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HW2, 3. Proof of Beta-Binominal conjugation.

1° we assume the pdf of Binominal (θ) follows Beta distribution

$\Rightarrow$ where $Beta(\theta|a,b) = \theta^{a-1}(1-\theta)^{b-1} \cdot \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} = \theta^{a-1}(1-\theta)^{b-1} \cdot \frac{1}{\beta(a,b)}$ ↙ Beta func.

and $\beta(a,b) = \int_0^1 \theta^{a-1}(1-\theta)^{b-1}\, d\theta.$

2° posterior $= \frac{\text{likelihood} \times \text{prior}}{\text{marginal}}$ $\Rightarrow p(\theta|x) = \frac{\overset{\text{binominal}}{P(X|\theta)} \cdot \overset{\text{Beta distribution}}{P(\theta)}}{P(X)}$

where m = num of 1
p = prob of 1

$$\Rightarrow f(\theta|x) = \frac{\cancel{\binom{N}{m}} p^m (1-p)^{N-m} \cdot p^{a-1}(1-p)^{b-1} \cancel{\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}}}{\int_0^1 \cancel{\binom{N}{m}} \theta^m (1-\theta)^{N-m} \theta^{a-1}(1-\theta)^{b-1} \cancel{\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}}\, d\theta}.$$

$$= \frac{p^m(1-p)^{N-m} \cdot p^{a-1}(1-p)^{b-1}}{\int_0^1 \theta^m (1-\theta)^{N-m} \theta^{a-1}(1-\theta)^{b-1}\, d\theta}$$

$$= \frac{p^{a+m-1}(1-p)^{b+N-m-1}}{\int_0^1 \theta^{a+m-1}(1-\theta)^{b+N-m-1}\, d\theta}.$$ ↙ $\beta(a+m, b+N-m)$

$= Beta(a+m, b+N-m)$ ⇝ same as the distribution of our prior $P(\theta)$ ∦