Select a lecture to view

Choose a lecture from the dropdown. The "Web" button takes you to the lecture page. This is designed to be viewed in fullscreen mode (F11 on most browsers). The "PDF" button takes you to a printable version.

10: Arc length and curvature | Web | PDF

Instructions for navigating slides are on the next page. To see them, swipe from right to left (if you have a touchscreen) or click the arrow that appears when you hover over the pig with your mouse. Keyboard navigation is also available, as explained in the Instructions page.

Go forward to enjoy the lecture. Come back or reload page to choose a new lecture to view.

Instructions

These slides should work with *any* modern browser: IE 9+, Safari 5+, Firefox 9+, Chrome 16+.

- Navigate with arrow keys; you may need to give the window focus by clicking outside the lecture frame (the pig) for key commands described throughout this slide to work properly.
- Press M to see a menu of slides. Press G to go to a specific slide. Press W to toggle scaling of the deck with the window. If scaling is off, slides will be 800 by 600; it is off by default.
- Use left click, middle click, right click or hold A, S, D on the keyboard and move the mouse to rotate, scale, or pan the object.

If your browser or hardware does not support WebGL, interacting with models will be *very* slow (and in general models can get CPU-intensive). Navigate to a slide away from any running model to stop model animation.

Lecture 10

Behind us

Differentiation and integration of vector-valued functions

Polar coordinates

Logistics

- Exam experience on Thursday

  Homework due tomorrow at 11 PM

Ahead

Today: arc length and curvature

("Today" is more than one lecture worth of material.)

Next: cleanup, test review

Read Sections 10.3, 13.3. *We don't cover everything in lecture or section. Failure to study might make you stupid.*

Questions!

Kangaroos++

A joey (baby kangaroo) is riding in her mother's pouch with a sophisticated inertial navigation system. She is too small to see out of the pouch, but her system records the velocity vector at any time.

The joey records the velocity at time t as

$$\mathbf{v}(t) = \langle 1, t, \sin(t) \rangle$$

The joey starts at the point $(0, 0, 1)$

Last time: the position is $\mathbf{f}(t) = \langle t, \frac{1}{2}t^2, 2 - \cos(t) \rangle$

New question: How long is the joey's path between $t = 0$ and $t = 15$?

Distance by accretion

How can we calculate the distance travelled by the joey from $t = a$ to $t = b$?

Linear approximation to function near time $t = s$: $\mathbf{L}(t) = \mathbf{f}(0) + t\mathbf{f}'(s)$

For a small change $\Delta(t)$, the distance travelled is thus approximately $\Delta(s) \approx |\mathbf{f}'(t)|\Delta(t)$

The resulting Riemann sums approximate the integral

$$s(t) = \int_a^b |\mathbf{f}'(t)| dt.$$

Important Note: when you reverse direction, the distance still adds up!

Practical distance, I

If we have $\mathbf{f}(t) = \langle x(t), y(t), z(t) \rangle$ then the distance from the starting point $t = a$ to a variable time T is

$$s(T) = \int_a^T \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt.$$

$$s(T) = \int_0^T \sqrt{1 + t^2 + \sin^2(t)}\, dt$$

Evaluating this function seems rather mysterious in general. Using numerical methods, we get a concrete answer for $T = 15$:

$$s(15) = 115.255\ldots.$$

Practical distance, I

If we have $\mathbf{f}(t) = \langle x(t), y(t), z(t) \rangle$ then the distance from the starting point $t = a$ to a variable time T is

$$s(T) = \int_a^T \sqrt{(x'(t)^2 + (y'(t))^2 + (z'(t))^2)}\, dt.$$

Try one:

What is the distance of the cylinder path $\mathbf{f}(t) = \langle \cos(t), \sin(t), t/2\pi \rangle$ from $t = 0$ to $t = T$?

Move on only when ready for the answer!

Practical distance, I

If we have $\mathbf{f}(t) = \langle x(t), y(t), z(t) \rangle$ then the distance from the starting point $t = a$ to a variable time T is

$$s(T) = \int_a^T \sqrt{(x'(t)^2 + (y'(t))^2 + (z'(t))^2)}\, dt.$$

Plugging/chugging:

$$\int_0^T \sqrt{\frac{4\pi^2 + 1}{4\pi^2}}\, dt = \frac{T\sqrt{4\pi^2 + 1}}{2\pi}$$

Distance vs arc length

Important: the integrals above give the total distance traveled.

If you never reverse direction, you are also computing the length of the path, not just your choice of traversal.

The physical length of the path is the arc length.

Practical arc length, II

Here's a funny special case. Suppose that $|\mathbf{f}'(t)| = 1$for all t. Then the distance traversed from time 0 to time T is

$$s(T) = \int_0^T |\mathbf{f}'(t)| dt = \int_0^T 1 dt = T.$$

In typical circumstances (called smooth: $\mathbf{f}'(t)$ is never 0), one can reparametrize the curve using arc length instead of t.

Example: for the unit circle, the arc length parametrization is

$$\langle x, y \rangle = \langle \cos(s), \sin(s) \rangle.$$

Meaning: the arc length as s goes from 0 to T is T!

Practical arc length, II

A parametric vector function $\mathbf{f}(t)$ is parametrized by arc length or an arc length parametrization if the arc length traced by $\mathbf{f}$ between $t = a$ and $t = b$ is $b - a$

$$\int_a^b |\mathbf{f}'(t)|dt = b - a$$

How do you find these things?

