Lecture 26

Behind us

- Taylor series

Final on 3/16 in KNE 130 1:30 PM to 4:20 PM (no exceptions or rescheduling)

Ahead

Today: making new Taylor series from old

Next: We have reached the end!

Read the Taylor notes sections 1, 2, 3, 4 and 5. *Instant infinite series, just add water.*

Questions!

Warm up brain

Here's a problem to try, to get your brain in the slippery place that lets us manipulate these series: find the Taylor expansion of $\frac{1}{2-x}$ on the interval $[0, 2]$.

Hint: let $y = \frac{1}{2}x$, so that the function becomes $\frac{1}{2-2y} = \frac{1}{2}\frac{1}{1-y}$. Yadda yadda yadda?

We can do all sorts of operations with Taylor series:

If $f(x) = \sum a_n x^n$ and $g(x) = \sum b_n x^n$ are the Taylor series expansion, and these expansions both converge on a given interval $[0, c]$, then the Taylor series for $f(x) + g(x)$ on the inverval $[0, c]$ is $\sum (a_n + b_n) x^n$.

If $f(x) = \sum a_n x^n$ is the Taylor series, convergent on $[0, c]$, then the derivative of $f(x)$ on the open interval $(0, c)$ is given by $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$.

If $f(x) = \sum a_n x^n$ is the Taylor series convergent on $[0, c]$, then an antiderivative is given by $\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$.

If $f(x) = \sum a_n x^n$ and $g(x) = \sum b_m x^m$ are both convergent Taylor series on $[0, c]$, then the Taylor series of $f(x)g(x)$ on that interval is given by $\sum_{n=0}^{\infty} (\sum_{p+q=n} a_p b_q) x^n$ (multiply like polynomials!).

Let's take a couple of these babies out for a spin.

Addition

Find the Taylor series for the function

$$f(x) = \frac{1}{x^2 - 3x + 2}$$

on the interval $[0, 1]$.
Thoughts?

Partial fractions:

$$\frac{1}{x^2 - 3x + 1} = \frac{1}{(2-x)(1-x)} = \frac{1}{2-x} - \frac{1}{1-x}.$$

Independent Taylor expansions: $\frac{1}{2-x} = \sum \frac{1}{2^{n+1}} x^n$; $\frac{1}{1-x} = \sum x^n$.

Assemble: $f(x) = \sum \frac{1}{2^{n+1}} x^n - \sum x^n = \sum (\frac{1}{2^{n+1}} - 1) x^n = \sum \frac{1-2^{n+1}}{2^{n+1}} x^n$.

Do one!

Compute the Taylor series of $\sin(x) + \cos(x)$ near $x = 0$.

Integration

Remember how integration is hard? If we have Taylor series at our disposal, we can often give some sort of expression for the integral of a function.

Example: calculate $\int_0^t \sin(x^2)dx$.

Well, $\sin(y) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} y^{2n+1}$, so $\sin(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{4n+2}$ (substitute x^2 for y).
This converges everywhere.

Integration

Thus,

$$\begin{aligned}
\int_0^t \sin(x^2)dx &= \int_0^t \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, x^{4n+2} dx \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+3)(2n+1)!}\, x^{4n+3}\Big|_0^t \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+3)(2n+1)!}\, t^{4n+3}.
\end{aligned}$$

This may (does!) look inscrutable, but the point is that it is a formula that we could use to compute that integral if we had to.

Try one!

Find an infinite series expansion for the integral $\int_1^t e^{1/t}dt$. For which t does it converge?

Tomorrow: *exam!*