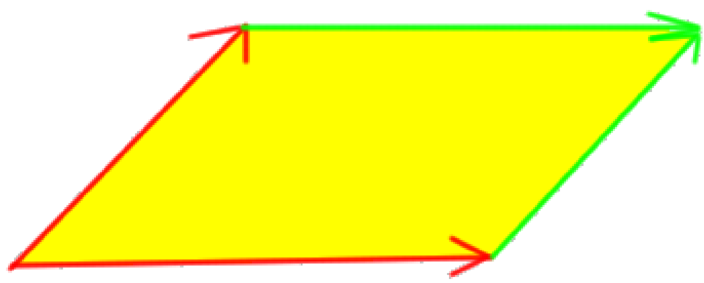
Cross products: mechanics

Warm up

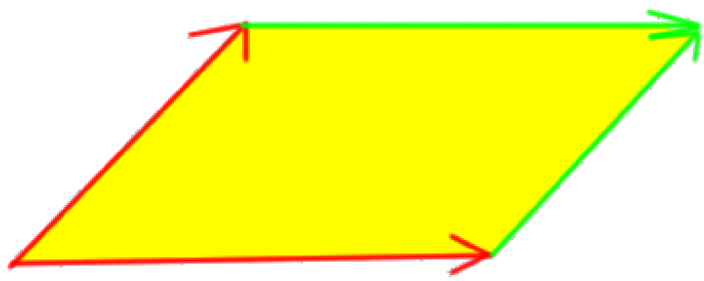
Question: two vectors span a parallelogram.

- What is the area of the parallelogram?
- What is a unit vector perpendicular to the parallelogram?



Warm up

Example: how about the parallelogram spanned by the two vectors $\langle 4, 0\rangle$ and $\langle 3, 4\rangle$ in the xy-plane?



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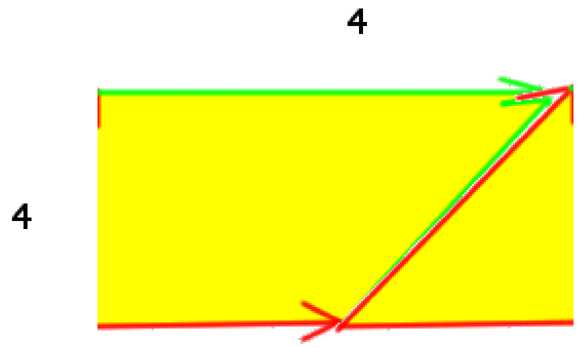

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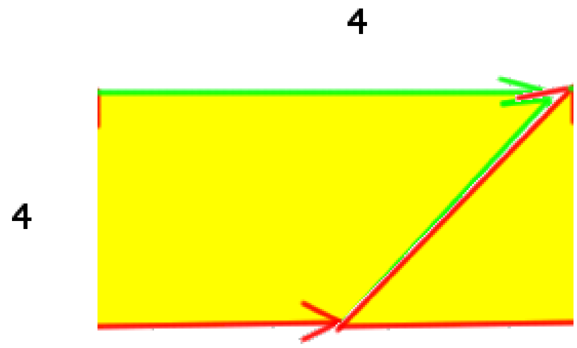

The area is thus $4 \cdot 4 = 16$.

Magic on the way

It turns out that there is a way to make a single vector encoding both the perpendicular direction and area of the parallelogram.

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Secret sauce: the cross product

Definition

The cross product of two vectors

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle, \mathbf{b} = \langle b_1, b_2, b_3 \rangle$$

is the vector

$$\mathbf{a} \times \mathbf{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle.$$

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Note: unlike the dot product, this *is* a vector!

Computing with a determinant

This is a slight abuse of notation, but we have

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

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Read about this in a textbook or the internet. Let's review it!

Computing with a determinant

$$
\begin{aligned}
\mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\
&= \mathbf{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}
\end{aligned}
$$

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$$\begin{aligned} &= \mathbf{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\ &= \mathbf{i}(a_2 b_3 - a_3 b_2) - \mathbf{j}(a_1 b_3 - a_3 b_1) \\ &\qquad + \mathbf{k}(a_1 b_2 - a_2 b_1) \end{aligned}$$

Example

$$\begin{aligned}\langle 1,1,0\rangle \times \langle 0,1,1\rangle \\ = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix}\end{aligned}$$

Example

$$
\begin{aligned}
\langle 1, 1, 0\rangle \times \langle 0, 1, 1\rangle \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} \\
&= \mathbf{i}\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}
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$$
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&= \mathbf{i}\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \\
&= \mathbf{i} - \mathbf{j} + \mathbf{k} = \langle 1, -1, 1\rangle
\end{aligned}
$$

Brain squeeze

Compute these cross products and draw the resulting vectors:

- $\langle 1, 2, 3 \rangle \times \langle 3, 6, 9 \rangle$
- $\mathbf{i} \times \mathbf{i}$
- $\mathbf{i} \times \mathbf{j}$
- $\mathbf{j} \times \mathbf{i}$
- $\mathbf{j} \times \mathbf{k}$
- $\mathbf{k} \times \mathbf{i}$