On the inexact scaled gradient projection method

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The problem, definitions and preliminaries results

The main problem

We want to present an inexact version of the scaled gradient projection method for constrained convex optimization problem as follows

$$\min\{f(x) : \; x \in C\}, \tag{1}$$

where C is a closed and convex subset of $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function.

Scaled Gradient Projection Method[1]

Step 0. Choose $\sigma, \tau \in (0,1)$, $0 < \alpha_{\min} \leq \alpha_{\max}$. Let $x^0 \in C$ and set $k = 0$;

Step 1. Choose $\alpha_k \in [\alpha_{\min}, \alpha_{\max}]$ and a positive definite matrix D_k and take $w^k \in C$ as

$$w^k := \mathcal{P}_C^{D_k}(x^k - \alpha_k D_k^{-1} \nabla f(x^k))$$

If $w^k = x^k$, then **stop**; otherwise,

Step 2. Choose τ_k and define the next iterate x^{k+1} as

$$x^{k+1} = x^k + \tau_k(w^k - x^k). \tag{2}$$

and go back to the Step 1.

[1] S. Bonettini, R. Zanella, and L. Zanni. "A scaled gradient projection method for constrained image deblurring". In: *Inverse Problems* 25.1 (2009), pp. 015002, 23.

Scaled Gradient Projection Method

Let D be a $n \times n$ positive definite matrix and $\| \cdot \|_D : \mathbb{R}^n \to \mathbb{R}$ be the norm defined by

$$\|d\|_D := \sqrt{\langle Dd, d \rangle}, \quad \forall d \in \mathbb{R}^n.$$

For a fixed constant $\mu \geq 1$, *denote by* $\mathcal{D}_\mu$ *the set of symmetric positive definite matrices* $n \times n$ *with all eigenvalues contained in the interval* $[\frac{1}{\mu}, \mu]$.

- $\mathcal{D}_\mu$ is compact;
- If $D \in \mathcal{D}_\mu$, it follows that D^{-1} also belongs to $\mathcal{D}_\mu$;
- $\forall D \in \mathcal{D}_\mu$, we obtain

$$\frac{1}{\mu}\|d\|^2 \leq \|d\|_D^2 \leq \mu\|d\|^2, \qquad \forall d \in \mathbb{R}^n.$$

Exact Projection

Definition

The exact projection of the point $v \in \mathbb{R}^n$ onto C with respect to the norm $\|\cdot\|_D$, denoted by $\mathcal{P}_C^D(v)$, is defined by

$$\mathcal{P}_C^D(v) := \arg\min_{z \in C} \|z - v\|_D^2.$$

Lemma

Let $v, w \in \mathbb{R}^n$. Then, $w = \mathcal{P}_C^D(v)$ if and only if $w \in C$ and

$$\langle D(v - w), y - w\rangle \leq 0,$$

for all $y \in C$.

Inexact Projections[2]

Definition
The feasible inexact projection mapping, with respect to the norm $\|\cdot\|_D$, onto C relative to a point $u \in C$ and forcing parameter $\zeta \in (0,1]$, denoted by $\mathcal{P}^D_{C,\zeta}(u,\cdot) : \mathbb{R}^n \rightrightarrows C$, is the set-valued mapping defined as follows

$$\mathcal{P}^D_{C,\zeta}(u,v) := \left\{ w \in C : \ \|w-v\|_D^2 \le \zeta \|\mathcal{P}^D_C(v) - v\|_D^2 + (1-\zeta)\|u-v\|_D^2 \right\}.$$

Each point $w \in \mathcal{P}^D_{C,\zeta}(u,v)$ is called a feasible inexact projection, with respect to the norm $\|\cdot\|_D$, of v onto C relative to u and forcing parameter $\zeta \in (0,1]$.

[2] Ernesto G. Birgin, José Mario Martínez, and Marcos Raydan. "Inexact spectral projected gradient methods on convex sets". In: *IMA J. Numer. Anal.* 23.4 (2003), pp. 539–559.

