Supporting Document of *A Robust Neural Control Design for Multi-drone Slung Payload Manipulation with Control Contraction Metrics*

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1 Derivation of the system dynamics

The time derivative of $\boldsymbol{B}_j$ is calculated below:

$$\dot{\boldsymbol{B}}_j = \begin{bmatrix} \mathbf{0}_{2\times 2} \\ -\frac{z_j^2 \underline{\boldsymbol{v}}_j^T + \underline{\boldsymbol{r}}_j^T \underline{\boldsymbol{v}}_j \underline{\boldsymbol{r}}_j^T}{z_j^3} \end{bmatrix}. \tag{1}$$

The inertial position of the payload is $\underline{\boldsymbol{x}}_p$, so the multirotor position, velocity, and acceleration in frame I can be defined as following:

$$\begin{aligned} &\underline{\boldsymbol{x}}_j = \underline{\boldsymbol{x}}_p + \underline{\boldsymbol{l}}_j, \quad \underline{\boldsymbol{u}}_j = \dot{\underline{\boldsymbol{x}}}_j = \underline{\boldsymbol{v}}_p + \boldsymbol{B}_j \underline{\boldsymbol{v}}_j, \\ &\underline{\boldsymbol{a}}_j = \dot{\underline{\boldsymbol{u}}} = \dot{\underline{\boldsymbol{v}}}_p + \dot{\boldsymbol{B}}_j \underline{\boldsymbol{v}}_j + \boldsymbol{B}_j \dot{\underline{\boldsymbol{v}}}_j \end{aligned} \tag{2}$$

We define the motion variable and the inertial motion variable as $\underline{\boldsymbol{u}} = \begin{bmatrix} \underline{\boldsymbol{v}}_p^T & \underline{\boldsymbol{v}}_1^T & \underline{\boldsymbol{v}}_2^T & \underline{\boldsymbol{v}}_3^T \end{bmatrix}^T$ and $\underline{\boldsymbol{u}}_I = \begin{bmatrix} \underline{\boldsymbol{v}}_p^T & \underline{\boldsymbol{u}}_1^T & \underline{\boldsymbol{u}}_2^T & \underline{\boldsymbol{u}}_3^T \end{bmatrix}^T$, where $\underline{\boldsymbol{v}}_p$ is the payload velocity and $\underline{\boldsymbol{u}}_j$ is the quadrotor velocity, all defined in the inertial frame. We follow Kane's method approach from [**?**] to calculate the partial velocity matrix as below:

$$\begin{aligned} \boldsymbol{U}^T &= \begin{bmatrix} \frac{\partial \underline{\boldsymbol{v}}_p}{\partial \underline{\boldsymbol{v}}_p^T} & \frac{\partial \underline{\boldsymbol{u}}_1}{\partial \underline{\boldsymbol{v}}_p^T} & \frac{\partial \underline{\boldsymbol{u}}_2}{\partial \underline{\boldsymbol{v}}_p^T} & \frac{\partial \underline{\boldsymbol{u}}_3}{\partial \underline{\boldsymbol{v}}_p^T} \\ \frac{\partial \underline{\boldsymbol{v}}_p}{\partial \underline{\boldsymbol{v}}_1^T} & \frac{\partial \underline{\boldsymbol{u}}_1}{\partial \underline{\boldsymbol{v}}_1^T} & \frac{\partial \underline{\boldsymbol{u}}_2}{\partial \underline{\boldsymbol{v}}_1^T} & \frac{\partial \underline{\boldsymbol{u}}_3}{\partial \underline{\boldsymbol{v}}_1^T} \\ \frac{\partial \underline{\boldsymbol{v}}_p}{\partial \underline{\boldsymbol{v}}_2^T} & \frac{\partial \underline{\boldsymbol{u}}_1}{\partial \underline{\boldsymbol{v}}_2^T} & \frac{\partial \underline{\boldsymbol{u}}_2}{\partial \underline{\boldsymbol{v}}_2^T} & \frac{\partial \underline{\boldsymbol{u}}_3}{\partial \underline{\boldsymbol{v}}_2^T} \\ \frac{\partial \underline{\boldsymbol{v}}_p}{\partial \underline{\boldsymbol{v}}_3^T} & \frac{\partial \underline{\boldsymbol{u}}_1}{\partial \underline{\boldsymbol{v}}_3^T} & \frac{\partial \underline{\boldsymbol{u}}_2}{\partial \underline{\boldsymbol{v}}_3^T} & \frac{\partial \underline{\boldsymbol{u}}_3}{\partial \underline{\boldsymbol{v}}_3^T} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{1}_{3\times 3} & \mathbf{1}_{3\times 3} & \mathbf{1}_{3\times 3} & \mathbf{1}_{3\times 3} \\ \mathbf{0}_{2\times 3} & \boldsymbol{B}_1^T & \mathbf{0}_{2\times 3} & \mathbf{0}_{2\times 3} \\ \mathbf{0}_{2\times 3} & \mathbf{0}_{2\times 3} & \boldsymbol{B}_2^T & \mathbf{0}_{2\times 3} \\ \mathbf{0}_{2\times 3} & \mathbf{0}_{2\times 3} & \mathbf{0}_{2\times 3} & \boldsymbol{B}_3^T \end{bmatrix}. \end{aligned} \tag{3}$$

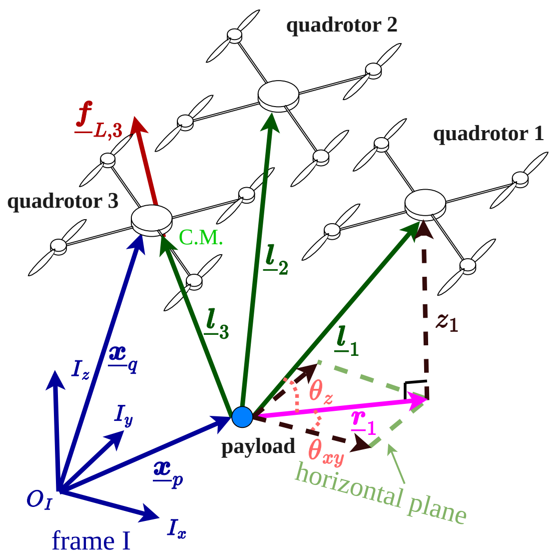


Figure 1: System geometry

Given that the inertial gravity is $\underline{\boldsymbol{g}}_I = \begin{bmatrix} 0 & 0 & -g \end{bmatrix}^T$ with g as the gravitational acceleration, the payload disturbance $\underline{\boldsymbol{\delta}}_p$, the multirotor disturbance $\underline{\boldsymbol{\delta}}_1$, $\underline{\boldsymbol{\delta}}_2$, $\underline{\boldsymbol{\delta}}_3$, and the multirotor three directional lift force $\underline{\boldsymbol{f}}_{L,1}$, $\underline{\boldsymbol{f}}_{L,2}$, $\underline{\boldsymbol{f}}_{L,3}$, the external and internal forces can be formulated:

$$\underline{\boldsymbol{f}}_e = \overbrace{\begin{bmatrix} m_p \underline{\boldsymbol{g}}_I \\ m_1 \underline{\boldsymbol{g}}_I \\ m_2 \underline{\boldsymbol{g}}_I \\ m_3 \underline{\boldsymbol{g}}_I \end{bmatrix}}^{\underline{\boldsymbol{f}}_{e,1}} + \overbrace{\begin{bmatrix} \mathbf{0}_{3\times 1} \\ \underline{\boldsymbol{f}}_{L,1} \\ \underline{\boldsymbol{f}}_{L,2} \\ \underline{\boldsymbol{f}}_{L,3} \end{bmatrix}}^{\underline{\boldsymbol{f}}_{e,2}} + \overbrace{\begin{bmatrix} \underline{\boldsymbol{\delta}}_p \\ \underline{\boldsymbol{\delta}}_1 \\ \underline{\boldsymbol{\delta}}_2 \\ \underline{\boldsymbol{\delta}}_3 \end{bmatrix}}^{\underline{\boldsymbol{f}}_{e,3}}, \quad \underline{\boldsymbol{f}}^{\star} = \begin{bmatrix} -m_p \dot{\underline{\boldsymbol{v}}}_p \\ -m_1 \underline{\boldsymbol{a}}_1 \\ -m_2 \underline{\boldsymbol{a}}_2 \\ -m_3 \underline{\boldsymbol{a}}_3 \end{bmatrix}. \tag{4}$$

We can decompose $\underline{\boldsymbol{f}}^{\star}$ in order to separate $\dot{\underline{\boldsymbol{u}}}$ and $\underline{\boldsymbol{u}}$, so the Kane's method can be performed accrodingly:

$$\begin{aligned} \underline{\boldsymbol{f}}^{\star} = & \overbrace{\begin{bmatrix} -m_p \mathbf{1}_{3\times 3} & \mathbf{0}_{3\times 2} & \mathbf{0}_{3\times 2} & \mathbf{0}_{3\times 2} \\ -m_1 \mathbf{1}_{3\times 3} & -m_1 \boldsymbol{B}_1 & \mathbf{0}_{3\times 2} & \mathbf{0}_{3\times 2} \\ -m_2 \mathbf{1}_{3\times 3} & \mathbf{0}_{3\times 2} & -m_2 \boldsymbol{B}_2 & \mathbf{0}_{3\times 2} \\ -m_3 \mathbf{1}_{3\times 3} & \mathbf{0}_{3\times 2} & \mathbf{0}_{3\times 2} & -m_3 \boldsymbol{B}_3 \end{bmatrix}}^{\underline{\boldsymbol{f}}_1^{\star}} \begin{bmatrix} \dot{\underline{\boldsymbol{v}}}_p \\ \dot{\underline{\boldsymbol{v}}}_1 \\ \dot{\underline{\boldsymbol{v}}}_2 \\ \dot{\underline{\boldsymbol{v}}}_3 \end{bmatrix} \\ & + \overbrace{\begin{bmatrix} \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 2} & \mathbf{0}_{3\times 2} & \mathbf{0}_{3\times 2} \\ \mathbf{0}_{3\times 3} & -m_1 \dot{\boldsymbol{B}}_1 & \mathbf{0}_{3\times 2} & \mathbf{0}_{3\times 2} \\ \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 2} & -m_2 \dot{\boldsymbol{B}}_2 & \mathbf{0}_{3\times 2} \\ \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 2} & \mathbf{0}_{3\times 2} & -m_3 \dot{\boldsymbol{B}}_3 \end{bmatrix}}^{\underline{\boldsymbol{f}}_2^{\star}} \begin{bmatrix} \underline{\boldsymbol{v}}_p \\ \underline{\boldsymbol{v}}_1 \\ \underline{\boldsymbol{v}}_2 \\ \underline{\boldsymbol{v}}_3 \end{bmatrix}. \end{aligned} \tag{5}$$

Since Kane's method states that $-\boldsymbol{U}^T \underline{\boldsymbol{f}}^\star = \boldsymbol{M}\dot{\underline{\boldsymbol{u}}} + \boldsymbol{C}\underline{\boldsymbol{u}}$, the inertial matrix $\boldsymbol{M}$ and the gyroscopic matrix $\boldsymbol{C}$, given the total system mass $m_t = m_p + m_1 + m_2 + m_3$, is calculated below:

$$\begin{aligned}\boldsymbol{M} &= -\boldsymbol{U}^T \underline{\boldsymbol{f}}_1^\star \\ &= \begin{bmatrix} m_t \mathbf{1}_{3\times3} & m_1\boldsymbol{B}_1 & m_2\boldsymbol{B}_2 & m_3\boldsymbol{B}_3 \\ m_1\boldsymbol{B}_1^T & m_1\boldsymbol{B}_1^T\boldsymbol{B}_1 & \mathbf{0}_{2\times2} & \mathbf{0}_{2\times2} \\ m_2\boldsymbol{B}_2^T & \mathbf{0}_{2\times2} & m_2\boldsymbol{B}_2^T\boldsymbol{B}_2 & \mathbf{0}_{2\times2} \\ m_3\boldsymbol{B}_3^T & \mathbf{0}_{2\times2} & \mathbf{0}_{2\times2} & m_3\boldsymbol{B}_3^T\boldsymbol{B}_3 \end{bmatrix},\end{aligned} \tag{6}$$

$$\boldsymbol{C} = -\boldsymbol{U}^T \underline{\boldsymbol{f}}_2^\star = \begin{bmatrix} \mathbf{0}_{3\times3} & m_1\dot{\boldsymbol{B}}_1 & m_2\dot{\boldsymbol{B}}_2 & m_3\dot{\boldsymbol{B}}_3 \\ \mathbf{0}_{2\times3} & m_1\boldsymbol{B}_1^T\dot{\boldsymbol{B}}_1 & \mathbf{0}_{2\times2} & \mathbf{0}_{2\times2} \\ \mathbf{0}_{2\times3} & \mathbf{0}_{2\times2} & m_2\boldsymbol{B}_2^T\dot{\boldsymbol{B}}_2 & \mathbf{0}_{2\times2} \\ \mathbf{0}_{2\times3} & \mathbf{0}_{2\times2} & \mathbf{0}_{2\times2} & m_3\boldsymbol{B}_3^T\dot{\boldsymbol{B}}_3 \end{bmatrix}. \tag{7}$$

The external forces of the system, meaning the gravitational force $\underline{\boldsymbol{f}}_g$, the actuation force from the lift of each quadrotor $\underline{\boldsymbol{f}}_a$, and the disturbances $\underline{\boldsymbol{f}}_\delta$, are calculated as below:

$$\begin{aligned}\underline{\boldsymbol{f}}_g &= \boldsymbol{U}^T \underline{\boldsymbol{f}}_{e,1} = \begin{bmatrix} M_t \underline{\boldsymbol{g}}_I \\ m_1\boldsymbol{B}_1^T\underline{\boldsymbol{g}}_I \\ m_2\boldsymbol{B}_2^T\underline{\boldsymbol{g}}_I \\ m_3\boldsymbol{B}_3^T\underline{\boldsymbol{g}}_I \end{bmatrix}, \\ \underline{\boldsymbol{f}}_a &= \boldsymbol{U}^T \underline{\boldsymbol{f}}_{e,2} = \begin{bmatrix} \underline{\boldsymbol{f}}_{L,1} + \underline{\boldsymbol{f}}_{L,2} + \underline{\boldsymbol{f}}_{L,3} \\ \boldsymbol{B}_1^T\underline{\boldsymbol{f}}_{L,1} \\ \boldsymbol{B}_2^T\underline{\boldsymbol{f}}_{L,2} \\ \boldsymbol{B}_3^T\underline{\boldsymbol{f}}_{L,3} \end{bmatrix}, \\ \underline{\boldsymbol{f}}_\delta &= \boldsymbol{U}^T \underline{\boldsymbol{f}}_{e,3} = \begin{bmatrix} \underline{\boldsymbol{\delta}}_p + \underline{\boldsymbol{\delta}}_1 + \underline{\boldsymbol{\delta}}_2 + \underline{\boldsymbol{\delta}}_3 \\ \boldsymbol{B}_1^T\underline{\boldsymbol{\delta}}_1 \\ \boldsymbol{B}_2^T\underline{\boldsymbol{\delta}}_2 \\ \boldsymbol{B}_3^T\underline{\boldsymbol{\delta}}_3 \end{bmatrix}.\end{aligned} \tag{8}$$

