Lecture 18

Behind us

- Local max/min
- We loved the Hessian

Ahead

Today: double integrals, I

Next: double integrals, II

Read Sections 15.1, 15.2, 15.3 *Or you will hate yourself forever.*

Questions!

How big is Mt. Rainier?

A large mining company has decided it wants to turn Mt. Rainier into pure profit by smashing and carting the whole thing away one rock at a time (for use in fishtanks). They ask the piglet of calculus to help them calculate the amount of rock they will end up with.

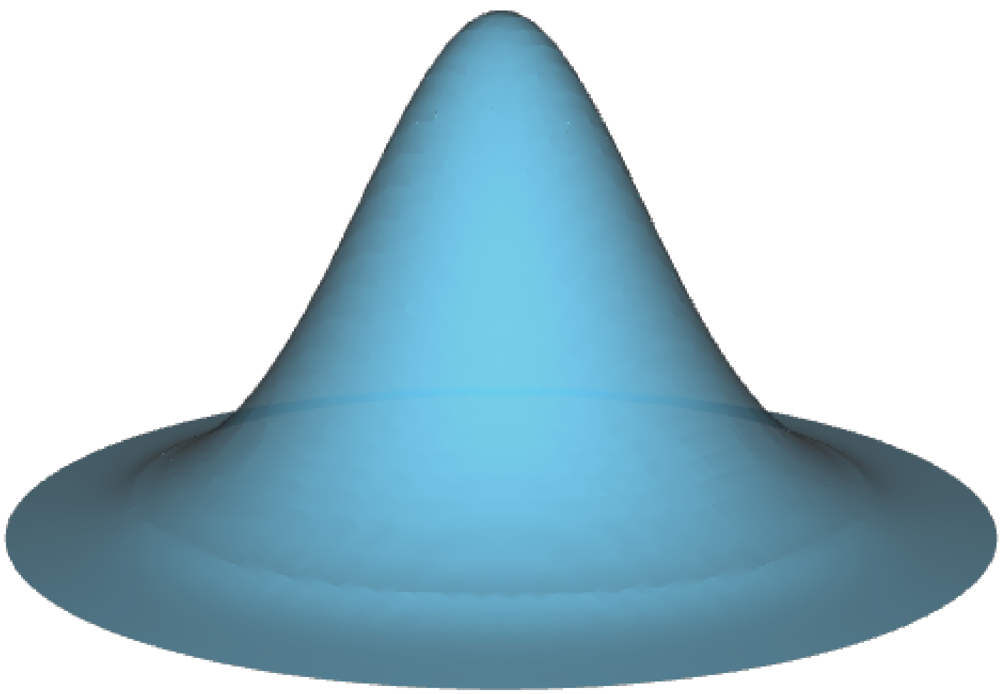
What are the piglet's options?

(Note: the piglet has access to the Puget Sound LIDAR Consortium's data set, thanks to Joe Marcus.)

