Lecture 15

Behind us

- Functions of multiple variables
- Contour plots/level curves
- Level surfaces

Ahead

Today: partial derivatives and tangent planes

Friday: differentials and optimization

Read Sections 14.3, 14.4, 14.7. *This is for your own safety.*

Homework due tomorrow at 11 PM

Questions!

Medical emergency

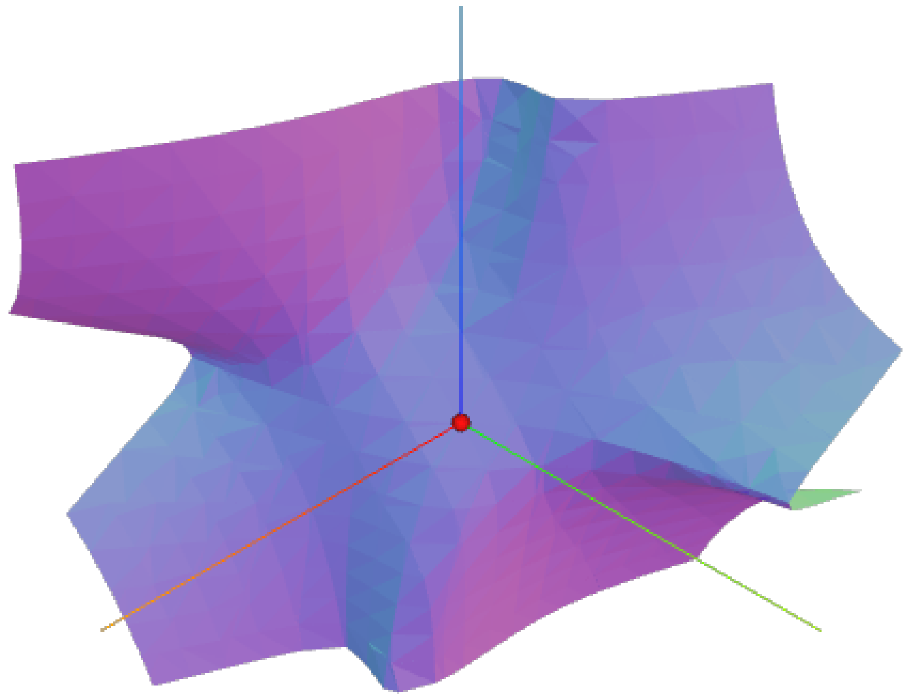
Nedwina has defective ankles, but like a true Seattleite she has decided to climb Mt. Rainier. Her doctor says she cannot walk on any surface whose angle with the xy-plane is greater than $\pi/3$.

Nedwina knows a nerd who modeled her favorite part of Rainier by the equation $x^3 + z^2(y^2 + y^3) + x + y + z = 0$.

Can Nedwina stand at the point $(0, 0, 0)$ without violating her doctor's orders?

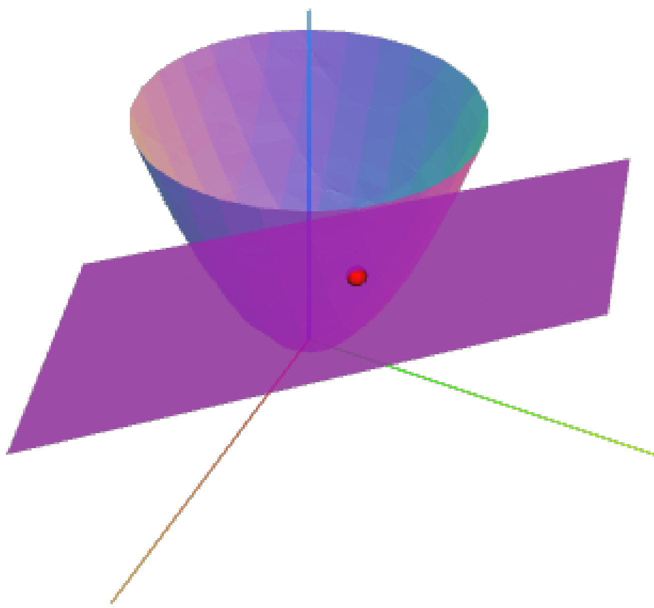
What is the best linear approximation to the surface around the point $(0, 0, 0)$?

