UNIVERSIDADE FEDERAL DE GOIÁS

INSTITUTO DE MATEMÁTICA E ESTATÍSTICA

An inexact scaled gradient projection method with applications in risk parity portfolios

Max Valério Lemes

Goiânia - Goiás

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Abstract

The purpose of this work is to present an inexact version of the scaled gradient projection method on a convex set, which is inexact in two sense. First, an inexact projection on the feasible set is computed, allowing for an appropriate relative error tolerance. Second, an inexact nonmonotone line search scheme is employed to compute a step size which defines the next iteration. It is shown that the proposed method has similar asymptotic convergence properties and iteration-complexity bounds as the usual scaled gradient projection method employing monotone line searches. Besides, it is presented an application of the method to build risk parity portfolios.

Keywords: Scaled gradient projection method, Feasible inexact projection, Constrained convex optimization, Portfolio optimization, Risk parity portfolio.

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Introduction

This work is devoted to the study of the *scaled gradient projection (SGP) method with nonmonotone line search* to solve general constrained convex optimization problems as follows

$$\min\{f(x) : \ x \in C\}, \tag{1}$$

where C is a closed and convex subset of $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function. Denotes by $f^* := \inf_{x\in C} f(x)$ the optimal value of (1) and by Ω^* its solution set, which we will assume to be nonempty unless the contrary is explicitly stated. Problem (1) is a basic issue of constrained optimization, which appears very often in various areas, including finance, machine learning, control theory, and signal processing, see for example [21, 22, 37, 48, 53, 68]. Recent problems considered in most of these areas, the datasets are large or high-dimensional and their solutions need to be approximated quickly with a reasonably accuracy. It is well known that SGP method with nonmonotone line search is among those that are suitable for this task, as will be explained below.

The gradient projection method is what first comes to mind when we start from the ideas of the classic optimization methods in an attempt to deal with problem (1). In fact, this method is one of the oldest known optimization methods to solve (1), the study of its convergence properties goes back to the works of Goldstein [41] and Levitin and Polyak [52]. After these works, several variants of it have appeared over the years, resulting in a vast literature on the subject, including [10, 11, 12, 34, 37, 44, 49, 60, 74]. Additional reference on this subject may be found in the recent review [18] and references therein. Among all the variants of the gradient projection method, the scaled version has been especially considered due to the flexibility provided in efficient implementations of the method; see [14, 5, 17, 19, 20]. In addition, its simplicity and easy implementation has attracted the attention of the scientific community that works on optimization over the years. This method usually uses only first-order derivatives, which makes it very stable from a numerical point of view and therefore quite suitable for solving large-scale optimization problems, see [56, 57, 68, 69]. At each current iteration, SGP method moves along the direction of the negative scaled gradient, and then projects the obtained point onto the constraint set. The current iteration and such projection define a feasible descent direction and a line search in this direction is performed to define the next iteration. In this way, the performance of the method is strongly related to each of the steps we have just mentioned. In fact, the scale matrix and the step size towards the negative scaled gradient are freely selected in order to improve the performance of SGP method but without increasing the cost of each iteration. Strategies for choosing both have their origins in the study of gradient method for unconstrained optimization, papers addressing this issues include but not

limited to [7, 19, 27, 28, 30, 38, 76, 26, 52]. It is worth mentioning that, for suitable choices of the scale matrix and the step size, SGP merges into the well known *spectral gradient method* extensively studied in [14, 13]. More details about selecting step sizes and scale matrices may be found in the recent review [18] and references therein.

In this work, we are particularly interested in the main stages that make up the SGP method, namely, in the projection calculation and in the line search employed. It is well known that the mostly computational burden of each iteration of the SGP method is in the calculation of the projection. Indeed, the projection calculation requires, at each iteration, the solution of a quadratic problem restricted to the feasible set, which may lead to a substantial increase in the cost per iteration if the number of unknowns is large. For this reason, it may not be justified to carry out exact projections when the iterates are far from the solution of the problem. In order to reduce the computational effort spent on projections, inexact procedures that become more and more accurate when approaching the solution, have been proposed, resulting in more efficient methods; see for example [14, 17, 40, 43, 67, 71, 62]. On the other hand, nonmonotone searches may improve the probability of finding an optimal global solution, in addition to potentially improving the speed of convergence of the method as a whole, see for example [25, 59, 70]. The concept of nonmonotone line search, that we will use here as a synonym for inexact line search, have been proposed first in [47], and later a new nonmonotone search was proposed in [75]. After these papers others nonmonotone searches appeared, see for example [3, 55]. In [66], an interesting general framework for nonmonotone line search was proposed, and more recently modifications of it have been presented in [45, 46].

The purpose of the present work is to present an inexact version of the SGP method, which is inexact in two sense. First, using a version of scheme introduced in [14] and also a variation of the one appeared in [71, Example 1], the inexact projection onto the feasible set is computed allowing an appropriate relative error tolerance. Second, using the inexact conceptual scheme for the line search introduced in [46, 66], a step size is computed to define the next iteration. More specifically, initially we show that the feasible inexact projection of [14] provides greater latitude than the projection of [71, Example 1]. In the first convergence result presented, we show that the SGP method using the projection proposed in [14] preserves the same partial convergence result as the classic method, that is, we prove that every accumulation point of the sequence generated by the SGP method is stationary for problem (1). Then, considering the inexact projection of [71, Example 1], and under mild assumptions, we establish full asymptotic convergence results and some complexity bounds. The presented analysis of the method is done using the general nonmonotone line search scheme introduced in [46]. In this way, the proposed method may be adapted to several line searches and, in particular, will allow obtaining several known versions of the SGP method as particular instances, including [10, 14, 49, 73]. Except for the particular case when we assume that the SGP method employs the nonmonotone line search introduced by [47], all other asymptotic convergence and complexity results are obtained without any assumption of compactness of the sub-level sets of the objective function. Finally, it is worth mentioning that the complexity results obtained for the SGP method with a general nonmonotone line search are

the same as in the classic case when the usual Armijo search is employed, namely, the complexity bound $\mathcal{O}(1/\sqrt{k})$ is unveil for finding ϵ-stationary points for problem (1) and, under convexity on f, the rate to find a ϵ-optimal functional value is $\mathcal{O}(1/k)$.

Key results from Chapters 2, 3, and 4 were compiled and published in January 2022, see [36].

In Chapter 1, some notations and basic results used throughout the work is presented. In particular, Chapter 2 is devoted to recall the concept of relative feasible inexact projection and some new properties about this concept are presented. Chapter 3 describes the SGP method with a general nonmonotone line search and some particular instances of it are presented. Partial asymptotic convergence results are presented besides a full convergence result and iteration-complexity bounds. Some numerical experiments are provided in Chapter 4. In Chapter 5, an application of SGP method to portfolio optimization is presented together with a backtest study. Finally, some concluding remarks are made in Chapter 6.

Chapter 1

Preliminaries

In this chapter, we introduce some notation and results used throughout our presentation.

First, we consider the index set $\mathbb{N} := \{0, 1, 2, \ldots\}$, the usual inner product $\langle \cdot, \cdot \rangle$ in $\mathbb{R}^n$, and the associated Euclidean norm $\|\cdot\|$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function and $C \subseteq \mathbb{R}^n$. The gradient ∇f of f is said to be *Lipschitz continuous* in C with constant $L > 0$ if

$$\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|,$$

for all $x, y \in C$. Combining this definition with the Fundamental Theorem of Calculus, we obtain the following result whose proof may be found in [12, Proposition A.24].

Lemma 1.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function and $C \subseteq \mathbb{R}^n$. Assume that ∇f is Lipschitz continuous in C with constant $L > 0$. Then,*

$$f(y) - f(x) - \langle \nabla f(x), y - x \rangle \leq (L/2)\|x - y\|^2,$$

for all $x, y \in C$.

Assume that C is a convex set. The function f is said to be *convex* on C, if

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle,$$

for all $x, y \in C$. We recall that a point $\bar{x} \in C$ is a *stationary point* for problem (1) if

$$\langle \nabla f(\bar{x}), x - \bar{x} \rangle \geq 0, \qquad \forall\, x \in C. \tag{1.1}$$

Consequently, if f is a convex function on C, then (1.1) implies that $\bar{x} \in \Omega^*$. We will now present some useful concepts for the analysis of the sequence generated by the scaled gradient method, for more details, see [24]. For that, let D be a $n \times n$ positive definite matrix and $\|\cdot\|_D : \mathbb{R}^n \to \mathbb{R}$ be the norm defined by

$$\|d\|_D := \sqrt{\langle Dd, d \rangle}, \quad \forall d \in \mathbb{R}^n. \tag{1.2}$$

For a fixed constant $\mu \geq 1$, *denote by $\mathcal{D}_\mu$ the set of symmetric positive definite matrices $n \times n$ with all eigenvalues contained in the interval $[\frac{1}{\mu}, \mu]$*. The set $\mathcal{D}_\mu$ is compact. Moreover, for each $D \in \mathcal{D}_\mu$, it follows that D^{-1} also belongs to $\mathcal{D}_\mu$. Furthermore, due to $D \in \mathcal{D}_\mu$, by (1.2), we obtain

$$\frac{1}{\mu}\|d\|^2 \leq \|d\|_D^2 \leq \mu\|d\|^2, \qquad \forall d \in \mathbb{R}^n. \tag{1.3}$$

Let us recall the the concept of sequence quasi-Fejér monotone to a set, introduced in [24].

Definition 1.2. *Let $(y^k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{R}^n$ and $(D_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{D}_\mu$. The sequence $(y^k)_{k\in\mathbb{N}}$ is said to be quasi-Fejér monotone to a set $W\subset\mathbb{R}^n$ with respect to $(D_k)_{k\in\mathbb{N}}$ if, there exists a sequence $(\eta_k)_{k\in\mathbb{N}}\subset[0,+\infty)$ such that $\sum_{k\in\mathbb{N}}\eta_k<\infty$ and for all $w\in W$, there exists a sequence $(\epsilon_k)_{k\in\mathbb{N}}\subset[0,+\infty)$ such that $\sum_{k\in\mathbb{N}}\epsilon_k<\infty$, and*

$$\|y_{k+1}-w\|^2_{D_{k+1}}\leq(1+\eta_k)\|y^k-w\|^2_{D_k}+\epsilon_k,$$

for all $k\in\mathbb{N}$.

The following lemma is useful to study the quasi-Fejér monotone sequence, its prove may be found in [61, Lemma 2.2.2].

Lemma 1.3. *Let $(\alpha_k)_{k\in\mathbb{N}}$, $(\eta_k)_{k\in\mathbb{N}}$ and $(\epsilon_k)_{k\in\mathbb{N}}$ be a sequences in $[0,+\infty)$ such that $\sum_{k\in\mathbb{N}}\eta_k<\infty$ and $\sum_{k\in\mathbb{N}}\epsilon_k<\infty$. Assume that $\alpha_{k+1}\leq(1+\eta_k)\alpha_k+\epsilon_k$, for all $k\in\mathbb{N}$. Then, $(\alpha_k)_{k\in\mathbb{N}}$ converges.*

The main property of quasi-Fejér monotone sequences is stated in the following. Its proof may be found in [24, Proposition 3.2 and Theorem 3.3]. For sake of completeness, we include it here.

Theorem 1.4. *Let $(y^k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{R}^n$ and $(D_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{D}_\mu$ such that $\lim_{k\to\infty}D_k=\bar{D}$. If $(y^k)_{k\in\mathbb{N}}$ is quasi-Fejér monotone to a nonempty set $W\subset\mathbb{R}^n$ with respect to $(D_k)_{k\in\mathbb{N}}$ then, for each $w\in W$, the sequence $(\|y^k-w\|_{D_k})_{k\in\mathbb{N}}$ converges. Furthermore, $(y^k)_{k\in\mathbb{N}}$ is bounded and, if each cluster point of $(y^k)_{k\in\mathbb{N}}$ belongs to W, then there exists $\bar{y}\in W$ such that $\lim_{k\to\infty}y^k=\bar{y}$.*

Proof. Take $w\in W$ and define the sequence $(\alpha_k)_{k\in\mathbb{N}}$, where $\alpha_k:=\|y^k-w\|_{D_k}$. Since $(y^k)_{k\in\mathbb{N}}$ is quasi-Fejér monotone to W, Lemma 1.3 implies that $(\alpha_k)_{k\in\mathbb{N}}$ converges. Now, by using the first inequality in (1.3), we have $\|y^k-w\|\leq\sqrt{\mu}\alpha_k$, for all $k\in\mathbb{N}$. Thus, $(y^k)_{k\in\mathbb{N}}$ is bounded. To prove the last statement, assume that $\bar{y},\hat{y}\in W$ are cluster points of $(y^k)_{k\in\mathbb{N}}$, and set $(y^{k_i})_{i\in\mathbb{N}}$ and $(y^{k_j})_{j\in\mathbb{N}}$ subsequences of $(y^k)_{k\in\mathbb{N}}$ such that $\lim_{j\to+\infty}y^{k_i}=\bar{y}$ and $\lim_{j\to+\infty}y^{k_j}=\hat{y}$. It follows from the first statement that $(\|y^k-\bar{y}\|_{D_k})_{k\in\mathbb{N}}$ and $(\|y^k-\hat{y}\|_{D_k})_{k\in\mathbb{N}}$ are convergent. Since $\lim_{k\to\infty}D_k=\bar{D}$, we have $\lim_{k\to\infty}\|\bar{y}\|_{D_k}=\|\bar{y}\|_{\bar{D}}$ and $\lim_{k\to\infty}\|\hat{y}\|_{D_k}=\|\hat{y}\|_{\bar{D}}$. Hence, due to

$$\langle y^k,D_k(\bar{y}-\hat{y})\rangle=\frac{1}{2}(\|y^k-\hat{y}\|^2_{D_k}-\|y^k-\bar{y}\|_{D_k}+\|\bar{y}\|_{D_k}-\|\hat{y}\|_{D_k}),$$

for all $k\in\mathbb{N}$, we conclude that the sequence $(\langle y^k,D_k(\bar{y}-\hat{y})\rangle)_{k\in\mathbb{N}}$ converges. Thus, taking into account that $\lim_{j\to+\infty}y^{k_i}=\bar{y}$, $\lim_{j\to+\infty}y^{k_j}=\hat{y}$ and $\lim_{k\to\infty}D_k=\bar{D}$ we obtain that

$$\langle\bar{y},\bar{D}(\bar{y}-\hat{y})\rangle=\lim_{i\to\infty}\langle y^{k_i},D_{k_i}(\bar{y}-\hat{y})\rangle=\lim_{j\to\infty}\langle y^{k_j},D_{k_j}(\bar{y}-\hat{y})\rangle=\langle\hat{y},\bar{D}(\bar{y}-\hat{y})\rangle.$$

Hence, using (1.3), we obtain

$$\frac{1}{\mu}\|\bar{y}-\hat{y}\|^2\leq\|\bar{y}-\hat{y}\|^2_{\bar{D}}=\langle\bar{y},\bar{D}(\bar{y}-\hat{y})\rangle-\langle\hat{y},\bar{D}(\bar{y}-\hat{y})\rangle=0,$$

which implies that $\bar{y}=\hat{y}$. Therefore, due to $(y^k)_{k\in\mathbb{N}}$ be bounded, we conclude that $(y^k)_{k\in\mathbb{N}}$ converges to $\bar{y}$. □

Chapter 2

Relative feasible inexact projections

In this chapter, we recall two concepts of relative feasible inexact projections onto a closed and convex set, and also present some new properties of them which will be used throughout this work. These concepts of inexact projections were introduced seeking to make the subproblem of computing the projections on the feasible set more efficient; see for example [14, 67, 71]. Before presenting the inexact projection concept that we will use, let us first recall the concept of exact projection with respect to a given norm. For that, *throughout this chapter we assume that* $D \in \mathcal{D}_\mu$.

Definition 2.1. *The exact projection of the point* $v \in \mathbb{R}^n$ *onto* C *with respect to the norm* $\|\cdot\|_D$*, denoted by* $\mathcal{P}_C^D(v)$*, is defined by*

$$\mathcal{P}_C^D(v) := \arg\min_{z \in C} \|z - v\|_D^2. \tag{2.1}$$

The next result characterizes the exact projection, its proof may be found in [8, Theorem 3.14].

Lemma 2.2. *Let* $v, w \in \mathbb{R}^n$*. Then,* $w = \mathcal{P}_C^D(v)$ *if and only if* $w \in C$ *and*

$$\langle D(v - w), y - w\rangle \leq 0, \tag{2.2}$$

for all $y \in C$.

Remark 2.3. It follows from Definition 2.1 that $w = \mathcal{P}_C^D(v)$ is the point of C more close to v with respect to the norm $\|\cdot\|_D$. On the other hand, by the Lemma 2.2, w is the unique point of C that for all $y \in C$ the angle θ among the vectors $v - w$ and $y - w$ is an obtuse angle. Figure 2.1 shows this fact considering the plane generated by points y, w and v.

Remark 2.4. It follows from Lemma 2.2 that $\|\mathcal{P}_C^D(v) - \mathcal{P}_C^D(u)\|_D \leq \|v - u\|_D$. Moreover, since $D \in \mathcal{D}_\mu$, by (1.3), we conclude that $\mathcal{P}_C^D(\cdot)$ is Lipschitz continuous with constant $L = \mu$. Furthermore, if $(D_k)_{k\in\mathbb{N}} \subset \mathcal{D}_\mu$, $\lim_{k\to+\infty} z^k = \bar{z}$, and $\lim_{k\to+\infty} D_k = \bar{D}$, then $\lim_{k\to+\infty} \mathcal{P}_C^{D_k}(z^k) = \mathcal{P}_C^{\bar{D}}(\bar{z})$, see [24, Proposition 4.2].

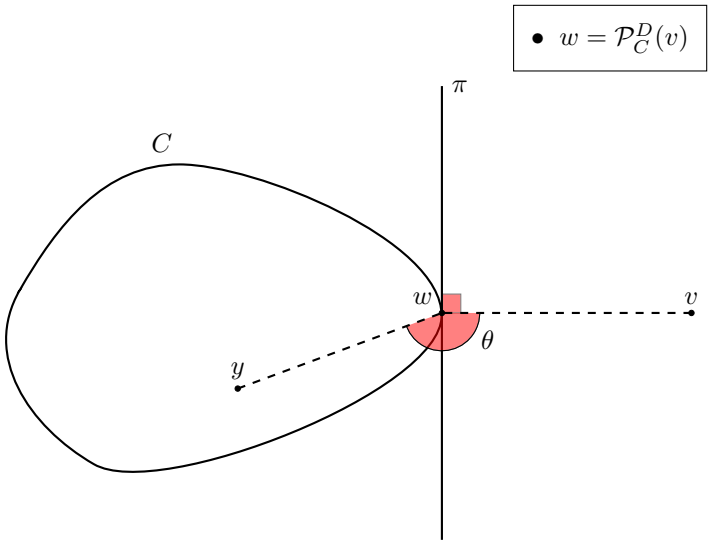


Figure 2.1: Exact projection of the point v onto C.

2.1 The first feasible inexact projection

In the following, we recall the concept of a feasible inexact projection with respect to $\|\cdot\|_D$ relative to a fixed point.

