Lines and planes: doing it

Planes described using vectors

If $\mathbf{u} = \langle \alpha, \beta, \gamma \rangle$ is perpendicular to plane P and $u = (a, b, c)$ is a point of P, we can describe P like this:

$$P = \{(x, y, z) : \langle x - a, y - b, z - c \rangle \cdot \mathbf{u} = 0\}$$

- Get a linear equation: $\alpha(x - a) + \beta(y - b) + \gamma(z - c) = 0$
- Any linear equation gives a plane.

Planes described using vectors

Describe the plane $x - 3y + 47z - 28 = 0$ using vectors.

- Normal vector: $\langle 1, -3, 47 \rangle$. How did I know?
- Trick: always just use the coefficients of x, y, and z
- To translate: find one solution by eyeballs. A solution: $(-16, 1, 1)$.
- So the plane is the set of endpoints of vectors $\mathbf{v}$ such that
$$(\mathbf{v} - \langle -16, 1, 1 \rangle) \cdot \langle 1, -3, 47 \rangle = 0.$$
- Another way to say it: it is what you get when you take the set of vectors perpendicular to $\langle 1, -3, 47 \rangle$ and translate them all by $\langle -16, 1, 1 \rangle$ (and then just keep the set of endpoints)

Practice

Describe the plane $3x - 4y - 5z = 6$ using vectors.

Who cares?

- Using this approach, you can prove that any plane is the set of solutions of a linear equation in x, y, z (see book!).
- This gives us a way to get a grip on the intersection of two planes.

Example

- Describe the intersection of the planes $x - 2y - z = 0$ and $2x - y + z = 6$.
- Perpendicular vectors: $\langle 1, -2, -1\rangle$ and $\langle 2, -1, 1\rangle$
- Common solution: $(2, 0, 2)$
- Thus, the line of intersection is the set of vectors $\mathbf{v}$ such that
$$(\mathbf{v} - \langle 2, 0, 2\rangle) \cdot \langle 1, -2, -1\rangle = 0$$
  and
$$(\mathbf{v} - \langle 2, 0, 2\rangle) \cdot \langle 2, -1, 1\rangle = 0.$$
- The vector $\mathbf{v} - \langle 2, 0, 2\rangle$ is perpendicular to both: cross product!
- The line is just the endpoints of vectors of the form
$$\langle 2, 0, 2\rangle + t\langle 1, -2, -1\rangle \times \langle 2, -1, 1\rangle,$$
  where t ranges over all scalars. Parametric equations!

Last step: expand cross product

To describe $\langle 2, 0, 2\rangle + t\langle 1, -2, -1\rangle \times \langle 2, -1, 1\rangle$, let's expand:

- $\langle 1, -2, -1\rangle \times \langle 2, -1, 1\rangle = \langle -3, -3, 3\rangle$
- So the line is given by the endpoints of the vectors $\{\langle 2, 0, 2\rangle + t\langle -3, -3, 3\rangle\}$
- Parametric form: $(x, y, z) = (2 - 3t, -3t, 2 + 3t)$
- As t varies, this traces out the line of intersection.

More practice

- Describe the intersection of $3x + 4y + 5z = 6$ and $y + z = 0$.
- Describe the intersection of $3x + 4y + 5z = 6$ and $6x + 8y + 10y = 12$.
- Describe the intersection of $3x + 4y + 5z = 6$ and $9x + 12y + 15z = 17$.