Loaded!

Please proceed to the calculus wonderland by pressing the right arrow key or clicking the right arrow that is visible when you move your mouse over the pig.

Instructions

These slides should work with *any* modern browser: IE 9+, Safari 5+, Firefox 9+, Chrome 16+.

- Navigate with arrow keys; you may need to give the window focus by clicking outside the lecture frame (the pig) for key commands described throughout this slide to work properly.
- Press M to see a menu of slides. Press G to go to a specific slide. Press W to toggle scaling of the deck with the window. If scaling is off, slides will be 800 by 600; it is off by default.
- Use left click, middle click, right click or hold A, S, D on the keyboard and move the mouse to rotate, scale, or pan the object.

If your browser or hardware does not support WebGL, interacting with models will be *very* slow (and in general models can get CPU-intensive). Navigate to a slide away from any running model to stop model animation.

Math 126

Lecture 4

Behind us

- Dot product
- Angles between vectors
- Projection onto a line

Homework due tomorrow at 11 PM

Questions?

Today

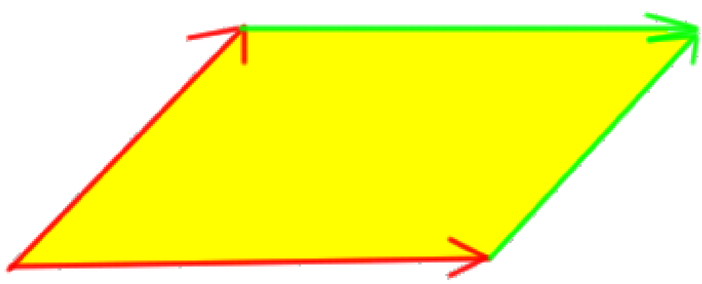
Cross products and the parallelepipeds they love

Read Section 12.4 of the book. *We will cover everything in lecture or section. That's OK.*

Warm up

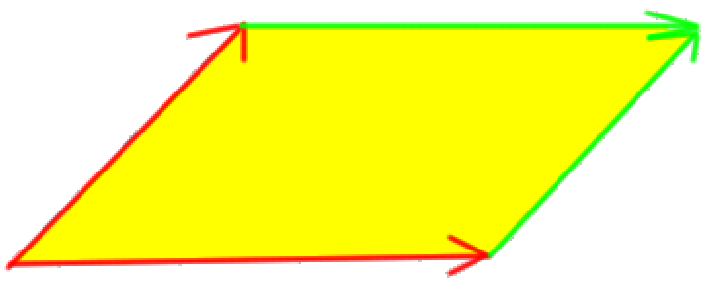
Question: two vectors span a parallelogram.

- What is the area of the parallelogram?
- What is a unit vector perpendicular to the parallelogram?



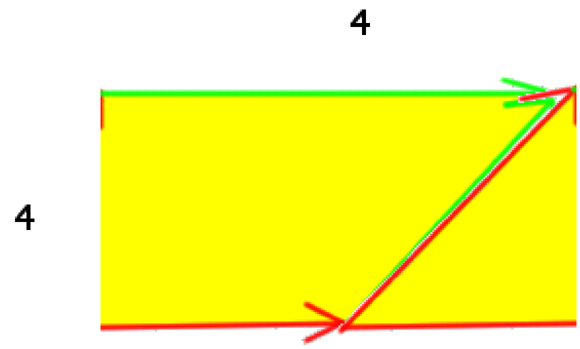
Warm up

Example: how about the parallelogram spanned by the two vectors $\langle 4, 0\rangle$ and $\langle 3, 4\rangle$ in the xy-plane?



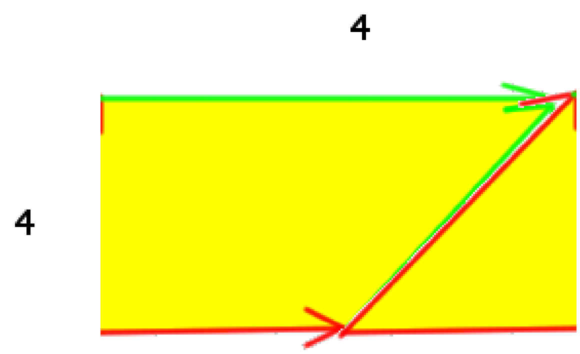
Warm up

Example: how about the parallelogram spanned by the two vectors $\langle 4, 0 \rangle$ and $\langle 3, 4 \rangle$ in the xy-plane?



Warm up

Example: how about the parallelogram spanned by the two vectors $\langle 4, 0 \rangle$ and $\langle 3, 4 \rangle$ in the xy-plane?