Inexact Projections[3]

Definition

The feasible inexact projection mapping, with respect to the norm $\|\cdot\|_D$, onto C relative to $u \in C$ and forcing parameter $\gamma \geq 0$, denoted by $\mathcal{R}^D_{C,\gamma}(u,\cdot) : \mathbb{R}^n \rightrightarrows C$, is the set-valued mapping defined as follows

$$\mathcal{R}^D_{C,\gamma}(u,v) := \left\{ w \in C : \ \langle D(v-w), y-w \rangle \leq \gamma \|w-u\|^2_D, \quad \forall\, y \in C \right\}.$$

Each point $w \in \mathcal{R}^D_{C,\gamma}(u,v)$ is called a feasible inexact projection, with respect to the norm $\|\cdot\|_D$, of v onto C relative to u and forcing parameter $\gamma \geq 0$.

[3] Fabiana R. de Oliveira, Orizon P. Ferreira, and Gilson N. Silva. "Newton's method with feasible inexact projections for solving constrained generalized equations". In: *Comput. Optim. Appl.* 72.1 (2019), pp. 159–177.

Inexact Projections

Lemma

Let $v \in \mathbb{R}^n$, $u \in C$, $\gamma \geq 0$ *and* $\zeta \in (0, 1]$. *If* $0 \leq \gamma < 1/2$ *and* $\zeta = 1 - 2\gamma$, *then*

$$\mathcal{R}^D_{C,\gamma}(u, v) \subset \mathcal{P}^D_{C,\zeta}(u, v).$$

Proposition

Let $v \in \mathbb{R}^n$, $u \in C$ *and assume that* C *is a bounded set. Then, for each* $0 < \gamma < 1/2$, *there exist* $0 < \zeta < 1$ *such that*

$$\mathcal{P}^D_{C,\zeta}(u, v) \subseteq \mathcal{R}^D_{C,\gamma}(u, v)$$

Lemma

Let $x \in C$, $\alpha > 0$ *and* $z(\alpha) = x - \alpha D^{-1}\nabla f(x)$. *Take* $w(\alpha) \in \mathcal{P}^D_{C,\zeta}(x, z(\alpha))$ *with* $\zeta \in (0,1]$. *Then, there hold*

(i) $\langle \nabla f(x), w(\alpha) - x\rangle \leq -\frac{1}{2\alpha}\|w(\alpha) - x\|^2_D + \frac{\zeta}{2\alpha}\left[\|\mathcal{P}^D_C(z(\alpha)) - z(\alpha)\|^2_D - \|x - z(\alpha)\|^2_D\right]$;

(ii) *the point* x *is stationary for problem* (1) *if, and only if,* $x \in \mathcal{P}^D_{C,\zeta}(x, z(\alpha))$;

(iii) *if* $x \in C$ *is a nonstationary point for problem* (1), *then* $\left\langle \nabla f(x), w(\alpha) - x \right\rangle < 0$. *Equivalently, if there exists* $\bar{\alpha} > 0$ *such that* $\left\langle \nabla f(x), w(\bar{\alpha}) - x \right\rangle \geq 0$, *then* x *is stationary for problem* (1).

Inexact scaled gradient method

InexProj-SGM employing nonmonotone line search

Step 0. Choose $\sigma, \zeta_{\min} \in (0,1)$, $0 < \alpha_{\min} \le \alpha_{\max}$ and $\mu \ge 1$. Let $x^0 \in C$, $\nu_0 \ge 0$ and set $k \leftarrow 0$.