Definition 2.5. *The feasible inexact projection mapping, with respect to the norm $\|\cdot\|_D$, onto C relative to a point $u \in C$ and forcing parameter $\zeta \in (0,1]$, denoted by $\mathcal{P}_{C,\zeta}^D(u,\cdot) : \mathbb{R}^n \rightrightarrows C$, is the set-valued mapping defined as follows*

$$\mathcal{P}_{C,\zeta}^D(u,v) := \left\{ w \in C : \ \|w-v\|_D^2 \leq \zeta \|\mathcal{P}_C^D(v) - v\|_D^2 + (1-\zeta)\|u-v\|_D^2 \right\}. \tag{2.3}$$

Each point $w \in \mathcal{P}_{C,\zeta}^D(u,v)$ is called a feasible inexact projection, with respect to the norm $\|\cdot\|_D$, of v onto C relative to u and forcing parameter $\zeta \in (0,1]$.

Remark 2.6. It follows from Definition 2.5 that $\mathcal{P}_{C,\zeta}^D(u,v)$ is a set generated by the intersection of C and a sphere centered in v and radius given by

$$\zeta \|\mathcal{P}_C^D(v) - v\|_D^2 + (1-\zeta)\|u-v\|_D^2.$$

If $\zeta = 1$, then $\mathcal{P}_{C,1}^D(u,v) = \{\mathcal{P}_C^D(v)\}$ is the exact projection of v onto C. However, when ζ is close to zero, the radius of sphere is close to $\|u-v\|_D$. The inexact condition appears when we consider points that do not minimize the distance from C to v, contrary to property (2.1). This situation is illustrated in Figure 2.2.

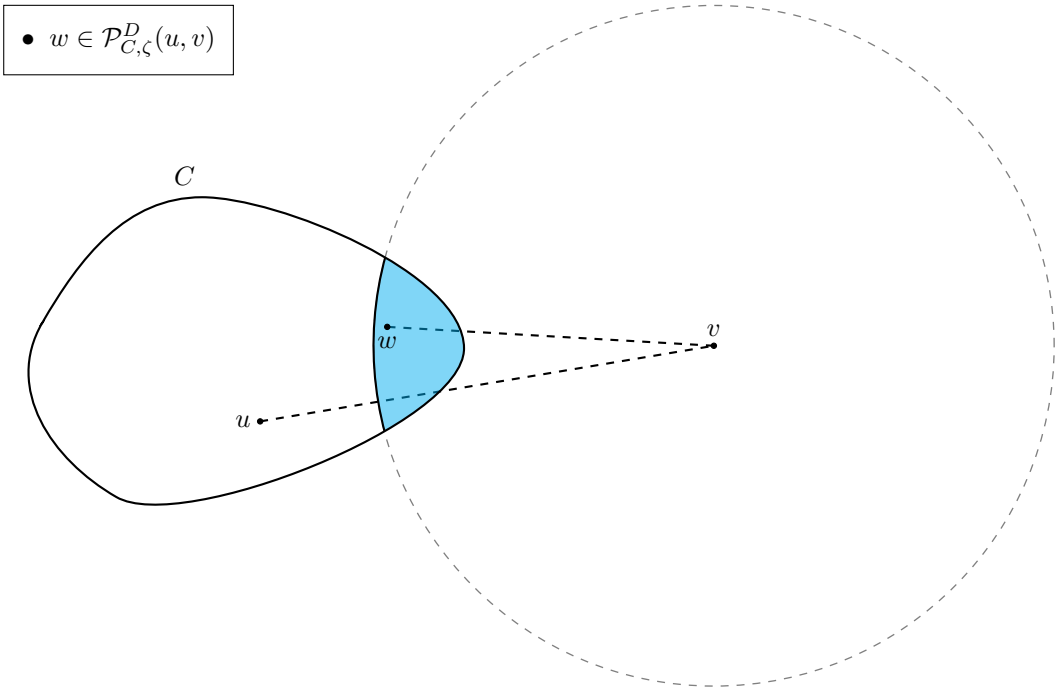


Figure 2.2: Feasible inexact projection of the point v onto C.

In the following, we show that the definition given above is nothing more than a reformulation of the concept of relative feasible inexact projection with respect to $\|\cdot\|_D$ introduced in [14].

Remark 2.7. Let $u \in C$, $v \in \mathbb{R}^n$ and D be an $n \times n$ positive definite matrix. Consider the quadratic function $Q : \mathbb{R}^n \to \mathbb{R}$ defined by

$$Q(z) := (1/2)\,\langle D(z-u), z-u\rangle + \langle D(u-v), z-u\rangle\,.$$

Thus, letting $\|\cdot\|_D$ be the norm defined by (1.2), some algebraic manipulations shows that

$$\|z-v\|_D^2 = 2Q(z) + \|u-v\|_D^2. \tag{2.4}$$

Hence, (2.4) and (2.1) implies that $\mathcal{P}^D_C(v) = \arg\min_{z\in C} Q(z)$. Let $\zeta \in (0,1]$. Thus, by using (2.4), after some calculations, we may see that the following inexactness condition introduced in [14],

$$w \in C, \qquad Q(w) \le \zeta Q(\mathcal{P}^D_C(v)),$$

is equivalent to find $w \in C$ such that

$$\|w-v\|_D^2 \le \zeta\|\mathcal{P}^D_C(v)-v\|_D^2 + (1-\zeta)\|u-v\|_D^2,$$

which corresponds to condition (2.3) in Definition 2.5.

The concept of feasible inexact projection in Definition 2.5 provides more latitude to the usual concept of exact projection (2.1). The next remark makes this more precise.

Remark 2.8. Let ζ be positive forcing parameter, $C \subset \mathbb{R}^n$ and $u \in C$ be as in Definition 2.5. First of all note that $\mathcal{P}^D_C(v) \in \mathcal{P}^D_{C,\zeta}(u,v)$. Therefore, $\mathcal{P}^D_{C,\zeta}(u,v) \neq \varnothing$, for all $u \in C$ and $v \in \mathbb{R}^n$. Consequently, the set-valued mapping $\mathcal{P}^D_{C,\zeta}(u,\cdot)$ as stated in (2.3) is well-defined. Moreover, for $\zeta = 1$, we have $\mathcal{P}^D_{C,1}(u,v) = \{\mathcal{P}^D_C(v)\}$. In addition, if $\underline{\zeta}$ and $\bar{\zeta}$ are forcing parameters such that $0 < \underline{\zeta} \le \bar{\zeta} \le 1$, then $\mathcal{P}^D_{C,\bar{\zeta}}(u,v) \subset \mathcal{P}^D_{C,\underline{\zeta}}(u,v)$.

Lemma 2.9. *Let $v \in \mathbb{R}^n$, $u \in C$ and $w \in \mathcal{P}^D_{C,\zeta}(u,v)$. Then,*

$$\langle D(v-w), y-w\rangle \leq \frac{1}{2}\|w-y\|_D^2 + \frac{1}{2}\left[\zeta\|\mathcal{P}_C^D(v)-v\|_D^2 + (1-\zeta)\|u-v\|_D^2 - \|y-v\|_D^2\right], \qquad y \in C.$$

Proof. Let $y \in C$. Since $2\langle D(v-w), y-w\rangle = \|w-y\|_D^2 + \|w-v\|_D^2 - \|v-y\|_D^2$, using (2.3) we have $2\langle D(v-w), y-w\rangle = \|w-y\|_D^2 + \zeta\|\mathcal{P}_C^D(v)-v\|_D^2 + (1-\zeta)\|u-v\|_D^2 - \|v-y\|_D^2$, which is equivalent to the desired inequality. □

2.2 The second feasible inexact projection

Next, we recall a second concept of relative feasible inexact projection onto a closed convex set, see [2, 29]. The definition is as follows.

Definition 2.10. *The feasible inexact projection mapping, with respect to the norm $\|\cdot\|_D$, onto C relative to $u \in C$ and forcing parameter $\gamma \geq 0$, denoted by $\mathcal{R}^D_{C,\gamma}(u,\cdot) : \mathbb{R}^n \rightrightarrows C$, is the set-valued mapping defined as follows*

$$\mathcal{R}^D_{C,\gamma}(u,v) := \left\{w \in C : \ \langle D(v-w), y-w\rangle \leq \gamma\|w-u\|_D^2, \quad \forall\ y \in C\right\}. \tag{2.5}$$

Each point $w \in \mathcal{R}^D_{C,\gamma}(u,v)$ is called a feasible inexact projection, with respect to the norm $\|\cdot\|_D$, of v onto C relative to u and forcing parameter $\gamma \geq 0$.

The concept of feasible inexact projection in Definition 2.10 also provides more latitude to the usual concept of exact projection. Next, we present some remarks about this concept.

Remark 2.11. Let $\gamma \geq 0$ be a forcing parameter, $C \subset \mathbb{R}^n$ and $u \in C$ be as in Definition 2.10. For all $v \in \mathbb{R}^n$, it follows from (2.5) and Lemma 2.2 that $\mathcal{R}^D_{C,0}(u,v) = \{\mathcal{P}^D_C(v)\}$ is the exact projection of v onto C. Moreover, $\mathcal{P}^D_C(v) \in \mathcal{R}^D_{C,\gamma}(u,v)$ concluding that $\mathcal{R}_{C,\gamma}(u,v) \neq \varnothing$, for all $u \in C$ and $v \in \mathbb{R}^n$. Consequently, the set-valued mapping $\mathcal{R}^D_{C,\gamma}(u,\cdot)$ as stated in (2.5) is well-defined.

We show now a geometric interpretation for the inexact projection defined in Definition 2.10.

Remark 2.12. Let $C \subset \mathbb{R}^n$, $u \in C$, $v \in \mathbb{R}^n$, $w \in \mathcal{R}^D_{C,\gamma}(u,v)$ be as stated in Definition 2.10 and $w_t = w + t(v-w)$, with $0 \leq t < 1$. It is easy to see that, for all $y \in C$,

$$y - w_t = y - w - t(v-w).$$

It follows from (2.5) that $\langle D(v-w), y-w\rangle \leq \gamma\|w-u\|_D^2$, so we have

$$\begin{aligned}\langle D(v-w_t), y-w_t\rangle &= (1-t)\langle D(v-w), y-w\rangle - t(1-t)\|v-w\|_D^2 \\ &\leq (1-t)\left[\gamma\|w-u\|_D^2 - t\|v-w\|_D^2\right].\end{aligned}$$

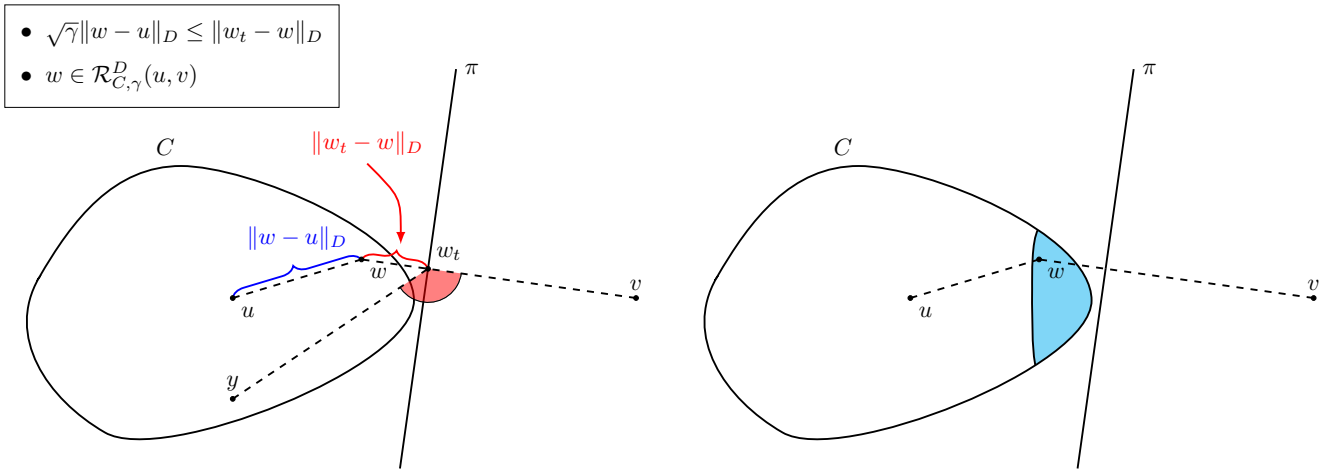
Since $0 \leq t < 1$,

$$\|w_t - w\|_D^2 = t^2\|v-w\|_D^2 \leq t\|v-w\|_D^2,$$

if

$$\gamma\|w-u\|_D^2 \leq \|w_t - w\|_D^2 \tag{2.6}$$

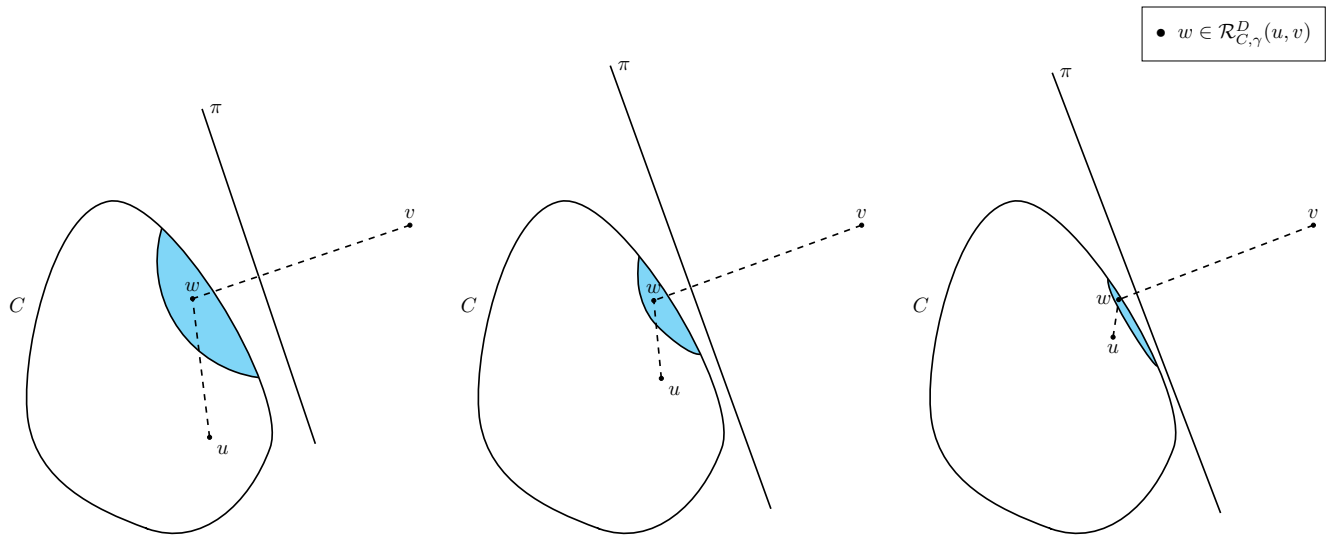
then $\langle D(v - w_t), y - w_t \rangle \leq 0$. In this case the inexact condition appears considering that there is w_t between w and v that satisfies (2.6) and consequently the condition (2.2). This situation is illustrated in Figure 2.3.



(a) Geometric interpretation. (b) Example of $\mathcal{R}^D_{C,\gamma}(u, v)$.

Figure 2.3: Geometric interpretation of projection $\mathcal{R}^D_{C,\gamma}(u, v)$.

In the following we presents some examples of regions given by inexact projection $\mathcal{R}^D_{C,\gamma}(u, v)$. Note that the projection $\mathcal{R}^D_{C,\gamma}(u, v)$ gets smaller as u gets closer to w.



(a) (b) (c)

Figure 2.4: Examples of regions given by inexact projection $\mathcal{R}^D_{C,\gamma}(u, v)$.

The next example shows the behavior of the projection $\mathcal{R}^D_{C,\gamma}(u, v)$ when the set C has a vertice and some flat parts.

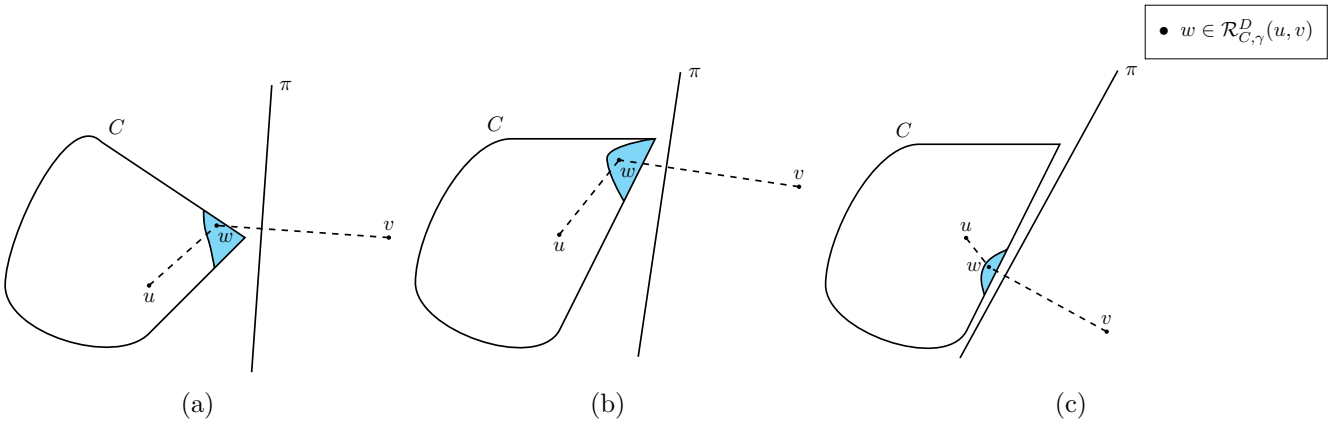


Figure 2.5: Examples of regions given by inexact projection $\mathcal{R}^D_{C,\gamma}(u,v)$.

The next lemma is a variation of [31, Lemma 6]. It will allow to relate Definitions 2.5 and 2.10.

Lemma 2.13. *Let $v \in \mathbb{R}^n$, $u \in C$, $\gamma \geq 0$ and $w \in \mathcal{R}^D_{C,\gamma}(u,v)$. Then, for all $0 \leq \gamma < 1/2$, the holds*

$$\|w - x\|_D^2 \leq \|x - v\|_D^2 + \frac{2\gamma}{1-2\gamma}\|u - v\|_D^2 - \frac{1}{1-2\gamma}\|w - v\|_D^2, \qquad \forall x \in C.$$

Proof. First note that $\|w-x\|_D^2 = \|x-v\|_D^2 - \|w-v\|_D^2 + 2\langle D(v-w), x-w\rangle$. Since $w \in \mathcal{R}^D_{C,\gamma}(u,v)$ and $x \in C$, combining the last equality with (2.5), we obtain

$$\|w - x\|_D^2 \leq \|x - v\|_D^2 - \|w - v\|_D^2 + 2\gamma\|w - u\|_D^2. \tag{2.7}$$

On the other hand, we also have $\|w - u\|_D^2 = \|u - v\|_D^2 - \|w - v\|_D^2 + 2\langle D(v - w), u - w\rangle$. Due to $w \in \mathcal{R}^D_{C,\gamma}(u,v)$ and $u \in C$, using (2.5) and considering that $0 \leq \gamma < 1/2$, we have

$$\|w - u\|_D^2 \leq \frac{1}{1-2\gamma}\|u - v\|_D^2 - \frac{1}{1-2\gamma}\|w - v\|_D^2.$$

Therefore, substituting the last inequality into (2.7), we obtain the desired inequality. □

2.3 The relationship among feasible inexact projections

In the following lemma, we present a relationship between Definitions 2.5 and 2.10.