The area is thus $4 \cdot 4 = 16$.

Magic on the way

It turns out that there is a way to make a single vector encoding both the perpendicular direction and area of the parallelogram.

Magic on the way

It turns out that there is a way to make a single vector encoding both the perpendicular direction and area of the parallelogram.

Secret sauce: the cross product

Definition

The cross product of two vectors

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle \ \mathbf{b} = \langle b_1, b_2, b_3 \rangle$$

is the vector

$$\mathbf{a} \times \mathbf{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle.$$

Definition

The cross product of two vectors

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle \ \mathbf{b} = \langle b_1, b_2, b_3 \rangle$$

is the vector

$$\mathbf{a} \times \mathbf{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle.$$

Note: unlike the dot product, this *is* a vector!

Computing with a determinant

This is a slight abuse of notation, but we have

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Computing with a determinant

This is a slight abuse of notation, but we have

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

You read about how to compute this in the book. Let's review it!

Computing with a determinant

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \mathbf{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Computing with a determinant

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\begin{aligned} &= \mathbf{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\ &= \mathbf{i}(a_2 b_3 - a_3 b_2) - \mathbf{j}(a_1 b_3 - a_3 b_1) \\ &\qquad + \mathbf{k}(a_1 b_2 - a_2 b_1) \end{aligned}$$

Example

$$
\begin{aligned}
\langle 1,1,0\rangle \times \langle 0,1,1\rangle \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix}
\end{aligned}
$$

Example

$$\begin{aligned}
\langle 1,1,0\rangle \times \langle 0,1,1\rangle
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} \\
&= \mathbf{i}\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}
\end{aligned}$$

Example

$$
\begin{aligned}
\langle 1,1,0\rangle \times \langle 0,1,1\rangle
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} \\
&= \mathbf{i}\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \\
&= \mathbf{i} - \mathbf{j} + \mathbf{k} = \langle 1,-1,1\rangle
\end{aligned}
$$

Brain squeeze

Compute these cross products and draw the resulting vectors:

- $\langle 1, 2, 3 \rangle \times \langle 3, 6, 9 \rangle$
- $\mathbf{i} \times \mathbf{i}$
- $\mathbf{i} \times \mathbf{j}$
- $\mathbf{j} \times \mathbf{i}$
- $\mathbf{j} \times \mathbf{k}$
- $\mathbf{k} \times \mathbf{i}$

Sweet Theorem, Cross Product Version

Given two vectors $\mathbf{a}$ and $\mathbf{b}$ with angle θ

the vector $\mathbf{a} \times \mathbf{b}$ is $\perp$ to both $\mathbf{a}$ and $\mathbf{b}$;

the list $\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b}$ satisfies the right-hand rule;

we have $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin(\theta)$;

$|\mathbf{a} \times \mathbf{b}|$ equals the area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$.

Sweet Sweet Theorem, Cross Product Version

Given two vectors $\mathbf{a}$ and $\mathbf{b}$ with angle θ

- the vector $\mathbf{a} \times \mathbf{b}$ is $\perp$ to both $\mathbf{a}$ and $\mathbf{b}$;
- the list $\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b}$ satisfies the right-hand rule;
- we have $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin(\theta)$;
- $|\mathbf{a} \times \mathbf{b}|$ equals the area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$.

Ties the cross product to area and orientation .

Do one

Example: find the area of the parallelogram spanned by $\langle 4, 0, 0 \rangle$ and $\langle 3, 4, 0 \rangle$

Compute the cross product!

> Can you predict the direction of the cross product vector without calculating anything?
>
> Can you predict the magnitude of the cross product without calculating anything?

I did it

Example: find the area of the parallelogram spanned by $\langle 4, 0, 0\rangle$ and $\langle 3, 4, 0\rangle$

Compute the cross product!

$$\langle 4, 0, 0\rangle \times \langle 3, 4, 0\rangle = \langle 0, 0, 16\rangle$$

Phew!

Hmmm.... Maybe drawing a picture of this (the parallelogram and the cross product) would help you cement this in your mind.

Exploratory probe

Suppose you model a surface with a computer: you can't do continuous or smooth things, so the surface is modeled as having a ton of faces (like the facets of a demonic gem). These faces might be described with coordinates in $\mathbf{R}^3$. How can you find the normal vector to one of the faces?

How many ways are there to make a normal vector? Is it unique? Can you use them to distinguish "inside" from "outside"?

Use the internet to find out if these things come up outside of math class!

Next time: *lines and planes!*