Step 1. Choose positive real numbers α_k and ζ_k and a positive definite matrix D_k such that

$$\alpha_{\min} \le \alpha_k \le \alpha_{\max}, \qquad 0 < \zeta_{\min} < \zeta_k \le 1, \qquad D_k \in \mathcal{D}_\mu.$$

Compute $w^k \in C$ as any feasible inexact projection with respect to the norm $\|\cdot\|_{D_k}$ of

$$z^k := x^k - \alpha_k D_k^{-1} \nabla f(x^k)$$

onto C relative to x^k with forcing parameter ζ_k, i.e.,

$$w^k \in \mathcal{P}_{C,\zeta_k}^{D_k}(x^k, z^k).$$

If $w^k = x^k$, then **stop** declaring convergence.

InexProj-SGM employing nonmonotone line search

Step 2. Set $\tau_{\rm trial} \leftarrow 1$. If

$$f\Big(x^k + \tau_{\rm trial}(w^k - x^k)\Big) \le f(x^k) + \sigma\tau_{\rm trial}\Big\langle \nabla f(x^k), w^k - x^k\Big\rangle + \nu_k, \tag{3}$$

then $\tau_k \leftarrow \tau_{\rm trial}$, define the next iterate x^{k+1} as

$$x^{k+1} = x^k + \tau_k(w^k - x^k), \tag{4}$$

and go to **Step 3**. Otherwise, choose $\tau_{\rm new} \in [\underline{\omega}\tau_{\rm trial}, \bar{\omega}\tau_{\rm trial}]$, set $\tau_{\rm trial} \leftarrow \tau_{\rm new}$, and repeat test (3).

Step 3. Take $\delta_{k+1} \in [\delta_{\min}, 1]$ and choose $\nu_{k+1} \in \mathbb{R}$ satisfying

$$0 \le \nu_{k+1} \le (1 - \delta_{k+1})\Big[f(x^k) + \nu_k - f(x^{k+1})\Big].$$

Set $k \leftarrow k + 1$ and go to **Step 1**.

Nonmonotone line search

Remarks
There are several ways of choosing ν_k

(i) If $\nu_k = 0$, the line search (4) is the well-known Armijo line search.

(ii) If $f_{\max} = \max\{f(x^{k-j}) \,|\, 0 \leq j \leq \min\{k, M\}\}$ and

$$\nu_k = f_{\max} - f(x^k) \tag{5}$$

the line search (4) is the same defined by Grippo, Lampariello and Lucidi[4].

[4] L. Grippo, F. Lampariello, and S. Lucidi. "A nonmonotone line search technique for Newton's method". In: *SIAM J. Numer. Anal.* 23.4 (1986), pp. 707–716.

Nonmonotone line search

(iii) Let $0 \leq \eta_{min} \leq \eta_{max} < 1$, $c_0 = f(x_0)$ and $q_0 = 1$. Choose $\eta_k \in [\eta_{min}, \eta_{max}]$ and set

$$q_{k+1} = \eta_k q_k + 1, \qquad c_{k+1} = (\eta_k q_k c_k + f(x^{k+1}))/q_{k+1}, \qquad \forall k \in \mathbb{N}.$$

If $\delta_{k+1} = 1/q_{k+1}$ and

$$\nu_k = c_k - f(x^k) \tag{6}$$

the line search (4) is the same defined by Zhang and Hager[5].

[5] H. Zhang and W. W. Hager. "A nonmonotone line search technique and its application to unconstrained optimization". In: *SIAM J. Optim.* 14.4 (2004), pp. 1043–1056.

Partial asymptotic convergence

Partial asymptotic convergence analysis

Lemma

There holds $0 \leq \delta_{k+1}\left[f(x^k) + \nu_k - f(x^{k+1})\right] \leq \left(f(x^k) + \nu_k\right) - \left(f(x^{k+1}) + \nu_{k+1}\right)$*, for all* $k \in \mathbb{N}$*. As consequence the sequence* $\left(f(x^k) + \nu_k\right)_{k\in\mathbb{N}}$ *is non-increasing.*

