\, u_n,$$

for all $y \in Y$. Here, $g_\alpha : \mathbb{R}^+ \to \mathbb{R}^+$, $g_\alpha(\sigma) \to \sigma^{-1}$ for $\sigma > 0$ and $\alpha \to 0$. Such an operator is indeed a regularization method, if there exists $C_\alpha < \infty$ such that $\|g_\alpha\|_\infty \le C_\alpha$.

If this is the case, then

$$\|R_\alpha\|^2 = \sum_{n=1}^{\infty} g_\alpha(\sigma_n)^2\,|\langle y, v_n\rangle|^2 \le C_\alpha^2\|y\|^2.$$

This means C_α is a bound for the operator R_α. Pointwise convergence of g_α implies R_α converges pointwise to A^+. Since $\|R_\alpha \le C_\alpha$, Proposition 11.19 (ii) can be replaced by $\lim_{\delta\to 0}\delta\cdot C_{\alpha(\delta)} = 0$. This is now a condition for R_α to be a regularization method.

(11.21) Example. (i) Truncated SVD: Consider $g_\alpha(\sigma) = \frac{1}{\sigma}\cdot \mathbb{1}_{\sigma>\alpha}$. Then $C_\alpha \le \frac{1}{\alpha}$ and $(\mathbb{R}_\alpha, \alpha)$ is a convergent regularization method if $\frac{\delta}{\alpha} \to 0$. The representation of the regularized solution is given by

$$x_\alpha := \mathbb{R}_\alpha y = \sum_{\delta_n \ge \alpha} \frac{1}{\sigma_n}\,\langle y, v_n\rangle\, u_n, \qquad y \in Y.$$

(ii) Lavrienhev Regularization: Set $g_\alpha(\delta) := \frac{1}{\sigma+\alpha}$ and then

$$x_\alpha := \mathbb{R}_\alpha y = \sum_{n} \frac{1}{\sigma_n + \alpha}\,\langle y, v_n\rangle\, u_n, \qquad y \in Y.$$

If A is positive semidefinite (which already implies ($\lambda_n = \sigma_n$, $u_n = v_n$ for all $n \in \mathbb{N}$), then one obtains

$$(A + \alpha I)x_\alpha = \sigma_n\,\langle y, v_n\rangle\, u_n = y.$$

Hence, x_α can be derived by solving a linear equation. Again, we have $C_\alpha \le \frac{1}{\alpha}$, thus, this represents a convergent regularizition method if $\frac{\delta}{\alpha} \to 0$.

(iii) Tikhonov Regularization: Define

$$g_\alpha = \frac{\sigma}{\sigma^2 + \alpha}$$

and $x_\alpha := R_\alpha y$ as before. Since $\sigma_n^2 + \alpha \ge 2\sigma_n\sqrt{\alpha}$, we have $g_\alpha(\sigma) = C_\alpha := (4\alpha)^{-\frac{1}{2}}$. This is convergent regularization method if $\frac{\delta}{\sqrt{\alpha}} \to 0$. Then x_α can be computed from

$$(A^*A + \alpha I)x = A^*y.$$

Thus, we can obtain x_α by solving a well-posed problem.

11.4 Tikhonov Regularization

(11.22) Remark. We now consider a non-linear equation of the form

$$F(x) = y,$$

where $F : X \to Y$ is a continuous non-linear operator. Problem: No SVD can be applied nor adjoint can be considered.

Recall that in Example 11.21 the Tikhonov regularization of a linear operator eequations solutions obey

$$(A^*A + \alpha I)x = A^*y^\delta.$$

This equation is the first order optimality condition for

$$\min_{x\in X} J_\alpha(x) := \min_{x\in X} \|Ax - y^\delta\|^2 + \alpha\|x\|^2$$

and $J_\alpha(x)$ is strictly convex, hence there exists a unique minimizer.

Proof.

$$J'\alpha(x) = 2A^*(Ax - y^\delta) + 2\alpha x$$

Hence $J'\alpha(x) = 0$, or equivalently $(A^*A + \alpha I)x = A^*y^\delta$. Then $J''_\alpha(x) = 2A^*A + 2\alpha$ which is strictly positive. Thus J_α is strictly convex. □

This gives us a way to extend to the nonlinear case:

(11.23) Definition. We shall call $\overline{x} \in X$ least squares solution of $F(x) = y$, if

$$\|F(\overline{x}) - y^\delta\| = \inf\{\|F(x) - y^\delta\| : x \in X\}.$$

a least squares solution x^+ is called x^*-minimal norm solution if

$$\|x^+ - x^*\| = \inf\{\|x - x^*\| : x \text{ is a least squares solution}\}.$$

(11.24) Theorem. *Let $F : X \to Y$ be continuous and weakly sequentially closed, that is, if $x_n \hookrightarrow x$ and $F(x_n) \hookrightarrow y$, then $F(x) = y$. Then there exists a minimizer $x_\alpha^\delta \in X$ of the functional*

$$J_\alpha(x) = \|F(x) - y^\delta\|^2 + \alpha\|x - x^*\|^2.$$

Proof. Define the following level sets

$$L_M = \{x \in X : J_\alpha(x) \leq M\}.$$

Since $J_\alpha(x^*) - y^\delta\|^2 < \infty$, L_M is non empty for sufficiently large M. Morevover, $x \in L_M$ then $\alpha\|x - x^*\|^2 \leq M$ and due to the triangle inequality

$$\|x\| \leq \|x^*\| + \sqrt{\frac{M}{\alpha}} =: R$$

Therefore L_M is contained in a ball of radius R. Since balls in X are compact with respect to the weak topology the sets L_M are weakly pre-compact. Since J_α is bounded from below by zero, its infimum is finite and thus there exists a minimizing sequence that converges to $\overline{x} \in X$. Moreover, the sequence $F(x_n)$ is bounded $\|F(x_n) - y^\delta\|^2 \leq M$ and hence there exists a weakly convergent subsequence (again denote by indices n) with $F(x_n) \hookrightarrow z \in Y$.