Lemma 2.14. *Let $v \in \mathbb{R}^n$, $u \in C$, $\gamma \geq 0$ and $\zeta \in (0, 1]$. If $0 \leq \gamma < 1/2$ and $\zeta = 1 - 2\gamma$, then*

$$\mathcal{R}^D_{C,\gamma}(u,v) \subset \mathcal{P}^D_{C,\zeta}(u,v).$$

Proof. Let $w \in \mathcal{R}^D_{C,\gamma}(u,v)$. Applying Lemma 2.13 with $x = \mathcal{P}^D_C(v)$ we have

$$\|w - \mathcal{P}^D_C(v)\|_D^2 \leq \|v - \mathcal{P}^D_C(v)\|_D^2 + \frac{2\gamma}{1-2\gamma}\|u - v\|_D^2 - \frac{1}{1-2\gamma}\|w - v\|_D^2,$$

After some algebraic manipulations in the last inequality we obtain that

$$\|w - v\|_D^2 \leq (1-2\gamma)\|v - \mathcal{P}^D_C(v)\|_D^2 + 2\gamma\|u - v\|_D^2 - (1-2\gamma)\|w - \mathcal{P}^D_C(v)\|_D^2.$$

Therefore, considering that $0 \leq \gamma < 1/2$ and $\zeta = 1 - 2\gamma$, the result follows from Definition 2.5.

□

Remark 2.15. Under the conditions of Lemma 2.14, there exists $0 \leq \gamma < 1/2$ and $\zeta = 1 - 2\gamma$ such that $\mathcal{P}^D_{C,\zeta}(u,v) \nsubseteq \mathcal{R}^D_{C,\gamma}(u,v)$. Indeed, let $\gamma = 3/8$, $\zeta = 1/4$, and $\bar{w} = \frac{1}{2}(\mathcal{P}^D_C(v) + u)$, then

$$\|\bar{w} - v\|^2_D = \frac{1}{4}\|\mathcal{P}^D_C(v) - v\|^2_D + \frac{1}{4}\|u - v\|^2_D + \frac{1}{2}\langle D(\mathcal{P}^D_C(v) - v), u - v\rangle.$$

Since $\mathcal{P}^D_C(v)$ is the exact projection of v, we have $\langle D(\mathcal{P}^D_C(v) - v), u - v\rangle \leq \|u - v\|^2_D$. Combining this inequality with the last equality and Definition 2.5, we conclude that $\bar{w} \in \mathcal{P}^D_{C,\zeta}(u,v)$. Now, letting $w_t = t\mathcal{P}^D_C(v) + (1-t)\bar{w}$ with $0 < t < 1$, after some algebraic manipulations we have

$$\langle D(v - \bar{w}), w_t - \bar{w}\rangle = t\|\bar{w} - u\|^2_D - \frac{t}{2}\langle D(v - \mathcal{P}^D_C(v)), u - \mathcal{P}^D_C(v)\rangle.$$

Thus, it follows from Lemma 2.2 that $\langle D(v - \bar{w}), w_t - \bar{w}\rangle \geq t\|\bar{w} - u\|^2_D$. Hence, taking $t > 3/8$ we conclude that $\bar{w} \notin \mathcal{R}^D_{C,\gamma}(u,v)$. Therefore, considering that $\bar{w} \in \mathcal{P}^D_{C,\zeta}(u,v)$, the statement follows.

It follows from Remark 2.15 that, in general, $\mathcal{P}^D_{C,\zeta}(u,v) \nsubseteq \mathcal{R}^D_{C,\gamma}(u,v)$. However, whenever C is a bounded set, we will show that for each fixed $0 \leq \gamma < 1/2$ there exist $0 < \zeta < 1$ such that $\mathcal{P}^D_{C,\zeta}(u,v) \subseteq \mathcal{R}^D_{C,\gamma}(u,v)$. For that, we first need the next lemma.

Lemma 2.16. *Let $v \in \mathbb{R}^n$, $u \in C$ and $0 < \gamma < 1/2$. Assume that C is a bounded set and take*

$$0 < \varepsilon < \frac{\gamma\|u - \mathcal{P}^D_C(v)\|^2_D}{1 - \gamma + \|v - \mathcal{P}^D_C(v)\|_D + 2\gamma\|u - \mathcal{P}^D_C(v)\|_D + \mathrm{diam}C}, \tag{2.8}$$

where diam$C$ *denotes the diameter of C. Then, $\{w \in C : \ \|w - \mathcal{P}^D_C(v)\|_D \leq \varepsilon\} \subset \mathcal{R}^D_{C,\gamma}(u,v)\}$.*

Proof. Take ε satisfying (2.8) and $w \in C$ such that $\|w - \mathcal{P}^D_C(v)\|_D \leq \varepsilon$. For all $z \in C$, we have

$$\begin{aligned}\langle D(v - w), z - w\rangle = {} & \langle D(v - \mathcal{P}^D_C(v)), z - \mathcal{P}^D_C(v)\rangle + \langle D(v - \mathcal{P}^D_C(v)), \mathcal{P}^D_C(v) - w\rangle \\ & + \langle D(\mathcal{P}^D_C(v) - w), z - \mathcal{P}^D_C(v)\rangle + \|\mathcal{P}^D_C(v) - w\|^2_D.\end{aligned}$$

Using Lemma 2.2, we have $\langle D(v - \mathcal{P}^D_C(v)), z - \mathcal{P}^D_C(v)\rangle \leq 0$. Thus, the last equality becomes

$$\langle D(v - w), z - w\rangle \leq \langle D(v - \mathcal{P}^D_C(v)), \mathcal{P}^D_C(v) - w\rangle + \langle D(\mathcal{P}^D_C(v) - w), z - \mathcal{P}^D_C(v)\rangle + \|\mathcal{P}^D_C(v) - w\|^2_D.$$

By using Cauchy-Schwarz inequality, we conclude from the last inequality that

$$\langle D(v - w), z - w\rangle \leq \|w - \mathcal{P}^D_C(v)\|_D \left(\|v - \mathcal{P}^D_C(v)\|_D + \|z - \mathcal{P}^D_C(v)\|_D\right) + \|w - \mathcal{P}^D_C(v)\|^2_D.$$

Since $\|w - \mathcal{P}^D_C(v)|_D \leq \varepsilon$ and $\|z - \mathcal{P}^D_C(v)\|_D \leq \mathrm{diam}C$, the last inequality implies that

$$\langle D(v - w), z - w\rangle \leq \varepsilon\left(\|v - \mathcal{P}^D_C(v)\|_D + \mathrm{diam}C\right) + \varepsilon^2, \tag{2.9}$$

On the other hand, if ε satisfies (2.8) then

$$\varepsilon\left(1 - \gamma + \|v - \mathcal{P}^D_C(v)\|_D + \mathrm{diam}C\right) + \gamma\varepsilon^2 < \gamma\|u - \mathcal{P}^D_C(v)\|^2_D - 2\gamma\varepsilon\|u - \mathcal{P}^D_C(v)\|_D + \gamma\varepsilon^2,$$

hence $\varepsilon\left(1 - \gamma + \|v - \mathcal{P}^D_C(v)\|_D + \mathrm{diam}C\right) + \gamma\varepsilon^2 < \gamma\left(\|u - \mathcal{P}^D_C(v)\|_D - \varepsilon\right)^2$. Since $\gamma, \varepsilon < 1$, we have $\varepsilon^2 < \varepsilon(1 - \gamma) + \gamma\varepsilon^2$ and we may conclude that

$$\varepsilon\left(\|v - \mathcal{P}^D_C(v)\|_D + \mathrm{diam}C\right) + \varepsilon^2 < \gamma\left(\|u - \mathcal{P}^D_C(v)\|_D - \varepsilon\right)^2.$$

It follows from (2.9) that

$$\langle D(v-w), z-w\rangle \leq \gamma \left(\|u-\mathcal{P}_C^D(v)\|_D - \varepsilon\right)^2. \tag{2.10}$$

Using again that $\|w-\mathcal{P}_C^D(v)|_D \leq \varepsilon$ and the triangular inequality, we have

$$0 < \|u-\mathcal{P}_C^D(v)\|_D - \varepsilon \leq \|u-\mathcal{P}_C^D(v)\|_D - \|w-\mathcal{P}_C^D(v)\|_D \leq \|u-w\|_D.$$

Hence, taking into account (2.10), we conclude that $\langle D(v-w), z-w\rangle \leq \gamma\|u-w\|_D^2$. Therefore, it follows from Definition 2.10 that $w \in \mathcal{R}_{C,\gamma}^D(u,v)$. □

Proposition 2.17. *Let $v \in \mathbb{R}^n$, $u \in C$ and assume that C is a bounded set. Then, for each $0<\gamma<1/2$, there exist $0<\zeta<1$ such that $\mathcal{P}_{C,\zeta}^D(u,v) \subseteq \mathcal{R}_{C,\gamma}^D(u,v)$.*

Proof. It follows from Lemma 2.16 that given $0<\gamma<1/2$ there exists $\varepsilon>0$ such that, for all $w \in C$ with $\|w-\mathcal{P}_C^D(v)\| \leq \varepsilon$, we have $w \in \mathcal{R}_\gamma^D(v)$. Otherwise, we may see in (2.3), when $\zeta \to 1$, the diameter of $C \cap \mathcal{P}_{C,\zeta}^D(u,v)$ tends to zero, then there exists ζ close to 1 such that $\operatorname{diam}(C \cap \mathcal{P}_{C,\zeta}^D(u,v)) < \varepsilon/2$, and $\mathcal{P}_{C,\zeta}^D(u,v) \subset \mathcal{R}_{C,\gamma}^D(u,v)$. □

Next, we present important properties of inexact projections, it will be useful in the sequel.

Lemma 2.18. *Let $x \in C$, $\alpha>0$ and $z(\alpha) = x-\alpha D^{-1}\nabla f(x)$. Take $w(\alpha) \in \mathcal{P}_{C,\zeta}^D(x,z(\alpha))$ with $\zeta \in (0,1]$. Then,*

(i) $\langle \nabla f(x), w(\alpha)-x\rangle \leq -\dfrac{1}{2\alpha}\|w(\alpha)-x\|_D^2 + \dfrac{\zeta}{2\alpha}\left[\|\mathcal{P}_C^D(z(\alpha))-z(\alpha)\|_D^2 - \|x-z(\alpha)\|_D^2\right]$;

(ii) the point x is stationary for problem (1) if and only if $x \in \mathcal{P}_{C,\zeta}^D(x,z(\alpha))$;

(iii) if $x \in C$ is a nonstationary point for problem (1), then $\left\langle \nabla f(x), w(\alpha)-x\right\rangle < 0$. *Equivalently, if there exists $\bar{\alpha}>0$ such that* $\left\langle \nabla f(x), w(\bar{\alpha})-x\right\rangle \geq 0$, *then x is stationary for problem (1).*

Proof. Since $w(\alpha) \in \mathcal{P}_{C,\zeta}^D(x,z(\alpha))$, applying Lemma 2.9 with $w=w(\alpha)$, $v=z(\alpha)$, $y=x$, and $u=x$, we conclude, after some algebraic manipulations, that

$$\langle D(z(\alpha)-w(\alpha)), x-w(\alpha)\rangle \leq \frac{1}{2}\|w(\alpha)-x\|_D^2 + \frac{\zeta}{2}\left[\|\mathcal{P}_C^D(z(\alpha))-z(\alpha)\|_D^2 - \|x-z(\alpha)\|_D^2\right].$$

Substituting $z(\alpha) = x-\alpha\nabla f(x)$ in the left hand side of the last inequality, some manipulations yield the inequality of item (i). For proving item (ii), we first assume that x is stationary for problem (1). In this case, (1.1) implies that $\langle \nabla f(x), w(\alpha)-x\rangle \geq 0$. Hence, due to $\|\mathcal{P}_C^D(z(\alpha)) - z(\alpha)\|_D \leq \|x-z(\alpha)\|_D$, item (i) implies

$$\frac{1}{2\alpha}\|w(\alpha)-x\|_D^2 \leq \frac{\zeta}{2\alpha}\left[\|\mathcal{P}_C^D(z(\alpha))-z(\alpha)\|_D^2 - \|x-z(\alpha)\|_D^2\right] \leq 0.$$

Since $\alpha>0$ and $\zeta \in (0,1]$, the last inequality yields $w(\alpha)=x$. Therefore, $x \in \mathcal{P}_{C,\zeta}^D(x,z(\alpha))$. Reciprocally, if $x \in \mathcal{P}_{C,\zeta}^D(x,z(\alpha))$, then Definition 2.5 implies that

$$\|x-z(\alpha)\|_D^2 \leq \zeta\|\mathcal{P}_C^D(z(\alpha))-z(\alpha)\|_D^2 + (1-\zeta)\|x-z(\alpha)\|_D^2.$$

Hence, $0 \leq \zeta \left(\|\mathcal{P}_C^D(z(\alpha)) - z(\alpha)\|_D^2 - (\|x - z(\alpha)\|_D^2 \right)$. Considering that $\zeta \in (0, 1]$ we have

$$\|x - z(\alpha)\|_D \leq \|\mathcal{P}_C^D(z(\alpha)) - z(\alpha)\|_D.$$

Thus, due to exact projection with respect to the norm $\|\cdot\|_D$ be unique and $z(\alpha) = x - D^{-1}\alpha\nabla f(x)$, we have $\mathcal{P}_C^D(x - \alpha D^{-1}\nabla f(x)) = x$. Hence, x is the solution of the constrained optimization problem $\min_{y \in C} \|y - z(\alpha)\|_D^2$, which taking into account that $\alpha > 0$ implies (1.1). Therefore, x is stationary point for problem (1). Finally, to prove item (iii), take x a nonstationary point for problem (1). Thus, by item (ii), $x \notin \mathcal{P}_{C,\zeta}^D(x, z(\alpha))$ and taking into account that $w(\alpha) \in \mathcal{P}_{C,\zeta}^D(x, z(\alpha))$, we conclude that $x \neq w(\alpha)$. Since $\|\mathcal{P}_C^D(z(\alpha)) - z(\alpha)\|_D \leq \|x - z(\alpha)\|_D$, $\alpha > 0$ and $\zeta \in (0, 1]$, it follows from item (i) that $\left\langle \nabla f(x), w(\alpha) - x \right\rangle < 0$ and the first sentence is proved. Finally, note that the second sentence is the contrapositive of the first sentence. □

Finally, it is worth mentioning that Definitions 2.5 and 2.10, introduced respectively in [14] and [29], are relative inexact concepts, while the concept introduced in [67, 71] is absolute.

2.4 Practical computation of inexact projections

In this section, for a given $v \in \mathbb{R}^n$ and $u \in C$, we discuss how to calculate a point $w \in C$ belonging to $\mathcal{P}_{C,\zeta}^D(u, v)$ or $\mathcal{R}_{C,\gamma}^D(u, v)$. We recall that Lemma 2.14 implies that $\mathcal{P}_{C,\zeta}^D(u, v)$ has more latitude than $\mathcal{R}_{C,\gamma}^D(u, v)$, i.e., $\mathcal{R}_{C,\gamma}^D(u, v) \subset \mathcal{P}_{C,\zeta}^D(u, v)$.

We begin our discussion by showing how a point $w \in \mathcal{P}_{C,\zeta}^D(u, v)$ may be calculated without knowing the point $\mathcal{P}_C^D(v)$. Considering that this discussion has already been covered in [14, Section 3, Algorithm 3.1], we will limit ourselves to giving a general idea of how this task is carried out; see also [17, Section 5.1]. The idea is to use an external procedure capable of computing two sequences $(c_\ell)_{\ell \in \mathbb{N}} \subset \mathbb{R}$ and $(w^\ell)_{\ell \in \mathbb{N}} \subset C$ satisfying the following conditions

$$c_\ell \leq \|\mathcal{P}_C^D(v) - v\|_D^2, \quad \forall \ell \in \mathbb{N}, \qquad \lim_{\ell \to +\infty} c_\ell = \|\mathcal{P}_C^D(v) - v\|_D^2, \qquad \lim_{\ell \to +\infty} w^\ell = \mathcal{P}_C^D(v). \tag{2.11}$$

In this case, if $v \notin C$, then we have $\|\mathcal{P}_C^D(v) - v\|_D^2 - \|u - v\|_D^2 < 0$. Hence, given an arbitrary $\zeta \in (0, 1)$, the second condition in (2.11) implies that there exists $\hat{\ell}$ such that

$$\|\mathcal{P}_C^D(v) - v\|_D^2 - \|u - v\|_D^2 < \zeta(c_{\hat{\ell}} - \|u - v\|_D^2).$$

Moreover, by using the last condition in (2.11), we conclude that there exists $\bar{\ell} > \hat{\ell}$ such that

$$\|w_{\bar{\ell}} - v\|_D^2 - \|u - v\|_D^2 < \zeta(c_{\bar{\ell}} - \|u - v\|_D^2), \tag{2.12}$$

which using the inequality in (2.11) yields

$$\|w_{\bar{\ell}} - v\|_D^2 < \zeta\|\mathcal{P}_C^D(v) - v\|_D^2 + (1 - \zeta)\|u - v\|_D^2.$$

Hence, Definition 2.5 implies that $w_{\bar{\ell}} \in \mathcal{P}_{C,\zeta}^D(u, v)$. Therefore, (2.12) may be used as a stopping criterion to compute a feasible inexact projection, with respect to the norm $\|\cdot\|_D$, of v onto C relative to u and forcing parameter $\zeta \in (0, 1]$. For instance, it follows from [14, Theorem 3.2,

Lemma 3.1] (see also [16]) that such sequences $(c_\ell)_{\ell\in\mathbb{N}} \subset \mathbb{R}$ and $(w^\ell)_{\ell\in\mathbb{N}} \subset C$ satisfying (2.11) may be computed by using *Dykstra's algorithm* [23, 33], whenever D is the identity matrix and the set $C = \cap_{i=1}^p C_i$, where C_i are closed and convex sets and the exact projection $\mathcal{P}_{C_i}^D(v)$ is easy to obtain, for all $i = 1, \ldots, p$.

We end this section by discussing how to compute a point $w \in \mathcal{R}_{C,\gamma}^D(u, v)$. For that, we apply the classical *Frank-Wolfe method*, also known as *conditional gradient method*, to minimize the function $\psi(z) := \|z - v\|^2/2$ onto the constraint set C with a suitable stop criteria depending of $u \in C$ and $\gamma \in (0, 1]$, see [9, 50]. To state the method we assume the existence of a linear optimization oracle (or simply LO oracle) capable of minimizing linear functions over the constraint set C, which is assumed to be compact. The Frank-Wolfe method is formally stated as follows.

Algorithm 2.1 Frank-Wolfe method to compute $w \in \mathcal{R}_{C,\gamma}^D(u, v)$

Input: $D \in \mathcal{D}_\mu$, $\gamma \in (0, 1]$, $v \in \mathbb{R}^n$ and $u \in C$.

Step 0. Let $w^0 \in C$ and set $\ell \leftarrow 0$.