Theorem

Assume that $\lim_{k\to+\infty} \nu_k = 0$*. Then, Algorithm InexProj-SGM stops in a finite number of iterations at a stationary point of problem* (1)*, or generates an infinite sequence* $(x^k)_{k\in\mathbb{N}}$ *for which every cluster point is stationary for problem* (1)*.*

Partial asymptotic convergence analysis

Proposition

If $\delta_{min} > 0$, *then* $\sum_{k=0}^{+\infty} \nu_k < +\infty$. *Consequently,* $\lim_{k \to +\infty} \nu_k = 0$.[6]

Remark

Armijo line search and nonmonotone line search strategy defined by (6) satisfies a condition $\delta_{min} > 0$. However, for the nonmonotone line search strategy proposed by (5), we can only guarantee that $\delta_{min} \geq 0$. Hence, we need deal with this case separately.

[6] Geovani N. Grapiglia and Ekkehard W. Sachs. "On the worst-case evaluation complexity of non-monotone line search algorithms". In: *Comput. Optim. Appl.* 68.3 (2017), pp. 555–577.

Partial asymptotic convergence analysis

Proposition

Assume that the sequence $(x^k)_{k\in\mathbb{N}}$ *is generated by Algorithm InexProj-SGM with the nonmonotone line search* (5), *i.e.,* $\nu_k = f_{\max} - f(x^k)$ *for all* $k \in \mathbb{N}$*. In addition, assume that the level set* $C_0 := \{x \in C : \ f(x) \leq f(x^0)\}$ *is bounded and* $\nu_0 = 0$*. Then,* $\lim_{k\to+\infty} \nu_k = 0$*.*

Full asymptotic convergence

Full asymptotic convergence analysis

We will prove, under suitable assumptions, the full convergence of the sequence $(x^k)_{k\in\mathbb{N}}$. For this end, we assume that in $\mathbf{Step\,1}$ of Algorithm InexProj-SGM:

A1. For all $k \in \mathbb{N}$, we take $w^k \in \mathcal{R}^{D_k}_{C,\gamma_k}(x^k, z^k)$ with $\gamma_k = (1-\zeta_k)/2$.

A2. For all $k \in \mathbb{N}$, we take $0 \leq \nu_k$ such that $\sum_{k=0}^{+\infty} \nu_k < +\infty$.

Armijo line search and nonmonotone line search strategies defined by (6) satisfies the assumption **A2**.

Full asymptotic convergence analysis

Lemma
For each $x \in C$, there holds

$$\|x^{k+1}-x\|^2_{D_k} \leq \|x^k-x\|^2_{D_k}+2\alpha_k\tau_k\left\langle \nabla f(x^k), x-x^k\right\rangle+\xi\left[f(x^k)-f(x^{k+1})+\nu_k\right], \quad \forall\, k \in \mathbb{N}.$$

where $\xi := \dfrac{2\alpha_{\max}}{\sigma} > 0.$

Corollary
Assume that f is a convex function. If $U := \left\{x \in C : f(x) \leq \inf_{k\in\mathbb{N}} \left(f(x^k)+\nu_k\right)\right\}$ is not empty, then $(x^k)_{k\in\mathbb{N}}$ converges to a stationary point of problem (1).

Full asymptotic convergence analysis

Theorem

If f *is a convex function and* $(x^k)_{k\in\mathbb{N}}$ *has no cluster points, then* $\Omega^* = \varnothing$, $\lim_{k\to\infty} \|x^k\| = +\infty$, *and* $\inf_{k\in\mathbb{N}} f(x^k) = \inf\{f(x) : x \in C\}$.

Full asymptotic convergence analysis

Corollary

If f *is a convex function and* $(x^k)_{k\in\mathbb{N}}$ *has at least one cluster point, then* $(x^k)_{k\in\mathbb{N}}$ *converges to a stationary point of problem* (1).

Full asymptotic convergence analysis

Theorem

Assume that f is a convex function and $\Omega^* \neq \varnothing$. Then, $(x^k)_{k\in\mathbb{N}}$ converge to an optimal solution of problem (1).