Because of the weak sequential closedness of F we must have $F(\overline{x}) = z$ and thus $J_\alpha(\overline{x}) = \lim_n J_\alpha(x_n) = \inf_{x\in X} J_\alpha(x)$, i.e. $x_\alpha^\delta := \overline{x}$ is a minimizer of J_α. □

(11.25) Theorem. *Let $F : X \to Y$ be continuous and weakly sequentially closed. Moreover, let $y_n \in Y$ such that $y_n \to y^\delta$ and let x_n be a sequence of minimizers of J_α with y^δ replaced by y_n. Then x_n has a weakly convergent subsequence and any accumulation point is a minimizer of J_α.*

Proof. Due to Theroem 11.24 there exists a sequence x_n corresponding to the y_n. Since

$$\|x_n - x^*\|^2 \leq \frac{1}{\alpha}\|F(x_n) - y_n\|^2 + \|x_n - x^*\|^2 \leq \|F(x^*) - y_n\|^2$$

and y_n converges to y^δ, x_n is contained in a ball with radius independent of n.

Due to the weak compactness, we can extract a convergent subsequence (again indexed by n). Now let x_N be an accumulation point of x_n, so $x_n \hookrightarrow x$. Since $\|F(x_n) - y_n\| \leq \|F(x^*) - y_n\|$ we conclude $F(x_n)$ is bounded and consequently there exists a subsequence of $F(x_n)$ with limit z and the weak sequential closedness of F sequres $F(x) = z$.

Finally from the weak lower semicontinuity of the square of norms in Hilbert spaces we conclude

$$\begin{aligned} J_\alpha(x) = \|F(x) - y^\delta\|^1 + \alpha\|x - x^*\|^2 &\leq \liminf_{n\to\infty} \|F(x_n) - y_n\|^2 + \alpha\|x_n - x^*\|^2 \\ &\leq \liminf_{n\to\infty} \|F\left(x_\alpha^d\right) - y_n\|^2 + \alpha\|x_\alpha^\delta - x^*\|^2 = \|F(x_\alpha^\delta) - y^\delta\|^2 + \alpha\|x_\alpha^\delta - x^*\| = J_\alpha(x_\alpha^\delta) \end{aligned}$$

where x_α^δ is the minimizer of J_α. □

(11.26) Theorem. *Let $y \in Y$ be such that there exists a x^*-minimal norm solution $x^+ \in X$ with $F(x^+) = y$. Let y^δ be noisy data of y and let x_α^δ be a regularized solution that obeys*

$$x_\alpha^\delta \in \arg\min\{\|F(x) - y^\delta\|^2 + \alpha\|x - x^*\|^2\}.$$

If $\alpha = \alpha(\delta, y^\delta)$ is chosen such that $\alpha \to 0$ and $\frac{\delta^2}{\alpha} \to 0$ as $\delta \to 0$. Then there exists a strongly convergent subsequence $x_{\alpha_n}^{\delta_n}$ (as $\delta_n \to 0$) and the limit of each convergent subsequence is an x^*-minimal norm solution of $F(x) = y$.

Sketch of the Proof. Since x_α^δ is a minimal norm solution we obtain

$$\|x_\alpha^\delta - x^*\| \leq \frac{\delta^2}{\alpha} + \|x^+ - x^*\|^2.$$

Since the first term converges to zero, it is bounded and in part $\|x_\alpha^\delta - x^*\|$ is uniformly bounded (with respect to δ) which allows us to extract a weakly convergent subsequence.

For $(x_{\alpha_n}^{\delta^n})$ being a weakly convergent subsequence with limit $\overline{x}$, the above estimate gives

$$\|\overline{x} - x^*\|^2 \leq \|x^+ - x^*\|^2$$

and $\|F(\overline{x}) - y\|^2 = 0$ and $\overline{x}$ is a minimal norm solution of $F(x) = y$. The final step is to show strong convergence of $(x_{\alpha_n}^{\delta_n}$ to $\overline{x}$.

□

11.5 Generalization of Tikhonov Regularization

(11.27) Remark. Before, we had considered

$$J_\alpha(x) = \|F(x) - y^\delta\|^2 + \alpha\|x\|^2.$$

Problem: We want to have a method that is more adapted to the data / what we expect.

Solution: Given a suitable functional that is non-negative $P : X \to \mathbb{R}^+$, consider the following generalization of the Thikhonov regularization

$$\widetilde{J}_\alpha = \|F(x) - y^\delta\|^2 + \alpha P(x).$$

Some possible choices include the total variation norm $P(\cdot) = \|\cdot\|_{TV}$, Sobolev-norms $P(\cdot) = \|\cdot\|_{H^k}$ and the L^1-norm $P(\cdot) = \|\cdot\|_{L^1}$. Further it is possible to chain these with operators e.g. $P(\cdot) = \|\Psi \cdot\|_{l^1}$ with the analysis operator $\Psi : H \to l^1$.