Toy model rides again

Let's think about a toy model given by

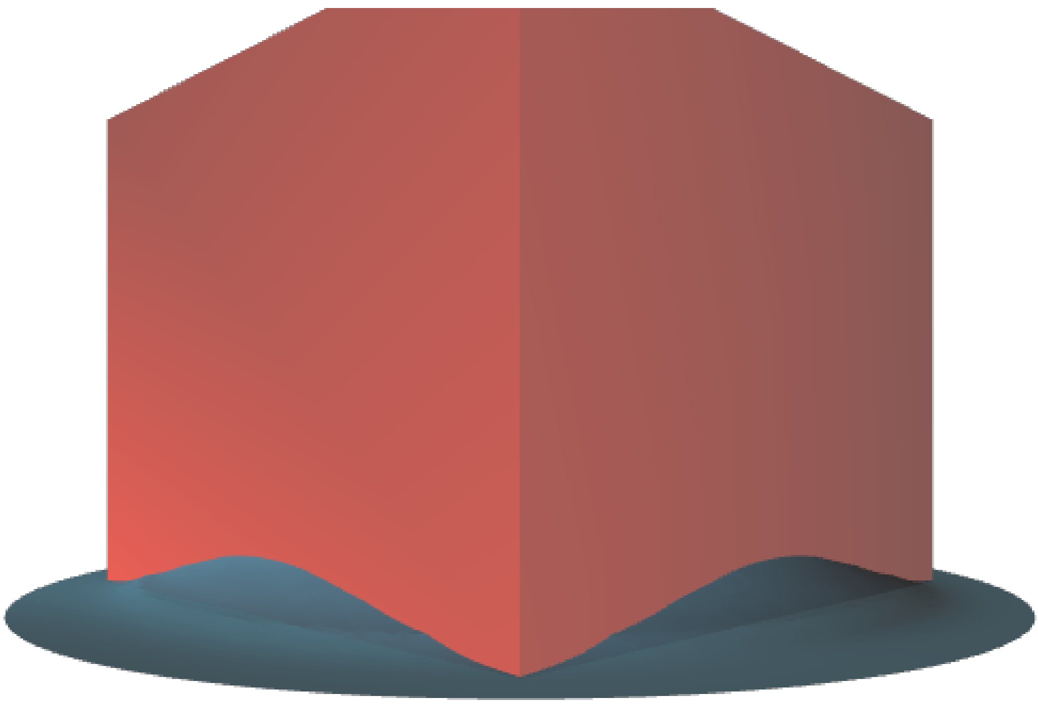
$$z = \sqrt{\frac{10 - x^2 - y^2}{e^{x^2+y^2}}}.$$



Crude approximation

Let's just try to get the volume over the 4 by 4 square centered at the origin.

That oughta do it.



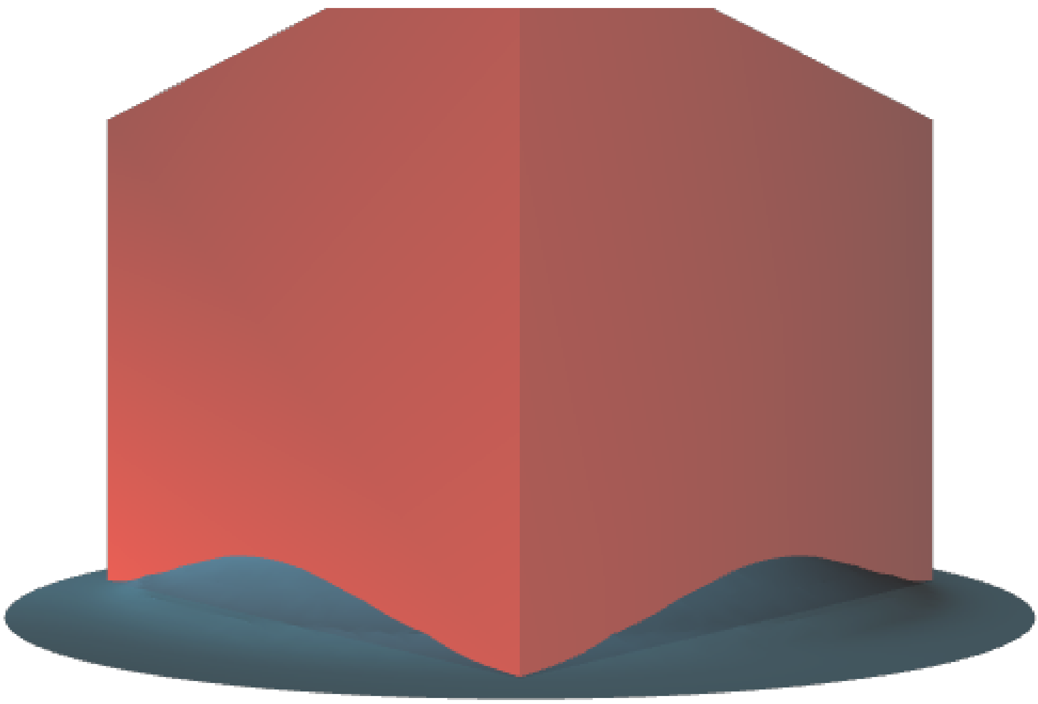
We can do better than that!