Iteration-complexity bound

Interation-complexity bound

Besides assuming that in $\mathbf{Step\,1}$ of Algorithm InexProj-SGM we take $(x^k)_{k\in\mathbb{N}}$ satisfying **A1** and **A2**, we also need the following assumption.

A3. The gradient ∇f of f is Lipschitz continuous with constant $L > 0$.

Lemma
The steepsize τ_k in Algorithm InexProj-SGM satisfies $\tau_k \geq \tau_{\min}$,

where

$$\tau_{\min} := \min\left\{1, \frac{\tau(1-\sigma)}{\alpha_{\max}\mu L}\right\}.$$

Interation-complexity bound

Theorem
For every $N \in \mathbb{N}$, the following inequality holds

$$\min\left\{\|w^k - x^k\| : \; k = 0, 1 \ldots, N-1\right\} \leq \sqrt{\frac{2\alpha_{\max}\mu\left(f(x^0) - f^* + \sum_{k=0}^{\infty}\nu_k\right)}{\sigma\tau_{\min}}}\frac{1}{\sqrt{N}}.$$

Theorem
Let f be a convex function on C. Then, for every $N \in \mathbb{N}$, there holds

$$\min\left\{f(x^k) - f^* : \; k = 0, 1 \ldots, N-1\right\} \leq \frac{\|x^0 - x^*\|_{D_0}^2 + \xi\left[f(x^0) - f^* + \sum_{k=0}^{\infty}\nu_k\right]}{2\alpha_{\min}\tau_{\min}}\frac{1}{N}.$$

Interation-complexity bound

Lemma

Let N_k be the number of function evaluations after $k \geq 1$ iterations of Algorithm InexProj-SGM. Then,

$$N_k \leq 1 + (k+1)\left[\frac{\log(\tau_{\min})}{\log(\tau)} + 1\right].$$

Interation-complexity bound

Theorem

For a given $\epsilon > 0$, the number of function evaluations in Algorithm InexProj-SGM are at most

$$1 + \left(\frac{2\alpha_{\max}\mu\left(f(x^0) - f^* + \sum_{k=0}^{\infty} \nu_k\right)}{\sigma\tau_{\min}} \frac{1}{\epsilon^2} + 1 \right) \left(\frac{\log(\tau_{\min})}{\log(\tau)} + 1 \right),$$

to compute x^k and w^k such that $\|w^k - x^k\| \leq \epsilon$.

Interation-complexity bound

Theorem

Let f be a convex function on C. For a given $\epsilon > 0$, the number of function evaluations in Algorithm InexProj-SGM are at most

$$1 + \left(\frac{\|x^0 - x^*\|_{D_0}^2 + \xi \left(f(x^0) - f^* + \sum_{k=0}^{\infty} \nu_k \right)}{2\alpha_{\min}\tau_{\min}} \frac{1}{\epsilon} + 1 \right) \left(\frac{\log(\tau_{\min})}{\log(\tau)} + 1 \right),$$

to compute x^k such that $f(x^k) - f^* \leq \epsilon$.

Numerical experiments

Numerical experiments

Given A and B two $m \times n$ matrices, with $m \geq n$, and $c \in \mathbb{R}$, we consider the matrix function $f : \mathbb{R}^{n\times n} \to \mathbb{R}$ given by:

$$f(X) := \frac{1}{2}\|AX - B\|_F^2 + \sum_{i=1}^{n-1}\left[c\left(X_{i+1,i+1} - X_{i,i}^2\right)^2 + (1 - X_{i,i})^2\right],$$

which combines a least squares term with a Rosenbrock-type function. $X_{i,j}$ stands for the ij-element of the matrix X and $\|\cdot\|_F$ denotes the Frobenius matrix norm, i.e., $\|A\|_F := \sqrt{\langle A, A\rangle}$ where the inner product is given by $\langle A, B\rangle = \mathrm{tr}(A^T B)$.