(11.28) Remark. We now consider the latter.

Idea: Let $(\psi_\lambda)_{\lambda\in\Lambda}$ be a representation system that provides (optimally) sparse approximation in the sense of decay of coefficients / error of best N-term approximation.

The l^1 promotes sparsity $\rightsquigarrow$ Compressed sensing.

(11.29) Example (cf. Example 11.2). (i) Missing data for images: $X = Y = L^2(\mathbb{R}^2)$. Assume the original image can be nicely represented using e.g. shearlets.

$$\min_f \| \underbrace{F(f)}_{f_k} - y^\delta \|^2 + \alpha \| (\langle f, \sigma_\eta \rangle)_\eta \|_{l^1}$$

(ii) Magneting resonance imaging: Solve

$$\min_f \| \widehat{f} - y^\delta \|^2 + \alpha \| (\langle f, \sigma_\eta \rangle)_\eta \|_{l^1}$$

(iii) Inverse scattering problem:

$$\min_f \| F(f) - y^\delta \|^2 + \alpha \| (\langle f, \sigma_\eta \rangle)_\eta \|_{l^1}$$

Where F involes the multistatic measurement operator.

12 Compressed Sensing

12.1 Main Idea of Compressed Sensing

Let us consider the system of linear equations $Ax = y$ with $A \in \mathbb{R}^{m\times n}$, $y \in \mathbb{R}^m$, $x \in \mathbb{R}^n$. In compressed sensing we regard this equation as *linear measurement process*, i.e. the vector x is encoded by m linear measurements $y \in \mathbb{R}^m$ using a know matrix A. The major challenge concerns the compressibility of x, i.e. the question: How small can the number of measurements m be such that we can still (uniquely) recover x from $Ax = y$ in an efficient way?

For the case $m < n$, this task is impossible to solve in general, since the system $Ax = y$ is then underdetermined. Hence, we need to impose further assumptions (prior knowledge) on the objective vector x. In many situations the vector x is not supported everywhere, i.e. has only very few non-zero entries compared to the dimension of the ambient space $\mathbb{R}^n$.

(12.1) Definition. We define $\|x\|_0 := |\operatorname{supp}(x)| := |\{j : x_j \neq 0\}|$, that is, the number of non-zero entries of $x \in \mathbb{R}^n$. A vector x is called **k-spars**, if $\|x\|_0 \leq k$.

A very natural approach is now to solve the optimization problem:

$$\min_{z\in\mathbb{R}^n} \|z\|_0, \text{s.t. } Az = y, \tag{P_0}$$

i.e. we are looking for the sparsest vector / solution of $Ax = y$. Unfortunately, this problem is numerically intractable if m and n are large, in particular we have the following result:

(12.2) Theorem. *The l^0-minimization problem (P_0) is NP-hard in general.*

Proof. See for example [1]. □

We instead consider the convex relaxation of this problem (P_0), which is known as **basis pursuit**:

$$\min_{z\in\mathbb{R}^n} \|z\|_1, \text{s.t. } Az = y. \tag{P_1}$$

This problem is convex and can be solved by linear programming.

The l^1-norm is used instead of the l^2-norm since the l^1 solution wil be sparser than the l^2 one.

Our major goals are to find a measurement matrix $A \in \mathbb{R}^{m\times n}$ such that we have:

(a) **Efficient recovery**: Every k-sparse vector $x \in \mathbb{R}^n$ is the unique solution of (P_1), with input $y = Ax$, $(k \ll n)$.

(b) **Strong compression**: The number of measurements m should be as small as possible.

12.2 Null Space Property

Notation: Let $T \subset [\![n]\!]n$ and $T^c = [\![n]\!] \setminus T$ in complement in $[\![n]\!]$. If $v \in \mathbb{R}^n$ is any vector, then we denote by v_T either the vector in $\mathbb{R}^{|T|}$, which contains the coordinates of v on T, or the vector in $\mathbb{R}^n$, which equals v on T and is zero elsewhere (on T^c). Moreover, for $A \in \mathbb{R}^{m\times n}$ we denote A_T the $m \times |T|$ submatrix containing the columns of A indexed by T.

(12.3) Definition. Let $A \in \mathbb{R}^{m\times n}$ and let $k \in [\![n]\!]$. Then A is said to have the **null space property** (NSP) of order k if $\|v_T\|_1 \le \|v_{T^c}\|_1$ for all $v \in \ker A$ and all $T \subset [\![n]\!]$ with $|T| \le k$.

(12.4) Remark. If $v \in \ker A$ is k-sparse with $T = \operatorname{supp}(v)$ and $A \in \mathbb{R}^{m\times n}$ has the NSP of order k, then $v = 0$ by definition.

Thus, the kernel of A does not contain vectors that are supported on a small set (of size at most k).

The following theorem shows that the NSP is a necessary and sufficient condition for efficient recovery.