Let's refine the grid! Vol: 50.5964

N 1

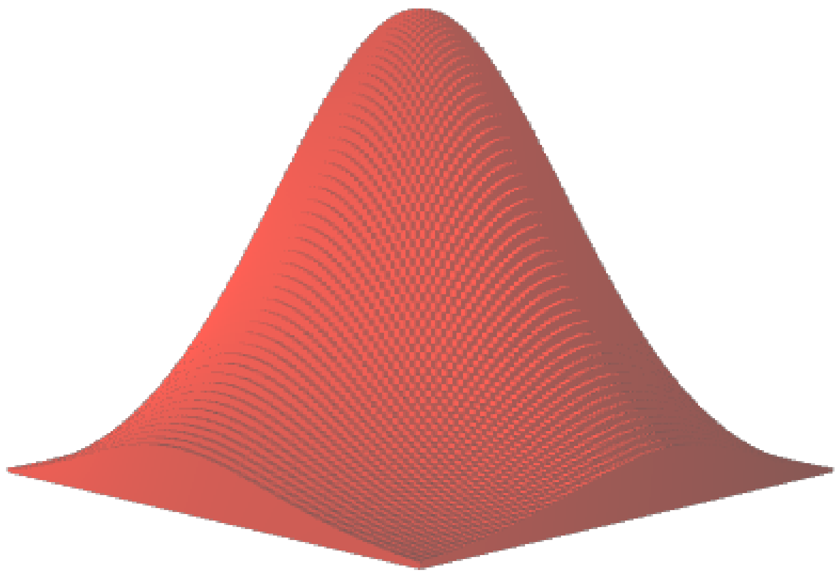
Close Controls

Change the number of blocks, observe the effect on the approximation. "N" means "N by N grid"



Keep going!

Just to show you, here is what happens for a 100 by 100 grid. Vol: 16.5899



That's a mighty fine approximation you have there. It would be a shame if anything happened to it....

Does this remind you of anything?

Are we just playing with blocks?

LEGO is Danish, but these things are _____.

Remember Riemann sums?

In your calculus youth, you learned about integration via Riemann sums.

Recall: given a function $f(x)$ defined on an interval $[a, b]$ and an integer n, we form the nth Riemann sum like this:

$$S_n = \sum_{i=1}^{n} f\left(a + (2i-1)\,\frac{b-a}{2n}\right)\frac{1}{n} = \sum_{i=1}^{n} f(x_i)\Delta(x)$$

The idea: divide the interval into n equal pieces, evaluate the function in the middle of each interval, and add up the areas of the resulting rectangles.

Stew in brain juices

Try to set up a similar thing for functions of two variables $f(x, y)$ over a rectangle $[a, b] \times [c, d]$: x goes from a to b and y goes from c to d.

Some components you might want to use to start:

$$\sum_{i=1}^{m}\sum_{j=1}^{n} \qquad \Delta(x) \qquad \Delta(y) \qquad f$$

This is not just a mindless exercise. How would you program a computer to make these approximations? Work backwards from there if you want!

Maybe I just worked some magic to make those models before.

Riemann sums over rectangles

Here's one possible answer:

Suppose $f(x, y)$ is defined and continuous on the rectangle
$$[a, b] \times [c, d] = \{(x, y) | a \leq x \leq b, c \leq y \leq d\}.$$
"The" nth Riemann sum of f is

$$S_n = \sum_{i=1}^{n} \sum_{j=1}^{n} f(x_{ij}, y_{ij}) \Delta(x) \Delta(y),$$

where (x_{ij}, y_{ij}) is a point in the ij-subrectangle
$$\left[a + (i-1)\,\frac{b-a}{n}\,, a + i\,\frac{b-a}{n}\right] \times \left[c + (j-1)\,\frac{d-c}{n}\,, c + \right.$$

Where the heck are the rectangles in there?!

The double integral appears

Clearly, in reasonable circumstances the approximations we get this way improve as the mesh gets finer and finer.

Let's give this limit a name!

