Lecture 24

Behind us

- Errors in linear approximation

Final on 3/16 in KNE 110 1:30 PM to 4:20 PM (no exceptions or rescheduling)

Ahead

Today: higher order approximation

Next: even higher order approximation

Read the Taylor notes sections 1, 2, and 3. *Please. Pretty please.*

Questions!

Nedwina and the piglet keep playing the market

It's still 1999, still before the bursting of the tech stock bubble, and Nedwina and the piglet of calculus have stock market fever.

(This is still not a good sign.)

Nedwina and the piglet realize that they need to do better than a linear approximation if they want to outsmart the Slimy Stock Snail. Recall that the best linear approximation is:

$$f(t) = 10000 + 3000t,$$

where t is the time in days from now. In addition, by analyzing the data, Nedwina and the piglet conclude that the best linear approximation to the first derivative of f is

$$g(t) = 3000 - 10t.$$

Is it ever enough?

Using their New and Improved Better Tasting Stock Toothpick (© 1999), how far off can N and the P be in six months? What additional information would help? Do they have enough juice?

What kinds of additional information could N and the P use to evaluate the accuracy of the new approximation?

Will it ever be enough?

Quadratic approximation

What kind of quadratic approximation do we get from our two pieces?

If $D(t)$ is the value of the DJIA, then $f(t) = 10000 + 3000t$ approximates $D(t)$ and $g(t) = 3000 - 10t$ approximates $D'(t)$.

Integrating $g(t)$ and accounting for the initial value (which we can compute as $f(0)$), we get an approximation function

$$h(t) = 10000 + 3000t - 5t^2.$$

This is the best quadratic approximation to our function.

General formula: the best quadratic approximation to $f(t)$ around the point $t = b$ is given by

$$T_2(x) = f(b) + f'(b)(x - b) + \frac{1}{2} f''(b)(x - b)^2.$$

Examples

Let's continue with the some examples from last time. Here are a bunch of functions with their quadratic approximations around $t = 0$:

Quadratic approx near 0: $T_2(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2$

- $f(t) = t^2$, best quadratic approximation near 0: $T_2(t) = t^2$
- $f(t) = t^3$, best quadratic approximation near 0: $T_2(t) = 0$ (!!)
- $f(t) = \sin(t)$, best quadratic approximation near 0: $T_2(t) = t$ (!?!)
- $f(t) = t^{367}$, best quadratic approximation near 0: $T_2(t) = 0$.

Do some!

Find the best quadratic approximations $f(b) + f'(b)(x - b) + \frac{1}{2} f''(b)(x - b)^2$ to the following functions around the points indicated:

- $f(x) = x^3$ around $b = 1$
- $f(x) = \sin(x)$ around $x = \pi/2$
- $f(x) = e^{-x^2}$ around $x = 0$

Error

We can estimate error just like we did last time.

The key: the *third* derivative! (Surprised?)

Theorem: If $|f'''(t)| \leq M$ for all t between b and c then

$$|f(c) - T_2(c)| \leq \frac{M}{6} |c - b|^3.$$

In our example: $T_2(c) = 10000 + 3000c - 5c^2$.

If $|f'''(t)| \leq M$ during the next half year, then we see that $T_2(c)$ is within $1004762M$ of $D(c)$.

Mind: blown

Why stop at quadratic approximations?

We can approximate to higher and higher order!

Let's fix a function $f(x)$ that is "infinitely differentiable" (every derivative has a derivative).

> Linear approx near b: $T_1(x) = f(b) + f'(b)(x-b)$
>
> Quadratic approx near b:
> $T_2(x) = f(b) + f'(b)(x-b) + \frac{1}{2}f''(b)(x-b)^2$
>
> Cubic approx near b:
> $T_3(x) = f(b) + f'(b)(x-b) + \frac{1}{2}f''(b)(x-b)^2 + \frac{1}{6}f'''(b)(x-b)^3$.
>
> General:
> $T_n(x) = f(b) + f'(b)(x-b) + \frac{1}{2}f^{(2)}(b)(x-b)^2 + \cdots + \frac{1}{n!}f^{(n)}(b)(x-$
> .

Say what?

Maybe we should refresh our memories about the notation.

> $f^{(n)}(b)$ is the nth derivative of f at b (so $f = f^{(0)}$, $f' = f^{(1)}$, $f'' = f^{(2)}$, etc.)
>
> $n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1$, pronounced "$n$ factorial", is the product of the integers from 1 to n (so $1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$,etc.), and by definition $0! = 1$.

The formula again: the nth Taylor polynomial of $f(x)$ around b is

$$T_n(x) = f(b) + f^{(1)}(b)(x-b) + \frac{1}{2} f^{(2)}(b)(x-b)^2 + \cdots + \frac{1}{n!} f^{(n)}(b)(x-b)$$

$$= \sum_{i=0}^{n} \frac{1}{i!} f^{(i)}(b)(x-b)^i$$

Example

Let's wrap the piglet's head around an example: $f(x) = \sin(x)$.