(12.5) Theorem. *Let $A \in \mathbb{R}^{m\times n}$ and let $k \in [\![n]\!]$. Then every k-sparse vector $x \in \mathbb{R}^n$ is the unique solution of (P_1) with input $y = Ax$ if and only if A has the null space property of order k.*

Proof. Let us assume that every k-sparse vector x is the unique solution of (P_1) with input $y = Ax$. Let $v \in \ker A$ and $T \in [\![n]\!]$ with $|T| \le k$. Then, v_T is k-sparse and therefore the unique solution of:

$$\min_{z\in\mathbb{R}^n} \|z\|_1, \text{s. t. } Az = Av_T.$$

Since, $A(-v_{T^c}) = A(v - v_{T^c}) = A(v_T)$. This particularly yields $\|v_T\|_1 \le \|v_{T^c}\|_1$ and A has the NSP of order k.

Conversly, let us assume that A has the NSP of order k. Let x be a k-sparse vector and $T = \operatorname{supp}(x)$.

We have to show that $\|x\|_1 < \|z\|_1$ for every $z \in \mathbb{R}^n$ different from x with $Ax = Az$. This follows immediatly from the NSP condition, in fact, for $(x - z) \in \ker A$:

$$\|x\|_1 \le \|x - z_T\|_1 + \|z_T\|_1 < \|(x - z)_{T^c}\|_1 + \|z_T\|_1 = \|z_{T^c}\| + \|z_T\|_1 = \|z\|_1. \qquad \square$$

(12.6) Remark. Theorem 12.5 implies that if A has NSP of order k, the solution of (P_0) can be determined by (P_1) for all k-sparse vectors. Thus, (P_0) is contained in the complexity class P when restricted to the set of k-sparse vectors.

12.3 Restricted Isometry Property

We want to answer the question, how to construct matrices that satisfy the NSP of low order.

(12.7) Definition. Let $A \in \mathbb{R}^{m\times n}$ and $k \in [\![n]\!]$. Then the restricted isometry constant $0 < \delta_k = \delta_k(A)$ of A of order k is the smallest constant such that

$$(1 - \delta_k)\|x\|_2^2 \le \|Ax\|_2^2 \le (1 + \delta_k)\|x\|_2^2$$

for all k-sparse vectors $x \in \mathbb{R}^n$. We say that A satisfies the **restricted isometry property** (RIP) of order k with constant $0 < \delta_k < 1$.

(12.8) Remark. The RIP states that A acts almost isometrically when restricted to k-sparse vectors, implying that distances and angles are almost preserved on this subset. Moreover, we trivially have $\delta_1(A) \le \delta_2(A) \le \dots$.

The following shows that a sufficiently small RIP constant implies the NSP.

(12.9) Theorem. *Let $A \in \mathbb{R}^{m\times n}$ and let $k \in \mathbb{N}$ with $k \le \frac{n}{2}$. If $\delta_{2k}(A) < \frac{1}{3}$, then A has the NSP of order k.*

Proof. Let $v \in \ker A$ and $T \subset [\![n]\!]$, $|T| \le k$. We shall show that

$$\|v_T\|_2 \le \frac{\delta_{2k}}{1 - \delta_k} \cdot \frac{\|v\|_1}{\sqrt{k}}. \tag{1}$$

If $\delta_k \le \delta_{2k} < \frac{1}{3}$, then by Hölder's inequality

$$\|v_T\|_1 \le \sqrt{k}\|v_T\|_2 \le \frac{\delta_{2k}}{1-\delta_k}\|v\|_1 < \frac{\frac{1}{3}}{1-\frac{1}{3}}\|v\|_1 = \frac{1}{2}\|v\|_1 = \frac{1}{2}\left(\|v_T\|_1 + \|v_{T^c}\|_1\right).$$

Hence $\|v_T\|_1 \le \|v_{T^c}\|_1$, which shows the NSP of order k.

It is left to prove (1). Observe the following: if $x, z \in \mathbb{R}^n$ are k-sparse with disjoint supports then $x \pm z$ is $2k$-sparse. Moreover, if $\|x\|_2^2 = \|z\|^2 = 1$, then $\|x \pm z\|_2^2 = 2$. Now the RIP applied to $x \pm z$ allows, combied with the polarization identity, allows

$$|\langle Ax, Az\rangle| = \frac{1}{4}\left|\|Ax + Az\|_2^2 - \|Ax - Az\|_2^2\right| \le \delta_{2k}.$$

Hence, if A has the RIP of order $2k$ and $x, z \in \mathbb{R}^n$ are k-sparse with disjoint supports, then we have:

$$|\langle Ax, Az\rangle| \le \delta_{2k}\|x\|_2\|z\|_2. \tag{2}$$

In order to show (1), let us asssume that $v \in \ker A$ fixed. It is now enough to consider $T = T_0$ where T_0 is the set that contains the k-largest entries of v in magnitude. Furthermore, we denote by T_1 the set of k largest entries of $v_{T_0^c}$ in magnitude and by T_2 the set of k largest entries of $v_{(T_0 \cup T_1)^c}$ in magnitude and so on.

