Lecture 19

Behind us

- The double integral: Riemann's revenge
- Conceptual equation: useful for implemention, but hard to compute by hand

Ahead

Today: double integrals over regions

Next: double integrals in polar coordinates

Read Sections 15.1, 15.2, 15.3 *Every time you don't study a kitten dies.*

Questions!

Yesterday, you saw

Iterated integrals

Given a continuous function $f(x,y)$ on a rectangle $[a,b]\times[c,d]$,

$$\iint_R f(x,y)dxdy = \int_c^d \left(\int_a^b f(x,y)dx\right)dy$$
$$= \int_a^b \left(\int_c^d f(x,y)dy\right)dx.$$

Example: for $f(x,y)=x+y$ and $R=[0,1]\times[0,1]$

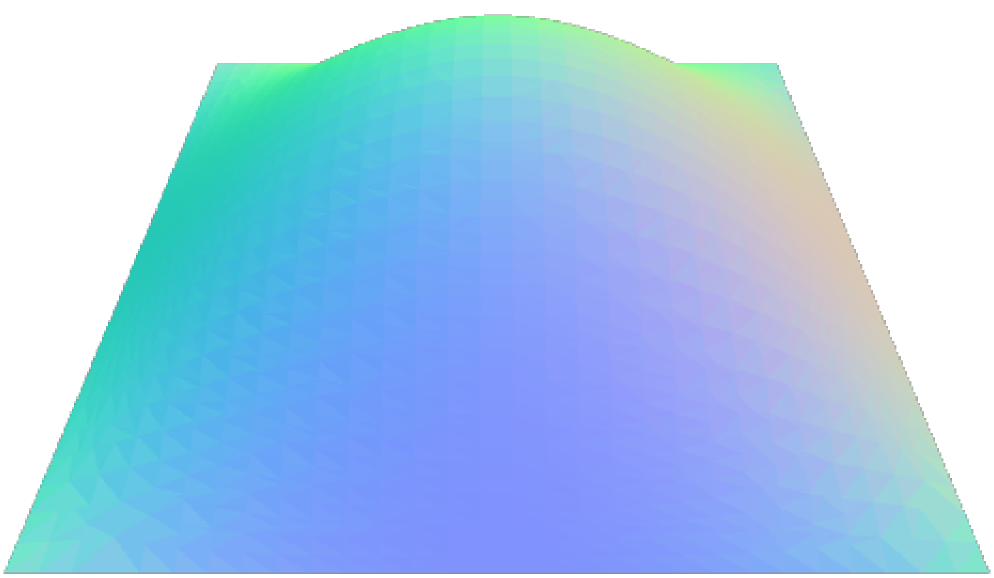
$$\iint_R f(x,y)dxdy = \int_0^1 \left(\int_0^1 (x+y)dx\right)dy = \int_0^1 \left(\frac{1}{2}x^2+yx\right)\Bigg|_0^1 dy$$
$$= \int_0^1 \frac{1}{2}+ydy = \frac{1}{2}y+\frac{1}{2}y^2\Bigg|_0^1 = 1.$$

Do one!

Remember our friend the egg carton? Equation:
$f(x, y) = \sin(x)\cos(y)$.

Calculate the double integral

$$\iint_{[0,\pi]\times[-\pi/2,\pi/2]} \sin(x)\cos(y)dxdy$$



What if we change the region to $[0, 2\pi] \times [0, 2\pi]$?

The piglet was excited

The piglet of calculus came knocking on your door at 2 AM waving a notebook in the air. She called you over and said:

"For any function $f(x, y)$ of the form $f(x, y) = g(x)h(y)$ and any rectangle $R = [a, b] \times [c, d]$, I know that

$$\iint_R f(x, y) dx dy = \left(\int_a^b g(x) dx\right) \cdot \left(\int_c^d h(y) dy\right).\text{''}$$

Was she right?

Iterating the piglet

Does our new tool help us understand the piglet's question?

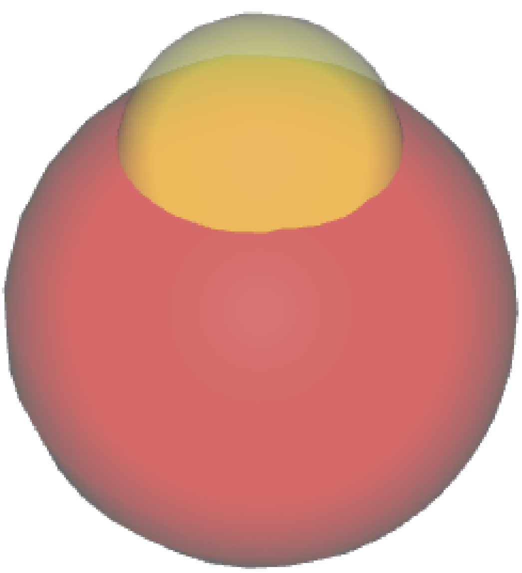
$$\iint_R g(x)h(y)dxdy = \left(\int_a^b g(x)dx\right) \cdot \left(\int_c^d h(y)dy\right).$$

Think about this for a few minutes.