Step 1. Use a LO oracle to compute an optimal solution z^ℓ and the optimal value s_ℓ^* as

$$z^\ell \in \arg\min_{z\in C} \langle w^\ell - v,\ z - w^\ell\rangle, \qquad s_\ell^* := \langle w^\ell - v,\ z^\ell - w^\ell\rangle. \tag{2.13}$$

If $-s_\ell^* \le \gamma\|w^\ell - u\|_D^2$, then define $w := w^\ell$ and **stop**.

Step 2. Compute α_ℓ and $w_{\ell+1}$ as

$$w_{\ell+1} := w^\ell + \alpha_\ell(z^\ell - w^\ell), \qquad \alpha_\ell := \min\left\{1, -s_\ell^*/\|z^\ell - w^\ell\|^2\right\}. \tag{2.14}$$

Set $\ell \leftarrow \ell + 1$, and go to Step 1.

Output: $w := w^\ell$.

Let us describe the main features of Algorithm 2.1, i.e., the Frank-Wolfe method applied to the problem $\min_{z\in C}\psi(z)$. In this case, (2.13) is equivalent to $s_\ell^* := \min_{z\in C}\langle\psi'(w^\ell),\ z - w^\ell\rangle$. Since ψ is convex, we have

$$\psi(z) \ge \psi(w^\ell) + \langle\psi'(w^\ell),\ z - w^\ell\rangle \ge \psi(w^\ell) + s_\ell^*,$$

for all $z \in C$. Define $w_* := \arg\min_{z\in C}\psi(z)$ and $\psi^* := \min_{z\in C}\psi(z)$. Letting $z = w_*$ in the last inequality, we obtain

$$\psi(w^\ell) \ge \psi^* \ge \psi(w^\ell) + s_\ell^*,$$

which implies that $s_\ell^* < 0$ whenever $\psi(w^\ell) \ne \psi^*$. Thus, we conclude that

$$-s_\ell^* = \langle v - w^\ell,\ z^\ell - w^\ell\rangle > 0 \ge \langle v - w_*,\ z - w_*\rangle,$$

for all $z \in C$. Therefore, if Algorithm 2.1 computes $w^\ell \in C$ satisfying $-s_\ell^* \le \gamma\|w^\ell - u\|_D^2$, then the method terminates. Otherwise, it computes the step size $\alpha_\ell = \arg\min_{\alpha\in[0,1]}\psi(w^\ell + \alpha(z^\ell - w^\ell))$ using exact minimization. Since z^ℓ, $w^\ell \in C$ and C is convex, we conclude from (2.14) that $w_{\ell+1} \in$

C, thus Algorithm 2.1 generates a sequence in C. Finally, (2.13) implies that $\langle v-w^\ell,\ z-w^\ell\rangle \leq -s_\ell^*$, for all $z \in C$. Considering that [9, Proposition A.2] implies that $\lim_{\ell\to+\infty} s_\ell^* = 0$ and taking into account the stopping criteria $-s_\ell^* \leq \gamma\|w^\ell - u\|_D^2$, we conclude that the output of Algorithm 2.1 is a feasible inexact projection $w \in \mathcal{R}_{C,\gamma}^D(u,v)$ i.e.,

$$\langle v - w,\ z - w\rangle \leq \gamma\|w^\ell - u\|_D^2,$$

for all $z \in C$.

Chapter 3

Inexact scaled gradient method

The aim of this chapter is to present an inexact version of SGP method, which inexactness are in two distinct senses. First, we use a version of the inexactness scheme introduced in [14], and also a variation of the one appeared in [71], to compute an inexact projection onto the feasible set allowing an appropriate relative error tolerance. Second, using the inexactness conceptual scheme for nonmonotones line search introduced in [45, 66], a step size is computed to define the next iterate. The statement of the conceptual algorithm is as follows.

Algorithm 3.1 SGP with inexact projection and nonmonotone line search

Step 0. Choose $\sigma, \zeta_{\min} \in (0,1)$, $\delta_{\min} \in [0,1)$, $0 < \underline{\omega} < \bar{\omega} < 1$, $0 < \alpha_{\min} \leq \alpha_{\max}$ and $\mu \geq 1$. Let $x^0 \in C$, $\nu_0 \geq 0$ and set $k \leftarrow 0$.

Step 1. Choose positive real numbers α_k and ζ_k, and a positive definite matrix D_k such that

$$\alpha_{\min} \leq \alpha_k \leq \alpha_{\max}, \qquad 0 < \zeta_{\min} < \zeta_k \leq 1, \qquad D_k \in \mathcal{D}_\mu. \tag{3.1}$$

Compute $w^k \in C$ as any feasible inexact projection with respect to the norm $\|\cdot\|_{D_k}$ of $z^k := x^k - \alpha_k D_k^{-1} \nabla f(x^k)$ onto C relative to x^k with forcing parameter ζ_k, i.e.,

$$w^k \in \mathcal{P}_{C,\zeta_k}^{D_k}(x^k, z^k). \tag{3.2}$$

If $w^k = x^k$, then **stop** declaring convergence.

Step 2. Set $\tau_{\text{trial}} \leftarrow 1$. If

$$f\big(x^k + \tau_{\text{trial}}(w^k - x^k)\big) \leq f(x^k) + \sigma \tau_{\text{trial}} \big\langle \nabla f(x^k), w^k - x^k \big\rangle + \nu_k, \tag{3.3}$$

then $\tau_k \leftarrow \tau_{\text{trial}}$, define the next iterate x^{k+1} as

$$x^{k+1} = x^k + \tau_k (w^k - x^k), \tag{3.4}$$

and go to **Step 3**. Otherwise, choose $\tau_{\text{new}} \in [\underline{\omega}\tau_{\text{trial}}, \bar{\omega}\tau_{\text{trial}}]$, set $\tau_{\text{trial}} \leftarrow \tau_{\text{new}}$, and repeat test (3.3).

Step 3. Take $\delta_{k+1} \in [\delta_{\min}, 1]$ and choose $\nu_{k+1} \in \mathbb{R}$ satisfying

$$0 \leq \nu_{k+1} \leq (1 - \delta_{k+1})\big[f(x^k) + \nu_k - f(x^{k+1})\big]. \tag{3.5}$$

Set $k \leftarrow k+1$ and go to **Step 1**.

Let us describe the main features of Algorithm 3.1. In Step 1, we first choose $\alpha_{\min} \leq \alpha_k \leq \alpha_{\max}$, $0 < \zeta_{\min} \leq \zeta_k < 1$, and $D_k \in \mathcal{D}_\mu$. Then, by using some (inner) procedure, such as those specified

in Chapter 2, we compute w^k as any feasible inexact projection of $z^k = x_k - \alpha_k D_k^{-1}\nabla f(x_k)$ onto the feasible set C relative to the previous iterate x^k with forcing parameter ζ_k. If $w^k = x^k$, then Lemma 2.18*(ii)* implies that x^k is a solution of problem (1). Otherwise, $w^k \neq x^k$ and Lemma 2.18*(i)* implies that $w^k - x^k$ is a descent direction of f at x^k, i.e., $\langle \nabla f(x^k), w^k - x^k\rangle < 0$. Hence, in Step 2, we employ a nonmonotone line search with tolerance parameter $\nu_k \geq 0$ to compute a step size $\tau_k \in (0, 1]$, and the next iterate is computed as in (3.4). Finally, due to (3.3) and $\delta_{k+1} \in [\delta_{\min}, 1]$, we have

$$0 \leq (1 - \delta_{k+1})\Big[f(x^k) + \nu_k - f(x^{k+1})\Big].$$

Therefore, the next tolerance parameter $\nu_{k+1} \in \mathbb{R}$ may be chosen satisfying (3.5) in Step 3, completing the iteration.

It is worth mentioning that the conditions in (3.1) allow combining several strategies for choosing the step sizes α_k and the matrices D_k to accelerate the performance of the classical gradient method. Strategies of choosing the step sizes α_k and the matrices D_k have their origin in the study of the gradient method for unconstrained optimization, papers dealing with this issue include but are not limited to [7, 28, 30, 38, 76], see also [19, 26, 27, 52]. More details about selecting step sizes α_k and matrices D_k may be found in the recent review [18] and references therein.

Below, we present some particular instances of the parameter $\delta_k \geq 0$ and the non-monotonicity tolerance parameter $\nu_k \geq 0$ in Step 3.

1. *Armijo line search*

   Taking $\nu_k \equiv 0$, the line search (3.3) is the well-known (monotone) Armijo line search, see [12, Section 2.3]. In this case, we may take $\delta_k \equiv 1$ in Step 3.

2. *Max-type line search*

   The earliest nonmonotone line search strategy was proposed in [47]. Let $M > 0$ be an integer parameter. In an iteration k, this strategy requires a step size $\tau_k > 0$ satisfying

   $$f\Big(x^k + \tau_k(w^k - x^k)\Big) \leq \max_{0\leq j\leq m_k} f(x^{k-j}) + \sigma\tau_k\Big\langle \nabla f(x^k), w^k - x^k\Big\rangle, \tag{3.6}$$

   where $m_0 = 0$ and $0 \leq m_k \leq \min\{m_{k-1} + 1, M\}$. To simplify the notations, we define

   $$f(x^{\ell(k)}) := \max_{0\leq j\leq m_k} f(x^{k-j}).$$

   In order to identify (3.6) as a particular instance of (3.3), we set

   $$\nu_k = f(x^{\ell(k)}) - f(x^k), \quad 0 = \delta_{\min} \leq \delta_{k+1} \leq [f(x^{\ell(k)}) - f(x^{\ell(k+1)})]/[f(x^{\ell(k)}) - f(x^{k+1})]. \tag{3.7}$$

   Parameters ν_k and δ_{k+1} in (3.7) satisfy the corresponding conditions in Algorithm 3.1, i.e., $\nu_k \geq 0$ and $\delta_{k+1} \in [\delta_{\min}, 1]$ (with $\delta_{\min} = 0$) satisfy (3.5). In fact, the definition of $f(x^{\ell(k)})$ implies that $f(x^k) \leq f(x^{\ell(k)})$ and hence $\nu_k \geq 0$. Due to $\langle \nabla f(x^k), w^k - x^k\rangle < 0$, it follows from (3.3) that $f(x^{\ell(k)}) - f(x^{k+1}) > 0$. Since $m_{k+1} \leq m_k + 1$, we conclude that

   $$f(x^{\ell(k)}) - f(x^{\ell(k+1)}) \geq 0.$$

Hence, since $f(x^{k+1}) \leq f(x^{\ell(k+1)})$, we obtain $\delta_{k+1} \in [0,1]$. Moreover, (3.5) is equivalent to

$$\delta_{k+1}[f(x^k) + \nu_k - f(x^{k+1})] \leq (f(x^k) + \nu_k) - (f(x^{k+1}) + \nu_{k+1}),$$

which in turn, taking into account that $\nu_k = f(x^{\ell(k)}) - f(x^k)$, is equivalent to second inequality in (3.7). Thus, (3.6) is a particular instance of (3.3) with ν_k and δ_{k+1} defined in (3.7). Therefore, Algorithm 3.1 has as a particular instance the inexact projected version of the scaled gradient method employing the nonmonotone line search (3.6). This version has been considered in [14]; see also [20, 72].

3. *Average-type line search*

Let us first recall the definition of the sequence of "cost updates' $(c_k)_{k\in\mathbb{N}}$ that characterizes the nonmonotone line search proposed in [75]. Let $0 \leq \eta_{\min} \leq \eta_{\max} < 1$, $c_0 = f(x_0)$ and $q_0 = 1$. Choose $\eta_k \in [\eta_{\min}, \eta_{\max}]$ and set

$$q_{k+1} = \eta_k q_k + 1, \qquad c_{k+1} = [\eta_k q_k c_k + f(x^{k+1})]/q_{k+1}, \qquad \forall k \in \mathbb{N}. \tag{3.8}$$

Some algebraic manipulations show that the sequence defined in (3.8) is equivalent to

$$c_{k+1} = (1 - 1/q_{k+1})c_k + f(x^{k+1})/q_{k+1}, \qquad \forall k \in \mathbb{N}. \tag{3.9}$$

Since (3.5) is equivalent to

$$f(x^{k+1}) + \nu_{k+1} \leq (1 - \delta_{k+1})(f(x^k) + \nu_k) + \delta_{k+1} f(x^{k+1}),$$

it follows from (3.9) that letting $\nu_k = c_k - f(x^k)$ and $\delta_{k+1} = 1/q_{k+1}$, Algorithm 3.1 becomes the inexact projected version of the scaled gradient method employing the nonmonotone line search proposed in [75]. Finally, considering that $q_0 = 1$ and $\eta_{\max} < 1$, the first equality in (3.8) implies that

$$q_{k+1} = 1 + \sum_{j=0}^{k} \prod_{i=0}^{j} \eta_{k-i} \leq \sum_{j=0}^{+\infty} \eta_{\max}^j = 1/(1 - \eta_{\max}).$$

In this case, due to $\delta_{k+1} = 1/q_{k+1}$, we may take $\delta_{\min} = 1 - \eta_{\max} > 0$ in Step 3. For gradient projection methods employing the nonmonotone Average-type line search see, for example, [6, 35, 73].

Remark 3.1. The general line search in Step 2 of Algorithm 3.1 with parameters δ_{k+1} and ν_k properly chosen in Step 3, also contains as particular cases the nonmonotone line searches that appeared in [3, 55], see also [45].

3.1 Partial asymptotic convergence analysis

The goal of this section is to present a partial convergence result for the sequence $(x^k)_{k\in\mathbb{N}}$ generated by Algorithm 3.1, namely, we will prove that every cluster point of $(x^k)_{k\in\mathbb{N}}$ is stationary for problem (1). For that, we state a result that is contained in the proof of [45, Theorem 4].

Lemma 3.2. *For all $k \in \mathbb{N}$,*

$$0 \leq \delta_{k+1}\Big[f(x^k) + \nu_k - f(x^{k+1})\Big] \leq \Big(f(x^k) + \nu_k\Big) - \Big(f(x^{k+1}) + \nu_{k+1}\Big).$$

As consequence the sequence $\Big(f(x^k) + \nu_k\Big)_{k\in\mathbb{N}}$ is non-increasing.

Next, we present our first convergence result. It is worth noting that, just as in the classical projected gradient method, we do not need to assume that f has a bounded sub-level set.

Proposition 3.3. *Assume that $\lim_{k\to+\infty} \nu_k = 0$. Then, Algorithm 3.1 stops in a finite number of iterations at a stationary point of problem (1), or generates an infinite sequence $(x^k)_{k\in\mathbb{N}}$ for which every cluster point is stationary for problem (1).*

Proof. First, assume that $(x^k)_{k\in\mathbb{N}}$ is finite. In this case, according to Step 1, there exists $k \in \mathbb{N}$ such that $x^k = w^k \in \mathcal{P}^{D_k}_{C,\zeta_k}(x^k, z^k)$, where $z^k = x^k - \alpha_k D_k^{-1}\nabla f(x^k)$, $0 < \bar{\zeta} < \zeta_k \leq 1$ and $\alpha_k > 0$. Therefore, applying Lemma 2.18*(ii)* with $x = x^k$, $\alpha = \alpha_k$ and $\zeta = \zeta_k$, we conclude that x^k is stationary for problem (1). Now, assume that $(x^k)_{k\in\mathbb{N}}$ is infinite. Let $\bar{x}$ be a cluster point of $(x^k)_{k\in\mathbb{N}}$ and $(x^{k_j})_{j\in\mathbb{N}}$ be a subsequence of $(x^k)_{k\in\mathbb{N}}$ such that $\lim_{j\to+\infty} x^{k_j} = \bar{x}$. Since C is closed and $(x^k)_{k\in\mathbb{N}} \subset C$, we have $\bar{x} \in C$. Moreover, since $\lim_{k\to+\infty} \nu_k = 0$, we have

$$\lim_{j\to+\infty} \Big(f(x^{k_j}) + \nu_{k_j}\Big) = f(\bar{x}).$$

Hence, considering that $\lim_{k\to+\infty} \nu_k = 0$ and Lemma 3.2 implies that $\Big(f(x^k) + \nu_k\Big)_{k\in\mathbb{N}}$ is non-increasing, we conclude that

$$\lim_{k\to+\infty} f(x^k) = \lim_{k\to+\infty} \Big(f(x^k) + \nu_k\Big) = f(\bar{x}).$$

On the other hand, due to $w^k \in \mathcal{P}^{D_k}_{C,\zeta_k}(x^k, z^k)$, where $z^k = x^k - \alpha_k \nabla f(x^k)$, Definition 2.5 implies

$$\|w^{k_j} - z^{k_j}\|^2_{D_k} \leq \zeta_{k_j}\|\mathcal{P}^{D_k}_C(z^{k_j}) - z^{k_j}\|^2_{D_k} + (1 - \zeta_{k_j})\|x^{k_j} - z^{k_j}\|^2_{D_k}. \tag{3.10}$$

Considering that $(\alpha_k)_{k\in\mathbb{N}}$ and $(\zeta_k)_{k\in\mathbb{N}}$ are bounded, $(D_k)_{k\in\mathbb{N}} \subset \mathcal{D}_\mu$, $(x^{k_j})_{j\in\mathbb{N}}$ converges to $\bar{x}$ and ∇f is continuous, the last inequality together Remark 2.4 and (1.3) imply that $(w^{k_j})_{j\in\mathbb{N}} \subset C$ is also bounded. Thus, we assume without loss of generality that $\lim_{j\to+\infty} w^{k_j} = \bar{w} \in C$. In addition, taking into account that $x^k \neq w^k$ for all $k = 0, 1, \ldots$, applying Lemma 2.18*(i)* with $x = x^k$, $\alpha = \alpha_k$, $z(\alpha) = z^k$ and $\zeta = \zeta_k$, we obtain that $\langle \nabla f(x^k), w^k - x^k\rangle < 0$, for all $k = 0, 1, \ldots$. Therefore, (3.3) and (3.4) imply that

$$0 < -\sigma\tau_k\Big\langle \nabla f(x^k), w^k - x^k\Big\rangle \leq f(x^k) + \nu_k - f(x^{k+1}), \qquad \forall\ k \in \mathbb{N}. \tag{3.11}$$

Now, due $\tau_k \in (0, 1]$, for all $k = 0, 1, \ldots$, we also assume without loss of generality that $\lim_{j\to+\infty} \tau_{k_j} = \bar{\tau} \in [0, 1]$. Therefore, since $\lim_{k\to+\infty} f(x^k) = f(\bar{x})$ and $\lim_{k\to+\infty} \nu_k = 0$, taking limit in (3.11) along the subsequences $(x^{k_j})_{j\in\mathbb{N}}$, $(w^{k_j})_{j\in\mathbb{N}}$ and $(\tau_{k_j})_{j\in\mathbb{N}}$ yields $\bar{\tau}\Big\langle \nabla f(\bar{x}), \bar{w} - \bar{x}\Big\rangle = 0$. We have two possibilities: $\bar{\tau} > 0$ or $\bar{\tau} = 0$. If $\bar{\tau} > 0$, then $\Big\langle \nabla f(\bar{x}), \bar{w} - \bar{x}\Big\rangle = 0$. Now, we assume that $\bar{\tau} = 0$. In this case, for all j large enough, there exists $0 < \hat{\tau}_{k_j} \leq \min\{1, \tau_{k_j}/\underline{\omega}\}$ such that

$$f\Big(x^{k_j} + \hat{\tau}_{k_j}(w^{k_j} - x^{k_j})\Big) > f(x^{k_j}) + \sigma\hat{\tau}_{k_j}\Big\langle \nabla f(x^{k_j}), w^{k_j} - x^{k_j}\Big\rangle + \nu_{k_j}. \tag{3.12}$$