If $f(x, y)$ is continuous on the rectangle $R = [a, b] \times [c, d]$ then the double integral of f over R is the limit

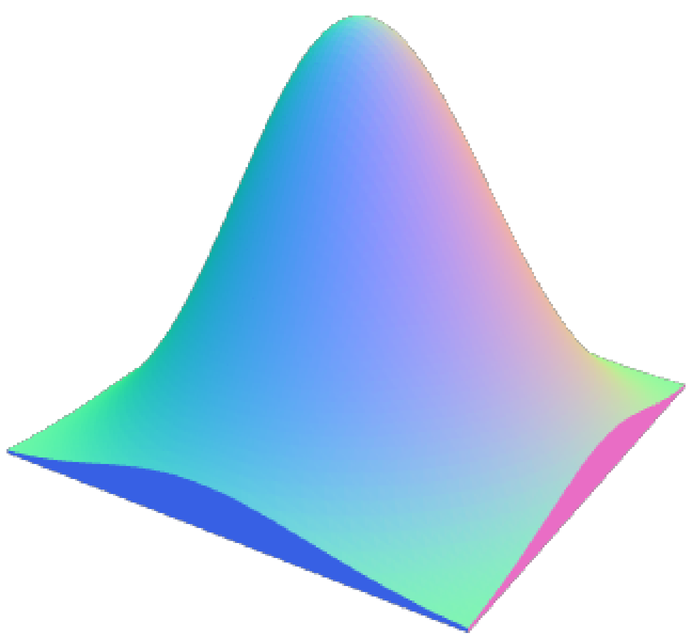
$$\iint_R f(x, y)dxdy = \lim_{n \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{n} f(x_{ij}, y_{ij})\Delta(x)\Delta(y).$$

You will see a slightly different formulation in your book. Compare them and decide if they're the same! (He divides the two directions into different numbers of parts, etc., etc.)

It computes volume

As you can guess from our model, the double integral computes a volume.

Theorem: given a function $f(x, y)$ continuous and non-negative over a rectangle R in $\mathbf{R}^2$, the volume of the solid between the xy-plane and the graph of f over R equals $\iint_R f dx dy$.



This baby has volume approximately 16.5895. Sweet. (How did the computer get this?)

How do you compute these?

Can we do a single example?

Here's one: $f(x,y)=1$ over the rectangle $R=[-1,1]\times[-1,1]$. For the nth Riemann sum, $\Delta(x)=\frac{1-(-1)}{n}=\frac{2}{n}$ and similarly for $\Delta(y)$.

Computing the Riemann sums:

$$
\begin{aligned}
S_n = \sum_{i=1}^{n}\sum_{j=1}^{n} 1\Delta(x)\Delta(y) = \left(\sum_{i=1}^{n}\Delta(x)\right)\left(\sum_{j=1}^{n}\Delta(y)\right) \\
&= \left(\sum_{i=1}^{n}\frac{2}{n}\right)\left(\sum_{j=1}^{n}\frac{2}{n}\right) \\
&= 2\cdot 2 = 4,
\end{aligned}
$$

so we get $\iint_R 1dxdy = 4$.

The piglet is excited

The piglet of calculus has come knocking on your door at 2 AM. She is waving a notebook in the air. She calls you over and says:

"For any function $f(x, y)$ of the form $f(x, y) = g(x)h(y)$ and any rectangle $R = [a, b] \times [c, d]$, I know that

$$\iint_R f(x, y) dx dy = \left(\int_a^b g(x) dx \right) \cdot \left(\int_c^d h(y) dy \right)."$$

Is she right?

To warm up, you could try $f(x, y) = x$ and then $f(x, y) = xy$.

If you settle the piglet's question, you might try your hand at computing $\iint_{[0,1]\times[0,2]} (x + y) dx dy$; this does not fit into the piglet's pattern, of course!

Germany + Italy = happy pig

Tomorrow in section you will learn about iterated integrals, an incredibly powerful method of computing these beasts.

Observation: we can try to split up the integral by dealing with the variables separately. If we choose the center points in each rectangle, we can write a Riemann sum and then start speculating wildly.

$$\sum_{i=1}^{n}\sum_{i=1}^{m} f(x_i, y_j)\Delta(x)\Delta(y) = \sum_{i=1}^{m}\Delta(x)\sum_{j=1} f(x_i, y_j)\Delta(y)$$

$$\approx \sum_{i=1}^{m}\Delta(x)\int_c^d f(x_i, y)dy \approx \int_a^b\left(\int_c^d f(x, y)dy\right)dx$$

Next time: *more integrals!*