By assumption we know $0 = A(v) = A(v_{T_0} + v_{T_1} + \ldots)$ which compbined with (2) yields:

$$\begin{aligned}\|v_{T_0}\|_2^2 &\le \frac{1}{1-\delta_k}\|Av_{T_0}\|_2^2 = \frac{1}{1-\delta_k}\langle Av_{T_0}, A(-v_{T_1}) + A(-v_{T_2}) + \ldots\rangle \\ &\le \frac{1}{1-\delta_k}\sum_{j\ge 1}\left|\langle Av_{T_0}, Av_{T_j}\rangle\right| \le \frac{d_{2k}}{1-\delta_k}\sum_{i\ge 1}\|v_{T_0}\|_2\|v_{T_j}\|_2.\end{aligned}$$

Hence

$$\begin{aligned}\sum_{j\ge 1}\|v_{T_j}\|_2 &= \sum_{j\ge 1}\left(\sum_{l\in T_j}|v_l|^2\right)^{\frac{1}{2}} \le \sum_{j\ge 1}\left(k\cdot\max_{l\in T_j}|v_l|^2\right)^{\frac{1}{2}} = \sum_{j\ge 1}\sqrt{k}\max_{l\in T_j}|v_l| \\ &\le \sum_{j\ge 1}\sqrt{k}\min_{l\in T_{j-1}}|v_l| \le \sum_{j\ge 1}\sqrt{k}\cdot\frac{1}{k}\sum_{l\in T_{j-1}}|v_l| = \frac{1}{\sqrt{k}}\|v\|_1.\end{aligned}$$

This yields (1) and completes the proof. □

12.4 Constructing RIP Matrices

Notation: We first recall some notations from probability theory that we will use from now on. Let $(\Omega, \Sigma, \mathbb{P})$ be a probabiliyt space. A measurable function $X : \Omega \to \mathbb{R}$ is called (real-valued) **random variable**. It's **expectation** is defined by

$$\mathbb{E}[X] := \int_\Omega X(\omega)\,\mathrm{d}\mathbb{P}(\omega)$$

and its **variance** by $\mathbb{V}[X] := \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] = \mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2$.

Recall *Markov's inequality*: For all $t > 0$ we have

$$\mathbb{P}\left[|X| > t\right] \le \frac{\mathbb{E}[|X|]}{t}.$$

A set of random variables $X_1, \ldots, X_n$ are called independent and identically distributed (i.i.d.) if they have the same distribution and

$$\mathbb{P}[X_1 < t_1, \ldots, X_n < t_n] = \prod_{i=1}^n \mathbb{P}[X_i < t_i],$$

for arbitrary $t_1, \dots, t_n \in \mathbb{R}$.

A random variabl is called **normal** (or **Gaussian**) if

$$\mathbb{P}[a \le x \le b) = \int_a^b \varphi(t)\,\mathrm{d}t := \frac{1}{\sqrt{2\pi\sigma^2}} \int_a^b e^{-\frac{(t-\mu)^2}{2\sigma^2}}\,\mathrm{d}t,$$

for fixed $\mu \in \mathbb{R}$ and $\sigma^2 \in (0, \infty)$. In this case we write $X \sim \mathcal{N}(\mu, \sigma^2)$. If $X \sim \mathcal{N}(0,1)$, then X is called **standard normal**.

It still remains to find matrices $A \in \mathbb{R}^{m\times n}$ with small RIP constants. Interestingly, it has turned out that the optimal number of required measurements is achived when choosing the entries of A randomly. In the following we will prove a result for **Gaussian matrices**, i.e.

$$A = \frac{1}{\sqrt{m}} \begin{pmatrix} w_{1,1} & \cdots & w_{1,n} \\ \vdots & \ddots & \vdots \\ w_{m,1} & \cdots & w_{m,n} \end{pmatrix} \quad \text{where} \quad w_{i,j} \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1), i \in [\![m]\!], j \in [\![n]\!]. \tag{G}$$

(12.10) Lemma (Concentration of measures). *Let $m \in \mathbb{N}$, $w_1, \dots, w_m \overset{i.i.d}{\sim} \mathcal{N}(0,1)$. Then for $0 < \varepsilon < 1$ we have*

$$\mathbb{P}\left[w_1^2 + \ldots + w_m^2 \ge (1+\varepsilon)m\right] \le e^{-\frac{m}{2}\left(\frac{\varepsilon^2}{2} - \frac{\varepsilon^3}{3}\right)}$$

and

$$\mathbb{P}\left[w_1^2 + \ldots + w_m^2 \le (1-\varepsilon)m\right] \le e^{-\frac{m}{2}\left(\frac{\varepsilon^2}{2} - \frac{\varepsilon^3}{3}\right)}$$

Proof. See notes on ISIS. $\square$

(12.11) Theorem. *Let $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{m\times n}$ be as in (G) . Then*

$$\mathbb{P}\left(\left|\|Ax\|_2^2 - \|x\|_2^2\right| \ge t\|x\|_2^2\right) \le 2e^{-\frac{m}{2}\left(\frac{t^2}{2} - \frac{t^3}{3}\right)} \le 2e^{-Cmt^2}$$

for $0 < t < 1$ and C is a universal constant.

