Loaded!

Please proceed to the calculus wonderland by pressing the right arrow key or clicking the right arrow that is visible when you move your mouse over the pig.

Instructions

These slides should work with *any* modern browser: IE 9+, Safari 5+, Firefox 9+, Chrome 16+.

- Navigate with arrow keys; you may need to give the window focus by clicking outside the lecture frame (the pig) for key commands described throughout this slide to work properly.
- Press M to see a menu of slides. Press G to go to a specific slide. Press W to toggle scaling of the deck with the window. If scaling is off, slides will be 800 by 600; it is off by default.
- Use left click, middle click, right click or hold A, S, D on the keyboard and move the mouse to rotate, scale, or pan the object.

If your browser or hardware does not support WebGL, interacting with models will be *very* slow (and in general models can get CPU-intensive). Navigate to a slide away from any running model to stop model animation.

Lecture 8

Behind us

- Curves via parametric equations
- Vector-valued functions

Coming up

Today: derivatives and integrals of vector functions

Next: explosive investigation of curvature and arclength

Read Sections 10.2, 13.2, and 10.3 of the book. *Failure to do so may result in lack of knowledge and general misery.*

Logistics

Midterm 1 is Feb 5 in section!

Questions!

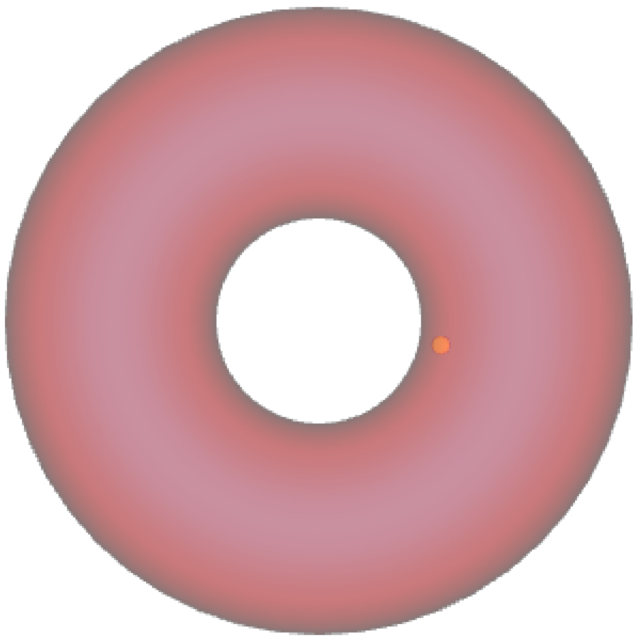
Question

A fun-loving electron is traveling in a spiral path around the surface of a torus.

> The radius of the torus (i.e., the radius of the circle at the center of the tube) is 2 and the diameter of the circular cross-section is 1.
>
> The electron starts at position $(1, 0, 0)$, travels at a constant angular velocity around the vertical axis of 1 radian per second, and its path winds up and around the torus 4 times before it returns to its starting position.

What is the position and velocity of the electron at time t?

Fun-loving electron in action



It moves!

Using the tools from last time, here is a parametric description of the motion:

$$(x, y, z) = (\cos(t)(2 - \cos(4t)), \sin(t)(2 - \cos(4t)), \sin(4t)$$

In vector form:

$$\mathbf{f}(t) = \langle \cos(t)(2 - \cos(4t)), \sin(t)(2 - \cos(4t)), \sin(4t) \rangle$$

What is the velocity? What *should* it be?

Velocity is the derivative of position, right?

We should have that the velocity of the electron is

$$\mathbf{v}(t) = \mathbf{f}'(t).$$

But what is this?

Classical definition of the derivative still works for vector-valued functions:

$$\mathbf{f}'(t) = \lim_{h \to 0} \frac{\mathbf{f}(t+h) - \mathbf{f}(t)}{h}$$

As usual, the derivative of the position vector is the velocity vector .

Calculating the derivative in practice

Given a vector function

$$\mathbf{f}(t) = \langle x(t), y(t), z(t) \rangle$$

the derivative is just

$$\mathbf{f}'(t) = \langle x'(t), y'(t), z'(t) \rangle,$$

the component-wise derivative .

The usual caveat applies: the derivative must exist for this to make sense! I.e., this formula works when all three derivatives exist, and when they don't neither does the derivative of $\mathbf{f}(t)$!

Practice

Given the formula

$$\mathbf{f}(t) = \langle \cos(t)(2 - \cos(4t)), \sin(t)(2 - \cos(4t)), \sin(4t) \rangle$$

for the motion of the electron on the torus,

> calculate the velocity of the electron as a function of t
>
> How does the speed vary over time?
>
> (What is the speed , anyway? The magnitude of the velocity vector? That sounds right....)

Help the Piglet

The piglet of calculus tried the question on the previous slide and got the following answer:

The derivative of $\mathbf{f}(t) = \langle \cos(t)(2 - \cos(4t)), \sin(t)(2 - \cos(4t)), \sin(4t) \rangle$ is:

$$\frac{d\mathbf{f}}{dt} = \langle -\sin(t)(2 - \cos(4t)) + 4\cos(t)\sin(4t), \\ 4\cos(t)\sin(4t), \sin(4t) \rangle$$

Every time the piglet enters the answer in webassign, it is marked wrong. What mistakes did the piglet make?

The piglet

Forgot to differentiate the third component!

Wrong: $\sin(4t)$

Right: $4\cos(4t)$

Used the product rule incorrectly in the second component (by just taking the product of the derivatives)!