Problem I[7]**:**

$$\begin{array}{ll} \min & f(X) \\ \text{s.t.} & X \in SDD^+, \\ & L \leq X \leq U, \end{array}$$

where SDD^+ is the cone of symmetric and diagonally dominant real matrices with positive diagonal, i.e.,

$$SDD^+ := \{X \in \mathbb{R}^{n\times n} \mid X = X^T,\ X_{i,i} \geq \sum_{j\neq i} |X_{i,j}|\ \forall i\},$$

L and U are given $n \times n$ matrices, and $L \leq X \leq U$ means that $L_{i,j} \leq X_{i,j} \leq U_{i,j}$ for all i, j.

[7] Ernesto G. Birgin, José Mario Martínez, and Marcos Raydan. "Inexact spectral projected gradient methods on convex sets". In: *IMA J. Numer. Anal.* 23.4 (2003), pp. 539–559.

Problem II[8,9]**:**

$$\begin{array}{ll} \min & f(X) \\ \text{s.t.} & X \in \mathbb{S}^n_+, \\ & \operatorname{tr}(X) = 1, \end{array}$$

where $\mathbb{S}^n_+$ is the cone of symmetric and positive semidefinite real matrices. The feasible set of Problem II was known as *spectrahedron* and appears in several interesting applications.

[8] Zeyuan Allen-Zhu et al. "Linear convergence of a Frank-Wolfe type algorithm over trace-norm balls". In: *Advances in Neural Information Processing Systems*. 2017, pp. 6191–6200.
[9] D.S. Gonçalves, M.A. Gomes-Ruggiero, and C. Lavor. "A projected gradient method for optimization over density matrices". In: *Optimization Methods and Software* 31.2 (2016), pp. 328–341.

Numerical experiments

We are interested in the spectral gradient version of the SPG method, so we set $D_k := I$ for all k, $\alpha_0 := \min(\alpha_{\max}, \max(\alpha_{\min}, 1/\|\nabla f(x^0)\|))$ and, for $k > 0$,

$$\alpha_k := \begin{cases} \min(\alpha_{\max}, \max(\alpha_{\min}, \langle s^k, s^k\rangle / \langle s^k, y^k\rangle)), & \text{if } \langle s^k, y^k\rangle > 0 \\ \alpha_{\max}, & \text{otherwise,} \end{cases}$$

where $s^k := X^k - X^{k-1}$, $y^k := \nabla f(X^k) - \nabla f(X^{k-1})$, $\alpha_{\min} = 10^{-10}$, and $\alpha_{\max} = 10^{10}$.

Concerning the stopping criterion, all runs were stopped at an iterate X^k declaring convergence if

$$\max_{i,j}(|X^k_{i,j} - W^k_{i,j}|) \leq 10^{-6},$$

where $W^k \in \mathcal{P}^{D_k}_{C,\zeta_k}(x^k, z^k)$.

