Dot products

Warm up

Let's start with a few natural questions about vectors that arise in the world.

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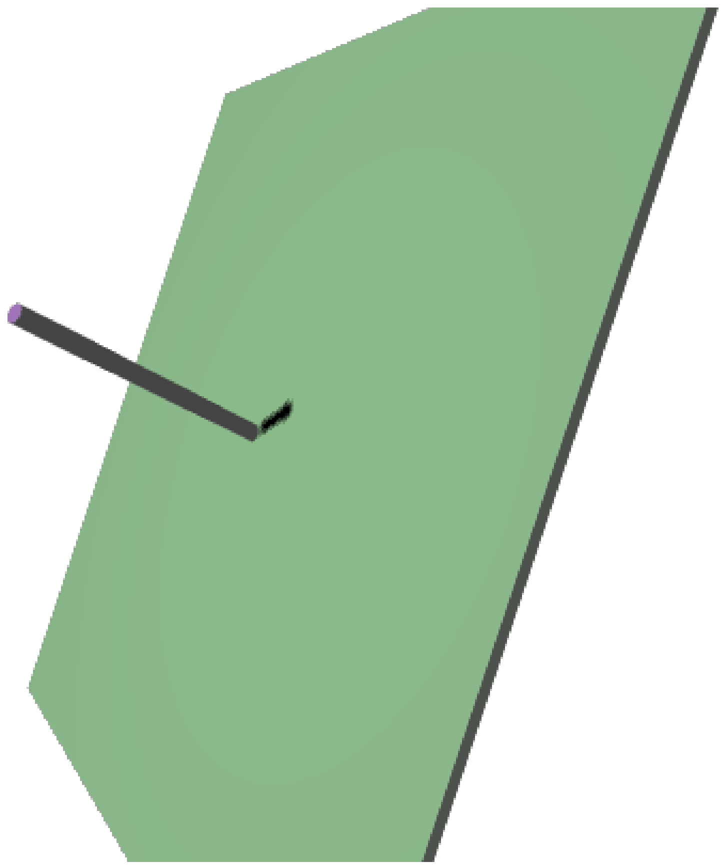
Question: a light shines perpendicularly down onto the plane $x + y + z = 0$. Can we calculate the shadow of the vector $\langle 3, 4, 5 \rangle$ in the plane? (Natural question for computer graphics, architecture, etc.)



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Similar: a light shines perpendicularly down onto the plane $z = 0$. How can we calculate the shadow of the vector $\langle 3, 4, 5 \rangle$ in the plane?

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Answer:

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Answer: projecting into the xy-plane just forgets about the z-component!

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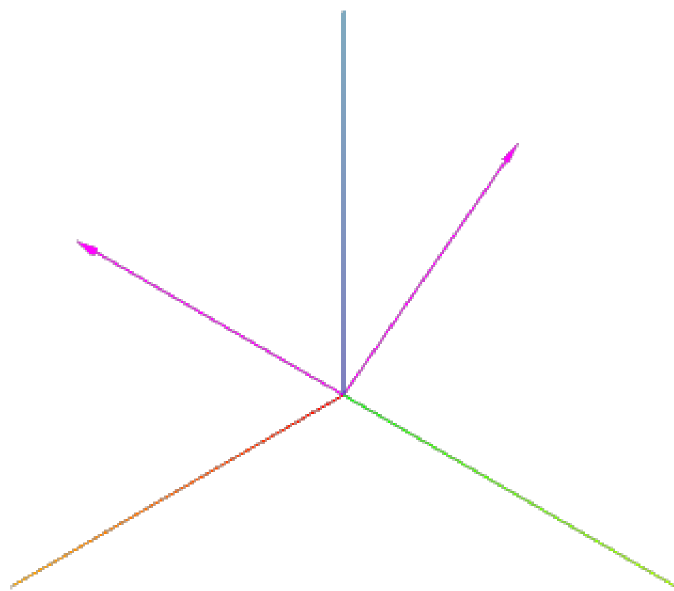
Question: given two vectors in $\mathbf{R}^3$, how can we compute the angle between them?



Today

We'll learn about a marvelous tool for answering these questions called the dot product.

As with components and addition of vectors, coordinates allow us to calculate the geometry.

As simple as they seem, the ideas we'll discuss here tap into one of the richest veins of ideas in mathematics.

Definition

The dot product of two vectors

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle \ \mathbf{b} = \langle b_1, b_2, b_3 \rangle$$

is the number

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

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Note: this is a number (scalar), not a vector!

Examples

$$\langle 1, 2, 3 \rangle \cdot \langle 3, 0, 1 \rangle = 1 \cdot 3 + 2 \cdot 0 + 3 \cdot 1 = 6$$
$$\langle 2, 5 \rangle \cdot \langle 5, -2 \rangle = 2 \cdot 5 + 5 \cdot (-2) = 0$$

Practice

Compute these dot products (build a data set for your inner child!):

- $\langle 1, 0, 0 \rangle \cdot \langle 2, 3, 7 \rangle$
- $\mathbf{i} \cdot \mathbf{j}$ (you know what $\mathbf{i}$ and $\mathbf{j}$ are because you read the book!)
- $\mathbf{i} \cdot \mathbf{i}$
- take a point (x, y, z) in the plane $x + y + z = 0$ and compute $\langle x, y, z \rangle \cdot \langle 1, 1, 1 \rangle$

Observations

$\mathbf{i}$ and $\mathbf{j}$ are perpendicular, and $\mathbf{i} \cdot \mathbf{j} = 0$

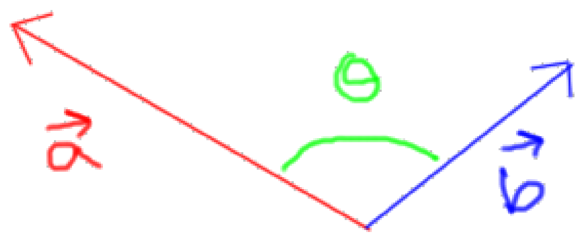
$\mathbf{i}$ has length 1, and $\mathbf{i} \cdot \mathbf{i} = 1$

Coincidence?

Sweet Theorem

Given two vectors $\mathbf{a}$ and $\mathbf{b}$, the angle θ between them satisfies

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\theta).$$



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Since by convention we always take the angle between two vectors to lie between 0 and π radians, this determines θ uniquely.

This theorem shows how the dot product relates to both angle and length

Let's see how we can leverage this for both angles and lengths.