Lecture 23

Behind us

- Area as a double integral
- Center of mass: squashed squirrel

You survived Midterm 2!!

Final on 3/16 in KNE 110 1:30 PM to 4:20 PM (no exceptions or rescheduling)

Ahead

Today: errors in linear approximation

Next: higher order approximation, Taylor polynomials

Read the Taylor notes sections 1, 2, and 3. *You never know, your professor might give a totally incorrect lecture!*

Questions!

Nedwina and the piglet play the market

It's 1999, right before the bursting of the tech stock bubble, and Nedwina and the piglet of calculus have stock market fever.

(This is not a good sign.)

Their friend, the Slimy Stock Snail, tells them that the best linear approximation to the Dow Jones Industrial Average is

$$f(t) = 10000 + 3000t,$$

where t is the time in days from now.

What could go wrong?

More precisely:

What kinds of additional information do Nedwina and the piglet need to evaluate the accuracy of the linear approximation?

Can they be sure that the projection will be within of the correct value in one year?

Examples

Let's start with some simple minded examples. Here are a bunch of functions with derivative 0 at $t = 0$:

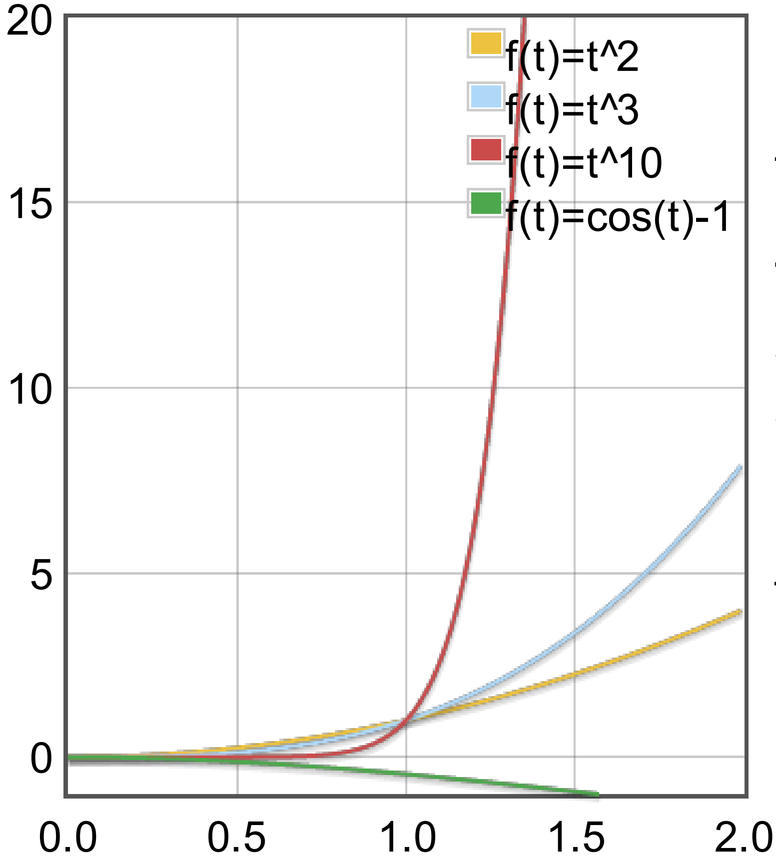
- $f(t) = t^2$
- $f(t) = t^3$
- $f(t) = \cos(t) - 1$
- $f(t) = t^{10}$

In each case, the best linear approximation around $t = 0$ is the function $L(t) = 0$.

How can we capture the error in this approximation at $t = 2$, i.e., how can we bound the absolute value $|L(1) - f(1)|$?

Photographic evidence

Here's a graph showing all of the functions between $t = 0$ and $t = 2$



None of the functions is especially close, but some are much worse than others.

The "rate of climb" of each graph seems to predict how far off the approximation is.

The pig is tingling: grasping the first derivative seems to help bound errors?

If you know more, you actually more, I

How about we start with a crudely wonderful estimate of values of the function:

Theorem: if there is a number M such that $|f'(x)| \leq M$ for all x between b and c then

$$|f(c) - f(b)| \leq M|c - b|.$$

Indication of proof:

$$|f(c) - f(b)| = |\int_b^c f'(x)dx| \leq |\int_b^c |f'(x)|dx| \leq |\int_b^c Mdx| = M|c - b|.$$

Example: $f(x) = \sin(x)$: $|f'(x)| \leq 1$ for all x, so $|f(c) - f(b)| \leq |c - b|$ for all b, c.

How good an estimate is this?

If you know more, you actually know more, II

If we know about more derivatives, we get *EVEN MORE*.

Given a differentiable function $f(t)$ and a point b where f is defined, the best linear approx to $f(t)$ near b is $L_b(t) = f(b) + f'(b)(t-b)$.

Theorem: if M is a number such that $|f''(x)| \leq M$ for all x between b and c then $|L_b(c) - f(c)| \leq \frac{1}{2} M|c-b|^2$.

Examples

$f(t) = t^{10}$: $|f''(t)| \leq 10 \cdot 9 \cdot 2^8 = 23040$ for $0 \leq t \leq 2$, so $|L_0(2) - f(2)| \leq 46080$ (ugh?)

$f(t) = \sin(t)$: $|f''(t)| \leq 1$ for all t, so for any value c we have $|L_b(c) - f(c)| \leq \frac{1}{2} |c - b|^2$.

When (for which b) is this estimate of the error in the linear approximation to $\sin$ good? When is it terrible?

Do one!

In our example: $L_0(t) = 10000 + 3000t$.

> What bound on $|f''(x)|$ for $0 \leq x \leq 730$ will ensure Nedwina and the piglet that the Stock Snail's estimate is within 5000 of the DJIA by the end of year 2 (day 730)?
>
> What bound on $|f''(x)|$ for $0 \leq x \leq 730$ will ensure that Nedwina and the piglet do not see the DJIA end up **below** 5000 by year 2 (day 730)?

Do one backwards!

Consider the function $f(t) = \cos(t)$. Find a number b such that on the interval $[\pi/2, c]$, the linear approximation $L_{\pi/2}(t)$ to $f(t)$ is always within 0.01 of $f(t)$.

Hint: try to bound the second derivative, and combine this with the theorem: $|f''(x)| \leq M$ implies $|L_b(x) - f(x)| \leq \frac{1}{2} M|x-b|^2$.

Do another one!

Do the same deal for the function $f(t) = 1/t$, approximated near $t = 1$ with an accuracy of 10^{-7}.

Theoretical question

Given a twice continuously differentiable function f, what if we know that for every value $t = b$, there is an interval $[b, c]$ such that $f''(x) = 0$ for $x \in [b, c]$? Can we say (or conjecture) anything about the structure of f?

Next time: *better approximations with polynomials of higher degree!*