Proof. At first, let $x \in \mathbb{R}^n$, $\|x\|_2 = 1$. Then by Lemma 12.10

$$\begin{aligned}\mathbb{P}\left(\left|\|Ax\|_2^2 - 1\right| > t\right) &= \mathbb{P}\left(\left|(w_{1,1}x_1 + \ldots + w_{1,n}x_n)^2 + \ldots + (w_{m,1}x_1 + \ldots + w_{m,n}x_n)^2 - m\right| > mt\right) \\ &= \mathbb{P}\left(\left|w_1^2 + \ldots + w_m^2 - m\right| > mt\right) \\ &= \mathbb{P}\left(w_1^2 + \ldots + w_m^2 \ge m(1+t)\right) + \mathbb{P}\left(w_1^2 + \ldots + w_m^2 \le m(1-t)\right) \\ &\le 2e^{-\frac{m}{2}\left(\frac{t^2}{2} - \frac{t^3}{3}\right)},\end{aligned}$$

where $w_1, \dots, w_m \sim \mathcal{N}(0,1)$ again. For an arbitrary $x \in \mathbb{R}^n$ we simply consider $\frac{x}{\|x\|}$ and apply the previous estimate. This proves the first inequality and the second follows with $C = \frac{1}{12}$ by algebraic manipulations. $\square$

(12.12) Lemma. *Let $t > 0$. There exists a set $m \subset \mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : \|x\|_2 = 1\}$ such that*

- $|M| \le \left(1 + \frac{2}{t}\right)^n$
- *For every $x \in \mathbb{S}^{n-1}$ there exists $z \in M$ such that $\|x - z\|_2 \le t$.*

Proof. Pick any $x^1 \in \mathbb{S}^{n-1}$. If $x^1, \dots, x^j \in \mathbb{S}^{n-1}$ were already chosen, then we pick $x^{j+1} \in \mathbb{S}^{n-1}$ such that $\|x^{j+1} - x^l\|_2 > t$ for all $l \in [\![j]\!]$. This procedure is then repeated as long as possible, i.e. until we end up with a set $M = \{x^1, \dots, x^N\} \subset \mathbb{S}^{n-1}$ such that for every $z \in \mathbb{S}^{n-1}$ there exists $j \in [\![N]\!]$ such that $\|x^j - z\| \le t$. This yields property (ii).

It is left to prove the upper bound for N, which is the cardinality of M. We will use a volume argument to prove this. By construction we have

$$\|x^j - x^l\|_2 > t \qquad \forall\, j, l \in [\![N]\!], j \ne l.$$

By the triange inequality the balls $B\left(x^j, \frac{t}{2}\right)$, $j \in [\![N]\!]$ are all disjoint and included in a 'larger' ball $B\left(0, 1+\frac{t}{2}\right)$. Comparing the volumes, we get

$$N\left(\frac{t}{2}\right)^n \cdot V \leq \left(1+\frac{t}{2}\right)^n V,$$

where V is the volume of the unit Ball in $\mathbb{R}^n$. Hence the inequality $|M| = N \leq \left(1+\frac{t}{2}\right)^n$ follows. □

With the concentration inequality in Theorem 12.11 and the entropy argument of Lemma 12.12 we are ready to state the main result on the RIP.

(12.13) Theorem. *Let $n \geq m \geq k$ and let $0 < \varepsilon < 1$ and $0 < \delta < 1$ such that*

$$m \geq C\delta^{-2}\left(k \ln\left(e\frac{n}{k}\right) + \ln\left(\frac{2}{\varepsilon}\right)\right)$$

where $C > 0$ is a universal constant. If $A \in \mathbb{R}^{m\times n}$ is Gaussian as defined in (G), then we have

$$\mathbb{P}\left(\delta_k(A) \leq \delta\right) \geq 1 - \varepsilon.$$

Proof. By Lemma 12.12 (with $t = \frac{1}{4}$) there is a set M such that $M \subset Z := \{z \in \mathbb{R}^n : \operatorname{supp}(z) \subset [\![k]\!], \|z\|_2 = 1\}$ and

- $|M| \leq q^k$
- $\min_{x\in M} \|x - z\| \leq \frac{1}{4}$ for all $z \in \mathbb{Z}$.

We shall show, that if $\left|\|Ax\|_2^2 - 1\right| \leq \frac{\delta}{2}$ for all $x \in M$, then $\left|\|Ax\|_2^2 - 1\right| \leq \delta$ for all $x \in \mathbb{Z}$.

For this purpose, let us proceed with the following bootstrap argument: Let $\gamma > 0$ be the smallest number such that

$$\left|\|Az\|_2^2 - 1\right| \leq \gamma \qquad \forall\, z \in Z.$$

Then we have $\left|\|Au\|_2^2 - \|u\|_2^2\right| \leq \gamma\|u\|_2^2$ for all $u \in \mathbb{R}^n$ with $\operatorname{supp} u \subset [\![k]\!]$ and by the polarization identity we have

$$|\langle Au, Av\rangle - \langle u, v\rangle| \leq \gamma\|u\|_2\|v\|_2 \qquad \forall\, u, v \in \mathbb{R}^n, \operatorname{supp}(u), \operatorname{supp}(v) \subset [\![k]\!].$$

For some fixed $z \in Z$ there exists $x \in M$ such that $\|x - z\| \leq \frac{1}{4}$. By the triange inequality we have

$$\left|\|Az\|_2^2 - 1\right| = \left|\|Ax\|_2^2 - 1 + \langle A(z+x), A(z-x)\rangle - \langle z+x, z-x\rangle\right| \leq \frac{\delta}{2} + \gamma\|z+x\|_2\|z-x\|_2 \leq \frac{\delta}{2} + \frac{\gamma}{2}.$$

As the supremum of the left hand side over all $z \in Z$ is equal to γ, we obtain $\gamma \leq \delta$ and the statement follows.