On the other hand, by the mean value theorem, there exists $\xi_{k_j} \in (0,1)$ such that

$$\langle \nabla f\Big(x^{k_j} + \xi_{k_j}\hat{\tau}_{k_j}(w^{k_j} - x^{k_j})\Big), \hat{\tau}_{k_j}(w^{k_j} - x^{k_j})\rangle = f\Big(x^{k_j} + \hat{\tau}_{k_j}(w^{k_j} - x^{k_j})\Big) - f(x^{k_j}).$$

Combining this equality with (3.12), and taking into account that $\nu_{k_j} \geq 0$, we have

$$\langle \nabla f\Big(x^{k_j} + \xi_{k_j}\hat{\tau}_{k_j}(w^{k_j} - x^{k_j})\Big), \hat{\tau}_{k_j}(w^{k_j} - x^{k_j})\rangle > \sigma\hat{\tau}_{k_j}\Big\langle \nabla f(x^{k_j}), w^{k_j} - x^{k_j}\Big\rangle,$$

for j large enough. Since $0 < \hat{\tau}_{k_j} \leq \min\{1, \tau_{k_j}/\underline{\omega}\}$, it follows that $\lim_{j\to\infty} \hat{\tau}_{k_j}\|w^{k_j} - x^{k_j}\| = 0$. Then, dividing both sides of the above inequality by $\hat{\tau}_{k_j} > 0$ and taking limits as j goes to $+\infty$, we conclude that

$$\langle \nabla f(\bar{x}), \bar{w} - \bar{x}\rangle \geq \sigma\langle \nabla f(\bar{x}), \bar{w} - \bar{x}\rangle.$$

Hence, due to $\sigma \in (0,1)$, we obtain $\langle \nabla f(\bar{x}), \bar{w} - \bar{x}\rangle \geq 0$. We recall that $\langle \nabla f(x^{k_j}), w^{k_j} - x^{k_j}\rangle < 0$, for all $j = 0, 1, \ldots$, which taking limit as j goes to $+\infty$ yields $\langle \nabla f(\bar{x}), \bar{w} - \bar{x}\rangle \leq 0$. Hence, we also have $\langle \nabla f(\bar{x}), \bar{w} - \bar{x}\rangle = 0$. Therefore, for any of the two possibilities, $\bar{\tau} > 0$ or $\bar{\tau} = 0$, we have $\langle \nabla f(\bar{x}), \bar{w} - \bar{x}\rangle = 0$. On the other hand, since $(\alpha_k)_{k\in\mathbb{N}}$ and $(\zeta_k)_{k\in\mathbb{N}}$ are bounded, we also assume without loss of generality that $\lim_{j\to+\infty} \alpha_{k_j} = \bar{\alpha} \in [\alpha_{\min}, \alpha_{\max}]$ and $\lim_{j\to+\infty} \zeta_{k_j} = \bar{\zeta} \in [\zeta_{\min}, 1]$. Thus, since Remark 2.4 implies that

$$\lim_{j\to+\infty} \mathcal{P}_C^{D_{k_j}}(z^{k_j}) = \mathcal{P}_C^{\bar{D}}(\bar{z}),$$

and considering that $\lim_{j\to+\infty} x^{k_j} = \bar{x} \in C$, $\lim_{j\to+\infty} w^{k_j} = \bar{w} \in C$, $\lim_{j\to+\infty} \tau_{k_j} = \bar{\tau} \in [0,1]$, $\lim_{j\to+\infty} D_{k_j} = \bar{D} \in \mathcal{D}_\mu$, taking limit in (3.10), we conclude that

$$\|\bar{w} - \bar{z}\|_{\bar{D}}^2 \leq \bar{\zeta}\|\mathcal{P}_C^{\bar{D}}(\bar{z}) - \bar{z}\|_{\bar{D}}^2 + (1 - \bar{\zeta})\|\bar{x} - \bar{z}\|_{\bar{D}}^2,$$

where $\bar{z} = \bar{x} - \bar{\alpha}\nabla f(\bar{x})$. Hence, Definition 2.5 implies that $\bar{w} \in \mathcal{P}_{C,\bar{\zeta}}^{\bar{D}}(\bar{x}, \bar{z})$, where $\bar{z} = \bar{x} - \bar{\alpha}\nabla f(\bar{x})$. Therefore, due to $\langle \nabla f(\bar{x}), \bar{w} - \bar{x}\rangle = 0$, we may apply second sentence in Lemma 2.18*(iii)* with $x = \bar{x}$, $z(\bar{\alpha}) = \bar{z}$ and $w(\bar{\alpha}) = \bar{w}$, to conclude that $\bar{x}$ is stationary for problem (1). □

The tolerance parameter ν_k that controls the non-monotonicity of the line search must be smaller and smaller as the sequence $(x^k)_{k\in\mathbb{N}}$ tends to a stationary point. Next corollary presents a general condition for this property, its proof may be found in [45, Theorem 4].

Corollary 3.4. *If $\delta_{\min} > 0$, then $\sum_{k=0}^{+\infty} \nu_k < +\infty$. Consequently, $\lim_{k\to+\infty} \nu_k = 0$.*

The Armijo and the nonmonotone Average-type line searches discussed in Chapter 3 satisfy the assumption of Corollary 3.4, i.e., $\delta_{\min} > 0$. However, for the nonmonotone Max-type line search, we may only guarantee that $\delta_{\min} \geq 0$. Hence, we may not apply Corollary 3.4 to conclude that $\lim_{k\to+\infty} \nu_k = 0$. In the next proposition, we will deal with this case separately.

Proposition 3.5. *Assume that the sequence $(x^k)_{k\in\mathbb{N}}$ is generated by Algorithm 3.1 with the nonmonotone line search (3.6), i.e., $\nu_k = f(x^{\ell(k)}) - f(x^k)$ for all $k \in \mathbb{N}$. In addition, assume that the level set $C_0 := \{x \in C : \ f(x) \leq f(x^0)\}$ is bounded and $\nu_0 = 0$. Then, $\lim_{k\to+\infty} \nu_k = 0$.*

Proof. First of all, note that $w^k \in \mathcal{P}^{D_k}_{C,\zeta_k}(x^k, z^k)$, where $z^k = x^k - \alpha_k D_k^{-1}\nabla f(x^k)$ and $D_k \in \mathcal{D}_\mu$. Thus, applying Lemma 2.18*(i)* with $x = x^k$, $w(\alpha) = w^k$, $z = z^k$ and $\zeta = \zeta_k$, we obtain

$$\|w^k - x^k\|^2 \leq -2\mu\alpha_{\max}\langle \nabla f(x^k), w^k - x^k\rangle, \qquad \forall k \in \mathbb{N}. \tag{3.13}$$

On the other hand, due to $f(x^{\ell(k)}) = f(x^k) + \nu_k$, Lemma 3.2 implies that $(f(x^{\ell(k)}))_{k\in\mathbb{N}}$ is non-increasing and

$$f(x^{k+1}) \leq f(x^{k+1}) + \nu_{k+1} \leq f(x^k) + \nu_k \leq f(x^0).$$

Hence, we have $(x^k)_{k\in\mathbb{N}} \subset C_0$ and, as a consequence, $(f(x^{\ell(k)}))_{k\in\mathbb{N}}$ converges. Note that $\ell(k)$ is an integer such that

$$k - m_k \leq \ell(k) \leq k. \tag{3.14}$$

Since $x^{\ell(k)} = x^{\ell(k)-1} + \tau_{\ell(k)-1}(w^{\ell(k)-1} - x^{\ell(k)-1})$, (3.6) implies that

$$f\Big(x^{\ell(k)}\Big) \leq f\Big(x^{\ell(\ell(k)-1)}\Big) + \sigma\tau_{\ell(k)-1}\Big\langle \nabla f(x^{\ell(k)-1}), w^{\ell(k)-1} - x^{\ell(k)-1}\Big\rangle,$$

for all $k > M$. In view of $(f(x^{\ell(k)}))_{k\in\mathbb{N}}$ be convergent, $\langle \nabla f(x^k), w^k - x^k\rangle < 0$ for all $k \in \mathbb{N}$, and taking into account that $\tau_k \in (0, 1]$, the last inequality together (3.13) implies that

$$\lim_{k\to+\infty} \tau_{\ell(k)-1}\|w^{\ell(k)-1} - x^{\ell(k)-1}\| = 0. \tag{3.15}$$

We proceed to prove that $\lim_{k\to+\infty} f(x^k) = \lim_{k\to+\infty} f(x^{\ell(k)})$. For that, set $\hat{\ell}(k) := \ell(k + M + 2)$. First, we prove by induction that, for all $j \geq 1$, the following two equalities hold

$$\lim_{k\to+\infty} \tau_{\hat{\ell}(k)-j}\|w^{\hat{\ell}(k)-j} - x^{\hat{\ell}(k)-j}\| = 0, \qquad \lim_{k\to+\infty} f(x^{\hat{\ell}(k)-j}) = \lim_{k\to+\infty} f(x^{\ell(k)}), \tag{3.16}$$

where we are considering $k \geq j - 1$. Assume that $j = 1$. Since $\{\hat{\ell}(k) : \ k \in \mathbb{N}\} \subset \{\ell(k) : \ k \in \mathbb{N}\}$, the first equality in (3.16) follows from (3.15). Hence, $\lim_{k\to+\infty} \|x^{\hat{\ell}(k)} - x^{\hat{\ell}(k)-1}\| = 0$. Since C_0 is compact and f is uniformly continuous on C_0, we have $\lim_{k\to+\infty} f(x^{\hat{\ell}(k)-1}) = \lim_{k\to+\infty} f(x^{\hat{\ell}(k)})$, which again using that $\{\hat{\ell}(k) : \ k \in \mathbb{N}\} \subset \{\ell(k) : \ k \in \mathbb{N}\}$ implies the second equality in (3.16). Assume that (3.16) holds for j. Again, due to

$$x^{\hat{\ell}(k)-j} = x^{\hat{\ell}(k)-j-1} + \tau_{\hat{\ell}(k)-j-1}(w^{\hat{\ell}(k)-j-1} - x^{\hat{\ell}(k)-j-1}),$$

it follows from (3.6) that

$$f\Big(x^{\hat{\ell}(k)-j}\Big) \leq f\Big(x^{\ell(\hat{\ell}(k)j-(j+1))}\Big) + \sigma\tau_{\hat{\ell}(k)-(j+1)}\Big\langle \nabla f(x^{\hat{\ell}(k)-(j+1)}), w^{\hat{\ell}(k)-(j+1)} - x^{\hat{\ell}(k)-(j+1)}\Big\rangle.$$

Similar argument used to obtain (3.15) yields $\lim_{k\to+\infty} \tau_{\hat{\ell}(k)-(j+1)}\|w^{\hat{\ell}(k)-(j+1)} - x^{\hat{\ell}(k)-(j+1)}\| = 0$. Thus, the first equality in (3.16) holds for $j+1$, which implies $\lim_{k\to+\infty} \|x^{\hat{\ell}(k)-j} - x^{\hat{\ell}(k)-(1+j)}\| = 0$. Again, the uniformly continuity of f on C_0 gives

$$\lim_{k\to+\infty} f(x^{\hat{\ell}(k)-(j+1)}) = \lim_{k\to+\infty} f(x^{\hat{\ell}(k)-j}),$$

which shows that the second equality in (3.16) holds for $j+1$. From (3.14) and $\hat{\ell}(k) := \ell(k+M+2)$, we obtain $\hat{\ell}(k) - k - 1 \leq M + 1$. Thus, taking into account that

$$x^{k+1} = x^{\hat{\ell}(k)} - \sum_{j=1}^{\hat{\ell}(k)-k-1} \tau_{\hat{\ell}(k)-j}\Big(w^{\hat{\ell}(k)-j} - x^{\hat{\ell}(k)-j}\Big),$$

it follows from the first inequality in (3.16) that $\lim_{k\to+\infty} \|x^{k+1} - x^{\hat{\ell}(k)}\| = 0$. Hence, due to f be uniformly continuous on C_0 and $(f(x^{\ell(k)}))_{k\in\mathbb{N}}$ be convergent, we conclude that

$$\lim_{k\to+\infty} f(x^k) = \lim_{k\to+\infty} f(x^{\hat{\ell}(k)}) = \lim_{k\to+\infty} f(x^{\ell(k)}),$$

and considering that $\nu_k = f(x^{\ell(k)}) - f(x^k)$ the desired results follows. □

Remark 3.6. Let $C_0 := \{x \in C : \ f(x) \leq f(x^0)\}$ be bounded and $(x^k)_{k\in\mathbb{N}}$ be generated by Algorithm 3.1 with the nonmonotone line search (3.6) with $\nu_0 = 0$. Then, combining Propositions 3.3 and 3.5, we conclude that $(x^k)_{k\in\mathbb{N}}$ is either finite terminating at a stationary point of problem (1), or infinite, and every cluster point of $(x^k)_{k\in\mathbb{N}}$ is stationary for problem (1). Therefore, we have an alternative proof for the result obtained in [14, Theorem 2.1].

Due to Proposition 3.3, *from now on we assume that the sequence* $(x^k)_{k\in\mathbb{N}}$ *generated by Algorithm 3.1 is infinite.*

3.2 Full asymptotic convergence analysis

The purpose of this section is to prove, under suitable assumptions, the full convergence of the sequence $(x^k)_{k\in\mathbb{N}}$. For this end, we need to be more restrictive with respect to the inexact projection in (3.2) and in the tolerance parameter that controls the non-monotonicity of the line search used in (3.3). More precisely, we assume that in Step 1 of Algorithm 3.1:

A1. For all $k \in \mathbb{N}$, we take $w^k \in \mathcal{R}^{D_k}_{C,\gamma_k}(x^k, z^k)$ with $\gamma_k = (1 - \zeta_k)/2$.

It is worth recalling that, taking the parameter $\gamma_k = (1 - \zeta_k)/2$, it follows from Lemma 2.14 that $\mathcal{R}^{D_k}_{C,\gamma_k}(x^k, z^k) \subset \mathcal{P}^{D_k}_{C,\zeta_k}(x^k, z^k)$. In addition, we also assume that in Step 2 of Algorithm 3.1:

A2. For all $k \in \mathbb{N}$, we take $0 \leq \nu_k$ such that $\sum_{k=0}^{+\infty} \nu_k < +\infty$.

It follows from Corollary 3.4 that the Armijo and the nonmonotone Average-type line searches discussed in Chapter 3 satisfy Assumption **A2**.

To prove the full convergence of the sequence $(x^k)_{k\in\mathbb{N}}$ satisfying **A1** and **A2** we consider an additional assumption on the sequence $(D_k)_{k\in\mathbb{N}} \subset \mathcal{D}_\mu$ as follows.

A3. For all $k \in \mathbb{N}$, $(1 + \eta_k)D_k - D_{k+1}$ is a positive semidefinite matrix, for some sequence $(\eta_k)_{k\in\mathbb{N}} \subset [0, +\infty)$ such that $\sum_{k\in\mathbb{N}} \eta_k < \infty$.

It is worth mentioning that Assumption **A3** has appeared in the study of the scaled gradient projection method, see, for example, [18]. Note that $D_k = I$ for all $k \in \mathbb{N}$, trivially satisfies **A3**.

We will begin establishing a basic inequality for $(x^k)_{k\in\mathbb{N}}$. To simplify notations, we define the constant

$$\xi := \frac{2\alpha_{\max}}{\sigma} > 0. \tag{3.17}$$

Lemma 3.7. *For each $x \in C$ and for all $k \in \mathbb{N}$, we have*

$$\|x^{k+1}-x\|^2_{D_{k+1}} \leq (1+\eta_k)\Big(\|x^k-x\|^2_{D_k}+2\alpha_k\tau_k\big\langle\nabla f(x^k),x-x^k\big\rangle+\xi\big[f(x^k)-f(x^{k+1})+\nu_k\big]\Big). \tag{3.18}$$

Proof. We know that

$$\|x^{k+1}-x\|^2_{D_k} = \|x^k-x\|^2_{D_k}+\|x^{k+1}-x^k\|^2_{D_k}-2\langle D_k(x^{k+1}-x^k),x-x^k\rangle,$$

for all $x \in C$ and $k \in \mathbb{N}$. Thus, using (3.4), we have

$$\|x^{k+1}-x\|^2_{D_k} = \|x^k-x\|^2_{D_k}+\tau_k^2\|w^k-x^k\|^2_{D_k}-2\tau_k\big\langle D_k(w^k-x^k),x-x^k\big\rangle, \qquad \forall\ k\in\mathbb{N}. \tag{3.19}$$

On the other hand, since $w^k \in \mathcal{R}^{D_k}_{C,\gamma_k}(x^k,z^k)$ with $z^k = x^k-\alpha_k D_k^{-1}\nabla f(x^k)$, it follows from Definition 2.10, with $y=x$, $D=D_k$, $u=x^k$, $v=z^k$, $w=w^k$, and $\gamma=\gamma_k$, that

$$\big\langle D_k(x^k-\alpha_k D_k^{-1}\nabla f(x^k)-w^k),x-w^k\big\rangle \leq \gamma_k\|w^k-x^k\|^2_{D_k}, \qquad \forall\ k\in\mathbb{N}.$$

Hence, after some algebraic manipulations in the last inequality, we have

$$-\big\langle D_k(w^k-x^k),x-x^k\big\rangle \leq \alpha_k\big\langle\nabla f(x^k),x-w^k\big\rangle-(1-\gamma_k)\|w^k-x^k\|^2_{D_k}.$$

Combining the last inequality with (3.19), we conclude that

$$\|x^{k+1}-x\|^2_{D_k} \leq \|x^k-x\|^2_{D_k}-\tau_k\big[2(1-\gamma_k)-\tau_k\big]\|w^k-x^k\|^2_{D_k}+2\tau_k\alpha_k\big\langle\nabla f(x^k),x-w^k\big\rangle. \tag{3.20}$$

Since $0 \leq \gamma_k < (1-\zeta_{\min})/2 < 1/2$ and $\tau_k \in (0,1]$, we have $2(1-\gamma_k)-\tau_k > \zeta_{\min} > 0$. Thus, it follows from (3.20) that

$$\|x^{k+1}-x\|^2_{D_k} \leq \|x^k-x\|^2_{D_k}+2\tau_k\alpha_k\big\langle\nabla f(x^k),x-w^k\big\rangle, \qquad \forall\ k\in\mathbb{N}.$$