The volume of an eyeball

Here's a simple model of an eyeball: take two spheres, say one of radius 1 and one of radius $1/2$ and translate the smaller one up by $3/4$.

Equations: $x^2 + y^2 + z^2 = 1$, $x^2 + y^2 + (z - 3/4)^2 = 1/4$



What is the total volume enclosed in the eyeball (including cornea)?

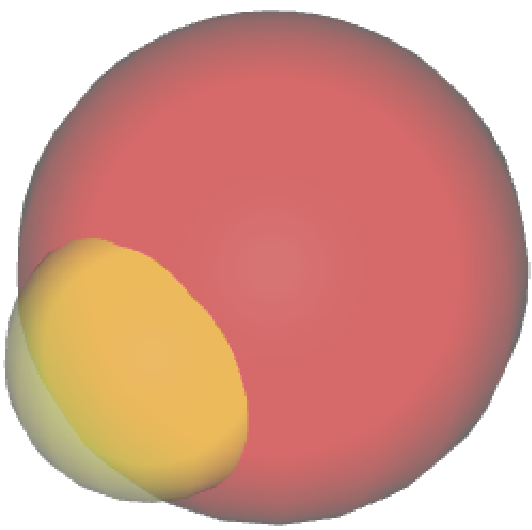
Hope (?): set up some kind of double integrals that calculate what we want.

Think very hard about actually calculating the integrals....

Find the sticking point

You know about double integrals over rectangles.

Now think about where your tools might fail in attacking the eyeball problem.



To get started, you might want to think about how you would like to use the magic of calculus to calculate the extra volume contained in the cornea. (Yes, this question is vague on purpose.)

The crux

The two spheres intersect in a circle.

How do we take double integrals over interesting regions?

Miracle? Math!

Here's something absolutely amazing.

When we take an iterated integral $\int_c^d \int_a^b f(x,y) dx dy$,

we can let a and b be functions of y!

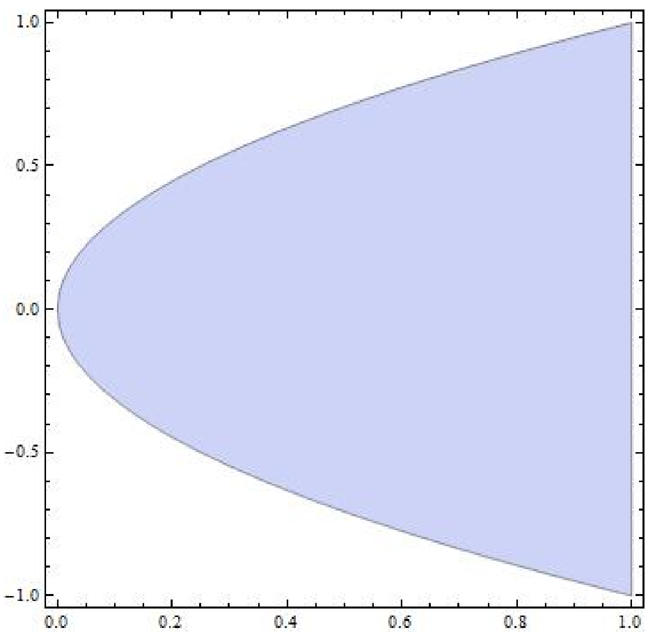
(And we can do the reverse if we integrate in the other order. Make the limits on the inner integral functions of the outer variable.)

After all, we're treating y like a constant when we integrate with respect to x.

All that happens is that the limits of the integral change when we change the (static) value of y.

Example

Let's return to the function $f(x, y) = x + y$, but let's integrate it over the region inside the parabola $x = y^2$ between $x = 0$ and $x = 1$.



The integral becomes
$\int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} (x + y) dy dx$.

Inner:
$\int_{-\sqrt{x}}^{\sqrt{x}} (x + y) dy = xy + \frac{1}{2} y^2 \Big|_{-\sqrt{x}}^{\sqrt{x}} = 2x^{3/2}$.

Outer: $\int_0^1 2x^{3/2} dx = \frac{4}{5} x^{5/2} \Big|_0^1 = \frac{4}{5}$

Piece of cake!

Do one

Let's warm up a bit for our eyeball problem.

What is the integral of the function $z = x + \frac{y}{e^{x^2+y^2}}$ over the unit disk?

If you are afraid, try doing the terms one at a time.

Standard operating procedure:

> Write the problem as a double integral.
>
> Choose a way of writing it as an iterated integral, expressing the limits of the inner integral in terms of the outer variable (!).
>
> See what happens. If you get stuck, try it the other way. Or subdivide the region and try again (less common).
>
> Remember: integration is HARD, so you cannot expect answers to come to you easily. If you find it hard, you are doing it right.

Return of the eyeball

Set up a calculation of the eyeball volume. You might want to use the formula for the volume of a sphere, but also one or more double integrals over regions.

Equations: $x^2 + y^2 + z^2 = 1$, $x^2 + y^2 + (z - 3/4)^2 = 1/4$

This time you have to think about your own picture. Dark alley, etc., etc.

Next time: *even more integrals!*