Lecture 17

Behind us

- Differentials
- Critical points

Ahead

Today: local max/min

Wednesday: double integrals, I

Read Sections 14.7, 15.1 *We cannot cover everything in lecture or section, but you will need it all for the rest of your lives!*

Homework due tomorrow at 11 PM

Questions!

From last time

Find all critical points ((a, b) such that $f_x(a, b) = 0 = f_y(a, b)$) for the following functions.

- $f(x, y) = x^2 + y^2$
- $f(x, y) = x^2 - y^2$
- $f(x, y) = \sin(x)\cos(y)$
- $f(x, y) = x^y$
- $f(x, y) = x^3 + y^3 + x + y$

Can you tell which of these critical points are local maxima or minima?

The critical points:

$f(x, y) = x^2 + y^2$: $(0, 0)$

$f(x, y) = x^2 - y^2$: $(0, 0)$

$f(x, y) = \sin(x)\cos(y)$: all $\left(m\,\frac{\pi}{2}, n\,\frac{\pi}{2}\right)$ with one of (m, n) odd and the other even (cool!)

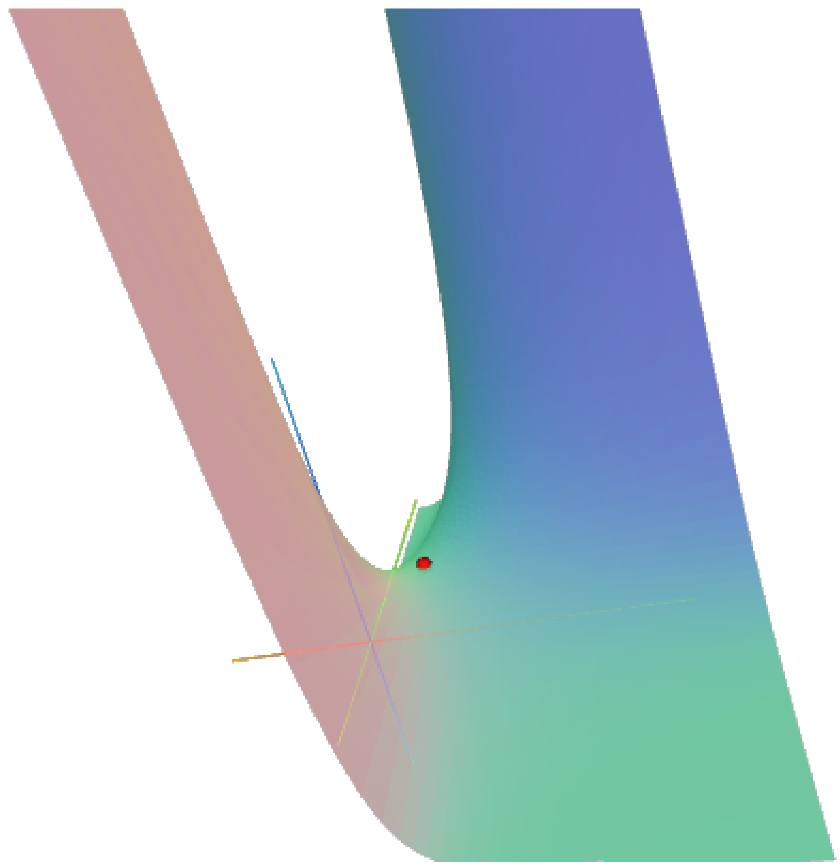
$f(x, y) = x^y$: $(1, 0)$

$f(x, y) = x^3 + y^3 + x + y$: none

What about local max/min?

exponential

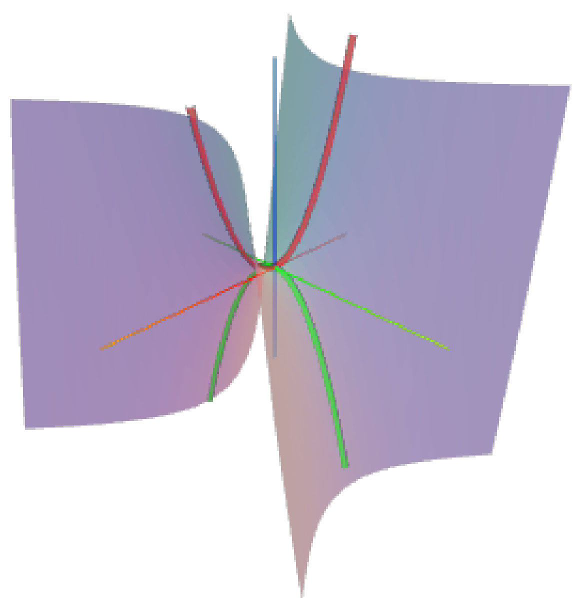
x^y



The hyperboloid: a pure example

It's hard to find a purer example of a critical point that is neither max nor min than the critical point of the hyperboloid.

Near that point the shape curves up in one direction and down in the other.
This is called a saddle point.



Contemplating the egg carton

The egg carton has an intriguing feature: sometimes the critical points are at local max/min, but sometimes they're not!

How can we analyze the difference?

Observation: near one of the funny points, the shape looks a like like a hyperboloid: going up in one direction and down in another.