Thus, considering that $\big\langle\nabla f(x^k),x-w^k\big\rangle = \big\langle\nabla f(x^k),x-x^k\big\rangle+\big\langle\nabla f(x^k),x^k-w^k\big\rangle$ and taking into account (3.3), we conclude that

$$\|x^{k+1}-x\|^2_{D_k} \leq \|x^k-x\|^2_{D_k}+2\tau_k\alpha_k\big\langle\nabla f(x^k),x-x^k\big\rangle+\frac{2\alpha_k}{\sigma}\big[f(x^k)-f(x^{k+1})+\nu_k\big], \tag{3.21}$$

for all $k \in \mathbb{N}$. On the other hand, applying Lemma 2.18*(iii)* with $x=x^k$, $\alpha=\alpha_k$, $D=D_k$, $w(\alpha)=w^k$, $z=z^k$ and $\zeta=\zeta_k$, we obtain $\langle\nabla f(x^k),w^k-x^k\rangle<0$, for all $k\in\mathbb{N}$. Therefore, it follows from (3.3) and (3.4) that

$$0 < -\sigma\tau_k\big\langle\nabla f(x^k),w^k-x^k\big\rangle \leq f(x^k)-f(x^{k+1})+\nu_k,$$

for all $k \in \mathbb{N}$. Hence, due to $0 < \alpha_k \leq \alpha_{\max}$, we have

$$\alpha_k[f(x^k) - f(x^{k+1}) + \nu_k] < \alpha_{\max}[f(x^k) - f(x^{k+1}) + \nu_k], \qquad \forall k \in \mathbb{N}.$$

Therefore, by combining of the last inequality with (3.17) and (3.21) we obtain that

$$\|x^{k+1} - x\|^2_{D_k} \leq \|x^k - x\|^2_{D_k} + 2\alpha_k\tau_k\Big\langle \nabla f(x^k), x - x^k\Big\rangle + \xi\Big[f(x^k) - f(x^{k+1}) + \nu_k\Big], \quad \forall\, k \in \mathbb{N}.$$

Since **A3** implies that $\|x^{k+1} - x\|^2_{D_{k+1}} \leq (1+\eta_k)\|x^{k+1} - x\|^2_{D_k}$ the desired inequality (3.18) follows. □

For proceeding with the analysis of the behavior of the sequence $(x^k)_{k\in\mathbb{N}}$, we define the following auxiliary set

$$U := \left\{x \in C : f(x) \leq \inf_{k\in\mathbb{N}} \Big(f(x^k) + \nu_k\Big)\right\}.$$

Corollary 3.8. *Assume that f is a convex function. If $U \neq \varnothing$, then $(x^k)_{k\in\mathbb{N}}$ converges to a stationary point of problem (1).*

Proof. Let $x \in U$. Since f is convex, we have

$$0 \geq f(x) - (f(x^k) + \nu_k) \geq \langle \nabla f(x^k), x - x^k\rangle - \nu_k,$$

for all $k \in \mathbb{N}$. Thus, $\langle \nabla f(x^k), x - x^k\rangle \leq \nu_k$, for all $k \in \mathbb{N}$. Using Lemma 3.7, and taking into account that $\tau_k \in (0, 1]$ and $0 < \alpha_{\min} \leq \alpha_k \leq \alpha_{\max}$, we obtain

$$\|x^{k+1} - x\|^2_{D_{k+1}} \leq (1+\eta_k)\|x^k - x\|^2_{D_k} + 2\alpha_{\max}\beta\nu_k + \xi\beta\Big[f(x^k) - f(x^{k+1}) + \nu_k\Big], \quad \forall\, k \in \mathbb{N},$$

where $\beta := 1 + \sup\{\eta_k :\ k \in \mathbb{N}\}$. Defining

$$\epsilon_k = 2\alpha_{\max}\beta\nu_k + \xi\beta\Big[f(x^k) - f(x^{k+1}) + \nu_k\Big],$$

we have

$$\|x^{k+1} - x\|^2_{D_{k+1}} \leq (1+\eta_k)\|x^k - x\|^2_{D_k} + \epsilon_k,$$

for all $k \in \mathbb{N}$. On the other hand, summing ϵ_k with $k = 0, 1, \ldots, N$ and using Corollary 3.4, we have

$$\sum_{k=0}^{N} \epsilon_k \leq 2\alpha_{\max}\beta \sum_{k=0}^{N} \nu_k + \xi\beta\left(f(x^0) - f(x) + \sum_{k=0}^{N+1} \nu_k\right) < +\infty, \qquad \forall N \in \mathbb{N}.$$

Hence, $\sum_{k=0}^{+\infty} \epsilon_k < +\infty$. Thus, it follows from Definition 1.2 that $(x^k)_{k\in\mathbb{N}}$ is quasi-Fejér monotone to U with respect to the sequence $(D_k)_{k\in\mathbb{N}}$. Since U is nonempty, it follows from Theorem 1.4 that $(x^k)_{k\in\mathbb{N}}$ is bounded, and therefore it has cluster points. Let $\bar{x}$ be a cluster point of $(x^k)_{k\in\mathbb{N}}$ and $(x^{k_j})_{j\in\mathbb{N}}$ be a subsequence of $(x^k)_{k\in\mathbb{N}}$ such that $\lim_{j\to\infty} x^{k_j} = \bar{x}$. Considering that f is continuous and $\lim_{k\to+\infty} \nu_k = 0$, we have $\lim_{j\to\infty}(f(x^{k_j}) + \nu_{k_j}) = f(\bar{x})$. On the other hand, Lemma 3.2 implies that $\Big(f(x^k) + \nu_k\Big)_{k\in\mathbb{N}}$ is non-increasing. Thus $\inf_{k\in\mathbb{N}}(f(x^k) + \nu_k) = \lim_{k\to\infty}(f(x^k) + \nu_k) = f(\bar{x})$. Hence, $\bar{x} \in U$, and Theorem 1.4 implies that $(x^k)_{k\in\mathbb{N}}$ converges to $\bar{x}$. The conclusion is obtained by using Proposition 3.3. □

Theorem 3.9. *If f is a convex function and $(x^k)_{k\in\mathbb{N}}$ has no cluster points, then $\Omega^* = \varnothing$, $\lim_{k\to\infty} \|x^k\| = +\infty$, and $\inf_{k\in\mathbb{N}} f(x^k) = \inf\{f(x) : x \in C\}$.*

Proof. Since $(x^k)_{k\in\mathbb{N}}$ has no cluster points, then $\lim_{k\to\infty} \|x^k\| = +\infty$. Assume by contradiction that $\Omega^* \neq \varnothing$. Thus, there exists $\tilde{x} \in C$, such that $f(\tilde{x}) \leq f(x^k)$ for all $k \in \mathbb{N}$. Therefore, $\tilde{x} \in U$. Using Corollary 3.8, we obtain that $(x^k)_{k\in\mathbb{N}}$ is convergent, contradicting that $\lim_{k\to\infty} \|x^k\| = \infty$. Therefore, $\Omega^* = \varnothing$. Now, we claim that $\inf_{k\in\mathbb{N}} f(x^k) = \inf\{f(x) : x \in C\}$. If $\inf_{k\in\mathbb{N}} f(x^k) = -\infty$, the claim holds. Assume by contraction that $\inf_{k\in\mathbb{N}} f(x^k) > \inf_{x\in C} f(x)$. Thus, there exists $\tilde{x} \in C$ such that $f(\tilde{x}) \leq f(x^k) \leq f(x^k) + \nu_k$, for all $k \in \mathbb{N}$. Hence, $U \neq \varnothing$. Using Corollary 3.8, we have that $(x^k)_{k\in\mathbb{N}}$ is convergent, contradicting again $\lim_{k\to\infty} \|x^k\| = +\infty$ and concluding the proof. □

Corollary 3.10. *If f is a convex function and $(x^k)_{k\in\mathbb{N}}$ has at least one cluster point, then $(x^k)_{k\in\mathbb{N}}$ converges to a stationary point of problem (1).*

Proof. Let $\bar{x}$ be a cluster point of the sequence $(x^k)_{k\in\mathbb{N}}$ and $(x^{k_j})_{j\in\mathbb{N}}$ be a subsequence of $(x^k)_{k\in\mathbb{N}}$ such that $\lim_{j\to+\infty} x^{k_j} = \bar{x}$. Considering that f is continuous and $\lim_{k\to+\infty} \nu_k = 0$, we have $\lim_{j\to\infty}(f(x^{k_j}) + \nu_{k_j}) = f(\bar{x})$. On the other hand, Corollary 3.4 implies that $(f(x^k) + \nu_k)_{k\in\mathbb{N}}$ is non-increasing. Hence, we have

$$\inf_{k\in\mathbb{N}}(f(x^k) + \nu_k) = \lim_{k\to\infty}(f(x^k) + \nu_k) = f(\bar{x}).$$

Therefore $\bar{x} \in U$. Using Corollary 3.8, we obtain that $(x^k)_{k\in\mathbb{N}}$ converges to a stationary point $\tilde{x} \in C$ of problem (1). □

Theorem 3.11. *Assume that f is a convex function and $\Omega^* \neq \varnothing$. Then, $(x^k)_{k\in\mathbb{N}}$ converges to an optimal solution of problem (1).*

Proof. If $\Omega^* \neq \varnothing$, then $U \neq \varnothing$. Therefore, Corollary 3.8 implies that $(x^k)_{k\in\mathbb{N}}$ converges to a stationary point of problem (1) and, due to f be convex, this point is also an optimal solution. □

3.3 Iteration-complexity bound

In the section, we preset some iteration-complexity bounds related to the sequence $(x^k)_{k\in\mathbb{N}}$ generated by Algorithm 3.1. For that, besides assuming **A1** and **A2**, we also need the following assumption.

A4. The gradient ∇f of f is Lipschitz continuous with constant $L > 0$.

For simple notations, we define the following positive constant

$$\tau_{\min} := \min\left\{1, \frac{\underline{\omega}(1-\sigma)}{\alpha_{\max}\mu L}\right\}. \tag{3.22}$$

Lemma 3.12. *The steepsize τ_k in Algorithm 3.1 satisfies $\tau_k \geq \tau_{\min}$.*

Proof. First, we assume that $\tau_k = 1$. In this case, we have $\tau_k \geq \tau_{\min}$ and the required inequality holds. Now, we assume that $\tau_k < 1$. Thus, it follows from (3.3) that there exists $0 < \hat{\tau}_k \leq \min\{1, \tau_k/\underline{\omega}\}$ such that

$$f\Big(x^k + \hat{\tau}_k(w^k - x^k)\Big) > f(x^k) + \sigma\hat{\tau}_k\Big\langle \nabla f(x^k), w^k - x^k\Big\rangle + \nu_k. \tag{3.23}$$

Considering that we are under assumption **A4**, we apply Lemma 1.1 to obtain

$$f\Big(x^k + \hat{\tau}_k(w^k - x^k)\Big) \leq f(x^k) + \hat{\tau}_k\Big\langle \nabla f(x^k), w^k - x^k\Big\rangle + \frac{L}{2}\hat{\tau}_k^2\|w^k - x^k\|^2. \tag{3.24}$$

Hence, the combination of (3.23) with (3.24) yields

$$(1-\sigma)\Big\langle \nabla f(x^k), w^k - x^k\Big\rangle + \frac{L}{2}\hat{\tau}_k\|w^k - x^k\|^2 > \frac{\nu_k}{\hat{\tau}_k}. \tag{3.25}$$

On the order hand, $w^k \in \mathcal{R}^{D_k}_{C,\gamma_k}(x^k, z^k)$ with $\gamma_k = (1-\zeta_k)/2$, where $z^k = x^k - \alpha_k D_k^{-1}\nabla f(x^k)$. Thus, applying Lemma 2.18*(i)* with $x = x^k$, $w(\alpha) = w^k$, $z = z^k$ and $\zeta = \zeta_k$, we obtain

$$\Big\langle \nabla f(x^k), w^k - x^k\Big\rangle \leq -\frac{1}{2\alpha_k}\|w^k - x^k\|^2_{D_k}.$$

Hence, considering that $\frac{1}{\mu}\|w^k - x^k\|^2 \leq \|w^k - x^k\|^2_{D_k}$ and $0 < \alpha_k \leq \alpha_{\max}$, the last inequality implies

$$\Big\langle \nabla f(x^k), w^k - x^k\Big\rangle \leq -\frac{1}{2\alpha_{\max}\mu}\|w^k - x^k\|^2.$$

The combination of the last inequality with (3.25) yields

$$\left(-\frac{(1-\sigma)}{2\alpha_{\max}\mu} + \frac{L}{2}\hat{\tau}_k\right)\|w^k - x^k\|^2 > \frac{\nu_k}{\hat{\tau}_k} \geq 0.$$

Thus, since $\hat{\tau}_k \leq \tau_k/\underline{\omega}$, we obtain $\tau_k \geq \underline{\omega}\hat{\tau}_k > \underline{\omega}(1-\sigma)/(\alpha_{\max}\mu L) \geq \tau_{\min}$ and the proof is concluded. □

Considering that $\mathcal{R}^{D_k}_{C,\gamma_k}(x^k, z^k) \subset \mathcal{P}^{D_k}_{C,\zeta_k}(x^k, z^k)$, it follows from Lemma 2.18*(ii)* that if $x^k \in \mathcal{R}^{D_k}_{C,\gamma_k}(x^k, z^k)$, then the point x^k is stationary for problem (1). Since $w^k \in \mathcal{R}^{D_k}_{C,\gamma_k}(x^k, z^k)$, the quantity $\|w^k - x^k\|$ may be seen as a measure of stationarity of the point x^k. In next theorem, we present an iteration-complexity bound for this quantity, which is a constrained inexact version of [45, Theorem 1].

Theorem 3.13. *Let $\tau_{\min}$ be defined in (3.22). Then, for every $N \in \mathbb{N}$, the following inequality holds*

$$\min\Big\{\|w^k - x^k\| : \ k = 0, 1\ldots, N-1\Big\} \leq \sqrt{\frac{2\alpha_{\max}\mu\left[f(x^0) - f^* + \sum_{k=0}^{\infty}\nu_k\right]}{\sigma\tau_{\min}}}\frac{1}{\sqrt{N}}.$$

Proof. Since $w^k \in \mathcal{R}^{D_k}_{C,\gamma_k}(x^k, z^k)$ with $\gamma_k = (1-\zeta_k)/2$, where $z^k = x^k - \alpha_k D_k^{-1}\nabla f(x^k)$, applying Lemma 2.18*(i)* with $x = x^k$, $w(\alpha) = w^k$, $z = z^k$ and $\zeta = \zeta_k$, and taking into account that $(1/\mu)\|w^k - x^k\|^2 \leq \|w^k - x^k\|^2_{D_k}$ and $0 < \alpha_k \leq \alpha_{\max}$, we obtain

$$\Big\langle \nabla f(x^k), w^k - x^k\Big\rangle \leq -\frac{1}{2\alpha_k}\|w^k - x^k\|^2_{D_k} \leq -\frac{1}{2\alpha_{\max}\mu}\|w^k - x^k\|^2.$$

The definition of τ_k and (3.3) imply

$$f(x^{k+1}) - f(x^k) \leq \sigma\tau_k \left\langle \nabla f(x^k), w^k - x^k \right\rangle + \nu_k.$$

The combination of the last two inequalities together with Lemma 3.12 yields

$$f(x^k) - f(x^{k+1}) + \nu_k \geq \sigma\tau_k \frac{1}{2\alpha_{\max}\mu} \|w^k - x^k\|^2 \geq \sigma\tau_{\min} \frac{1}{2\alpha_{\max}\mu} \|w^k - x^k\|^2.$$

Hence, performing the sum of the above inequality for $k = 0, 1, \ldots, N-1$, we conclude that

$$\sum_{k=0}^{N-1} \|w^k - x^k\|^2 \leq \frac{2\alpha_{\max}\mu \left[f(x^0) - f(x^{N+1}) + \sum_{k=0}^{N} \nu_k \right]}{\sigma\tau_{\min}} \leq \frac{2\alpha_{\max}\mu \left[f(x^0) - f^* + \sum_{k=0}^{\infty} \nu_k \right]}{\sigma\tau_{\min}},$$

which implies the desired result. □

Next we present some results regarding the number of function evaluations performed by Algorithm 3.1. Note that the computational cost associated to each (outer) iteration involves a gradient evaluation, the computation of a (inexact) projection, and evaluations of f at different trial points. Thus, we must consider the function evaluations at the rejected trial points.

Lemma 3.14. *Let N_k be the number of function evaluations after $k \geq 0$ iterations of Algorithm 3.1. Then, $N_k \leq 1 + (k+1)[\log(\tau_{\min})/\log(\bar{\omega}) + 1]$.*

Proof. Let $j(k) \geq 0$ be the number of inner iterations in Step 2 of Algorithm 3.1 to compute the step size τ_k. Thus, $\tau_k \leq \bar{\omega}^{j(k)}$. Using Lemma 3.12, we have $0 < \tau_{\min} \leq \tau_k$ for all $k \in \mathbb{N}$, which implies that $\log(\tau_{\min}) \leq \log(\tau_k) = j(k)\log(\bar{\omega})$, for all $k \in \mathbb{N}$. Hence, due to $\log(\bar{\omega}) < 0$, we have $j(k) \leq \log(\tau_{\min})/\log(\bar{\omega})$. Therefore,

$$N_k = 1 + \sum_{\ell=0}^{k} (j(\ell) + 1) \leq 1 + \sum_{i=0}^{k} \left(\frac{\log(\tau_{\min})}{\log(\bar{\omega})} + 1 \right) = 1 + (k+1)\left(\frac{\log(\tau_{\min})}{\log(\bar{\omega})} + 1 \right),$$

where the first equality follows from the definition of N_k. □

Theorem 3.15. *For a given $\epsilon > 0$, Algorithm 3.1 computes x^k and w^k such that $\|w^k - x^k\| \leq \epsilon$ using, at most,*

$$1 + \left(\frac{2\alpha_{\max}\mu \left[f(x^0) - f^* + \sum_{k=0}^{\infty} \nu_k \right]}{\sigma\tau_{\min}} \frac{1}{\epsilon^2} + 1 \right) \left(\frac{\log(\tau_{\min})}{\log(\bar{\omega})} + 1 \right)$$

function evaluations.

Proof. The proof follows straightforwardly from Theorem 3.13 and Lemma 3.14. □

Theorem 3.16. *Let f be a convex function on C. For a given $\epsilon > 0$, the number of function evaluations performed by Algorithm 3.1 is, at most,*

$$1 + \left(\frac{\|x^0 - x^*\|_{D_0}^2 + \xi \left[f(x^0) - f^* + \sum_{k=0}^{\infty} \nu_k \right]}{2\alpha_{\min}\tau_{\min}} \frac{1}{\epsilon} + 1 \right) \left(\frac{\log(\tau_{\min})}{\log(\bar{\omega})} + 1 \right),$$

to compute x^k such that $f(x^k) - f^* \leq \epsilon$.

Proof. The proof follows straightforwardly from Theorem 3.17 and Lemma 3.14. □ We ended this section with a theorem about iteration-complexity bound for the sequence $\left(f(x^k)\right)_{k\in\mathbb{N}}$ when f is convex.