Using this observation, the remainder of the proof follows by a simple union bound:

$$\begin{aligned}\mathbb{P}(\delta_k(A) > \delta) &\leq \sum_{\substack{T\subset[\![n]\!]\\|T|\leq k}} \mathbb{P}\left(\exists\, z \in \mathbb{R}^n : \operatorname{supp}(z) \subset T, \|z\|_2 = 1 \text{ and } \left|\|Az\|_2^2 - 1\right| > \delta\right)\\ &= \binom{n}{k}\mathbb{P}\left(\exists\, z \in Z : \left|\|Az\|_2^2 - 1\right| > \delta\right) \leq \binom{n}{k}\mathbb{P}\left(\exists\, z \in M : \left|\|Az\|_2^2 - 1\right| > \frac{\delta}{2}\right).\end{aligned}$$

To estimate the latter term we use Theorem 12.11:

$$\mathbb{P}(\delta_k(A) > \delta) \leq q^k \binom{n}{k} 2e^{-2m\delta^2}.$$

hence it is enough to show that the last quantitiy is at most ε if $m \geq c \cdot \delta^2\left(k \cdot \ln\left(e\frac{n}{k}\right) + \ln\left(\frac{2}{\varepsilon}\right)\right)$.

if m is chosen as in the theorem. This follows by straight forward algebraic manipulations and the fact that $\binom{n}{k} \leq \frac{n^k}{k!} \leq \left(\frac{en}{k}\right)^k$.

□

The following corollary shows that our goals (a) and (b) can be met with only $m \approx C' \cdot k \cdot \ln\left(e\frac{n}{2k}\right)$, which is significantly smaller than the dimension of $\mathbb{R}^n$ (for n very large).

(12.14) Corollary. *Let $n \geq m \geq k \geq 1$ with $k \leq \frac{n}{2}$ and $m \geq C' \cdot k \cdot \ln\left(e\frac{n}{2k}\right)$, where C' is some universal constant. If further $A \in \mathbb{R}^{m\times n}$ is as in (G), then the following holds with probability of at least $1-2e^{-\frac{m}{2C'}}$:*

Every k sparse vector $x \in \mathbb{R}^n$ can be recover by the basis pursuit, i.e. x is the unique solution of (P_1) with input $y = Ax$.

Proof. At first, we apply Theorem 12.13 for $2k$ and some $\delta < \frac{1}{3}$. Putting $\varepsilon = 2e^{-\delta^2\frac{m}{2C'}}$ we obtain that $A \in \mathbb{R}^{m\times n}$ has the RIP with $\delta_{2k} < \frac{1}{3}$ with probability of at least $1 - 2e^{\frac{m}{2C'}}$. Then we apply Theorem 12.9 and Theorem 12.5 to conclude the statement. □

(12.15) Remark. Unfortunately, there is no deterministic construction of measurement matrices known which achieves the bound Corollary 12.14.

12.5 Stability and Robustness

The following two features have to be taken into account when studying the recoverability of sparse signals.

(i) **Stability**: We also want to recover or at least approximate signals $x \in \mathbb{R}^n$ that are compressible, meaning that their best k-term approximation error decreases quickly.

(ii) **Robustness**: We would like to recover sparse or compressible vectors from noisy measurements. As a basic model one usually assumes the measurements are of the form $y = Ax + e$ where e is small in some sense.

We will now focus on the recovery properties of a slightly different problem than (P_1). For $\eta \geq 0$ we consider the convex optimization problem

$$\begin{aligned}&\min\nolimits_{z\in\mathbb{R}^n} \|z\|_1,\\ &\text{s. t. } \|y - Az\|_z \leq \eta.\end{aligned} \tag{P_1,η}$$

(12.16) Remark. If $\eta = 0$, then (P_1, η) is the same as (P_1).

(12.17) Theorem. *Let $\delta_{2k} < \sqrt{2} - 1$ and $\|e\|_2 \leq \eta$. Then the solution $\widehat{x}$ of (P_1, η) satisfies*

$$\|x - \widehat{x}\|_2 \leq C \cdot \frac{\sigma_k(x)_1}{\sqrt{k}} + D\eta.$$

where C and D are universal constants.

Proof. First, recall that if A has the RIP of order $2k$ and $u, v \in \Sigma_k$ are two vectors with disjoint support, then we have as in the proof of Theorem 12.9

$$|\langle Au, Av\rangle| \leq \delta_{2k}\|u\|_2\|v\|_2.$$

Let us put $h := x - \widehat{x}$ and let us define $T_0 \subset [\![n]\!]$ to be the index set containing the locations of the k largest entries of x in modulus. Furthermore, let $T_1 \subset T_0^c$ be the set containing the next k largest entries and so on.

