Lecture 20

Behind us

- Double integrals over regions
- Volumes of eyeballs (now with cornea!)

Ahead

Today: double integrals with polar sauce

Next: center of mass

Read Sections 15.4 and 15.5. *It burns, it burns!*

Midterm 2 on 3/5 in section

Final on 3/16

Questions!

Return of the eyeball

We had a model for an eyeball given by two equations:

$x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + \left(z - \frac{3}{4}\right)^2 = \frac{1}{4}$.

Circle of intersection: $z = \frac{7}{8}$, $x^2 + y^2 = \frac{15}{64}$.

A double integral that calculates the extra volume added inside the cornea: let R be the disk bounded by the circle $x^2 + y^2 = \frac{15}{64}$

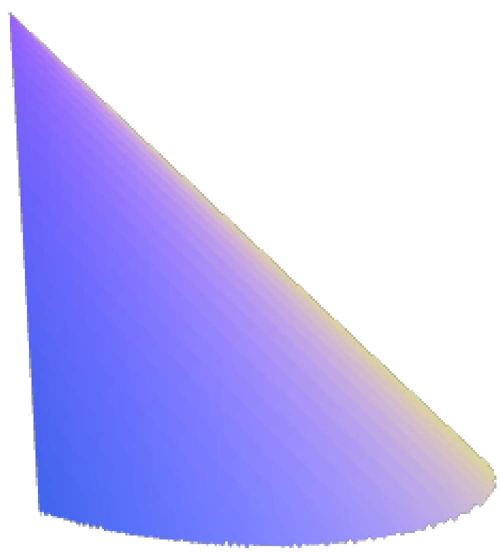
$$\iint_R \frac{3}{4} + \sqrt{\frac{1}{4} - x^2 - y^2} - \sqrt{1 - x^2 - y^2}\, dA.$$

How can we compute something so crazy-looking?!

Let's try a simpler problem first.

Shark fins

Nedwina is studying the biomechanics of the shark, and part of her project requires her to calculate the volume and center of mass of the dorsal fin.



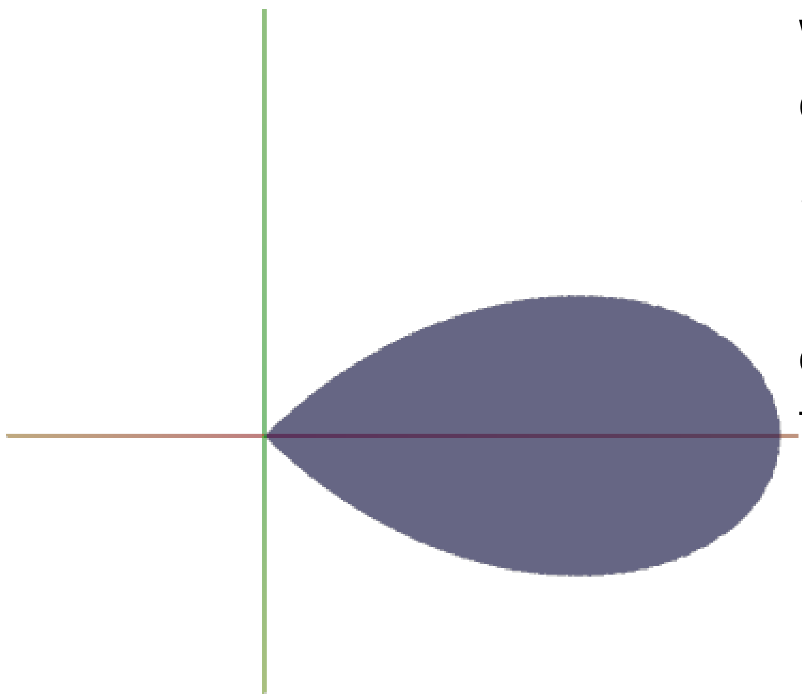
Polar shark

The outer surface of the fin is described by the equation

$$z = \cos(2\theta) - r$$

where $-\pi/4 \leq \theta \leq \pi/4$ and $0 \leq r \leq \cos(2\theta)$.

The region at the base of the fin is the teardrop shape inside the graph of $r = \cos(2\theta)$, $-\pi/4 \leq \theta \leq \pi/4$



We would like to compute the integral $\iint_R z dA$.

Do we need to re-express everything in terms of x and y?

Polar coordinates are coordinates, too

We can perform double integration in polar coordinates!

Tricky point: we need to pay careful attention to dA.

In Cartesian coordinates: $dA = dxdy$.

In polar coordinates: $dA = rdrd\theta$.

A "small area element" in polar coordinates has area $rdrd\theta$. Let's go to the tablet for a brief explanation!

Nedwina's shark

In our example, we can thus do this:

$$\iint_R z dA = \int_{-\pi/4}^{\pi/4} \int_0^{\cos(2\theta)} zrdrd\theta = \int_{-\pi/4}^{\pi/4} \int_0^{\cos(2\theta)} (\cos(2\theta) - r) rdrd\theta$$

$$= \int_{-\pi/4}^{\pi/4} \frac{1}{2}\cos(2\theta)r^2 - \frac{1}{3}r^3 \Big|_0^{\cos(2\theta)} d\theta = \int_{-\pi/4}^{\pi/4} \frac{1}{6}\cos^3(2\theta) d\theta = \frac{1}{12}\int_{-\pi/2}^{\pi/2} \mathrm{co}$$

$$= \frac{1}{12}\int_{-\pi/2}^{\pi/2} \cos(u)(1 - \sin^2(u)) du = \frac{1}{12}(\sin(u) - \frac{1}{3}\sin^3(u))\Big|_{-\pi/2}^{\pi/2} = \frac{1}{9}.$$

(Look at this slide later for the details. It is a standard integration problem once you change the variables!)

Eyeballs, again? (Better than brussels sprouts.)

Our setup: let R be the disk bounded by the circle $x^2 + y^2 = \frac{15}{64}$

$$\iint_R \frac{3}{4} + \sqrt{\frac{1}{4} - x^2 - y^2} - \sqrt{1 - x^2 - y^2}\, dA.$$

Do it!

(The numbers might get ugly. That's ok: eyeballs are messy.)

Next time: *area and mass!*