After compensating the quadrotor's weights by setting $\underline{\boldsymbol{f}}_{L,j} = -m_j\underline{\boldsymbol{g}}_I + \delta\underline{\boldsymbol{f}}_{L,j}$, we can get $\underline{\boldsymbol{f}}_g + \underline{\boldsymbol{f}}_a = \underline{\boldsymbol{f}}_{g,p} + \boldsymbol{H}\underline{\boldsymbol{\zeta}}$ where

$$\begin{aligned}\underline{\boldsymbol{f}}_{g,p} &= \begin{bmatrix} m_p\underline{\boldsymbol{g}}_I \\ \mathbf{0}_{6\times1} \end{bmatrix} \\ \boldsymbol{H} &= \begin{bmatrix} \mathbf{1}_{3\times3} & \mathbf{1}_{3\times3} & \mathbf{1}_{3\times3} \\ \boldsymbol{B}_1^T & \mathbf{0}_{2\times3} & \mathbf{0}_{2\times3} \\ \mathbf{0}_{2\times3} & \boldsymbol{B}_2^T & \mathbf{0}_{2\times3} \\ \mathbf{0}_{2\times3} & \mathbf{0}_{2\times3} & \boldsymbol{B}_3^T \end{bmatrix} \\ \underline{\boldsymbol{\zeta}} &= \begin{bmatrix} \delta\underline{\boldsymbol{f}}_{L,1} \\ \delta\underline{\boldsymbol{f}}_{L,2} \\ \delta\underline{\boldsymbol{f}}_{L,3} \end{bmatrix}\end{aligned} \tag{9}$$

Also, we can set $\underline{\boldsymbol{f}}_\delta = \boldsymbol{H}_\delta\underline{\boldsymbol{\delta}}$, where

$$
\begin{aligned}
\boldsymbol{H}_\delta &= \begin{bmatrix} \mathbf{1}_{3\times3} & \mathbf{1}_{3\times3} & \mathbf{1}_{3\times3} & \mathbf{1}_{3\times3} \\ \mathbf{0}_{2\times3} & \boldsymbol{B}_1^T & \mathbf{0}_{2\times3} & \mathbf{0}_{2\times3} \\ \mathbf{0}_{2\times3} & \mathbf{0}_{2\times3} & \boldsymbol{B}_2^T & \mathbf{0}_{2\times3} \\ \mathbf{0}_{2\times3} & \mathbf{0}_{2\times3} & \mathbf{0}_{2\times3} & \boldsymbol{B}_3^T \end{bmatrix} \\
\underline{\boldsymbol{\delta}} &= \begin{bmatrix} \underline{\boldsymbol{\delta}}_p \\ \underline{\boldsymbol{\delta}}_1 \\ \underline{\boldsymbol{\delta}}_2 \\ \underline{\boldsymbol{\delta}}_3 \end{bmatrix}.
\end{aligned} \tag{10}
$$

2 Stability Proof of UDE for the system

Below shows the proof for Theorem 1

Proof. We define a Lyapunov function V_e to show that the estimation errors of the effective disturbances in (**??**) converge:

$$
V_e = \frac{1}{2} c_T \tilde{\underline{\boldsymbol{\delta}}}_T^2 + \frac{1}{2} \sum_{j=1}^{N} \Big[c_T \lambda_T N/(2\kappa_j) + c_j/N \Big] \tilde{\underline{\boldsymbol{\delta}}}_j^2 \tag{11}
$$