As $\widehat{x}$ is a solution of (P_1, η) we get

$$\|Ah\|_2 = \|A(x - \widehat{x})\|_2 \leq \|Ax - y\|_2 + \|y - A\widehat{x}\| \leq 2\eta.$$

Moreover, since $\widehat{x}$ is a solution of (P_1, η) we get

$$\begin{aligned}\|h_{T_0^c}\|_1 &= \|(x+h)_{T_0^c} - x_{T_0^c}\|_1 + \|(x+h)_{T_0} - h_{T_0}\|_1 - \|x_{T_0}\|_1.\\ &\leq \|(x+h)_{T_0^c}\|_1 + \|x_{T_0^c}\|_1 + \|(x+h)_{T_0}\|_1 + \|h_{T_0}\|_1 - \|x_{T_0}\|_1.\\ &= \|x+h\|_1 + \|x_{T_0^c}\|_1 + \|h_{T_0}\|_1 - \|x_{T_0}\|_1.\\ &\leq \|x\|_1 + \|x_{T_0^c}\|_1 + \|h_{T_0}\|_1 - \|x_{T_0}\|_1.\\ &= \|h_{T_0}\|_1 + 2\|x_{T_0^c}\|_1 \leq \sqrt{k}\|h_{T_0}\|_2 + 2\sigma_k(x)_1.\end{aligned}$$

Using this inequality together with the estimate obtained in the proof of Theorem 12.1 we get

$$\sum_{j\geq 2} \|h_{T_j}\|_2 \leq k^{-\frac{1}{2}} \|h_{T_0^c}\|_1 \leq \|h_{T_0}\|_2 + 2k^{-\frac{1}{2}} \sigma_k(x)_1.$$

We now use the RIP and the above inequalities plus the fact that

$$h_{T_0}\|_2 + \|h_{T_1}\|^2 \leq \sqrt{2}\|h_{T_0\cup T_1}\|_2$$

to get

$$\begin{aligned}(1-\delta_{2k})\|h_{T_0\cup T_1}\|_2^2 \leq \|Ah_{T_0\cup T_1}\|_2^2 \leq \langle Ah_{T_0\cup T_1}, Ah\rangle - \left\langle Ah_{T_0\cup T_1}, \sum_{j\geq 2} Ah_{T_j}\right\rangle \\ \leq \|Ah_{T_0\cup T_1}\|_2\|Ah\|_2^2 + \sum_{j\geq 2}\left|\left\langle Ah_{T_0}, Ah_{T_j}\right\rangle\right| + \sum_{j\geq 2}\left|\left\langle Ah_{T_1}, Ah_{T_j}\right\rangle\right| \\ \leq 2\eta\sqrt{1+\delta_{2k}}\|h_{T_0\cup T_1}\|_2 + \delta_{2k}(\|h_{T_0}\|_2 + \|h_{T_1}\|_2)\cdot\|h_{T_j}\|_2\end{aligned}$$

We divide this inequality by

$$(1-\delta_{2k})\|h_{T_0\cup T_1}\|_2,$$

replace $\|h_{T_0}\|_2$ by $\|h_{T_0\cup T_1}\|_2$ and subtrackt $\sqrt{2}\frac{\delta_{2k}}{1-\delta_{2k}}\|h_{T_0\cup T_1}\|_2$ to arrive at

$$\|h_{T_0\cup T_1}\|_2 \leq (1-\rho)^{-1}(\alpha\eta + 2\rho k^{-\frac{1}{2}}\sigma_k(x)_1),$$

where

$$\alpha = \frac{2\sqrt{1+\delta_{2k}}}{1-\delta_{2k}}$$

$$\rho = \frac{\sqrt{\delta_{2k}}}{1-\delta_{2k}}.$$

We conclude the proof by using the bound for $\sum_{j\geq 2}\|h_{T_j}\|_2$

$$\begin{aligned}\|h\| \leq \|h_{(T_0\cup T_1)^c}\|_2 + \|h_{T_0\cup T_1}\|_2 \leq \sum_{j\geq 2}\|h_{T_j}\|_2 + \|h_{T_0\cup T_1}\|_2 \\ \leq 2\|h_{T_0\cup T_1}\|_2 + 2k^{-\frac{1}{2}}\sigma_k(x)_1 \leq C\frac{\sigma_k(x)_1}{\sqrt{k}} + D\eta\end{aligned}$$

with $C = 2(1-\rho)^{-1}\alpha$ and $D = 2(1+\rho)(1-rho)^{-1}$. □

12.6 Optimality of bounds

When recovering k-sparse vectors, one obviously needs at least $m \geq k$ linear measurements. Indeed, even if the support of the vector is known, at least k measurements are needed to identify the values of the non-zero entries. Thus the bound obtained in Theorem 12.13 can only be improved by a logarithmic factor and is essentialy optimal.

The following result, unfortunately, states that this is not possible

(12.18) Theorem. *Let $k \leq m < n$ and $A \in \mathbb{R}^{m\times n}$ be any measurement matrix and let $\Delta : \mathbb{R}^m \to \mathbb{R}^n$ be some recovery map such that for some constant $C > 0$ we have*

$$\|x - \Delta(Ax)\|_2 \leq C \cdot \frac{\sigma_k(x)_1}{\sqrt{k}}.$$

Then $m \geq C' k \ln(e\frac{k}{n})$ with some constant C' that depends on C.

(12.19) Remark (General remarks).

- Compressed sensing with redundent dictionaries
- Analysis formulation

$$\begin{aligned}&\min_{x\in\mathbb{R}^n} \|\Psi x\|_1 \\ &\text{s.t. } \|y - Ax\|_2 \leq \delta,\end{aligned}$$

where Ψ is an analysis operator.

- Non convex

$$\begin{aligned}&\min_{x\in\mathbb{R}^n} \|x\|_p\\ &\text{s. t. } \|y-Ax\|_2 \leq \delta,\end{aligned}$$

with $p \in (0,1)$. Some of the results for l^1 can be transferred to l^p.

References

[1] S. Foucart, H. Ranhut, *A mathematical introduction to compressive sensing.*

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