Theorem 3.17. *Let f be a convex function on C. Assume that the sequence $(D_k)_{k\in\mathbb{N}}$ satisfies* **A3** *with $\eta_k \equiv 0$. Then, for every $N \in \mathbb{N}$,*

$$\min\left\{f(x^k) - f^* : \ k = 0, 1 \ldots, N-1\right\} \leq \frac{\|x^0 - x^*\|^2_{D_0} + \xi\left[f(x^0) - f^* + \sum_{k=0}^{\infty}\nu_k\right]}{2\alpha_{\min}\tau_{\min}}\frac{1}{N}.$$

Proof. Using the first inequality in (3.1) and Lemma 3.12, we have $2\alpha_{\min}\tau_{\min} \leq 2\alpha_k\tau_k$, for all $k \in \mathbb{N}$. We also know form the convexity of f that $\langle \nabla f(x^k), x^* - x^k\rangle \leq f^* - f(x^k)$, for all $k \in \mathbb{N}$. Thus, applying Lemma 3.7 with $x = x^*$ and taking into account that $\eta_k \equiv 0$, after some algebraic manipulations, we conclude

$$2\alpha_{\min}\tau_{\min}\left[f(x^k) - f^*\right] \leq \|x^k - x^*\|^2_{D_k} - \|x^{k+1} - x^*\|^2_{D_{k+1}} + \xi\left[f(x^k) - f(x^{k+1}) + \nu_k\right] \quad k = 0, 1, \ldots.$$

Hence, performing the sum of the above inequality for $k = 0, 1, \ldots, N-1$, we obtain

$$2\alpha_{\min}\tau_{\min}\sum_{k=0}^{N-1}\left[f(x^k) - f^*\right] \leq \|x^0 - x^*\|^2_{D_0} - \|x^{N+1} - x^*\|^2_{D_N} + \xi\Big[f(x^0) - f(x^{N+1}) + \sum_{k=0}^{N-1}\nu_k\Big].$$

Thus, $2\alpha_{\min}\tau_{\min}N\min\{f(x^k) - f^* : \ k = 0, 1 \ldots, N-1\} \leq \|x^0 - x^*\|^2_{D_0} + \xi[f(x^0) - f^* + \sum_{k=0}^{N-1}\nu_k]$, which implies the desired inequality. □

Remark 3.18. For suitable choices of the scale matrix and the step size, SGP merges into the well known *spectral gradient method*. Therefore, all obtained results in this section also hold to this method.

Chapter 4

Numerical experiments

This chapter presents some numerical experiments in order to illustrate the potential advantages of considering inexact schemes in the SGP method. We will discuss inexactness associated with both the projection onto the feasible set and the line search procedure.

Given A and B two $m \times n$ matrices, with $m \geq n$, and $c \in \mathbb{R}$, we consider the matrix function $f : \mathbb{R}^{n \times n} \to \mathbb{R}$ given by:

$$f(X) := \frac{1}{2}\|AX - B\|_F^2 + \sum_{i=1}^{n-1} \left[c\left(X_{i+1,i+1} - X_{i,i}^2\right)^2 + (1 - X_{i,i})^2 \right], \tag{4.1}$$

which combines a least squares term with a Rosenbrock-type function. Throughout this chapter, $X_{i,j}$ stands for the ij-element of the matrix X and $\|\cdot\|_F$ denotes the Frobenius matrix norm, i.e., $\|A\|_F := \sqrt{\langle A, A\rangle}$ where the inner product is given by $\langle A, B\rangle = \operatorname{tr}(A^T B)$. The test problems consist of minimizing f in (4.1) subject to two different feasible sets, as described below. We point out that interesting applications in many areas emerge as constrained least squares matrix problems, see [14] and references therein. In turn, the Rosenbrock term was added in order to make the problems more challenging.

Problem I:

$$\begin{array}{ll} \min & f(X) \\ \text{s.t.} & X \in SDD^+, \\ & L \leq X \leq U, \end{array}$$

where SDD^+ is the cone of symmetric and diagonally dominant real matrices with positive diagonal, i.e.,

$$SDD^+ := \{X \in \mathbb{R}^{n \times n} \mid X = X^T,\ X_{i,i} \geq \sum_{j \neq i} |X_{i,j}|\ \forall i\},$$

L and U are given $n \times n$ matrices, and $L \leq X \leq U$ means that $L_{i,j} \leq X_{i,j} \leq U_{i,j}$ for all i, j. The feasible set of Problem I was considered, for example, in the numerical tests of [14].

Problem II:

$$\begin{array}{ll} \min & f(X) \\ \text{s.t.} & X \in \mathbb{S}_+^n, \\ & \operatorname{tr}(X) = 1, \end{array}$$

where $\mathbb{S}_+^n$ is the cone of symmetric and positive semidefinite real matrices and $\operatorname{tr}(X)$ denotes the trace of X. The feasible set of Problem II was known as *spectrahedron* and appears in several interesting applications see, for example, [4, 42] and references therein.

It is easy to see that the feasible set of Problem I is a closed and convex set and the feasible set of Problem II is a compact and convex set. As discussed in Section 2.4, the Dykstra's alternating algorithm and the Frank-Wolfe algorithm may be used to calculate inexact projections. The choice of the most appropriate method depends on the structure of the feasible set under consideration. For Problem I, we used the Dykstra's algorithm described in [14], see also [63]. In this case, $SDD^+ = \cap_{i=1}^n SDD_i^+$, where

$$SDD_i^+ := \{X \in \mathbb{R}^{n\times n} \mid X = X^T,\ X_{i,i} \geq \sum_{j\neq i} |X_{i,j}|\} \text{ for all } i = 1, \ldots, n,$$

and the projection of a given $Z \in \mathbb{R}^{n\times n}$ onto SDD^+ consists of cycles of projections onto the convex sets SDD_i^+. Here an iteration of the Dykstra's algorithm should be understood as a complete cycle of projections onto all SDD_i^+ sets and onto the box $\{X \in \mathbb{R}^{n\times n} \mid L \leq X \leq U\}$. Recall that this scheme provides an inexact projection as in Definition 2.5. Now consider Problem II. It is well known that calculating an exact projection onto the spectrahedron (i.e., onto the feasible set of Problem II) requires a complete spectral decomposition, which may be prohibitive specially in the large scale case. In contrast, the computational cost of an iteration of the Frank-Wolfe algorithm described in Algorithm 2.1 is associated by an extreme eigenpair computation, see, for example, [50]. Unfortunately, despite its low cost per-iteration, the Frank-Wolfe algorithm suffers from a slow convergence rate. Thus, we considered a variant of the Frank-Wolfe algorithm proposed in [4], which improves the convergence rate and the total time complexity of the classical Frank-Wolfe method. This algorithm specialized for the projection problem over the spectrahedron is carefully described in [1]. Without attempting to go into details, it replaces the top eigenpair computation in Frank-Wolfe with a top-p (with $p \ll n$) eigenpair computation, where p is an algorithmic parameter automatically selected. The total number of computed eigenpairs may be used to measure the computational effort to calculate projections. We recall that a Frank-Wolfe type scheme provides an inexact projection as in Definition 2.10.

We notice that Problems I and II may be seen as particular instances of the problem (1) in which the number of variables is $(n^2 + n)/2$. This mean that they may be solved by using Algorithm 3.1. We are especially interested in the spectral gradient version [14, 13] of the SGP method, which is often associated with large-scale problems [15]. For this, we implemented Algorithm 3.1 considering $D_k := I$ for all k, $\alpha_0 := \min(\alpha_{\max}, \max(\alpha_{\min}, 1/\|\nabla f(x^0)\|))$ and, for $k > 0$,

$$\alpha_k := \begin{cases} \min(\alpha_{\max}, \max(\alpha_{\min}, \langle s^k, s^k\rangle / \langle s^k, y^k\rangle)), & \text{if } \langle s^k, y^k\rangle > 0 \\ \alpha_{\max}, & \text{otherwise,} \end{cases}$$

where $s^k := X^k - X^{k-1}$, $y^k := \nabla f(X^k) - \nabla f(X^{k-1})$, $\alpha_{\min} = 10^{-10}$, and $\alpha_{\max} = 10^{10}$. We set $\sigma = 10^{-4}$, $\underline{\tau} = 0.1$, $\bar{\tau} = 0.9$, $\mu = 1$ and $\nu_0 = 0$. Parameter $\delta_{\min}$ was chosen according to the line search used (see Chapter 3), while parameter $\zeta_{\min}$ depends on the inexact projection scheme considered.

In the line search scheme (Step 2 of Algorithm 3.1), if a step size τ_{trial} is not accepted, then τ_{new} is calculated using one-dimensional quadratic interpolation employing the safeguard $\tau_{\text{new}} \leftarrow \tau_{\text{trial}}/2$ when the minimum of the one-dimensional quadratic lies outside $[\underline{\omega}\tau_{\text{trial}}, \bar{\omega}\tau_{\text{trial}}]$, see, for example,

[58, Section 3.5]. Concerning the stopping criterion, all runs were stopped at an iterate X^k declaring convergence if

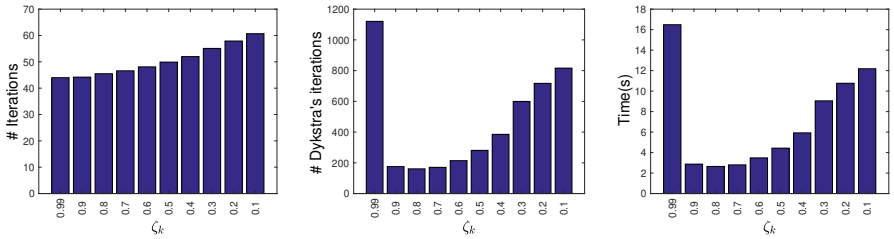
$$\max_{i,j}(|X^k_{i,j} - W^k_{i,j}|) \leq 10^{-6},$$

where W^k is as in (3.2). Our codes are written in Matlab and are freely available at https://github.com/maxlemes/SGP. All experiments were run on a macOS 10.15.7 with 3.7GHz Intel Core i5 processor and 8GB of RAM.

4.1 Influence of the inexact projection

We begin the numerical experiments by checking the influence of the forcing parameters that control the degree of inexactness of the projections in the performance of the method. In this first battery of tests, we used Armijo line searches, see Chapter 3.

We generated 10 instances of Problem I using $n = 100$, $m = 200$, and $c = 10$. The matrices A and B were randomly generated with elements belonging to $[-1, 1]$. We set $L \equiv 0$ and $U \equiv \infty$ as in [14]. For each instance, the starting point X^0 was randomly generated with elements belonging to $[0, 1]$, then it was redefined as $(X^0 + (X^0)^T)/2$ and its diagonal elements were again redefined as $2\sum_{j\neq i}^{n} X_{i,j}$, ensuring a feasible starting point. Figure 4.1 shows the average number of iterations, the average number of Dykstra's iterations, and the average CPU time in seconds needed to reach the solution for different choices of ζ_k, namely, $\zeta_k = 0.99$, 0.9, 0.8, 0.7, 0.6, 0.5, 0.4, 0.3, 0.2, and 0.1 for all k. Remember that *smaller values* of ζ_k imply *more inexact* projections. As expected, the number of iterations tended to increase as ζ_k decreased, see Figure 4.1(a). On the other hand, the computational cost of an outer iteration (which may be measured by the number of Dykstra's iterations) tends to decrease when considering smaller values of ζ_k. This suggests a trade-off, controlled by parameter ζ_k, between the number and the cost per iteration. Figure 4.1(b) shows that values for ζ_k close to 0.8 showed better results, which is in line with the experiments reported in [14]. Finally, as may be seen in Figure 4.1(c), the CPU time was shown to be directly proportional to the number Dykstra's iterations.

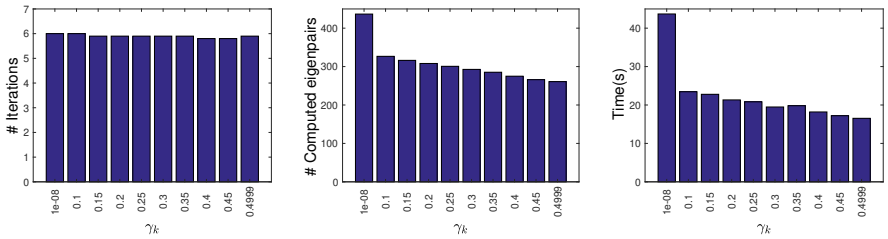


(a) (b) (c)

Figure 4.1: Results for 10 instances of Problem I using $n = 100$, $m = 200$, and $c = 10$. Average number of: (a) iterations; (b) Dykstra's iterations; (c) CPU time in seconds needed to reach the solution for different choices of ζ_k.

Although Algorithm 3.1 is given only in terms of parameter ζ_k, we will directly consider parameter γ_k for Problem II in which inexact projections are computed according to Definition 2.10. We randomly generated 10 instances of Problem II with $n = 800$, $m = 1000$, and $c = 100$. Matrices A

and B were obtained similarly to Problem I. In turn, a starting point X^0 was randomly generated with elements in the interval $[-1, 1]$, then it was redefined to be $X^0(X^0)^T/\text{tr}(X^0(X^0)^T)$, resulting in a feasible initial guess. Figure 4.2 shows the average number of iterations, the average number of computed eigenpairs, and the average CPU time in seconds needed to reach the solution for different constant choices of γ_k ranging from 10^{-8} to 0.4999. Now, *higher values* of γ_k imply *more inexact* projections. Note that for appropriate choices of ζ_k, the adopted values of γ_k fulfill Assumption A1 of Section **??**. Concerning the number of iterations, as may be seen in Figure 4.2(a), the algorithm was not very sensitive to the choice of parameter γ_k. Hence, since higher values of γ_k imply cheaper iterations, the number of computed eigenpairs and the CPU time showed to be inversely proportional to γ_k, see Figures 4.2(b)–(c). Thus, our experiments suggest that the best value for γ_k seems to be 0.4999.



(a) (b) (c)

Figure 4.2: Results for 10 instances of Problem II using $n = 800$, $m = 1000$, and $c = 100$. Average number of: (a) iterations; (b) computed eigenpairs; (c) CPU time in seconds needed to reach the solution for different choices of γ_k.

4.2 Influence of the line search scheme

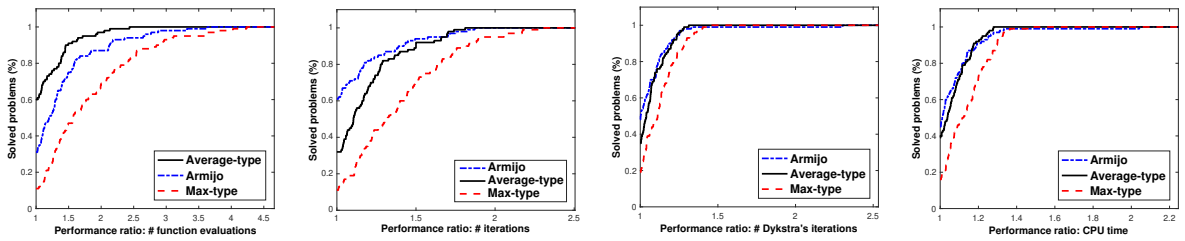
The following experiments compare the performance of the algorithm with different strategies for computing the step sizes. We considered the Armijo, the Average-type, and the Max-type line searches discussed in Chapter 3. Based on our numerical experience, we employed the fixed value $\eta_k = 0.85$ for the Average-type line search and $M = 5$ for the Max-type line search. According to the results of the previous section, we used the fixed forcing parameters $\zeta_k = 0.8$ and $\gamma_k = 0.4999$ to compute inexact projections for Problems I and II, respectively.

We randomly generated 100 instances of each problem as described in Section 4.1. The dimension of the problems and the parameter c in (4.1) were also taken arbitrarily. For Problem I, we choose $100 \leq n \leq 800$ and $10 \leq c \leq 50$, whereas for Problem II, we choose $10 \leq n \leq 200$ and $100 \leq c \leq 1000$. In both cases, we set $m = 2n$. We compare the strategies with respect to the number of function evaluations, the number of (outer) iterations, the total computational effort to calculate projections (measured by the number of Dykstra's iterations and computed eigenpairs for Problems I and II, respectively), and the CPU time. The results are shown in Figures 4.3 and 4.4 for Problems I and II, respectively, using performance profiles [32].

For Problem I, with regard to the number of function evaluations, the algorithm with the Average-type line search was the most efficient among the tested strategies. In a somewhat surprising way, in this set of test problems, the Armijo strategy was better than the Max-type line

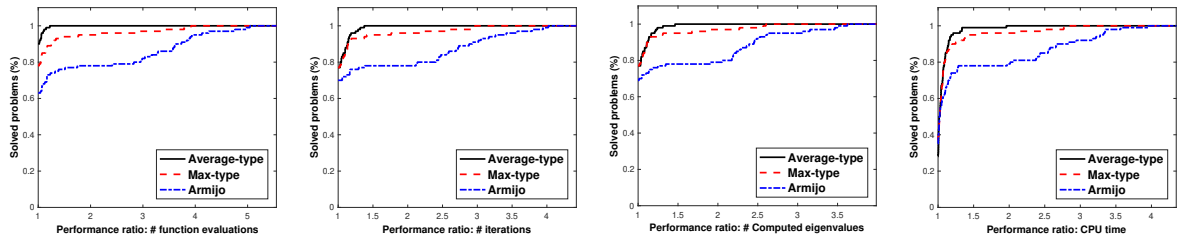
search, see Figure 4.3(a). On the other hand, as may be seen in Figure 4.3(b), the Armijo strategy required fewer iterations than the nonmonotone strategies. As expected, this was reflected in the number of Dykstra's iterations and the CPU time, see Figures 4.3(c)–(d). We conclude that, with respect to the last two criteria, the Armijo and Average-type strategies had similar and superior performances to the Max-type strategy.

Now, concerning Problem II, Figure 4.4 shows that the nonmonotone strategies outperformed the Armijo strategy in all the comparative criteria considered. Again, the Average-type strategy seems to be superior to the Max-type strategy.



(a) (b) (c) (d)

Figure 4.3: Performance profiles for Problem I considering Algorithm 3.1 with the Armijo, the Average-type, and the Max-type line searches strategies using as performance measurement: (a) number of function evaluations; (b) number of (outer) iterations; (c) number of Dykstra's iterations; (d) CPU time.



(a) (b) (c) (d)

Figure 4.4: Performance profiles for Problem II considering Algorithm 3.1 with the Armijo, the Average-type, and the Max-type line searches strategies using as performance measurement: (a) number of function evaluations; (b) number of (outer) iterations; (c) number of computed eigenpairs; (d) CPU time.

From all the above experiments, we conclude that the nonmonotone line searches tend to require fewer objective function evaluations. However, this does not necessarily mean computational savings, since there may be an increase in the number of iterations. In this case, optimal efficiency of the algorithm comes from a compromise between those two conflicting tendencies. Overall, the use of nonmonotone line search techniques is mainly justified when the computational effort of an iteration is associated with the cost of evaluating the objective function.

Chapter 5

Applications

5.1 Risks Measures

In this section we discuss different ways of measuring the risk of a portfolio.

Definition 5.1. *A portfolio with n assets, $A_1, A_2, \ldots, A_n$, is a vector $x^\top = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, each coordinate x_i being the weight of the capital invested in asset A_i.*

In financial science, Risk is the probability of losing capital. Considering the loss as a random variable, we may calculate the standard deviation (volatility) of its distribution. Nobel laureate in economics, Harry Markowitz in 1952 (see [54]) assumed volatility as a risk measure under the hypothesis that the returns has a normal distribution.

Other risk measures are expressed in terms of loss distribution percentiles. An upper percentile of the loss distribution is called *Value-at-Risk* (here VaR_β)[1]. For instance, with probability 95%, the loss is smaller than $\text{VaR}_{0.95}$. The popularity of VaR_β is mostly related to being a simple way to understand high loss representations. VaR_β may be estimated and managed quite efficiently, when the underlying risk factors are normally (log-normally) distributed.