Influence of the inexact projection

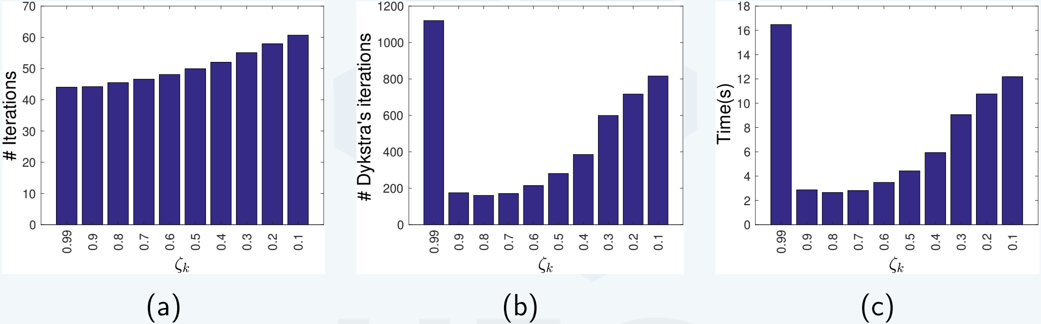
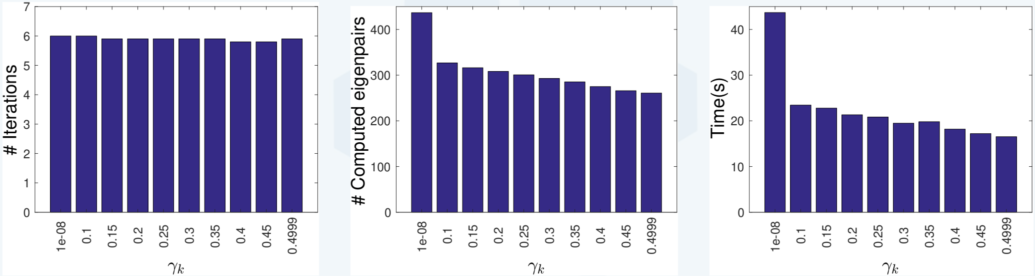


Figure 1: Results for 10 instances of Problem I using $n = 100$, $m = 200$, and $c = 10$. Average number of: (a) iterations; (b) Dykstra's iterations; (c) CPU time in seconds needed to reach the solution for different choices of ζ_k.

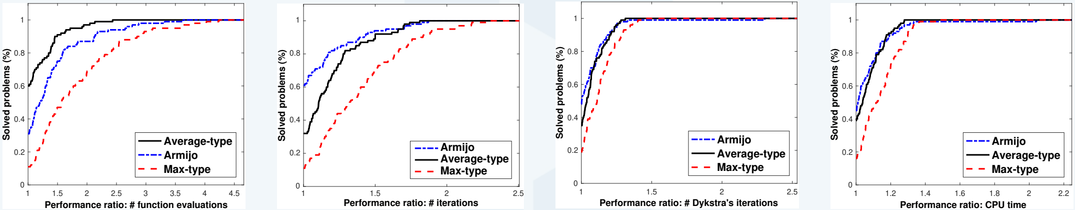
Influence of the inexact projection



(a) (b) (c)

Figure 2: Results for 10 instances of Problem II using $n = 800$, $m = 1000$, and $c = 100$. Average number of: (a) iterations; (b) computed eigenpairs; (c) CPU time in seconds needed to reach the solution for different choices of γ_k.

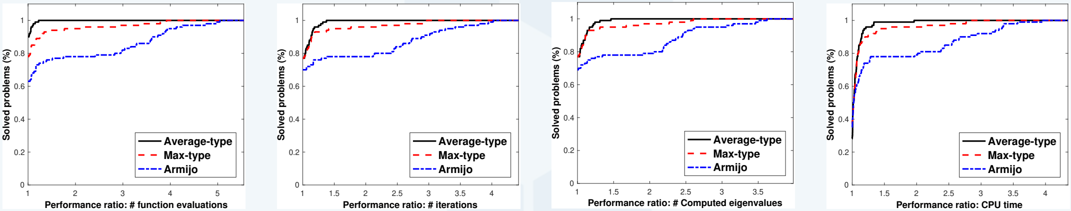
Influence of the line search scheme



(a) (b) (c) (d)

Figure 3: Performance profiles for Problem I considering the SPG method with the Armijo, the Average-type, and the Max-type line searches strategies using as performance measurement: (a) number of function evaluations; (b) number of (outer) iterations; (c) number of Dykstra's iterations; (d) CPU time.

Influence of the line search scheme



(a) (b) (c) (d)

Figure 4: Performance profiles for Problem II considering the SPG method with the Armijo, the Average-type, and the Max-type line searches strategies using as performance measurement: (a) number of function evaluations; (b) number of (outer) iterations; (c) number of computed eigenpairs; (d) CPU time.

Thank you!