What happened for functions of one variable?

> The sign of the second derivative ruled the roost.
>
> Positive second derivative means local min, negative means local max.
>
> Vanishing second derivative left us alone with our wits.

Multivariable fix?

There is an additional piece of information needed to account for both variables.
This is not magic, but it looks like before you know enough theory:

Definition: the Hessian determinant of $f(x, y)$ is the function

$$D(x, y) = f_{xx} f_{yy} - f_{xy}^2.$$

Here, as you learned from the book, f_{xx}, f_{yy}, and f_{xy} denote second derivatives.

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} \qquad f_{yy} = \frac{\partial^2 f}{\partial y^2} \qquad f_{xy} = \frac{\partial^2 f}{\partial y \partial x}$$

The order of the subscripts is the order in which the derivatives are taken. Be careful: the "denominator" of the partial derivatives will have the variables in the reverse order!

The second derivative test

Here's the love:

Suppose f has continuous second partial derivatives and (a, b) is a critical point. Then

> if $D > 0$ and $f_{xx}(a, b) > 0$ then (a, b) is a local minimum;
> if $D > 0$ and $f_{xx}(a, b) < 0$ then (a, b) is a local maximum;
> if $D < 0$ then (a, b) is a saddle point.

Achtung! This does not cover all cases!

For example, if $D = 0$ we get no love at all from this.

Example:
$f(x, y) = x^2 + y^2$

Unique critical point: $(0, 0)$

$D(x, y) = f_{xx} f_{yy} - f_{xy}^2 = 2 \cdot 2 - 0 = 4 > 0$

$f_{xx} = 2 > 0$

Conclusion: local minimum! (Phew!!)

Do one!

Describe which critical points of $f(x, y) = \sin(x)\cos(y)$ are local minima, local maxima, and saddle points.

$D = f_{xx} f_{yy} - f_{xy}^2$

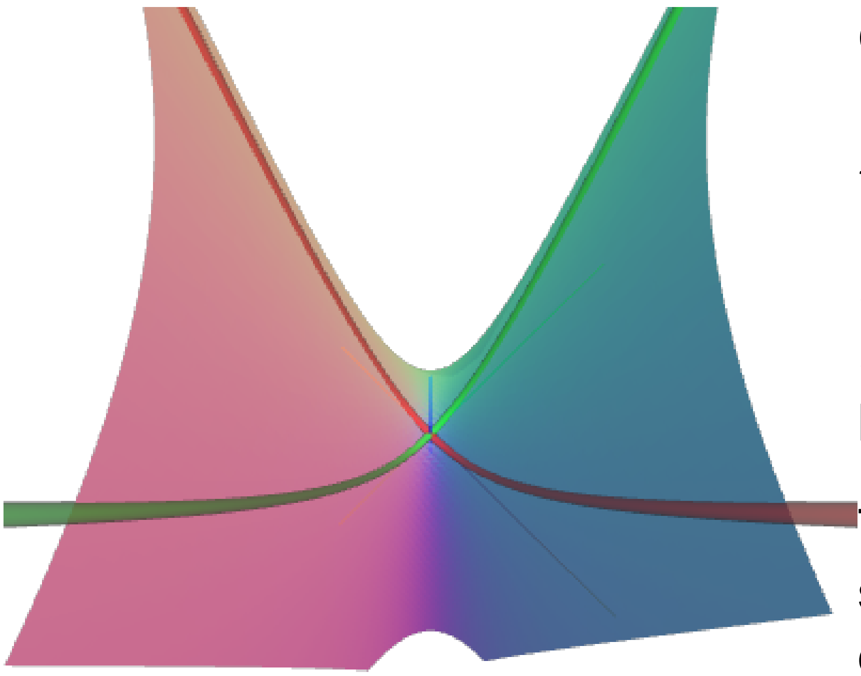
$D > 0$, $f_{xx} < 0$ implies local max (but not other way)

$D > 0$, $f_{xx} > 0$ implies local min (but not other way)

$D < 0$ means saddle point

Second derivatives alone are not enough

Consider the function $f(x, y) = x^2 + 4xy + y^2$.



Critical point: $(0, 0)$

$D(0, 0) = -12$

$f_{xx}(0, 0) = 2$ and $f_{yy}(0, 0) = 2$, both positive

The graph shows why: the saddle is oriented differently!

Another one for fun

Consider the function $f(x, y) = x^y$ with critical point $(1, 0)$.

What is the Hessian determinant $D(1, 0)$?

Is $(1, 0)$ a local max, a local min, or neither?

Is there an "easy" explanation for the previous part?

This matters in reality, too

Here is an example of a problem that you might approach using the methods we've developed today: what is the smallest distance between the point $(1, 2, 3)$ and the paraboloid $z = x^2 + y^2$?

> Function to be minimized:
> $d(x, y, z) = (x-1)^2 + (y-2)^2 + (z-3)^2$ (the square of the distance; why?)
>
> Relation: $z = x^2 + y^2$
>
> So now trying to minimize:
> $f(x, y) = (x-1)^2 + (y-2)^2 + (x^2 + y^2 - 3)^2$
>
> You can try this in the comfort of your room.

Similar: look at Example 6 in Section 14.7 of your book.

Next time: *integrals!*