An alternative risk measure is known as the *Conditional Value-at-Risk* (CVaR_β)[2]. The CVaR_β risk measure is closely related to VaR_β. For continuous distributions, CVaR_β is defined as the conditional expected loss, under the condition that it exceeds VaR_β, see Rockafellar and Uryasev (see [64]). For continuous distributions, this risk measure is also known as *Expected Shortfall*, *Mean Excess Loss*, *Mean Shortfall*, or *Tail Value-at-Risk*.

Remark 5.2. Since loss may be seen as negative return, if the return of portfolio has a normal distribution, it follows that both VaR_β and CVaR_β may be rewritten by

$$-\mu(x) + c\sigma(x)$$

where $\mu(x)$ and $\sigma(x)$ denote the return mean value and volatility, associated with portfolio x, and c is a constant (see [65, Chapter 2]). If the portfolio manager has very optimistic forecasts, component $\mu(x)$ may reduce the risk measure substantially. This explains why omitting the mean component has been adopted as a standard practice by the asset management industry.

[1] VaR_β is the percentile of the loss distribution.

[2] CVaR_β is the average loss in the $100(1-\beta)\%$ worst case scenarios.

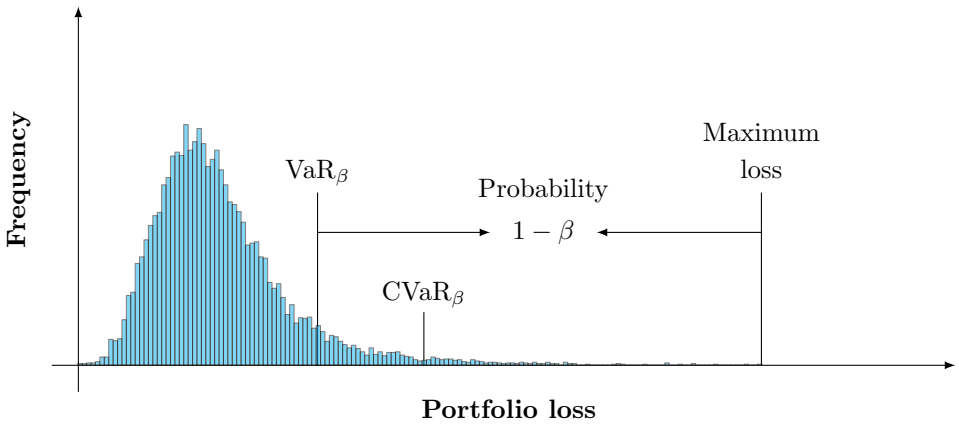


Figure 5.1: Portfolio Loss Distribution, VaR and CVaR.

5.2 Portfolio optimization

In this section we present some ways to build a portfolio. In a universe of n assets let $x^\top = (x_1, x_2, \ldots, x_n)$ be a portfolio and by R_i denote the return of asset i. If $R^\top = (R_1, R_2, \ldots, R_n)$ is the random vector of assets, then the return of the portfolio x is given by:

$$R(x) = x_1 R_1 + x_2 R_2 + \cdots x_n R_n = x^\top R.$$

If we denote by μ and Σ the vector of expected returns and the covariance matrix of asset returns respectively, we deduce that the expected return of portfolio x equals:

$$\mu(x) = x^\top \mu.$$

Its variance is given by:

$$\sigma^2(x) = x^\top \Sigma x.$$

The minimum variance portfolio (MVP) of Markowitz (see [54]), consists of minimize variance with fully invested capital, achieved by solving the problem:

$$\begin{aligned} &\min_x \left\{ x^\top \Sigma x \right\}, \\ &\text{s.t.} \ \left\{ \mathbf{1}^\top x = 1. \right. \end{aligned} \tag{5.1}$$

where $\mathbf{1}^\top = (1, 1, \ldots, 1)$.

A portfolio optimization problem is defined when we want to minimize (or maximize) a performance measure subject to a set of constraints. Here are some examples of performance measures and constraints:

- **Performance Measures**
  - Expected return: $\mu(x)$;
  - Volatility: $\sigma(x)$;

– Sharpe Ratio (SR): expected return per unit of risk

$$\mathrm{SR}(x) = \frac{x^\top \mu - r_f}{\sigma(x)}$$

where r_f is the risk-free rate (e.g. treasury bill interest rate);

– Information Ratio (IR): Sharpe Ratio with $r_f = 0$;

– VaR_β (Value at Risk): loss quantile;

– CVaR_β (Conditional Value at Risk): expected loss value above VaR_β.

- **Constraints**

  – Capital constraint: $\mathbf{1}^\top x = 1$;

  – Long-only constraint: $x \geq 0$;

  – Self-financial constraint: $\mathbf{1}^\top x = 0$;

  – Holding constraint: $\mathbf{I} \leq x \leq \mathbf{J}$ where $\mathbf{I}, \mathbf{J} \in \mathbb{R}^n$ are lower and upper portfolio bounds.

  – Leverage constraint: $\|x\|_1 \leq K$.

Some known portfolio optimization problems are:

Global Mean Variance Portfolio (GMVP)

Maximizing the expected return $\mu(x)$ for a fixed value ν of the volatility $\sigma(x)$

$$\begin{aligned} &\min_x \{-\mu(x)\}, \\ \text{s.t. } &\begin{cases} \sigma^2(x) = \nu, \\ \mathbf{1}^\top x = 1 \end{cases} \end{aligned} \tag{5.2}$$

Maximum Sharpe Ratio Portfolio (MSRP)

Maximize Sharpe ratio with self-financial constraints

$$\begin{aligned} &\max_x \Big\{\mathrm{SR}(x)\Big\}, \\ \text{s.t. } &\begin{cases} \mathbf{1}^\top x = 0, \\ \|x\|_1 \leq K, \end{cases} \end{aligned}$$

the latter constraint being a leverage restriction. Performances and constraints may also be combine as

$$\begin{aligned} &\min_x \Big\{\lambda \mathrm{CVaR}_\beta(x) - \mu(x)\Big\}, \\ \text{s.t. } &\begin{cases} \mathbf{1}^\top x = 1, \\ \|x\|_1 \leq K. \end{cases} \end{aligned}$$

Here, λ is a parameter controlling the investor risk-aversity.

Furthermore, we may consider portfolio optimization problems as multi-objective problems: Let $f : \mathbb{R}^n \to \mathbb{R}^2$ be the function $f(x) = \Big(\mathcal{P}(x), \mathcal{R}(x)\Big)$, where $\mathcal{P}(x)$ is a performance measure

(that we want to minimize) of portfolio x and $\mathcal{R}(x)$ is a risk measure of x. We want to solve the problem

$$\min_x \Big\{ f(x) = \Big(\mathcal{P}(x), \mathcal{R}(x)\Big) \Big\},$$
$$\text{s.t.} \ \Big\{ \mathbf{1}^\top x = 1.$$

In all these problems we want to optimize a performance measure. In the remainder of this chapter we present a new approach with a risk allocation perspective.

5.3 Risk Parity

Markowitz's portfolio was never fully embraced by practitioners, among other reasons since, considers only the overall risk of the portfolio, disregarding diversification risks (i.e., concentrates too much risk in few assets, which is undesirable as observed during the 2008 financial crisis). In order to avoid this problem, one solution are to consider the risk parity portfolio.

Risk parity is an approach to portfolio management that focuses on risk, rather than capital allocation. The risk parity approach asserts that, when asset allocations are adjusted to the same risk level, the portfolio may achieve a higher Sharpe ratio as well as being more resistant to market downturns.

While the minimum variance portfolio (MVP) tries to minimize the variance (with the disadvantage the portfolio risk may be concentrated in only a few assets), the risk parity portfolio distributes the weights, so that the risk contributions of all assets (or asset class, such as bonds, stocks, real estate, etc.) are the same.

In order to define a risk-based allocation strategy, it is necessary to define how the risk of each asset affects the overall risk of the portfolio. Let $x^\top = (x_1, x_2, \ldots, x_n)$ be a portfolio with n assets and $\mathcal{R}(x)$ be a differentiable, homogeneous risk measure of x. We have

$$\mathcal{R}(x) = \frac{d}{d\lambda}\mathcal{R}(\lambda x) = \sum_{i=1}^{n} x_i \frac{\partial \mathcal{R}(x)}{\partial x_i}.$$

We define the risk contribution of asset i as

$$\mathcal{RC}_i(x) = x_i \frac{\partial \mathcal{R}(x)}{\partial x_i}.$$

Therefore, the risk may be written as follows

$$\mathcal{R}(x) = \sum_{i=1}^{n} \mathcal{RC}_i(x)$$

which is known as **Euler's Allocation Principle**.

The volatility as well as CVaR_β satisfies the properties required by the risk measure, $\mathcal{R}$, defined above. However, VaR_β satisfies these properties only in the Gaussian case. When asset returns follows a Normal Distribution, by Remark 5.2, we have that $\mathcal{R}(x) = \sigma(x) = \sqrt{x^\top \Sigma x}$, it follows that

$$\mathcal{RC}_i(x) = x_i \frac{\partial \sigma(x)}{\partial x_i} = \frac{x_i (\Sigma x)_i}{\sqrt{x^\top \Sigma x}}.$$

It may be interesting for the investor to choose different risk levels for different assets. In this case the investor choose the percentage of risk, b_i, that each asset should have in the portfolio, that is,

$$\mathcal{RC}_i(x) = b_i \mathcal{R}(x), \tag{5.3}$$

$b_i \geq 0$ for all i, and $\mathbf{1}^\top b = 1$, with $b^\top = (b_1, b_2, \ldots, b_n)$. A portfolio distribution based on equation (5.3) is called a *risk budgeting portfolio* (RBP). When $b_i = 1/n$, for all i, the distribution is called *risk parity portfolio* (RPP) or *equal risk portfolio* (ERP).

In general, find the risk budgeting portfolio consists in solving the following non-linear system

$$\begin{aligned} &\mathcal{RC}_i(x) = b_i \mathcal{R}(x), \\ &\text{s.t.} \begin{cases} b_i \geq 0, \\ x_i \geq 0, \\ \mathbf{1}^\top b = 1, \\ \mathbf{1}^\top x = 1. \end{cases} \end{aligned}$$

Kaya and Lee (see [51]) show that, in the Gaussian case, the risk budgeting portfolio may be found solving the optimization problem

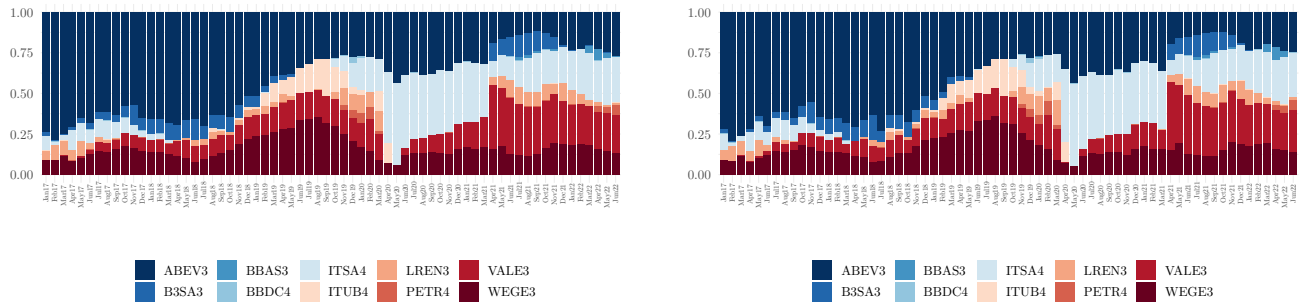
$$\begin{aligned} &\min_{x \geq \mathbf{0}} \{-b^\top \ln(x)\}; \\ &\text{s.t.} \begin{cases} \sigma^2(x) \leq \sigma_0, \\ \mathbf{1}^\top x = 1, \end{cases} \end{aligned} \tag{5.4}$$

σ_0 being a volatility target.

5.4 A Backtest Study

In this section we present an application for Algorithm 3.1, a backtest study of a portfolio build with the 10 biggest companies on the brazilian market: Ambev S.A. (ABEV3), B3 S.A. (B3SA3), Banco do Brasil S.A. (BBAS3), Banco Bradesco S.A. (BBDC4), Itaúsa S.A. (ITSA3), Itaú Unibanco S.A. (ITUB4), Lojas Renner S.A. (LREN3), Petróleo Brasileiro S.A (PETR4), Vale S.A. (VALE3) and WEG S.A. (WEGE3), in parentheses we have the tickers of each company. We did a backtest, building 2 portfolios with monthly rebalance. One was the minimum variance portfolio (MVP), and the other the risk parity portfolio (RPP), obtained solving optimization problems (5.1) and (5.4), respectively. Data pertaining to the period between January 2016 and June 2022, was taken from Yahoo Finance (`http://finance.yahoo.com`). For rebalancing, the volatilities were calculated using data from the previous 12 months. 2016 year data was used only for past volatilities calculations.

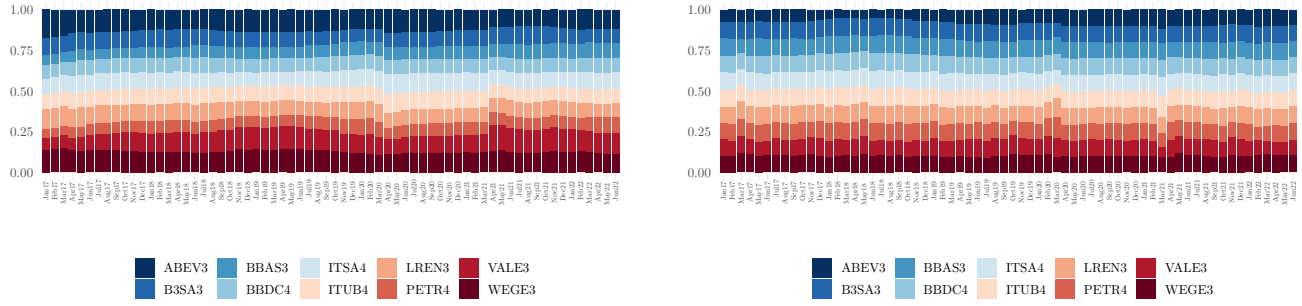
Generally, MVP is rarely used in practice, one of the reasons resides in the fact that the weights (and consequently the risks), may be concentrated in a few assets. Figure 5.2 shows the weight distribution as well as the risk contribution of stocks during the MVP backtest study. We note that the MVP concentrates capital in few assets and consequently also concentrates risk in these assets.



(a) Weigths (b) Risks

Figure 5.2: Monthly distribution of MVP.

A RPP has the weights distributed with the purpose of having a uniform risk distribution, Figure 5.3 shows the distribution of weights and risks in our backtest study.



(a) Weigths (b) Risks

Figure 5.3: Monthly distribution of RPP.

The total return of the RPP was greater than the MVP in the period studied. While MVP had a total return of 52%, the accumulated RPP return was 119% (44% higher than the MVP).

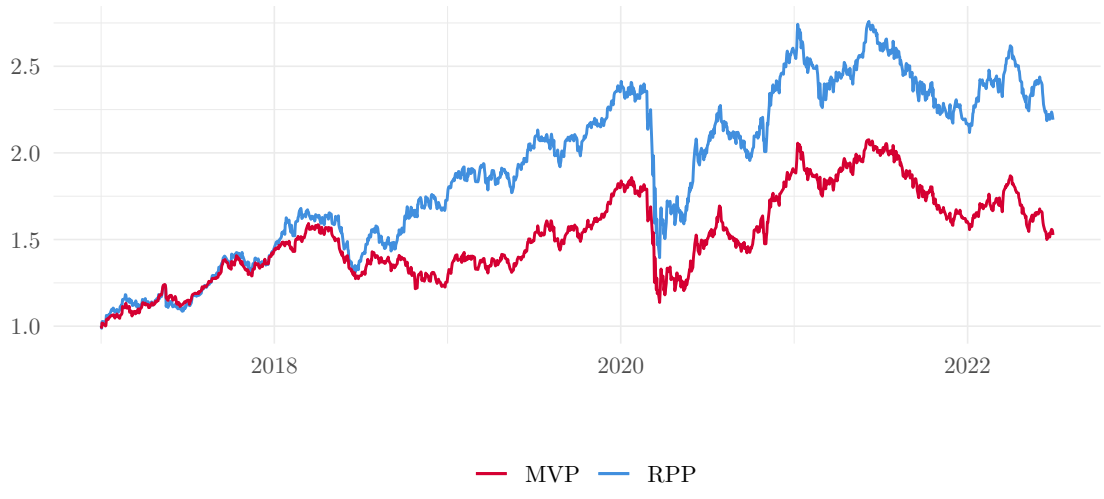


Figure 5.4: MVP and RPP accumulated returns: January 2017 − June 2022.

The MVP volatility was slightly smaller than the RPP. However, considering the zero risk-free rate, the RRP Sharpe ratio was bigger than the MVP.

	MVP	RPP
Annualized Return	0.0785	0.1505
Annualized Std Dev	0.2419	0.2637
Annualized Sharpe (Rf=0%)	0.3246	0.5708

Table 5.1: Annualized returns: January 2017 - June 2022.

We remind that after 2020, due to the COVID-19 pandemic, the volatility of all assets has exploded, so we separate two period in our study, from 2017 until 2019 and after 2020. We may observe that the RPP performed much better (135% gain) than the MVP (79% gain) in the period between January 2017 and December 2019. Besides, MVP had a great annualized Sharpe ratio of 1.20, however, lower than RPP annualized Sharpe ratio (1.58).

	MVP	RPP
Annualized Return	0.2107	0.3212
Annualized Std Dev	0.1744	0.2027
Annualized Sharpe (Rf=0%)	1.2087	1.5851

Table 5.2: Annualized returns: January 2017 - December 2019.

(a) January 2017 − December 2019.

(b) January 2020 − June 2022.

Figure 5.5: MVP and RPP accumulated returns.

From January 2020 to June 2022, MVP exhibited an accumulated 15.2% loss, while RPP reduced the loss to only 6.8%. Despite the loss in the period, the RPP improved the protection of the invested capital.

	MVP	RPP
Annualized Return	-0.0631	-0.0278
Annualized Std Dev	0.3046	0.3227
Annualized Sharpe (Rf=0%)	-0.2070	-0.0860

Table 5.3: Annualized returns: January 2020 - June 2022.

In our study, the RPP had a superior performance when compared to the MVP, both in high and low moments. Clearly, this study is not conclusive, but simply an example of the use of Algorithm 3.1.

Chapter 6

Conclusions

In this work, we study the SGP method to solve constrained convex optimization problems employing inexact projections onto the feasible set and a general nonmonotone line search. We expect that this work will contribute to the development of research in this field, mainly to solve large-scale problems when the computational effort of an iteration is associated with the projections onto the feasible set and the cost of evaluating the objective function. Indeed, the idea of using the inexactness in the projection as well as in the line search, instead of the exact ones, is particularly interesting from a computational point of view. In particular, it is noteworthy that the Frank-Wolfe method has a low computational cost per iteration resulting in high computational performance in different classes of compact sets, see [39, 50]. An issue that deserves attention is the search for new efficient methods such as the Frank-Wolfe's and Dykstra's methods that generate inexact projections.

Data availability

The codes supporting the numerical experiments are freely available in the Github repository, https://github.com/maxlemes/SGP, and the data used in backtest study are available in https://github.com/maxlemes/teseAcademica.

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