Here's the model with $(0, 0, 0)$ indicated



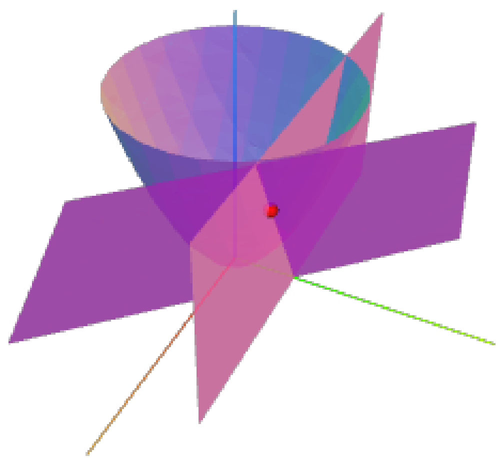
A simpler situation

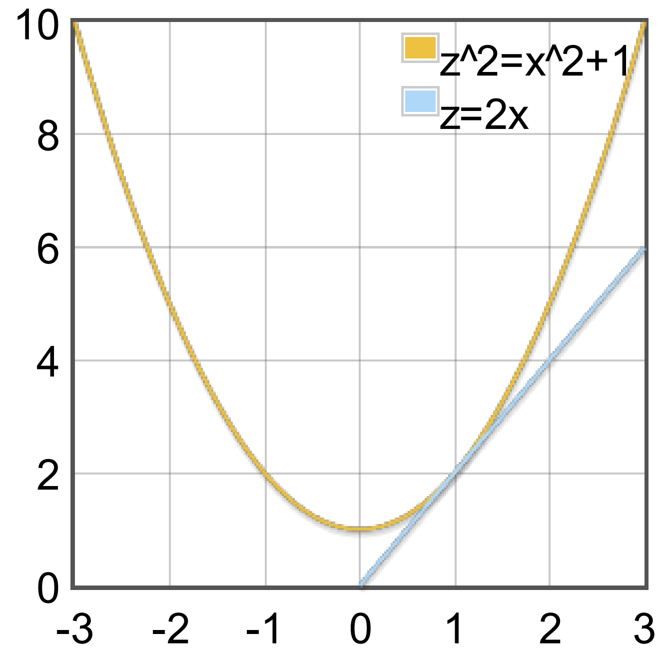
Find the tangent plane to the paraboloid $z = x^2 + y^2$ at the point $(1, 1, 2)$.



A silly observation

If we slice the whole package with a plane, we go from a surface and tangent plane to a curve and tangent line!





Partial derivatives

This process of restricting attention to one variable and taking derivatives "with respect to that variable" is called taking partial derivatives.

Formal definition: given a function of two variables $f(x, y)$, the partial derivatives with respect to x and y at a point (a, b) are defined as follows.

$$\frac{\partial f}{\partial x} = f_x(a, b) = \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h}$$

$$\frac{\partial f}{\partial y} = f_y(a, b) = \lim_{h \to 0} \frac{f(a, b+h) - f(a, b)}{h}$$

Rules

- Treat one variable as a constant
- Differentiate with respect to the other one
- Rejoice

Example

For example, consider the function $f(x, y) = x^2 + y^2$. To compute the partial derivative with respect to x, we treat y as a constant and do the usual one-variable differentiation:

$$\frac{\partial f}{\partial x} = 2x.$$

Do some! Compute the partial derivatives with respect to x and y of these functions.

$$f(x, y) = \sin(x)\cos(y)$$
$$f(x, y) = x^y$$
$$f(x, y) = y^5$$

Tangent planes to graphs

One of the main uses of partial derivatives is in producing tangent planes.

Theorem: for a function $f(x, y)$, the tangent plane to the graph $z = f(x, y)$ at the point (a, b, c) is given by the equation

$$z - c = f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

Example: $f(x, y) = x^2 + y^2$, $(a, b, c) = (1, 1, 2)$, get the plane

$$z - 2 = 2(x - 1) + 2(y - 1).$$

Why does this make sense?

> If we take a vertical or horizontal trace, the tangent plane should give the tangent line. Does it?
>
> The partial derivatives seem to capture just a vertical and horizontal trace. Is that really enough?

Try one!

Calculate the tangent plane to the graph of the function
$$f(x, y) = \sin(x)\cos(y)$$
at the point $(a, b, \sin(a)\cos(b))$.

General formula for $f(x, y)$ at (a, b, c):

$$z - c = f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

Big but

Our nerd's Rainier model does not express z as a function of x and y:

$$x^3 + z^2(y^2 + y^3) + x + y + z = 0$$

What should we do?

> Think of z as an implicit function of x and y!
>
> Do you remember how implicit differentiation works?

Implicit partial derivatives

Follow the procedure you learned in the days of yore. Let's focus on partial derivatives with respect to x.

> Pretend z is a function of x and y is a constant, so we have a partial derivative $\partial z/\partial x$.
>
> Use the product rule, chain rule, etc., to take the derivative of the relation.
>
> Solve for $\partial z/\partial x$ in terms of x, y, z.

Example: for the cone $z^2 = x^2 + y^2$ we get $2zz_x = 2x$, so for $z \neq 0$ we have

$$\frac{\partial z}{\partial x} = \frac{x}{z}\,.$$

Tangent plane to cone

Continuing the cone example, at a point (a, b, c) of the circular cone $z^2 = x^2 + y^2$, we have $z_x(a, b) = a/c$ and $z_y(a, b) = b/c$. Thus, the tangent plane is

$$z - c = \frac{a}{c}(x - a) + \frac{b}{c}(y - b),$$

or, equivalently,

$$c(z - c) = a(x - a) + b(y - b).$$

For kicks, let's expand that:

$$cz - c^2 = ax - a^2 + by - b^2.$$

Since $c^2 = a^2 + b^2$, we can simplify this to $cz = ax + by$. Nice!

Bizarre note: if we scale a, b, and c by the same number, the tangent plane does not change. Is there a reason for this?

Your turn!

Find the tangent plane to Mt. Rainier at $(0,0,0)$.

The equation is $x^3 + z^2(y^2 + y^3) + x + y + z = 0$.

Is the plane too steep for Nedwina?

Next time: *differentials, optimization!*