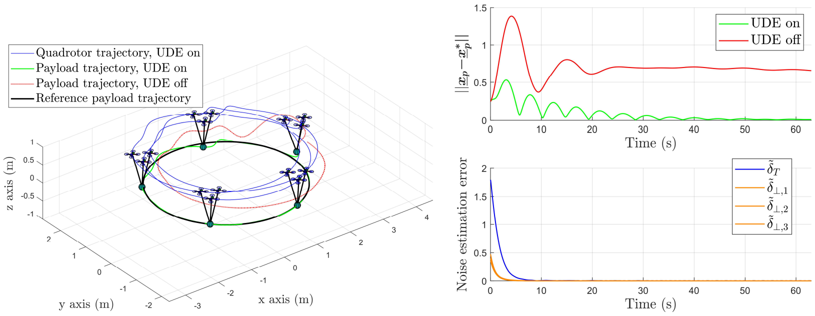
where c_T and c_j are positive constants. According to the error dynamics defined in (**??**) and (**??**), we obtain the time derivative of V_e as:

$$
\begin{aligned}
\dot{V}_e = &-c_T \lambda_T \tilde{\underline{\boldsymbol{\delta}}}_T^T \tilde{\underline{\boldsymbol{\delta}}}_T - c_T \lambda_T \sum_{j=1}^{N} \tilde{\underline{\boldsymbol{\delta}}}_{\perp,j}^T \tilde{\underline{\boldsymbol{\delta}}}_T \\
&- \sum_{j=1}^{N} \Big[c_T \lambda_T N/(2\kappa_j) + c_j/N \Big] \tilde{\underline{\boldsymbol{\delta}}}_j^T \mathfrak{B}_j \tilde{\underline{\boldsymbol{\delta}}}_{\perp,j}
\end{aligned} \tag{12}
$$

According to (**??**), $\mathfrak{B}_j \underline{\boldsymbol{l}}_j = \underline{\mathbf{0}}$, we have $\tilde{\underline{\boldsymbol{\delta}}}_j^T \mathfrak{B}_j = \tilde{\underline{\boldsymbol{\delta}}}_{\perp,j}^T \mathfrak{B}_j$. Using Lemma **??**, we conclude that the time derivative of the Lyapunov function is

$$
\begin{aligned}
\dot{V}_e = &\sum_{j=1}^{N} -\Big(c_T \lambda_T \tilde{\underline{\boldsymbol{\delta}}}_T^T \tilde{\underline{\boldsymbol{\delta}}}_T + c_T \lambda_T N \tilde{\underline{\boldsymbol{\delta}}}_{\perp,j}^T \tilde{\underline{\boldsymbol{\delta}}}_T \\
&+ (c_T \lambda_T N^2/2) \tilde{\underline{\boldsymbol{\delta}}}_{\perp,j}^T \tilde{\underline{\boldsymbol{\delta}}}_{\perp,j} + c_j \kappa_j \tilde{\underline{\boldsymbol{\delta}}}_{\perp,j}^T \tilde{\underline{\boldsymbol{\delta}}}_{\perp,j} \Big)/N \\
\leq &\sum_{j=1}^{N} -\Big[\frac{1}{2} c_T \lambda_T \tilde{\underline{\boldsymbol{\delta}}}_T^T \tilde{\underline{\boldsymbol{\delta}}}_T + c_j \kappa_j \tilde{\underline{\boldsymbol{\delta}}}_{\perp,j}^T \tilde{\underline{\boldsymbol{\delta}}}_{\perp,j} + \\
&\frac{1}{2} c_T \lambda_T \tilde{\underline{\boldsymbol{\delta}}}_T^T + c_T \lambda_T N \tilde{\underline{\boldsymbol{\delta}}}_{\perp,j}^T \tilde{\underline{\boldsymbol{\delta}}}_T + (c_T \lambda_T N^2/2) \tilde{\underline{\boldsymbol{\delta}}}_{\perp,j}^T \tilde{\underline{\boldsymbol{\delta}}}_{\perp,j} \Big]/N \\
\leq &\sum_{j=1}^{N} -\Big[\frac{1}{2} c_T \lambda_T \tilde{\underline{\boldsymbol{\delta}}}_T^T \tilde{\underline{\boldsymbol{\delta}}}_T + c_j \kappa_j \tilde{\underline{\boldsymbol{\delta}}}_{\perp,j}^T \tilde{\underline{\boldsymbol{\delta}}}_{\perp,j} \Big]/N \leq 0
\end{aligned} \tag{13}
$$