- Start with a parametrization: $\mathbf{f}(t)$.
- Hypothetical reparametrization: $t = t(s)$
- Chain rule: $|\mathbf{f}'(t)| = |\mathbf{f}'(t)t'(s)|$; want $t'(s) = |\mathbf{f}'(t(s))|^{-1}$ for all s

Practical arc length, II

Example: circle of radius 2, parametrized by $\mathbf{f}(t) = \langle 2\cos(t), 2\sin(t)\rangle$

$$\mathbf{f}'(t) = \langle -2\sin(t), 2\cos(t)\rangle$$

$$|\mathbf{f}'(t)| = \sqrt{4\sin^2(t) + 4\cos^2(t)} = 2$$

So: solve $t'(s) = 1/2$, one solution is $t = s/2$

End result: arc length parametrization of circle of radius 2 is
$\mathbf{f}(s) = \langle 2\cos(s/2), 2\sin(s/2)\rangle$

Find the arc length parametrization of the circle of radius r.

Unit tangent vectors

Something remarkable happens when parametrizing curves by arc length:
every tangent vector has length 1.

We can do this more generally whenever we know that $\mathbf{f}'(t) \neq 0$

The unit tangent vector to the curve $\mathbf{f}(t)$ at a point $t = a$ is

$$\mathbf{T}(a) = \frac{\mathbf{f}'(a)}{|\mathbf{f}'(a)|}.$$

Example: for the joey with $\mathbf{f}'(t) = \langle 1, t, \sin(t), \rangle$ the unit tangent vector at time a is

$$\mathbf{T}(a) = \frac{1}{\sqrt{1 + t^2 + \sin^2(t)}} \langle 1, t, \sin(t) \rangle.$$

Practice

Compute the unit tangent to the parabola $y = x^2$ at the point (a, a^2).

Hint:
one way to do this is to parametrize the path first, say using $x = t$. Then calculate $\mathbf{f}'(t)/|\mathbf{f}'(t)|$. What if you parametrize (half of) the parabola as $\langle \sqrt{t}, t \rangle$ instead?

The benefits of all of this

Why work with arc length and unit tangents?

Capture intrinsic geometric information,
not artifacts of the choices we made in our description.

Recovers delicate static information about the shape.

For example: $\mathbf{T}'(t) \cdot \mathbf{T}(t) = 0$I.e., $\mathbf{T}'(t)$ is perpendicular to $\mathbf{T}(t)$.

Indeed,

$$0 = \frac{d}{dt}1 = \frac{d}{dt}|\mathbf{T}(t)| = \frac{d}{dt}(\mathbf{T}(t) \cdot \mathbf{T}(t)) = 2\mathbf{T}(t) \cdot \mathbf{T}'(t).$$

Meaning: $\mathbf{T}'(t)$, unlike the acceleration in general, is always changing the tangent to the curve in the most efficient way.

Curvature

The curvature of the smooth parametric curve $\mathbf{f}(t)$ is defined to be

$$\kappa(t) = \left|\frac{d\mathbf{T}(t)}{ds}\right|,$$

where s is the arc length function.

Since $s'(t) = |\mathbf{f}'(t)|$, we also have $\kappa(t) = |\mathbf{T}'(t)|/|\mathbf{f}'(t)|$

A lot to digest!

When $\mathbf{f}$ is already paramterized by arc length, this simplifies to

$$\kappa(s) = |\mathbf{f}''(s)|,$$

the acceleration.

In practice, this is *not* how you will compute it (because you won't have paths parametrized by arc length most of the time).

Examples

***The Circle of Radius r* and *The Parabola Fights Back*.**

The equations parametrizing with arc length: $\mathbf{f}(s) = \langle r\cos(s/r), r\sin(s/r)\rangle$

Thus,

$$\kappa(s) = \left|\left\langle -\frac{1}{r}\cos\left(\frac{s}{r}\right), -\frac{1}{r}\sin\left(\frac{s}{r}\right)\right\rangle\right| = \frac{1}{r}\,.$$

Makes sense:
the curvature of a circle of large radius is small.
After all, the path is basically a straight line!

What about for something like the parabola? Try it. You might consider using $\mathbf{f}(t) = \langle t, t^2\rangle$ and $\kappa = |\mathbf{T}'|/|\mathbf{f}'|$ together with your calculation of $\mathbf{T}$ from before.
What a mess!

Curvature in practice

Mathematicians have thought about this one pretty hard, and here is what turns out to happen:

The curvature of the smooth path parametrized by $\mathbf{f}(t)$ is

$$\kappa(t) = \frac{|\mathbf{f}'(t) \times \mathbf{f}''(t)|}{|\mathbf{f}'(t)|^3}.$$

Your book contains a proof!

We can dispatch the parabola $\mathbf{f}(t) = \langle t, t^2 \rangle$

$$\kappa(t) = \frac{|\langle 1, 2t, 0\rangle \times \langle 0, 2, 0\rangle|}{(1+4t^2)^{\frac{3}{2}}} = \frac{2}{(1+4t^2)^{\frac{3}{2}}}.$$

Do one!

Recall the motion of the joey: $\mathbf{f}(t) = \langle t, \frac{1}{2}t^2, 2 - \cos(t) \rangle$

Calculate the limit of the curvature as $t \to \infty$.

Formulas for curvature:

$$\kappa(t) = \left| \frac{d\mathbf{T}}{ds} \right| = \frac{|\mathbf{T}'(t)|}{|s'(t)|} = \frac{|\mathbf{f}'(t) \times \mathbf{f}''(t)|}{|\mathbf{f}'(t)|^3}$$

Next time: *test review!*