Wrong: $4\cos(t)\sin(4t)$

Right: $\cos(t)(2-\cos(4t))+4\sin(t)\sin(4t)$

Anything else?

Have you ever made mistakes like these?

A new problem

A joey (baby kangaroo) is riding in her mother's pouch. She has a smartphone with an accelerometer that continuously reports the acceleration vector. Her friend wrote an app that calculates the velocity at any moment in time t. She is too small to see out of the pouch, but, like all infant kangaroos, she is interested in calculating her position as a function of time. Sadly, her somewhat stupid friend never figured out how to get the position before he started a new app that mimics the hilarious sounds that giraffes make when embarrassed.

> The joey records the velocity at time t as
> $$\mathbf{v}(t) = \langle 1, t, \sin(t) \rangle$$
> The joey starts at the point $(0, 0, 1)$
>
> Where is the joey at time t?

Integrate!

Just like one variable calculus: reconstruct position from velocity by integrating .

Riemann sums: $\sum \mathbf{v}(t_i)\Delta(t)$

Practical:

$$\int \langle x(t), y(t), z(t)\rangle dt = \left\langle \int x(t)dt, \int y(t)dt, \int z(t)dt \right\rangle$$

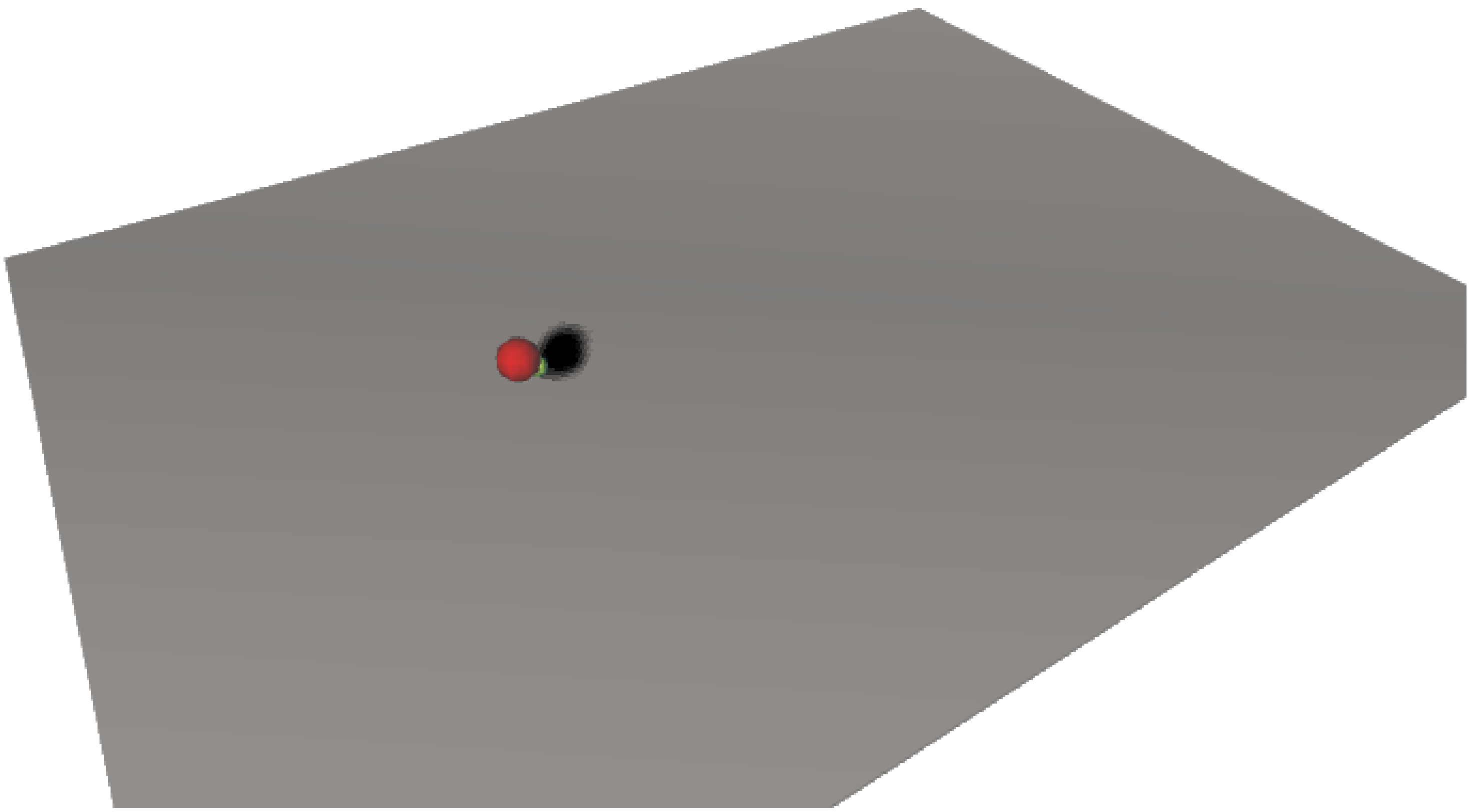
Example:

$$\int \langle 1, t, t^2\rangle dt = \langle a, b, c\rangle + \left\langle t, \frac{1}{2}\, t^2, \frac{1}{3}\, t^3 \right\rangle$$

The constant of integration is now a vector!

Help the joey

Using vector integration, compute the path that the joey takes, starting at $t = 0$. Initial position: $(0, 0, 1)$, velocity $\langle 1, t, \sin(t) \rangle$. A looping animation (for $t = 0$ to $t = 15$ in slo-mo):



Next time: *curvature and arclength!*