Linear approximation at $b = 0$: $T_1(x) = x$

Quadratic approx at $b = 0$: $T_2(x) = x$

Cubic: $T_3(x) = x - \frac{1}{6}x^3$

Quartic: $T_4 = x - \frac{1}{6}x^3$

Quintic: $T_5 = x - \frac{1}{6}x^3 + \frac{1}{5!}x^5$

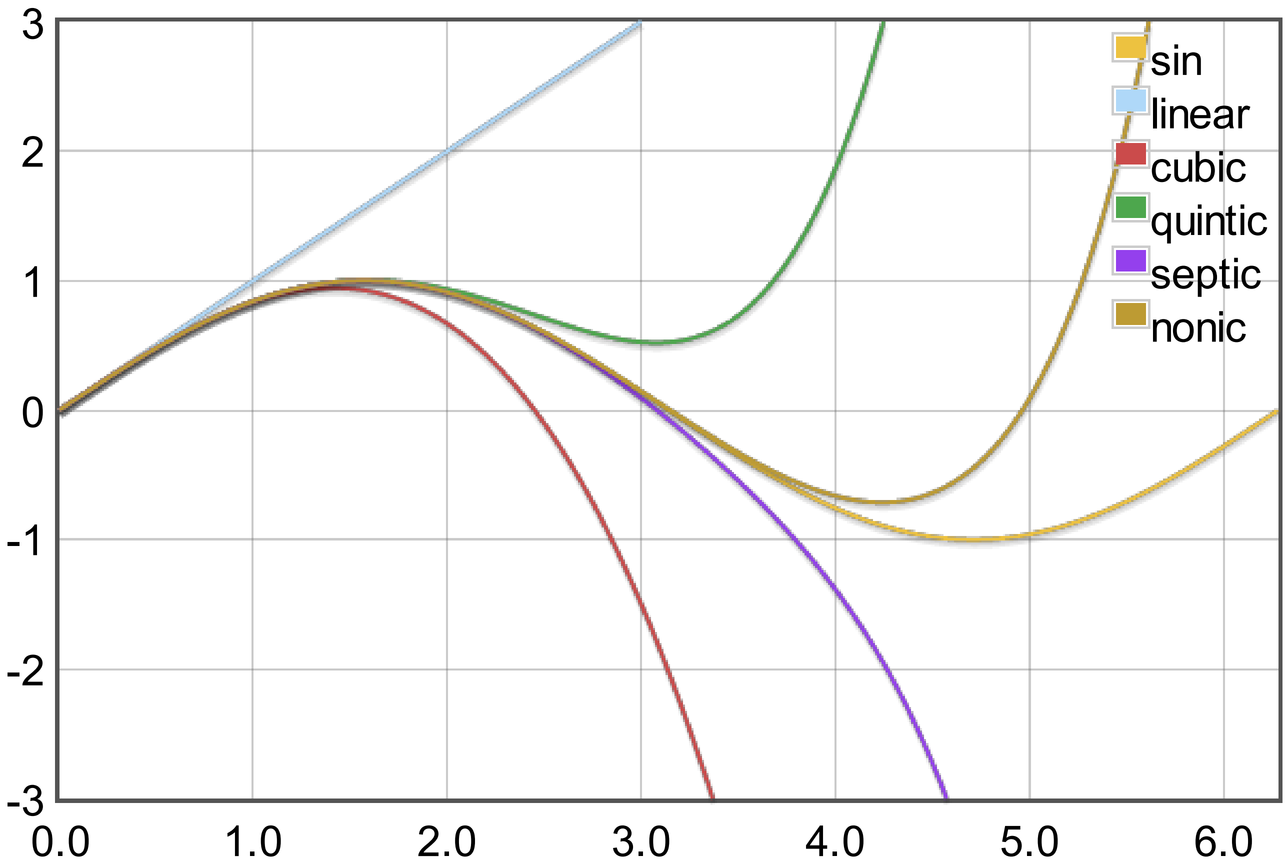
Septic: $T_7 = x - \frac{1}{6}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7$

General, for $n = 2m + 1$ odd: $T_n = \sum_{i=0}^{m} \frac{(-1)^i}{(2i+1)!}x^{2i+1}$

(don't worry!)

Photographic evidence

Here's a graph showing the approximations of odd degree up to 9 between $t = 0$ and $t = 2\pi$



Approximations of different orders get bad in different ways: end up too high or too low. Is there a pattern?

Anything else to observe?

Try some!

Find formulas for T_n as a sum (with $\sum$, some limits, etc.) for the following functions:

$f(x) = \cos(x)$ near $b = \pi$

$f(x) = e^x$ near $b = 0$

Challenge:

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

near $b = 0$.

Next time: *all of the orders at once as an infinite sum of love!*