Since $\dot{V}_e$ is negative semi-definite and V_e is positive definite, hence the disturbance estimation errors $\tilde{\underline{\boldsymbol{\delta}}}_T$ and $\tilde{\underline{\boldsymbol{\delta}}}_{\perp,j}$ are bounded. Therefore, the total control forces to the



(a) Circular trajectory

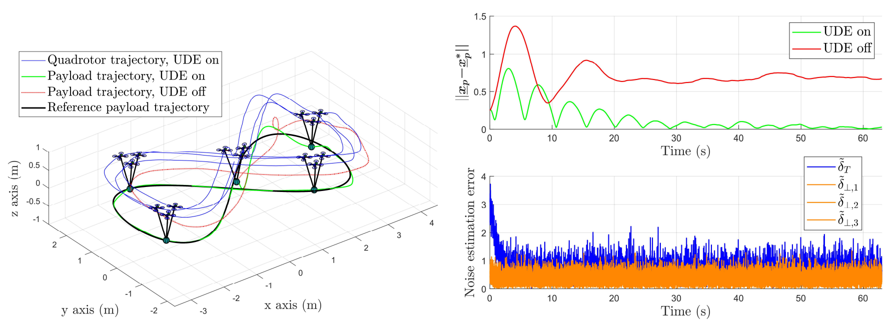
(b) Circular path UDE error

Figure 2: Circular path MATLAB simulation performance plots, (a): comparison with UDE on and off; (b): estimated disturbance error with respect to time when using UDE.

entire system are bounded due to (**??**). Using the dynamics of the estimation error in (**??**) and (**??**), we conclude that $\dot{\tilde{\underline{\boldsymbol{\delta}}}}_T$ and $\dot{\tilde{\underline{\boldsymbol{\delta}}}}_j$ are bounded. Hence, $\ddot{V}_e$ is bounded and $\dot{V}_e$ is uniformly continuous. According to Barbalat's Lemma, $\dot{V}_e \to 0$ as $t \to \infty$, and we conclude that $\tilde{\underline{\boldsymbol{\delta}}}_T \to \underline{\mathbf{0}}$ and $\tilde{\underline{\boldsymbol{\delta}}}_{\perp,j} \to \underline{\mathbf{0}}$ as $t \to \infty$. $\square$

3 Matlab plots for circular and figure-8 trajectory tracking

The performance of the circular and figure-8 path following is demonstrated in Fig.2 and Fig.3. The disturbance force is a summation of constant noise and stochastic noise $\underline{\boldsymbol{\delta}} = \underline{\boldsymbol{\delta}}_c + \underline{\boldsymbol{\delta}}_s$, where $\underline{\boldsymbol{\delta}}_c = \begin{bmatrix} 0.3 & -0.2 & 0.5 & 0.3 & \dots & 0.3 \end{bmatrix}^T \in \mathbb{R}^{12}$. Only a constant noise is used in circular path following to ensure perfect tracking when the noise level is constant. A stochastic noise of $\underline{\boldsymbol{\delta}}_s \sim \mathscr{U}(0,1) \cdot 0.3$ is used in figure-8 path following. Accuracy is significantly improved in both cases with the UDE turned on, and the system can quickly converge to a bounded neighbourhood of the reference trajectory for the figure-8 configuration. For the circular path, the noise converges to 0, satisfying Assumption **??**.



(a) Figure-8 trajectory

(b) Figure-8 path UDE error

Figure 3: Figure-8 path MATLAB simulation performance plots, (a): comparison with UDE on and off; (b): estimated disturbance error with respect to time when using UDE.